THE JOURNAL OF PHILOSOPHY
VOLUME CXVI, NO. 6, JUNE 2019

A THEORY OF CONDITIONAL ASSERTION*

According to one tradition, uttering an indicative conditional involves a special kind of speech act: a conditional assertion.

> An affirmation of the form 'if *p* then *q*' is [not] an affirmation of a conditional [but] a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent. . . . [If] the antecedent turns out to have been false, our conditional affirmation is as if it had never been made.[1]

The slogan above is intuitive. Quine took it to be the usual, "everyday" attitude. Yet one way or another, conditional assertion is not one of the dominant theories of conditionals today. This is for many reasons, perhaps best summed up here:

> We believe these are mostly problems with understanding the view in the first place—understanding just what conditional assertions are, how they operate, and how they interact with the common machinery of cur-

* This research was supported by Research Grants Council Grant #23602118.

[1] W. V. Quine, *Methods of Logic*, rev. ed. (New York: Holt, 1959), p. 12. For further discussion, see G. H. von Wright, *Logical Studies* (New York: Humanities Press, 1957), p. 131; and J. L. Mackie, *Truth, Probability, and Paradox: Studies in Philosophical Logic* (Oxford: Clarendon Press, 1973). For developments of the view, see Richard C. Jeffrey, "On Indeterminate Conditionals," *Philosophical Studies*, XIV, 3 (April 1963): 37–43; Nuel D. Belnap Jr., "Conditional Assertion and Restricted Quantification," *Noûs*, IV, 1 (February 1970): 1–12; Nuel D. Belnap Jr., "Restricted Quantification and Conditional Assertion," in Hugues Leblanc, ed., *Truth, Syntax, and Modality: Proceedings of the Temple University Conference on Alternative Semantics* (Amsterdam, the Netherlands: North-Holland Publishing Company, 1973), pp. 48–75; Ruth Manor, "A Semantic Analysis of Conditional Assertion," *Journal of Philosophical Logic*, III, 1–2 (March 1974): 37–52; Stephen J. Barker, "Towards a Pragmatic Theory of 'If'," *Philosophical Studies*, LXXIX, 2 (August 1995): 185–211; Peter Milne, "Bruno de Finetti and the Logic of Conditional Events," *British Journal for the Philosophy of Science*, XLVIII, 2 (June 1997): 195–232; and Keith DeRose, "Can It Be That It Would Have Been Even Though It Might Not Have Been?," *Philosophical Perspectives*, XIII (1999): 387–413, among others.

0022-362X/19/1606/293–318

> rent philosophy of language—machinery that was not set up with such a speech act in view. . . .It can also seem problematic what kind of logic conditionals could have if they were devices of conditional assertion.[2]

The crucial task in giving an account of conditional assertion is to model the intuitive ideas above using the tools of contemporary semantics and pragmatics. Several attempts to do so have been judged "immediately implausible" or incoherent.[3] For example, several leading accounts of conditional assertion imply that when the antecedent of a conditional is false, the conditional fails to express a proposition.[4] Here, conditional assertion is analogous to handing someone a sealed envelope containing a record of the consequent, with instructions to open it only if the antecedent is true.[5] Yet there are many cases where this seems wrong. Consider:

(1) If you don't finish mowing the yard this afternoon, you can't go to the mall after dinner.

Lycan observes that the addressee learns something immediately upon hearing this conditional.[6] They need not wait to learn whether the antecedent is true to determine whether anything has been expressed. Moreover, they cannot control whether the speaker fails to express a proposition simply by mowing the lawn.

It is unclear, however, whether the problems above are essential to the theory of conditional assertion. In fact, a wide range of theorists have adopted the slogan of conditional assertion in some way or other. Several theorists have offered implementations of the ideas

[2] See Keith DeRose and Richard E. Grandy, "Conditional Assertions and 'Biscuit' Conditionals," *Noûs*, XXXIII, 3 (September 1999): 405–20, at p. 408.

[3] See, for example, Michael Dummett, *Frege: Philosophy of Language* (London: Duckworth, 1973); and William G. Lycan, "Conditional-Assertion Theories of Conditionals," in Judith Thomson and Alex Byrne, eds., *Content and Modality: Themes from the Philosophy of Robert Stalnaker* (Oxford: Oxford University Press, 2006), pp. 148–64.

[4] Belnap, "Conditional Assertion and Restricted Quantification," *op. cit.*, develops a version of this theory in possible worlds semantics. The following authors all explore a view where the conditional is truth-valueless when the antecedent is false: Jeffrey, "On Indeterminate Conditionals," *op. cit.*; Manor, "A Semantic Analysis of Conditional Assertion," *op. cit.*; Robert Stalnaker, "Indicative Conditionals," *Philosophia*, V, 3 (July 1975): 269–86, note 2; Michael McDermott, "On the Truth Conditions of Certain 'If'-Sentences," *The Philosophical Review*, CV, 1 (January 1996): 1–37; Janneke Huitink, "Partial Semantics for Iterated *If*-Clauses," in Arndt Riester and Torgrim Solstad, eds., *Proceedings of Sinn und Bedeutung 13* (Stuttgart, Germany: OPUS, 2009), pp. 203–16; and Janneke Huitink, "Domain Restriction by Conditional Connectives" (July 2009), https://www.semanticsarchive.net/Archive/zg2MDM4M/Huitink-domainrestriction.pdf.

[5] See Jeffrey, "On Indeterminate Conditionals," *op. cit.*

[6] Lycan, "Conditional-Assertion Theories of Conditionals," *op. cit.*

above that involve a failure of truth value in the case when the antecedent is false.[7] Others instead implement the crucial ideas behind conditional assertion using a "No Truth Value" (NTV) theory.[8] Stalnaker argues that the material conditional itself satisfies most of the goals of conditional assertion, and then argues that his own theory of the indicative does too.[9] Starr shows how dynamic conditionals can accomplish much of the same.[10]

To summarize: conditional assertion is the usual, everyday account of conditionals; and it is immediately implausible. It has been largely ignored by the literature; and is also a component of every leading theory of conditionals. With this much disagreement on what conditional assertion even is, it is worth taking a step back and searching for a more precise definition.

This is a goal worth pursuing, for at least a few reasons. First, the phenomenon of conditional assertion stands at the intersection of semantics and pragmatics. The intuitive idea behind conditional assertion suggests that conditionals are in some way importantly different than other bits of language. Instead of having straightforward truth conditions, conditionals are understood in terms of the characteristic effects that they have on the commitments of conversational participants. Ultimately, we will argue that the vague, pragmatic characterization of conditional assertion with which we began can be regimented in a surprisingly precise way, which leaves room for just a small family of well-behaved semantics for the conditional. This paper will thus offer a new method of inquiring into the meaning of an expression. Instead of starting with our intuitions about when various sentences are true or false, or when various inferences are true or false, we start merely with a vague sense of how the commitments of conversational participants are affected by conditional talk. From this starting point, we will land within an interesting class of potential meanings for the conditional.

[7] Jeffrey, "On Indeterminate Conditionals," *op. cit.*; Belnap, "Conditional Assertion and Restricted Quantification," *op. cit.*; Manor, "A Semantic Analysis of Conditional Assertion," *op. cit.*; and McDermott, "On the Truth Conditions of Certain 'If'-Sentences," *op. cit.*

[8] Dorothy Edgington, "On Conditionals," *Mind*, CIV, 414 (April 1995): 235–329; and Barker, "Towards a Pragmatic Theory of 'If'," *op. cit.*

[9] Robert Stalnaker, "Conditional Propositions and Conditional Assertions," in Andy Egan and Brian Weatherson, eds., *Epistemic Modality* (New York: Oxford University Press, 2011), pp. 227–48.

[10] William B. Starr, "Indicative Conditionals, Strictly," unpublished manuscript (February 2014).

Second, conditional assertion is one member of a family of philosophically interesting conditional states, including conditional belief, conditional intention, conditional questions, and conditional promises.[11] A good theory of conditional assertion might well serve as the first step in a broader theory of these other conditional attitudes. As we will see, the model we rely on in what follows is general enough to be of use in modeling these other states as well.

This paper offers a new theory of conditional assertion. We first develop a formal framework that can model speech acts like conditional assertion. This framework is ecumenical, usable even by those who are skeptical of a semantic treatment of speech acts. All we assume is that we have a set of contexts, that agents are committed to certain claims in context, and that utterances change the context. We can then use this model to provide a precise characterization of what conditional assertion is.

Our characterization of conditional assertion is weak enough to be consistent with a broad range of theories. It does not have any immediately paradoxical conclusions, such as denying that conditionals with false antecedents express propositions. Nonetheless, our theory is still strong enough to be interesting. Our first result is that relying on just a few background assumptions, we can prove that any theory of conditional assertion in our sense implies the validity of several inferences involving conditionals. First, conditional assertion requires the validity of Modus Ponens. Second, conditional assertion requires the validity of one of the paradoxes of material implication: that the negation of the antecedent implies the conditional.

With these logical results under our belt, we turn to semantic implementations of conditional assertion. First, we consider whether the conditional-assertion theory can be implemented within a truth-conditional theory of conditionals. We will see that in this setting, conditional assertion requires that the conditional is almost truth functional. The conditional is true whenever the antecedent is false, and false whenever the antecedent is true and the consequent is false. But the conditional's truth is unsettled when the antecedent and consequent are both true.

[11] For conditional belief, see Robert Stalnaker, *Inquiry* (Cambridge, MA: MIT Press, 1984); Peter Gardenförs, *Knowledge in Flux: Modeling the Dynamics of Epistemic States* (Cambridge, MA: MIT Press, 1988); and Hannes Leitgeb, *The Stability of Belief: How Rational Belief Coheres with Probability* (Oxford: Oxford University Press, 2017). For conditional intentions, see J. P. W. Cartwright, "Conditional Intention," *Philosophical Studies*, LX, 3 (November 1990): 233–55; Luca Ferrero, "Conditional Intentions," *Noûs*, XLIII, 4 (December 2009): 700–41; and Kirk Ludwig, "What Are Conditional Intentions?," *Methode: Analytic Perspectives*, IV, 6 (2015): 30–60. For conditional questions, see William B. Starr, "What 'If'?," *Philosophers' Imprint*, XIV, 10 (April 2014): 1–27.

After considering truth-conditional implementations, we turn to dynamic semantics.[12] We identify a new family of dynamic conditionals that satisfy the requirements of conditional assertion. On all these theories, the conditional tests whether learning the antecedent suffices to accept the consequent. However, these theories differ in what happens when this test fails. Extant dynamic conditionals predict that in cases of failure, we reach an absurd state.[13] By relaxing this assumption, we generate a family of conditional-assertion theories. We let failure of the test create a state containing all worlds from the context in which the antecedent is false, and then some. The result is a new kind of dynamic meaning.

I. CONDITIONAL ASSERTION

We model a formal language $\mathscr{L}$ consisting of an ordinary propositional language $\mathscr{L}^0$, supplemented by an indicative-conditional operator $\rightarrow$. We focus on conditionals that do not themselves contain another conditional.[14]

> *Definition 1.* Let $\mathscr{A}$ be the set of atomic sentences α. Let $\mathscr{L}^0$ be a language containing $\mathscr{A}$ and closed under negation $\neg$ and conjunction $\wedge$. Let $\mathscr{L}$ be the smallest set that contains $\mathscr{L}^0$ and that, for any sentences A and B in $\mathscr{L}^0$, contains the indicative conditional $\mathsf{A} \rightarrow \mathsf{B}$ and its negation $\neg(\mathsf{A} \rightarrow \mathsf{B})$. Let $\vee$ and $\supset$ abbreviate $\neg(\neg\mathsf{A} \wedge \neg\mathsf{B})$ and $\neg(\mathsf{A} \wedge \neg\mathsf{B})$.

We assume that utterances take place in context, which we represent with the variable s. We let $[\cdot]$ be a function modeling the effect of performing a speech act in context. So an utterance of A in s generates

[12] For some defenders of dynamic semantics, see Frank Veltman, "Defaults in Update Semantics," *Journal of Philosophical Logic*, XXV, 3 (June 1996): 221–61; Jeroen Groenendijk, Martin Stokhof, and Frank Veltman, "Coreference and Modality," in Shalom Lappin, ed., *Handbook of Contemporary Semantic Theory* (Oxford: Blackwell, 1996), pp. 179–213; David I. Beaver, *Presupposition and Assertion in Dynamic Semantics* (Stanford, CA: CSLI Publications, 2001); Anthony S. Gillies, "Epistemic Conditionals and Conditional Epistemics," *Noûs*, XXXVIII, 4 (December 2004): 585–616; and Malte Willer, "Dynamics of Epistemic Modality," *The Philosophical Review*, CXXII, 1 (January 2013): 45–92. For some nearby views, see Seth Yalcin, "Epistemic Modals," *Mind*, CXVI, 464 (November 2007): 983–1026; Eric Swanson, "How Not to Theorize about the Language of Subjective Uncertainty," in Egan and Weatherson, eds., *Epistemic Modality*, *op. cit.*, pp. 249–69; Eric Swanson, "The Application of Constraint Semantics to the Language of Subjective Uncertainty," *Journal of Philosophical Logic*, XLV, 2 (April 2016): 121–46; and Sarah Moss, "On the Semantics and Pragmatics of Epistemic Vocabulary," *Semantics and Pragmatics*, VIII, 5 (2015): 1–81.

[13] Gillies, "Epistemic Conditionals and Conditional Epistemics," *op. cit.*

[14] The language below allows embeddings of conditionals under negation. This may seem worrisome for some defenders of conditional assertion, since it requires negating a conditional speech act. Fortunately, none of our results rely especially on this feature of the language. In particular, in Fact 1 the inference $\mathsf{A}; \neg\mathsf{B} \models \neg(\mathsf{A} \rightarrow \mathsf{B})$ could be replaced with $\mathsf{A}; \mathsf{A} \rightarrow \mathsf{B} \models \mathsf{B}$.

a new context $s[\mathsf{A}]$. If A is a declarative sentence (in $\mathscr{L}^0$), $s[\mathsf{A}]$ models the effect of asserting A in s. By contrast, $s[\mathsf{A} \rightarrow \mathsf{B}]$ models the effect of conditionally asserting B given A. Finally, we assume that within any context the relevant agents are committed to some claims. We use $\models$ to express this relation, so that $s \models \mathsf{A}$ just in case the participants of context s are committed to A. Finally, we call a context absurd just in case it is committed to every claim.[15]

Definition 2. Let s be a context. Let C be the set of contexts. Let $[\cdot]$ be a function from sentences in $\mathscr{L}$ to functions from C to C. Let $\models$ be a relation between contexts s and sentences A in $\mathscr{L}$. s is absurd just in case $s \models \mathsf{A}$ for every $\mathsf{A} \in \mathscr{L}$.

To start with, we make no assumptions about the exact structure of contexts. For now, our goal is to take the intuitive idea behind conditional assertion and see how far this idea alone takes us toward a precise theory of conditionals.

Given the framework we have adopted, the theory of conditional assertion can be expressed as two constraints on $[\mathsf{A} \rightarrow \mathsf{B}]$. Suppose that $\mathsf{A} \rightarrow \mathsf{B}$ is uttered in context s. This moves us to state $s[\mathsf{A} \rightarrow \mathsf{B}]$. The theory of conditional assertion tells us what happens if the antecedent A turns out true, and also what happens if the antecedent turns out false. We model these two scenarios using $[\mathsf{A}]$ and $[\neg\mathsf{A}]$. So the conditional-assertion theory tells us something about both $s[\mathsf{A} \rightarrow \mathsf{B}][\mathsf{A}]$ and $s[\mathsf{A} \rightarrow \mathsf{B}][\neg\mathsf{A}]$.

In the former case, the theory says that we are committed to the consequent. Since we are using $\models$ to model commitment, this means that $s[\mathsf{A} \rightarrow \mathsf{B}][\mathsf{A}] \models \mathsf{B}$. In other words: given any body of information, if the conditional and its antecedent are added to the information, then the new information contains the consequent.

In the second case, where the antecedent turns out false, our original conditional assertion is "as if it had never been made." We model this last idea through the identity $s[\mathsf{A} \rightarrow \mathsf{B}][\neg\mathsf{A}] = s[\neg\mathsf{A}]$. In other words: when a state is updated with the conditional and then the negation of its antecedent, the state might as well just have been updated with the antecedent's negation immediately. We can also think about this last requirement more generally as a thesis about what it is to learn a conditional. No matter what information an agent has, if they first learn the conditional and then learn that the antecedent

[15] Here we rely on many of the formal tools introduced within update semantics. For some representative samples, see Veltman, "Defaults in Update Semantics," *op. cit.*; Beaver, *Presupposition and Assertion in Dynamic Semantics, op. cit.*; and Gillies, "Epistemic Conditionals and Conditional Epistemics," *op. cit.*

is false, they arrive in the same state as if they had just learned that the antecedent were false from the beginning. Learning $\neg\mathsf{A}$ screens off learning $\mathsf{A} \to \mathsf{B}$.[16] Putting these two ideas together, we reach the following theory:

Definition 3. $\to$ is a conditional-assertion operator iff:

1. $s[\mathsf{A} \to \mathsf{B}][\mathsf{A}] \models \mathsf{B}$.
2. $s[\mathsf{A} \to \mathsf{B}][\neg\mathsf{A}] = s[\neg\mathsf{A}]$.

In the rest of this paper, we explore the implications of the definition above for the theory of conditionals. But before we do this, it is worth flagging a few points where we could have made a different choice.

First, when the antecedent turns out true, we said that we are committed to the consequent. But here we might replace the notion of commitment with that of actually uttering the consequent. Then we would say that uttering a conditional has the same effect as uttering the consequent whenever the antecedent turns out true. That gives us the equation $s[\mathsf{A} \to \mathsf{B}][\mathsf{A}] = s[\mathsf{A}][\mathsf{B}]$. This condition is significantly stronger than our own. In particular, we show later that our own condition is compatible with the failure of $\mathsf{A} \wedge \mathsf{B}$ to imply $\mathsf{A} \to \mathsf{B}$. This inference's validity is guaranteed by the stronger condition here.

Second, in both clauses above we analyzed the notion of the antecedent turning out true or false in a particular way: in terms of updating the context with the antecedent. However, one might distinguish the bare truth or falsity of the antecedent from its being learned. In that case, one might say that $s[\mathsf{A} \to \mathsf{B}] \models \mathsf{B}$ whenever A is true, and $s[\mathsf{A} \to \mathsf{B}] = s$ whenever A is false. Something like this idea

[16] Opponents of conditional assertion have sometimes observed that any screening-off condition like the above faces an immediate challenge. See Lycan, "Conditional-Assertion Theories of Conditionals," *op. cit.* For example, suppose that a certain politician utters (i):

(i) If Congress passes a health-care bill, I will sign it.

Even if Congress fails to pass a health-care bill, (i) appears to still have an effect on the context. For example, (i) may influence what future bills may be brought before Congress. One natural way for defenders of conditional assertion to deal with such cases is by appealing to secondary effects of assertion (Robert Stalnaker, "Assertion," in Peter Cole, ed., *Syntax and Semantics, Volume 9: Pragmatics* (New York: Academic Press, 1978), pp. 315–32). In addition to the ordinary effects of uttering a conditional, any competent conversational interlocutor can also infer that when a conditional like (i) is uttered, the speaker takes themselves to satisfy the normative requirements on uttering. For example, in the case of (i) this requires the speaker to have certain intentions with respect to health-care repeal in general. Even if the possession of such intentions is not part of the actual meaning of the conditional above, such information can still be gleaned from an utterance of it.

is suggested by Belnap, who argued that the proper interpretation of our second condition requires the conditional to fail to express a proposition:

> If the conditional asserted something according to semantics, we could not on the pragmatic level treat its utterance as if it had never happened.[17]

We are now in a position to see why this is too strong. The conditional can assert something when the antecedent is false, as long as whatever it asserts is screened off by the assertion of the antecedent's negation. In this case, the conditional assertion might as well never have happened.

One concern with the gappy theory above, compared with our own, is that it requires a conditional's effect on context to depend on something other than the context itself: the antecedent's actual truth value. This idea is hard to make sense of in the model above. In addition, it is hard to know how conversational participants could implement this idea when s leaves the antecedent unsettled. That is, this idea violates Stalnaker's uniformity principle, on which a sentence should have the same effect on context at every world consistent with the information of the context.[18]

Our definition of conditional assertion is quite different from some of the traditional accounts, especially those involving truth-value gaps. Ultimately, the results below serve as our argument for thinking of conditional assertion this way, rather than as it has been interpreted before. Whether or not this is what others have had in mind when talking about conditional assertion, we show that our definition is fruitful, characterizing a new and well-behaved region of semantic and logical space.

Before jumping in to the details of the theory, it is worth flagging a few benefits of the account right from the start. First, the account avoids the "initial implausibility" objection to earlier theories of conditional assertion.[19] Our definition in no way requires that the conditional is meaningless, truth-valueless, or fails to express a proposition when the antecedent is false. Rather, we understand the falsity condition in a dramatically different way, as a screening-off condition. That learning $\neg\mathsf{A}$ screens off the information in $\mathsf{A} \to \mathsf{B}$ is perfectly

[17] Belnap, "Restricted Quantification and Conditional Assertion," *op. cit.*, p. 49. See also Belnap, "Conditional Assertion and Restricted Quantification," *op. cit.*

[18] Stalnaker, "Assertion," *op. cit.*

[19] For example, those in Jeffrey, "On Indeterminate Conditionals," *op. cit.*; and Belnap, "Conditional Assertion and Restricted Quantification," *op. cit.*

consistent with the fact that we can use conditionals to communicate information even when as a matter of fact the antecedent is false.

Another initial benefit of the theory is that it is in several senses ecumenical. First, several proponents of conditional assertion have accepted an NTV view of conditionals, denying that they express propositions that can be assigned a truth value. While the theory above is consistent with conditionals expressing propositions, it does not require this. So our theory provides a neutral framework within which propositional and NTV proponents alike can investigate the consequences of conditional assertion.

The theory is also ecumenical in another way. Conditional assertion has primarily been thought about within the theory of speech acts. But our set of contexts above could be anything. In particular, we can interpret contexts simply as the possible beliefs that a rational agent might possess. Then $[\cdot]$ is the rule by which an agent updates her beliefs over time, and $s \models \mathsf{A}$ says that an agent in state s believes A. Under this interpretation, the theory above defines an epistemic concept of conditional learning, rather than a pragmatic concept of conditional assertion. The results below double as a characterization of this special kind of learning.

Finally, the theory is ecumenical within the theory of pragmatics. One natural goal with conditional assertion is to provide a general account of the conditional that connects conditional assertion with other speech acts like conditional promising, questioning, and commanding.[20] Again, because our definition of a context is completely neutral, our framework can also be used to model these other conditional speech acts. For example, we might represent a context as a stack of questions under discussion, as a to-do list of commands, or with all of this information.[21] In each case, our definition of conditional assertion above generalizes straightforwardly to other conditional speech acts. For example, on the definition above a conditional command would add the consequent's command to the to-do list of a state when the antecedent is learned, but would have no effect on the to-do list of a context if the antecedent's negation is learned.

We now have a precise statement of what conditional assertion is, along with a rough sense of how it compares to a few alternatives. In

[20] See Barker, "Towards a Pragmatic Theory of 'If'," *op. cit.*; and Lycan, "Conditional-Assertion Theories of Conditionals," *op. cit.*, for discussion, among others.

[21] See Craige Roberts, "Information Structure in Discourse: Towards an Integrated Formal Theory of Pragmatics," *Semantics and Pragmatics*, v, 6 (2012): 1–69; Paul Portner, "Imperatives and Modals," *Natural Language Semantics*, xv, 4 (December 2007): 351–83; and Sarah Murray and William B. Starr, "The Structure of Communicative Acts," *Linguistics and Philosophy* (forthcoming), for examples.

the rest of this paper, we explore the implications of this model for the logic, truth conditions, and conversational dynamics of conditionals.

II. THE LOGIC OF CONDITIONAL ASSERTION

Our first goal is to investigate the logic of conditional-assertion operators. In particular, we introduce a few minimal assumptions and use them to show that any conditional-assertion operator must validate several inferences, including Modus Ponens and the False Antecedent inference.

Our model contains a rule for updating contexts with an utterance, and also a representation of the commitments that each context contains. These two features of our model allow us to formulate a definition of entailment. In particular, let us say that an argument is valid just in case we are committed to its conclusion whenever we utter the premises.[22]

Definition 4. $\mathsf{A}_1, \ldots, \mathsf{A}_n \models \mathsf{B}$ iff $\forall s\ s[\mathsf{A}_1] \ldots [\mathsf{A}_n] \models \mathsf{B}$.

The model above is ecumenical. It need not be interpreted as a semantic theory of conditionals. We could instead think of it merely as a formal pragmatics, describing how various speech acts affect the information of speakers. In that case, we can think of entailment as a formal model of how information changes in systematic ways when speech acts are performed. That being said, we also show in later sections that the model can be interpreted truth conditionally. In that setting, this definition coincides with preservation of truth.

To prove our results, we need to add a few more assumptions. Basically, we assume that the fragment of our language not containing conditionals is conservative, obeying a variety of structural rules that bring it close to behaving truth conditionally. But we make no assumptions about the behavior of conditionals. First, our entailment relation allows us to make a few well-behavedness assumptions about negation. Without accepting any specific semantics for negation, we can commit to the law of noncontradiction. In our framework, this requires that any state is committed to anything after being updated with both of A and $\neg\mathsf{A}$. In addition, we assume that any coherent context that is not committed to A will remain coherent after an utterance of $\neg\mathsf{A}$.

Definition 5. $\neg$ is well behaved only if:

[22] See Johan van Benthem, *Exploring Logical Dynamics* (Stanford, CA: CSLI Publications, 1996), for discussion. Our results below also hold for another popular definition of entailment, that an argument is valid just in case whenever we are committed to all the premises, we are committed to the conclusion.

1. $\mathsf{A}; \neg\mathsf{A} \models \mathsf{B}$.
2. If $s \not\models \mathsf{A}$, then $s[\neg\mathsf{A}]$ is not absurd.

We also need several assumptions about updating. First, we rely on a strong principle linking commitment and updating. We suppose that the participants of a context are committed to something just in case uttering it leaves their state unchanged. Second, we suppose that updating a context always gives us more information. More precisely, we suppose that we can never get back to s by uttering two sentences A and B, at least one of which changes the state.[23]

Definition 6. $\models$ and $[\cdot]$ are well behaved only if:

1. $s[\mathsf{A}] = s$ iff $s \models \mathsf{A}$.
2. If $s[\mathsf{A}] \neq s$ or $s[\mathsf{A}][\mathsf{B}] \neq s[\mathsf{A}]$, then $s[\mathsf{A}][\mathsf{B}] \neq s$.

Finally, we focus on sentences that are well behaved in a few ways. First, we only consider antecedents and consequents that preserve commitment under updating.[24] Second, we focus on antecedents that are idempotent, so that any context is committed to them after updating with them.

Definition 7.

1. A is persistent iff for any s, B, whenever $s \models \mathsf{A}$, $s[\mathsf{B}] \models \mathsf{A}$.
2. A is idempotent iff for any s, $s[\mathsf{A}] \models \mathsf{A}$.

With all of these assumptions in place, we now show that the conditional-assertion theory imposes a significant amount of truth functionality on the conditional. To see why, note that we can use our definition of entailment above to express a version of truth functionality.[25] The conditional is truth functional just in case a commitment to whether A and B each hold is sufficient to generate a commitment on whether $\mathsf{A} \rightarrow \mathsf{B}$:

Definition 8. $\rightarrow$ is truth functional iff:

1. $\mathsf{A}; \mathsf{B} \models \mathsf{A} \rightarrow \mathsf{B}$ or $\mathsf{A}; \mathsf{B} \models \neg(\mathsf{A} \rightarrow \mathsf{B})$, and

[23] See Jeroen Groenendijk and Martin Stokhof, "Two Theories of Dynamic Semantics," in Jan van Eijck, ed., *Logics in AI: European Workshop JELIA '90 Amsterdam, The Netherlands, September 10–14, 1990 Proceedings*, Lecture Notes in Computer Science, vol. 478 (Berlin: Springer-Verlag, 1991), pp. 55–64; and Daniel Rothschild and Seth Yalcin, "On the Dynamics of Conversation," *Noûs*, LI, 1 (March 2017): 24–48, for discussion.

[24] See Veltman, "Defaults in Update Semantics," *op. cit.*, for discussion. We do not require that the conditional or its negation is persistent.

[25] See Kit Fine, "The Possibility of Vagueness," *Synthese*, CXCIV, 10 (October 2017): 3699–725, for discussion.

2. $\neg A; B \models A \rightarrow B$ or $\neg A; B \models \neg(A \rightarrow B)$, and
3. $A; \neg B \models A \rightarrow B$ or $A; \neg B \models \neg(A \rightarrow B)$, and
4. $\neg A; \neg B \models A \rightarrow B$ or $\neg A; \neg B \models \neg(A \rightarrow B)$.

This notion of truth functionality does not assume that the conditional has truth conditions. Rather, this is a commitment-theoretic notion of truth functionality, requiring that our commitment to the conditional supervenes on our commitments to its parts. Again, we show later that all the key notions here can be interpreted truth conditionally, in which case the definition above coincides with the usual one.

We now show that any conditional-assertion operator is at least three quarters truth functional. A commitment to $\neg A$ implies a commitment to $A \rightarrow B$, and a commitment to A and to $\neg B$ implies a commitment to $\neg(A \rightarrow B)$. Nonetheless, this leaves open the possibility that a state could be committed to A and to B while remaining agnostic on whether $A \rightarrow B$.

Fact 1. Suppose $\models$, $[\cdot]$, and $\neg$ are well behaved, A and $\neg B$ are persistent, and $\neg A$ is idempotent. Suppose $\rightarrow$ is a conditional-assertion operator. Then:

1. $A; \neg B \models \neg(A \rightarrow B)$.
2. $\neg A \models A \rightarrow B$.

(For proofs of the main facts in what follows, see the appendix.)

The first condition in the above is basically equivalent to the validity of Modus Ponens. While we can also show that Modus Ponens must be valid for any conditional-assertion operator, we stick with the formulation above because of its connection with truth functionality.

Fact 1 shows that conditional assertion places surprising demands on the logic of conditionals. After all, the False Antecedent inference from $\neg A$ to $A \rightarrow B$ is controversial. Consider:

(2) a. The butler did it.
 b. ??So: if the butler didn't do it, then the gardener did.

This inference is one of the signature properties of the material conditional. Nonetheless, conditional assertion does not imply collapse to the material conditional. While the False Antecedent inference is required of any conditional-assertion operator, the True Consequent inference is not. In particular, the truth of the antecedent and consequent need not imply the truth of the corresponding conditional. This And-to-If inference is controversial. For example, following McDermott, imagine that a coin is tossed twice, and Sally bets that it will land heads both times.[26] Suppose the coin does land heads twice. In this scenario (3) seems false:

(3) If at least one heads came up, Sally won.

After all, in any scenario in which one coin landed heads and the other landed tails, Sally loses the bet. This is just one of many counterexamples to And to If in the literature. Others have focused on cases where A and B are irrelevant to one another, or where B is extremely unlikely in the presence of A.[27] Moreover, Lycan objects to the theories of conditional assertion developed by Jeffrey and Belnap precisely because they validate And to If.[28] We have now seen that this objection does not extend to all theories of conditional assertion.

The theory of conditional assertion imposes heavy demands on the logic of conditionals. But what about the semantics? In the rest of this paper, we consider which semantic theories of conditionals are consistent with conditional assertion. To do so, we assign progressively richer meanings to the conditional. We start within a truth-conditional framework. Later, we let the conditional have a dynamic meaning. This allows us to explore the prospects for nonpropositional theories of conditional assertion.

III. THE TRUTH CONDITIONS OF CONDITIONAL ASSERTION

We now explore the space of conditional-assertion theories for truth-conditional operators. To do so, we now interpret contexts as sets of possible worlds.[29]

Definition 9. Let a world w assign a truth value to every atomic sentence. Let a context s be a set of worlds.

Once contexts are sets of worlds, we can model assertion and conditional assertion in terms of narrowing down this set. Since we are assuming the conditional is truth conditional, we can assign every sentence a set of possible worlds where it is true:

[26] Michael McDermott, "True Antecedents," *Acta Analytica*, XXII, 4 (December 2007): 333–35.

[27] For discussion, see, among others, David Lewis, *Counterfactuals* (Oxford: Blackwell, 1973), p. 27; Jonathan Bennett, "Counterfactuals and Possible Worlds," *Canadian Journal of Philosophy*, IV, 2 (December 1974): 381–402; Kit Fine, Critical Notice of *Counterfactuals*, *Mind*, LXXXIV, 1 (1975): 451–58; Alan Penczek, "Counterfactuals with True Components," *Erkenntnis*, XLVI, 1 (January 1997): 79–85; McDermott, "True Antecedents," *op. cit.*; Aidan McGlynn, "The Problem of True–True Counterfactuals," *Analysis*, LXXII, 2 (April 2012): 276–85; and Chaoan He, "Conjunction, Connection and Counterfactuals," *Erkenntnis*, LXXXI, 4 (August 2016): 705–19.

[28] Lycan, "Conditional-Assertion Theories of Conditionals," *op. cit.*

[29] See Stalnaker, "Assertion," *op. cit.*

Definition 10. Let $[\![\cdot]\!]$ assign every sentence in $\mathscr{L}$ a set of possible worlds.

Then we can introduce a truth-conditional update function $[\cdot]_1$, which lets an utterance of any sentence narrow down a context to the worlds where it is true.[30]

Definition 11. $s[\mathsf{A}]_1 = s \cap [\![\mathsf{A}]\!]$.

Say that $\rightarrow$ is propositional just in case $[\![\mathsf{A} \rightarrow \mathsf{B}]\!]$ is defined for any A, B in $\mathscr{L}^0$ and updating with $\mathsf{A} \rightarrow \mathsf{B}$ amounts to intersecting with $[\![\mathsf{A} \rightarrow \mathsf{B}]\!]$ (that is, just in case $[\mathsf{A} \rightarrow \mathsf{B}] = [\mathsf{A} \rightarrow \mathsf{B}]_1$). Our task now is to determine what constraints are imposed on $[\![\cdot]\!]$ if we require $\rightarrow$ to be a conditional-assertion operator.

Our earlier result, Fact 1, has a corollary in the propositional domain. We can show the conditional is a propositional conditional-assertion operator just in case (i) $\mathsf{A} \rightarrow \mathsf{B}$ is true at any world where A is false, and (ii) $\mathsf{A} \rightarrow \mathsf{B}$ is false at any world where A is true and B is false.

Fact 2. If $\rightarrow$ is propositional and $\neg$ is well behaved, then $\rightarrow$ is a conditional-assertion operator iff:

1. $[\![\mathsf{A}]\!] \cap [\![\neg\mathsf{B}]\!] \cap [\![\mathsf{A} \rightarrow \mathsf{B}]\!] = \emptyset$.
2. $[\![\neg\mathsf{A}]\!] \subseteq [\![\mathsf{A} \rightarrow \mathsf{B}]\!]$.

Stalnaker observes that updating with the material conditional achieves the goals of the conditional-assertion theory.[31] We have now seen that the material conditional is not alone in doing so. Rather, there is a family of truth-conditional theories with this property.

In this section, we figured out the truth conditions of any conditional-assertion operator, assuming that it has truth conditions. But perhaps we should not model conditional assertion using truth conditions. Fortunately, this does not require that we give up on semantically interpreting the conditional. Rather, in the next section we uncover a rich family of conditional-assertion operators within the framework of dynamic semantics. On these accounts, conditionals do not express propositions. Updating with them is not a matter of simply learning which world one inhabits.

[30] Throughout, we omit subscripts on our update functions whenever possible.
[31] Stalnaker, "Conditional Propositions and Conditional Assertions," *op. cit.*

IV. THE DYNAMICS OF CONDITIONAL ASSERTION

In this section, we will explore the prospects for modeling conditional assertion using dynamic conditionals. There are at least two reasons why this is a natural goal to pursue. First, we saw above that many have associated the theory of conditional assertion with the idea that conditionals do not express propositions.[32] Within the model of information exchange developed above, nonpropositionality can be articulated precisely. We will see below that the conditional counts as nonpropositional when learning it cannot be reduced to learning that one inhabits a fixed set of possibilities. As we will see, this is the characteristic property of conditionals in dynamic semantics.

Second, we saw in the last section that the material conditional is a particularly natural choice of conditional-assertion operator. This raises the question of whether there are other well-behaved conditional-assertion operators. But in general, dynamic conditionals tend to be closely related to the material conditional. In a series of papers, Gillies has shown that dynamic conditionals can recover some of the signature benefits of the material conditional. For example, Gibbard showed that in a classical setting, only the material conditional can validate Import Export, the principle that $\mathsf{A} \rightarrow (\mathsf{B} \rightarrow \mathsf{C})$ is equivalent to $(\mathsf{A} \wedge \mathsf{B}) \rightarrow \mathsf{C}$.[33] Gillies shows that once we move to a dynamic semantics, we can define a "test" conditional (discussed below) that validates Import Export but has a different meaning than the material conditional.[34] This operation checks whether a context would be committed to the consequent after being updated with the antecedent. While the test conditional and the material conditional are commitments of the same contexts, they induce different updates on them, in a way that shows up under further operations like negation. For this reason, the test conditional does not validate the paradoxical Material Negation inference, that $\neg(\mathsf{A} \rightarrow \mathsf{B})$ implies $\mathsf{A} \wedge \neg\mathsf{B}$. Similarly, in a propositional framework only the material conditional can validate the Deduction Theorem. By contrast, as we will review in what follows, once we introduce nonpropositional meanings into our framework, we can validate the Deduction Theorem without accepting a material analysis.

[32] See, for example, Edgington, "On Conditionals," *op. cit.*; Barker, "Towards a Pragmatic Theory of 'If'," *op. cit.*; and DeRose, "Can It Be That It Would Have Been Even Though It Might Not Have Been?," *op. cit.*

[33] Allan Gibbard, "Two Recent Theories of Conditionals," in William L. Harper, Robert Stalnaker, and Glenn Pearce, eds., *Ifs: Conditionals, Belief, Decision, Chance, and Time* (Boston: D. Reidel, 1981), pp. 211–47.

[34] Anthony S. Gillies, "On Truth-Conditions for *If* (but Not Quite Only *If*)," *The Philosophical Review*, CXVIII, 3 (July 2009): 325–49.

We will see below that this pattern is repeated in the case of conditional assertion. The move away from propositional to nonpropositional meanings opens up the space for dynamic conditionals that are still conditional-assertion operators, without being truth functional in the ways we demarcated above. Interestingly, conditional assertion interacts richly with extant work on dynamic conditionals. When we combine the demands of conditional assertion with the validity of the Deduction Theorem, we can dramatically constrain the space of possible meanings for the conditional, so that we accept either the material analysis or a new kind of dynamic conditional I call 'test-like'.

To explore dynamic theories of conditionals, we retain some of the assumptions from the previous section. We again assume that contexts are sets of worlds. But now we only semantically associate declarative sentences with sets of worlds.

Definition 12. Let $[\![\cdot]\!]$ assign every sentence in $\mathscr{L}^0$ a set of possible worlds.

Then we introduce an update function $[\cdot]_2$ that behaves conservatively for ordinary assertions but not for conditional assertions. $[\cdot]_2$ lets ordinary assertions update s intersectively.

Definition 13. If $\mathsf{A} \in \mathscr{L}^0$, then $s[\mathsf{A}]_2 = s \cap [\![\mathsf{A}]\!]$.

Crucially, we do not require that $[\mathsf{A} \rightarrow \mathsf{B}]_2$ itself update the context in a traditional, truth-conditional manner. That is, we do not assume the existence of some fixed proposition p where for any state s, $s[\mathsf{A} \rightarrow \mathsf{B}]_2 = s \cap p$. Rather, we allow $[\mathsf{A} \rightarrow \mathsf{B}]_2$ to be any function whatsoever from sets of worlds to sets of worlds. Now we explore what constraints are imposed on $[\mathsf{A} \rightarrow \mathsf{B}]_2$ by conditional assertion.

With Fact 1, we saw that the validity of Modus Ponens and False Antecedent are necessary for conditional assertion. Then with Fact 2 we saw that their joint validity is also sufficient for conditional assertion in a truth-conditional framework. This sufficiency claim fails when the conditional is genuinely dynamic, not amounting to intersection with a fixed set of worlds. Consider the following dynamic conditional, where $\mathsf{A} \rightarrow \mathsf{B}$ tests s to see whether $s[\mathsf{A}]$ is committed to B:[35]

$$s[\mathsf{A} \rightarrow \mathsf{B}] = \{w \in s \mid s[\mathsf{A}] \models \mathsf{B}\}.$$

This semantics validates Modus Ponens and False Antecedent. However, it rejects the screening-off component of conditional assertion. For a wide range of contexts, $s[\mathsf{A} \rightarrow \mathsf{B}][\neg\mathsf{A}] \neq s[\neg\mathsf{A}]$. For example,

[35] Gillies, "Epistemic Conditionals and Conditional Epistemics," *op. cit.*

suppose s contains a world w where A and $\neg\mathsf{B}$, and a world v where $\neg\mathsf{A}$. $s[\neg\mathsf{A}]$ is not empty, since it contains v. But $s[\mathsf{A} \to \mathsf{B}][\neg\mathsf{A}]$ is empty, since $s[\mathsf{A} \to \mathsf{B}]$ is empty. Summing up, the test conditional behaves like a conditional-assertion operator whenever the test is passed; but when the test fails, it departs from the requirements of conditional assertion.

We can say more about the class of meanings that are not conditional-assertion operators. The dynamic conditional above is an example of a test. Tests are the paradigmatic example of a dynamic meaning. They have been proposed as the meaning not only of conditionals, but also of possibility and necessity modals.[36] To be more precise, a context-change potential is a test just in case it always returns either the input or the absurd state.

Definition 14.

1. $[\mathsf{A}]$ is a test iff for every state s: $s[\mathsf{A}] = s$ or $s[\mathsf{A}] = \emptyset$.
2. $\to$ is a test operator iff for any sentences A and B: $[\mathsf{A} \to \mathsf{B}]$ is a test.

The dynamic conditional above is a test operator, but it is not a conditional-assertion operator. This is no coincidence: no operator is both a test and a conditional-assertion operator.

Fact 3. If $\to$ is a test operator, then $\to$ is not a conditional-assertion operator.

Here is a general diagnosis of why tests are not conditional-assertion operators. When a test fails, the absurd state results. This means that any test semantics for the conditional has the potential to remove worlds from the state where the antecedent is false. But in this case, $\neg\mathsf{A}$ does not screen off $\mathsf{A} \to \mathsf{B}$. So the test is not a conditional-assertion operator. If we want to find an interestingly dynamic conditional-assertion operator, we need a new type of dynamic meaning.[37]

Tests are not conditional-assertion operators because they can eliminate worlds where the antecedent is false. This points us in the direction of characterizing the class of conditional-assertion operators.

[36] Veltman, "Defaults in Update Semantics," *op. cit.*

[37] Other extant dynamic conditionals in the literature also fail to be conditional-assertion operators, for example, the conditional in Jeffrey Sanford Russell and John Hawthorne, "General Dynamic Triviality Theorems," *The Philosophical Review*, CXXV, 3 (July 2016): 307–39, which is just like the test conditional above except that it behaves like the material conditional in cases where $s[\mathsf{A}]$ does not support $\neg\mathsf{B}$.

The first requirement of this class is simply that $s[\neg \mathsf{A}] \subseteq s[\mathsf{A} \rightarrow \mathsf{B}]$. This guarantees that $s[\mathsf{A} \rightarrow \mathsf{B}][\neg \mathsf{A}]$ and $s[\neg \mathsf{A}]$ are identical.

To produce a conditional-assertion operator, we must meet one more requirement: that $s[\mathsf{A} \rightarrow \mathsf{B}][\mathsf{A}] \models \mathsf{B}$. To achieve this goal, let us consider $s[\mathsf{A}]$ rather than $s[\neg \mathsf{A}]$. Here, we must guarantee that after updating with $[\mathsf{A} \rightarrow \mathsf{B}]$, s has entirely excluded $s[\mathsf{A}][\neg \mathsf{B}]$. To achieve this result, we let $s[\mathsf{A} \rightarrow \mathsf{B}]$ be the union of two updates: $s[\neg \mathsf{A}]$, and some privileged portion of $s[\mathsf{A}][\mathsf{B}]$. Any component of $s[\mathsf{A}][\mathsf{B}]$ will do, as long as $s[\mathsf{A}][\neg \mathsf{B}]$ is eliminated.

To implement these ideas, we need one more tool. Say that f is a generalized selection function just in case f takes s, $[\mathsf{A}]$, and $[\mathsf{B}]$ as input, and returns some subset of $s[\mathsf{A}][\mathsf{B}]$. These generalized selection functions take a context and two propositions as input, and return some selected worlds in the context where both propositions are true. Generalized selection functions give us a precise way to think about the privileged portion of $s[\mathsf{A}][\mathsf{B}]$. Then we can represent updating with the conditional in terms of the union of $s[\neg \mathsf{A}]$ with $f(s, [\mathsf{A}], [\mathsf{B}])$.[38]

Definition 15.

1. A generalized selection function f is a function from a context and two context-change potentials to a new context, where $f(s, [\mathsf{A}], [\mathsf{B}]) \subseteq s[\mathsf{A}][\mathsf{B}]$.
2. $\rightarrow$ is selective iff $s[\mathsf{A} \rightarrow \mathsf{B}] = s[\neg \mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}])$.

Selective conditionals split up the job of conditional assertion into two steps. To guarantee that $s[\mathsf{A} \rightarrow \mathsf{B}][\mathsf{A}]$ is committed to B, the conditional only includes a region of $s[\mathsf{A}]$ that survives update with $[\mathsf{B}]$, selected by f. To guarantee that the conditional is screened off by the negated antecedent, the conditional also includes all of the context that survives update with the negated antecedent. It turns out that $\rightarrow$ is a conditional-assertion operator just in case it is selective in this way.

Fact 4. $\rightarrow$ is a conditional-assertion operator iff $\rightarrow$ is selective.

[38] This formal tool allows us to require some sort of connection between A and B. This bears some resemblance to previous work in Roderick M. Chisholm, "The Contrary-to-Fact Conditional," *Mind*, LV, 220 (October 1946): 289–307; Nelson Goodman, "The Problem of Counterfactual Conditionals," this JOURNAL, XLIV, 5 (Feb. 27, 1947): 113–28; J. L. Mackie, "Counterfactuals and Causal Laws," in R. J. Butler, ed., *Analytical Philosophy* (Oxford: Basil Blackwell, 1962), pp. 66–80; Nicholas Rescher, *Hypothetical Reasoning* (Amsterdam: North-Holland Publishing Company, 1964); and especially Dov M. Gabbay, "A General Theory of the Conditional in Terms of a Ternary Operator," *Theoria*, XXXVIII, 3 (December 1972): 97–104, who explicitly introduces an operator that takes both A and B as input, and delivers a set of worlds relevant to the evaluation of the conditional. Our own theory departs significantly from all of the above, since it allows that the conditional is true whenever the antecedent is false.

We now have a characterization of conditional assertion in dynamic semantics. This result is in a sense less general than Fact 1, our logical characterization of conditional assertion, since it assumes that contexts are sets of worlds. For example, in a later section we develop a different model of contexts, and introduce a conditional-assertion operator in that setting. The resulting operator is not selective but still satisfies Modus Ponens and False Antecedent. But while our latest result is weaker than Fact 1 in one sense, in another sense it is stronger. It provides necessary and sufficient conditions for conditional assertion, while Fact 1 provided only necessary conditions. The result is also more general than Fact 2, which governed the truth-conditional case. In particular, each conditional-assertion operator in the truth-conditional setting is also a selective operator, corresponding to a particular choice of selector. To guarantee truth conditionality, all that is required is that the selector be a function of $s \cap [\![\mathsf{A}]\!]$ and $s \cap [\![\mathsf{B}]\!]$.

Importantly, however, there are also many nonpropositional selective conditionals. To see how they work, let us return to the test semantics above. There, we saw that the test semantics fails the screening-off component of conditional assertion whenever the test for $s[\mathsf{A}] \models \mathsf{B}$ is failed. A natural thought is then to change the test semantics for this failure condition. In the semantics above, failure of the test results in $\emptyset$, which eliminates $\neg\mathsf{A}$ worlds from s. We can modify the meaning above by letting failure of the test result in a different state. In particular, we can let it result in $s[\neg\mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}])$, our selective output from before. When the conditional has this form, guaranteeing that the context is committed to the consequent after updating with the antecedent, but otherwise updating with $s[\neg\mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}])$, let us call it *test-like*.

In fact, once a conditional is selective, we can derive this last operation by constraining f in a particular way. To achieve this result, call a selector test-like when it returns all of $s[\mathsf{A}][\mathsf{B}]$ whenever $s[\mathsf{A}]$ is committed to B. When $s[\mathsf{A}]$ is committed to B, s is identical to $s[\neg\mathsf{A}] \cup s[\mathsf{A}][\mathsf{B}]$. This implies that test-like selectors allow the conditional to return the entire state whenever $s[\mathsf{A}] \models \mathsf{B}$.

Definition 16.

1. f is test-like iff $f(s, [\mathsf{A}], [\mathsf{B}]) = s[\mathsf{A}][\mathsf{B}]$ whenever $s[\mathsf{A}] \models \mathsf{B}$.
2. $\rightarrow$ is test-like iff $s[\mathsf{A} \rightarrow \mathsf{B}] = \begin{cases} s & \text{if } s[\mathsf{A}] \models \mathsf{B} \\ s[\neg\mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}]) & \text{otherwise.} \end{cases}$

Fact 5. $\rightarrow$ is test-like just in case $\rightarrow$ is selective and f is test-like.

Test-like operators retain much of the test semantics above. In particular, whenever $s[\mathsf{A}] \models \mathsf{B}$, test-like operators act like tests, returning the input state. But they are also selective, which implies that they are conditional-assertion operators. Test-like operators are thus a new type of dynamic meaning, designed to satisfy the demands of conditional assertion.

To better understand the family of test-like operators, let us explore a few representative examples. Test-like operators are a special case of selective operators. In a similar way, the material conditional is a special case of a test-like operator. To see why, note that as f returns more worlds, the conditional becomes weaker. The weakest test-like operator makes f the identity function. This operator never removes any worlds from $s[\mathsf{A}][\mathsf{B}]$.

$$s[\mathsf{A} \to \mathsf{B}] = \begin{cases} s & \text{if } s[\mathsf{A}] \models \mathsf{B} \\ s[\neg\mathsf{A}] \cup s[\mathsf{A}][\mathsf{B}] & \text{otherwise.} \end{cases}$$

This last operator is simply the material conditional in disguise, assuming $s[\mathsf{A} \supset \mathsf{B}] = s[\neg\mathsf{A}] \cup s[\mathsf{A}][\mathsf{B}]$. After all, either $s[\mathsf{A}]$ supports B or not. In the former case, $s[\mathsf{A} \to \mathsf{B}] = s[\mathsf{A} \supset \mathsf{B}] = s$. In the latter case, $s[\mathsf{A} \to \mathsf{B}]$ simply eliminates all $\mathsf{A} \wedge \neg\mathsf{B}$ possibilities from s. But this is exactly what $[\mathsf{A} \supset \mathsf{B}]$ does in that case.

At the other extreme, the strongest test-like operator always removes every world in $s[\mathsf{A}][\mathsf{B}]$. This conditional looks more like the test semantics from earlier:

$$s[\mathsf{A} \to \mathsf{B}] = \begin{cases} s & \text{if } s[\mathsf{A}] \models \mathsf{B} \\ s[\neg\mathsf{A}] & \text{otherwise.} \end{cases}$$

This gives us a conditional-assertion operator, but it is a strange one. Suppose s contains some $\mathsf{A} \wedge \mathsf{B}$ worlds and some $\mathsf{A} \wedge \neg\mathsf{B}$ worlds. Then $s[\mathsf{A} \to \mathsf{B}]$ removes all the $\mathsf{A} \wedge \mathsf{B}$ worlds. This result seems too strong. So it seems that we want some choice of test-like operator in between these two extremes.

Not every test-like operator is nonpropositional, since the material conditional is test-like. Nonetheless, we can show that any choice of test-like f other than the identity function will result in a nonpropositional theory of $\to$. For these operators, updating is not just intersecting with a fixed set of possible worlds.

Fact 6. If $\to$ is test-like and $[\to] \neq [\supset]$, then $\to$ is not propositional.

For any such f, there will be some witnessing s, A, and B where $f(s, [\mathsf{A}], [\mathsf{B}]) \neq s[\mathsf{A}][\mathsf{B}]$. This means that there will be some world w removed from $s[\mathsf{A}][\mathsf{B}]$ by f. This w is an A and B world. However, now consider the context $\{w\}$. $\{w\} \models [\mathsf{A} \to \mathsf{B}]$, since $\{w\}$ passes the test imposed by $\mathsf{A} \to \mathsf{B}$. This means that whether $[\mathsf{A} \to \mathsf{B}]$ removes w from a context depends on the structure of that entire context. We cannot model the effect of $[\mathsf{A} \to \mathsf{B}]$ in terms of intersecting s with a fixed set of possible worlds.

While test-like conditionals are essentially dynamic, they are only dynamic in a limited way. It turns out that any test-like conditional can be factorized into a static and a dynamic component. First, note that a standard semantics for disjunction within update semantics consists of updating the state with each disjunct, and unioning the result, so that $s[\mathsf{A} \vee \mathsf{B}] = s[\mathsf{A}] \cup s[\mathsf{B}]$. This semantics for disjunction is not essentially dynamic. In particular, whenever both A and B are propositional, so is $\mathsf{A} \vee \mathsf{B}$. Since test-like conditionals are all selective, we know that we can express their meaning with the identity $s[\mathsf{A} \to \mathsf{B}] = s[\neg\mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}])$. Given the semantics for disjunction above, that means that we can define any conditional-assertion operator in terms of disjunction. In particular, we could introduce into our object language a new connective $\overline{\wedge}$ that takes two sentences as input and returns the result of selecting some worlds in s where both claims are true. For example, this operator might be the meaning of *because*, which entails its inputs and expresses some connection between them. Then we can define the conditional in terms of $\vee$ and $\overline{\wedge}$:

Definition 17.

1. $s[\mathsf{A} \vee \mathsf{B}] = s[\mathsf{A}] \cup s[\mathsf{B}]$.
2. $s[\mathsf{A} \mathbin{\overline{\wedge}} \mathsf{B}] = f(s, [\mathsf{A}], [\mathsf{B}])$.

Fact 7. If $\to$ is selective, then $[\mathsf{A} \to \mathsf{B}] = [\neg\mathsf{A} \vee (\mathsf{A} \mathbin{\overline{\wedge}} \mathsf{B})]$.

Whenever $\to$ is selective, and hence whenever $\to$ is a conditional-assertion operator, it can be expressed as a disjunction of $\neg\mathsf{A}$ and $\mathsf{A} \mathbin{\overline{\wedge}} \mathsf{B}$. This means that for test-like operators, all of the nonpropositionality can be factorized into $\overline{\wedge}$. This in turn means that all of the essentially dynamic component of the conditional's meaning is contributed by the selector.

We have now identified a subclass of selective conditional operators. We know that these are all conditional-assertion operators, and that with the exception of the material conditional they are all essentially dynamic. Above, we said that these test-like operators retain something from the test semantics: an account of what happens when $s[\mathsf{A}]$ is committed to B. This feature from the test semantics is important, because it corresponds to the Ramsey Test. Here, consider Ramsey's famous idea connecting conditional belief and learning:

> If two people are arguing 'If *p* will *q*?'... they are adding *p* hypothetically to their stock of knowledge and arguing on that basis about *q*.[39]

[39] F. P. Ramsey, *Philosophical Papers*, ed. D. H. Mellor (New York: Cambridge University Press, 1990), p. 155.

We can represent this idea in our own framework as yet another requirement on $[\mathsf{A} \to \mathsf{B}]$. In particular, let us say that an agent believes $\mathsf{A} \to \mathsf{B}$ just in case she would be committed to B if she learned A. Now let us interpret belief in terms of our commitment relation $\models$, and let us interpret learning using $[\cdot]$. Putting these together, we reach the following requirement:

Definition 18. $\to$ is Ramseyan just in case $s \models \mathsf{A} \to \mathsf{B}$ iff $s[\mathsf{A}] \models \mathsf{B}$.

According to this version of the Ramsey test, a state supports $\mathsf{A} \to \mathsf{B}$ just in case it would support B if it was updated with A.

In a truth-conditional setting, the requirement above is quite strong. In fact, in a truth-conditional setting the material conditional is the unique Ramseyan operator.[40] To see why, note that the above formulation of the Ramsey Test is equivalent to the Deduction Theorem, given our definition of entailment. It guarantees that B follows from any premise combined with A just in case that premise itself implies $\mathsf{A} \to \mathsf{B}$.

By contrast, in a dynamic setting there are a variety of Ramseyan operators. For example, consider again the test operator from Gillies.[41] This operator is Ramseyan, because it does not allow an agent to believe $\mathsf{A} \to \mathsf{B}$ if her current information does not support B when updated with A. However, the test operator says more than just this. It also says that whenever $s[\mathsf{A}]$ does not support B, s is actually committed to the negation of $\mathsf{A} \to \mathsf{B}$.

Now consider our test-like operators. It turns out that when we focus on selective operations, the test-like operators precisely characterize the Ramsey Test.

Fact 8. If $\to$ is selective, then $\to$ is test-like iff $\to$ is Ramseyan.

We saw earlier that the selective operators are just the conditional-assertion operators. This means that once one accepts the theory of conditional assertion, satisfaction of the Ramsey Test is equivalent to accepting a test-like theory of the conditional. On the other hand, we already saw before that the Ramsey Test does not guarantee conditional-assertion operatorhood, since the test conditional satisfies the former but not the latter. Likewise, there are many conditional-assertion operators that are not Ramseyan, since there are selective operators that are not test-like.[42]

[40] See Gillies, "On Truth-Conditions for *If* (but Not Quite Only *If*)," *op. cit.*, for discussion.

[41] Gillies, "Epistemic Conditionals and Conditional Epistemics," *op. cit.*

[42] The discussion above is not the last word on selective conditionals. There are many more interesting constraints on selection to consider. For example, one constraint worth considering is to make f a function of $s[\mathsf{A}][\mathsf{B}]$, rather than of s, $[\mathsf{A}]$, and $[\mathsf{B}]$ individually.

We have now introduced a preferred family of conditional-assertion operators and characterized it in terms of the Ramsey Test. It turns out that test-like operators have another independently plausible structural property: they are idempotent, so that $s[\mathsf{A} \to \mathsf{B}]$ always supports $\mathsf{A} \to \mathsf{B}$. Some dynamic meanings lack this property, while others possess it.[43]

Fact 9. If $\to$ is test-like, then $\to$ is idempotent.

Conditional-assertion operators are precisely those that are selective. Within this family of operators, we have now located two desirable properties: satisfaction of the Ramsey Test and idempotence. Possession of the former property is equivalent to being test-like. Each of these properties in turn guarantees idempotence.

We started this section with a negative result: the quintessential example of a dynamic meaning, the test, cannot be a conditional-assertion operator. To model conditional assertion in dynamic semantics, we then developed a new kind of dynamic meaning. To start, we introduced the notion of a selective operator, proving this was equivalent to conditional assertion. Then we developed the notion of a test-like operator. This turned out to be a special case of a selective operator, and was genuinely nonpropositional. We saw that test-like operators therefore occupy the intersection of two plausible constraints: conditional assertion and the Ramsey Test. In this way, the test-like operators retain one of the signature benefits of the test semantics for conditionals.

V. CONCLUSION

In this paper we explored the consequences of an intuitive picture of what it is to utter a conditional. Along the way, we offered a new methodology for investigating the meaning of the conditional. Most work on conditionals proceeds by starting with data concerning the truth conditions or valid inferences of the conditional, and goes on to design a semantics that predicts this data. By contrast, we started with an intuitive idea about what it is to learn a conditional: crucially, that learning the antecedent is false screens off learning the conditional. We went on to characterize several families of meanings for the conditional that are consistent with that theory of learning. Ultimately, we used our model of conditional assertion to uncover a new kind of conditional meaning: the family of test-like operators.

[43] Say $\to$ is idempotent iff $\mathsf{A} \to \mathsf{B}$ is idempotent for any A and B.

APPENDIX

Definition 3. $\to$ is a conditional-assertion operator iff:

1. $s[\mathsf{A} \to \mathsf{B}][\mathsf{A}] \models \mathsf{B}$.
2. $s[\mathsf{A} \to \mathsf{B}][\neg\mathsf{A}] = s[\neg\mathsf{A}]$.

Fact 1. Suppose $\models$, $[\cdot]$, and $\neg$ are well behaved, A and $\neg\mathsf{B}$ are persistent, and $\neg\mathsf{A}$ is idempotent. Then if $\to$ is a conditional-assertion operator, then:

1. $\mathsf{A}; \neg\mathsf{B} \models \neg(\mathsf{A} \to \mathsf{B})$.
2. $\neg\mathsf{A} \models \mathsf{A} \to \mathsf{B}$.

Proof. Suppose for reductio that $\mathsf{A}; \neg\mathsf{B} \not\models \neg(\mathsf{A} \to \mathsf{B})$, and suppose that $\to$ is a conditional-assertion operator. Then there is some s where $s[\mathsf{A}][\neg\mathsf{B}] \not\models \neg(\mathsf{A} \to \mathsf{B})$. So by the well-behavedness of negation, we know that $s[\mathsf{A}][\neg\mathsf{B}][\mathsf{A} \to \mathsf{B}]$ is not absurd. But $s[\mathsf{A}] \models \mathsf{A}$, and so by the persistence of A, we know that $s[\mathsf{A}][\neg\mathsf{B}][\mathsf{A} \to \mathsf{B}] \models \mathsf{A}$. So $s[\mathsf{A}][\neg\mathsf{B}][\mathsf{A} \to \mathsf{B}][\mathsf{A}] = s[\mathsf{A}][\neg\mathsf{B}][\mathsf{A} \to \mathsf{B}]$. So by condition (i) of $\to$ being a conditional-assertion operator we can infer that $s[\mathsf{A}][\neg\mathsf{B}][\mathsf{A} \to \mathsf{B}] \models \mathsf{B}$. But $s[\mathsf{A}][\neg\mathsf{B}] \models \neg\mathsf{B}$, and so by the persistence of $\neg\mathsf{B}$ $s[\mathsf{A}][\neg\mathsf{B}][\mathsf{A} \to \mathsf{B}] \models \neg\mathsf{B}$. So $s[\mathsf{A}][\neg\mathsf{B}][\mathsf{A} \to \mathsf{B}]$ is absurd after all. Contradiction.

For the last condition, take some context s. Since $\neg\mathsf{A}$ is idempotent, we know that $s[\neg\mathsf{A}] \models \neg\mathsf{A}$. But by what we have proved above, we know that whenever $s \models \neg\mathsf{A}$, we have that $s \models \mathsf{A} \to \mathsf{B}$. So $s[\neg\mathsf{A}] \models \mathsf{A} \to \mathsf{B}$. □

Fact 2. If $\to$ is propositional and $\neg$ is well behaved, then $\to$ is a conditional-assertion operator iff:

1. $[\![\mathsf{A}]\!] \cap [\![\neg\mathsf{B}]\!] \cap [\![\mathsf{A} \to \mathsf{B}]\!] = \emptyset$.
2. $[\![\neg\mathsf{A}]\!] \subseteq [\![\mathsf{A} \to \mathsf{B}]\!]$.

Proof. If $\to$ is propositional, then the two requirements of conditional-assertion operators are as follows. First, for any set of worlds s, $s \cap [\![\mathsf{A} \to \mathsf{B}]\!] \cap [\![\mathsf{A}]\!]$ is a subset of $[\![\mathsf{B}]\!]$. Second, for any set of worlds s, $s \cap [\![\mathsf{A} \to \mathsf{B}]\!] \cap [\![\neg\mathsf{A}]\!]$ is equal to $s \cap [\![\neg\mathsf{A}]\!]$. Further, if negation is well behaved then $[\![\neg\mathsf{A}]\!] = W - [\![\mathsf{A}]\!]$. Given this assumption, the two requirements here are equivalent to those above. □

Fact 3. If $\to$ is a test operator, then $\to$ is not a conditional-assertion operator.

Proof. Let c be a context $\{w, v\}$, where $w(\mathsf{A}) = 0$, $v(\mathsf{A}) = 1$, and $v(\mathsf{B}) = 0$. If $\to$ is a test operator, then $c[\mathsf{A} \to \mathsf{B}]$ is either s or $\emptyset$. Suppose the former. Then $c[\mathsf{A} \to \mathsf{B}][\mathsf{A}] \not\models \mathsf{B}$, so $\to$ is not a conditional-assertion operator. Suppose the latter. Then $c[\mathsf{A} \to \mathsf{B}][\neg\mathsf{A}] = \emptyset \neq \{w\} = c[\neg\mathsf{A}]$. So $\to$ is not a conditional-assertion operator. □

Definition 15.

1. A generalized selection function f is a function from a context and two context-change potentials to a new context, where $f(s, [\mathsf{A}], [\mathsf{B}]) \subseteq s[\mathsf{A}][\mathsf{B}]$.
2. $\rightarrow$ is selective iff $s[\mathsf{A} \rightarrow \mathsf{B}] = s[\neg\mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}])$.

Fact 4. $\rightarrow$ is a conditional-assertion operator iff $\rightarrow$ is selective.

Proof. Suppose $\rightarrow$ is selective. Then $s[\mathsf{A} \rightarrow \mathsf{B}][\mathsf{A}] \models \mathsf{B}$, since the only A worlds in $s[\mathsf{A} \rightarrow \mathsf{B}]$ are B worlds. In addition, $s[\mathsf{A} \rightarrow \mathsf{B}][\neg\mathsf{A}] = s[\neg\mathsf{A}]$ because $s[\mathsf{A} \rightarrow \mathsf{B}]$ includes all of $s[\neg\mathsf{A}]$.

Now suppose $\rightarrow$ is a conditional-assertion operator. It suffices to show that for any s, $s[\mathsf{A} \rightarrow \mathsf{B}] \subseteq s[\neg\mathsf{A}] \cup s[\mathsf{A}][\mathsf{B}]$. In that case, we can construct a selector f so that $f(s, [\mathsf{A}], [\mathsf{B}]) = s[\mathsf{A} \rightarrow \mathsf{B}] - s[\neg\mathsf{A}]$. So suppose $w \in s[\mathsf{A} \rightarrow \mathsf{B}]$ but $w \notin s[\neg\mathsf{A}] \cup s[\mathsf{A}][\mathsf{B}]$. Then $w \in s[\mathsf{A}][\neg\mathsf{B}]$. But this means that $s[\mathsf{A} \rightarrow \mathsf{B}][\mathsf{A}] \not\models \mathsf{B}$. □

Definition 16.

1. f is test-like iff $f(s, [\mathsf{A}], [\mathsf{B}]) = s[\mathsf{A}][\mathsf{B}]$ whenever $s[\mathsf{A}] \models \mathsf{B}$.
2. $\rightarrow$ is test-like iff $s[\mathsf{A} \rightarrow \mathsf{B}] = \begin{cases} s & \text{if } s[\mathsf{A}] \models \mathsf{B} \\ s[\neg\mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}]) & \text{otherwise.} \end{cases}$

Fact 5. $\rightarrow$ is test-like just in case $\rightarrow$ is selective and f is test-like.

Proof. Suppose $\rightarrow$ is selective and f is test-like. Now suppose $s[\mathsf{A}] \models \mathsf{B}$. Then $f(s, [\mathsf{A}], [\mathsf{B}]) = s[\mathsf{A}][\mathsf{B}]$, and so $s[\mathsf{A} \rightarrow \mathsf{B}] = s$. □

Fact 6. If $\rightarrow$ is test-like and $[\rightarrow] \neq [\supset]$, then $\rightarrow$ is not propositional.

Proof. Suppose $\rightarrow$ is test-like and not the material conditional. Then $f(s, [\mathsf{A}], [\mathsf{B}])$ must eliminate a $\mathsf{A} \wedge \mathsf{B}$ world w in some case. But since $\rightarrow$ is test-like, $\{w\} \models \mathsf{A} \rightarrow \mathsf{B}$. So $w \in f(\{w\}, [\mathsf{A}], [\mathsf{B}])$. So $s[\mathsf{A} \rightarrow \mathsf{B}] \neq \bigcup \big\{ \{w\}[\mathsf{A} \rightarrow \mathsf{B}] \mid w \in s \big\}$. □

Definition 17.

1. $s[\mathsf{A} \vee \mathsf{B}] = s[\mathsf{A}] \cup s[\mathsf{B}]$.
2. $s[\mathsf{A} \barwedge \mathsf{B}] = f(s, [\mathsf{A}], [\mathsf{B}])$.

Fact 7. If $\rightarrow$ is selective, then $[\mathsf{A} \rightarrow \mathsf{B}] = [\neg\mathsf{A} \vee (\mathsf{A} \barwedge \mathsf{B})]$.

Proof. $s[\mathsf{A} \rightarrow \mathsf{B}] = s[\neg\mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}]) = s[\neg\mathsf{A}] \cup s[\mathsf{A} \barwedge \mathsf{B}] = s[\neg\mathsf{A} \vee (\mathsf{A} \barwedge \mathsf{B})]$. □

Definition 18. $\rightarrow$ is Ramseyan just in case $s \models \mathsf{A} \rightarrow \mathsf{B}$ iff $s[\mathsf{A}] \models \mathsf{B}$.

Fact 8. If $\rightarrow$ is selective, then $\rightarrow$ is test-like iff $\rightarrow$ is Ramseyan.

Proof. Suppose $\rightarrow$ is selective, and that $\rightarrow$ is test-like. Suppose $s[\mathsf{A}] \models \mathsf{B}$. Then $s \models \mathsf{A} \rightarrow \mathsf{B}$ by the first case of being a test-like operator. Now suppose $s[\mathsf{A}] \not\models \mathsf{B}$. In that case $s[\mathsf{A} \rightarrow \mathsf{B}] = s[\neg\mathsf{A}] \cup f(s[\mathsf{A}][\mathsf{B}])$. Since $s[\mathsf{A}] \not\models \mathsf{B}$, we know that $s[\mathsf{A}][\neg\mathsf{B}] \neq \emptyset$, and so $s[\neg\mathsf{A}] \cup s[\mathsf{A}][\mathsf{B}] \subset s$. Since $f(s[\mathsf{A}][\mathsf{B}]) \subset s[\mathsf{A}][\mathsf{B}]$, this implies that $s[\mathsf{A} \rightarrow \mathsf{B}] = s[\neg\mathsf{A}] \cup f(s[\mathsf{A}][\mathsf{B}]) \subset s$, and so $s \not\models \mathsf{A} \rightarrow \mathsf{B}$.

Suppose that $\rightarrow$ is selective and Ramseyan. Now suppose that $s[\mathsf{A}] \models \mathsf{B}$. We must show that $f(s, [\mathsf{A}], [\mathsf{B}]) = s[\mathsf{A}][\mathsf{B}]$. Since $\rightarrow$ is Ramseyan and $s[\mathsf{A}] \models \mathsf{B}$, we know that $s \models \mathsf{A} \rightarrow \mathsf{B}$. This means that $s = s[\neg\mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}])$. But if $\rightarrow$ is not test-like, then $f(s, [\mathsf{A}], [\mathsf{B}])$ is not $s[\mathsf{A}][\mathsf{B}]$, which means that some $\mathsf{A} \wedge \mathsf{B}$ world is eliminated from s by $f(s, [\mathsf{A}], [\mathsf{B}])$. This last step assumes that $s[\mathsf{A}]$ is nonempty, but if this condition fails then $f(s, [\mathsf{A}], [\mathsf{B}])$ is trivially identical to $s[\mathsf{A}][\mathsf{B}]$. But if some $\mathsf{A} \wedge \mathsf{B}$ world is eliminated from s by $f(s, [\mathsf{A}], [\mathsf{B}])$, it cannot be contained in $s[\neg\mathsf{A}]$. So it is not in $s[\neg\mathsf{A}] \cup f(s, [\mathsf{A}], [\mathsf{B}])$, and so $s \not\models \mathsf{A} \rightarrow \mathsf{B}$. $\square$

Fact 9. If $\rightarrow$ is test-like, then $\rightarrow$ is idempotent.

Proof. If $\rightarrow$ is selective, then $s[\mathsf{A} \rightarrow \mathsf{B}]$ contains only worlds where either $\neg\mathsf{A}$, or $\mathsf{A} \wedge \mathsf{B}$. This means that $s[\mathsf{A} \rightarrow \mathsf{B}][\mathsf{A}] \models \mathsf{B}$. Now suppose $\rightarrow$ is test-like. This then implies that $s[\mathsf{A} \rightarrow \mathsf{B}] \models \mathsf{A} \rightarrow \mathsf{B}$, since $s[\mathsf{A} \rightarrow \mathsf{B}]$ satisfies the condition for support required by test-like conditionals. Since $s[\mathsf{A} \rightarrow \mathsf{B}] \models \mathsf{A} \rightarrow \mathsf{B}$ for arbitrary s, A, and B, we know that $\rightarrow$ is idempotent. $\square$

SIMON GOLDSTEIN

Lingnan University