The Logic of Hyperlogic
Part A: Foundations*

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Abstract. Hyperlogic is a hyperintensional system designed to regiment metalogical claims (e.g., "Intuitionistic logic is correct" or "The law of excluded middle holds") into the object language, including within embedded environments such as attitude reports and counterfactuals. This paper is the first of a two-part series exploring the logic of hyperlogic. This part presents a minimal logic of hyperlogic and proves its completeness. It consists of two interdefined axiomatic systems: one for classical consequence (truth preservation under a classical interpretation of the connectives) and one for "universal" consequence (truth preservation under any interpretation). The sequel to this paper explores stronger logics that are sound and complete over various restricted classes of models as well as languages with hyperintensional operators.

A1 Introduction

Philosophers of logic debate about metalogical claims like the following:

(1) Classic logic is correct.

(2) The law of excluded middle holds.

(3) Some contradiction entails everything.

Such metalogical claims can also felicitously occur in embedded environments. One illustration involves claims such as (4)–(6), which describe what holds *according to* certain logics.

(4) According to intuitionistic logic, the law of excluded middle doesn't hold.

(5) In strong Kleene logic, nothing is valid.

(6) Everything that is intuitionistically valid is classically valid.

Other illustrations of the embeddability of metalogical claims come from attitude verbs, conditionals, and modals:

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(7) Inej believes intuitionistic logic is correct.

(8) If intuitionistic logic were correct, excluded middle would fail.

(9) It might be that there are true contradictions.

Although such metalogical claims are loaded with theoretical terms whose nature is philosophically contentious, these are all perfectly intelligible claims of English. Given this, it is natural to investigate the semantic analysis of such claims. To do this, we need to answer two questions. First, how do we regiment metalogical claims into the object language so that they can be assigned a compositional semantic value? Second, how do we assign compositional semantic values to such regimentations?

Recently, Kocurek (2021b) has developed a hyperintensional system that offers answers to both questions called **hyperlogic**. To regiment metalogical claims, hyperlogic utilizes a combination of several different devices: a multigrade entailment operator $\rhd$; propositional quantifiers $\forall p$ and $\exists p$ (Fine, 1970) to regiment laws of logic; and terms and operators borrowed from hybrid logic (Areces and ten Cate, 2006; Braüner, 2017), such as nominals ($l_1, l_2, l_3, \ldots$) to regiment claims about which logic is correct, and operators @ to regiment "according to" claims. To illustrate, here is how we could regiment (1)–(6) in hyperlogic.[1]

(1) Classic logic is correct.

$$cl$$

(2) The law of excluded middle holds.

$$\forall p(\rhd(p \vee \neg p))$$

(3) Some contradiction entails everything.

$$\exists p\, \forall q((p \wedge \neg p) \rhd q)$$

(4) According to intuitionistic logic, the law of excluded middle doesn't hold.

$$@_{il} \neg \forall p(\rhd(p \vee \neg p))$$

[1] As Kocurek (2021b, fn. 9) points out, there are multiple regimentations of (4) depending on how we interpret the 'not' in 'does not hold' (classically or intuitionistically). Fortunately, hyperlogic can regiment both readings (see Definition A2.5 for expressing classical negation in the scope of "according to" operators).

(5) In strong Kleene logic, nothing is valid.

$$@_{k3} \neg \exists p(\rhd p)$$

(6) Everything that is intuitionistically valid is classically valid.

$$\forall p(@_{il}(\rhd p) \rightarrow @_{cl}(\rhd p))$$

To assign compositional semantic values to metalogical claims, hyperlogic introduces a shiftable convention parameter—a "hyperconvention"—into points of evaluation. This parameter determines the interpretation of the logical connectives (as well as $\rhd$).[2] The semantic value of a formula is a set of world-hyperconvention pairs. While metalogical claims may express a trivial possible worlds proposition relative to a hyperconvention, they can have nontrivial semantic values that hyperintensional environments can exploit.

My aim in this paper is not to defend hyperlogic as a semantic theory for metalogical claims. Rather, my aim is to address the following question: given that hyperlogic is designed to reason about other logics, what, if anything, can we say about logical consequence within hyperlogic itself? In other words, what is the logic of hyperlogic?

At first, one might suspect the logic of hyperlogic is entirely uninteresting. How much could be valid in a framework with the expressive resources to talk about other logics? As it turns out, however, this initial impression is mistaken. To show this, I present a sound and complete proof system for hyperlogic. It involves two separate axiomatic systems that are recursively defined in terms of one another, each representing different kinds of consequence: one represents ordinary classical consequence (truth preservation relative to a classical interpretation of the connectives) while the other represents "universal" consequence (truth preservation relative to *any* interpretation of the connectives). This dual proof system contains rules for moving back and forth between these axiomatic systems. The result is an elegant, tractable, and nontrivial logic for hyperlogic.

[2] One could interpret this parameter as determining the Kaplanian character of the connectives (Kaplan, 1977). Alternatively, one could interpret it as the content of the connectives determined by their character given a particular conversational context. Officially, hyperlogic is neutral on what determines the interpretation of the connectives on a particular occasion of use. In particular, it is compatible with contextualist, relativist, expressivist, and even objectivist views about the connectives. What hyperlogic requires is simply the ability of hyperintensional operators to shift the hyperconvention parameter. To keep things simple, we will set aside issues around context-sensitivity so that we don't have to add the context parameter to the index.

There are several reasons independent of the semantics of metalogical claims to be interested in the logic of hyperlogic. For one, the main semantic innovation of hyperlogic, viz., to add a shiftable convention parameter for interpreting the logical connectives, is behind several "conventionalist" approaches to hyperintensionality in the literature, which model hyperintensional environments as convention-shifting operators (cf. Muskens 1991; Williamson 2009; Locke 2019; Kocurek and Jerzak 2021; Muñoz 2020).[3] This contrasts with approaches that introduce incomplete and/or inconsistent states (impossible worlds, truthmakers, situations, etc.) into standard possible worlds frameworks.[4] Even if one thinks these conventionalist approaches are ultimately mistaken, one might still wonder how many hyperintensional phenomena can be explained in terms of it. Hyperlogic presents an encouraging answer for conventionalists about hyperintensionality.

In addition, hyperlogic provides a simple logic for "according to". For example, the following sounds fine to say:

(10) Pluto is not a planet, but according to the folk definition of 'planet', Pluto is a planet.

The phrase "according to the folk definition of 'planet'" seems to, in some sense, shift the interpretation of 'planet' mid-sentence so that the second 'Pluto is a planet' is interpreted via the folk definition of 'planet' (Kocurek et al., 2020, p. 8). If so, it's natural to ask how this operator works and what logical principles govern it. As we'll see, hyperlogic offers a simple yet attractive answer to these questions.

Finally, hyperlogic may provide insight into the problem of logical omniscience. Stalnaker (1976a,b, 1984) famously analyzed the content of an agent's mental state as a set of possible worlds, viz., those at which what the agent (actually) believes is true. While this view has its merits, it infamously

[3] The idea that some operators could, in principle, shift conventions is due to Einheuser (2006). See Kocurek et al. 2020 for further defense of this claim. There is some resemblance between these "conventionalist" approaches to hyperintensionality and two-dimensional semantics (Stalnaker, 1978; Davies and Humberstone, 1980; Einheuser, 2006). One could replace the convention parameter with a world-as-actual parameter that determines the interpretation of the logical connectives and achieve much of the same effect. However, tying the convention parameter to the world-as-actual parameter has some undesirable consequences; e.g., we may want to shift these parameters separately. Furthermore, it's unlikely that the world-as-actual will always determine a unique convention. See Kocurek and Jerzak 2021, fn. 18 for discussion.

[4] See, e.g., Mares 1997; Nolan 1997; Fine 2012; Krakauer 2012; Brogaard and Salerno 2013; Jago 2014; Kment 2014; Weiss 2017; Berto et al. 2018; Berto and Jago 2019; Priest 2018; Leitgeb 2019; French et al. 2020; Sedlár 2021. For overview, see Berto and Nolan 2021; Kocurek 2021a.

predicts that agents' beliefs are closed under logical entailment. There is a vast disagreement in the literature over how to address this problem.[5] Hyperlogic potentially provides a novel and attractive solution by analyzing mental content in terms of sets of world-hyperconvention pairs instead of sets of worlds. This new picture can validate certain modest closure principles without requiring beliefs be closed under classical consequence. It can thus avoid at least certain forms of logical omniscience while preserving the main features that initially motivated the Stalnakerian picture.

This paper is the first in a two-part series on the logic of hyperlogic. Part A focuses on a very general system for hyperlogic, which places no restrictions on the class of models. The logic of this system is fairly weak, and therefore constitutes a kind of minimal hyperlogic upon which stronger hyperlogics can be based. Part B explores stronger logics of this sort, as well as the logic of hyperlogic enriched further with hyperintensional operators.

Here is an outline of what is to come in this part. In §A2, I give a brief overview of the syntax and semantics of hyperlogic. In §A3, I present, and prove the completeness of, a proof system for the fragment of hyperlogic without propositional quantifiers. In §A4, I extend these results to the language of hyperlogic with propositional quantifiers. I conclude in §A5.

A2 Hyperlogic: Syntax and Semantics

We start by reviewing the syntax, semantics, and consequence relation(s) of hyperlogic as presented in Kocurek 2021b. In §A2.1, we introduce the language of hyperlogic. In §A2.2, we clarify the notion of a hyperconvention and use it to state a semantics for hyperlogic. In §A2.3, we identify two notions of consequence in hyperlogic and explain their relation.

A2.1 Syntax

The language of hyperlogic is an extension of the language of standard propositional modal logic. We start with an infinite stock of propositional variables $\mathsf{Prop} = \{p_1, p_2, p_3, \ldots\}$, the usual boolean connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$), and modal operators ($\Box$, $\Diamond$). To reduce on clutter, we define $(\phi \leftrightarrow \psi)$ as $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$ rather than treat $\leftrightarrow$ as a primitive connective. Nothing in what follows would substantively change if we primitively introduced $\leftrightarrow$ (or other sentential connectives) into our language.

[5] For discussion, see Soames 1987, 2008; Berto 2010; Ripley 2012; Bjerring 2013; Jago 2015; Bjerring and Schwarz 2017; Yalcin 2018; Bjerring and Skipper 2019; Hawke et al. 2019; Skipper and Bjerring 2020; Elga and Rayo 2021.

Definition A2.1 (Base Language $\mathcal{L}^{\mathbf{0}}$).

$$\phi ::= p \mid \neg\,\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid \Box\,\phi \mid \Diamond\,\phi.$$

The full language of hyperlogic extends $\mathcal{L}^{0}$ in three ways. I will introduce each extension separately so that fragments of the full language can be studied independently.

First, hyperlogic adds an "entailment" operator $\triangleright$, where $(\phi_1, \ldots, \phi_n \triangleright \psi)$ represents the claim that $\phi_1, \ldots, \phi_n$ (in that order) entail ψ. This operator is left-multigrade, meaning it can take any finite number (possibly zero) of arguments on the left. We could make $\triangleright$ right-multigrade as well (e.g., to represent multiple-conclusion logics) without substantively affecting the results presented in what follows. But for notational ease, we assume a fixed arity of 1 on the right.

Definition A2.2 (Entailment Language $\mathcal{L}^{\mathbf{E}}$).

$$\phi ::= p \mid \neg\,\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid \Box\,\phi \mid \Diamond\,\phi \mid (\phi, \ldots, \phi \triangleright \phi).$$

Second, hyperlogic adds propositional quantifiers $\forall p$ and $\exists p$ that bind into sentence position (Fine, 1970). When combined with the entailment operator, we can regiment laws of logic as universal entailment claims. For instance, we can regiment the law of double negation elimination as $\forall p(\neg\neg p \triangleright p)$.

Definition A2.3 (Quantified Language $\mathcal{L}^{\mathbf{Q}}$).

$$\phi ::= p \mid \neg\,\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid \Box\,\phi \mid \Diamond\,\phi \mid \forall p\,\phi \mid \exists p\,\phi.$$

Finally, hyperlogic adds operators similar to those found in hybrid logic. Hybrid logic extends propositional modal logic with (i) **state terms** $\sigma_1, \sigma_2, \sigma_3, \ldots$ (including state variables and state "nominals", i.e., constants), which double as terms denoting worlds and as atomic formulas that hold at their denotation; (ii) for each state term σ, an "according to" operator $@_\sigma$, which resets the world of evaluation to the world denoted by σ; and (iii) for each state variable s, a binding operator $\downarrow s$, which reassigns the denotation of s to the current world of evaluation (Areces and ten Cate, 2006; Braüner, 2017). Informally, we can read s as "s is actual", $@_s$ as "according to s,. . . ", and $\downarrow s$ as "where s is the current world,. . . ".

Instead of hybrid operators for worlds, hyperlogic introduces hybrid operators for interpretations of the base language, including the logical connectives. Thus, it introduces an infinite stock of **interpretation variables** $\mathsf{IVar} = \{i_1, i_2, i_3, \ldots\}$ and **interpretation nominals** $\mathsf{INom} = \{l_1, l_2, l_3, \ldots\}$. We single out a designated nominal cl to stand for a classical (**S5**) interpretation of the connectives. An **interpretation term** is a member of $\mathsf{ITerm} := \mathsf{IVar} \cup \mathsf{INom}$. We use ι, κ, λ, etc. as metavariables over interpretation terms.

Definition A2.4 (Hybrid Language $\mathcal{L}^{\mathsf{H}}$).

$$\phi ::= p \mid \iota \mid \neg\,\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid \Box\,\phi \mid \Diamond\,\phi \mid @_\iota\,\phi \mid {\downarrow}\, i.\phi.$$

Informally, we can read ι as "ι is correct", $@_\iota$ as "according to ι,. . . ", and $\downarrow i$ as "where i is the current interpretation,. . . ".

The binder $\downarrow$ allows us to define the following as abbreviations for the connectives under their classical interpretation.

Definition A2.5 (Rigidly Classical Connectives). Where i is not in ϕ or ψ:

$$\begin{array}{rclrcl}
\sim\phi & := & \downarrow i.\, @_{cl}\, \neg\, @_i\, \phi & (\phi \,\&\, \psi) & := & \downarrow i.\, @_{cl}(@_i\, \phi \wedge @_i\, \psi) \\
\blacksquare\,\phi & := & \downarrow i.\, @_{cl}\, \Box\, @_i\, \phi & (\phi + \psi) & := & \downarrow i.\, @_{cl}(@_i\, \phi \vee @_i\, \psi) \\
\blacklozenge\,\phi & := & \downarrow i.\, @_{cl}\, \Diamond\, @_i\, \phi & (\phi \supset \psi) & := & \downarrow i.\, @_{cl}(@_i\, \phi \rightarrow @_i\, \psi) \\
\bot & := & (p \,\&\, \sim p) & (\phi \equiv \psi) & := & ((\phi \supset \psi) \,\&\, (\psi \supset \phi)).
\end{array}$$

These "connectives" are interpreted classically even at nonclassical interpretations and even within the scope of "according to" operators. We will make extensive use of these rigidly classical connectives throughout, as it is in large part thanks to them that hyperlogic has a nontrivial logic.

These three extensions can be freely combined: $\mathcal{L}^{\mathsf{QE}}$ is the quantified entailment language, $\mathcal{L}^{\mathsf{QH}}$ is the quantified hybrid language, and $\mathcal{L}^{\mathsf{HE}}$ is the hybrid entailment language. For convenience, we define the full language of hyperlogic as $\mathcal{H} := \mathcal{L}^{\mathsf{QHE}}$.

Definition A2.6 (Substitution). We adopt the usual notions of "free" and "bound" variables (where i is bound by $\downarrow i$ and p is bound by $\forall p$ and $\exists p$). We say ι_2 is **free for** ι_1 in ϕ if no free occurrence of ι_1 in ϕ is in the scope of $\downarrow \iota_2$. In that case, we write $\phi[\iota_2/\iota_1]$ for the result of replacing every free occurrence of ι_1 in ϕ with ι_2. Similarly, ψ is **free for** p in ϕ if no free occurrence of p in ϕ is in the scope of $\forall q$ or $\exists q$ where q occurs free in ψ, or a binder $\downarrow i$ where i occurs free in ψ. If ψ is free for p in ϕ, we write $\phi[\psi/p]$ for the result of replacing every free occurrence of p in ϕ with ψ. Simultaneous substitution $\phi[\psi_1/p_1, \ldots, \psi_n/p_n]$ is defined likewise.

A2.2 Semantics

The main semantic innovation behind hyperlogic is to relativize truth to a "hyperconvention", i.e., a maximally specific interpretation of the base language. More precisely, a hyperconvention specifies a space of (coarse-grained) possible worlds propositions, assigns each propositional variable to a possible worlds proposition in the space, and assigns each connective in $\mathcal{L}^0$ (or $\mathcal{L}^\mathsf{E}$) to an operation on propositions, i.e., a function from some proposition(s) to a proposition.[6]

Definition A2.7 (Hyperconvention). Let $W \neq \varnothing$ and $\pi \subseteq \wp W$. A **π-hyperconvention for $\mathcal{L}^0$ (over W)** is a function c with domain $\{\neg, \Box, \Diamond, \wedge, \vee, \rightarrow\} \cup \mathsf{Prop}$ such that:

(i) $c(p) \in \pi$ for all $p \in \mathsf{Prop}$
(ii) $c(\triangle)\colon \pi^n \to \pi$ for each n-ary operator $\triangle \in \{\neg, \Box, \Diamond, \wedge, \vee, \rightarrow\}$.

A **π-hyperconvention for $\mathcal{L}^\mathsf{E}$ (over W)** adds $\triangleright$ to the domain, where:

(iii) $c(\triangleright)\colon \pi^{<\omega} \times \pi \to \pi$.

We call π the **proposition space** for c. We write π_c for the π that c is defined over and $\triangle_c$ (with infix notation) for $c(\triangle)$. A **hyperconvention for $\mathcal{L}$ (over W)** is a π-hyperconvention for $\mathcal{L}$ over W for some $\pi \subseteq \wp W$. We let $\mathbb{H}_W^{(\mathcal{L})}$ be the set of all hyperconventions for $\mathcal{L}$ over W. Throughout, I use "hyperconvention" to mean "hyperconvention for $\mathcal{L}^\mathsf{E}$" if the language under discussion contains $\triangleright$, and "hyperconvention for $\mathcal{L}^0$" otherwise.

[6] Note that, in Definition A2.7, π_c is unique; e.g., it's the unique domain of $c(\neg)$, which is a total function.

At the outset, we place no constraints on which operations can be assigned to a connective by a hyperconvention. The task of exploring how things change when we impose such constraints is taken up in Part B.

Just as a proposition is typically modeled as a set of worlds, a "convention" is modeled as a set of hyperconventions.

Definition A2.8 (Convention). An **convention** is a nonempty set of hyperconventions. We let $\mathbb{C}_W^{(\mathcal{L})}$ be the set of conventions (for $\mathcal{L}$) over W.

We can think of a logic as a special type of convention that only concerns the interpretation of the connectives (and $\triangleright$, if present). Here, we need not take a stand on what features of entailment are essential to logic: the hyperconvention semantics can accommodate a range of views on this matter.

To define our models, we need to introduce the notions of an index and an index proposition. In the hyperconvention semantics, truth is evaluated relative to an *index*, i.e., a world-hyperconvention pair.

Definition A2.9 (Index). Given a set H of hyperconventions over W, an **index** over H is a pair $\langle w, c\rangle$ where $w \in W$ and $c \in H$. We let $\mathbb{I}_H = W \times H$ be the set of indices over H.

As a formal semantics, hyperlogic is neutral on how to understand what an index represents. Kocurek (2021b, p. 13682) interprets indices as worlds "under descriptions". On this picture, logic is not a feature of the world but a feature of our representation of it (cf. Kocurek and Jerzak 2021). We could, however, instead hold that logic is genuinely part of the world. In that case, an index $\langle w, c\rangle$ represents a (perhaps logically impossible) world, where the w component determines all the nonsemantic facts while c determines the semantic facts.

Given this notion of an index, there are now three relevant notions of "proposition" to consider. First, there's the standard, coarse-grained notion of a proposition as a set of worlds, which is what hyperconventions assign to propositional variables, and operations on which they assign to connectives. Call this the **intension** of a formula relative to a hyperconvention. Second, there's a fine-grained notion of a proposition as a set of indices, which is the **compositional semantic value** of a formula. (Thus, semantic values are more fine-grained than intensions.) Finally, there is an intermediate notion of a "visible" index proposition, i.e., a function from hyperconventions to intensions in their proposition space. More precisely:

Definition A2.10 (Index Proposition). Given a set of hyperconventions H over W, an **index proposition** over H is a set of indices $A \subseteq \mathbb{I}_H$. Where $A \subseteq \mathbb{I}_H$, let $A(c) := \{w \in W \mid \langle w, c \rangle \in A\}$. An index proposition A is **visible** if $A(c) \in \pi_c$ for all $c \in H$. We let $\mathbb{P}_H$ be the set of *visible* index propositions over H. We use $X, Y, Z, \ldots$ for worlds propositions and $P, Q, R, \ldots$ for visible index propositions.

Since a propositional variable's intension relative to a hyperconvention is always a world proposition from that hyperconvention's proposition space (i.e., $c(p) \in \pi_c$), the (fine-grained) semantic value of a propositional variable is always a visible index proposition. Propositional quantifiers, therefore, range over over visible index propositions (see Kocurek 2021b, p. 13677).

We are now ready to define our models and semantics more explicitly. A model in this semantics specifies (i) a set of states (or "worlds"), (ii) a set of conventions for interpretation terms to denote, (iii) a set of (visible) propositions for quantifiers to range over, and (iv) a valuation function.

Definition A2.11 (Hyperframes and Hypermodels). A **hyperframe** is a triple of the form $\mathcal{F} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}} \rangle$, where:

- $W \neq \varnothing$ is a **state space**
- $D_{\mathbb{C}} \subseteq \mathbb{C}_W$ is a **convention domain**; we define $D_{\mathbb{H}} := \bigcup D_{\mathbb{C}}$ to be the **hyperconvention domain** (in other words, $D_{\mathbb{H}}$ is the set of hyperconventions that appear somewhere in $D_{\mathbb{C}}$)
- $D_{\mathbb{P}} \subseteq \mathbb{P}_{D_{\mathbb{H}}}$ is a **proposition domain** such that:
  (i) for all $p \in \mathsf{Prop}$, $P_p \in D_{\mathbb{P}}$, where $P_p(c) = c(p)$ for all $c \in D_{\mathbb{H}}$
  (ii) for all $c \in D_{\mathbb{H}}$ and all $X \in \pi_c$, there is a $P \in D_{\mathbb{P}}$ such that $P(c) = X$.

A **valuation for** $\mathcal{F}$ is a mapping V such that:

- $V(p) \in D_{\mathbb{P}}$ for each $p \in \mathsf{Prop}$
- $V(l) \in D_{\mathbb{C}}$ for each $l \in \mathsf{INom}$
- $V(i) \in D_{\mathbb{C}} \cup \{\{c\} \mid c \in D_{\mathbb{H}}\}$ for each $i \in \mathsf{IVar}$.

A **hypermodel based on** $\mathcal{F}$ is a tuple $\mathcal{M} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}}, V \rangle$ where V is a valuation for $\mathcal{F}$.

Following Kocurek (2021b), we only impose two minimal constraints on proposition domains at the outset (Part B will explore others). These minimal constraints effectively rule out distinct yet indiscernible hyperconventions (i.e., they ensure the soundness of PII in Table A6 in §A4.1). It would be interesting to see how the logic of hyperlogic changes if we drop those constraints. But I have yet to find a completeness proof that does without them, so I leave that task aside.[7]

Definition A2.12 (Semantics). Where x is a (propositional or interpretation) variable and v is a possible value for that variable, let V^x_v be the valuation like V except that $V^x_v(x) = v$, and let $\mathcal{M}^x_v$ be $\langle W, D_{\mathbb{C}}, D_{\mathbb{P}}, V^x_v \rangle$. Then:

$$
\begin{array}{lcl}
\mathcal{M}, w, c \Vdash p & \Leftrightarrow & w \in V(p)(c) \;\; \text{[i.e., } \langle w, c \rangle \in V(p)\text{]} \\
\mathcal{M}, w, c \Vdash \iota & \Leftrightarrow & c \in V(\iota) \\
\mathcal{M}, w, c \Vdash \triangle(\phi_1, \ldots, \phi_n) & \Leftrightarrow & w \in c(\triangle)([\![\phi_1]\!]^{\mathcal{M},c}, \ldots, [\![\phi_n]\!]^{\mathcal{M},c}) \\
\mathcal{M}, w, c \Vdash \forall p\, \phi & \Leftrightarrow & \text{for all } P \in D_{\mathbb{P}}\text{: } \mathcal{M}^p_P, w, c \Vdash \phi \\
\mathcal{M}, w, c \Vdash \exists p\, \phi & \Leftrightarrow & \text{for some } P \in D_{\mathbb{P}}\text{: } \mathcal{M}^p_P, w, c \Vdash \phi \\
\mathcal{M}, w, c \Vdash @_\iota\, \phi & \Leftrightarrow & \text{for all } c' \in V(\iota)\text{: } \mathcal{M}, w, c' \Vdash \phi \\
\mathcal{M}, w, c \Vdash \downarrow i.\phi & \Leftrightarrow & \mathcal{M}^i_{\{c\}}, w, c \Vdash \phi,
\end{array}
$$

where $\triangle \in \{\neg, \wedge, \vee, \rightarrow, \Box, \Diamond, \triangleright\}$ and $[\![\phi]\!]^{\mathcal{M},c} = \{v \in W \mid \mathcal{M}, v, c \Vdash \phi\}$. If Γ is a set of formulas, we write $\mathcal{M}, w, c \Vdash \Gamma$ to mean that $\mathcal{M}, w, c \Vdash \gamma$ for all $\gamma \in \Gamma$. When $\mathcal{M}$ is clear from context, we drop mention of it.

Note, the righthand side of the semantic clause for $\triangle$ should be read as requiring that $c(\triangle)([\![\phi_1]\!]^{\mathcal{M},c}, \ldots, [\![\phi_n]\!]^{\mathcal{M},c})$ is defined, i.e., each $[\![\phi_i]\!]^{\mathcal{M},c}$ is in the proposition space of c. If $[\![\phi_i]\!]^{\mathcal{M},c} \notin \pi_c$ for some ϕ_i, then $\mathcal{M}, w, c \nVdash \triangle(\phi_1, \ldots, \phi_n)$ regardless of w. In other words, if $c(\triangle)([\![\phi_1]\!]^{\mathcal{M},c}, \ldots, [\![\phi_n]\!]^{\mathcal{M},c})$ is undefined, then $[\![\triangle(\phi_1, \ldots, \phi_n)]\!]^{\mathcal{M},c} = \varnothing$ (but still defined).

Also, following Kocurek (2021b), we interpret iterated @-operators as redundant. Thus, $@_{\iota_1} @_{\iota_2} \phi$ is equivalent to $@_{\iota_2} \phi$. This is how such operators standardly work in hybrid logic and it simplifies the semantics and logic greatly. This equivalence could be questioned, though, and it would be worth investigating a more general semantics where it doesn't hold. Doing so is beyond the scope of this paper, however.

[7] Such constraints are necessary to validate certain plausible quantificational inferences. For example, if we dropped constraint (ii), the inference from $p \vee \neg p$ to $\exists q(p \vee q)$ would fail (even classically).

Which logics can be represented as a hyperconvention on this semantics? Kocurek (2021b) proves a result that provides an answer to this question.[8] Say a logic **L** over $\mathcal{L}^0$ is a set of pairs of the form $\langle\langle\phi_1, \ldots, \phi_n\rangle, \psi\rangle$ where $\phi_1, \ldots, \phi_n, \psi \in \mathcal{L}^0$ (we allow the first element to be the empty tuple $\langle\rangle$). Intuitively, if $\langle\langle\phi_1, \ldots, \phi_n\rangle, \psi\rangle \in \mathbf{L}$, then $\phi_1, \ldots, \phi_n$, in that order, entail ψ in **L**. Say a logic **L** is **representable** by a hyperconvention c over W if for any hyperframe $\mathcal{F} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}}\rangle$ where $c \in D_{\mathbb{H}}$, there is a hypermodel $\mathcal{M} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}}, V\rangle$ based on $\mathcal{F}$ such that:

$$[\![(\phi_1, \ldots, \phi_n) \rhd \psi]\!]^{\mathcal{M},c} = \{w \in W \mid \langle\langle\phi_1, \ldots, \phi_n\rangle, \psi\rangle \in L\}\,.$$

Proposition A2.13 (Representation). Any logic is representable by a hyperconvention over W given $|W| \geqslant \aleph_0$.

This means that so long as the state space of a hypermodel is sufficiently large, one can represent any finitary logic over that state space. This includes many of the familiar logics in the literature (intuitionistic logic, Kleene's logic, paraconsistent logics, quantum logic, etc.).[9]

A2.3 Consequence

There are two notions of consequence we can define in the hyperconvention semantics. First, there is a classical notion of consequence, i.e., truth-preservation relative to a classical interpretation of the connectives. Second, there is a "universal" notion of consequence, i.e., consequence no matter how we interpret the connectives.[10]

[8] Proposition A2.13 is a modest strengthening of Kocurek's result, which only states that $[\![(\phi_1, \ldots, \phi_n) \rhd \psi]\!]^{\mathcal{M},c} = W$ iff $\langle\langle\phi_1, \ldots, \phi_n, \psi\rangle\rangle \in L$ but doesn't say that $[\![(\phi_1, \ldots, \phi_n) \rhd \psi]\!]^{\mathcal{M},c} = \varnothing$ if $\langle\langle\phi_1, \ldots, \phi_n, \psi\rangle\rangle \notin L$. The proof of Proposition A2.13 is an easy generalization of the proof of the weaker result (see Kocurek 2022).

[9] As an anonymous referee points out, this result excludes logics where the premises have more structure than an ordered tuple (Belnap, 1982; Read, 1988; Slaney, 1990; Restall, 2000; Logan, 2022) and logics characterized in terms of "hypersequents", i.e., sequences of sequents (Avron, 1996; Restall, 2006; Poggiolesi, 2008). It is an open question whether, and to what extent, these logics could be included in the present framework by suitably generalizing the syntax.

[10] In principle, we could similarly define other nonclassical notions of consequence by introducing more designated nominals (il for intuitionistic logic, $k3$ for strong Kleene logic, etc.). In that case, we could define rigidly nonclassical connectives using these designated nominals (e.g., $\neg_{il}\,\phi := {\downarrow} k.\,@_{il}\,\neg @_k\,\phi$ where k isn't in ϕ). One complication, however, is that many nonclassical logics violate structural laws (e.g., monotonicity, commutativity,

To define these notions more precisely, we need to define the notion of a "classical" interpretation of the connectives.[11]

Definition A2.14 (Classical Hyperconvention). Given a hyperframe $\mathcal{F} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}} \rangle$, a hyperconvention $c \in \mathbb{H}_W$ is **classical for** $\mathcal{F}$ if:

(i) $[\![\phi]\!]^{\mathcal{M},c} \in \pi_c$ for every $\phi \in \mathcal{H}$ and every $\mathcal{M}$ based on $\mathcal{F}$

(ii) for all $X, Y \in \pi_c$:

$$\neg_c X = \overline{X} \qquad X \vee_c Y = X \cup Y \qquad \Box_c X = \{w \in W \mid X = W\}$$
$$X \wedge_c Y = X \cap Y \qquad X \rightarrow_c Y = \overline{X} \cup Y \qquad \Diamond_c X = \{w \in W \mid X \neq \varnothing\}$$

(iii) for all $X_1, \ldots, X_n, Y \in \pi_c$:

$$(X_1, \ldots, X_n \rhd_c Y) := \{w \in W \mid X_1 \cap \cdots \cap X_n \subseteq Y\}\,.$$

A convention is **classical for** $\mathcal{F}$ if all of its member are. A (hyper)convention is **classical for** $\mathcal{M}$ if it's classical for the hyperframe $\mathcal{M}$ is based on. Finally, $\mathcal{M}$ is **classical** if $V(cl)$ is classical for $\mathcal{M}$.

Note that classical hyperconventions interpret $\Box$ and $\Diamond$ as universal **S5** modals. I suspect that the proofs in §A3–§A4 can survive if only require $\Box$ and $\Diamond$ obey a normal modal logic, assuming we make corresponding adjustments to the axioms (see fn. 16 for one possible strategy). But I won't take up this question here, as the proofs are already complex enough even assuming $\Box$ and $\Diamond$ are universal modals.

etc.). So in defining nonclassical notions of consequence, one must take care not to define them as mere truth-preservation over hyperconventions in the denotation of the designated nominals. Moreover, there are subtle issues implementing hybrid logic in a nonclassical setting. For example, Braüner and de Paiva (2006) observe that there are multiple ways to develop intuitionistic hybrid logic. Standefer (2020) makes a similar observation about relevant logic. In fact, Standefer observes that the "naïve" way of adding an "actually" operator (a close relative of hybrid operators) to Routley-Meyer semantics for relevant logic results in violations of relevance; e.g., it would validate $q \rightarrow @(p \rightarrow p)$, which "flagrantly violates relevance intuitions" (p. 254). A similar observation would apply here: the "naïve" way of developing relevant (Routley-Meyer) hyperlogic would validate $q \rightarrow @_{cl}(p \rightarrow p)$. So revising the semantics of the hybrid operators to accommodate one's preferred nonclassical logic may require some work. (E.g., following Standefer, we might try introducing accessibility relations for $@_{cl}$ to avoid these violations of relevance.)

[11] Note, Definition A2.14 differs from Kocurek's definition in the addition of clause (i). We need clause (i) to ensure classical hyperconventions satisfy substitution instances of classical theorems: e.g., if $[\![@_l\, p]\!]^c \notin \pi_c$, then $[\![@_l\, p]\!]^c \vee_c \neg_c [\![@_l\, p]\!]^c$ is undefined and so $[\![@_l\, p \vee \neg @_l\, p]\!]^c = \varnothing$. With that said, clause (i) does not mean the proposition space is the full powerset $\wp\, W$ (i.e., π_c could be a proper boolean subalgebra of $\wp\, W$).

It is straightforward to verify the following:

Proposition A2.15 (Classical Connectives). If $\mathcal{M}$ is classical, then:

$$\begin{array}{lcl}
\mathcal{M}, w, c \Vdash \sim \phi & \Leftrightarrow & \mathcal{M}, w, c \nVdash \phi \\
\mathcal{M}, w, c \Vdash \phi \,\&\, \psi & \Leftrightarrow & \mathcal{M}, w, c \Vdash \phi \text{ and } \mathcal{M}, w, c \Vdash \psi \\
\mathcal{M}, w, c \Vdash \phi + \psi & \Leftrightarrow & \mathcal{M}, w, c \Vdash \phi \text{ or } \mathcal{M}, w, c \Vdash \psi \\
\mathcal{M}, w, c \Vdash \phi \supset \psi & \Leftrightarrow & \mathcal{M}, w, c \Vdash \phi \text{ only if } \mathcal{M}, w, c \Vdash \psi \\
\mathcal{M}, w, c \Vdash \blacksquare \phi & \Leftrightarrow & \text{for all } w' \in W\text{: } \mathcal{M}, w', c \Vdash \phi \\
\mathcal{M}, w, c \Vdash \blacklozenge \phi & \Leftrightarrow & \text{for some } w' \in W\text{: } \mathcal{M}, w', c \Vdash \phi.
\end{array}$$

Henceforth, I will only consider classical hypermodels: *when I say 'hypermodel', I always mean 'classical hypermodel'*.

Definition A2.16 (Consequence). Where $\Gamma \subseteq \mathcal{H}$ and $\phi \in \mathcal{H}$:

- Γ **classically entails** ϕ, written $\Gamma \vDash \phi$, if for any (classical) hypermodel $\mathcal{M} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}}, V \rangle$, any $w \in W$, and any $\underline{c \in V(cl)}$:

$$\mathcal{M}, w, c \Vdash \Gamma \quad \Rightarrow \quad \mathcal{M}, w, c \Vdash \phi.$$

- Γ **universally entails** ϕ, written $\Gamma \Vdash \phi$, if for any (classical) hypermodel $\mathcal{M} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}}, V \rangle$, any $w \in W$, and any $\underline{c \in D_{\mathbb{H}}}$:

$$\mathcal{M}, w, c \Vdash \Gamma \quad \Rightarrow \quad \mathcal{M}, w, c \Vdash \phi.$$

Classical/universal validity, equivalence, etc. are defined likewise.

Kocurek (2021b, Theorem 8) proves the following:

Proposition A2.17 (Embedding Consequence). Let $\Gamma \subseteq \mathcal{H}$ and $\phi \in \mathcal{H}$. Where l is an interpretation nominal, let $@_l\, \Gamma = \{@_l\, \gamma \mid \gamma \in \Gamma\}$.

(a) Assume l (distinct from cl) does not occur anywhere in Γ or in ϕ. Then $\Gamma \Vdash \phi$ iff $@_l\, \Gamma \vDash @_l\, \phi$.

(b) $\Gamma \vDash \phi$ iff $cl, \Gamma \Vdash \phi$.

Proposition A2.17 essentially gives us a method of moving back and forth between classical and universal consequence. This will be the key to developing our proof system of hyperlogic in the next section.

A3 Completeness for the Quantifier-Free Fragments

We now present some soundness and completeness results for the quantifier-free languages $\mathcal{L}^{\mathsf{H}}$ and $\mathcal{L}^{\mathsf{HE}}$. In §A3.1, we present an axiomatization for consequence in $\mathcal{L}^{\mathsf{H}}$. In §A3.2, we prove this system is sound and complete. In §A3.3, we extend the axiomatization to $\mathcal{L}^{\mathsf{HE}}$.

A3.1 Proof Systems for Classical and Universal Reasoning

The axiomatizations for classical and universal consequence in $\mathcal{L}^{\mathsf{H}}$ involves a kind of double recursion: they are not defined as separate systems with their own axioms and rules but rather interdefined with rules for moving between the two (cf. Curry 1952; Indrzejczak 1997, 1998; Humberstone 2016, §7.5).

When reasoning within hyperlogic, it is often useful to switch back and forth between classical reasoning and universal reasoning, as these notions of consequence obey different rules. Consider the rule of necessitation:

If ϕ is provable, then $\Box\,\phi$ is provable.

This rule is classically sound since classical hyperconventions interpret $\Box$ as a normal modal operator. But the rule is not universally sound: while $@_{cl}(p \vee \neg\, p)$ is universally valid, $\Box\, @_{cl}(p \vee \neg\, p)$ is not since a hyperconvention could interpret $\Box$ abnormally. By contrast, consider the corresponding rule for @:

If ϕ is provable, then $@_\iota\,\phi$ is provable.

This rule is not classically sound: while $p \vee \neg\, p$ is classically valid, $@_\iota(p \vee \neg\, p)$ is not since ι may denote a nonclassical convention. Yet the rule is universally sound: if ϕ holds on any hyperconvention, then ϕ holds at every hyperconvention in the convention denoted by ι, i.e., $@_\iota\,\phi$ holds.

For this reason, we will introduce two interdefined proof systems: $\vdash$ (for classical provability) and $\Vdash$ (for universal provability). We call the collection of these two proof systems **H**, the minimal hyperlogic in $\mathcal{L}^{\mathsf{H}}$.[12] Before giving the axioms and rules (Tables A1–A2), let me explain some of the notation used to state them.

First, because the deduction theorem is classically sound ($\phi_1, \ldots, \phi_n \vDash \psi$ iff $\vDash (\phi_1 \wedge \cdots \wedge \phi_n) \rightarrow \psi$), we can simply define $\phi_1, \ldots, \phi_n \vdash \psi$ as shorthand

[12] Technically, we should subscript $\vdash$ and $\Vdash$ to the proof system **H** to distinguish it from later proof systems. But we drop this subscript throughout for readability.

for $\vdash (\phi_1 \wedge \cdots \wedge \phi_n) \to \psi$. However, the deduction theorem is not universally sound: $\phi_1, \ldots, \phi_n \Vdash \psi$ does not imply $\Vdash (\phi_1 \wedge \cdots \wedge \phi_n) \to \psi$ since nothing of that form is universally valid. So the "axioms" for $\Vdash$ have formulas on the left.[13]

Second, we introduce the following abbreviations (where i isn't ι):

$$\begin{array}{rclcrcl} |\iota|_1 & := & @_\iota \downarrow i.\, @_\iota\, i & \quad & \iota = \kappa & := & @_\iota\, \kappa \,\&\, @_\kappa\, \iota \\ \iota \in \kappa & := & |\iota|_1 \,\&\, @_\iota\, \kappa & \quad & \phi = \psi & := & \blacksquare(\phi \equiv \psi). \end{array}$$

The truth conditions of these abbreviations reduce to the following:[14]

$$\begin{array}{lcl} w, c \Vdash |\iota|_1 & \Leftrightarrow & |V(\iota)| = 1 \\ w, c \Vdash \iota \in \kappa & \Leftrightarrow & \text{for some } c' \colon V(\iota) = \{c'\} \text{ where } c' \in V(\kappa) \\ w, c \Vdash \iota = \kappa & \Leftrightarrow & V(\iota) = V(\kappa) \\ w, c \Vdash \phi = \psi & \Leftrightarrow & [\![\phi]\!]^c = [\![\psi]\!]^c . \end{array}$$

We'll use $\neq$ to abbreviate $\sim(\cdots = \cdots)$ (e.g., $\iota \neq \kappa$ abbreviates $\sim(\iota = \kappa)$).

Some further notational conventions: We write $\dashv\vdash$ and $\dashv\Vdash$ for coprovability. Where $\phi_1, \ldots, \phi_n$ are formulas, we write $\widehat{\phi}$ for $(\phi_1 \,\&\, \cdots \,\&\, \phi_n)$. We use $\star$ as a metavariable over unary connectives ($\{\neg, \Box, \Diamond\}$), $\circ$ over binary connectives ($\{\wedge, \vee, \to\}$), and $\triangle$ over connectives of any arity. (This will generally be clear from context.) The rigidly classical counterparts of $\star$, $\circ$, and $\triangle$ are designated as $\bigstar$, $\bullet$, and $\blacktriangle$ respectively. (For example, if $\star = \neg$, then $\bigstar = \sim$; if $\circ = \wedge$, then $\bullet = \&$; etc.)

Tables A1–A2 contain the basic axioms and rules for each proof system. A proof is just a list of statements of the form $\phi_1, \ldots, \phi_n \vdash \psi$ or $\phi_1, \ldots, \phi_n \Vdash \psi$, each line of which is either an axiom or follows from previous lines via a rule. By induction on the length of proofs, both proof systems satisfy the following substitution principles.

Lemma A3.1 (Term Substitution). If $\vdash \phi$ and ι' is free for ι in ϕ, where ι is any interpretation term besides cl, then $\vdash \phi[\iota'/\iota]$. Similarly for $\Vdash$.

[13] A related deduction theorem is universally sound: $\phi_1, \ldots, \phi_n \Vdash \psi$ iff $\Vdash (\phi_1 \,\&\, \cdots \,\&\, \phi_n) \supset \psi$. For technical reasons, however, it is easier to state the axioms without appeal to this "universal deduction theorem". This universal deduction theorem is derivable in **H**, so nothing is lost in this choice.

[14] For $|\iota|_1$, note that $V(\iota)$ is nonempty by Definition A2.8. Thus, $\mathcal{M}, w, c \Vdash @_\iota \downarrow i.\, @_\iota\, i$ iff for all $c', c'' \in V(\iota)$, $\mathcal{M}^i_{\{c'\}}, w, c'' \Vdash i$ (i.e., $c'' \in \{c'\}$), which holds iff $|V(\iota)| = 1$.

$\mathbf{H}_{\vdash}$	
Axioms	
S5	$\vdash \phi$ where ϕ is a substitution instance of an **S5**-theorem
Cl	$\vdash cl$
Bool	$\vdash \sim\phi \leftrightarrow \neg\phi$
	$\vdash (\phi \mathbin{\&} \psi) \leftrightarrow (\phi \wedge \psi)$
	$\vdash \blacksquare\phi \leftrightarrow \Box\phi$
Rigid	$\vdash @_\iota \kappa \rightarrow \Box @_\iota \kappa$
Rules	
MP	if $\vdash \phi$ and $\vdash \phi \rightarrow \psi$, then $\vdash \psi$
Nec	if $\vdash \phi$, then $\vdash \Box\phi$
U2C	if $\phi_1, \ldots, \phi_n \Vdash \psi$, then $\phi_1, \ldots, \phi_n \vdash \psi$

Table A1: Axioms and rules for classical provability in $\mathcal{L}^{\mathsf{H}}$

$\mathbf{H}_{\Vdash}$	
Axioms	
$\text{Elim}_{@}$	$\iota, @_\iota \phi \Vdash \phi$
Ref	$\Vdash @_\iota \iota$
Red	$@_\kappa @_\iota \phi \dashv\Vdash @_\iota \phi$
SubId	$(\iota = \kappa), \phi \Vdash \phi'$ where ϕ' is the result of replacing any number of occurrences of ι that κ is free for in ϕ with κ
$\text{Idle}_{\downarrow}$	$\downarrow i.\, @_i \phi \dashv\Vdash \phi$ where i is not free in ϕ
$\text{Vac}_{\downarrow}$	$\downarrow i.\phi \dashv\Vdash \phi$ where i is not free in ϕ
Rep	$\phi = \psi \Vdash \star\phi = \star\psi$ where $\star \in \{\neg, \Box, \Diamond\}$
	$\phi_1 = \psi_1, \phi_2 = \psi_2 \Vdash (\phi_1 \circ \phi_2) = (\psi_1 \circ \psi_2)$ where $\circ \in \{\wedge, \vee, \rightarrow\}$
Rules	
Struct	all the standard structural rules (reflexivity, monotonicity, transitivity, commutativity, etc.) apply to $\Vdash$
C2U	if $\phi_1, \ldots, \phi_n \vdash \psi$, then $cl, \phi_1, \ldots, \phi_n \Vdash \psi$
$\text{Gen}_{@}$	if $\phi_1, \ldots, \phi_n \Vdash \psi$, then $@_\iota \phi_1, \ldots, @_\iota \phi_n \Vdash @_\iota \psi$
$\text{Gen}_{\downarrow}$	if $\lvert i \rvert_1, i, \phi_1, \ldots, \phi_n \Vdash \psi$, then $\downarrow i.\phi_1, \ldots, \downarrow i.\phi_n \Vdash \downarrow i.\psi$

Table A2: Axioms and rules for universal provability in $\mathcal{L}^{\mathsf{H}}$

Lemma A3.2 (Uniform Substitution). If $\vdash \phi$ and ψ is free for p in ϕ, then $\vdash \phi[\psi/p]$. Similarly for $\Vdash$.

Tables A3–A4 contain some useful theorems and derivable rules. Their derivations are left as exercises for the reader.[15] Throughout, I suppress mention of axioms corresponding to classical propositional reasoning (Struct, MP, Ded) and of RE, which is implicitly used frequently. I likewise suppress mention of S5 unless the application involves specifically modal reasoning.[16] Also, by the U2C rule, all of the axioms for $\Vdash$ can be imported into $\vdash$, so I use the same labels for both versions.

$\mathbf{H}_{\vdash}$ (theorems and derivable rules)	
Theorems	
Dist$_@$	$\iota \in cl \vdash (@_\iota \star \phi \leftrightarrow \star @_\iota \phi)$
	$\iota \in cl \vdash (@_\iota(\phi \circ \psi) \leftrightarrow (@_\iota \phi \circ @_\iota \psi))$
Bool	$\bigstar \phi \dashv\vdash \star \phi$
	$(\phi \bullet \psi) \dashv\vdash (\phi \circ \psi)$
	$(\phi = \psi) \dashv\vdash \Box(\phi \leftrightarrow \psi)$
Rigid	$\iota \vdash \Box \iota$
	$\neg @_\iota \kappa \vdash \Box \neg @_\iota \kappa$
	$\lvert\iota\rvert_1 \vdash \Box \lvert\iota\rvert_1$
Rep	$\lvert\iota\rvert_1, \Box(@_\iota \phi \leftrightarrow @_\iota \psi) \vdash \Box(@_\iota \star \phi \leftrightarrow @_\iota \star \psi)$
	$\lvert\iota\rvert_1, \Box(@_\iota \phi \leftrightarrow @_\iota \phi'), \Box(@_\iota \psi \leftrightarrow @_\iota \psi') \vdash \Box(@_\iota(\phi \circ \psi) \leftrightarrow @_\iota(\phi' \circ \psi'))$
Derivable Rules	
RK	if $\phi_1, \ldots, \phi_n \vdash \psi$, then $\Box \phi_1, \ldots, \Box \phi_n \vdash \Box \psi$
Gen$^{\vdash}_{\downarrow}$	if $\lvert i\rvert_1, i, \phi_1, \ldots, \phi_n \vdash \psi$, then $\downarrow i.\phi_1, \ldots, \downarrow i.\phi_n \vdash \downarrow i.\psi$

Table A3: Some useful theorems and derivable rules for $\vdash$ in $\mathbf{H}$

[15] Solutions can be found here: `https://philpapers.org/archive/KOCSTQ.pdf`.

[16] Many appeals to the S5 axiom apply to any normal modal logic. Other appeals to S5 could be dispensed with if we introduced a primitive $=$ operator into the language, rather than defining it in terms of $\blacksquare$ and $\equiv$. Indeed, we could define $\blacksquare$ in terms of $=$ as follows: $\blacksquare \phi := (\phi = (\phi + \sim \phi))$. This suggests that we could weaken Definition A2.14 so that classical conventions only need to interpret $\Box$ and $\Diamond$ as normal modalities if we extend the language with a primitve $=$ and add the appropriate axioms governing $=$ (e.g., to ensure $\blacksquare$ validates the T axiom, we would need $\phi = \psi \Vdash \phi \equiv \psi$). We would also need to revise some of the axioms, e.g., Bool. Verifying these changes would result in a sound and complete proof system is left for future research.

$\mathbf{H}_{\Vdash}$ (theorems and derivable rules)

Theorems

S5	$\Vdash \phi$ where ϕ is a substitution instance of an **S5**-theorem whose connectives are replaced with their rigidly classical counterparts
Intro$_{@}$	$\iota, \lvert\iota\rvert_1, \phi \Vdash @_\iota \phi$
DA$_{@}$	$\lvert\iota\rvert_1, @_\iota \downarrow i.\phi \Vdash @_\iota \phi[\iota/i]$ where ι is free for i in ϕ
	$\lvert\iota\rvert_1, @_\iota \phi[\iota/i] \Vdash @_\iota \downarrow i.\phi$ where ι is free for i in ϕ
VE$_{\downarrow}$	$\downarrow i.\phi \dashv\Vdash \downarrow j.\phi[j/i]$ where j is free for i in ϕ and j is not free in $\downarrow i.\phi$
Dist$_{@}$	$\lvert\iota\rvert_1 \Vdash @_\iota \bigstar \phi \equiv \bigstar @_\iota \phi$
	$\lvert\iota\rvert_1 \Vdash @_\iota(\phi \bullet \psi) \equiv (@_\iota \phi \bullet @_\iota \psi)$
Dist$^{+}_{@}$	$@_\iota \blacksquare \phi \dashv\Vdash \blacksquare @_\iota \phi$
	$@_\iota(\phi \,\&\, \psi) \dashv\Vdash (@_\iota \phi \,\&\, @_\iota \psi)$
VDist$_{@}$	$@_\kappa(@_\iota \phi \bullet \psi) \dashv\Vdash (@_\iota \phi \bullet @_\kappa \psi)$
Intro$_{\&}$	$\phi, \psi \Vdash (\phi \,\&\, \psi)$
Elim$_{\&}$	$(\phi \,\&\, \psi) \Vdash \phi$ and $(\phi \,\&\, \psi) \Vdash \psi$
Bool	$@_\iota\, cl \Vdash @_\iota \star \phi \equiv @_\iota \bigstar \phi$
	$@_\iota\, cl \Vdash @_\iota(\phi \circ \psi) \equiv @_\iota(\phi \bullet \psi)$

Derivable Rules

Ded	$\phi_1, \ldots, \phi_n, \phi \Vdash \psi$ iff $\phi_1, \ldots, \phi_n \Vdash \phi \supset \psi$
Nec	if $\Vdash \phi$, then $\Vdash \blacksquare \phi$
RE	if $\phi \dashv\Vdash \phi'$, then $\psi \dashv\Vdash \psi'$ where ψ' is the result of replacing some occurrences of ϕ with ϕ' in ψ

Table A4: Some useful theorems and derivable rules for $\Vdash$ in **H**

A3.2 Soundness and Completeness

We now set out to prove that **H** is sound and complete in $\mathcal{L}^{\mathsf{H}}$—that is, $\vdash$ is sound and complete for classical consequence in $\mathcal{L}^{\mathsf{H}}$ and $\Vdash$ is sound and complete for universal consequence in $\mathcal{L}^{\mathsf{H}}$. The proof of soundness is straightforward, though it requires two lemmas (established by induction on formulas), which we'll appeal to later.

Lemma A3.3 (Unused Variables). For any $\mathcal{F} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}} \rangle$, any $w \in W$, any $c \in D_{\mathbb{H}}$, and any $\mathcal{M}$ and $\mathcal{M}'$ based on $\mathcal{F}$, if V and V' agree on all free variables in ϕ (including propositional variables), then $\mathcal{M}, w, c \Vdash \phi$ iff $\mathcal{M}', w, c \Vdash \phi$.

Lemma A3.4 (Partial Substitution). For any $\mathcal{M} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}}, V\rangle$, if $V(\iota_1) = V(\iota_2)$ and ι_2 is free for ι_1 in ϕ, then $[\![\phi]\!]^{\mathcal{M}} = [\![\phi']\!]^{\mathcal{M}}$, where ϕ' is the result of replacing some occurrences of ι_1 with ι_2 in ϕ.

We write $\Gamma \vdash \phi$ to mean "$\gamma_1, \ldots, \gamma_n \vdash \phi$ is a theorem of **H** for some $\gamma_1, \ldots, \gamma_n \in \Gamma$". Similarly for $\Gamma \Vdash \phi$.

Theorem A3.5 (Soundness in $\mathcal{L}^{\mathsf{H}}$). For all $\Gamma \subseteq \mathcal{L}^{\mathsf{H}}$ and $\phi \in \mathcal{L}^{\mathsf{H}}$:
(a) If $\Gamma \vdash \phi$, then $\Gamma \vDash \phi$.
(b) If $\Gamma \Vdash \phi$, then $\Gamma \Vdash \phi$.

We start by proving completeness for classical consequence via a canonical model construction. We then pair this completeness result with Proposition A2.17 to bootstrap our way to completeness for universal consequence.

Throughout, let $\mathsf{Prop}^+ = \{p_1^+, p_2^+, p_3^+, \ldots\}$ be some new propositional variables, let $\mathsf{INom}^+ = \{l_1^+, l_2^+, l_3^+, \ldots\}$ be some new interpretation nominals, and let $\mathcal{L}^{\mathsf{H}+}$ be the expansion of $\mathcal{L}^{\mathsf{H}}$ with these new terms. In proofs, I will use $\lightning$ ("contradiction") to signal the end of a reductio argument. Also, by "consistent", I mean classically consistent, i.e., $\Gamma \nvdash \bot$ (where $\bot := (p \mathbin{\&} \sim p)$ (Definition A2.5)).

Definition A3.6 (Lindenbaum Set). A set $\Gamma \subseteq \mathcal{L}^{\mathsf{H}+}$ is **Lindenbaum** if it is a maximal consistent set satisfying the following constraints:
(i) Γ^+ is **nominalized**: there is an $l_\Gamma \in \mathsf{INom}^+$ such that $l_\Gamma, |l_\Gamma|_1 \in \Gamma^+$
(ii) Γ^+ **witnesses** $\neg$**@s**: if $\neg @_\iota \phi \in \Gamma^+$, then there is an $l_\iota \in \mathsf{INom}^+$ such that $(l_\iota \in \iota) \in \Gamma^+$ and $\neg @_{l_\iota} \phi \in \Gamma^+$
(iii) Γ^+ **differentiates terms**: if $(\iota \neq \kappa) \in \Gamma^+$ where $|\iota|_1, |\kappa|_1 \in \Gamma^+$, then there is a $p^+ \in \mathsf{Prop}^+$ such that $(@_\iota p^+ \neq @_\kappa p^+) \in \Gamma^+$.

Lemma A3.7 (Lindenbaum). If $\Gamma \subseteq \mathcal{L}^{\mathsf{H}}$ is consistent, then there is a Lindenbaum set $\Gamma^+ \subseteq \mathcal{L}^{\mathsf{H}+}$ where $\Gamma \subseteq \Gamma^+$.

Proof: Set $l_\Gamma = l_1^+$. Enumerate the $\mathcal{L}^{\mathsf{H}+}$-formulas as $\phi_1, \phi_2, \phi_3, \ldots$. We define a sequence of sets $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$. First, $\Gamma_1 = \Gamma \cup \{l_\Gamma, |l_\Gamma|_1\}$. Next, $\Gamma_k' = \Gamma_k \cup \{\phi_k\}$ if $\Gamma_k, \phi_k \nvdash \bot$; otherwise, $\Gamma_k' = \Gamma_k$. Lastly, let's say l^+ or p^+ is "unused" if it is the first member of INom^+ or Prop^+

that has yet to appear in the construction. Then:

$$\Gamma_{k+1} = \begin{cases} \Gamma'_k \cup \{l^+ \in \iota, \neg @_{l^+} \psi\} & \text{if } \phi_k \in \Gamma'_k \text{ and } \phi_k = \neg @_\iota \psi \text{ where } \\ & l^+ \text{ is unused} \\ \Gamma'_k \cup \{@_\iota p^+ \neq @_\kappa p^+\} & \text{if } \phi_k \in \Gamma'_k \text{ and } \phi_k = (\iota \neq \kappa) \wedge |\iota|_1 \wedge \\ & |\kappa|_1 \text{ where } p^+ \text{ is unused} \\ \Gamma'_k & \text{otherwise.} \end{cases}$$

Let $\Gamma^+ = \bigcup_{k \geqslant 1} \Gamma_k$. The proof that Γ^+ is maximal is standard. The fact that Γ^+ satisfies (i)–(iii) follows from the construction of Γ^+. We just need to show Γ^+ is consistent. It suffices to show that each Γ_k is consistent. By "i is fresh", I mean it occurs nowhere in the relevant formulas.

Base case. Suppose Γ_1 is inconsistent. Thus, by Elim$_\&$, $l_\Gamma, |l_\Gamma|_1 \vdash \neg \hat{\gamma}$ for some $\gamma_1, \ldots, \gamma_n \in \Gamma_1$. By Lemma A3.1, where i is fresh, $i, |i|_1 \vdash \neg \hat{\gamma}$. By Gen$_\downarrow$, $\vdash \downarrow i. \neg \hat{\gamma}$. By Vac$_\downarrow$, $\vdash \neg \hat{\gamma}$, $\lightning$.

Inductive step. Suppose Γ_k is consistent. By construction, Γ'_k is consistent. Suppose for reductio Γ_{k+1} is inconsistent. That means $\Gamma_{k+1} \neq \Gamma'_k$, which means $\phi_k \in \Gamma_{k+1}$ where either $\phi_k = \neg @_\iota \psi$ or $\phi_k = (\iota \neq \kappa) \wedge |\iota|_1 \wedge |\kappa|_1$. Assume throughout that the contradiction is derivable from $\gamma_1, \ldots, \gamma_n \in \Gamma_k$.

Suppose $\phi_k = \neg @_\iota \psi$. Let l^+ be the witness introduced into Γ_{k+1}. Observe that $@_\iota |l^+|_1 \dashv\vdash |l^+|_1$ by Red (recall: $|l^+|_1 := @_{l^+} \downarrow i. @_{l^+} i$). Thus, where i is fresh:

$$\begin{array}{rl} \hat{\gamma}, l_\Gamma, |l_\Gamma|_1, \neg @_\iota \psi, l^+ \in \iota \vdash @_{l^+} \psi & \\ \hat{\gamma}, l_\Gamma, |l_\Gamma|_1, \neg @_\iota \psi, i \in \iota \vdash @_i \psi & \text{Lemma A3.1} \\ \hat{\gamma}, l_\Gamma, |l_\Gamma|_1, \neg @_\iota \psi, |i|_1, @_i \iota \vdash @_i \psi & \text{C2U, Intro}_\& \\ @_{l_\Gamma} \hat{\gamma}, @_{l_\Gamma} cl, |l_\Gamma|_1, @_{l_\Gamma} \neg @_\iota \psi, |i|_1, @_i \iota \Vdash @_i \psi & \text{Gen}_@\text{, Ref, Red} \\ @_{l_\Gamma} \hat{\gamma}, @_{l_\Gamma} cl, |l_\Gamma|_1, @_{l_\Gamma} \neg @_\iota \psi, \iota \Vdash \psi & \text{Gen}_\downarrow\text{, Idle}_\downarrow\text{, Vac}_\downarrow \\ @_{l_\Gamma} \hat{\gamma}, @_{l_\Gamma} cl, |l_\Gamma|_1, @_{l_\Gamma} \neg @_\iota \psi \Vdash @_\iota \psi & \text{Gen}_@\text{, Ref, Red} \\ @_{l_\Gamma} \hat{\gamma}, @_{l_\Gamma} cl, |l_\Gamma|_1, @_{l_\Gamma} \neg @_\iota \psi \vdash @_\iota \psi & \text{U2C} \\ \hat{\gamma}, l_\Gamma, |l_\Gamma|_1 \vdash @_\iota \psi & \text{Intro}_@\text{, Cl, } \lightning. \end{array}$$

Suppose $\phi_k = (\iota \neq \kappa) \wedge |\iota|_1 \wedge |\kappa|_1$. Where p^+ is the witness introduced into Γ_{k+1}:

$$\hat{\gamma}, |\iota|_1, |\kappa|_1, @_\iota p^+ \neq @_\kappa p^+ \vdash (\iota = \kappa) \qquad \text{Bool}$$

$$
\begin{array}{rll}
\widehat{\gamma}, |\iota|_1, |\kappa|_1, @_\iota p^+ \neq @_\kappa p^+ \vdash @_\iota p^+ = @_\kappa p^+ & \text{SubId} \\
\widehat{\gamma}, |\iota|_1, |\kappa|_1 \vdash @_\iota p^+ = @_\kappa p^+ & \text{Bool} \\
\widehat{\gamma}, |\iota|_1, |\kappa|_1 \vdash @_\iota \iota = @_\kappa \iota & \text{Lemma A3.2} \\
\widehat{\gamma}, |\iota|_1, |\kappa|_1 \vdash @_\kappa \iota & \text{Ref, S5} \\
\widehat{\gamma}, |\iota|_1, |\kappa|_1 \vdash @_\iota \kappa & \text{similarly} \\
\widehat{\gamma}, |\iota|_1, |\kappa|_1 \vdash (\iota = \kappa) & \text{Intro}_{\&}, \lightning.
\end{array}
$$

■

Throughout, let Γ be a Lindenbaum set, and let $\mathsf{ITerm}^+ = \mathsf{ITerm} \cup \mathsf{INom}^+$.

Definition A3.8 (Canonical State Space). The **canonical state space** of Γ is the set W_Γ of all Lindenbaum sets Δ where for all $\phi \in \mathcal{L}^{\mathsf{H}+}$, if $\Box\phi \in \Gamma$, then $\phi \in \Delta$.

Lemma A3.9 (Existence). If $\Box\phi \notin \Delta \in W_\Gamma$, then $\phi \notin \Delta'$ for some $\Delta' \in W_\Gamma$.

Proof: Let $\Delta^{-\Box} = \{\psi \mid \Box\psi \in \Delta\}$. Observe $\Delta^{-\Box} \cup \{\neg\phi\}$ is consistent by RK. By Rigid and S5, $\Delta^{-\Box}$ is also nominalized (since $l_\Gamma \in \Gamma$) and differentiates terms (since Γ differentiates terms). Moreover, any $\Delta' \supseteq \Delta^{-\Box}$ will continue to have these properties (for differentiation of terms, note that either $(\iota = \kappa) \in \Delta^{-\Box}$ or $(\iota \neq \kappa) \in \Delta^{-\Box}$ for any ι and κ). So we just need to show $\Delta^{-\Box} \cup \{\neg\phi\}$ can be consistently extended to witness $\neg @$s. One can then extend this set into a maximal consistent one.

Enumerate all formulas of the form $\neg @_\iota \psi$ as $\chi_1, \chi_2, \chi_3, \ldots$. Define the formulas $\delta_0, \delta_1, \delta_2, \ldots$ as follows: $\delta_0 := \neg\phi$; given δ_n is defined so that $\Delta^{-\Box}, \delta_0, \ldots, \delta_n \nvdash \bot$, let $\delta_{n+1} := \chi_{n+1} \to (l^+ \in \iota \wedge \neg @_{l^+} \psi)$ where $\chi_{n+1} = \neg @_\iota \psi$ and l^+ is the first from INom^+ such that $\Delta^{-\Box}, \delta_0, \ldots, \delta_n, \chi_{n+1} \to (l^+ \in \iota \wedge \neg @_{l^+} \psi) \nvdash \bot$. Given there always is such a l^+, $\Delta^* = \Delta^{-\Box} \cup \{\delta_0, \delta_1, \delta_2, \ldots\}$ will consistently witness $\neg @$s.

Suppose for reductio that $\delta_1, \ldots, \delta_n$ are defined but there is no l^+ meeting the above condition. Thus, for all l^+, there are some $\gamma_1, \ldots, \gamma_m \in \Delta^{-\Box}$ such that $\widehat{\gamma} \vdash \widehat{\delta} \to \neg(\chi_{n+1} \to (l^+ \in \iota \wedge \neg @_{l^+} \psi))$. By RK, $\Box\widehat{\gamma} \vdash \Box(\widehat{\delta} \to \neg(\chi_{n+1} \to (l^+ \in \iota \wedge \neg @_{l^+} \psi)))$. Since $\Box\widehat{\gamma} \in \Delta$, that means $\Box(\widehat{\delta} \to \chi_{n+1}), \Box(\widehat{\delta} \to \neg(l^+ \in \iota \wedge \neg @_{l^+} \psi)) \in \Delta$. So if $\Box(\widehat{\delta} \to \neg \chi_{n+1}) \in \Delta$, then $\Box \neg \widehat{\delta} \in \Delta$, and so $\Delta^{-\Box}, \delta_0, \ldots, \delta_n \vdash \bot$, $\lightning$. Hence, $\Diamond(\widehat{\delta} \wedge \chi_{n+1}) \in \Delta$, i.e., $\Diamond(\widehat{\delta} \wedge \neg @_\iota \psi) \in \Delta$. Now, where $l_\Delta, |l_\Delta|_1 \in \Delta$, the following are $\vdash$-provable from $l_\Delta, |l_\Delta|_1$:

$$\begin{aligned} \Diamond(\widehat{\delta} \wedge \neg @_\iota \psi) &\leftrightarrow \Diamond \neg(@_{l_\Delta} \neg \widehat{\delta} \vee @_\iota \psi) && \text{Rigid, Intro}_@\text{, S5} \\ &\leftrightarrow \blacklozenge \sim(@_{l_\Delta} \neg \widehat{\delta} + @_\iota \psi) && \text{Bool} \\ &\leftrightarrow \blacklozenge \sim @_\iota(@_{l_\Delta} \neg \widehat{\delta} + \psi) && \text{VDist}_@ \\ &\leftrightarrow \neg @_\iota \blacksquare(@_{l_\Delta} \neg \widehat{\delta} + \psi) && \text{S5, Dist}^+_@\text{, Bool.} \end{aligned}$$

Since Δ witnesses $\neg @$s, there is an l^+ such that $l^+ \in \iota, \neg @_{l^+} \blacksquare(@_{l_\Delta} \neg \widehat{\delta} + \psi) \in \Delta$. Reversing the provable equivalence above, we get $\Diamond(\widehat{\delta} \wedge (l^+ \in \iota \wedge \neg @_{l^+} \psi)) \in \Delta$. But $\Box(\widehat{\delta} \to \neg(l^+ \in \iota \wedge \neg @_{l^+} \psi)) \in \Delta$, so Δ is inconsistent, $\lightning$. ■

Corollary A3.10 (Plenitude). For all $\Delta \in W_\Gamma$ and all ϕ:
(a) $\Box \phi \in \Delta$ iff $\phi \in \Delta'$ for all $\Delta' \in W_\Gamma$.
(b) $\Diamond \phi \in \Delta$ iff $\phi \in \Delta'$ for some $\Delta' \in W_\Gamma$.

Definition A3.11 (Definable Sets). Where $X \subseteq W_\Gamma$ and $\phi \in \mathcal{L}^{\mathsf{H}+}$, we define $[X]_\iota := \{\phi \in \mathcal{L}^{\mathsf{H}+} \mid X = \{\Delta \in W_\Gamma \mid @_\iota \phi \in \Delta\}\}$.

Lemma A3.12 (Replacement of Equivalent Definitions). Where $|\iota|_1 \in \Gamma$:
(a) If $@_\iota \star \phi \in \Delta$ for some $\phi \in [X]_\iota$, then $@_\iota \star \phi \in \Delta$ for all $\phi \in [X]_\iota$.
(b) If $@_\iota(\phi \circ \psi) \in \Delta$ for some $\phi \in [X]_\iota$ and $\psi \in [Y]_\iota$, then $@_\iota(\phi \circ \psi) \in \Delta$ for all $\phi \in [X]_\iota$ and $\psi \in [Y]_\iota$.

Proof: We just prove (a) for illustration. Suppose $\phi \in [X]_\iota$ is such that $@_\iota \star \phi \in \Delta$. Let $\psi \in [X]_\iota$. Thus, for all $\Delta' \in W_\Gamma$, $@_\iota \phi \in \Delta'$ iff $\Delta' \in X$ iff $@_\iota \psi \in \Delta'$. By Corollary A3.10, $\Box(@_\iota \phi \leftrightarrow @_\iota \psi) \in \Gamma$. By Rep (since $|\iota|_1 \in \Gamma$), $\Box(@_\iota \star \phi \leftrightarrow @_\iota \star \psi) \in \Gamma$. By Definition A3.8, $@_\iota \star \phi \leftrightarrow @_\iota \star \psi \in \Delta$. Hence, $@_\iota \star \psi \in \Delta$. ■

Definition A3.13 (Canonical Hyperconventions). Where $|\kappa|_1 \in \Gamma$, the **canonical κ-hyperconvention** c_κ over W_Γ is defined as follows:

(i) $\pi_{c_\kappa} = \wp\, W_\Gamma$

(ii) $c_\kappa(p) = \{\Delta \in W_\Gamma \mid @_\kappa\, p \in \Delta\}$

(iii) If $@_\kappa\, cl \in \Gamma$, then each $c_\kappa(\triangle)$ is defined as in Definition A2.14. Otherwise, $c_\kappa(\triangle)(X_1, \ldots, X_n)$ is the following set:

$$\{\Delta \in W_\Gamma \mid \exists \phi_1 \in [X_1]_\kappa \cdots \exists \phi_n \in [X_n]_\kappa \colon\ @_\kappa \triangle(\phi_1, \ldots, \phi_n) \in \Delta\}\,.$$

For any $\iota \in \mathsf{ITerm}^+$, define the **canonical ι-convention** as $C_\iota := \{c_\kappa \mid (\kappa \in \iota) \in \Gamma\}$.

Note that C_ι is well-defined by the following lemma:

Lemma A3.14 (Identity for Canonical Hyperconventions). Where $|\kappa|_1, |\lambda|_1 \in \Gamma$, $c_\kappa = c_\lambda$ iff $(\kappa = \lambda) \in \Gamma$. Thus, if $c_\kappa = c_\lambda$, then $(\kappa \in \iota) \in \Gamma$ iff $(\lambda \in \iota) \in \Gamma$.

> *Proof*: The left-to-right direction follows since Γ differentiates terms. The right-to-left direction follows from SubId and Corollary A3.10.■

Finally, C_ι is always nonempty: if $|\iota|_1 \in \Gamma$, then $(\iota \in \iota) \in \Gamma$; and if $\neg\, |\iota|_1 \in \Gamma$, i.e., $\neg\, @_\iota \downarrow i.\, @_\iota\, i \in \Gamma$, then since Γ witnesses $\neg\, @$s, $(l^+ \in \iota) \in \Gamma$ for some l^+.

Definition A3.15 (Canonical Hypermodel). We define the **canonical hypermodel** of Γ as $\mathcal{M}_\Gamma = \langle W_\Gamma, D_{\mathbb{C}\Gamma}, D_{\mathbb{P}\Gamma}, V_\Gamma\rangle$ where:

- $D_{\mathbb{C}\Gamma} = \{C_\iota \mid \iota \in \mathsf{ITerm}^+\}$ (and so, $D_{\mathbb{H}\Gamma} = \{c_\kappa \mid |\kappa|_1 \in \Gamma\}$)
- $D_{\mathbb{P}\Gamma} = \mathbb{P}_{D_{\mathbb{H}\Gamma}}$
- $V_\Gamma(p) = \{\langle \Delta, c_\kappa\rangle \mid \Delta \in c_\kappa(p)\}$ for each $p \in \mathsf{Prop}$
- $V_\Gamma(\iota) = C_\iota$ for each $\iota \in \mathsf{ITerm}^+$.

Lemma A3.16 (Canonical Classical Convention). C_{cl} is classical.

Lemma A3.17 (Truth). $\mathcal{M}_\Gamma, \Delta, c_\kappa \Vdash \phi$ iff $@_\kappa\, \phi \in \Delta$.

Proof: By induction on the structure of formulas. The base cases follow from Definitions A3.13 and A3.15 and Corollary A3.10.
IH. Suppose the claim holds for ϕ and ψ. Observe that $\phi \in [\llbracket\phi\rrbracket^{c_\kappa}]_\kappa$.
Connectives. We just do the $\neg$-case. Suppose first that $@_\kappa\, cl \notin \Gamma$. Then:

$$\begin{array}{llll} \Delta, c_\kappa \Vdash \neg\phi & \Leftrightarrow & \Delta \in c_\kappa(\neg)(\llbracket\phi\rrbracket^{c_\kappa}) & \\ & \Leftrightarrow & \exists\psi \in [\llbracket\phi\rrbracket^{c_\kappa}]_\kappa :\ @_\kappa \neg\psi \in \Delta & \text{Definition A3.13} \\ & \Leftrightarrow & @_\kappa \neg\phi \in \Delta & \text{Lemma A3.12.} \end{array}$$

Suppose now $@_\kappa\, cl \in \Gamma$. Then:

$$\begin{array}{llll} \Delta, c_\kappa \Vdash \neg\phi & \Leftrightarrow & \Delta, c_\kappa \nVdash \phi & \text{Lemma A3.16} \\ & \Leftrightarrow & @_\kappa\, \phi \notin \Delta & \text{IH} \\ & \Leftrightarrow & \neg @_\kappa\, \phi \in \Delta & \text{maximality} \\ & \Leftrightarrow & @_\kappa \neg\phi \in \Delta & \text{Dist}_@ \text{ (since } |\kappa|_1, @_\kappa\, cl \in \Delta). \end{array}$$

$@_\iota$ case. Since $V_\Gamma(\iota) = \{c_\lambda \mid (\lambda \in \iota) \in \Gamma\}$ by Definition A3.15:

$$\begin{array}{llll} \Delta, c_\kappa \Vdash @_\iota\, \phi & \Leftrightarrow & \text{for all } \lambda\text{: } (\lambda \in \iota) \in \Gamma \Rightarrow \Delta, c_\lambda \Vdash \phi & \\ & \Leftrightarrow & \text{for all } \lambda\text{: } (\lambda \in \iota) \in \Gamma \Rightarrow @_\lambda\, \phi \in \Delta & \text{IH} \\ & \Leftrightarrow & @_\iota\, \phi \in \Delta & (*) \\ & \Leftrightarrow & @_\kappa @_\iota\, \phi \in \Delta & \text{Red.} \end{array}$$

For (*): Suppose first that $@_\iota\, \phi \notin \Delta$. Since Δ witnesses $\neg @_\iota$, $(l_\iota \in \iota), \neg @_{l_\iota} \phi \in \Delta$ for some l_ι (so $\lambda = l_\iota$ is our counterexample). Conversely, suppose $@_\iota\, \phi \in \Delta$. If $(\lambda \in \iota) \in \Gamma$, then $@_\lambda\, \iota \in \Delta$ by Rigid, and so by Elim$_@$, Gen$_@$, and Red, $@_\lambda\, \phi \in \Delta$.
$\downarrow i$ case. By VE$_\downarrow$, we may assume WLOG that κ is free for i in ϕ.

$$\begin{array}{llll} \mathcal{M}_\Gamma, \Delta, c_\kappa \Vdash \downarrow i.\phi & \Leftrightarrow & (\mathcal{M}_\Gamma)^i_{\{c_\kappa\}}, \Delta, c_\kappa \Vdash \phi & \\ & \Leftrightarrow & \mathcal{M}_\Gamma, \Delta, c_\kappa \Vdash \phi[\kappa/i] & \text{Lemmas A3.3 and A3.4} \\ & \Leftrightarrow & @_\kappa\, \phi[\kappa/i] \in \Delta & \text{IH} \\ & \Leftrightarrow & @_\kappa \downarrow i.\phi \in \Delta & \text{DA}_@. \end{array}$$

■

$\mathbf{H}_{\rhd}$	
All the axioms and rules in **H**, plus:	
Def$_{\rhd}$	$\vdash (\phi_1, \ldots, \phi_n \rhd \psi) \leftrightarrow \Box((\phi_1 \wedge \cdots \wedge \phi_n) \rightarrow \psi)$
Rep$_{\rhd}$	$(\phi_1 = \phi_1'), \ldots, (\psi = \psi') \Vdash (\phi_1, \ldots, \phi_n \rhd \psi) = (\phi_1', \ldots, \phi_n' \rhd \psi')$

Table A5: Axioms and rules for provability in $\mathcal{L}^{\mathsf{HE}}$

Theorem A3.18 (Completeness in $\mathcal{L}^{\mathsf{H}}$). Where $\Gamma \subseteq \mathcal{L}^{\mathsf{H}}$ and $\phi \in \mathcal{L}^{\mathsf{H}}$:
(a) If $\Gamma \vDash \phi$, then $\Gamma \vdash \phi$.
(b) If $\Gamma \Vdash \phi$, then $\Gamma \Vdash \phi$.

Proof: For (a), suppose $\Gamma \nvdash \bot$. By Lemma A3.7, we can extend Γ to a Lindenbaum set Γ^+. By Lemma A3.17, $\Gamma^+, c_{l_\Gamma} \Vdash \phi$ iff $@_{l_\Gamma} \phi \in \Gamma^+$, which holds (by Intro$_@$ and Elim$_@$) iff $\phi \in \Gamma^+$. Hence, $\Gamma^+, c_{l_\Gamma} \Vdash \Gamma$. By Definition A3.15, $c_{l_\Gamma} \in V_\Gamma(cl)$ since $@_{l_\Gamma} cl \in \Gamma$ by Intro$_@$ and Cl. Hence, Γ is classically satisfiable.

For (b), suppose $\Gamma \Vdash \phi$. Introduce a new interpretation nominal l to the language. By Proposition A2.17, $@_l \Gamma \vDash @_l \phi$. By (a), $@_l \Gamma \vdash @_l \phi$. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ be such that $@_l \gamma_1, \ldots, @_l \gamma_n \vdash @_l \phi$. By Lemma A3.1, where i is fresh, $@_i \gamma_1, \ldots, @_i \gamma_n \vdash @_i \phi$. By C2U, $cl, @_i \gamma_1, \ldots, @_i \gamma_n \Vdash @_i \phi$. By Gen$_@$ (with $@_{cl}$), Ref, and Red, $@_i \gamma_1, \ldots, @_i \gamma_n \Vdash @_i \phi$. Hence, by Gen$_\downarrow$ and Idle$_\downarrow$, $\gamma_1, \ldots, \gamma_n \Vdash \phi$. Thus, $\Gamma \Vdash \phi$. ∎

A3.3 Axioms for $\rhd$

It is straightforward to extend **H** into $\mathcal{L}^{\mathsf{HE}}$. The two additional axioms needed are stated in Table A5. The resulting system is called $\mathbf{H}_{\rhd}$. To prove the completeness of $\mathbf{H}_{\rhd}$, we simply amend Definition A3.13 so that $c_\kappa(\rhd)(X_1, \ldots, X_n, Y)$ is defined as:

$$\{\Delta \in W_\Gamma \mid \exists \phi_1 \in [X_1]_\kappa \cdots \exists \phi_n \in [X_n]_\kappa\, \exists \psi \in [Y]_\kappa \colon\ @_\kappa(\phi_1, \ldots, \phi_n \rhd \psi) \in \Delta\}\,.$$

The proof goes through as in §A3.2, adding the relevant inductive steps for $\rhd$.

A4 Completeness with Propositional Quantifiers

We now extend these results to languages with propositional quantifiers. In §A4.1, we state the additional axioms and rules governing quantifiers. In §A4.2, we revise the proof of completeness from §A3.2. In §A4.3, we consider how these results are affected when $\triangleright$ is introduced.

A4.1 Axiomatizing Quantifiers

The new axioms and rules for the quantifiers are stated in Table A6. We call the resulting system **QH**. This axiomatization makes use of the following abbreviations (where p is not free in ϕ):

$$\begin{aligned} \mathsf{E}\phi \quad &:= \quad \exists p(p = \phi) \\ (\triangle_\kappa = \triangle_\lambda) \quad &:= \quad \forall p_1 \forall q_1 \cdots \forall p_n \forall q_n [(@_\kappa p_1 = @_\lambda q_1 \,\&\, \cdots \,\&\, @_\kappa p_n = @_\lambda q_n) \supset \\ &\qquad (@_\kappa \triangle(p_1, \dots, p_n) = @_\lambda \triangle(q_1, \dots, q_n))]. \end{aligned}$$

If $V(\kappa) = \{c_1\}$ and $V(\lambda) = \{c_2\}$, the truth conditions reduce to the following:

$$\begin{aligned} w, c \Vdash \mathsf{E}\phi \quad &\Leftrightarrow \quad [\![\phi]\!]^c \in \pi_c \\ w, c \Vdash \triangle_\kappa = \triangle_\lambda \quad &\Leftrightarrow \quad c_1(\triangle) = c_2(\triangle). \end{aligned}$$

Intuitively, $\mathsf{E}\phi$ says that ϕ denotes a world proposition that "exists" according to the current hyperconvention. This formula is not trivially satisfied: if, say, $W \notin \pi_c$, then $\mathcal{M}, w, c \nVdash \mathsf{E}(p \supset p)$ since for no $Q \in D_{\mathbb{P}}$ does $Q(c) = W$.

Note that $\text{Elim}_\forall$ does not allow substituting *any* ψ for p (even if p is free for ψ), since ψ need not denote an existent proposition according to a hyperconvention. For example, if $\varnothing \notin \pi_c$, then $\mathcal{M}, w, c \Vdash \forall p \,\blacklozenge\, p$ (since $P(c) \in \pi_c$ for any $P \in \mathbb{P}_{D_{\mathbb{H}}}$) even though $\mathcal{M}, w, c \nVdash \blacklozenge(q \,\&\, {\sim} q)$ (since $[\![(q \,\&\, {\sim} q)]\!]^{\mathcal{M},c} = \varnothing$ regardless of c). However, since $V(p)$ is always a visible proposition ($V(p)(c) \in \pi_c$), instantiation with other propositional variables is allowed. Note also the PII axiom, which is effectively the principle of the identity of indiscernibles for hyperconventions: if c_1 and c_2 have the same proposition space, and interpret the propositional variables and connectives the same way, then $c_1 = c_2$. The soundness of PII is ensured by the two minimal constraints on proposition domains in Definition A2.11.[17]

[17] Note that, where $V(\iota) = \{c_1\}$ and $V(\kappa) = \{c_2\}$, the premise $\forall p(@_\iota p = @_\kappa p)$ of PII simultaneously guarantees that $\pi_{c_1} = \pi_{c_2}$ and that $c_1(p) = c_2(p)$ for all p. If, for instance, $X \in \pi_{c_1} - \pi_{c_2}$, then by constraint (ii) in Definition A2.11, there is a $P \in D_{\mathbb{P}}$ such that $P(c_1) = X$. And $P(c_2) \neq X$ since $P(c_2) \in \pi_{c_2}$. If instead $c_1(p) \neq c_2(p)$, then $P_p(c_1) \neq P_p(c_2)$, where $P_p \in D_{\mathbb{P}}$ by constraint (i). Either way, there is a $P \in D_{\mathbb{P}}$ such that $\mathcal{M}^p_P, w, c \nVdash @_\iota p = @_\kappa p$.

QH

All the axioms and rules in **H**, plus:

$K_\forall$	$\vdash \forall p(\phi \to \psi) \to (\forall p\, \phi \to \forall p\, \psi)$
$Dual_\forall$	$\sim \forall p\, \phi \dashv\Vdash \exists p \sim \phi$
$Elim_\forall$	$\forall p\, \phi \Vdash \phi[q/p]$ where q is free for p in ϕ
$Vac_\forall$	$\phi \Vdash \forall p\, \phi$ where p does not occur free in ϕ
ClEx	$\vdash \mathsf{E}\phi$
OpEx	$\mathsf{E} \triangle(\phi_1, \ldots, \phi_n) \dashv\Vdash (\mathsf{E}\phi_1 \,\&\, \cdots \,\&\, \mathsf{E}\phi_n)$
TrEx	$\triangle(\phi_1, \ldots, \phi_n) \Vdash (\mathsf{E}\phi_1 \,\&\, \cdots \,\&\, \mathsf{E}\phi_n)$
PII	$\lvert\iota\rvert_1\,, \lvert\kappa\rvert_1\,, \forall p(@_\iota\, p = @_\kappa\, p), \{\triangle_\iota = \triangle_\kappa\}_\triangle \Vdash (\iota = \kappa)$
$BF_\blacksquare$	$\forall p \blacksquare \phi \Vdash \blacksquare \forall p\, \phi$
$BF_@$	$\forall p\, @_\iota\, \phi \Vdash @_\iota \forall p\, \phi$
$BF_\downarrow$	$\forall p \downarrow i.\phi \Vdash \downarrow i. \forall p\, \phi$
$Gen_\forall$	if $\Vdash \phi$, then $\Vdash \forall p\, \phi$

Table A6: Axioms and rules for provability in $\mathcal{L}^{\mathsf{QH}}$

The proofs of Lemmas A3.3 and A3.4 remain unchanged. In addition, we have the following (which is needed to prove the soundness of $Elim_\forall$):

Lemma A4.1 (Propositional Relabeling). If q is free for p in ϕ, then $[\![\phi[q/p]]\!]^{\mathcal{M}} = [\![\phi]\!]^{\mathcal{M}^p_{V(q)}}$. Similarly for simultaneous substitution.

Theorem A4.2 (Soundness in $\mathcal{L}^{\mathsf{QH}}$). Where $\Gamma \subseteq \mathcal{L}^{\mathsf{QH}}$ and $\phi \in \mathcal{L}^{\mathsf{QH}}$:
(a) If $\Gamma \vdash \phi$, then $\Gamma \vDash \phi$.
(b) If $\Gamma \Vdash \phi$, then $\Gamma \Vdash\!\!\!= \phi$.

Table A7 contains some useful derivable theorems and rules. The proof of Lemma A3.1 still goes through. However, Lemma A3.2 no longer holds in **QH**: e.g., $\vdash \forall p(p \to @_i\, p) \to (q \to @_i\, q)$ yet $\nvdash \forall p(p \to @_i\, p) \to (cl \to @_i\, cl)$.[18] Instead, only a restricted form of Lemma A3.2 holds (fortunately, this suffices):

[18] To see why the latter is unsound, suppose $V(i) = \{c\}$ where c is not classical and $\pi_c = \{\{w\}\}$. Then where $c^* \in V(cl)$, $w, c^* \Vdash \forall p(p \to @_i\, p)$ and $w, c^* \Vdash cl$; yet $w, c^* \nVdash @_i\, cl$.

Lemma A4.3 (Propositional Substitution). If $\vdash \phi$ and q is free for p in ϕ, then $\vdash \phi[q/p]$.

QH (theorems and derivable rules)

Theorems	
$\text{Dual}_\forall$	$\neg \forall p\, \phi \dashv\vdash \exists p \neg \phi$
$\text{VDist}_\exists$	$\exists p(\phi \wedge \psi) \dashv\vdash (\phi \wedge \exists p\, \psi)$ where p does not occur free in ϕ
$\text{VE}_\forall$	$\forall p\, \phi \vdash \forall q\, \phi[q/p]$ where q is free for p in ϕ and q does not occur free in $\forall p\, \phi$
$\text{K}_\forall$	$\forall p(\phi \supset \psi), \forall p\, \phi \Vdash \forall p\, \psi$
$\text{Intro}_\exists$	$\phi[q/p] \Vdash \exists p\, \phi$ where q is free for p in ϕ
NecEx	$\mathsf{E}\phi \Vdash \blacksquare\, \mathsf{E}\phi$
	$\sim \mathsf{E}\phi \Vdash \blacksquare \sim \mathsf{E}\phi$
$\text{BF}_\blacklozenge$	$\blacklozenge\, \exists p\, \phi \Vdash \exists p\, \blacklozenge\, \phi$
$\text{BF}^+_@$	$\lvert\iota\rvert_1, @_\iota\, \exists p\, \phi \Vdash \exists p\, @_\iota\, \phi$
$\text{CBF}_@$	$@_\iota\, \forall p\, \phi \Vdash \forall p\, @_\iota\, \phi$
Derivable Rules	
$\text{RK}_\forall$	if $\phi_1, \ldots, \phi_n \vdash \psi$, then $\forall p\, \phi_1, \ldots, \forall p\, \phi_n \vdash \forall p\, \psi$

Table A7: Some useful theorems and derivable rules in **QH**

A4.2 Completeness

Now for completeness. The lemmas from §A3.2 whose proof need revision are Lemmas A3.7, A3.9, A3.14, A3.16 and A3.17. Throughout, let Prop^+ and INom^+ be as before, and let $\mathcal{L}^{\mathsf{QH}+}$ be the expansion of $\mathcal{L}^{\mathsf{QH}}$ with these new terms but *without* propositional quantifiers binding elements of Prop^+ (so members of Prop^+ are treated as propositional "constants").

Definition A4.4 (Henkin Set). A set $\Gamma \subseteq \mathcal{L}^{\mathsf{QH}+}$ is **Henkin** if it is a maximal consistent set that is nominalized, witnesses $\neg\,@$s, and:
(iii) Γ^+ **witnesses** $\exists$**s**: if $\exists p\, \phi \in \Gamma^+$, then there is a $p \in \mathsf{Prop}^+$ not in ϕ such that $\phi[p^+/p] \in \Gamma^+$.

Note that Henkin sets do necessarily not differentiate terms in the sense of Definition A3.6. We don't want to assume that if $(\iota \neq \kappa) \in \Gamma^+$, then

$(@_\iota\, p^+ \neq @_\kappa\, p^+) \in \Gamma^+$ for some p^+ since Γ^+ might contain $\forall p(@_\iota\, p = @_\kappa\, p)$. In that case, Γ^+ would have to contain $(\triangle_\iota \neq \triangle_\kappa)$ for some $\triangle$.

Lemma A4.5 (Henkin). If $\Gamma \subseteq \mathcal{L}^{\mathsf{QH}}$ is consistent, then there is a Henkin set $\Gamma^+ \subseteq \mathcal{L}^{\mathsf{QH}+}$ where $\Gamma \subseteq \Gamma^+$.

Proof: Proof is as before except we revise the definition of Γ_{k+1}:

$$\Gamma_{k+1} = \begin{cases} \Gamma'_k & \text{if } \phi_k \notin \Gamma'_k\text{; otherwise:} \\ \Gamma'_k \cup \{l^+ \in \iota, \neg @_{l^+}\, \psi\} & \text{if } \phi_k = \neg @_\iota\, \psi \text{ and } l^+ \text{ is unused} \\ \Gamma'_k \cup \{\psi[p^+/p]\} & \text{if } \phi_k = \exists p\, \psi \text{ and } p^+ \text{ is unused} \\ \Gamma'_k & \text{otherwise} \end{cases}$$

We need to show that if Γ'_k is consistent and $\phi_k = \exists p\, \psi$, then Γ_{k+1} is consistent. Suppose otherwise. That means for some $\gamma_1, \ldots, \gamma_n \in \Gamma_k$, where $q \in \mathsf{Prop}$ is fresh:

$$\begin{array}{rll} \widehat{\gamma}, \exists p\, \psi \vdash \neg\, \psi[p^+/p] & \\ \widehat{\gamma}, \exists p\, \psi \vdash \neg\, \psi[q/p] & \text{Lemma A4.3} \\ \forall q\, \widehat{\gamma}, \forall q\, \exists p\, \psi \vdash \forall q \neg\, \psi[q/p] & \text{RK}_\forall \\ \widehat{\gamma}, \exists p\, \psi \vdash \forall q \neg\, \psi[q/p] & \text{Vac}_\forall \\ \widehat{\gamma}, \exists p\, \psi \vdash \forall p \neg\, \psi & \text{VE}_\forall \\ \widehat{\gamma} \vdash \neg \exists p\, \psi & \text{Dual}_\forall, \lightning. \end{array}$$

■

Lemma A4.6 (Existence (Revised)). Suppose $\Box\, \phi \notin \Delta \in W_\Gamma$. Then there is a $\Delta' \in W_\Gamma$ such that $\phi \notin \Delta'$.

Proof: As before, $\Delta^{-\Box} \cup \{\neg\, \phi\}$ is guaranteed to be consistent and nominalized. Enumerate all sentences of the form $\neg @_\iota\, \psi$ or of the form $\exists q\, \psi$ as $\chi_1, \chi_2, \chi_3, \ldots$. We define a sequence of formulas $\delta_0, \delta_1, \delta_2, \ldots$ as before except the definition depends on the form of χ_{n+1}. If $\chi_{n+1} = \neg @_\iota\, \psi$, define δ_{n+1} as in Lemma A3.9. If $\chi_{n+1} = \exists q\, \psi$, then define $\delta_{n+1} := \chi_{n+1} \to \psi[p^+/q]$, where p^+ is the first propositional variable such that $\Delta^{-\Box}, \delta_0, \ldots, \delta_n, \chi_{n+1} \to \psi[p^+/q] \nvdash \bot$. As before, it suffices to show that there always is such a p^+. Suppose $\delta_0, \ldots, \delta_n$ are defined but there is no p^+ meeting the above condition. Reasoning as before, this means that $\Box(\widehat{\delta} \to \chi_{n+1}) \in \Delta$ and $\Box(\widehat{\delta} \to \neg\,\psi[p^+/q]) \in \Delta$ for all p^+. Once again, it must be that $\Diamond(\widehat{\delta} \wedge \chi_{n+1}) \in \Delta$. Let p be fresh. By $\mathrm{VE}_\forall$, $\Diamond(\widehat{\delta} \wedge \exists p\, \psi[p/q]) \in \Delta$. By $\mathrm{VDist}_\exists$, $\Diamond\, \exists p(\widehat{\delta} \wedge \psi[p/q]) \in \Delta$. By $\mathrm{BF}_\blacklozenge$ and Bool, $\exists p\, \Diamond(\widehat{\delta} \wedge \psi[p/q]) \in \Delta$. Since Δ witnesses $\exists$s, there is a p^+ such that $\Diamond(\widehat{\delta} \wedge \psi[p^+/q]) \in \Delta$, contradicting the fact that $\Box(\widehat{\delta} \to \neg\,\psi[p^+/q]) \in \Delta$, $\lightning$. ■

The proofs of the intermediate lemmas through Lemma A3.12 remain intact. To continue, we revise the definition of a canonical hyperconvention (Definition A3.13).

Definition A4.7 (Canonical Hyperconventions (Revised)). Let $|\kappa|_1 \in \Gamma$. Define the **canonical κ-hyperconvention** c_κ over W_Γ as follows:

(i) $\pi_{c_\kappa} = \{X \mid \exists p^+ \in \mathsf{Prop}^+ \colon\ p^+ \in [X]_\kappa\}$

(ii) $c_\kappa(p) = \{\Delta \in W_\Gamma \mid @_\kappa\, p \in \Delta\}$

(iii) $c_\kappa(\triangle)(X_1, \ldots, X_n)$ is defined as follows (regardless of whether $@_\kappa\, cl \in \Gamma$):

$$\{\Delta \in W_\Gamma \mid \exists \phi_1 \in [X_1]_\kappa \cdots \exists \phi_n \in [X_n]_\kappa \colon\ @_\kappa \triangle(\phi_1, \ldots, \phi_n) \in \Delta\}\,.$$

The definition of C_ι is as before.

Since π_{c_κ} is no longer the full powerset, we must ensure that this does define a hyperconvention in that the outputs of c_κ are always within π_{c_κ}.

Lemma A4.8 (Canonical Hyperconventions are Hyperconventions). Let $|\kappa|_1 \in \Gamma$.

(a) If $@_\kappa\, cl \in \Gamma$ and $[X]_\kappa \neq \varnothing$, then $X \in \pi_{c_\kappa}$.

(b) $c_\kappa(p) \in \pi_{c_\kappa}$.

(c) $c_\kappa(\triangle)(X_1, \ldots, X_n) \in \pi_{c_\kappa}$ for any $X_1, \ldots, X_n \in \pi_{c_\kappa}$.

Proof: For (a), let $\phi \in [X]_\kappa$. By ClEx and C2U, $cl, \kappa \Vdash \mathsf{E}\phi$. By $\mathrm{Gen}_@$ and Ref, $@_\kappa\, cl \Vdash @_\kappa\, \mathsf{E}\phi$. By U2C, $@_\kappa\, cl \vdash @_\kappa\, \mathsf{E}\phi$. Thus, $@_\kappa\, \mathsf{E}\phi \in \Gamma$. By $\mathrm{BF}^+_@$, $\exists p(p =_\kappa \phi) \in \Gamma$. By $\exists$-witnessing, there is a $p^+ \in \mathsf{Prop}^+$ such that $(p^+ =_\kappa \phi) \in \Gamma$. By $\mathrm{Dist}_@$, $(@_\kappa\, p^+ = @_\kappa\, \phi) \in \Gamma$. By Corollary A3.10, $p^+ \in [X]_\kappa$. Hence, $X \in \pi_{c_\kappa}$.

For (b), $\exists q\, \Box(@_\kappa\, q \leftrightarrow @_\kappa\, p) \in \Gamma$ by Intro$_\exists$. Since Γ witnesses $\exists$s, there is a $p^+ \in \mathsf{Prop}^+$ such that $\Box(@_\kappa\, p^+ \leftrightarrow @_\kappa\, p) \in \Gamma$. By Corollary A3.10, $c_\kappa(p) = \{\Delta \in W_\Gamma \mid @_\kappa\, p^+ \in \Delta\}$. Hence, $p^+ \in [c_\kappa(p)]_\kappa$, i.e., $c_\kappa(p) \in \pi_{c_\kappa}$.

For (c), we just show the $\star$-case. Let $X \in \pi_{c_\kappa}$. Suppose first $@_\kappa\, cl \in \Gamma$. Then for some $\phi \in \mathcal{L}^{\mathsf{QH}+}$, $\phi \in [X]_\kappa$, i.e., $X = \{\Delta \in W_\Gamma \mid @_\kappa\, \phi \in \Delta\}$. Thus, by Lemma A3.12, $c_\kappa(\star)(X) = \{\Delta \in W_\Gamma \mid @_\kappa \star \phi \in \Delta\} \in \pi_{c_\kappa}$. Suppose instead $@_\kappa\, cl \notin \Gamma$. So for some $p^+ \in \mathsf{Prop}^+$, $p^+ \in [X]_\kappa$, i.e., $X = \{\Delta \in W_\Gamma \mid @_\kappa\, p^+ \in \Delta\}$. Thus, $c_\kappa(\star)(X) = \{\Delta \in W_\Gamma \mid @_\kappa \star p^+ \in \Delta\}$ by Lemma A3.12. By OpEx, $\mathsf{BF}^+_@$, and Dist$_@$, $\exists q(q =_\kappa \star p^+) \in \Gamma$. By witnessing $\exists$s, there is a $q^+ \in \mathsf{Prop}^+$ such that $(q^+ =_\kappa \star p^+) \in \Gamma$. Hence, $c_\kappa(\star)(X) = \{\Delta \in W_\Gamma \mid @_\kappa\, q^+ \in \Delta\} \in \pi_{c_\kappa}$. ∎

We must also verify Lemma A3.14 still holds in order for C_ι to be well-defined.

Lemma A4.9 (Identity for Canonical Hyperconventions (Revised)). Where $|\kappa|_1 \in \Gamma$ and $|\lambda|_1 \in \Gamma$:

$$c_\kappa = c_\lambda \quad \Leftrightarrow \quad (\kappa = \lambda) \in \Gamma.$$

Proof: The right-to-left direction is as before. For the left-to-right direction, suppose $(\kappa \neq \lambda) \in \Gamma$. By PII, either (i) $\forall p(@_\kappa\, p = @_\lambda\, p) \notin \Gamma$ or (ii) $(\triangle_\kappa = \triangle_\lambda) \notin \Gamma$ for some $\triangle \in \{\neg, \wedge, \vee, \rightarrow, \Box, \Diamond\}$. We'll show that either way, $c_\kappa \neq c_\lambda$.

Suppose (i). By Dual$_\forall$ and witnessing $\exists$s, there is a p^+ such that $(@_\kappa\, p^+ \neq @_\lambda\, p^+) \notin \Gamma$. Thus, $c_\kappa(p^+) \neq c_\lambda(p^+)$, since:

$$c_\kappa(p^+) = \{\Delta \in W_\Gamma \mid @_\kappa\, p^+ \in \Delta\} \neq \{\Delta \in W_\Gamma \mid @_\lambda\, p^+ \in \Delta\} = c_\lambda(p^+).$$

Suppose instead (ii). I'll just do the $\star$ case to illustrate. By Dual$_\forall$ and witnessing $\exists$s, $(@_\kappa\, p^+ = @_\lambda\, q^+) \supset (@_\kappa \star p^+ = @_\lambda \star q^+) \notin \Gamma$ for some p^+ and q^+. Hence, $(@_\kappa\, p^+ = @_\lambda\, q^+) \in \Gamma$, and so $c_\kappa(p^+) = c_\lambda(q^+)$. Moreover, $(@_\kappa \star p^+ = @_\lambda \star q^+) \notin \Gamma$. Thus, by Lemma A3.12:

$$\begin{aligned} c_\kappa(\star)(c_\kappa(p^+)) &= \{\Delta \in W_\Gamma \mid @_\kappa \star p^+ \in \Delta\} \\ &\neq \{\Delta \in W_\Gamma \mid @_\lambda \star q^+ \in \Delta\} = c_\lambda(\star)(c_\lambda(q^+)) = c_\lambda(\star)(c_\kappa(p^+)). \end{aligned}$$

Hence, $c_\kappa(\star) \neq c_\lambda(\star)$. ∎

Definition A4.10 (Canonical Hypermodel (Revised)). The **canonical hypermodel** of Γ is the hypermodel $\mathcal{M}_\Gamma = \langle W_\Gamma, D_{\mathbb{C}\Gamma}, D_{\mathbb{P}\Gamma}, V_\Gamma \rangle$ is defined as in Definition A3.15, except where $\mathsf{Prop}^* = \mathsf{Prop} \cup \mathsf{Prop}^+$:

$$D_{\mathbb{P}\Gamma} = \{P \in \mathbb{P}_{D_{\mathbb{H}\Gamma}} \mid \exists p^* \in \mathsf{Prop}^* \,\forall c_\kappa \in D_{\mathbb{H}\Gamma}\colon\ p^* \in [P(c_\kappa)]_\kappa \}$$

It is easy to check that $D_{\mathbb{P}\Gamma}$ satisfies conditions i and ii from Definition A2.11. Note also that $V_\Gamma(p) \in D_{\mathbb{P}\Gamma}$ by Lemma A4.8(b).

Lemma A4.11 (Canonical Classical Convention (Revised)). Where $c_\kappa \in C_{cl}$:

$$\begin{array}{lll} \neg_{c_\kappa} X = \overline{X} & X \wedge_{c_\kappa} Y = X \cap Y & \Box_{c_\kappa} X = \{w \in W \mid X = W\} \\ X \vee_{c_\kappa} Y = X \cup Y & X \rightarrow_{c_\kappa} Y = \overline{X} \cup Y & \Diamond_{c_\kappa} X = \{w \in W \mid X \neq \varnothing\}\,. \end{array}$$

Proof: We show $c_\kappa(\neg)(X) = \overline{X}$ for illustration. Suppose $p^+ \in [X]_\kappa$. By Lemma A3.12 and $\text{Dist}_@$, $c_\kappa(\neg)(X) = \{\Delta \in W_\Gamma \mid \neg\, @_\kappa\, p^+ \in \Delta\}$. By maximal consistency, $c_\kappa(\neg)(X) = \{\Delta \in W_\Gamma \mid @_\kappa\, p^+ \notin \Delta\}$. Hence, $c_\kappa(\neg)(X) = \overline{X}$ by Definition A3.11. ■

Lemma A4.12 (Truth). $\mathcal{M}_\Gamma, \Delta, c_\kappa \Vdash \phi$ iff $@_\kappa\, \phi \in \Delta$.

Proof: The proof is the same as before, except now we must tweak the connectives case and also deal with the quantifier cases. For the connectives, I'll just do the $\neg$-case. The proof is the same except when $[\![\phi]\!]^{c_\kappa} \notin \pi_{c_\kappa}$. In that case, $\Delta, c_\kappa \nVdash \neg\, \phi$ (see page 11). Thus, we must show that $@_\kappa \neg\, \phi \notin \Delta$. Since $[\![\phi]\!]^{c_\kappa} \notin \pi_{c_\kappa}$, there is no $p^+ \in [[\![\phi]\!]^{c_\kappa}]_\kappa$, i.e., no p^+ such that $\Box(@_\kappa\, p^+ \leftrightarrow @_\kappa\, \phi) \in \Gamma$. By $\exists$-witnessing, $\neg\, \exists p\, \Box(@_\kappa\, p \leftrightarrow @_\kappa\, \phi) \in \Gamma$. By Bool, $\text{Dist}_@$, and $\text{BF}^+_@$, $\neg\, @_\kappa\, \mathsf{E}\phi \in \Gamma$. By OpEx, $\neg\, @_\kappa\, \mathsf{E} \neg\, \phi \in \Gamma$. By NecEx, $\Box \neg\, @_\kappa\, \mathsf{E} \neg\, \phi \in \Gamma$. By TrEx and RK, $\Box \neg\, @_\kappa \neg\, \phi \in \Gamma$. Hence, $\neg\, @_\kappa \neg\, \phi \in \Delta$, and so $@_\kappa \neg\, \phi \notin \Delta$.

For the quantifiers, here's the $\forall$-case (the $\exists$-case is similar):

$$
\begin{array}{rcll}
\mathcal{M}_\Gamma, \Delta, c_\kappa \Vdash \forall p\, \phi & \Leftrightarrow & \text{for all } P \in D_{\mathbb{P}\Gamma} \colon (\mathcal{M}_\Gamma)^p_P, \Delta, c_\kappa \Vdash \phi & \\
& \Leftrightarrow & \text{for all } p^* \in \mathsf{Prop}^* \colon (\mathcal{M}_\Gamma)^p_{V_\Gamma(p^*)}, \Delta, c_\kappa \Vdash \phi & \text{def. of } D_{\mathbb{P}\Gamma} \\
& \Leftrightarrow & \text{for all } p^* \in \mathsf{Prop}^* \colon \mathcal{M}_\Gamma, \Delta, c_\kappa \Vdash \phi[p^*/p] & \text{Lemma A4.1} \\
& \Leftrightarrow & \text{for all } p^* \in \mathsf{Prop}^* \colon\ @_\kappa\, \phi[p^*/p] \in \Delta & \text{IH} \\
& \Leftrightarrow & \neg \exists p \neg @_\kappa\, \phi \in \Delta & (*) \\
& \Leftrightarrow & \forall p\, @_\kappa\, \phi \in \Delta & \text{Dual}_\forall \\
& \Leftrightarrow & @_\kappa \forall p\, \phi \in \Delta & \text{BF}_@, \text{CBF}_@.
\end{array}
$$

The left-to-right direction of the (*) step follows from $\exists$-witnessing, while the right-to-left direction follows from $\text{Intro}_\exists$ and VE. ■

Lemma A4.13 (Closure for Canonical Classical Convention). Let $c_\kappa \in C_{cl}$. Then $[\![\phi]\!]^{\mathcal{M},c_\kappa} \in \pi_{c_\kappa}$ for any $\phi \in \mathcal{L}^{\mathsf{QH}+}$ and any $\mathcal{M}$ based on $\langle W_\Gamma, D_{\mathbb{C}\Gamma}, D_{\mathbb{P}\Gamma}\rangle$.

Proof: Let $c_\kappa \in C_{cl}$, let $\phi \in \mathcal{L}^{\mathsf{QH}+}$, let $\mathcal{M} = \langle W_\Gamma, D_{\mathbb{C}\Gamma}, D_{\mathbb{P}\Gamma}, V\rangle$. Let $q_1, \ldots, q_n$ be the free propositional variables in ϕ. By definition of $D_{\mathbb{P}\Gamma}$, there are some $q_1^*, \ldots, q_n^* \in \mathsf{Prop}^*$ such that $V(q_i)(c_\kappa) = \{\Delta \in W_\Gamma \mid @_\kappa\, q_i^* \in \Delta\} = V_\Gamma(q_i^*)(c_\kappa)$ for all $c_\kappa \in D_{\mathbb{H}\Gamma}$. By Lemmas A4.1 and A4.12, $[\![\phi]\!]^{\mathcal{M},c_\kappa} = \{\Delta \in W_\Gamma \mid @_\kappa\, \phi[q_1^*/q_1, \ldots, q_n^*/q_n] \in \Delta\}$. Hence, $\phi[q_1^*/q_1, \ldots, q_n^*/q_n] \in [[\![\phi]\!]^{\mathcal{M},c_\kappa}]_\kappa$, and so $[\![\phi]\!]^{\mathcal{M},c_\kappa} \in \pi_{c_\kappa}$. ■

From here, the proof of completeness is the same. (In particular, Lemmas A4.11 and A4.13 show that C_{cl} is classical.) Thus:

Theorem A4.14 (Completeness in $\mathcal{L}^{\mathsf{QH}}$). Where $\Gamma \subseteq \mathcal{L}^{\mathsf{QH}}$ and $\phi \in \mathcal{L}^{\mathsf{QH}}$:

(a) If $\Gamma \vDash \phi$, then $\Gamma \vdash \phi$.
(b) If $\Gamma \Vdash\!\!= \phi$, then $\Gamma \Vdash \phi$.

A4.3 Axioms for $\rhd$

In adding $\rhd$ to the language, the main complication involves PII. Since $\rhd$ can take any number of arguments on the left, we cannot state "$\rhd_\kappa = \rhd_\lambda$" as a single formula. In fact, completeness is not possible in $\mathcal{H}$ as it stands, since consequence is not compact in $\mathcal{H}$. In particular, $\Gamma \vDash (\iota = \kappa)$, where

$\Gamma := \{|\iota|_1, |\kappa|_1, \forall p(@_\iota p = @_\kappa p)\} \cup \{\triangle_\iota = \triangle_\kappa\}_\triangle \cup \{\triangleright_\iota =_n \triangleright_\kappa\}_{n\in\omega}$. Yet $\Gamma_0 \not\models (\iota = \kappa)$ for each finite $\Gamma_0 \subseteq \Gamma$. Still, as we'll see in Part B, there are natural restricted classes of hypermodels on which PII is sound as is.[19] Over such classes, completeness can be restored.

A5 Conclusion

Hyperlogic is a hyperintensional system that is designed to regiment, and facilitate reasoning about, metalogical claims within the object language. This is achieved by introducing a multigrade entailment operator, propositional quantifiers, and modified hybrid operators into the language. To interpret these claims, we introduced hyperconventions, i.e., maximally specific interpretations, into points of evaluation. While one might suspect that the logic of hyperlogic is uninteresting, as we've seen, this suspicion is incorrect. We presented dual axiomatic systems for both classical and universal consequence in a number of fragments of hyperlogic and proved their soundness and completeness.

The minimal logic of hyperlogic explored in this paper is fairly weak and assumes next to nothing about the possible interpretations of the connectives. It also does not yet include hyperintensional operators like belief operators or counterfactuals. In Part B of this series, we begin to fill these gaps by exploring stronger logics that can be obtained by imposing various restrictions on the class of hypermodels and also by adding hyperintensional operators to the language.

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[19] One constraint examined in Part B is that any two hyperconventions be discernible via some visible proposition in the domain, i.e., $P(c_1) \neq P(c_2)$ for some $P \in D_{\mathbb{P}}$. This constraint is not satisfied by every hypermodel: it can be that $P(c_1) = P(c_2)$ for all $P \in D_{\mathbb{P}}$ while still $c_1 \neq c_2$ if, say, $c_1(\triangle) \neq c_2(\triangle)$ for some $\triangle$ (the constraints from Definition A2.11 only ensure that if $P(c_1) = P(c_2)$ for all $P \in D_{\mathbb{P}}$, then $\pi_{c_1} = \pi_{c_2}$ and $c_1(p) = c_2(p)$ for all p; see footnote 17). Requiring that there be some $P \in D_{\mathbb{P}}$ that discerns c_1 from c_2 would allow us to even drop the $\{\triangle_\iota = \triangle_\kappa\}_\triangle$ premises from PII.

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