

On the Mathematical Representation of Spacetime: A Case Study in Historical–Phenomenological Desedimentation

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Abstract: This essay is a contribution to the historical phenomenology of science, taking as its point of departure Husserl’s later philosophy of science and Jacob Klein’s seminal work on the emergence of the symbolic conception of number in European mathematics during the late sixteenth and seventeenth centuries. Since neither Husserl nor Klein applied their ideas to actual theories of modern mathematical physics, this essay attempts to do so through a case study of the concept of “spacetime.” In §1, I sketch Klein’s account of the emergence of the symbolic conception of number, beginning with Vieta in the late sixteenth century. In §2, through a series of historical illustrations, I show how the principal impediment to assimilating the new symbolic algebra to mathematical physics, namely, the dimensionless character of symbolic number, is overcome via the translation of the traditional language of ratio and proportion into the symbolic language of equations. In §§3–4, I critically examine the concept of “Minkowski spacetime,” specifically, the purported analogy between the Pythagorean distance formula and the Minkowski “spacetime interval.” Finally, in §5, I address the question of whether the concept of Minkowski spacetime is, as generally assumed, indispensable to Einstein’s general theory of relativity.

Keywords: spacetime; Edmund Husserl; Albert Einstein; Jacob Klein; Minkowski; symbolic mathematics; relativity.

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“The only reason he [the relativist] can do his calculations is because he first stripped his datum of any qualitative character and considered it as an abstract number.”

Émile Meyerson, *The Relativistic Deduction*²

Einstein’s popular exposition of the theory of relativity includes a humorous characterization of the process of setting up a Cartesian coordinate grid. He imagines a marble table top and a set of rigid rods, cut to length and laid out on the table top in squares: “It is a veritable wonder,” Einstein remarks, “that we can carry out this business without getting into the greatest of difficulties. We only need to think of the following”:

If at any moment three squares meet at a corner, then two sides of the fourth square are already laid, and, as a consequence, the arrangement of the remaining two sides of the square is already completely determined. But I am now no longer able to adjust the quadrilateral so that its diagonals may be equal. If they are equal of their own accord, then this is an especial favor of the marble slab and of the little rods, about which I can only be thankfully surprised. We must needs experience many such surprises if the construction is to be successful.³

So you better hope the rods line up, and thank your maker, or perhaps the rods themselves, if they do.

Without reading too much into Einstein’s tongue-in-cheek remark about thanking the rods, we should not fail to note the substantive issue raised in this passage, one upon which Hermann Weyl would soon press Einstein more forcefully in connection with the latter’s continued reliance upon such gifts in his theory of gravity.⁴ Einstein, while conceding the questionable validity of the very concept of rigid rods and clocks, nevertheless insisted that actual rods and clocks “work,” so to speak, enabling us to subject our theories to empirical test.⁵ Therefore, while ideally we would not have to depend upon such “elaborate appliances,” as Eddington called them, from the perspective of the practicing physicist the concept of the rigid rod and clock remains indispensable.

Weyl was unimpressed by Einstein’s defense of rigid rods and clocks. His objection, as he put it, had nothing to do with the actual behavior of rods and clocks, but was instead a philosophically inspired objection based on Weyl’s study of Edmund Husserl’s transcendental phenomenology, specifically, Husserl’s *Ideas Pertaining to a Pure Phenomenology*, volume one, of 1913. Weyl argued that general

2. Émile Meyerson, *The Relativistic Deduction*, trans. David A. Sipfle and Mary-Alice Sipfle (Dordrecht: D. Reidel, 1985; first published 1925), 111.

3. Albert Einstein, *Relativity: The Special and General Theories*, trans. Robert W. Lawson, with an introduction by Nigel Calder (New York: Random House, 2006 [1916]), 76–7.

4. On the exchange between Weyl and Einstein, see Thomas Ryckman, *The Reign of Relativity* (Oxford: Oxford University Press, 2005), chapters 4–6.

5. See, for instance, “Geometry and Experience” (1921), trans. W. Perrett and G. B. Jeffery, in *Einstein, Ideas and Opinions* (New York: Modern Library, 1994 [1954]), 257–9.

relativity was marred by a blemish imported via Riemannian geometry, namely, the illegitimate congruent displacement of vectors in respect of length. The very concept of congruence, maintained Weyl, is physically meaningful only in the immediately intuitable, infinitesimal region of an ideal observer. Accordingly, Riemannian geometry lacks for what Husserl terms an “eidetic intuition” or experiential insight into the meaning of congruence (“intuition” in the phenomenological sense always intending something *conceptual*, the experientially given, essential aspect of an object). Thus, for Weyl, an intuitively intelligible conception of “congruence” must base distant congruence on directly intuitable local congruence.⁶ Only by attending to the phenomenological context of Weyl’s criticism can we make sense of his otherwise puzzling assertion, skeptically regarded by Einstein, that the actual behavior of rods and clocks, or atoms and their spectral lines, or the like, has “nothing to do with the ideal process of congruent displacement.”⁷ Indeed, the parallel with Einstein’s own treatment of simultaneity can hardly be lost on the attentive reader of the kinematical section of Einstein’s 1905 paper on special relativity. Einstein there argues that simultaneity is an essentially *local* concept, such that physical intelligibility accrues to the concept of distant simultaneity only indirectly, based on its originary local sense. Einstein could well have asserted in his 1905 paper on special relativity that the “ideal constitution” of simultaneity, its phenomenological essence or eidetic structure, has “nothing to do with” the empirical behavior of clocks, light beams, or any other particular physical phenomenon.

Weyl’s reinterpretation of general relativity represents an attempt at phenomenological *reconstruction*. In this essay, I wish to broaden that phenomenological approach by considering the theory of relativity in light of the historical phenomenological perspective set forth by Husserl in his last major work, *The Crisis of the European Sciences* of 1936. Here philosophy of science, understood by Husserl now as historical phenomenology of science, has for its task bringing the concepts of modern mathematical physics to intuitive clarity through a historical “desedimentation,” as it were, apart from which, argues Husserl, “science as given in its present-day form ... is mute as a development of meaning.”⁸ According to the Husserl of *Crisis*, that is to say, the very sense of the concepts of modern mathematical physics is constituted by a series of historical accretions of meaning or “sedimentations” which can only be brought to light by a philosophical analysis, grounded in historical research, through which such sedimented meanings

6. See, for instance, Hermann Weyl “Gravitation and Electricity” (1918), trans. Perrett and Jeffery, in Albert Einstein, H. A. Lorentz, H. Weyl and H. Minkowski, *The Principle of Relativity* (New York: Dover, 1952), 201–16.

7. Quoted in Ryckman, *Reign of Relativity*, 87.

8. Edmund Husserl, *The Crisis of European Sciences and Transcendental Phenomenology* (originally unfinished and unpublished), trans. David Carr (Evanston, IL: Northwestern University Press, 1970), 58. The term “desedimentation,” never actually used by Husserl, was first used by Jacques Derrida. See Burt C. Hopkins, *The Origin of the Logic of Symbolic Mathematics: Edmund Husserl and Jacob Klein* (Bloomington, IN: Indiana University Press, 2011), 71.

are “reactivated.” Elaborating on the theme, Husserl reprehends the “surreptitious substitution,” as he sees it, of symbolic mathematical abstractions for intuitable physical realities, which substitution amounts to a kind of “reification of method,” where the means of representing the world is mistaken for the thing represented:

Mathematics and mathematical science, as a garb of ideas, or the garb of symbols of the symbolic mathematical theories, encompasses everything which, for scientists and the educated generally, represents the life-world, dresses it up as ‘objectively actual and true’ nature. It is through the garb of ideas that we take for true being what is actually a method ...”⁹

To bring to intuitive clarity this “garb of ideas,” via a phenomenological investigation grounded historically, is the task laid out by Husserl’s later philosophy of science.

Husserl did not carry out the historically grounded investigation he regarded as a necessary corrective to the reification of method in modern mathematical physics. However, even before Husserl’s *Crisis* was published, his interpreter Jacob Klein had already carried out such an investigation, at least in part, in his groundbreaking study *Greek Mathematical Thought and the Origin of Algebra*, where Klein traces the transformation of the conception of number in European mathematics, inaugurated by Vieta in the late sixteenth century.¹⁰ Klein, for his part, while emphasizing the implications of his work on symbolic mathematics for our philosophical understanding of modern mathematical science, did not extend his investigations to natural science itself, and in particular to the assimilation of symbolic mathematics to mathematical physics. His investigations thus leave unfinished the task of a phenomenological account of modern mathematical physics. No such phenomenological account as a whole will be attempted in this paper, but rather simply a case study, in outline form, of the concept of “spacetime” originated by Hermann Minkowski, in which space and time are merged into a single (four-dimensional) continuum with time treated as if it were a geometrical dimension.¹¹

It bears underlining that the task of “desedimentation” envisioned by Husserl, and actually carried out by Klein for the modern symbolic concept of number, remains a *philosophical* rather than historical task *per se*. That is, we attempt to uncover the physical or intuitive sense of a concept, to the degree possible, by tracing its genesis in the directly experienced “life world.” An alternate possibility, of course, is a concept’s “exploding,” as it were, into incoherence. In the present study, I demonstrate that when subjected to this type of analysis, the concept of “spacetime,” at least as conventionally understood by physicists and philosophers of science as a single geometrical continuum (“semi-Riemannian manifold”), does

9. Husserl, *Crisis*, 51.

10. Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (New York: Dover, 1968 [1934–36]).

11. The generalized version of Minkowski spacetime in Einstein’s theory of gravity is customarily referred to as a “semi-Riemannian manifold.”

in fact explode into incoherence. At the same time, however, the analysis will lend greater intelligibility to Einstein's special and general theories of relativity.

The essay consists of five sections. First, I sketch Klein's seminal contribution on the emergence of the symbolic conception of number beginning with Vieta in the late sixteenth century. In §2, through a series of historical illustrations, I show how the principal impediment to assimilating the new symbolic algebra to mathematical physics, namely, the dimensionless character of symbolic number, is overcome via the translation of the traditional language of ratio and proportion into the symbolic language of equations. In §3–4, I critically examine the concept of "Minkowski spacetime," highlighting the essential role of symbolic-algebraic representation in the construction of this concept, and analyzing the supposed analogy between the Pythagorean distance formula and the Minkowski "spacetime interval." Finally, in §5, I consider whether the concept of Minkowski spacetime is, as generally assumed, indispensable to Einstein's general theory of relativity.

I embark upon the following study in the spirit of Heinrich Hertz, when he remarked regarding his critique of the foundations of the science of mechanics that he was driven not by any perceived clash between its predictions and experimental results, but rather by the desire "to rid myself of the oppressive feeling that to me its elements were not free from things obscure and unintelligible."¹²

1 Klein on Symbolic Number

According to Jacob Klein, in *Greek Mathematical Thought*, the concept of "number" in European mathematics undergoes a decisive change in intelligibility with Vieta's *Analytical Art* of 1591.¹³ I briefly summarize Klein's thesis. In the Greek or "pre-modern" mathematical tradition, if you will, the concept of number accords without exception with Euclid's definition in Book VII of the *Elements*: "A number (*arithmos*) is a multitude composed of units."¹⁴ That is to say, a number *is* an assemblage of countable items themselves, and must accordingly always be a determinate amount of a specific *kind* of units:

The fundamental phenomenon which we should never lose sight of in determining the meaning of *arithmos* is counting, or more exactly, the *counting-off*.

12. Heinrich Hertz, *Principles of Mechanics*, trans. D. E. Jones and J. T. Walley (New York: Dover, 1956), 33.

13. Vieta's precedence, for Klein, lies in his reinterpretation of Diophantus' algebra, such that the Diophantine "species" is rendered a symbolic "generalized number" upon which numerical calculations can be performed. Klein also notes, however, that Vieta's reinterpretation of ancient algebra, or Greek "logistic," presupposes an implicitly symbolic conception of number already existing in the sixteenth century. The definitive study on Klein's interpretation of symbolic number is Burt C. Hopkins, *The Origin of the Logic of Symbolic Mathematics: Edmund Husserl and Jacob Klein*. For a shorter treatment, see Hopkins, "Jacob Klein on François Vieta's Establishment of Algebra as the General Analytical Art," *Graduate Faculty Philosophy Journal* 25:2 (2004): 51–85.

14. Euclid, *The Elements*, trans. T. L. Heath (New York: Dover, 1956), vol. 2, 277.

of some number of things. These things, however different they may be, are taken as uniform when counted; they are, for example, either *apples*, or apples and pears which are counted as *fruit*, or apples, pears, and plates which are counted as “*objects*.” Insofar as these things underlie the counting process they are understood as *of the same kind*. . . . Thus the *arithmos* indicates in each case a *definite number of definite things*.¹⁵

This direct or natural understanding of number is reflected in our commonsense manner of speaking, as when, for example, we refer to a “dozen eggs” and by “dozen” mean not “the number 12,” but the eggs themselves as a countable collection of units. The identical conception governs also Greek arithmetical *science*, the difference being that for the latter, we deal with a field of “pure” units or monads rather than particular kinds like apples, fruit, or such:

For even a “pure” number, i.e., a number of “pure” units, is no less “concrete” or “specified” than a number of apples. What distinguishes such a number is in both cases its twofold determinateness: it is, first of all, a number of objects *determined in such and such a way*, and it is, secondly, *just so and so many* of these objects.¹⁶

Accordingly, regardless of whatever questions might be raised regarding the mode of being of the pure monads, the sense of “number” remains the same: a countable collection of determinate units.

For modern algebra beginning with Vieta, by contrast, number is recast as a symbolic entity defined by its general relationships to other numbers in a symbolic calculus. One might go so far as to say that to be a number in the modern sense is to be the possible value of an algebraic variable. As Klein demonstrates, the modern symbolic conception of number is not simply “more abstract,” but is rather a symbolic construction upon an abstraction, through which the abstract concept of number, as found, for instance, in Aristotle, is rendered a “number” in its own right (Vieta’s “species”):

As soon as “general number” is conceived and represented in the medium of species as an “object” in itself, that is, symbolically, the modern concept of “number” is born. Usually its development is explained by a reference to its ever increasing “abstractness.” But this facile and easily misunderstood manner of speaking leaves its true and complicated structure completely in the dark. The modern concept of “number,” as it underlies symbolic calculi, is itself, as is that which it intends, *symbolic in nature—it is identical with Vieta’s concept of species*.¹⁷

Of decisive importance for our investigation is that numbers in the sense of Vietan “species” are now dimensionless entities upon which numerical operations

15. Klein, *Greek Mathematical Thought*, 46.

16. *Ibid.*, 48. For the Platonic tradition, the pure units, which serve as the basis for a science of arithmetic, are separately existing, non-sensible monads; whereas for the Aristotelian tradition they are abstracted, “neutral” monads.

17. *Ibid.*, 175–6.

are nevertheless performed as if they were actual quantities in their own right. In Klein's words,

While every *arithmos* intends *immediately the things or the units themselves* whose number it happens to be, his [Vieta's] letter sign intends *directly the general character of being a number* which belongs to every possible number, that is to say, it intends "number in general" immediately, but the things or units which are at hand in each number only mediately. In the language of the schools: the letter sign designates the intentional object of a "second intention" (*intentio secunda*), namely of a concept which itself directly intends another *concept* and not a *being*. Furthermore—and this is the truly decisive turn—this general character of number or, what amounts to the same thing, this "general number" in all its indeterminateness, that is, its merely possible determinateness, is accorded a certain independence which permits it to be the subject of "calculational" operations.¹⁸

This is the conception of number that we take for granted today.

Although it is often said that that Greek mathematics recognized only the "natural numbers," as if the Greek conception of number simply referred to a subset of our own, such an assertion betrays a failure of historical perspective. The Greeks knew nothing of "natural numbers" in our sense of the term, for the modern conception of "natural number" is already symbolic-dimensionless from the outset. Greek mathematics had no more conception of "the number 4" in our sense than it did of "the number -4." Yet only through the natural conception of number that governed Greek mathematics can an *indirect* sense can accrue to the dimensionless, symbolic numbers employed in the equations of modern mathematical physics. For this reason, the modern symbolic-algebraic conception of number should not be regarded simply as advance on the Greek conception, although, to be sure, it is that in an obvious sense; but more significantly, the modern conception represents a *transformation* of the Greek conception, founded upon the latter and carrying it implicitly as a sedimented meaning-constituent, while nevertheless transgressing the very intelligibility of the Greek conception.

Having established the conceptual structure of "number" in the modern symbolic sense, Klein next demonstrates how Descartes succeeds in transforming geometry into a symbolic representation of equations.¹⁹ The procedure is explicitly outlined by Descartes himself in his *Discourse on Method* (1637):

Nor did I have any intention of trying to learn all the special sciences commonly called 'mathematics' [i.e., astronomy, music, and optics]. For I saw that, despite the diversity of their objects, they agree in considering nothing but the various relations or proportions that hold between these objects. And so I thought it best only to examine such proportions in general ... Next I observed that in order to know these proportions I would need sometimes to consider them separately, and sometimes merely to keep them in mind

18. Klein, *Greek Mathematical Thought*, 174.

19. Klein, *Greek Mathematical Thought*, 197–211.

or understand many together. And I thought that to better consider them separately I should suppose them to hold between lines ... But in order to keep them in mind or understand several together, I thought it necessary to designate them by the briefest possible symbols. In this way I would take over all that is best in geometrical analysis and in algebra, using the one to correct all the defects of the other.²⁰

For Descartes, equations are therefore symbolic abbreviations for proportions among ratios, which latter can themselves be symbolized by lines (or geometrical figures in general). More specifically, Descartes' particular innovation is to employ line lengths as symbolic representations of symbolic dimensionless numbers, and then to use these "coordinate systems," as we call them, to "graph" equations.

Consider, if you will, the following diagrams:

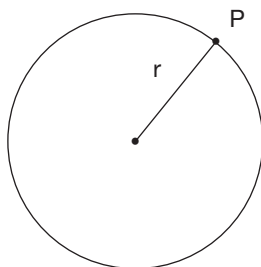


Figure 1 "Geometrical Circle"

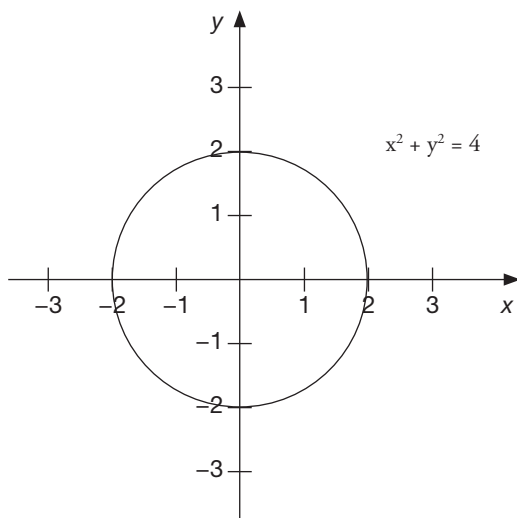


Figure 2 "Symbolic Circle"

20. Descartes, *Discourse on Method*, Part Two, in *The Philosophical Writings of Descartes*, trans. John Cottingham *et al.*, 3 vols. (Cambridge: Cambridge University Press, 1985–91), vol. 1, 120–21.

Let us regard Figure 1 as a circle drawn with pencil and compass, maybe from the books of Euclid or Apollonius. This drawn figure directly represents a geometrical circle, a figure in space. It is not a graph. Figure 2 is a symbolic representation or graph of an equation, which itself represents generalized quantitative relationships between symbolic dimensionless numbers. Coincidentally, the graph of the equation in Figure 2 also has the shape of a circle in space, even though it is not a circle *per se*, but rather a graph of the equation. Such graphs, the plot of a circle, for example, must accordingly be understood as symbolic representations of equations (so $r^2 = x^2 + y^2$), which in turn themselves symbolically represent generalized relationships between symbolic dimensionless numbers, which themselves represent whatever it is we are actually talking about, perhaps a geometrical circle (but perhaps not).

Notice that “geometry” occupies two distinct levels here, functioning at one level as a symbolic representation of generalized quantitative relationships, and at another as a possible subject matter for such representation. I trust the reader will agree that there is ample potential for Husserlian “sedimentation” in these layers of symbolic representation. Klein’s claim in *Greek Mathematical Thought*, the full argument for which I will not review here, is that Descartes implicitly identifies the symbolic circle with the real circle in space; and more generally, that Descartes has actually created a “symbolic space” and implicitly *identified* that symbolic space with physical space, in the sense that he attributes to physical space the formal properties of symbolic space. Still more provocatively, Klein suggests that this Cartesian “symbolic space,” now surreptitiously identified with physical space, subsequently becomes the proximate object of modern mathematical physics. A prime candidate here, of course, would be “Minkowski space,” a symbolic space used for the representation of a theoretical “spacetime.”

Two decisive and closely related features of modern mathematics can be traced to the symbolic conception of number emerging in European science in the sixteenth and seventeenth centuries. First, while “pre-modern,” non-symbolic mathematics refers directly to things, such that arithmetic is directly “about” collections of countable objects and geometry is directly about figures in space, the proximate object of modern symbolic mathematics is general symbolic *possibilities*. Second, and by consequence, modern formal mathematical systems are self-referential in a distinctive sense. That is to say, such systems are in the first place “about themselves”; only indirectly are they correlated, or “coordinated,” if you like, to things in the world. It is for this reason that the assimilation of symbolic mathematics to physics raises a unique set of questions regarding modern mathematical physics and its relationship to the experienced physical world. At the most general level, the symbolic conception of number is emblematic of a symbolic way of thinking that determines modern science and even, Klein suggests, modern civilization itself.²¹ At a more specific level, however, we can ask how the symbolic conception

21. “Thus, our own life does not belong to us. We appear to be in the most direct contact with the world around us, but in reality the vast machinery of our society permits us to perceive the world

of number actually functions in the mathematics of modern physics. What are the conditions for the employment of symbolic number, and how are mathematical concepts in physics, previously expressed by means of the traditional mathematics of ratio and proportion, reformulated in terms of symbolic algebra? It is to this more specific task that we will be devoting ourselves in the next section.

2 The Physical Sense of Equations in Modern Mathematical Physics

If the symbolic conception of number is to be deployed in physics, there exists an obvious obstacle to be overcome; namely, how to represent the inherently *dimensional* quantities of the physical world by means of the *dimensionless* symbolic numbers employed in algebraic “formulae.” Clearly, the equations or formulae of modern mathematical physics can have an indirect meaning only. We have only to observe that algebraic formulae equate *quantities*, which require unit conventions for their physical sense, while proportions directly equate *relations* (ratios) between quantities, with no necessary reference to unit conventions. Moreover, the equations of physics employ arithmetical operations that are nonsensical if performed directly on dimensional quantities. For instance, in the formula for momentum $p = mv$, what could it mean to “multiply” a quantity of mass by a quantity of velocity? Euclid’s definition of multiplication characterizes the operation, in a perfectly intelligible way, as *repeated addition*: “A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.”²² Thus, three kilograms taken four times is twelve kilograms, but there is no more sense to saying “three kilograms taken four feet per second times” than there is to saying “three apples taken four oranges times.” Strictly speaking, that is, *quantities* cannot intelligibly be multiplied together. We perform such algebraic operations by regarding the quantities as symbolic dimensionless numbers, multiplying them together, and then “plugging” the result back into physically intelligible units (e.g., given $3 \text{ kg} \times 4 \text{ m/sec}$ we multiply “ 3×4 ” and then plug “12” into the units of momentum [kg-m/sec]). The operation therefore cannot be even carried out except by means of the modern, symbolic conception of number.

The traditional understanding of physics in terms of ratio and proportion is plainly at work in Galileo. Since Galileo neither knew what an equation was nor

only through generally accepted views. The directness of our contact with the world is of the same symbolic character as the concepts we use to understand it. We can comprehend how our whole social and economic system, which we term Capitalism, and which is, in its origins, closely connected to the modern idea of knowledge and science, has acquired such symbolic unreality.” Jacob Klein, “Modern Rationalism,” in Klein, *Lectures and Essays*, ed. Robert W. Williamson and Elliott Zuckerman (Annapolis, MD: St John’s College Press, 1985), 64.

22. Euclid, Book VII, Definition 15, 278.

operated with the symbolic conception of number at all, we will misunderstand his mathematical conception if, as is often done, we translate the method of ratio and proportion he employs into algebraic equations. Thus, when Galileo says, for instance, in *Two New Sciences* (1638), that if a body accelerates uniformly, “the spaces run through in any times whatever are to each other as the duplicate ratio of their times; that is, as the squares of those times,”²³ he does not mean by “the square of time” that we are to “multiply” time by itself, a senseless operation in any case. Multiplication, recall, insofar as we have any conception of it that could be intelligibly applied to the dimensional quantities of physics, is *repeated addition*. While we are used to performing the operation “2 seconds times 2 seconds,” for example, by multiplying “ 2×2 ” and then appending the units “ sec^2 ,” we obviously have no direct physical conception of a “square second.”

Galileo himself uses *geometrical squares* (built on line lengths which themselves represent quantities of time) to represent *duplicate* or *compound* ratios of time intervals.²⁴ Note that in the method employed by Galileo, the geometrical squares function *representatively* rather than as objects of representation. For clearly, Galileo’s law of accelerated motion, which we are accustomed to expressing algebraically in the form $s = at^2/2$, has no more essentially to do with geometrical squares than it does with multiplying time by itself.²⁵ Likewise, while our algebraic expression t^2 might appear, at first glance, to represent some physical quantity of interest, it actually represents no physical quantity at all, but rather represents

23. Galileo, *Two New Sciences*, Third Day, “On Naturally Accelerated Motion,” Proposition II, Theorem II, trans. Stillman Drake (Toronto: Wall and Emerson, 2000 [1638]), 166.

24. A duplicate ratio, for readers unfamiliar with the term, is a species of “compound ratio” whereby a ratio is compounded with itself. A compound ratio, in turn, is a ratio formed by combining two or more ratios in a prescribed way. The compounding of ratios is defined geometrically in Euclid, VI. 23, but since compounding is not a geometrical operation *per se*, I will here suggest a more general definition for our purposes: Given the ratio $a:b$ and the ratio $b:c$, the compounded ratio is $a:c$ (where a , b , and c are any quantities of the same kind). For illustrative purposes, to compound 6:5 and 4:3, we first rewrite those ratios as 24:20 and 20:15 (thereby obtaining the requisite middle term b). Compounding yields 24:15 or 8:5. Thus the compounded ratio is the ratio formed by the extremes when we take the antecedent of one ratio as the consequent of that ratio with which the former is to be compounded. Accordingly, in our example we took 4 (the antecedent of 4:3), and made it the consequent of a 6:5 ratio (which, of course, required our rewriting the original 4:3 as 20:15). In the Galilean example, where time is represented by the length of the side of a square, the ratio of the area of the square to the unit of area gives the duplicate ratio. Note that strictly speaking, compounding is not possible unless we are dealing with ratios all of whose terms are homogeneous in dimension. For example, while we can place the ratio 6 meters to 5 meters in proportion with the ratio 6 seconds to 5 seconds, we cannot “compound” these ratios, since the compounded ratio would be, *per impossibile*, 8 meters to 5 seconds. This latter operation is nevertheless often referred to as “compounding,” since it translates into algebra as multiplication of fractions, just as does the compounding of ratios in the proper sense of the term.

25. With this example we might contrast, for instance, the Pythagorean Theorem expressed algebraically, in which case the squared terms genuinely represent physical quantities, namely, areas of actual geometrical squares. Pythagorean squares are *squares*, not duplicate ratios, while Galileo’s squares are representations of duplicate ratios of times.

indirectly a *relation* (ratio) between physical quantities. Let us term such algebraic artifacts “pseudo-quantities” or *symbolic quantities*, since they function as symbolic abbreviations for relations between actual quantities. We will see that such symbolic quantities are generated whenever ratios are treated as if they were fractional numbers and then subjected to numerical operations such as multiplication, division, and so forth.²⁶

Isaac Newton, writing some fifty years after Galileo, is willing to countenance the multiplication and division of heterogeneous physical quantities, although he clearly regards such operations as a kind of abbreviated notation for ratios and proportions. For instance, at the outset of the *Principia* (1687), Newton characterizes the concept of “quantity of motion” in traditional terms of ratio and proportion: “Therefore, in a body twice as large, with equal velocity, it [the motion] is double, and with double velocity, quadruple” (“ideoque in corpore duplo maiore, aequali cum velocitate, duplus est, & dupla cum velocitate quadruplus”).²⁷ On the other hand, in his scholium to the Laws of Nature, Newton obtains the quantity of motion by “multiplying” (“ducere . . . in”)²⁸ the quantity of matter or mass by the velocity, significantly adding, in the third edition of 1726, the qualifying phrase “as I thus shall say” (“ut ita dicam”), as if to assure the reader, “I know you can’t really do this.”²⁹

26. The distinction I have drawn between physical quantities and symbolic quantities has precedent in J. B. Stallo’s *The Concepts and Theories of Modern Physics* (1881), where Stallo complains of “the error respecting the true nature of arithmetical and algebraic quantities [which] has become next to ineradicable by reason of the inveterate use of the word ‘quantity’ for the purpose of designating indiscriminately both extended objects or forms of extension and the abstract numerical units or aggregates by which their metrical relations are determined . . . The use of letters as algebraic symbols, i.e., as representatives of numbers, is in itself a serious (though, perhaps unavoidable) infirmity of mathematical notation. In the simple formula, for instance, expressive of the velocity of a moving body in terms of space and time ($v = \frac{s}{t}$), the letters have a tendency to suggest to the mathematician that he has before him direct representatives of the things or elements with which he deals, and not merely their ratios expressible in numbers. In every algebraic operation the use of letters obscures the real nature, both of the processes and of the results, and tends to strengthen ontological prepossessions.” J. B. Stallo, *The Concepts and Theories of Modern Physics*, third edition (1888), ed. Percy W. Bridgman (Cambridge, MA: Harvard University Press, 1960), 275, 277.

27. Newton, *Principia*, p. 1 (my translation)..

28. “Ducere in” is geometrical language; the construction of a rectangle with sides representing mass and velocity respectively is implied. See Newton, *Arithmetica Universalis*, trans. Raphson, 1720, facsimile (http://www3.babson.edu/archives/newton_collection/UniversalArithmetick1720.pdf), 4–5. This text was originally published in 1707 without Newton’s approval, and Newton’s name did not appear on the English translation of 1720. A second Latin edition appeared in 1722. Although in this text Newton explicitly dismisses the traditional Euclidean or intuitive conception of number (“multitude of units”) in favor of the Vietan symbolic conception (“species”), he nevertheless attempts, at least for negative numbers (“less than nothing”), an intuitive accounting by means of an example from “human affairs,” remarking that we designate our possessions “affirmative goods” (positive) and our debts “negative” ones.

29. *Ibid.*, 23. A similar instance is Proposition LXXXVIII, Theorem XLV, where the multiplication of physical quantities is qualified, in the third edition, by the phrase “si ita loquar” (212). Guicciardini plausibly suggests that even though Newton uses the language of proportion in the

While as early as 1699, Pierre Varignon expressed velocity algebraically as a “ratio” of distance and time ($v = \frac{ds}{dt}$), fully a century later, Laplace, in his *Celestial Mechanics* (1798), is still at pains to justify such use of non-homogeneous ratios in physics:

Time and space, being heterogeneous quantities, cannot be *directly* compared with each other; therefore an interval of time, such as the *second* is taken for the unit of time and a given space, such as the *meter*, is taken for the unit of space; then space and time are expressed as abstract numbers, denoting how many measures of these particular species each of them contains, and they may then be compared with each other. In this manner the velocity is expressed by the ratio of two abstract numbers, and its *unit* is the velocity of a body which describes one meter in a second.³⁰

Note that the “abstract numbers” of which Laplace speaks here, which alone render the procedure workable, are not simply *abstractions* per se, but rather *symbolic dimensionless numbers* (in Klein’s sense). To paraphrase: Time and space, being heterogeneous quantities, have no ratio (which is to say, that no quantity of space can be a multiple of any quantity of time, or vice versa). Therefore, we define a unit of time and a unit of space, and treat the ratios of measured times and spaces to their respective units as if those ratios were dimensionless (“abstract”) numbers. These dimensionless numbers, rendered symbolically “homogeneous” with one another, can now be placed in ratio. That ratio is then to be rewritten as a fractional number, and the ratio of this fractional number to unity will be proportional to the ratio of the velocity to unity. We can then rewrite the proportion as an equality. Thus, velocity expressed as “distance over time,” as we say, is understood by Laplace here as a symbolic abbreviation for a proportion (“The ratio of velocity to its unit is jointly proportional to the ratio of the distanced traversed to its unit and the ratio of the unit of time to the time elapsed”). In general, equations in mathematical physics were understood throughout the eighteenth century as symbolic abbreviations of this kind.³¹

Principia, the proportions contained therein should generally be understood as equations, since Newton analyzes single quantities at specific points, whereas a genuine proportion requires two ratios and therefore four terms. Guicciardini carefully analyzes Proposition VI of Book I, and Huygens’ criticism thereof in the latter’s notes on the *Principia*. Newton in Proposition VI concludes that the quantity of force at a particular point “will be inversely as the solid $\frac{SP^2QT^2}{QR}$.” The locution “be inversely as” suggests a proportion, but since we find no ratio of two forces, the sense appears to be rather $F = \frac{QR}{SP^2QT^2}$, and Guicciardini cites Huygens’ critical comment: “He [Newton] says that the centripetal force in P is reciprocally as the solid ... In order to say reciprocal, it is necessary to give or to conceive another point p, in which the centripetal force can be compared to the centripetal force which is at P.” Niccolò Guicciardini, *Reading the Principia* (Cambridge: Cambridge University Press, 1999), 125–35. The most plausible interpretation, it seems to me, is that while Newton may indeed often be thinking mathematically in terms of equations, he regards the *physical meaning* of these equations in terms of ratio and proportion.

30. Quoted in John Roche, *The Mathematics of Measurement* (London: Athlone Press, 1998), 138.

31. See Roche, *Mathematics of Measurement*, chapter 7.

To the degree we fail to recognize the equations of physics as symbolic abbreviations for proportions among directly intelligible physical quantities, the proximate object of our physics is rendered a *symbolic entity*, which we are nevertheless wont to take for the physical world itself. One symptom of such symbolic reification is our inevitably imagining, for instance, that the equations of mathematical physics are *definitions* of the associated concepts. We learn at the beginning of an elementary physics textbook, for example, that “force” is defined as “a push or a pull.” But once we get used to the formula $F=ma$, where ma is conceived as if it were a physical quantity in its own right rather than a symbolic abbreviation for the conjoint proportion $F_1:F_2 :: m_1:m_2$ and $F_1:F_2 :: a_1:a_2$, we can easily imagine that force is “defined as” ma . Our original push and pull, by means of which a measure of physical intelligibility accrued to the concept of force, thus finds itself “sedimented” in that concept, and we regard ourselves as having attained the deeper insight that force “is” the symbolic entity ma . But, of course, force cannot “be” ma , since ma is merely a symbolic quantity, not a physical quantity. Whatever Newton’s own qualms about the concept of force, he never reduces the *definition* of force (“an action exerted upon a body, in order to change its state”³²) to its *quantification* (“The change of motion is proportional to the motive force impressed”³³).

It is worth observing, to conclude this section, that while the *prima facie* problematic employment of symbolic dimensionless numbers in *physics* is still on Laplace’s radar, so to speak, in 1798, symbolic number itself, if the passage quoted above is any indication, has already been effectively sedimented in Western mathematical history by the end of the eighteenth century. By the close of the nineteenth century, however, not just symbolic number, but “symbolic nature” will have been rendered part of a sedimented history. The concept of spacetime, we shall see, is a notable legacy of this sedimented history.

We are, of course, far from regretting either the development of modern symbolic mathematics or its assimilation into mathematical physics. Quite the contrary, symbolic mathematics not only enables calculations that cannot be carried out with the traditional mathematics of ratio and proportion, but also reveals physical relationships in nature that could not otherwise be discovered. However, with the obvious advantages of symbolic mathematics comes an inevitable *loss of meaning* due to its reliance on numerical operations carrying no direct (or sometimes even evident *indirect*) physical sense. Such is the case, I shall now argue, with Hermann Minkowski’s “four-dimensional” interpretation of Einstein’s special theory of relativity.

32. Newton, *Principia*, 2.

33. *Ibid.*, 13.

3 “Minkowski Spacetime”³⁴

In his famous essay “Space and Time” (1908), Hermann Minkowski presents for the first time the concept of “spacetime” in the sense of a *single continuum* of space and time.³⁵ Since there is already a kind of union of space and time in pre-Minkowski special relativity (as originally published by Einstein in 1905), however, it behooves us to give the notion of “merging” space and time in a single continuum some added precision. In pre-Minkowski special relativity, the unification of space and time is simply the interdependence of space and time expressed by the Lorentz equations (i.e., distance and time variables respectively in one inertial reference frame are dependent on both variables in another inertial reference frame³⁶). However, this special relativistic unification of space and time has nothing to do with merging space and time into a single continuum as per the concept of the Minkowski “spacetime line element.”³⁷ The necessary distinctions here are often glossed over. For instance, Roberto Torretti remarks that

though the notion of an n -manifold ... was first conceived by Bernhard Riemann in the 1850s, one can now, with the benefit of hindsight, reasonably maintain that spacetime was already handled as a 4-manifold when Galileo plotted spaces against times in his discussion of the motion of projectiles.³⁸

Such an assertion is at least misleading, since the mere fact that Galileo represents time as the length of a line on a diagram no more implies that he conceives of space and time as a single “spacetime manifold” than my plotting the amount of beer I drink against time on a graph entails my thinking of a unified “beertime manifold.” If all that is meant by “four-dimensional manifold” is that we can represent points (events) in space and time by means of four numbers (coordinates), the concept of a “four-dimensional manifold” is innocuous enough; but the notion of a “manifold” in the Riemannian sense suggests the notion of a *single geometrical continuum* defined by a line element. There is certainly no such notion in Galileo.

Michael Friedman more helpfully points out that the minimal notion of a four-dimensional spacetime manifold (set of events defined by four coordinates) in no wise implies any unification of space and time such as we associate with relativity

34. Minkowski spacetime has been a subject of philosophical interest since Einstein employed it in the formulation of his general theory of relativity, and there now exists a sizable body of literature that takes the validity of the concept for granted, focusing instead on its scientific and philosophical significance. Since our analysis is situated on a different conceptual level, where the validity of the concept cannot be taken for granted, we shall refrain from engaging this body of literature unless a specific occasion should arise.

35. Hermann Minkowski, “Space and Time,” in Einstein *et al.*, *The Principle of Relativity*, 75–96.

36. I speak here of the transformation of space and time intervals themselves, not *coordinate* transformations per se. That is, while the so-called Galilean transformation for *coordinates* is $x' = x - vt$ and $t' = t$, the transformation for distance and time intervals themselves is simply $x' = x$ and $t' = t$.

37. Although Minkowski does not actually use either the term “line element” or “spacetime interval” in the 1908 paper, the concept is clearly intended.

38. Roberto Torretti, *Relativity and Geometry* (Oxford: Pergamon Press, 1983), 22.

theory: “As we shall see, we effect a relativistic unification of space and time only if we view space-time as a four-dimensional *semi-Riemannian* manifold.”³⁹ In other words, “spacetime” could be a four dimensional manifold in the minimal sense without being a *single continuum*.⁴⁰ But even Friedman is too imprecise for our purposes, since he fails to distinguish the two senses of “relativistic unification” we have noted, namely, Einstein’s original 1905 (pre-Minkowski) version of special relativity, on the one hand, and Minkowski’s (1908) on the other. Apart from Minkowski’s conception of the “interval” or four-dimensional line element, there is no sense in speaking of a “semi-Riemannian manifold” at all, since it is a fundamental principle of Riemannian geometry that any geometry is defined by its line element.

Einstein himself often uses the term “continuum” in an imprecise way, as, for instance, when he remarks in the “Autobiographical Reflections” of the Schillp volume that the idea of a “four-dimensional continuum” is not something newly introduced by the special theory of relativity, since Newton also employed a four-dimensional continuum. The difference, Einstein explains, is that the Newtonian four-dimensional continuum “falls naturally into a three-dimensional and a one-dimensional (time), so that the four-dimensional point of view does not force itself upon one as *necessary*.”⁴¹ However, the “necessity” (in special relativity) to which Einstein adverts in the preceding passage is not the necessity of a “single continuum” in the Minkowskian sense, but rather simply the necessity, as Einstein puts it, of a “formal dependence between the way in which the space coordinates, on the one hand, and the temporal coordinates, on the other, have to enter into the natural laws.” But this latter “necessity” (formal interdependence of space and time variables), which after all simply reflects the interdependence of space and time in the special theory of relativity, is inherent to the Lorentz transformation itself, with or without the Minkowski formalism. Indeed, the significance of the Minkowski formalism, according to Einstein here, is merely that

before Minkowski’s investigation it was necessary to carry out a Lorentz-transformation on a law in order to test its invariance under such transformations; he, on the other hand, succeeded in introducing a formalism such that the mathematical form of the law itself guarantees its invariance under Lorentz-transformations.⁴²

Such “automatic covariance” is, to be sure, a noteworthy mathematical technique, but it in no wise implies that space and time themselves have been unified in a single continuum.

39. Michael Friedman, *Foundations of Spacetime Theories* (Princeton, NJ: Princeton University Press, 1983), 34.

40. I shall conform to custom by referring to time as a “dimension,” even though strictly speaking the term “dimension” is a metaphor originating in the geometrical representation of time.

41. Paul Arthur Schillp, ed., *Albert Einstein: Philosopher–Scientist* (Evanston, IL: Library of Living Philosophers, 1949), 57–9.

42. *Ibid.*, 59.

square on the distance between two events and the square on the distance light would travel in the time interval between the events. It is all space, not “spacetime.” However, Minkowski reminds the reader that by an appropriate choice of units, the velocity of light can be rendered $c = 1$, yielding $t^2 - x^2 = t'^2 - x'^2$, which does augment the Pythagorean resemblance, at least visually.⁴⁴ Moreover, if we replace the time variable t with $\sqrt{-1} t$, such that $-t^2 - x^2 = -t'^2 - x'^2$ (or with requisite algebraic manipulation $t^2 + x^2 = t'^2 + x'^2$), the expression takes on an even more Pythagorean appearance.⁴⁵ This last intervention (substitution of $\sqrt{-1} t$ for t) is pivotal, since otherwise we have no transformation equations, as a function of the rotation angle from K to K' , for the individual space and time variables; and without a transformation equation we could have no grounds whatsoever for deeming the construction a representation of a *single* “spacetime continuum.”⁴⁶ As physicist David Bohm observes (of the analogous Euclidean/Pythagorean transformation):

Without such a transformation we would hardly even be justified in regarding the three dimensions as united into a single space as ‘continuum’ (e.g., in an arbitrary graph, in which one physical quantity such as temperature is plotted against another, such as pressure, there is no such unification).⁴⁷

Of course, it is difficult to see what *physical* basis or justification there could be for substituting $\sqrt{-1} t$ for t , besides the desire to obtain a notational resemblance with the Pythagorean line element formula. Today we employ hyperbolic trigonometric functions instead, thus obviating the need for $\sqrt{-1} t$.⁴⁸ The use of hyperbolic functions, however, comes at the price of reverting to $t^2 - x^2 = t'^2 - x'^2$, which again detracts from the Pythagorean “look” of the equation. And the expression appears still less Pythagorean if we restore, as we must, the units of the velocity of light. For even if we designate c unit velocity and, as a matter of convenience, drop its units, those units are still there—*adjusting* our units such that c is unity has nothing whatsoever to do with *dropping* those units and treating c as a dimensionless number. If we adjust our units, c will equal *one unit of distance per one unit of time*. Clearly, if we really wish to drop the units of light velocity, we can do so without first adjusting them to unity, and if we wish to so adjust them, we can do so without dropping them. These two operations have absolutely nothing to do with one another.

What can fairly be called Minkowski’s algebraic “sleight of hand,” then, merely enhances the visual–notational resemblance between the Lorentz-invariant

44. Minkowski, “Space and Time,” 88.

45. *Ibid.*

46. The transformations are: $x = x' \cos \theta - t' \sin \theta$ and $t = x' \sin \theta + t' \cos \theta$ exactly analogous, at least notationally, to the Euclidean $x = x' \cos \theta - y' \sin \theta$ and $y = x' \sin \theta + y' \cos \theta$.

47. David Bohm, *The Special Theory of Relativity* (London: Routledge, 1996; first published 1965), 148.

48. The equivalent hyperbolic transformations are $t = t' \cosh \beta - x' \sinh \beta$ and $x = x' \cosh \beta - t' \sinh \beta$ (where $\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$).

quadratic $c^2t^2 - x^2$ and the Pythagorean theorem (when the latter is expressed algebraically). The significance of this visual-notational resemblance for the plausibility of Minkowski's theory should not be underestimated. Indeed, at the earliest stage (1910) in his acceptance of the Minkowski formalism, Einstein comments that "the formal analogy of the transformations (a_1) [$x^2 + y^2 = x'^2 + y'^2$] and (a_2) [$c^2t^2 - x^2 = c^2t'^2 - x'^2$] jumps out at the eyes [my italics]."⁴⁹ And we are susceptible to forgetting that the *physical meaning* of the Lorentz invariant $c^2t^2 - x^2$ is wholly determined by the transformation equation from which it is originally derived. It carries no independent meaning. The Pythagorean invariant $x^2 + y^2$, by contrast, has no essential connection to any transformation equation whatsoever. The analogy between the two must therefore appear dubious, especially given Minkowski's notational machinations. Furthermore, a Euclidean coordinate rotation can in fact be physically performed. That is, I can lay out a Cartesian grid made of wire mesh in my backyard, pick two points on the ground and measure the distance between them, then rotate the wire mesh, transform the coordinates, and calculate the same distance by means of the Pythagorean Theorem. However, no "time axis" can be rotated physically, but merely a line *representing* time, rotated *on a graph*. That is to say, the so-called "spacetime rotation," upon which the plausibility of the Minkowski "interval" so depends, is merely a *symbolic rotation*, not a physical rotation. Einstein himself never tired of emphasizing that frames of reference consist of actual physical objects (rods and clocks).

3.2 The concept of "four-vector"

On the basis of the supposed analogy, described above, between the Pythagorean distance formula and the Lorentzian quadratic invariant, Minkowski introduces the notion of "four-vectors" in spacetime, all specific instances of which are tied to the concept of the four-dimensional interval.⁵⁰ We must ask whether Minkowski's "four-vectors" can be regarded as "vectors" in any physically intelligible sense. A vector, in the usual sense, is a single physical or geometrical quantity with a direction in space. Such a quantity might be a directed line segment, for instance, a velocity, a force, or the like. A vector thus conceived is a *single* quantity which, given a system of reference, we *may* elect to resolve into its "components." Thus, for a displacement vector in space we typically have the x , y , and z components, related to their counterparts x' , y' , and z' in some other system of reference by a transformation equation. Observe that the converse does not follow; that is, given a set of quantities related by a transformation equation, it is not necessarily the case that those quantities are "components" of a vector in the proper sense of the term

49. "*saute aux yeux*" (the article appeared in 1910 in French translation only). "Le principé de relativité et ses consequences dans la physique moderne," in *The Collected Papers of Albert Einstein*, vol. 3, ed. Martin J. Klein *et al.* (Princeton, NJ: Princeton University Press, 1993), 155–76, 169.

50. Minkowski, "Space and Time," 84ff.

(single directed physical or geometrical quantity). Nevertheless, it is customary to regard any such set of quantities, in a purely formal or mathematical sense, as a “vector.”⁵¹

Let us keep these two concepts of “vector” distinct by terming the latter variety (in which the so-called “components” are not necessarily components of any single directed quantity) an analytical or “symbolic” vector. A symbolic vector, that is to say, is simply a set of quantities subject to a transformation law. Taking the so-called Minkowski “four-velocity” (spacetime distance per unit of proper time or $\frac{dx_{\mu}}{d\tau}$) as an example, we can immediately see that Minkowski’s four-vectors are analytical or symbolic vectors rather than vectors in the physical or geometrical sense. For the time component ($\frac{dt}{d\tau}$) is simply the ratio between coordinate time and proper time (or, if you like, the dimensionless number $= \frac{c}{\sqrt{c^2-v^2}}$, itself a symbolic representation of a ratio of velocities). Since a ratio is not itself a physical or geometrical quantity, but a relation between quantities, $\frac{dt}{d\tau}$ cannot be a vector component in the proper (physical) sense. By contrast, the spatial components $\frac{x}{\tau}$, $\frac{y}{\tau}$, $\frac{z}{\tau}$ indeed comprise a vector ($v\beta$ since $\tau = \frac{t}{\beta}$), but not a four-dimensional vector in “spacetime,” but rather the usual three-dimensional velocity in space, rendered Lorentz covariant via the correction factor β .

From our perspective, then, it is an irony that Minkowski is often cited as having departed from Einstein’s algebraic approach in favor of a “geometrical” or vectorial approach, given that Minkowski’s very notion of “four-vector” is essentially tied to the method of representation in terms of Cartesian components. Indeed, special relativity could not have been originally formulated by Einstein in terms of Minkowskian “four vectors,” since the vectorial approach requires that the Lorentz transformation be *already in hand*. The time component $\frac{dt}{d\tau}$ of the Minkowski “four velocity,” as we said, is simply the relativistic factor $\beta = \frac{c}{\sqrt{c^2-v^2}}$, which can in no wise be originally *derived* from the Minkowski formalism.⁵²

51. Thus Einstein himself, in his Princeton lectures (1921), remarks, “The ensemble of three quantities, defined for every system of Cartesian coordinates, and which transforms as the components of an interval, is called a vector. ... We can thus get at the meaning of the concept of a vector without referring to a geometrical representation.” Albert Einstein, *The Meaning of Relativity* (Princeton, NJ: Princeton University Press, 1988 [1922], 11). Einstein here defines a “vector” as a set of three quantities transforming as an interval. But the mere fact that a set of quantities *transforms* as an interval is not a sufficient condition for regarding it as an interval; or, for that matter, as a set of components of any vector *quantity* at all. Einstein is here speaking of what we will call an “analytical” or “symbolic” vector.

52. At least one commentator has noted this feature of Minkowski’s formulation. Alberto Martínez observes, “Physicists did not merely neglect the use of vector algebra to formulate relativity theory; ordinary vector analysis could not be used to derive relativity theory. This is remarkable because physicists believed that vector algebra and coordinate algebra were equivalent, in that either could be used to obtain the same results.” Alberto Martínez, *Kinematics: The Lost Origins of Einstein’s Relativity* (Baltimore, MD: Johns Hopkins University Press, 2009), 381. Martínez adds, “A related issue concerns the sense in which early relativity received a vectorial interpretation. After the works of Poincaré and Einstein, theorists reformulated relativistic kinematics in terms of “four-dimensional vectors.” This movement was impelled by Minkowski, who pursued a symmetric interpretation of

One final feature of Minkowski's presentation merits brief mention, namely, the concept of "proper time" (τ) itself. As Minkowski observes, the integral $\int dt$ along the path of a particle, with the proper time τ measured in the frame of the particle (i.e., by a clock traveling with the particle) is proportional to the integral $\int \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}$. We note here that, clearly, this proportion holds whether or not $\sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}$ is regarded as a "spacetime interval." Therefore, the successful employment of the proportion, for whatever purpose, can in no wise constitute evidence for the physical validity of the concept of Minkowski spacetime.

To be sure, we have no reason to object to Minkowski's four-dimensional "spacetime," especially given its proven fruitfulness in the history of relativity theory, so long as the latter is interpreted strictly as a formal-mathematical construction. However, Minkowski further suggests that he has uncovered the true "absolute world," the only world "given by phenomena."⁵³ Too often since Minkowski this assertion has been accepted at face value, even though no such proposition is actually demonstrated or even defended in Minkowski's essay. The idea that the "single continuum" view of space and time is, in Minkowski's words, "forced upon us" by phenomena seems predicated rather on the philosophical assumption that if the metrical properties of space and time are determined relative to the velocity of a reference frame, then space and time as previously understood (qua distinct continua) must not be "real." But however powerful Minkowski's four-dimensional approach proves itself from a purely mathematical point of view, and however fruitful for relativity theory it has proved itself historically, it must appear of dubious value as an intelligible representation of the *physical* world. For the very notion of "spacetime" hinges *entirely* on a strained analogy between the relativistic invariant $c^2 t^2 - x^2$ and the Pythagorean distance theorem. A secure judgment on this matter, however, must be based on more careful consideration of the physical sense of the relativistic quadratic invariant itself.

4 Genetic Explication in Terms of Ratio and Proportion of the Purported Analogy between Space and "Spacetime"

If we discard the physically gratuitous features of Minkowski's presentation, we are left with the special relativistic transformation equation itself: $c^2 t^2 - x^2 = c^2 t'^2 - x'^2$. This equation, and in particular the squared terms, must bear the entire weight of the supposed analogy between space and spacetime. When we represent the line element of space (Euclidean or non-Euclidean) algebraically, the squared terms

the parameter t analogous to the coordinates x, y, z . In Minkowski's interpretation, the concept of a vector summarized coordinate-analytic notions. Previously, vector theorists had advocated the priority of vectors by conceiving them as consisting *fundamentally* of direction and magnitude and only incidentally as expressible in terms of Cartesian coordinates" (384–5).

53. Minkowski, "Space and Time," 83.

must be understood as symbolic quantities which refer, ultimately, to geometrical squares. For instance, the line element of three-dimensional Euclidean space in tensor form $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ ($g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$), represents the Pythagorean Theorem algebraically, such that ds^2 designates a geometrical square on a hypotenuse of length s , and each squared term $dx^1 dx^1$, $dx^2 dx^2$, and $dx^3 dx^3$, designates a geometrical square on a side of length dx^1 , dx^2 , and dx^3 respectively. Even if we generalize the tensor to accommodate non-Euclidean spaces, we still necessarily represent (small) geometrical squares. There is no geometrical line element without the Pythagorean Theorem, and there is no Pythagorean Theorem without geometrical squares.

Of course, squared terms in the equations of physics as a rule are not indirect representations of geometrical squares. Thus, if we wish to physically interpret the so-called “spacetime interval” $c^2 t^2 - x^2$, it will be helpful to first derive this expression (the origin of which, we recall, lies in the Lorentz transformation rather than in geometry), non-algebraically in terms of ratio and proportion. This is surprisingly easy to do. Consider reference bodies A and B, in uniform motion relative to one another and coinciding in place and time at event D (readers may easily furnish their own diagram if needed). Some time after event D, a light pulse is emitted from A and overtakes B on its way to hitting a mirror (event E), and reflecting back past B on its way to finally arriving back at A.⁵⁴ For the light pulse’s trip *to* the mirror, we form the ratio between B’s reception time and A’s emission time (event D being zero reference time), and since the trip back to A is symmetrical, the same ratio will obtain for the reception time at A and the “emission” time at B (that is, the time the reflected pulse passes B on its way back to A). If we designate the time interval between events D and E as either t_A or t_B appropriately, the distance between these events as x_A or x_B , and the time it takes light to travel from either body A or body B to event E as T_A or T_B , then for the “to” trip the transmission time at A is $t_A - T_A$, and the reception time at B is $t_B - T_B$. The light pulse is then reflected at event E and passes B at $t_B + T_B$, being received back by A at $t_A + T_A$. Thus, the ratio of $(t_B + T_B)$ to $(t_A + T_A)$ is proportional to the ratio of $(t_A - T_A)$ to $(t_B - T_B)$; or, in proportion notation, $(t_B + T_B) : (t_A + T_A) :: (t_A - T_A) : (t_B - T_B)$. Note that this proportion represents the *relations* obtaining between time intervals in two inertial reference frames, and designates no invariant or absolute physical *quantity*.

To convert the proportion to an algebraic equation, we first rewrite T_A and T_B respectively as symbolic fractional numbers: $\frac{x_A}{c}$ (the time it takes light to go from reference body A to event E) and $\frac{x_B}{c}$ (the time it takes light to go from reference body B to event E). Then we rewrite the original ratios as fractions and the original proportion as an equality: $\frac{(t_B + \frac{x_B}{c})}{(t_A + \frac{x_A}{c})} = \frac{(t_A - \frac{x_A}{c})}{(t_B - \frac{x_B}{c})}$. Rearranging these terms, we obtain

54. I borrow this example from Bondi, who as a matter of course treats it algebraically; Hermann Bondi, *Relativity and Common Sense* (Mineola, NY: Dover, 1964), 116–18. Note that it does not matter whether we assume a frame of reference relative to which everything occurs on a single line, though it is easier to visualize things if we do.

$(t_A + \frac{x_A}{c})(t_A - \frac{x_A}{c}) = (t_B + \frac{x_B}{c})(t_B - \frac{x_B}{c})$, and multiplying through by c produces $(ct_A + x_A)(ct_A - x_A) = (ct_B + x_B)(ct_B - x_B)$. Further reduction yields the familiar Lorentzian transformation in its general form: $c^2t_A^2 - x_A^2 = c^2t_B^2 - x_B^2$. Clearly, the squared terms in the Lorentz transformation are *notational artifacts* that result from treating ratios as symbolic fractional numbers and multiplying them together. That is, these squared terms are symbolic representations of compounded ratios. It follows that the resemblance between the Lorentzian invariant $c^2t^2 - x^2$ and the Pythagorean line element $x^2 + y^2$ is strictly notational, and indeed merely visual; for the squared terms in the Pythagorean line element symbolically represent actual geometrical squares, not compounded ratios. There is, therefore, little basis for invoking any physical/geometrical analogy between space and so-called “spacetime,” least of all one that could underwrite the concept of a “spacetime interval.” The formal analogy at issue, that is, obtains simply between two forms of representation by means of coordinate transformation, not between the things represented by those transformations.

In light of this result, the following three propositions would appear to hold for the metrical properties of special relativistic space and time:

1. The metrics of space and time are *heterogeneous*; whereas were it physically real, the “spacetime metric” would necessarily be homogeneous and thus expressible solely in units of “spacetime.” Thus, the special theory of relativity implies no *single continuum* of spacetime.
2. The metrics of space and time are *relativistic*; whereas were it physically real, the “spacetime interval” would be an invariant or “absolute” quantity.
3. The metrics of space and time are *interdependent* or “entangled”; whereas in Newtonian mechanics, the metrical properties of “absolute space” and “absolute time” are independent of one another.

5 Minkowski Spacetime and the General Theory of Relativity

We have cast doubt, in the preceding analysis, on the physical intelligibility of the concept of Minkowski spacetime (the merging of space and time in a single continuum). This very concept, however, is by consensus indispensable to the general theory of relativity. Such an interpretation has the authority of Einstein himself, who, in his popular treatment of the general theory, for instance, famously remarks that general relativity “would perhaps have gotten no farther than its long clothes” if it had not been for Minkowski’s innovation.⁵⁵ On the question of the precise nature of general relativity’s reliance on the Minkowski approach, the “maximal” interpretation, as it were, is that Einstein’s appropriation of Minkowski commits us to regarding the physical world itself as a single geometrical continuum or “semi-Riemannian manifold.” More modest would be the assertion that while the

55. Einstein, *Relativity: The Special and General Theories*, 54.

Minkowski “single continuum” approach is indispensable to the *mathematical formulation* of general relativity, this does not commit us to regarding the physical world itself as a single continuum. The minimal claim, which I will urge here, is that it is solely the quadratic invariant $c^2t^2 - x^2$ that is essential to the formulation of general relativity, not the concept of Minkowski spacetime itself; and that Minkowski’s contribution lies rather in his having seen a significance to the proportionality between this quadratic invariant and proper time τ . The *concept* of Minkowski spacetime, I shall now demonstrate, is not only physically incoherent, as sufficiently demonstrated in the preceding section, but also both mathematically and physically superfluous to the general theory of relativity.

Turning our attention to Einstein’s own presentation of the general theory, we first note in passing that the famous “rotating disk” thought experiment described in Einstein’s 1916 paper, on the basis of which the connection between gravity and the so-called “curvature of spacetime” is deduced, has no relation to the idea of a unification of space and time beyond that already effected by the special theory of relativity. The non-Euclidean metric of *space* is derived from special relativistic length contraction of the measuring rod along the circumference, but not along the diameter, of the disk, while the gravitational effect on time is deduced from special relativistic “time dilation” at the periphery, but not at the center, of the disk.⁵⁶ Up to this point in Einstein’s paper, then, we have nothing that could be termed “spacetime” in the relevant sense. Only when the “purely mathematical

56. In his definitive treatment of the role of the rotating disk in the development of general relativity, John Stachel emphasizes that for Einstein, the rotating disk thought experiment showed the impossibility of employing Euclidean geometry in a gravitational field. John Stachel, “The Rigidly Rotating Disk as the ‘Missing Link’ in the History of General Relativity,” in *Einstein and the History of General Relativity*, ed. Stachel and Howard (Boston, MA: Birkhäuser, 1989), 48–62. Stachel further remarks that “Minkowski’s four-dimensional formulation played an important role in Einstein’s considerations at this point” (58). The significance of Minkowski for Einstein, according to Stachel, resided in Einstein’s realization that a theory of gravity based on the notion of a non-Euclidean metric requires a generalization of Gauss’s two-dimensional surface theory (that is, Gauss’s geometry of curved surfaces), which could be accomplished by a suitable generalization of Minkowski’s four-dimensional “flat” spacetime. Stachel is surely correct that this was Einstein’s thinking, the coherence of which we analyze in this section. The claim that a single “spacetime continuum” is a necessary condition for the formulation of a theory of gravity incorporating the results of the rotating disk thought experiment is one I in fact dispute. It bears mention as well that there is some potentially misleading terminology in play here. We have already noted (pages 168–9 above) the ambiguity of the term “four-dimensional,” which in the context of the Minkowski approach must be taken in the sense of a single continuum defined by a spacetime line element. However, when we say that according to general relativity, gravity arises from (or actually is) the “curvature of spacetime,” we are employing at least two layers of metaphor. In the first place, the very term “curved *space*” is a metaphor lifted from the Gaussian theory of curved surfaces, based on the appropriation of Gauss’s mathematics to describe the metrical properties of space itself. “Curved space” in this sense is simply a metaphor for “non-Euclidean.” But when time is involved as well, and we speak of “curved spacetime,” then an additional layer of metaphor has been added, since to call *time* “non-Euclidean” is to speak of time as if it were a spatial phenomenon. “Non-Euclidean time” is a metaphorical way of communicating the idea that the metrical properties of time are affected by gravity.

task,” as Einstein puts it, of finding generally covariant equations through the formation of tensors arises does the Minkowski interval (“linear element”) make its appearance.⁵⁷ Clearly, neither this “linear element” nor any “four-dimensional continuum” is implied by anything Einstein has said thus far in the paper, nor is it obvious that any such concept is required for the formation of the requisite tensors.

In his “Notes on the Origin of the General Theory of Relativity” (1934), Einstein presents two related reasons for the indispensability of the “spacetime line-element” to the general theory of relativity. The first is the need for a reformulation of the law of inertia in terms of the concept of a geodesic line, that is, a “straightest line” analogous to the classical conception of the “shortest line” between two points: “A material point, which is acted on by no force, will be represented in four-dimensional space by a straight line, that is to say, by a shortest line, or more correctly, an extremal line.”⁵⁸ The second reason follows immediately upon the first. The determination of such an extremal or straightest line requires a metric or line element conforming to a generally covariant transformation. That is to say, we must generalize the “quasi-Euclidean” Minkowski line-element such that the restriction to inertial reference frames is removed. Riemannian geometry is to supply this line element:

This concept [extremal distance] presupposes that of the length of a line element, that is to say, a metric. In the special theory of relativity, as Minkowski had shown, this metric was a quasi-Euclidean one, i.e., the square of the “length” ds of a line element was a certain quadratic function of the differentials of the coordinates. If other coordinates are introduced by means of a non-linear transformation, ds^2 remains a homogeneous function of the differentials of the coordinates, but the coefficients of this function ($g_{\mu\nu}$) cease to be constant and become certain functions of the coordinates. In mathematical terms this means that physical (four-dimensional) space has a Riemannian metric.⁵⁹

Einstein forthwith sums up the entire line of thought as clearly as one could want:

The timelike extremal lines of this metric furnish the law of motion of a material point which is acted on by no force apart from the forces of gravity. The coefficients ($g_{\mu\nu}$) at the same time describe the gravitational field with reference to the coordinate system selected.⁶⁰

In Einstein’s admirably lucid account, then, we require (1) a metrical description of space and time in terms of the $g_{\mu\nu}$ or metric tensor, and (2) a law of motion

57. Einstein, “The Foundation of the General Theory of Relativity,” in *The Principle of Relativity*, 119.

58. “Notes on the Origin of the General Theory of Relativity,” in Einstein, *Ideas and Opinions*, 316.

59. *Ibid.*, 316–17.

60. *Ibid.*, 317.

(timelike extremal line). This raises two questions for our consideration: (1) Is the quadratic form requisite for the desired generally covariant transformation of space and time variables a *geometrical* quadratic form per se? Or is it the case merely that Riemannian geometry may be used to *represent* this transformation? And (2), what is the physical meaning of the term “straightest line” given that *time* is involved rather than space alone? That is, is a “straight line” in so-called “four-dimensional spacetime” a *line* anywhere but on a graph?

Regarding the first of these questions, it can be demonstrated that the coefficients ($g_{\alpha\beta}$) in the quadratic form $g_{\alpha\beta}(x)dx^\alpha dx^\beta$ transform as a covariant tensor of second rank. As such, however, the tensor is merely an *analytical entity* having no relationship to geometry or “four-vectors” per se. That is to say, if we define a covariant tensor of rank two in a purely analytical way as follows:

$$A'_{\alpha\beta} = \left(\frac{\partial x_\gamma}{\partial x'_\alpha} \right) \left(\frac{\partial x_\delta}{\partial x'_\beta} \right) A_{\gamma\delta}$$

then there is no reason to suppose that the variables, which transform as indicated, represent anything geometrical. To determine whether they do, we must first ascertain whether the particular quadratic form for which this tensor supplies coefficients is derived from geometry.

The transformations for dt' and dx' respectively, when the restriction to inertial reference frames is removed, will take the form $dt' = A dt + B dx$, and $dx' = C dt + D dx$, where A , B , C , and D are position-dependent coefficients (partial differential functions of the space and time variables) rather than constants. These transformations are entailed by the original Lorentz transformation itself (applied to a small region of space and a small interval of time), not by Riemannian geometry. For the desired generally covariant transformation, we obtain, by substitution into the Lorentz transformation $c^2 dt^2 - dx^2 = c^2 dt'^2 - dx'^2$:

$$g'_{11} c^2 dt'^2 + g'_{22} dx'^2 + 2g'_{12} c dt' dx' = g_{11} c^2 dt^2 + g_{22} dx^2 + 2g_{12} c dt dx$$

or more compactly $g'_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} dx^\mu dx^\nu$. Note that we have as yet no reason to regard $g_{\mu\nu} dx^\mu dx^\nu$ as anything other than one side of a transformation equation, even though it does notationally resemble the line element of differential geometry. Furthermore, while the quadratic differential form (line element) of differential *geometry* is obtained by substitution into the Pythagorean Theorem, the quadratic differential form above is obtained, as we saw, by substitution into the Lorentz transformation (which is why the metric “signature” is different, with a minus sign appearing where the Pythagorean Theorem has a plus sign).

Clearly, the mere use of tensors does not entail a geometrical subject matter, and merely coining the term “semi-Riemannian” to account for the change of signs in no way alters the essentials of the case. As John Norton reminds us, “Ricci and Levi-Civita’s x_1, \dots, x_n were *variables* and not necessarily geometric coordinates. They were at pains to emphasize that what was then called infinitesimal geometry

was just one of the many possible applications of their calculus.”⁶¹ Indeed, Levi-Civita opens his 1923 monograph, *The Absolute Differential Calculus*, with this observation regarding the use of “geometrical terminology”:

In analytical geometry it frequently happens that complicated algebraic relationships represent simple geometric properties. In some of these cases, while the algebraic relationships are not easily expressed in words, the use of geometrical language, on the contrary, makes it possible to express the equivalent geometrical relationships clearly, concisely, and intuitively. Further, geometrical relationships are often easier to discover than are the corresponding analytical properties, so that geometrical terminology offers not only an illuminating means of exposition, but also a powerful instrument of research.⁶²

Note the three domains identified in the passage above: (1) geometry itself; (2) analysis or algebra; and (3) geometry as a *form of representation* (“geometrical language” or “geometrical terminology”). Clearly, the expressed idea is that there are instances where the *general* analytical relationships at issue are obscure, such that we can advantageously represent these “analytical properties,” themselves intended to represent the geometrical relationships under study, by means of a “geometrical language.” Presumably, this point would remain in force were the general analytical procedure (algebra) used to study something other than geometry; that is, it might still be advantageous to use geometry to represent the analytical properties under study. The analytical procedure itself, then, has nothing essentially to do with geometry, even though it may both be represented by means of geometry, and used to study geometry.

Levi-Civita here echoes Descartes with remarkable exactitude by distinguishing the distinct levels at which geometry may come into play. As Descartes remarks of his own symbolic method, although we are developing a general analytical–symbolic calculus that can be used to investigate geometrical or other subject matter, it is convenient to use geometry as a means of representing that calculus, whether or not the subject matter under study is itself geometrical in character:

We have as much reason to abstract propositions from geometrical figures, if the problem has to do with these, as we have from any other subject matter. The only figures that we need to reserve for this purpose are rectilinear and rectangular surfaces, or straight lines, which we also call figures, because, as we noted above, these are just as good as surfaces in assisting us to imagine an object that is really extended. Lastly, these same figures must serve to represent sometimes continuous magnitudes, sometimes a set or a number.⁶³

61. John Norton, “General Covariance and the Foundations of General Relativity: Eight Decades of Dispute,” *Rep. Prog. Phys.* (1993), 799–800.

62. Tullio Levi-Civita, *The Absolute Differential Calculus*, trans. Marjorie Long (Mineola, NY: Dover, 1977), 1.

63. Descartes, *Rules for the Direction of the Mind*, Rule 14, in *The Philosophical Writings of Descartes*, vol. 1, 65.

Taken on their own terms, the tensors of Ricci and Levi-Civita simply regard the transformation of variables by invariance, as Levi-Civita again notes in the section of the same monograph entitled “Algebraic Foundations of the Absolute Differential Calculus: Effect on Some Analytical Entities of a Change in Variables”:

Consider n independent variables x_1, x_2, \dots, x_n , which we shall as usual denote collectively as x , and suppose a transformation applied to them which leads to another set of n independent variables ... The geometrical name for this operation is of course *change of coordinates* ...⁶⁴

Such “analytical entities,” which may, but need not be, interpreted geometrically, are simply algebraic expressions which transform by invariance. The answer to our initial question above, then, is no, the general relativistic quadratic form $g_{\mu\nu}(x) dx^\mu dx^\nu$ is not a *geometrical* form per se, although it may be represented via Riemannian geometry regarded as a “symbolic space.” The quadratic form is derived from the Lorentz transformation rather than geometry, and the coefficients $g_{\mu\nu}$ have no intrinsic connection to geometry.⁶⁵

Let us proceed to the second question raised above, which regards the law of motion. We are entitled to suppose that the principle of maximal proper time, derived from the special theory of relativity, still holds in general relativity. Or at least this hypothesis must be made if we are to employ Einstein’s principle of equivalence, according to which we are to extend our analysis of the restricted case where special relativity holds in a finite region to the general case (presence of gravitational source masses) where it does not:

We now make the assumption, which readily suggests itself, that this covariant system of equations also defines the motion of the point in a gravitational field in the case when there is no system of reference ... with respect to which the special theory of relativity holds good in a finite region.⁶⁶

Moreover, we know already, from the special theory of relativity, that for free body motion the proper time τ is proportional to the Lorentz quadratic invariant $\sqrt{(ct)^2 - x^2}$, which implies that for the equation of motion or geodesic, the expression $c^2 t^2 - x^2$ is maximized.⁶⁷ Clearly, the interpretation of this expression as a “line

64. Levi-Civita, *Absolute Differential Calculus*, 61.

65. To be sure, *space* in general relativity may be with justification called a “Riemannian manifold” (not “semi-Riemannian”), since the non-Euclidean metric of space is obtained by substitution into the Pythagorean Theorem. But the ten independent coefficients $g_{\mu\nu}$ of the metric tensor in general relativity are simply position-dependent, partial differential functions of the space and time variables, which latter in no sense need be merged into a single geometrical continuum.

66. Einstein, “Foundation of the General Theory of Relativity,” 143.

67. If we assume the law of inertia, we can deduce by means of the Lorentz transformation the principle of “extremal proper time” for the geodesic of flat space and time. If a clock occupies the same spatial point in inertial reference frame A at two different times, for instance, then were it to undergo any acceleration during this time interval, the clock would experience a time dilation effect such that the proper time interval would be less than if the clock remained at rest. Thus, the law of

element” or spacetime interval is superfluous, since all that matters for determining the inertial trajectory of a free body is this symbolic quantity’s proportionality to proper time. Thus, for the geodesic of general relativity, the foregoing yields

$$c \int_A^B d\tau = \int_A^B \sqrt{g_{11}c^2 dt^2 + g_{22}dx^2 + g_{12}dx^2 + 2g_{12}c dt dx}$$

or, more compactly, $c \int_A^B d\tau = \int_A^B \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$. And since proper time is maximized, the proportional symbolic quantity $\int_A^B \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$ is also maximized, yielding the desired gravitational trajectory $\delta \int_A^B \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = 0$.

To reiterate, none of the preceding requires our interpreting $\sqrt{g_{\mu\nu} dx^\mu dx^\nu}$ as a line element or as anything essentially geometrical. We are liable to be misled, of course, when we express this proportion symbolically in the form of an equation and drop the units of light velocity ($\int_A^B d\tau = \int_A^B \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$), for then we can easily imagine that proper time “is” the symbolic quantity $\int_A^B \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$, and that the proper time registered on a clock therefore “measures” a spacetime interval. But “proper time” is and can only be simply *time*, and therefore it cannot *measure* “spacetime.” The very confusion at work here, with which the literature on spacetime is replete, is the conflation of *measurement* and *representation*. Any quantity, of whatever kind, can be *measured* solely in units homogeneous with the quantity itself. If I am talking about how much money I have, the unit of measure must be an amount of money. To be sure, I can *represent* a quantity by means of units heterogeneous with the quantity, as I would do were I to draw a graph in which the length of a bar represented how much money I have. But this would not be to *measure* my wealth in units of length. Granted that the invariability of the velocity of light in special relativity entails a determinate relationship between distance and time, such that we can measure distance indirectly by means of time (as we do with radar, for instance), or vice versa, but strictly speaking this is not to measure distance in *units* of time. Were there such a thing in the physical world as the spacetime interval, it could be *measured* solely in units of “spacetime.”

Clearly, the law of motion (general relativistic “geodesic”), as we have derived it above, has nothing whatsoever to do with any concept of “four-dimensional straight line,” being determined rather by the principle of maximal proper time (read off a clock), in keeping with the law of inertia. Nevertheless, the gravitational trajectory can be represented geometrically as a straight (or “straightest”) line by means of the now-preferred method of tangent vector parallel transport. This method is based on the following considerations: Given a time axis, we can graph the motion of a physical particle by drawing a line through a series of plotted dots, each dot standing for the body’s position in space at an instant of time. For a free body uninfluenced by gravity, such a line will be straight in Cartesian coordinates.

inertia implies that the proper time will assume the maximum value in inertial reference frame A. The preceding, of course, applies as well to any other inertial reference frame.

Therefore, if we draw a vector tangent to this line, that vector will obviously lie on the line and be parallel to itself (that is, point always in the same direction) at any point on the line. The vector's direction (angle) with respect to the time axis represents the speed of the moving particle. Thus, our body is making a straight line *in three-dimensional space* and, since this body is maintaining a constant speed, the graphed line on a “spacetime diagram” is also straight.

Clearly, it is only when time is represented *on a graph* that can we speak of the trajectory of a free body as a “straightest” or geodesic line. To speak of a “straightest line” in spacetime is therefore to employ a *metaphor* from geometry. As noted in our discussion of Minkowski's 1908 essay, the so-called “four-velocity” is an analytical or symbolic vector, not a physical vector, with the time component ($\frac{dt}{dt}$) being a symbolic representation of a ratio of velocities (the dimensionless number $\frac{c}{\sqrt{c^2-v^2}}$ or β). Thus, if we take the traditional law of inertia in three-dimensional terms ($\frac{dv}{dt} = 0$, or component-wise $\frac{d^2x}{dt^2} = 0$, $\frac{d^2y}{dt^2} = 0$, $\frac{d^2z}{dt^2} = 0$), it trivially follows that $\frac{d^2x}{dt^2} = 0$, $\frac{d^2y}{dt^2} = 0$, $\frac{d^2z}{dt^2} = 0$, and $\frac{d^2t}{dt^2} = 0$ (or in the preferred indexical notation, $\frac{d^2x_\mu}{dt^2} = 0$). But all the physically relevant information is contained in the standard three-dimensional acceleration terms $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2z}{dt^2}$.

Extending the notion of parallel transport to the general case where the special theory of relativity does not hold in a finite region (that is, extending it to gravitational fields generated by source masses), we obtain the law of motion (gravitational geodesic) in terms of parallel transport of the tangent “four-velocity” vector:

$$\frac{d^2x_\mu}{dt^2} + \Gamma_{\alpha\beta}^{\mu} \frac{dx_\alpha}{dt} \frac{dx_\beta}{dt} = 0$$

or

$$\frac{d^2x_\mu}{dt^2} = -\Gamma_{\alpha\beta}^{\mu} \frac{dx_\alpha}{dt} \frac{dx_\beta}{dt}$$

The term containing $\Gamma_{\alpha\beta}^{\mu}$ may be regarded as a “correction factor” for the presence of gravitational source masses, since the $\Gamma_{\alpha\beta}^{\mu}$ (Christoffel symbols), constructed from the first derivatives of the $g_{\mu\nu}$, vanish for the special case of constant $g_{\mu\nu}$, where no source masses are present. In the general case, however, unlike the preceding case where the special theory of relativity held in a finite region, the time term $\frac{d^2t}{dt^2}$ carries essential information, not because it is a vector component in the geometrical or physical sense, but rather simply because it supplies the ratio between t and τ , which is no longer a constant ratio since the $g_{\mu\nu}$ are no longer constant. Therefore, although the method of tangent vector parallel transport is perfectly functional in a purely mathematical sense, we must conclude that extremal proper time, which carries a direct physical meaning, is to be preferred. Moreover, we have confirmed our initial thesis that the concept of the Minkowski “interval” is both physically and mathematically superfluous to the general theory of relativity.

6 Concluding Remarks

Our desedimentation of the concept of “spacetime” has yielded a twofold result. The concept itself has been revealed in its inherent incoherence, while the governing idea of the general theory of relativity (that gravity is to be understood in terms of inertial motion where the metrical properties of space and time are affected by source masses), has disclosed itself in a more intuitively coherent way. The theory of relativity, both special and general, gains significantly in physical intelligibility when interpreted apart from the concept of “Minkowski spacetime.”

But why should this result matter, if the experimental predictions of the theory remain unaltered? I want to suggest three respects in which it does matter. The first regards the aim of scientific understanding itself. It is sometimes suggested that the only criterion for a mathematical formalism is whether it “works,” and since the Minkowski formalism does in fact work, such criticisms of the concept of spacetime as I have offered are otiose. But at a minimum, and prior to consideration of whatever philosophical issues may arise in connection with such an “instrumental” view of science, we should remind ourselves that whether something “works” depends on what one is trying to do with it. If we are trying to *understand* the world, and not just predict its course or manipulate it technologically, then the intuitive coherence of physical concepts is paramount. Genuine science is simply not served by the reification of symbolic mathematical entities as if they were physical realities.

A second reason concerns the need for conceptual clarity in the philosophy of space and time. Philosophers tend to assume, as a starting point in the philosophy of space and time, the validity of the concept of Minkowski spacetime. The result is no end of philosophical mischief, such as the idea that change is an “illusion,” that we have no free will, that time is “static,” and so forth. Einstein himself was not immune to such ill-considered metaphysical extrapolations, as when he poignantly attempted to console the family of his recently deceased friend Michele Besso with the thought that in mathematical physics the distinction between past, present, and future is no more than a “stubbornly persistent illusion.” If only spacetime physics could be a cure for grief or the fear of death. Unfortunately, the Minkowski “geometrical” approach gives the erroneous impression that mathematical physics supports philosophical theories of static or “block” time. As Meyerson observed already in 1925,

It should be noticed that if space and time are henceforth to be more or less merged into a single continuum, this change will clearly work to the advantage of space Let us observe, moreover, that this already follows from the very fact that the construction at which one arrives is a *geometry*. And one need only open an exposition of the doctrine to note that, where time is concerned, the writer always speaks of one dimension, obviously conceived as spatial, while no attempt is ever made to represent the properly spatial dimensions in terms of time.⁶⁸

68. Meyerson, *Relativistic Deduction*, 72.

In whatever way we regard philosophically the paradoxes of time in the special theory of relativity, only a kind of bewitchment by symbolic mathematics could make us imagine that such paradoxes have been resolved through the concept of spacetime.

Thirdly, I believe scientists themselves have an interest in the critique of Minkowski spacetime. A number of philosophically-minded scientists, for example David Bohm and Ilya Prigogine, to mention just two, have expressed reservations about the manner in which time is conceived in contemporary mathematical physics.⁶⁹ More recently, Lee Smolin, in his critical assessment of the state of contemporary theoretical physics (*The Trouble With Physics*, 2006), suggests that the principal impediment to a successful unification of general relativity and quantum mechanics is the static representation of time. Smolin describes the origin of this problem in seventeenth-century mechanics, when Galileo and Descartes discovered that “you could draw a graph, with one axis being space and the other being time,” such that “time is represented as if it were a dimension of space.”⁷⁰ Concludes Smolin, “We have to find a way to *unfreeze time*” Of course, neither Galileo nor Descartes ever imagined that representing time geometrically meant that time itself was a geometrical phenomenon. Minkowski’s “spacetime” interpretation of the special theory of relativity, on the other hand, is the very example of Smolin’s concern, since here time is not merely represented by means of space, but actually conceived to be a geometrical dimension.

Along the same lines as Smolin, physicist Joy Christian, a theorist of quantum gravity, seeks to break what he calls

the spell of the “block” view of time which is widely thought to be an inevitable byproduct of Einstein’s special relativity. According to this “block” view, since in the Minkowski picture time is as “laid out” a priori as space, and since space clearly does not seem to “flow,” what we perceive as a “flow of time,” or becoming,” must be an illusion. Worse still, in Einstein’s theory, the relativity of simultaneous events demands that what is “now” for one inertial observer cannot be the same, in general, for another. Therefore, to accommodate “nows” of all possible observers, events must exist a priori, all at once, across the whole span of time. As Weyl once so aptly put it, “The objective world simply is, it does not happen.”⁷¹

A good beginning toward breaking this “spell” would be to discard the Minkowski conception altogether.

69. See David R. Griffin, ed., *Physics and the Ultimate Significance of Time* (Albany, NY: State University of New York, 1986).

70. Lee Smolin, *The Trouble With Physics* (Boston, MA: Houghton Mifflin, 2006), 256–7. To be precise, while Galileo does use such a form of representation, although of course not with Cartesian axes, Descartes to my knowledge never does so.

71. Joy Christian, “Passage of Time in a Planck Scale Rooted Local Inertial Structure,” Cornell University archive (http://xxx.lanl.gov/PS_cache/gr-c/pdf/0308/0308028v4.pdf), 2003, 12.

Beyond the theory of spacetime itself, it could be that other areas of contemporary mathematical physics are similarly impeded by the reification of symbolic mathematical entities. If so, the genetic approach originally suggested by Husserl, pioneered for the case of modern symbolic mathematics by Klein, and concretely demonstrated in this essay for the concept of “spacetime,” might make a contribution to clarifying concepts in those areas as well.

Acknowledgments

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