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Conditional Random Quantities and Compounds of Conditionals

Abstract. In this paper we consider conditional random quantities (c.r.q.'s) in the setting of coherence. Based on betting scheme, a c.r.q. $X|H$ is not looked at as a restriction but, in a more extended way, as $XH + \mathbb{P}(X|H)H^c$; in particular (the indicator of) a conditional event $E|H$ is looked at as $EH + P(E|H)H^c$. This extended notion of c.r.q. allows algebraic developments among c.r.q.'s even if the conditioning events are different; then, for instance, we can give a meaning to the sum $X|H + Y|K$ and we can define the iterated c.r.q. $(X|H)|K$. We analyze the conjunction of two conditional events, introduced by the authors in a recent work, in the setting of coherence. We show that the conjoined conditional is a *conditional random quantity*, which may be a conditional event when there are logical dependencies. Moreover, we introduce the negation of the conjunction and by applying De Morgan's Law we obtain the disjoined conditional. Finally, we give the lower and upper bounds for the conjunction and disjunction of two conditional events, by showing that the usual probabilistic properties continue to hold.

Keywords: Conditional events, Conditional random quantities, Coherence, Iterated conditioning, Import–Export principle, Conjunction, Disjunction, Lower/upper prevision bounds.

1. Introduction

Probabilistic reasoning under coherence allows a consistent treatment of uncertainty in many applications of statistics, economy, decision theory and artificial intelligence; it allows one to manage incomplete probabilistic assignments in a situation of vague or partial knowledge (see e.g. [4,6–8,14,17,18,29,42]); in particular, it is useful for a flexible numerical approach to inference rules in nonmonotonic reasoning and for the psychology of uncertain reasoning (see, e.g., [5,24,26–28,34,35,47–50,56,57]). We recall that nonmonotonic reasoning and conditional logics are important topics in

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artificial intelligence (see e.g. [2,21,37,38,53,55]). In probability theory and in probability logic a relevant problem, discussed by many authors working in artificial intelligence, is that of suitably defining the logical operations of conjunction and disjunction among conditional events. In this context we recall the pioneering paper written in 1935 by de Finetti [19], where it was proposed a three-valued logic for conditional events coinciding with that one of Lukasiewicz. An interesting survey of the contributions by different authors (such as Adams, Belnap, Calabrese, de Finetti, Dubois, van Fraassen, McGee, Goodmann, Lewis, Nguyen, Prade, Schay) to the study of three-valued logics and compounds of conditionals has been given in [45]; an extensive study of conditionals has been made in [22]; see also [44]. In the many works concerning logical operations among conditional events, the conjunction and disjunction have been usually defined as suitable conditionals; see e.g. [2,11–13,21,36,37,52]. Our perspective is different because we work in a quantitative framework rather than in a logical one: in our approach the conjunction and disjunction of two conditional events are (not conditional events but) *conditional random quantities*. We recall that usually in the literature a c.r.q. $X|H$ is looked at as a restriction of X to H, with $X|H$ undefined when H is false. In our approach, using the conditional prevision $\mathbb{P}(X|H)$, we look at $X|H$ as an extended quantity which coincides with the restriction when H is true and is equal to $\mathbb{P}(X|H)$ when H is false. In this way, based on the betting scheme of de Finetti ([20]), in agreement with [40,41] the c.r.q. $X|H$ may be interpreted as the amount that you receive in a bet on X conditional on H, if you agree to pay $\mathbb{P}(X|H)$. In our paper we formally define the equality between two c.r.q.'s; in particular we show that $X|HK$ coincides with $[XH + \mathbb{P}(X|HK)H^c]|K$. Moreover, among other results, by the extended notion of c.r.q. we obtain a meaning for numerical operations, such as the sum $X|H + Y|K$, and a natural definition for a notion of iterated c.r.q. $(X|H)|K$, which coincides with $X|HK$ if $H \subseteq K$, or $K \subseteq H$. Then, we develop in the framework of coherence a theory for the compounds of conditionals connected with that ones proposed in [39,44]. We suitably define the conjunction $(A|H) \wedge (B|K)$ of two conditional events $A|H, B|K$, which in the case of some logical dependencies may be a conditional event. We introduce the negation for conjoined conditionals; then, by De Morgan's Law, we define the disjunction of two conditional events. Finally, we obtain the lower and upper bounds for the coherent extensions of a probability assessment (x, y) on $\{A|H, B|K\}$ to their conjunction $(A|H) \wedge (B|K)$ and their disjunction $(A|H) \vee (B|K)$. Interestingly, the usual probabilistic properties continue to hold in terms of previsions.

2. Preliminary Notions and Results

In this section we recall some basic notions and results on coherence for conditional prevision assessments. In our approach an event A represents an uncertain fact described by a (non ambiguous) logical proposition; hence we look at A as a two-valued logical entity which can be true (T), or false (F). The indicator of A, denoted by the same symbol, is a two-valued numerical quantity which is 1, or 0, according to whether A is true, or false. The sure event is denoted by Ω and the impossible event is denoted by $\emptyset$. Moreover, we denote by $A \wedge B$, or simply AB, (resp., $A \vee B$) the logical conjunction (resp., logical disjunction). The negation of A is denoted A^c. Given any events A and B, we simply write $A \subseteq B$ to denote that A logically implies B, that is AB^c is the impossible event $\emptyset$; an equivalent notation is $A \vDash B$. We recall that n events are logically independent when the number m of constituents, or possible worlds, generated by them is 2^n (in general $m \leq 2^n$).

2.1. Coherent Conditional Prevision Assessments

Given a prevision function $\mathbb{P}$ defined on an arbitrary family $\mathcal{K}$ of finite c.r.q.'s, consider a finite subfamily $\mathcal{F}_n = \{X_i|H_i,\, i \in J_n\} \subseteq \mathcal{K}$, where $J_n = \{1, \ldots, n\}$, and the vector $\mathcal{M}_n = (\mu_i,\, i \in J_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$ is the assessed prevision for the c.r.q. $X_i|H_i$. With the pair $(\mathcal{F}_n, \mathcal{M}_n)$ we associate the random gain $G = \sum_{i \in J_n} s_i H_i (X_i - \mu_i)$; moreover, we set $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$ and we denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n$. Then, using the *betting scheme* of de Finetti, we have

Definition 1. The function $\mathbb{P}$ defined on $\mathcal{K}$ is coherent if and only if, $\forall n \geq 1$, $\forall\, \mathcal{F}_n \subseteq \mathcal{K}$, $\forall\, s_1, \ldots, s_n \in \mathbb{R}$, it holds that: $min\; \mathcal{G}_{\mathcal{H}_n} \;\leq 0 \leq max\; \mathcal{G}_{\mathcal{H}_n}$.

Given a family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, for each $i \in J_n$ we denote by $\{x_{i1}, \ldots, x_{ir_i}\}$ the set of possible values for the restriction of X_i to H_i; then, for each $i \in J_n$ and $j = 1, \ldots, r_i$, we set $A_{ij} = (X_i = x_{ij})$. Of course, for each $i \in J_n$, the family $\{H_i^c, A_{ij}H_i, j = 1, \ldots, r_i\}$ is a partition of the sure event Ω, with $A_{ij}H_i = A_{ij}$, $\bigvee_{j=1}^{r_i} A_{ij} = H_i$. Then, the constituents generated by the family $\mathcal{F}_n$ are (the elements of the partition of Ω) obtained by expanding the expression $\bigwedge_{i \in J_n}(A_{i1} \vee \cdots \vee A_{ir_i} \vee H_i^c)$. We set $C_0 = H_1^c \cdots H_n^c$ (it may be $C_0 = \emptyset$); moreover, we denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Hence $\bigwedge_{i \in J_n}(A_{i1} \vee \cdots \vee A_{ir_i} \vee H_i^c) = \bigvee_{h=0}^{m} C_h$. With each C_h, $h \in J_m$, we associate a vector $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}$, $j = 1, \ldots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq H_i^c$; with C_0 it is associated $Q_0 = \mathcal{M}_n = (\mu_1, \ldots, \mu_n)$. We illustrate by an example how to compute the constituents C_h's and the associated point Q_h's for a pair of c.r.q.'s.

EXAMPLE 1. Given any r.q. $X \in \{1, \ldots, 6\}$, let be $H = (X \in \{2, 4, 6\})$ and $K = (X \in \{4, 5, 6\})$. Now, let us consider the family $\{X_1|H_1, X_2|H_2\} = \{X|H, X|K\}$. We have: $A_{11} = (X = 2)$, $A_{12} = (X = 4)$, $A_{13} = (X = 6)$, $A_{21} = (X = 4)$, $A_{22} = (X = 5)$, $A_{23} = (X = 6)$. Then $(A_{11} \vee A_{12} \vee A_{13} \vee H_1^c) \wedge (A_{21} \vee A_{22} \vee A_{23} \vee H_2^c) = C_0 \vee C_1 \vee \cdots \vee C_4$, where $C_0 = H^c K^c = (X \in \{1, 3\})$, $C_1 = A_{11} H_2^c = (X = 2)$, $C_2 = A_{12} A_{21} = (X = 4)$, $C_3 = A_{13} A_{23} = (X = 6)$, $C_4 = H_1^c A_{22} = (X = 5)$. Defining $\mathbb{P}(X|H) = \mu$, $\mathbb{P}(X|K) = \nu$, the associated points Q_h's are: $Q_0 = (\mu, \nu)$, $Q_1 = (2, \nu)$, $Q_2 = (4, 4)$, $Q_3 = (6, 6)$, $Q_4 = (\mu, 5)$.

Denoting by $\mathcal{I}_n$ the convex hull of $Q_1, \ldots, Q_m$, the condition $\mathcal{M}_n \in \mathcal{I}_n$ amounts to the existence of a vector $(\lambda_1, \ldots, \lambda_m)$ such that: $\sum_{h \in J_m} \lambda_h Q_h = \mathcal{M}_n$, $\sum_{h \in J_m} \lambda_h = 1, \lambda_h \geq 0 \quad \forall\, h$; in other words, $\mathcal{M}_n \in \mathcal{I}_n$ is equivalent to the solvability of the system (Σ), associated with $(\mathcal{F}_n, \mathcal{M}_n)$, given below.

$$(\Sigma) \quad \sum_{h \in J_m} \lambda_h q_{hi} = \mu_i,\ i \in J_n\,;\ \sum_{h \in J_m} \lambda_h = 1\,;\ \lambda_h \geq 0,\ \ h \in J_m. \qquad (1)$$

Given the assessment $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, let S be the set of solutions $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of system (Σ) defined in (1). Then, assuming the system (Σ) solvable, that is $S \neq \emptyset$, we define:

$$I_0 = \left\{ i : \max_{\Lambda \in S} \sum_{h : C_h \subseteq H_i} \lambda_h = 0 \right\},\ \mathcal{F}_0 = \{X_i|H_i, i \in I_0\},\ \mathcal{M}_0 = (\mu_i,\, i \in I_0).$$

Then, the following theorem can be proved ([8, Thm 3])

THEOREM 1. [Operative characterization of coherence] *A conditional prevision assessment* $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ *on the family* $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$ *is coherent if and only if the following conditions are satisfied:*

(i) the system (Σ) *defined in (1) is solvable;*

(ii) if $I_0 \neq \emptyset$*, then* $\mathcal{M}_0$ *is coherent.*

Coherent conditional previsions and non dominance w.r.t. proper scoring rules have been investigated in [9]. Comparative previsions, qualitative criteria of coherence and weak dominance have been studied in [46].

3. Deepenings on Conditional Random Quantities

In this section we deepen some aspects on c.r.q.'s. We consider an extended notion for c.r.q.'s; we define the equality between two c.r.q.'s and we give a condition under which two c.r.q.'s are equal. Then, we examine a non trivial case where such a condition is satisfied; moreover, by linearity of prevision, we give a simple proof of the general compound prevision theorem. We also recall a notion of iterated conditioning for c.r.q.'s, which has a rationale in terms of economic transactions [32]. We recall that, given an event $H \neq \emptyset$ and a r.q. X, based on the betting metaphor the conditional prevision $\mathbb{P}(X|H)$ is defined as the amount you agree to pay, by knowing that you will receive the amount X if H is true, or you will receive back the amount you payed if H is false, because the bet is called off.

Remark 1. Usually, in literature (see e.g. [3]) the c.r.q. $X|H$ is looked at as the *restriction* of X to H; that is a r.q. whose possible values are that ones of X compatible with H, while $X|H$ is undefined when H is false. Thus, $X|H$ cannot be interpreted as the amount you receive when you pay $\mathbb{P}(X|H)$. This interpretation is obtained by suitably defining $X|H$ when H is false, as made by some authors; see e.g. [15,32,40,41]. In Lad ([40], [41, Section 3.2]) $X|H$ is defined as the random quantity $X|H = XH + \mathbb{P}(X|H)H^c$, by looking in this way at the conditional prevision $\mathbb{P}(X|H)$ as the prevision of the "new object" $X|H$. As we can see, formally this extended notion of $X|H$ also depends on the conditional prevision $\mathbb{P}(X|H)$. In [15] a systematic study has been developed on conditional events, where among other things it has been shown that, for the indicator $T(E|H) = EH + tH^c$ of a conditional event $E|H$, the truth-value t satisfies all the properties of a conditional probability. This study has been extended in the setting of possibility theory in [10]. Thus, the third value of a conditional event is not strictly related to a particular measure of uncertainty (see [16]). Our semantics is probabilistic, then in our approach the third value is a conditional probability. A similar comment applies for c.r.q.'s studied in this paper; of course, our approach could be generalized by using other measures of uncertainty. In [15] also a generalization to a c.r.q. $Y|K$ has been considered by setting $Y|K = Y \cdot K + t^* K^c$, where $Y \cdot K$ is looked at as a suitable restriction of Y to K. Then, taking as Y the conditional event $E|H$, the relation $(E|H)|K = E|HK$ is obtained, which is in agreement with the so-called Import–Export Principle ([44]). Concerning the approach of Lad, we could observe that it may be ambiguous to use the same symbol $X|H$ to denote both the restriction and the quantity $XH + \mathbb{P}(X|H)H^c$. In Sect. 3.1 we analyze this aspect.

3.1. An Extended Notion for Conditional Random Quantities

We introduce below our extended notion of conditional random quantity.

Definition 2. Given an event $H \neq \emptyset$ and a r.q. X, let $\mathbb{P}(X|H)$ be the conditional prevision of $X|H$. We set $X|^*H = XH + \mathbb{P}(X|H)H^c$.

We observe that $X|^*H = X|H$, when H is true, and $X|^*H = \mathbb{P}(X|H)$, when H is false. In particular, for the indicator of a conditional event $A|H$ (usually denoted by the same symbol) we have $A|^*H = AH + P(A|H)H^c$. The choice of $P(A|H)$, as third value of the indicator when H is false, has been considered in other works (see e.g. [17,25,38,40,41,44]). In the next result we show that $\mathbb{P}(X|^*H) = \mathbb{P}(X|H)$ and $(X|^*H)|^*H = X|^*H$.

Proposition 1. *Given any event $H \neq \emptyset$, any r.q. X, and any coherent assessment $\mathbb{P}(X|H) = \mu$, we have:*

(a) The extension $\mathbb{P}(X|^*H) = \mu^*$ is coherent if and only if $\mu^* = \mu$;

(b) $(X|^*H)|^*H = X|^*H$.

Proof. (a) The random gain associated with the assessment (μ, μ^*) is

$$G = s_1H(X-\mu) + s_2(X|^*H - \mu^*) = s_1H(X-\mu) + s_2(XH + \mu H^c - \mu^*),$$

with s_1, s_2 arbitrary real numbers. By choosing $s_1 = 1, s_2 = -1$, we have

$$G = H(X-\mu) - (XH + \mu H^c - \mu^*) = -\mu H - \mu H^c + \mu^* = \mu^* - \mu.$$

As G is constant, by coherence, it must be $G = 0$, that is: $\mu^* = \mu$.

(b) We have $(X|^*H)|^*H = (XH + \mu H^c)|^*H = (XH + \mu H^c)H + \mu^* H^c = XH + \mu H^c = X|^*H$. ■

Remark 2. Given an event $H \neq \emptyset$ and a finite r.q. X, let $V_H = \{x_1, \ldots, x_r\}$ be the set of possible values of X restricted to H. By Definition 1, the assessment $\mathbb{P}(X|H) = \mu$ is coherent if and only if $min\, V_H \leq \mu \leq max\, V_H$. In particular, if $V_H = \{c\}$, then by coherence it must be $\mu = c$ and hence $X|^*H = cH + cH^c = c$. Of course, for $X = H$ (resp. $X = H^c$) it holds that $\mu = P(H|H) = 1$ (resp. $\mu = P(H^c|H) = 0$); hence $H|^*H = 1$, $H^c|^*H = 0$.

We define below the notion of equality for two c.r.q.'s $X|H$ and $Y|K$.

Definition 3. Given any events $H \neq \emptyset$, $K \neq \emptyset$, and any r.q.'s X, Y, let Π be the set of the coherent prevision assessments $\mathbb{P}(X|H) = \mu, \mathbb{P}(Y|K) = \nu$. We define $X|^*H$ and $Y|^*K$ equal if and only if

$$XH + \mu H^c = YK + \nu K^c, \quad \text{for every } (\mu,\nu) \in \Pi. \tag{2}$$

As shown in Theorem 4, $X|^*H = Y|^*K$ implies $\mu = \nu$, $\forall (\mu,\nu) \in \Pi$.

EXAMPLE 2. We discuss a critical example[1] on the equality of c.r.q.'s. Given a partition $\{A_1, \ldots, A_5\}$ of Ω, we set $H = A_1 \vee A_2 \vee A_3$, $K = A_2 \vee A_3 \vee A_4 \vee A_5$, $X = A_1 + 2A_2 - A_3 + xA_4 + yA_5$ and $Y = zA_1 + 2A_2 - A_3 + A_4 + A_5$, with x, y, z arbitrary real numbers. We observe that the c.r.q.'s $X|H$ and $Y|K$, looked at as restrictions, do not coincide; moreover, $X|^*H$ and $Y|^*K$ do not coincide too. We remark that as specified in Definition 3, in order that $X|^*H$ and $Y|^*K$ coincide, the equality (2) must be satisfied for every $(\mu, \nu) \in \Pi$. We have $XH = A_1 + 2A_2 - A_3$ and $YK = 2A_2 - A_3 + A_4 + A_5$; then defining $\mu = \mathbb{P}(X|H)$, $\nu = \mathbb{P}(Y|K)$, with (μ, ν) coherent, it follows

$$X|^*H = A_1 + 2A_2 - A_3 + \mu(A_4 + A_5), \;\; Y|^*K = 2A_2 - A_3 + A_4 + A_5 + \nu A_1.$$

The assessment $(\mu, \nu) = (\frac{2}{3}, \frac{3}{4})$, being associated with the uniform distribution on the partition, is coherent and $A_1 + 2A_2 - A_3 + \frac{2}{3}(A_4 + A_5) \neq 2A_2 - A_3 + A_4 + A_5 + \frac{3}{4}A_1$. Therefore, by Definition 3, $X|^*H \neq Y|^*K$. Of course, as a kind of 'local coincidence', it may happen that $A_1 + 2A_2 - A_3 + \mu(A_4 + A_5) = 2A_2 - A_3 + A_4 + A_5 + \nu A_1$, for some coherent assessment (μ, ν). For instance, this coincidence holds for $(\mu, \nu) = (1, 1)$, which is obtained by assessing $P(A_i) = \frac{1}{5}$, $i = 1, 2, 4$, $P(A_3) = \frac{1}{10}$, $P(A_5) = \frac{3}{10}$.

REMARK 3. As shown by Proposition 1, $X|^*H$ and $X|H$ coincide when H is true and have the same prevision, the unique difference being that $X|^*H$ is defined when H is false. Moreover, defining $\mathbb{P}(X|H) = \mu$, the random gains G and G^*, associated with $X|H$ and $X|^*H$, coincide because

$$G^* = s(X|^*H - \mu) = s(XH + \mu H^c - \mu) = s(XH - \mu H) = sH(X - \mu) = G.$$

More in general, in Definition 1, for the random gain G it holds that

$$G = \sum_{i \in J_n} s_i H_i (X_i - \mu_i) = \sum_{i \in J_n} s_i (X_i H_i + \mu_i H_i^c - \mu_i) = \sum_{i \in J_n} s_i (X_i|^*H_i - \mu_i) = G^*.$$

Then, *in what follows we identify* $X|H$ and $X|^*H$; in other words, from now on the symbol $X|H$ denotes the quantity $XH + \mathbb{P}(X|H)H^c$.

We also remark that, by this extended notion of c.r.q., given a prevision assessment $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ on a family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, the points Q_h's associated with the constituents C_h's represent the possible values of the random vector $(X_1|H_1, \ldots, X_n|H_n)$. Moreover, we can give a meaning, for instance, to the expression $X|H + Y|K$, which is identified with $X|^*H + Y|^*K$. As a consequence, we can look at the sum $\mathbb{P}(X|H) + \mathbb{P}(Y|K)$ as the prevision of $X|H + Y|K$ (see Theorem 2 below), and so on.

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[1] We thank an anonymous referee for this example.

3.2. Some Algebraic Developments

The next result shows that the additivity of prevision for our extended notion of c.r.q. is preserved and that $X|H+Y|K=(X|H+Y|K)|(H\vee K)$.

THEOREM 2. *Given any events $H\neq\emptyset, K\neq\emptyset$ and any r.q.'s X, Y, we have:*

(i) $\mathbb{P}(X|H+Y|K)=\mathbb{P}[(X|H+Y|K)|(H\vee K)]=\mathbb{P}(X|H)+\mathbb{P}(Y|K)$;

(ii) $X|H+Y|K=(X|H+Y|K)|(H\vee K)$.

PROOF. (i) Given any coherent assessment $\mathbb{P}(X|H)=\mu, \mathbb{P}(Y|K)=\nu$, we have $X|H+Y|K=XH+\mu H^c+YK+\nu K^c$. We set $\mathbb{P}(X|H+Y|K)=\eta$, $\mathbb{P}[(X|H+Y|K)|(H\vee K)]=\beta$; then, the random gain associated with the prevision assessment (μ,ν,η) is

$$G=s_1H(X-\mu)+s_2K(Y-\nu)+s_3(XH+\mu H^c+YK+\nu K^c-\eta),$$

with s_1, s_2, s_3 arbitrary real numbers, and for $s_1=s_2=1, s_3=-1$ we have

$$G=H(X-\mu)+K(Y-\nu)-(XH+\mu H^c+YK+\nu K^c-\eta)=\eta-\mu-\nu.$$

As G is constant, by coherence it must be $G=0$, i.e. $\eta=\mu+\nu$. Moreover, the random gain associated with the assessment (μ,ν,β) is

$$G=s_1H(X-\mu)+s_2K(Y-\nu)+s_3(H\vee K)(XH+\mu H^c+YK+\nu K^c-\beta),$$

and, denoting by $\mathcal{G}_{H\vee K}$ the set of values of G restricted to $H\vee K$, for $s_1=s_2=1, s_3=-1$ we have

$$\begin{aligned}G&=H(X-\mu)+K(Y-\nu)-(H\vee K)(XH+\mu H^c+YK+\nu K^c-\beta)=\\&=-\mu H-\nu K-\mu H^cK-\nu HK^c+\beta(H\vee K)=(\beta-\mu-\nu)(H\vee K).\end{aligned}$$

As $\mathcal{G}_{H\vee K}=\{\beta-\mu-\nu\}$, by coherence $\beta-\mu-\nu=0$; i.e. $\beta=\mu+\nu=\eta$.

(ii) Given any coherent assessment $\mathbb{P}(X|H)=\mu, \mathbb{P}(Y|K)=\nu$, we have $(X|H+Y|K)|(H\vee K)=(XH+\mu H^c+YK+\nu K^c)(H\vee K)+(\mu+\nu)H^cK^c=XH+\mu(H^cK+H^cK^c)+YK+\nu(HK^c+H^cK^c)=X|H+Y|K$. ■

Below, we consider linear combinations of c.r.q.'s.

THEOREM 3. *Given any real quantities a, b, any events $H\neq\emptyset, K\neq\emptyset$, any r.q.'s X, Y, and any coherent assessment $\mathbb{P}(X|H)=\mu, \mathbb{P}(Y|K)=\nu$, we have $(aX)|H+(bY)|K=a(X|H)+b(Y|K)=aX|H+bY|K$.*

PROOF. By linearity of prevision, $\mathbb{P}[(aX)|H]=a\mu$, $\mathbb{P}[(bY)|K]=b\nu$; then, $(aX)|H=(aX)H+(a\mu)H^c=a(XH+\mu H^c)=a(X|H)$; moreover $(bY)|K=\cdots=b(Y|K)$ and we can simply write $aX|H$ and $bY|K$. By Theorem 2, we have $\mathbb{P}[(aX)|H+(bY)|K]=\mathbb{P}[(aX)|H]+\mathbb{P}[(bY)|K]=a\mu+b\nu$; then $(aX)|H+(bY)|K=(aXH+a\mu H^c+bYK+b\nu K^c)(H\vee K)+(a\mu+b\nu)H^cK^c=aXH+a\mu H^c+bYK+b\nu K^c=a(X|H)+b(Y|K)=aX|H+bY|K$. ■

In the next result we show that two c.r.q.'s $X|H$ and $Y|K$ are equal if, for each constituent contained in $H \vee K$, the values of $X|H$ and $Y|K$ coincide.

THEOREM 4. *Given any events $H \neq \emptyset$, $K \neq \emptyset$, and any r.q.'s X, Y, let Π be the set of the coherent prevision assessments $\mathbb{P}(X|H) = \mu, \mathbb{P}(Y|K) = \nu$. Then, the statements (i) and (ii) below hold.*

(i) Assume that, for every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \vee K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.

(ii) For every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \vee K$ is true if and only if $X|H = Y|K$.

PROOF. (i) Assume that, for every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ associated with the constituent C_h coincide, for each $C_h \subseteq H \vee K$; then for each $(\mu, \nu) \in \Pi$, by choosing $s_1 = 1, s_2 = -1$ in the random gain, we have

$$\begin{aligned} G &= H(X - \mu) - K(Y - \nu) = (X|H - \mu) - (Y|K - \nu) = \\ &= (X|H - Y|K) + (\nu - \mu). \end{aligned}$$

Then, by the hypothesis, $\mathcal{G}_{H \vee K} = \{\nu - \mu\}$ and by coherence it must be $\nu - \mu = 0$; that is $\nu = \mu$, $\forall (\mu, \nu) \in \Pi$.

(ii) By hypothesis, we have $(XH + \mu H^c)(H \vee K) = (YK + \nu K^c)(H \vee K)$; moreover, from condition (i), $\mu = \nu$ for every $(\mu, \nu) \in \Pi$; then

$$\begin{aligned} X|H &= XH + \mu H^c = (XH + \mu H^c)(H \vee K) + (XH + \mu H^c)H^c K^c = \\ &= (YK + \nu K^c)(H \vee K) + \mu H^c K^c = YK + \nu H K^c + \nu H^c K^c = Y|K. \end{aligned}$$

Vice versa, $X|H = Y|K$ trivially implies $X|H = Y|K$ when $H \vee K$ is true. ∎

By Definition 3 and Theorem 4 it immediately follows

COROLLARY 1. *Given any event $H \neq \emptyset$ and any r.q.'s X and Y, assume that $XH = YH$. Then $X|H = Y|H$.*

In Corollary 1 the values $\mathbb{P}(X|H)$ and $\mathbb{P}(Y|H)$ do not play any role. We illustrate now a non trivial case where two c.r.q.'s are equal; then, by linearity of prevision we obtain the formula $\mathbb{P}(XH|K) = P(H|K)\mathbb{P}(X|HK)$.

THEOREM 5. *Given any events $H \neq \emptyset, K \neq \emptyset$, and any r.q. X, we have:*

(i) for all coherent assessments $\mathbb{P}(X|HK) = y$, $\mathbb{P}[(XH + yH^c)|K] = \mu$, it holds that $y = \mu$ and $X|HK = (XH + yH^c)|K$;

(ii) for all coherent assessments $P(H|K) = x$, $\mathbb{P}(X|HK) = y$, $\mathbb{P}(XH|K) = z$, it holds that $z = xy$.

Proof. (i) Given any coherent assessment (y, μ), we have

$$X|HK = XHK + y(HK)^c = XHK + yK^c + yH^cK = \\ = (XH + yH^c)K + yK^c$$

and $(XH+yH^c)|K = (XH+yH^c)K+\mu K^c$. We observe that $HK \vee K = K$ and that $X|HK = (XH + yH^c)|K$ when K is true. Then, by Theorem 4, $y = \mu$ for every (y, μ) coherent and $X|HK = (XH + yH^c)|K$.

(ii) Given any coherent assessment (x, y, z) on $\{H|K, X|HK, XH|K\}$, by condition (i) and by linearity of prevision we have

$$y = \mathbb{P}(X|HK) = \mathbb{P}[(XH + yH^c)|K] = \\ = \mathbb{P}(XH|K) + yP(H^c|K) = z + y(1 - x),$$

from which it follows: $z = xy$, that is: $\mathbb{P}(XH|K) = P(H|K)\mathbb{P}(X|HK)$, which represents the (general) *compound prevision theorem.* ■

We illustrate condition (i) of Theorem 5 by the example below.

Example 3. Consider a r.q. $X \in \{1, \ldots, 6\}$, with $P(X = h) = p_h, h = 1, \ldots, 6$, with $H = (X \in \{2, 4, 6\})$ and $K = (X \in \{4, 5, 6\})$. We have $HK = (X \in \{4, 6\})$ and $\mathbb{P}(X|HK) = y = \mathbb{P}[(XH+yH^c)|K] = \cdots = \frac{4p_4+6p_6}{p_4+p_6}$, when $p_4 + p_6 > 0$; moreover, when $p_4 + p_6 = 0$, using coherence, we can still verify that $\mathbb{P}(X|HK) = y = \mathbb{P}[(XH + yH^c)|K] \in [4, 6]$. Finally, we remark that $X|HK = (XH + yH^c)|K = X$, for $X \in \{4, 6\}$; $X|HK = (XH + yH^c)|K = y$, for $X \in \{1, 2, 3, 5\}$; hence $X|HK = (XH + yH^c)|K$. We observe that to check the equality $X|HK = [XH + \mathbb{P}(X|HK)\, H^c]|K$ does not make sense if we look at the c.r.q.'s as restrictions.

3.3. Iterated Conditioning

We recall below the notion of iterated c.r.q. $(X|H)K$ given in [32]; such a notion is consistent with that one given for conditional events in [33] and will be used in Section 4.

Definition 4. Given any events H, K, with $H \neq \emptyset, K \neq \emptyset$, and a r.q. X, we define $(X|H)|K = [XH + \mathbb{P}(X|H)H^c]|K$.

If X is any event A, we have $(A|H)|K = [AH + P(A|H)H^c]|K$; then

$$\mathbb{P}[(A|H)|K] = P(A|HK)P(H|K) + P(A|H)P(H^c|K),$$

which coincides with a result given in [44]. In condition (i) of Theorem 5 the value y is the prevision of $X|HK$; hence $X|HK = [XH+\mathbb{P}(X|HK)H^c]|K \neq (X|H)|K = [XH + \mathbb{P}(X|H)H^c]|K$, with $(X|H)|K = X|HK = (X|K)|H$, if $H \subseteq K$, or $K \subseteq H$ (see [32, Proposition 1]); then, given any event A it holds that $(A|H)|(H \vee K) = A|H$. Moreover $A|HK \neq (A|H)|K$; therefore, in agreement with [2,39], in our approach the Import–Export Principle ([44]) does not hold. For instance, assume that $K = H^c \vee A$ (material conditional associated with $A|H$)and $AH = \emptyset$, so that $P(A|H) = 0$. Then the Import–Export Principle cannot be applied because $A|HK = A|AH = A|\emptyset$; on the contrary, as $H^c \vee A = H^c$, by Definition 4 we have

$$(A|H)|K = (A|H)|(H^c \vee A) = (A|H)|H^c = (AH + 0 \cdot H^c)|H^c = 0|H^c = 0;$$

therefore $\mathbb{P}[(A|H)|K] = P(A|H) = 0$, while $P(H^c \vee A)$ could be high. A probabilistic analysis of weak and strong inferences from the material conditional $A^c \vee B$ to the associated conditional $B|A$ has been given in [28].

4. Conjunction of Conditional Events

Some authors look at the conditional "if A then C", denoted $A \to C$, as the event $A^c \vee C$ (*material conditional*), but, since some years, many authors look at $A \to C$ as the conditional event $C|A$ (see e.g. [28,48,51]). Compounds of conditionals have been studied by many researchers in many fields, such as mathematics, philosophical logic, artificial intelligence, nonmonotonic reasoning, psychology. A very general discussion of the different aspects which concern conditionals has been given in [22,45]. Recently, a probabilistic theory of conditionals has been proposed by Kaufmann in [39]. In such a paper the author obtains, by a complex procedure, probabilistic formulas which suggest how to assign values to conditionals. Given a conditional $A \to C$ and the associated conditional event $C|A$, with $P(A) > 0$, Kaufmann shows that, by defining the truth value of $A \to C$ as:

$$V(A \to C) = \begin{cases} 1, & AC \;\; true \\ 0, & AC^c \;\; true \\ P(C|A), & A^c \;\; true \end{cases}$$

it follows: $P(A \to C) = P(C|A) = \frac{P(AC)}{P(A)}$.

Moreover, assuming $P(A \vee C) > 0$, Kaufmann obtains for the conjoined conditional $(A \to B) \wedge (C \to D)$ the formula

$$P[(A \to B) \wedge (C \to D)] = \frac{P(ABCD) + P(B|A)P(A^cCD) + P(D|C)P(ABC^c)}{P(A \vee C)}.$$

Based on this result, Kaufmann suggests a natural way of defining the values of conjoined conditionals. Another relevant theory of compounds of conditionals is given by McGee in [44], where also the problem of what would be fair betting odds on conjunctions of conditionals is investigated. The papers of Kaufmann and McGee contain very nice results on compounds of conditionals. We obtain (and generalize) such results in a direct and simpler way in the setting of coherence; we also show that some well known probabilistic properties, which hold in the classical setting, are preserved.

A basic aspect: if we only assess $P(B|A) = x, P(D|C) = y$, how can we check the consistency of the extension $P[(A \to B) \wedge (C \to D)] = z$?

In our setting $(A \to B) \wedge (C \to D)$ is looked at as a *conditional random quantity* $(B|A) \wedge (D|C)$; hence, we assess the *prevision* (and not the probability) of the conjoined conditional. Moreover, we can manage without problems the case where the denominator $P(A \vee C)$, in the formula of Kaufmann, is zero and, starting with the assessment $P(B|A) = x,\ P(D|C) = y$, we can determine the values of $z = \mathbb{P}[(B|A) \wedge (D|C)]$ which coherently extend the assessment (x, y) on $\{B|A, D|C\}$ to the c.r.q. $(B|A) \wedge (D|C)$.

Conjunction of conditionals in the setting of coherence. We introduce the notion of conjunction, by first giving some logical and probabilistic remarks. Given any events A, B, H, with $H \neq \emptyset$, let us consider the conjunctions AB and $(A|H) \wedge (B|H) = AB|H$. We recall that the indicator AB coincides with both the minimum and the product of the indicators; i.e. $AB = min\,\{A, B\} = A \cdot B$. Moreover, $AB|H = min\,\{A, B\}|H = (A \cdot B)|H$. If we assess $P(A|H) = x, P(B|H) = y$, then

$$A|H = AH + xH^c = \begin{cases} A, & \text{if } H = 1, \\ x, & \text{if } H = 0, \end{cases} \quad B|H = BH + yH^c = \begin{cases} B, & \text{if } H = 1, \\ y, & \text{if } H = 0. \end{cases}$$

We set $Z = min\,\{A|H, B|H\} = min\,\{AH + xH^c, BH + yH^c\}$; we have $Z \in \{1, 0, x, y\}$ and, defining $\mathbb{P}(Z|H) = z$, we have $Z|H = ZH + zH^c$, with $Z|H \in \{1, 0, z\}$. We observe that $ZH = ABH$; then, by Corollary 1, we have $Z|H = AB|H$. In other words, the c.r.q.'s $min\,\{A|H, B|H\}|H$ and $AB|H$ are equal.[2] Then, as $H = H \vee H$, we have

$$(A|H) \wedge (B|H) = min\,\{A|H, B|H\}\,|\,H = min\,\{A|H, B|H\}\,|\,(H \vee H). \quad (3)$$

Formula (3) suggests how to define, still taking the *minimum*, the notion of conjunction (already given in [33]) for $A|H$ and $B|K$, with $K \neq H$.

Definition 5. *(Conjunction)* Given any pair of conditional events $A|H$ and $B|K$, and any coherent assessment $P(A|H) = x, P(B|K) = y$, we define

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[2]In particular, for $B = A$, in agreement with Definition 4 we have $Z = A|H$, $Z|H = (A|H)|H = A|H$, $z = x$; the equality $(A|H)|H = A|H$ still holds from the viewpoint of iterated conditionals introduced in [33].

$$(A|H) \wedge (B|K) = min\,\{A|H, B|K\}\,|\,(H \vee K). \tag{4}$$

Notice that, defining $Z = min\,\{A|H, B|K\}$, the conjunction $(A|H) \wedge (B|K)$ is the c.r.q. $Z\,|\,(H \vee K)$. Moreover, defining $T = A|H \cdot B|K$ we have $Z \neq T$, while by Corollary 1 it holds that $Z\,|\,(H \vee K) = T\,|\,(H \vee K)$. Then

$$(A|H) \wedge (B|K) = (A|H \cdot B|K)\,|\,(H \vee K). \tag{5}$$

Interpretation with the betting scheme.[3] By assessing $\mathbb{P}[(A|H) \wedge (B|K)] = z$, you agree to pay the amount z and to receive the amount $min\,\{A|H, B|K\}$ when the disjunction $H \vee K$ is true, or to receive back the amount z when the bet is *called off* ($H \vee K$ false). That is, you pay z and you receive

$$(A|H) \wedge (B|K) = \begin{cases} 1, & AHBK \;\; true, \\ 0, & A^cH \vee B^cK \;\; true, \\ x, & H^cBK \;\; true, \\ y, & AHK^c \;\; true, \\ z, & H^cK^c \;\; true; \end{cases}$$

therefore, *operatively*, for $(A|H) \wedge (B|K)$ we obtain the *representation*

$$(A|H) \wedge (B|K) \;=\; 1 \cdot AHBK + x \cdot H^cBK + y \cdot AHK^c + z \cdot H^cK^c, \tag{6}$$

with (x, y, z) coherent. Then, by *linearity* of prevision, it follows

$$\mathbb{P}[(A|H) \wedge (B|K)] = z = P(AHBK) + xP(H^cBK) + \\ + yP(AHK^c) + zP(H^cK^c),$$

and we obtain: $zP(H \vee K) = P(AHBK) + xP(H^cBK) + yP(AHK^c)$.
In particular, if $P(H \vee K) > 0$, as in [39,44], we obtain

$$\mathbb{P}[(A|H) \wedge (B|K)] = \frac{P(AHBK) + P(A|H)P(H^cBK) + P(B|K)P(AHK^c)}{P(H \vee K)}.$$

REMARK 4. We observe that, in case of some logical dependencies, the conjunction may be a conditional event; for instance, it may be verified that for $K = AH$ we have $(A|H) \wedge (B|AH) = (A|H \cdot B|AH)|H = AB|H$; moreover, assuming $A|H \subseteq B|K$, where $\subseteq$ denotes the inclusion relation of Goodman and Nguyen, it holds that $(A|H) \wedge (B|K) = A|H$ (see [33]).

Import–Export Principle and Lewis' triviality result. We observe that, as in our approach the Import–Export Principle does not hold, we avoid the counter-intuitive consequences related to Lewis' triviality result ([43]; see also [54, Section 3.7], where Lewis' triviality result and its consequences for

[3] Adams [1] found problematic this interpretation for the conjunction of conditionals. Conjoined conditionals and conditionals bets have been also investigated in [44].

logics for indicative conditionals have been discussed). We first prove that the probability of $A|H$ can be disintegrated w.r.t. a partition $\{B, B^c\}$.

THEOREM 6. *Given any events A, H, B, with $H \neq \emptyset$, $B \neq \emptyset$, it holds that*

$$P(A|H) = \mathbb{P}[(A|H)|B]P(B) + \mathbb{P}[(A|H)|B^c]P(B^c). \tag{7}$$

PROOF. For the sake of simplicity we set $P(A|H) = x$. Then, by Definition 4 we have: $\mathbb{P}[(A|H)|B]P(B) + \mathbb{P}[(A|H)|B^c]P(B^c) =$
$\mathbb{P}[(AH + xH^c)|B]P(B) + \mathbb{P}[(AH + xH^c)|B^c]P(B^c) =$
$P(AH|B)P(B) + xP(H^c|B)P(B) + P(AH|B^c)P(B^c) + xP(H^c|B^c)P(B^c) =$
$P(AH) + xP(H^c) = P(A|H)P(H) + P(A|H)P(H^c) = P(A|H)$. ■

Of course, the previous result still holds for a partition $\{B_1, \ldots, B_n\}$. In particular, formula (7) holds for $B = A$; but, as in our approach in general $\mathbb{P}[(A|H)|A] \neq P(A|HA) = 1$ and $\mathbb{P}[(A|H)|A^c] \neq P(A|HA^c) = 0$, we have

$$P(A|H) = \mathbb{P}[(A|H)|A]P(A) + \mathbb{P}[(A|H)|A^c]P(A^c) \neq P(A).$$

On the contrary, if the Import–Export Principle were valid, we would have $\mathbb{P}[(A|H)|A]P(A) + \mathbb{P}[(A|H)|A^c]P(A^c) = P(A)$ (Lewis' triviality result).
A critical analysis related to the disintegration of conditional probabilities is given in [23]. We recall that, given two conditional events $A|H, B|K$ the iterated conditional $(B|K)|(A|H)$ has been defined in [33] as

$$(B|K)|(A|H) = (A|H) \wedge (B|K) + \mathbb{P}[(B|K)|(A|H)]A^c|H,$$

where the prevision of $(B|K)|(A|H)$ is defined in agreement with the betting metaphor. In the same paper it has been proved that

$$\mathbb{P}[(B|K) \wedge (A|H)] = \mathbb{P}[(B|K)|(A|H)]P(A|H).$$

In particular, for $K = \Omega$, we have $\mathbb{P}[(A|H) \wedge B] = \mathbb{P}[(A|H)|B]P(B)$. Then, assuming $P(A|H) > 0$, we obtain the following Bayes formula for *iterated conditionals*: $\mathbb{P}[B|(A|H)] = \frac{\mathbb{P}[(A|H) \wedge B]}{P(A|H)} = \frac{\mathbb{P}[(A|H)|B]P(B)}{\mathbb{P}[(A|H)|B]P(B) + \mathbb{P}[(A|H)|B^c]P(B^c)}$.

5. Lower and Upper Bounds for $(A|H) \wedge (B|K)$

We will now determine the coherent extensions of the assessment (x, y) on $\{A|H, B|K\}$ to the conjunction $(A|H) \wedge (B|K)$. We recall that the extension $z = P(AB|H)$ of the assessment (x, y) on $\{A|H, B|H\}$, with A, B, H logically independent, is coherent if and only if

$$max\{x + y - 1, 0\} \leq z \leq min\{x, y\}.$$

The next theorem shows that the same result holds for $(A|H) \wedge (B|K)$.

THEOREM 7. *Given any coherent assessment* (x, y) *on* $\{A|H, B|K\}$, *with* A, H, B, K *logically independent, and with* $H \neq \emptyset, K \neq \emptyset$, *the extension* $z = \mathbb{P}[(A|H) \wedge (B|K)]$ *is coherent if and only if the Fréchet-Hoeffding bounds are satisfied, that is*

$$max\{x + y - 1, 0\} = z' \;\leq\; z \;\leq\; z'' = min\{x, y\}. \tag{8}$$

PROOF. First of all we observe that, by logical independence of A, H, B, K, the assessment (x, y) is coherent for every $(x, y) \in [0, 1]^2$. We will determine the values z', z'' by the geometrical approach described in Section 2. The constituents associated with the family $\mathcal{F} = \{A|H,\, B|K,\, (A|H) \wedge (B|K)\}$ and contained in $H \vee K$ are: $C_1 = AHBK$, $C_2 = AHB^cK$, $C_3 = A^cHBK$, $C_4 = A^cHB^cK$, $C_5 = AHK^c$, $C_6 = A^cHK^c$, $C_7 = H^cBK$, $C_8 = H^cB^cK$. The associated points Q_h's are: $Q_1 = (1, 1, 1)$, $Q_2 = (1, 0, 0)$, $Q_3 = (0, 1, 0)$, $Q_4 = (0, 0, 0)$, $Q_5 = (1, y, y)$, $Q_6 = (0, y, 0)$, $Q_7 = (x, 1, x)$, $Q_8 = (x, 0, 0)$. Considering the convex hull $\mathcal{I}$ of $Q_1, \ldots, Q_8$, the coherence of the prevision assessment $\mathcal{M} = (x, y, z)$ on $\mathcal{F}$ requires that the condition $\mathcal{M} \in \mathcal{I}$ be satisfied, which amounts to the solvability of the following system

$$(\Sigma) \qquad \mathcal{M} = \sum_{h=1}^{8} \lambda_h Q_h,\; \sum_{h=1}^{8} \lambda_h = 1,\; \lambda_h \geq 0, \forall\, h.$$

We observe that $Q_5 = yQ_1 + (1 - y)Q_2$, $Q_6 = yQ_3 + (1 - y)Q_4$, $Q_7 = xQ_1 + (1 - x)Q_3$, $Q_8 = xQ_2 + (1 - x)Q_4$; then, the convex hull $\mathcal{I}$ is the tetrahedron with vertices Q_1, Q_2, Q_3, Q_4. Thus, (Σ) is equivalent to

$$(\Sigma') \qquad \mathcal{P} = \sum_{h=1}^{4} \lambda'_h Q_h,\; \sum_{h=1}^{4} \lambda'_h = 1,\; \lambda'_h \geq 0, \forall\, h,$$

with $\lambda'_1 = \lambda_1 + y\lambda_5 + x\lambda_7$, $\lambda'_2 = \lambda_2 + (1-y)\lambda_5 + x\lambda_8$, $\lambda'_3 = \lambda_3 + y\lambda_6 + (1-x)\lambda_7$, $\lambda'_4 = \lambda_4 + (1-y)\lambda_6 + (1-x)\lambda_8$. Then, $\mathcal{M} \in \mathcal{I}$ if and only if (Σ') is solvable. We observe that (Σ') can be written as

$$(\Sigma') \quad \lambda'_1 + \lambda'_2 = x,\; \lambda'_1 + \lambda'_3 = y,\; \lambda'_1 = z,\; \lambda'_1 + \lambda'_2 + \lambda'_3 + \lambda'_4 = 1,\; \lambda'_h \geq 0, \forall\, h\,;$$

that is

$$(\Sigma') \quad \lambda'_1 = z,\; \lambda'_2 = x - z,\; \lambda'_3 = y - z,\; \lambda'_4 = z - (x + y - 1),\; \lambda'_h \geq 0, \forall\, h.$$

(Σ') is solvable if and only if $max\{x + y - 1, 0\} \leq z \leq min\{x, y\}$. Moreover, the vector $(\lambda_1, \ldots, \lambda_8) = (\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4, 0, 0, 0, 0)$, where $(\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4)$ is any solution of (Σ'), is a solution of (Σ) such that

$$\sum_{r:C_r \subseteq H} \lambda_r = \sum_{r:C_r \subseteq K} \lambda_r = \sum_{r:C_r \subseteq H \vee K} \lambda_r = 1 > 0,$$

and hence $I_0 = \emptyset$; then, by Theorem 1, the solvability of (Σ) is also sufficient for the coherence of $\mathcal{M}$. Therefore, the extension $\mathbb{P}[(A|H) \wedge (B|K)] = z$ of the assessment (x, y), with $(x, y) \in [0, 1]^2$, is coherent if and only if $z \in [z', z'']$, where $z' = max\,\{x + y - 1, 0\}$ and $z'' = min\,\{x, y\}$. ∎

We recall that for the quasi conjunction of $A|H$ and $B|K$, defined as the conditional event $\mathcal{C}(A|H, B|K) = (AH \vee H^c) \wedge (BK \vee K^c)|(H \vee K)$, only the inequality on the lower bound holds; indeed, the extension $\gamma = P[\mathcal{C}(A|H, B|K)]$ of (x, y) is coherent if and only if $\gamma' \leq \gamma \leq \gamma''$, where $\gamma' = z' = max\{x + y - 1, 0\}$ and $\gamma'' = \frac{x+y-2xy}{1-xy}$ if $(x, y) \neq (1, 1)$; $\gamma'' = 1$ if $(x, y) = (1, 1)$. We observe that: $\gamma'' \geq max\{x, y\} \geq min\{x, y\} = z''$.

Quasi conjunction is a basic notion in nonmonotonic reasoning ([2,21,31,34]). A probabilistic analysis of the lower and upper bounds for the quasi conjunction, in terms of t-norms and t-conorms, has been given in [30,35].

6. Negation and Disjunction

Given any coherent assessment (x, y, z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$, it holds that $(A|H) \wedge (B|K) \in \{1, 0, x, y, z\} \subset [0, 1]$. We recall that for conditional events the negation is usually defined as $(E|H)^c = E^c|H = (1-E)|H$. In our approach we have $(1 - E)|H = 1 - E|H$; hence $(E|H)^c = 1 - E|H$. Then, for the negation of conjunction we give the definition below.

Definition 6. Given any conditional events $A|H, B|K$, the negation of $(A|H) \wedge (B|K)$ is defined as $[(A|H) \wedge (B|K)]^c = 1 - (A|H) \wedge (B|K)$.

We observe that $[(A|H) \wedge (B|K)]^c = 1 - min\{A|H, B|K\}\,|\,(H \vee K) = (1 - min\{A|H, B|K\})\,|\,(H \vee K) = max\{A^c|H, B^c|K\}\,|\,(H \vee K)$. Then, based on the relation $A \vee B = (A^c B^c)^c$ (De Morgan's Law), for the notion of disjunction we give the definition below.

Definition 7. Given any conditional events $A|H, B|K$, the disjunction $(A|H) \vee (B|K)$ is defined as: $(A|H) \vee (B|K) = [(A^c|H) \wedge (B^c|K)]^c$.

We observe that $(A|H) \vee (B|K) = 1 - min\{A^c|H, B^c|K\}\,|\,(H \vee K) = (1 - min\{A^c|H, B^c|K\})\,|\,(H \vee K) = max\{A|H, B|K\}\,|\,(H \vee K)$. If we assess $P(A|H) = x, P(B|K) = y, \mathbb{P}[(A|H) \vee (B|K)] = \gamma$, then

$$(A|H) \vee (B|K) = 1 \cdot (AH \vee BK) + x \cdot H^c B^c K + y \cdot A^c H K^c + \gamma \cdot H^c K^c.$$

Prevision sum rule. The classical formula $P(A \vee B) = P(A) + P(B) - P(AB)$ still holds for the conjunction and disjunction of conditional events. Indeed, by recalling Theorem 2 and the properties of iterated conditioning, we have

$(A|H) \vee (B|K) + (A|H) \wedge (B|K) = [max\{A|H, B|K\} + min\{A|H, B|K\}]|(H \vee K) = = (A|H + B|K)|(H \vee K) = (A|H)|(H \vee K) + (B|K)|(H \vee K) = A|H + B|K$. Thus

$$\mathbb{P}[(A|H) \vee (B|K)] = \mathbb{P}(A|H) + \mathbb{P}(B|K) - \mathbb{P}[(A|H) \wedge (B|K)].$$

Finally, assuming A, H, B, K logically independent, from (8), γ is a coherent extension of (x, y) if and only if $max\{x, y\} \leq \gamma \leq min\{x + y, 1\}$, i.e.

$$max\{P(A|H), P(B|K)\} \leq \mathbb{P}[(A|H) \vee (B|K)] \leq min\{P(A|H)+P(B|K), 1\}.$$

7. Conclusions

We have considered c.r.q.'s and conditional previsions in the setting of coherence. In agreement with the approach of other authors, a c.r.q. $X|H$ has been looked at as the quantity $XH + \mathbb{P}(X|H)H^c$, which assumes the value $\mathbb{P}(X|H)$ when H is false. We have defined the equality among c.r.q.'s, by showing that $X|HK$ coincides with $[XH + \mathbb{P}(X|HK)H^c]|K$; this equality, by linearity of prevision, allows to directly obtain the compound prevision theorem. By this extended notion of c.r.q., we have obtained some algebraic developments; for instance, we have given a meaning to the sum $X|H+Y|K$; then, the sum of previsions $\mathbb{P}(X|H) + \mathbb{P}(Y|K)$ can be read as the prevision of the sum, $\mathbb{P}(X|H + Y|K)$. We have examined the notion of iterated c.r.q. $(X|H)|K$, which does not coincide with the c.r.q. $X|HK$; in particular, given any event A, the iterated conditional $(A|H)|K$ does not coincide with the conditional event $A|HK$; thus, in our approach the Import–Export Principle is not valid and we avoid the counter-intuitive consequences related to Lewis' triviality result. We have studied the conjunction of two conditional events and we have interpreted the prevision of a conjoined conditional by the betting scheme. Then, using iterated conditionals, we have obtained a disintegration formula for conditional probabilities and a kind of Bayes formula. We have defined the negation of the conjunction of conditionals and by De Morgan's Law the associated disjunction. Differently from other authors, in our coherence-based approach, the result of the conjunction, or the disjunction, is (not a conditional, but) a c.r.q., for which we have determined the lower and upper prevision bounds. We have shown that the usual probabilistic properties continue to hold in terms of previsions.

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References

[1] Adams, E., The logic of conditionals, *Inquiry* 8(1–4):166–197, 1965.

[2] Adams, E., *The Logic of Conditionals*, Reidel, Dordrecht, 1975.

[3] Berti, P., E. Regazzini., and P. Rigo, Well calibrated, coherent forecasting systems, *Theory of Probability & Its Applications* 42(1):82–102, 1998.

[4] Biazzo, V., and A. Gilio, On the linear structure of betting criterion and the checking of coherence, *Annals of Mathematics and Artificial Intelligence* 35(1–4):83–106, 2002.

[5] Biazzo, V., A. Gilio, T. Lukasiewicz., and G. Sanfilippo, Probabilistic logic under coherence: complexity and algorithms, *Annals of Mathematics and Artificial Intelligence* 45(1–2):35–81, 2005.

[6] Biazzo, V., A. Gilio., and G. Sanfilippo, Coherence checking and propagation of lower probability bounds, *Soft Computing* 7(5):310–320, 2003.

[7] Biazzo, V., A. Gilio., and G. Sanfilippo, On the checking of g-coherence of conditional probability bounds, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 11(Suppl 2):75–104, 2003.

[8] Biazzo, V., A. Gilio., and G. Sanfilippo, Generalized coherence and connection property of imprecise conditional previsions, in *Proceedings of IPMU'08*, Malaga, Spain, 2008, pp. 907–914.

[9] Biazzo, V., A. Gilio., and G. Sanfilippo, Coherent conditional previsions and proper scoring rules, in S. Greco et al. (eds.), *Advances in Computational Intelligence*, vol. 300 of *CCIS*, Springer-Verlag, Berlin, 2012, pp. 146–156.

[10] Bouchon-Meunier, B., G. Coletti., and C. Marsala, Independence and possibilistic conditioning, *Annals of Mathematics and Artificial Intelligence* 35(1–4):107–123, 2002.

[11] Bruno, G., and A. Gilio, Confronto fra eventi condizionati di probabilità nulla nell'inferenza statistica bayesiana, *Rivista di Matematica per le Scienze Economiche e Sociali* 2:141–152, 1985.

[12] Calabrese, P., An algebraic synthesis of the foundations of logic and probability, *Information Sciences* 42(3):187–237, 1987.

[13] Capotorti, A., and B. Vantaggi, A general interpretation of conditioning and its implication on coherence, *Soft Computing* 3(3):148–153, 1999.

[14] Capotorti, A., A. Lad., and G. Sanfilippo, Reassessing accuracy rates of median decisions, *The American Statistician* 61(2):132–138, 2007.

[15] Coletti, G., and R. Scozzafava, Conditioning and inference in intelligent systems, *Soft Computing* 3(3):118–130, 1999.

[16] Coletti, G., and R. Scozzafava, From conditional events to conditional measures: a new axiomatic approach, *Annals of Mathematics and Artificial Intelligence* 32(1–4):373–392, 2001.

[17] Coletti, G., and R. Scozzafava, *Probabilistic Logic in a Coherent Setting*, Kluwer, Dordrecht, 2002.

[18] Coletti, G., R. Scozzafava., and B. Vantaggi, Inferential processes leading to possibility and necessity, *Information Sciences* 245:132–145, 2013.

[19] de Finetti, B., La logique de la probabilité, in *Actes du Congrès International de Philosophie Scientifique, Paris, 1935*, Hermann et C.ie, Paris, 1936, pp. IV 1–IV 9.

[20] de Finetti, B., *Teoria delle probabilitá*, 2 vols., Ed. Einaudi, Torino, 1970.

[21] Dubois, D., and H. Prade, Conditional objects as nonmonotonic consequence relationships, *IEEE Transactions on Systems, Man, and Cybernetics* 24(12):1724 –1740, 1994.

[22] Edgington, D., On conditionals, *Mind* 104(414):235–329, 1995.

[23] Edgington, D., Estimating conditional chances and evaluating counterfactuals, *Studia Logica*, this issue.

[24] Fugard, A. J. B., N. Pfeifer, B. Mayerhofer., and G. D. Kleiter, How people interpret conditionals: Shifts toward the conditional event, *Journal of Experimental Psychology: Learning, Memory, and Cognition* 37(3):635–648, 2011.

[25] Gilio, A., Criterio di penalizzazione e condizioni di coerenza nella valutazione soggettiva della probabilità, *Bollettino della Unione Matematica Italiana* 4B(3, Serie 7):645–660, 1990.

[26] Gilio, A., Probabilistic reasoning under coherence in system P, *Annals of Mathematics and Artificial Intelligence* 34(1–3):5–34, 2002.

[27] Gilio, A., Generalizing inference rules in a coherence-based probabilistic default reasoning, *International Journal of Approximate Reasoning* 53(3):413–434, 2012.

[28] Gilio, A., and D. Over, The psychology of inferring conditionals from disjunctions: a probabilistic study, *Journal of Mathematical Psychology* 56(2):118–131, 2012.

[29] Gilio, A., and S. Ingrassia, Totally coherent set-valued probability assessments, *Kybernetika* 34(1):3–15, 1998.

[30] Gilio, A., and G. Sanfilippo, Quasi Conjunction and p-entailment in nonmonotonic reasoning, in C. Borgelt et al., (eds.), *Combining Soft Computing and Statistical Methods in Data Analysis*, vol. 77 of *AISC*, Springer, Heidelberg, 2010, pp. 321–328.

[31] Gilio, A., and G. Sanfilippo, Quasi conjunction and inclusion relation in probabilistic default reasoning, in W. Liu (ed.), *ECSQARU 2011*, vol. 6717 of *LNCS*, Springer, Heidelberg, 2011, pp. 497–508.

[32] Gilio, A., and G. Sanfilippo, Conditional random quantities and iterated conditioning in the setting of coherence, in L. C. van der Gaag (ed.), *ECSQARU 2013*, vol. 7958 of *LNCS*, Springer, Heidelberg, 2013, pp. 218–229.

[33] Gilio, A., and G. Sanfilippo, Conjunction, disjunction and iterated conditioning of conditional events, in *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*, vol. 190 of *AISC*, Springer, Heidelberg, 2013, pp. 399–407.

[34] Gilio, A., and G. Sanfilippo, Probabilistic entailment in the setting of coherence: The role of quasi conjunction and inclusion relation, *International Journal of Approximate Reasoning* 54(4):513–525, 2013.

[35] Gilio, A., and G. Sanfilippo, Quasi conjunction, quasi disjunction, t-norms and t-conorms: probabilistic aspects, *Information Sciences* 245:146–167, 2013.

[36] Gilio, A., and R. Scozzafava, Conditional events in probability assessment and revision, *IEEE Transactions on Systems, Man, and Cybernetics* 24(12):1741 –1746, 1994.

[37] Goodman, I. R., H. T. Nguyen., and E. A. Walker, *Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning*, North-Holland, Amsterdam, 1991.

[38] Jeffrey, R., Matter-of-fact conditionals, *Proceedings of the Aristotelian Society, Supplementary Volume* 65:161–183, 1991.

[39] Kaufmann, S., Conditionals right and left: probabilities for the whole family, *Journal of Philosophical Logic* 38:1–53, 2009.

[40] Lad, F., Coherent prevision as a linear functional without an underlying measure space: the purely arithmetic structure of conditional quantities, in G. Coletti et al. (eds.), *Mathematical Models for Handling Partial Knowledge in Artificial Intelligence*, Plenum Press, New York, 1995, pp. 101–112.

[41] Lad, F., *Operational Subjective Statistical Methods*, Wiley, New York, 1996.

[42] Lad, F., G. Sanfilippo., and G. Agró, Completing the logarithmic scoring rule for assessing probability distributions, *AIP Conference Proceedings* 1490(1):13–30, 2012.

[43] Lewis, D., Probabilities of conditionals and conditional probabilities, *Philosophical Review* 85(3):297–315, 1976.

[44] McGee, V., Conditional probabilities and compounds of conditionals, *Philosophical Review* 98(4):485–541, 1989.

[45] Milne, P., Bruno de Finetti and the Logic of Conditional Events, *British Journal for the Philosophy of Science* 48(2):195–232, 1997.

[46] Pedersen, A. P., An extension theorem and a numerical representation theorem for qualitative comparative expectations, *Studia Logica*, this issue.

[47] Pfeifer, N., Reasoning about uncertain conditionals, *Studia Logica*, this issue.

[48] Pfeifer, N., Experiments on aristotle's thesis: Towards an experimental philosophy of conditionals, *The Monist* 95(2):223–240, 2012.

[49] Pfeifer, N., and G. D. Kleiter, Inference in conditional probability logic, *Kybernetika* 42:391–404, 2006.

[50] Pfeifer, N., and G. D. Kleiter, Framing human inference by coherence based probability logic, *Journal of Applied Logic* 7(2):206–217, 2009.

[51] Pfeifer, N., and G. D. Kleiter, The conditional in mental probability logic, in M. Oaksford, and N. Chater (eds.), *Cognition and Conditionals: Probability and Logic in Human Thought*, Oxford University Press, Oxford, 2010, pp. 153–173.

[52] Schay, G., An algebra of conditional events, *Journal of Mathematical Analysis and Applications* 24:334–344, 1968.

[53] Thorn, P. D., and G. Schurz, A Utility based evaluation of logico-probabilistic systems, *Studia Logica*, this issue.

[54] Unterhuber, M., *Possible Worlds Semantics for Indicative and Counterfactual Conditionals? A Formal-Philosophical Inquiry into Chellas-Segerberg Semantics*, Ontos Verlag (Logos Series), Frankfurt, 2013.

[55] Unterhuber, M., and G. Schurz, Completeness and Correspondence in Chellas-Segerberg Semantics, *Studia Logica*, this issue.

[56] Wallmann, C., and G. D. Kleiter, Exchangeability in probability logic, in S. Greco et al. (eds.), *Advances in Computational Intelligence*, vol. 300 of *CCIS*, Springer-Verlag, Berlin, 2012, pp. 157–167.

[57] Wallmann, C., and G. D. Kleiter, Probability Propagation in Generalized Inference Forms, *Studia Logica*, this issue.

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