Updating knowledge using subsets

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ABSTRACT. *Larry Moss and Rohit Parikh used subset semantics to characterize a family of logics for reasoning about knowledge. An important feature of their framework is that subsets always decrease based on the assumption that knowledge always increases. We drop this assumption and modify the semantics to account for logics of knowledge that handle arbitrary changes, that is, changes that do not necessarily result in knowledge increase, such as the update of our knowledge due to an action. We present a system which is complete for subset spaces and prove its decidability.*

KEYWORDS: *knowledge update, bimodal logic, logic of knowledge, subset semantics.*

DOI:10.3166/JANCL.21.427–441

1. Introduction

Subset logic is a bimodal logic that combines two modal operators, one corresponding to knowledge, and one corresponding to effort, and models the increase of knowledge after a larger amount of resources has been spent in acquiring it. Subset logic has been introduced by Moss and Parikh who also established the basic results (Moss & Parikh, 1992; Dabrowski et al., 1996). A great deal of further research has been devoted to characterizing the underlying structure of subsets using axioms of this logic. For example, the system "topologic" has been found complete with respect to topological spaces (Georgatos, 1993). Variants of this logic have also been developed to address knowledge after program termination and time passing (Heinemann, 1999; Heinemann, 2007).

The main novelty of subset logic is its semantics, where, after fixing a space of subsets of a set of worlds, sentences are interpreted over a pair (x, U), where x is the actual world the agent resides in, and U is the view the agent has. The agent's view consists of those worlds the agent considers possible. We can represent effort by restricting the agent's view. Restricting the view means that the agent cancels out some of the alternatives, and, as a result, increase of knowledge occurs. In this paper, we would like to explore the possibility of using subsets of worlds to model any kind of change, rather than change that corresponds to increase of knowledge. Consider the following example.

EXAMPLE 1. — Imagine a room with a table and two cubes: a red one and a blue one. The agent is outside the room but knows that one of the cubes is on the table and one is on the floor. Now, suppose that the agent instructs a robot to enter the room and place the red cube on the floor. The update of the agent's knowledge base after the robot's action can be modeled as follows. Denote the sentence "the red cube is on the floor" with r, and similarly with b for the blue cube. The initial view is $U = \{w_1, w_2\}$ where $w_1 = \{r \wedge \neg b\}$ and $w_2 = \{\neg r \wedge b\}$. After the robot is instructed to place the red cube on the floor the first possibility persists while the second possibility turns into $w_2' = \{r \wedge b\}$. The resulting view of the agent is $U' = \{w_1, w_2'\}$. □

Note that, in the above example, U' is not a subset of U and change does not result in an increase but rather in an update of knowledge (in the sense of (Katsuno & Mendelzon, 1991)). In particular, the sentence

$$\mathsf{K}(r \leftrightarrow \neg b)$$

is true at U but false in U'. Further, the resulting subset is determined by the transformation of its components. Before instructing the robot we do not know whether the red cube is on the floor:

$$\neg\mathsf{K}\neg r$$

but after instructing the robot (call this action a) the following holds:

$$[a]\mathsf{K}r.$$

Our modification is threefold:

– We consider subsets that do not necessarily form a topological space but rather they are determined by the accessibility relations. This is not contrary to the basic intuition behind the Moss-Parikh logics but rather complementary. We do not want to express the structure of subsets but rather the structure of actions that restrict the agent's view to those subsets.

– We consider changes that do not necessarily result in a smaller subset. Frequently, we reason with defeasible knowledge or we jump to conclusions as in non-monotonic reasoning. Other times, we need to revise our beliefs. In such cases, the resulting epistemic state is not a refinement but rather a transformation of the original one.

– We make explicit the accessibility relations that bring about different forms of change. Such actions can be the result of a program, a game move, pieces of information about a changing world, or, simply, the passage of time.

We view this logic as a tool for studying the transformations of knowledge in a more general setting much like dynamic epistemic logic (van Ditmarsch et al., 2007), although we restrict our attention to a single agent (for a multiple agent approach using subset logic see (Heinemann, 2008)). Systems that combine two or more modal logics are nothing new (Kracht & Wolter, 1991; Gabbay & Shehtman, 1998). Their theory, in the simple cases, is straightforward. Similarly, basic results such as completeness for Kripke models and decidability are straightforward once those have been obtained for the individual logics separately (see (Gabbay et al., 2003)). However, our completeness and decidability results refer *not* to the usual frame models with accessibility relations but, instead, we use subset models.

In the next section, we define the subset logic $\mathbf{SC}$ for reasoning about change and prove completeness for subset semantics via an appropriate translation to/from Kripke models. Then, we prove a normal form theorem and use it to sketch a proof of completeness in Section 3 and a proof of decidability in Section 4. We conclude with a sketch of a logic that incorporates actions.

2. Syntax and semantics

We follow the notation of (Moss & Parikh, 1992).

Our language is bimodal and propositional. We start with a countable set Atom of *atomic formulas* containing two distinguished elements $\top$ and $\bot$. Then the *language* $\mathcal{L}$ is the least set such that $\mathsf{Atom} \subseteq \mathcal{L}$ and closed under the following rules:

$$\frac{\phi, \psi \in \mathcal{L}}{\phi \wedge \psi \in \mathcal{L}} \qquad \frac{\phi \in \mathcal{L}}{\neg\phi, \Box\phi, \mathsf{K}\phi \in \mathcal{L}}$$

Notice that, in contrast to example 1, the language does not contain actions. We make use of a single $\Box$ modality but all results extend to a multi-modal setting (see Section 5). The intended interpretation of the $\Box$ modality is that of necessity, meaning, what is true in all worlds which are possible outcomes of the change of the current world. However, in the bimodal setting, $\Box$ acquires a second dimension. Change of the current world can be indeterminate, that is, it can be described with several worlds accessible from the current one, and we make no assumption on the properties of this change (e.g. symmetry, transitivity, etc). Therefore, $\Box$ is a just a modality obeying normality. What turns the $\Box$ modality in to *necessity during update* is not change of truth, but change of knowledge. All possible changes (to talk about a particular change one needs a richer language, e.g. actions) of the knowledge base K is a priori specified, in the sense that it it described with a formula ϕ, and results to a unique knowledge base $K \circ \phi$, the update of K with ϕ. So change at the knowledge level is deterministic. The update part of $\Box$ will be described, semantically, with subsets and, syntactically, with combination axioms (axioms involving both $\Box$ and K).

The interpretation of the language using subsets follows:

DEFINITION 2. — *Let X be a set, R a binary relation on X,* i.e.*, $R \subseteq X \times X$ called* accessibility*, and $\mathcal{O}$ a subset of the powerset of X,* i.e. *$\mathcal{O} \subseteq \mathcal{P}(X)$ such that $X \in \mathcal{O}$. We denote the set $\{(x, U) : x \in X, U \in \mathcal{O},$ and $x \in U\} \subseteq X \times \mathcal{O}$ by $X \dot{\times} \mathcal{O}$. For each $U \in \mathcal{O}$, let U^R be the set of the elements accessible from U, that is, the set $\{y : (x, y) \in R, x \in U\}$. The set $\mathcal{O}$ will be called R-*closed *if whenever $U \in \mathcal{O}$ then $U^R \in \mathcal{O}$.*

Let $\mathcal{O}$ be R-closed, then the triple $\langle X, R, \mathcal{O}\rangle$ will be called a subset frame*. A* model *is a quadruple $\langle X, R, \mathcal{O}, i\rangle$, where $\langle X, R, \mathcal{O}\rangle$ is a subset frame and i a map from* Atom *to $\mathcal{P}(X)$ with $i(\top) = X$ and $i(\bot) = \emptyset$ called* initial interpretation.

DEFINITION 3. — *The* satisfaction relation *$\models_{\mathcal{M}}$, where $\mathcal{M}$ is the model $\langle X, \mathcal{O}, R, i\rangle$, is a subset of $(X \dot{\times} \mathcal{O}) \times \mathcal{L}$ defined recursively by (we write $x, U \models_{\mathcal{M}} \phi$ instead of $((x, U), \phi) \in \models_{\mathcal{M}}$):*

$$
\begin{array}{lll}
x, U \models_{\mathcal{M}} A & \textit{iff} & x \in i(A), \textit{ where } A \in \mathsf{Atom} \\
x, U \models_{\mathcal{M}} \phi \wedge \psi & \textit{iff} & x, U \models_{\mathcal{M}} \phi \textit{ and } x, U \models_{\mathcal{M}} \psi \\
x, U \models_{\mathcal{M}} \neg\phi & \textit{iff} & x, U \not\models_{\mathcal{M}} \phi \\
x, U \models_{\mathcal{M}} \mathsf{K}\phi & \textit{iff} & \textit{for all } y \in U,\ y, U \models_{\mathcal{M}} \phi \\
x, U \models_{\mathcal{M}} \Box\phi & \textit{iff} & \textit{for all } y \in X \textit{ such that } (x, y) \in R,\ y, U^R \models_{\mathcal{M}} \phi.
\end{array}
$$

If $x, U \models_{\mathcal{M}} \phi$ for all (x, U) belonging to $X \dot{\times} \mathcal{O}$ then ϕ is valid *in $\mathcal{M}$, denoted by $\mathcal{M} \models \phi$.*

We abbreviate $\neg\Box\neg\phi$ and $\neg\mathsf{K}\neg\phi$ by $\Diamond\phi$ and $\mathsf{L}\phi$ respectively. We have that

$$
\begin{array}{ll}
x, U \models_{\mathcal{M}} \mathsf{L}\phi & \text{if there exists } y \in U \text{ such that } y, U \models_{\mathcal{M}} \phi \\
x, U \models_{\mathcal{M}} \Diamond\phi & \text{if there exists } y \in X \text{ such that } (x, y) \in R \text{ and } y, U^R \models_{\mathcal{M}} \phi.
\end{array}
$$

The axiom system $\mathbf{SC}$ consists of axiom schemes 1 through 8 and rules of table 1 (see page 431). We will write $\vdash_{\mathbf{SC}} \phi$ iff ϕ is a theorem of $\mathbf{SC}$.

Observe that we require that $\Box$ satisfies the **K** (normality) axiom and K satisfies the **S5** axioms. We have two interaction axioms:

Axiom 7 is the *cross axiom*, a standard axiom for subset logic, a propositional analogue of the Barcan formula and the "perfect recall" of (Schmidt & Tishkovsky, 2008) for a single modality.

Axiom 8 together with Axiom 7 implies that the accessibility relation of $\Box$ is a function (deterministic) on subsets, that is, from the epistemic point of view, it implies that possible knowledge is necessary.

Observe that the cornerstone axiom of subset logic:

$$(A \to \Box A) \wedge (\neg A \to \Box \neg A), \text{ for } A \in \mathsf{Atom}$$

is not valid in $\mathbf{SC}$. It stipulates that the non-epistemic facts true in the world of an agent will remain true as the only change we allow is epistemic. The actual state of an agent remains always the same although the agent's view may change. In contrast, the logic we axiomatize allows arbitrary changes including changes to the agent's actual state and this axiom is no longer valid.

Table 1. *Axioms and Rules of* $\mathbf{SC}$

Axioms

1) All propositional tautologies
2) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
3) $\mathsf{K}(\phi \to \psi) \to (\mathsf{K}\phi \to \mathsf{K}\psi)$
4) $\mathsf{K}\phi \to \phi$
5) $\mathsf{K}\phi \to \mathsf{K}\mathsf{K}\phi$
6) $\phi \to \mathsf{K}\mathsf{L}\phi$
7) $\mathsf{K}\Box\phi \to \Box\mathsf{K}\phi$
8) $\Diamond\mathsf{K}\phi \to \mathsf{K}\Box\phi$

Rules

$$\frac{\phi \to \psi, \phi}{\psi}\text{ MP}$$

$$\frac{\phi}{\mathsf{K}\phi}\text{ K-Necessitation} \qquad \frac{\phi}{\Box\phi}\ \Box\text{-Necessitation}$$

The following holds:

THEOREM 4. — *The axioms and rules of* $\mathbf{SC}$ *are sound with respect to subset frames.*

PROOF 5. — The proof is straightforward and we show only soundness for Axiom 8. Suppose $x, U \models \Diamond\mathsf{K}\phi$. This implies that there is a w such that xRw and $w, U^R \models \mathsf{K}\phi$. Now, let $y \in U$ and yRz. We need to show that $z, U^R \models \phi$ which follows from $w, U^R \models \mathsf{K}\phi$. ■

We may choose to interpret $\mathbf{SC}$ on Kripke frames. Let $(W, R_\Box, R_\mathsf{K})$ be a frame, where W is a set of worlds with two binary accessibility relations $R_\Box$ and R_K on W. Then the above axiom system corresponds to the following first order properties:

1) R_K is an equivalence relation,
2) $\forall x, y, w(xR_\Box w \wedge wR_\mathsf{K} y \to \exists w'(xR_\mathsf{K} w' \wedge w' R_\Box y))$(see Fig. 1), and

$$\begin{tikzcd} x \arrow[r, dashed, "R_{\mathsf{K}}"] \arrow[d, "R_{\Box}"'] & w' \arrow[d, dashed, "R_{\Box}"] \\ w \arrow[r, "R_{\mathsf{K}}"'] & y \end{tikzcd}$$

Figure 1. *First-order property corresponding to Axiom 7*

$$\begin{tikzcd} x \arrow[r, "R_{\mathsf{K}}"] \arrow[d, "R_{\Box}"'] & w' \arrow[d, "R_{\Box}"] \\ w \arrow[r, dashed, "R_{\mathsf{K}}"'] & y \end{tikzcd}$$

Figure 2. *First-order property corresponding to Axiom 8*

3) $\forall x, y, w, w'(xR_{\mathsf{K}}w' \wedge w'R_{\Box}y \wedge xR_{\Box}w \rightarrow wR_{\mathsf{K}}y)$ (see Fig. 2).

Property 1 corresponds to the $\mathbf{S5}$ set of axioms for K, Property 2 corresponds to Axiom 7, and Property 3 corresponds to Axiom 8. We will show the last correspondence (Axiom 8 is the least known). Suppose Property 3 does not hold, *i.e.* there exist x, y, w, w' with $xR_{\mathsf{K}}w$, $wR_{\Box}y$, $xR_{\Box}w'$ but $\neg w'R_{\mathsf{K}}w$. Consider a valuation i such that $i(A) = \{y\}$, for some atomic A. In a model based on i, we have $z \models \neg A$ for all $z \neq y$. So, by our initial assumption, we have $w' \models \mathsf{K}\neg A$. Therefore, $x \models \Diamond\mathsf{K}\neg A$ but $x \not\models \mathsf{K}\Box\neg A$. So Axiom 8 fails. The other direction is straightforward.

To prove (strong) completeness of the logic for Kripke models is straightforward because $\mathbf{SC}$ is the fusion of $\mathbf{T}$ and $\mathbf{K}$ augmented by Axioms 7 and 8 which are Sahlqvist formulas (Sahlqvist, 1975; Blackburn et al., 2002).

We can show completeness for subset models by translating any Kripke model based on a frame as above in to an equivalent subset model. Let $(W, R_{\Box}, R_{\mathsf{K}}, i)$ be a Kripke model whose underlying frame satisfies Properties 1-3 as above, then let $(W, \mathcal{O}, R_{\Box}, i)$ be the subset model, where $\mathcal{O}$ consists of the equivalence classes of R_{K}, that is, the set of subsets $\mathcal{O}$ is a partition of W[1]. We have the following

1. This translation, due to one of the reviewers, resulted in a considerable simplification of the completeness proof over the original one sketched in the next section.

LEMMA 6. — *The partition $\mathcal{O}$ is $R_{\Box}$-closed.*

PROOF 7. — Let $U \in \mathcal{O}$. We will show that $U^{R_{\Box}}$ is a R_{K}-equivalence class. Let $x, y \in U^{R_{\Box}}$ and assume that $x', y' \in U$ are such that $x' R_{\Box} x$ and $y' R_{\Box} y$. By Property 3, we have $x R_{\mathsf{K}} y$. To show that $U^{R_{\Box}}$ is maximal, let $x R_{\mathsf{K}} y$ with $x \in U^{R_{\Box}}$. Let $x' \in U$ such that $x' R_{\Box} x$. By Property 2, $x' R_{\mathsf{K}} \circ R_{\Box} y$, and, therefore, $y \in U^{R_{\Box}}$. ■

Furthermore, the subset model defined by this translation is equivalent to the starting Kripke model in the following sense:

LEMMA 8. — *For all $w \in W$ and $\phi \in \mathcal{L}$, we have that*

$$w \models \phi \textit{ iff } w, U \models \phi,$$

where U is the R_{K}-equivalence class of w.

PROOF 9. — By induction on ϕ. The atomic and boolean cases are straightforward. Suppose $w \models \Box\phi$. Let $w R_{\Box} w'$. We have that $w' \models \phi$ so $w', U^{R_{\Box}} \models \phi$, by induction hypothesis, and therefore $w, U \models \Box\phi$. The converse is similar as is the $\mathsf{K}\phi$ case. ■

Further, there is a simple reduction of a subset space model $(X, \mathcal{O}, R, i)$ to an equivalent Kripke model (see for example Section 4.1 in (Georgatos, 1993))

$$(X \dot{\times} \mathcal{O}, R_{\Box}, R_{\mathsf{K}}, i'),$$

where

- $(x, U) R_{\Box} (y, W)$ iff xRy and $W = U^R$,
- $(x, U) R_{\mathsf{K}} (y, W)$ iff $W = U$, and
- $i'(A) = \{(x, U) : x \in i(A)\}$.

Similarly, we have that

$$x, U \models \phi \text{ iff } (x, U) \models \phi.$$

The above two translations are not, in general, inverses of each other. The translation from a Kripke model to a subset model always produces a partition on the worlds. Those translations combine to show the following

THEOREM 10. — *The axiom system* $\mathbf{SC}$ *is strongly complete with the class of R-closed subset models.*

3. Normal form

In this section, we will show that $\mathbf{SC}$ possesses a normal form (Theorem 15). We will sketch a translation of the canonical model to a subset model based using the normal form and, in the next section, we will employ the normal form again to prove the finite model property. We will need the following

LEMMA 11. — *The following are theorems of* $\mathbf{SC}$.

1) $\Diamond(\phi \wedge \mathsf{K}\psi) \leftrightarrow \Diamond\phi \wedge \Diamond\mathsf{K}\psi$
2) $\Box(\phi \vee \mathsf{K}\psi) \leftrightarrow \Box\phi \vee \Box\mathsf{K}\psi$
3) $\Box\mathsf{K}\phi \leftrightarrow \Diamond\bot \vee \mathsf{K}\Box\phi$
4) $\Box\mathsf{L}\phi \leftrightarrow \Diamond\bot \vee \mathsf{L}\Diamond\phi$

PROOF 12. — For Case 1, the one implication is straightforward. For the other

1. $\Diamond\phi \wedge \Diamond\mathsf{K}\psi \rightarrow \Diamond\phi \wedge \Box\mathsf{K}\psi$ by Axioms 7 and 8
2. $\Diamond\phi \wedge \Box\mathsf{K}\psi \rightarrow \Diamond(\phi \wedge \mathsf{K}\psi)$ in a normal system.

Similarly for Case 2.

Both Cases 3 and 4 follow from normality, Axiom 7, in one direction, and Axiom 8, in the opposite direction. ■

DEFINITION 13. — *Let $\mathcal{L}^{\Box} \subseteq \mathcal{L}$ be the set of formulas generated by the following rules:*

$$\mathsf{Atom} \subseteq \mathcal{L}^{\Box} \qquad \frac{\phi, \psi \in \mathcal{L}^{\Box}}{\phi \wedge \psi \in \mathcal{L}^{\Box}} \qquad \frac{\phi \in \mathcal{L}^{\Box}}{\neg\phi, \Box\phi \in \mathcal{L}^{\Box}}$$

Let $\mathcal{L}^{\mathsf{K}}$ be the set $\{\mathsf{K}\phi, \mathsf{L}\phi | \phi \in \mathcal{L}^{\Box}\}$.

DEFINITION 14. — *We say that $\phi \in \mathcal{L}$ is in* prime normal form *(PNF) if it has the form*

$$\psi \wedge \mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_i$$

where $\psi, \psi', \psi_i \in \mathcal{L}^{\Box}$ and n is finite. ϕ is in disjunctive normal form *(DNF) if it has the form $\bigvee_{i=1}^{m} \phi_i$, where each ϕ_i is in PNF and m is finite.*

We shall omit the cardinality of (finite) conjunctions and disjunctions, writing, e.g., $\bigvee_i \phi_i$ instead of $\bigvee_{i=1}^{n} \phi_i$. Suppose that ϕ is a formula in the following form

$$\bigwedge_i \left(\psi_i \vee \mathsf{L}\psi_i' \vee \bigvee_j \mathsf{K}\psi_i^j \right),$$

where $\psi_i, \psi_i', \psi_i^j \in \mathcal{L}^{\Box}$. We shall call such a form *conjunctive normal form* (CNF). Using the distributive laws, we may show that DNF and CNF are effectively interchangeable up to equivalence.

THEOREM 15 (DNF). — *For every $\phi \in \mathcal{L}$, there is (effectively) a ψ in DNF such that*

$$\vdash_{\mathbf{SC}} \phi \leftrightarrow \psi.$$

PROOF 16. — *By induction on the logical structure of ϕ.*

– If $\phi = A$, where A is atomic, the result is immediate because the set of atomic formulas belongs to $\mathcal{L}^{\Box}$ so A is equivalent to a formula in PNF.

– Suppose that $\phi = \neg\psi$. Then, by induction hypothesis, ψ is equivalent to a formula in DNF, which implies that ϕ is equivalent to a formula in CNF which is equivalent to a formula in DNF, as noted just above.

– If $\phi = \psi \vee \chi$ then ϕ is equivalent to a disjunction of two formulas in DNF, i.e. *is itself in DNF.*

– If $\phi = \mathsf{K}\psi$ then ψ is equivalent to a formula in CNF, and hence ϕ is equivalent to a formula of the following form

$$\bigwedge_i \mathsf{K}\left(\chi_i \vee \mathsf{L}\chi_i' \vee \bigvee_j \mathsf{K}\chi_i^j\right),$$

since K distributes over conjunctions. Now, since $\mathsf{K}(\phi \vee \mathsf{K}\psi) \leftrightarrow \mathsf{K}\phi \vee \mathsf{K}\psi$ and $\mathsf{KL}\chi \leftrightarrow \mathsf{L}\chi$ are theorems of **S5***, the above formula is equivalent to*

$$\bigwedge_i \left(\mathsf{L}\chi_i' \vee \left(\mathsf{K}\chi_i \vee \bigvee_j \mathsf{K}\chi_i^j\right)\right),$$

which is equivalent to a formula in CNF.

– If $\phi = \Box\psi$ then, by induction hypothesis, ψ is equivalent to a formula in CNF, and hence ϕ is equivalent to a formula of the following form

$$\bigwedge_i \Box\left(\chi_i \vee \mathsf{L}\chi_i' \vee \bigvee_j \mathsf{K}\chi_i^j\right),$$

since $\Box$ distributes over conjunctions. By repeated applications of Lemma 11.2, the above formula is equivalent to

$$\bigwedge_i \left(\Box\chi_i \vee \Box\mathsf{L}\chi_i' \vee \bigvee_j \Box\mathsf{K}\chi_i^j\right). \tag{1}$$

Using Lemmas 11.4 and 11.3, (1) is equivalent to

$$\bigwedge_i \left(\Box\chi_i \vee \Diamond\bot \vee \mathsf{L}\Diamond\chi_i' \vee \bigvee_j \mathsf{K}\Box\chi_i^j\right), \tag{2}$$

which is in CNF. ■

We will show how one can construct a subset model equivalent to the canonical model. This syntactic construction is more nuanced from the one in the previous section as it identifies those sets of maximal consistent theories that correspond to a single point. All proofs for the rest of this section are omitted.

The *canonical model* of $\mathbf{SC}$ is the structure

$$\mathcal{C} = \left(S, \{\xrightarrow{\Diamond}, \xrightarrow{\mathsf{L}}\}, v\right),$$

where

$$\begin{array}{l} S = \{s \subseteq \mathcal{L} : s \text{ is } \mathbf{SC}\text{-maximal consistent}\}, \\ s \xrightarrow{\Diamond} t \text{ iff } \{\phi \in \mathcal{L} : \Box\phi \in s\} \subseteq t, \\ s \xrightarrow{\mathsf{L}} t \text{ iff } \{\phi \in \mathcal{L} : \mathsf{K}\phi \in s\} \subseteq t, \\ v(A) = \{s \in S : A \in s\}, \end{array}$$

along with the usual satisfaction relation (defined inductively):

$$\begin{array}{lll} s \models_{\mathcal{C}} A & \text{iff} & s \in v(A) \\ s \not\models_{\mathcal{C}} \bot & & \\ s \models_{\mathcal{C}} \neg\phi & \text{iff} & s \not\models_{\mathcal{C}} \phi \\ s \models_{\mathcal{C}} \phi \wedge \psi & \text{iff} & s \models_{\mathcal{C}} \phi \text{ and } s \models_{\mathcal{C}} \psi \\ s \models_{\mathcal{C}} \Box\phi & \text{iff} & \text{for all } t \in S,\ s \xrightarrow{\Diamond} t \text{ implies } t \models_{\mathcal{C}} \phi \\ s \models_{\mathcal{C}} \mathsf{K}\phi & \text{iff} & \text{for all } t \in S,\ s \xrightarrow{\mathsf{L}} t \text{ implies } t \models_{\mathcal{C}} \phi. \end{array}$$

We write $\mathcal{C} \models \phi$, if $s \models_{\mathcal{C}} \phi$ for all $s \in S$.

A canonical model exists for all consistent bimodal systems with the normality axiom scheme for each modality. We have the following well known theorems (see (Chellas, 1980), or (Goldblatt, 1992).)

THEOREM 17 (TRUTH THEOREM). — *For all $s \in S$ and $\phi \in \mathcal{L}$,*

$$s \models_{\mathcal{C}} \phi \qquad \textit{iff} \qquad \phi \in s.$$

THEOREM 18 (COMPLETENESS THEOREM). — *For all $\phi \in \mathcal{L}$,*

$$\mathcal{C} \models \phi \qquad \textit{iff} \qquad \vdash_{\mathbf{SC}} \phi.$$

We will make use of the following sets:

DEFINITION 19. — *For all elements s of the canonical model S, let*

$$s^{\Box} = s \cap \mathcal{L}^{\Box} \qquad s^{\mathsf{K}} = s \cap \mathcal{L}^{\mathsf{K}}$$

and

$$S^{\Box} = \{s^{\Box} : s \in S\} \qquad S^{\mathsf{K}} = \{s^{\mathsf{K}} : s \in S\}.$$

Let $T \subseteq \mathcal{L}^{\Box}(\mathcal{L}^{\mathsf{K}})$*. We say* T *is an* $\mathcal{L}^{\Box}$*(*$\mathcal{L}^{\mathsf{K}}$*)*-theory *if* $T \in S^{\Box}$ *(*$T \in S^{\mathsf{K}}$*).*

Let f be a map from S to $S^{\Box} \dot{\times} S^{\mathsf{K}}$ defined by $f(s) = (s^{\Box}, s^{\mathsf{K}})$. It is straightforward to show that the map f is 1-1 and onto. So, f has an inverse defined by $f^{-1}(T, T') = s(T, T')$, where $s(T, T') \in S$ is the unique maximal consistent extension of $T \cup T'$. Therefore, the worlds of the canonical model may be split in two components that will be used to construct a point and a subset in the following definition.

DEFINITION 20. — *Let* $\mathcal{C} = \left(S, \{\xrightarrow{\Diamond}, \xrightarrow{\mathsf{L}}\}, v\right)$ *be the canonical model of* $\mathbf{SC}$ *and let* $S^{\Box}$*,* S^{K} *be as in Definition 19. The* standard subset model *is defined with*

$$(X, \mathcal{O}, R, v),$$

where

$$X = \{x_T : T \in S^{\Box}\},$$

i.e.*, there is a point* x_T *for each* $\mathcal{L}^{\Box}$*-theory* T*,*

$$\mathcal{O} = \{U_{T'} : T' \in S^{\mathsf{K}}\},$$

i.e., there is a subset $U_{T'}$ *for each* $\mathcal{L}^{\mathsf{K}}$*-theory* T'*, with membership relation defined by*

$$U_{T'} = \{x_T : T \cup T' \textit{ consistent}\},$$

$$x_{T_1} R x_{T_2} \quad \textit{iff} \quad T_1 \xrightarrow{\Diamond} T_2,$$

and

$$v(A) = \{x_T : A \in T\}.$$

Given $U_T \in \mathcal{O}$, we need to show that $U_T^R \in \mathcal{O}$. For each $T \in S^{\mathsf{K}}$, let

$$T^R = \{\mathsf{K}\phi : \mathsf{K}\Box\phi \in T\} \cup \{\mathsf{L}\psi : \mathsf{L}\Diamond\psi \in T\}.$$

It is easy to show that if T^R is consistent then $T^R \in S^{\mathsf{K}}$. We have the following

LEMMA 21. — *For all* $U_T \in \mathcal{O}$*, we have*

$$U_T^R = U_{T^R}.$$

As a corollary, $\mathcal{O}$ is R-closed and so the canonical subset model is well defined. We can show the following (using induction)

THEOREM 22. — *For all* $(x_T, U_{T'}) \in X \dot{\times} \mathcal{O}$*, we have*

$$x_T, U_{T'} \models \phi \quad \textit{iff} \quad \phi \in s(T, T').$$

4. Decidability

In this section, we will show that the logic $\mathbf{SC}$ is decidable by showing that it possesses the finite model property: if a formula is satisfiable then it is satisfiable in a finite subset model, *i.e.*, a subset model that has a finite number of points and therefore a finite number of subsets.

The proof relies upon both the normal form theorem of the previous section as well as a filtration.

To this end, let ϕ be a formula and $(X, \mathcal{O}, R, i)$ a subset model in which ϕ is satisfied, that is, there exist $x \in X$ and $U \in \mathcal{O}$ such that $x, U \models \phi$. The Normal form theorem allows us to assume that ϕ is in DNF, *i.e.* a has the form

$$\bigvee_i \left(\chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right),$$

where $\chi_i, \chi_i', \chi_i^j \in \mathcal{L}^{\Box}$. Denote the set $\{\chi : \chi$ is a subformula of $\chi_i, \chi_i', \chi_i^j : i = 1, \ldots, n\}$ with $\mathcal{L}_\phi^{\Box}$ and $\{\mathsf{L}\chi_i^j : i = 1, \ldots, n\}$ with $\mathcal{L}_\phi^{\mathsf{K}}$.

We will now define a filtration on $(X, \mathcal{O}, i)$ as follows. Let $\sim_\phi$ be an equivalence relation on X defined by $x \sim_\phi y$, for $x, y \in X$, when $x, U \models \chi$ iff $y, U \models \chi$ for all $\chi \in \mathcal{L}_\phi^{\Box}$ and some $U \in \mathcal{O}$. Note here that a straightforward induction shows that the satisfaction of a formula χ in $\mathcal{L}^{\Box}$ is independent of the subset, that is, $x, V \models \chi$ for some $V \in X$ iff $x, U \models \chi$ for all $U \in X$. Denote the equivalence class of x under $\sim_\phi$ with x_ϕ, the set $\{x_\phi : x \in U\}$ with U_ϕ, and the set of all equivalence classes with X_ϕ. Now consider the subset model $(X_\phi, \mathcal{P}(X_\phi), R_\phi, i_\phi)$, where, for all atomic A in $\mathcal{L}_\phi^{\Box}$,

$$i_\phi(A) = \{x_\phi : x \in i(A)\}$$

and, for all $\Box b \in \mathcal{L}_\phi^{\Box}$ and some $U, V \in \mathcal{O}$,

$$x_\phi R_\phi y_\phi \text{ iff } x, U \models \Box b \text{ then } y, V \models b.$$

Observe that "for some $U, V \in \mathcal{O}$" above implies "for all $U, V \in \mathcal{O}$", as $\Box b \in \mathcal{L}_\phi^{\Box}$, and that the powerset $\mathcal{P}(X_\phi)$ is R_ϕ-closed.

We have the following

LEMMA 23. — *For all $\psi \in \mathcal{L}_\phi^{\Box}$, we have*

$$x, U \models \psi \textit{ iff } x_\phi, U_\phi \models \psi.$$

PROOF 24. — The atomic case follows from the definition of i_ϕ. The boolean cases are straightforward. Now, suppose ψ is of the form $\Box\chi$. Let $x, U \models \Box\chi$. To show that $x_\phi, U_\phi \models \Box\chi$. Let $x_\phi R_\phi y_\phi$. By definition of R_ϕ, we have $y, V \models \chi$ for some $V \in \mathcal{O}$.

By induction hypothesis, we have $y_\phi, V_\phi \models \chi$. As mentioned above $y_\phi, W \models \chi$ for all $W \subseteq X_\phi$ since $\chi \in \mathcal{L}_\phi^\Box$. In particular, $y_\phi, U_\phi^{R_\phi} \models \chi$. For the other direction, suppose $x_\phi, U_\phi \models \Box\chi$ and let xRy. We have $x_\phi R_\phi y_\phi$, and so $y_\phi, U_\phi^{R_\phi} \models \chi$. As $\chi \in \mathcal{L}_\phi^\Box$, we have $y_\phi, (U^R)_\phi \models \chi$ ($y_\phi \in (U^R)_\phi$ because $y \in U^R$). By induction hypothesis, $y, U^R \models \chi$. ■

This extends to all subformulas of ϕ by the following

LEMMA 25. — *For all ψ, where ψ is a subformula of ϕ, we have*

$$x, U \models \psi \textit{ iff } x_\phi, U_\phi \models \psi.$$

PROOF 26. — If $\psi \in \mathcal{L}_\phi^\Box$, the lemma follows from the previous lemma. The boolean cases are straightforward. Now, suppose ψ belongs to $\mathcal{L}_\phi^{\mathsf{K}}$, *i.e.* is of the form $\mathsf{L}\chi$ where $\chi \in \mathcal{L}_\phi^\Box$. Let $x, U \models \mathsf{L}\chi$. We must show that $x_\phi, U_\phi \models \mathsf{L}\chi$. There exists $y \in U$ such that $y, U \models \chi$. By induction hypothesis we have $y_\phi, U_\phi \models \chi$, so $x_\phi, U_\phi \models \mathsf{L}\chi$. For the other direction, suppose $x_\phi, U_\phi \models \mathsf{L}\chi$. There exists $y_\phi \in U_\phi$ such that $y_\phi, U_\phi \models \chi$ so, by induction hypothesis, $y, U \models \chi$ and therefore $x, U \models \mathsf{L}\chi$. ■

As a consequence, the logic $\mathbf{SC}$ satisfies the finite model property with respect to the class of $\mathbf{SC}$-models. The main result of this section follows.

COROLLARY 27. — *The logic* $\mathbf{SC}$ *is decidable.*

5. Conclusion

We have presented a variant of the Moss-Parikh Subset Logic that handles arbitrary changes along with a completeness and decidability result. Our presentation has made use of a single update ($\Box$) modality but extending the language, semantics, and subsequent results to a multi-modal setting is straightforward. For the sake of completeness we will briefly mention how this can be done but we will omit all details.

First, the language will be augmented with a set Act of symbols corresponding to sorts of changes. A change can be a result of an action but not necessarily so (for example, time passing). As a result, we need to include in the language formulas of the form $[a]\phi$. On the semantics side, models will be equipped with the set $\{R_a : a \in \mathsf{Act}\}$ of binary relations on X. For each $U \in \mathcal{O}$ and $a \in \mathsf{Act}$, let U^{R_a} be the set of the elements accessible from U, that is, the set $\{y : (x,y) \in R_a, x \in U\}$. The set $\mathcal{O}$ will be called R_a*-closed* if whenever $U \in \mathcal{O}$ then $U^{R_a} \in \mathcal{O}$. If $\mathcal{O}$ is R_a-closed for each $a \in \mathsf{Act}$, then the triple $\langle X, \{R_a\}_{a \in \mathsf{Act}}, \mathcal{O}\rangle$ will be called a *action subset frame* and proceed similarly for the definition of the model. Satisfaction now will include the case

$$x, U \models_{\mathcal{M}} [a]\phi \quad \text{if} \quad \text{for all } y \in X \text{ such that } (x,y) \in R_a,\ y, U^{R_a} \models_{\mathcal{M}} \phi.$$

Now, all results including decidability and the normal form theorem lift to the extended language in a straightforward way as actions do not interact with each other. This extended language allow us to express the original Moss-Parikh modality using a modality $[U]$ for each $U \in \mathcal{O}$, whose semantics are given by the relation

$$R_U = \{(x, x) : x \in U\}.$$

The update operator we introduced in this paper is an interesting addition to the already extensive arsenal of subset logic. We believe that such an addition is very useful as a building block to an epistemic logic that handles change in various forms. As an example, update can be combined with the public announcement operator of dynamic epistemic logic, and/or with a family of knowledge operators corresponding to knowledge of multiple agents (see (Heinemann, 2008) for a proposal within the framework of subset logic).

The combination $\mathbf{PDL}+\mathbf{K5}$ of propositional dynamic logic with a logic of knowledge has been extensively studied in (Schmidt & Tishkovsky, 2008). These results do not carry over in our system because of the interaction axiom

$$\langle a \rangle \mathsf{K}\phi \to \mathsf{K}[a]\phi$$

which is stronger than the ones (NL and CR) considered in (Schmidt & Tishkovsky, 2008). Adding a calculus of action in the manner of dynamic logic, or interpreting the modalities in a temporal context is perhaps the most promising extension of this logic.

Acknowledgements

I am grateful to the referees for several corrections and helpful suggestions. Support for this project was provided by a PSC-CUNY Award, jointly funded by The Professional Staff Congress and The City University of New York.

6. References

Blackburn, P., de Rijke, M., & Venema, Y. (2002). *Modal logic*. Cambridge tracts in theoretical computer science. Cambridge University Press.

Chellas, B. F. (1980). *Modal Logic: An Introduction*. Cambridge: Cambridge University Press.

Dabrowski, A., Moss, L. S., & Parikh, R. (1996). Topological reasoning and the logic of knowledge. *Annals of Pure and Applied Logic*, 78(1-3), 73–110.

van Ditmarsch, H., van der Hoek, W., & Kooi, B. (2007). *Dynamic Epistemic Logic*. Springer, 1st edition.

Gabbay, D., Kurucz, A., Wolter, F., & Zakharyaschev, M. (2003). *Many-dimensional modal logics: theory and applications*. Studies in Logic, 148. Elsevier Science.

Gabbay, D. M. & Shehtman, V. B. (1998). Products of modal logics, part 1. *Logic Journal of the IGPL*, 6(1), 73–146.

Georgatos, K. (1993). *Modal logics for topological spaces*. PhD thesis, City University of New York.

Goldblatt, R. I. (1992). *Logics of time and computation*. CSLI lecture notes. Center for the Study of Language and Information.

Heinemann, B. (1999). Temporal aspects of the modal logic of subset spaces. *Theoretical Computer Science*, 224(1-2), 135–155.

Heinemann, B. (2007). A PDL-like logic of knowledge acquisition. In V. Diekert, M. V. Volkov, & A. Voronkov (Eds.), *CSR*, volume 4649 of *Lecture Notes in Computer Science* (pp. 146–157).: Springer.

Heinemann, B. (2008). Topology and knowledge of multiple agents. In H. Geffner, R. Prada, I. M. Alexandre, & N. David (Eds.), *IBERAMIA*, volume 5290 of *Lecture Notes in Computer Science* (pp. 1–10).: Springer.

Katsuno, H. & Mendelzon, A. (1991). On the difference between updating a knowledge base and revising it. In J. F. Allen, R. Fikes, & E. Sandewall (Eds.), *KR'91: Principles of Knowledge Representation and Reasoning* (pp. 387–394). San Mateo, California: Morgan Kaufmann.

Kracht, M. & Wolter, F. (1991). Properties of independently axiomatizable bimodal logics. *Journal of Symbolic Logic*, 56(4), 1469–1485.

Moss, L. S. & Parikh, R. (1992). Topological reasoning and the logic of knowledge. In Y. Moses (Ed.), *Proceedings of the Fourth Conference (TARK 1992)* (pp. 95–105).

Sahlqvist, H. (1975). Correspondence and completeness in the first- and second-order semantics for modal logic. In S. Kanger (Ed.), *Proceedings of the Third Scandinavian Logic Symposium*, volume 82 of *Studies in Logic and the Foundations of Mathematics* (pp. 110–143). Amsterdam: North-Holland.

Schmidt, R. A. & Tishkovsky, D. (2008). On combinations of propositional dynamic logic and doxastic modal logics. *Journal of Logic, Language and Information*, 17(1), 109–129.