Constructing and Classifying Five-Dimensional Black Holes Using Integrability

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Abstract

In this thesis we look at the problem of finding and classifying stationary and biaxisymmetric solutions in five-dimensional theories of gravity, using particular hidden symmetries. We consider three theories: the electrostatic sector of Einstein-Maxwell, vacuum gravity and minimal supergravity (Einstein-Maxwell gravity with a Chern-Simons term).

For electrostatic solutions to Einstein-Maxwell theory, the equations on the metric and Maxwell field possess a $SL(2,\mathbb{R})$ symmetry. This allows one to derive transformations which either charge a solution or immerse it in an electric Melvin background. By considering a neutral static black lens seed and performing these two transformations with appropriately tuned transformation parameters, we construct the first example of a regular black lens in Einstein Maxwell theory with topologically trivial asymptotics.

For vacuum gravity we consider asymptotically flat solutions. The vacuum Einstein equations are integrable in the sense that they can be reformulated as the integrability condition for an auxiliary linear system of PDEs. Taking these PDEs, one can integrate them over the event horizons, the axes of symmetry and infinity. By carefully considering continuity conditions between these solutions, one may actually solve for metric data on the horizons and the axes in terms of some geometrically defined moduli, subject to a set of polynomial constraints. This represents a very useful tool for answering the existence problem, reducing it to the much more tractable question of whether a particular system of polynomials (subject to some inequalities) has any solutions. Using this polynomial system we provide a constructive uniqueness proof for the Kerr (using analogous four-dimensional results), Myers-Perry and black ring solutions. We also prove, through a combination of analytic and numerical methods, that the "simplest" $L(n,1)$ black lens cannot exist by showing that it must possess a conical singularity on one of the axes.

Finally we consider the case of asymptotically flat solutions in minimal supergravity. As with the vacuum, this is an integrable theory and so a similar analysis can be performed with exactly analogous results, although with rather more complicated polynomial systems determining the existence of solutions. A notable feature of minimal supergravity, not present in the vacuum theory, is the existence of regular solitons - in this context these are non-trivial solutions without black hole regions. We begin the exploration of the moduli space of these solitons by first studying the case of flat space.

Declaration

I have written this thesis based on my own work in collaboration with my supervisor James Lucietti. No part of this thesis has been submitted for any other degree or qualification. Chapter 2 is based on my paper [1], Chapters 3 and 4 are based on the papers [2, 3], authored by myself and my supervisor.

(*Fred Tomlinson, August 2021*)

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Lay Summary

This thesis is about gravity and black holes in higher dimensions.

Our modern understanding of gravity is encapsulated in Einstein's theory of General Relativity (GR), formulated roughly 100 years ago. This theory is founded upon the idea that absolute notions of space and time should instead be replaced by the unifying concept of "spacetime". The guiding principle behind this theory can be concisely stated in the words of John Wheeler: "Spacetime tells matter how to move; matter tells spacetime how to curve". What this means is that spacetime has a structure which affects the motion of matter. This structure is then itself a dynamical object, meaning that it can be changed by the presence of matter.

One of the most striking predictions of GR is the existence of black holes. These are very dense regions of spacetime which have a boundary known as an event horizon. Any matter (or light) that crosses this horizon towards the black hole can never escape. Although these objects were initially posited purely from a theoretical standpoint, they were later experimentally observed in a variety of ways. The most accurate measurements have been made over the last 10 years, based on observations of the so-called gravitational waves emitted from two colliding black holes.

The previous two paragraphs give a description of gravity in 4 spacetime dimensions (3 spatial and 1 time dimension). However there is nothing particular to 4 dimensions about the mathematical formulation of GR and it is straightforward to consider this theory with any number of spatial dimensions. Whilst these descriptions are not directly physically relevant - we live in 4 dimensions - they turn out to be valuable theories on a mathematical level. A prime example of this is the holographic principle. This principle underlies numerous results which show that the equations for GR in D dimensions can actually help us understand particle physics in $D - 1$ dimensions.

Since we are interested in particle physics in our familiar 4 dimensions of spacetime, it is natural to look at gravity in 5 dimensions and in particular we will look at black holes. In 4 dimensions an isolated, equilibrium, rotating black hole must have a (squashed) spherical horizon and can be fully described by just two parameters which describe its mass and how much it is rotating. This is known as the no-hair theorem and it means that all black holes with the same parameters must have an identical gravitational field; there can be no other distinguishing features. However in 5 dimensions there is much more room for black hole variety. For example, along with the analogue of the spherical 4 dimensional black hole, there are also black holes which have a "doughnut" shaped horizon, black rings. If a black ring is positioned around a spherical black hole, this creates an even stranger object known as the black Saturn.

The aim of this thesis is to explore this zoo of black objects in order to understand the full range of possible black holes in five dimensions.

Contents

Chapter 1

Introduction

Black holes are regions of spacetime with sufficiently strong gravitational forces that prevent even light from escaping. There has been great interest in studying such objects for over a hundred years since the development of the theory of general relativity (GR) (1915) and the discovery of the Schwarzschild metric for a static black hole spacetime (1916). The next leap in understanding came with the discovery of the Kerr metric [4] (1963) which describes the much more complicated situation of a rotating black hole. Shortly afterwards the charged generalisation was derived, the Kerr-Newman metric [5] (1965). Following on from these results was the work of Penrose and Hawking on singularity theorems in the late '60s [6, 7, 8]. Up to this point there was some scepticism about the physical existence of black holes. The singularity theorems settled this conclusively by demonstrating that in fact black holes are a true physical prediction of GR, which one would expect to form under generic conditions. This work was eventually awarded a Nobel prize over 50 years later in 2020. Another important advance in the mathematical understanding of black holes was in the derivation of various uniqueness results in the late '60s and early '70s (see [9] for a review), the culmination of which was the no-hair theorem. This will be discussed in much greater detail shortly, but essentially this theorem guarantees that, at equilibrium, black holes can be characterised uniquely with just their mass, angular momentum and electromagnetic charges.

Around the same time, astronomers were beginning to identify black holes based on indirect observations, beginning with the identification of Cygnus X-1 in 1971 [10, 11]. Since then there have been huge advances in our understanding of black holes as astrophysical objects. Of particular note is the landmark work of LIGO in the first measurement of the gravitational wave signature from two merging black holes [12] in 2016.

The position of black holes in modern theoretical physics has grown in importance over the last few decades as a powerful probe of quantum gravity. There are two principal aspects to this. Firstly black holes need a theory of quantum gravity to be understood properly. Black holes therefore provide an almost unique opportunity to study a situation where both GR describing gravity and quantum field theories (QFTs) describing the other forces come into conflict. A notable example of this conflict is given by the black hole information paradox [13]. Roughly speaking this is the fact that information can seemingly disappear into a black hole, clashing with the unitarity postulate of QFTs which necessitates that information instead be conserved.

Secondly black holes provide a useful test-bed for ideas from AdS/CFT [14] and, more generally, holography. Broadly speaking these provide an equivalence between D-dimensional "bulk" gravity theories and $D-1$-dimensional "boundary" QFTs. Therefore by understanding black holes in the bulk one gains insight into the corresponding configuration in the boundary theory.

Holography also clearly provide a motivation for studying higher-dimensional black holes, at least in the five-dimensional case, which would correspond to matter theories in four-dimensional space. In order to fully explore this higher-dimensional case however, it is instructive to first consider the four-dimensional case, which is rather better understood.

The reviews [15, 16, 9] were useful in writing this chapter and provide further details on some of the results that will be discussed.

1.1 Four-dimensional electrovacuum black holes

We will begin by considering a four dimensional spacetime (M, g, F) satisfying the equations of Einstein-Maxwell gravity. The action is given by[1]

$$S = \int_M R \star 1 - 2F \wedge \star F, \tag{1.1}$$

where we will use the fact that the field strength $F = \mathrm{d}A$ (at least locally). This theory will serve as a useful grounding to understand the various five-dimensional theories that we will consider later on, namely the vacuum, electrovacuum and minimal supergravity theories.

It is a natural question to ask what the full space of solutions for this theory is - this is the classification problem. To make this problem more manageable we impose certain simplifying assumptions. We will take the spacetime to be stationary and asymptotically flat (AF), here and throughout the thesis (with the notable exception of when we look at Melvin asymptotics in Chapter 2):

Asymptotically Flat: This condition states that the spacetime approaches Minkowski space at "infinity". More precisely we will assume that the asymptotic region of M is diffeomorphic to $\mathbb{R} \times \mathbb{R}^3 \setminus B$ where B is a closed ball centred at the origin of the $\mathbb{R}^3$. Using this, one can then construct a Cartesian coordinate system x^A $(A, B = 0, \dots, 3)$ in the asymptotic region of M. Introducing a flat metric on this space η_{AB} and a radial coordinate $r = x^i x^i$ $(i, j = 1, \dots, 3)$, g and A (in some gauge) must satisfy

$$g_{AB} = \eta_{AB} + O_2(r^{-1}), \qquad A_A = O_2(r^{-1}), \tag{1.2}$$

where if $f = O_m(r^n)$, then $\partial^k f = O(r^{n-k})$, for all integers $k,\ 0 \leq k \leq m$.

Using flat space as the choice of limiting spacetime is a natural choice for the asymptotic behaviour of our space M. This condition also allows one to model an "isolated body" in general relativity as is done in Newtonian gravity or classical electromagnetism. There are of course other possible asymptotic conditions, for example de Sitter, anti de Sitter (AdS) or Kaluza-Klein. However it is notable that known solutions possessing these other asymptotics tend to have parameters which can vary the exact asymptotic behaviour of the spacetime. By taking an appropriate limit, one can often recover an AF solution. For example taking the infinite limit of an AdS radius in an asymptotically AdS space gives an AF space. This demonstrates that a good understanding of AF spacetimes is still useful in understanding spacetimes with non flat asymptotics.

Stationary: We recall that stationary spacetimes possess a Killing vector field (KVF) k, compatible with A, such that k is asymptotically timelike. This means that k satisfies

$$\mathscr{L}_k g = \mathscr{L}_k A = 0. \tag{1.3}$$

[1] Here and throughout this thesis we take $c = 1, G = (16\pi)^{-1}$ but otherwise adopt the conventions of Wald's textbook [17].

This also tells us that the isometry group of g must have an $\mathbb{R}$ subgroup.

Stationary spacetimes represent equilibrium states, that is spacetimes that have settled down and exhibit time-independent features. In contrast to these solutions one could instead consider the larger class of dynamical spacetimes where there is not necessarily a stationary KVF. Although these are more generic and obviously of great astrophysical interest (in understanding black hole mergers for example), very little is known about exact solutions in this context and instead approximate and numerical methods must be employed. For this reason and since we are interested in exact solutions, we will assume stationarity throughout.

Asymptotic flatness allows us to define a couple of useful invariants which can be associated to a spacetime. The mass of an AF, stationary spacetime is given by the standard Komar integral

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} \star \mathrm{d}k \tag{1.4}$$

where S^2_∞ represents the 2-sphere at infinity. Now recall that if in addition to the stationary KVF the spacetime possesses a spacelike KVF m with topologically S^1 orbits which is compatible with k and A, then the spacetime is *axisymmetric*. In other words

$$[m,k] = 0, \qquad \mathscr{L}_m g = \mathscr{L}_m A = 0. \tag{1.5}$$

Using this axial KVF one can write down the Komar integral for the angular momentum

$$J = \frac{1}{16\pi}\int_{S^2_\infty} \star \mathrm{d}m. \tag{1.6}$$

Using asymptotic flatness we can clarify exactly what we mean by black hole spacetimes. An AF spacetime can be conformally compactified, adding in a new conformal boundary and bringing the asymptotic region into a finite region of coordinate space. The important features of the boundary for our purposes are future and past null infinity, denoted $\mathcal{I}^+$ and $\mathcal{I}^-$ respectively - these provide start and endpoints for null curves in the spacetime. One can then define the domain of outer communication (DOC) of a spacetime to be given by the intersection of the causal past of $\mathcal{I}^+$ with the causal future of $\mathcal{I}^-$. We can also define the black hole region as the complement of the causal past of $\mathcal{I}^+$ in M. From the definition this is a subset of the complement of the DOC in M, and if it is non-empty then we call the spacetime a black hole spacetime. These definitions are illustrated in Figure 1.1, the Penrose diagram for the Schwarzschild spacetime.

Another important assumption that we will make throughout the thesis is that horizons we consider are non-degenerate; this means that the surface gravities associated to them should be non-zero. This is for the most part a technical assumption, in the sense that many of the results we present can be proved in the degenerate cases as well. However these proofs normally require treating the degenerate case on a completely different footing to the non-degenerate cases and so complicate the analysis somewhat.

An important solution satisfying these assumptions (AF, stationary, non-degenerate horizon) is the Kerr-Newman spacetime [5], the charged generalisation of the Kerr spacetime. It can be written in Boyer-Lindquist coordinates as

$$\begin{aligned} g &= -\Sigma^{-1}(\Delta - a^2\sin^2\theta)\mathrm{d}t^2 - 2a\sin^2\theta\Sigma^{-1}(r^2+a^2-\Delta)\mathrm{d}t\mathrm{d}\phi \\ &\quad + \sin^2\theta\Sigma^{-1}[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\mathrm{d}\phi^2 + \Sigma\Delta^{-1}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 \\ A &= -\Sigma^{-1}\left[Qr(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi) + P\cos\theta(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)\right] \end{aligned} \tag{1.7}$$

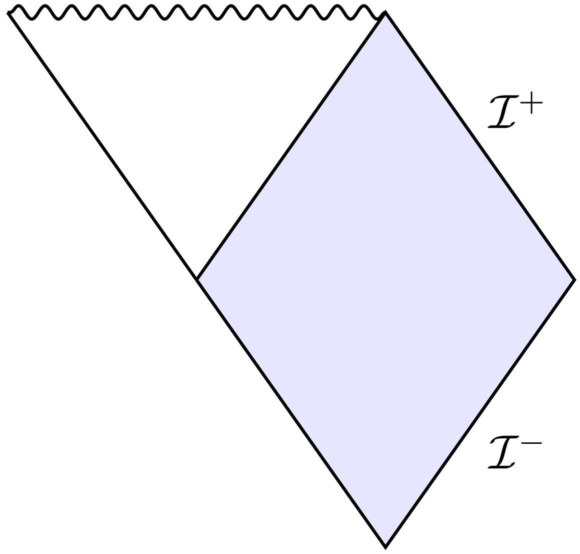

Figure 1.1: The Penrose diagram for the Schwarzschild spacetime. $\mathcal{I}^+$ and $\mathcal{I}^-$ represent future and past null infinities respectively. The DOC corresponds to the blue diamond, the black hole region corresponds to the white triangle and the wiggly line represents the singularity of the spacetime.

where θ, ϕ are angles on the 2-sphere, $r > 0$ and

$$\Sigma = r^2 + a^2 \cos^2 \theta, \qquad \Delta = r^2 - 2Mr + a^2 + e^2, \qquad e = \sqrt{Q^2 + P^2}. \tag{1.8}$$

M is the mass, the angular momentum $J = aM$, Q is the electric charge and P is the magnetic charge. The parameters are constrained to obey $M^2 > a^2 + e^2$ with extremal black holes found in the limit that the inequality becomes an equality. Physically this represents an isolated, spherical, rotating black hole with charge. Note that this solution is also axisymmetric around the axis of rotation and ∂_ϕ gives the KVF associated to this symmetry.

This in fact represents the most general family of solutions under our assumption gives certain global regularity conditions; we will call a stationary spacetime *well-defined* if: i) the DOC is globally hyperbolic, ii) k has complete orbits, iii) any cross section of the black hole horizon is compact, iv) there exists an acausal, connected, hypersurface which is asymptotic to a constant time slice in (1.2) and whose boundary is a cross section of the horizon (see [9] for full details). For this classification result (also known as the no-hair theorem), we must also assume that (M, g, F) is analytic (though it is believed that this assumption is ultimately unnecessary). Then the theorem can be stated as follows:

Theorem 1. *Analytic, well-defined, stationary and AF black hole spacetimes in Einstein-Maxwell gravity with connected, non-degenerate horizons have a DOC which is isometric to a member of the Kerr-Newman family of solutions.*

The purpose of the remainder of this section will be to describe the main steps in the proof of this theorem. This will serve as a useful precursor to the discussion of the higher-dimensional case and ultimately vacuum gravity and minimal supergravity in five dimensions. The proof of the uniqueness theorem follows a rather different path depending on whether the black hole is rotating or not, corresponding to either the generic stationary or static case.

1.1.1 Static uniqueness

First we consider the case where the black hole is not rotating i.e. when $k^2|_{\mathcal{H}} = 0$, where $\mathcal{H}$ is the black hole horizon. One can use this to demonstrate that the stationary KVF k must be hypersurface orthogonal which shows that the spacetime is static. This is known as the staticity theorem [18, 19], and its proof involves combining various differential identities with Stokes' theorem. The fact that non-rotating black holes must be part of a static spacetime is an intuitive, albeit non-trivial, statement. Note that this has not been proved in the degenerate case and in fact violations of this theorem in different theories of gravity are known when the horizons can be degenerate (the BMPV spacetime [20] provides an example of this in five-dimensional minimal supergravity [21]).

We've now taken information defined on the black hole horizon (i.e. the fact that it is non-rotating) and proved the global statement that the spacetime is static. There are a couple of ways to complete the uniqueness proof. The classic method by Israel proceeds by first showing that the DOC must be spherically symmetric [22, 23]. The proof that the DOC must be isometric to Reissner–Nordström solution is then a consequence of Birkoff's theorem. A more recent alternative to Israel's theorem is provided by Bunting and Masood-ul-Alam [24, 25]. This method involves glueing together a spatial slice of the DOC to itself along the black hole horizon. Under a conformal transformation this resultant glued space is complete and has zero mass, therefore using the positive mass theorem one can show that it is isometric to flat Euclidean space. One can then show that the conformal rescaling must be such that the original space is the Reissner–Nordström solution.

1.1.2 Generic stationary uniqueness

Now consider the case where the black hole is rotating i.e. $k^2|_{\mathcal{H}} \neq 0$. Topological censorship states that any curve starting and ending in the asymptotic region can be deformed to a curve lying completely in the asymptotic region [26]. This relies on the Einstein equations together with the null energy condition. A simple consequence of this theorem is that the DOC is simply connected. This can be used to prove the horizon topology theorem [27, 28], which states that the only allowed topologies for black hole horizons are 2-spheres. Following on from this one can prove the rigidity theorem [29]. This establishes two fundamental results, first that there is an additional KVF m making the spacetime axisymmetric. Second one can show that the horizon is in fact a Killing horizon with normal equal to $k + \Omega m$ for each horizon component where Ω is the angular velocity of the corresponding component (this result justifies the existence of the surface gravities we have discussed previously). A key part in the proof of this theorem is the use of analyticity to take a local result about axisymmetry and turn it into a global statement. These three theorems apply equally to the static case above and we shall shortly discuss the generalisation of these two theorems to the higher dimensional cases where a richer structure emerges.

The next crucial step is to construct some new global coordinates in the DOC adapted to the symmetries of the solution known as the Weyl-Papapetrou coordinates. In these coordinates the metric can be separated into two blocks (this is proved by the circularity theorem [30, 31, 32]). One of these blocks corresponds to conformally flat (ρ, z) coordinates (essentially cylindrical polars), the other block corresponds to a time and angle coordinate associated to the symmetries (we give the form of the metric in terms of these coordinates for the general D case explcitly in (1.12)). One can then compare the metrics for two solutions possessing the same mass (1.4) and angular momentum (1.6) in these coordinates. By using complicated nonlinear integral expressions, the differences between components of the metrics are seen to vanish and so the two metrics must describe the same spacetime. These integral expressions were initially written down by Robinson just for the vacuum case [33]. It was later realised by Mazur [34] that there was in fact deep structure underlying this result coming from the hidden $SL(2)$ symmetry in vacuum gravity. By exploiting a similar $SU(2,1)$ symmetry in Einstein-Maxwell gravity, Mazur was able to derive a corresponding identity in this more general setting. Using this identity completes the proof of the uniqueness in the rotating case.

The fact that vacuum and electrovacuum theories in four dimensions possess these hidden symmetries is closely related to the fact that they are integrable theories. Broadly speaking this means that the Einstein equations can be rewritten as the integrability condition of a pair of linear PDEs. We will discuss this and many of the ideas described above in greater detail in the context of higher dimensions in the coming sections.

1.1.3 Comments on the no-hair theorem

There are two important respects in which Theorem 1 can be strengthened: First, in the case that the horizon is rotating the non-degeneracy assumption can be dropped. Second, in the case that the horizon is non-rotating, using the method of Bunting and Masood-ul-Alam discussed [24], one can relax the connectedness assumption (in which case one finds that the Reissner–Nordström is the unique solution).

The question mark over the existence of rotating black holes with more than a single horizon component is an interesting gap in the no-hair theorem. Important results have been developed that go some way towards addressing this. Notably Weinstein has established a uniqueness theorem for h horizons [35] in terms of $4h-1$ parameters, using results from the theory of harmonic maps. Roughly speaking these parameters come from the three uniqueness parameters for each horizon (i.e. mass, angular momentum and charge) in addition to the $h-1$ lengths of the "struts" connecting adjacent horizons. Together with this, Weinstein gave a non-constructive proof of the existence of such solutions up to possible conical singularities arising from these struts.

In order to address the existence of fully regular solutions one must use different methods. One such method relies on the integrability structure of Einstein-Maxwell gravity and has been pursued by Varzugin [36, 37] and Meinel and Neugebauer [38]. It turns out that one can construct these solutions of Weinstein on a portion of the axis of symmetry for a general h-horizon solution and moreover derive various constraints on the uniqueness parameters (although these are difficult to solve even for $h=2$). This has been used as part of the proof that there are no solutions with $h=2$ in the vacuum case [39], and to narrow down the most general candidate solution for $h=2$ in the electrovacuum case [40]. We will discuss this constructive approach based on integrability in much greater detail throughout the thesis. Indeed the content of Chapter 3 is essentially an attempt to extend and refine this method in the five-dimensional vacuum setting.

1.2 Black holes in higher dimensions

In the previous section we've been considering GR in four dimensions; now we will extend this to an arbitrary number of dimensions $D \geq 4$. There are a few reasons why this is an interesting and useful thing to do. Firstly as we've discussed previously, holography sets up a correspondence between D dimensional gravity theories and $D-1$ dimensional QFTs. Therefore in principle studying five-dimensional gravity theories can help shed some light on theories of matter in four dimensions. Another reason to take higher dimensional black holes seriously is that they can act as a toy model for black holes in four dimensions. For example the first calculations accounting for the microscopic origin of the Bekenstein-Hawking entropy for a black hole were done for certain black holes in five dimensions [41]. It is still an open problem how to do this generically in four dimensions. Studying gravity in other dimensions can also be instructive because of its differences rather than similarities. An example of this, which we shall soon cover in more detail, is that the naive extension of the no-hair theorem to $D > 4$ breaks down, a somewhat surprising result which shows us that this is really a property unique to four dimensions.

To begin with we will consider a number of the features of four-dimensional gravity we've described in the previous section and describe what happens to them in D dimensions. We will again consider an AF, stationary spacetime with non-degenerate horizons with a general matter content (satisfying the dominant energy condition). Then the following theorems hold:

Topological Censorship: Global hyperbolicity already provides some information about the topology of the DOC, namely that $\langle\langle M \rangle\rangle \cong \mathbb{R} \times \Sigma$ where Σ is a Cauchy surface. In addition to this the D-dimensional topological censorship theorem states that any curve starting and ending in the asymptotic region can be continuously deformed to a curve that lies entirely within the asymptotic region [26, 42], just as in the 4-dimensional case. This theorem is equivalent to the fact that the DOC (and so Σ) is simply connected [43].

Horizon Topology: For $D = 4$ we saw that any black hole horizons (strictly their cross-sections) must necessarily have S^2 topology - this is Hawking's topology theorem [27]. The generalisation to arbitrary dimension is more complicated: the D-dimensional horizon topology theorem states that the so-called Yamabe invariant associated to each horizon component is positive [44, 45, 46]. This is an invariant associated to the smooth structure of a space defined in terms of the average scalar curvatures over the components for various metrics (see above papers for the exact construction of this invariant). For $D = 4$ this Yamabe invariant is essentially the Euler characteristic of the horizon, positivity of which implies that the horizon must have spherical topology, recovering the four-dimensional result. For $D = 5$ this restriction implies that the horizon topology must be a connected sum of arbitrary discrete quotients of S^3 and $S^2 \times S^1$ ring spaces. We see that non-spherical horizon topologies are now possible in 5 dimensions and as D grows larger the constraints placed on the horizon topology become weaker.

Rigidity Theorem: The electrovacuum rigidity theorem in four dimensions ensures that the black hole horizon is a Killing horizon and that the spacetime is axisymmetric (assuming the spacetime is analytic) [29]. The higher dimensional generalisation gives exactly the same result [47, 48], guaranteeing exactly one extra compatible axial KVF. We note that this is a less restrictive additional symmetry as D gets larger. In principle an AF spacetime in D dimensions can have up to $\lfloor (D-1)/2 \rfloor$ axial KVFs. This bound comes from the dimension of the Cartan subgroup of the rotation group $SO(D-1)$. For $D = 4$ this bound simply gives 1, so the rigidity theorem gives as many KVFs as possible. For any $D > 4$, this is no longer the case and so rigidity guarantees relatively little symmetry.

Clearly these theorems are most restrictive in the case $D = 4$. This means that for $D > 4$, various novel features emerge such as the existence of (non-extremal) multi-black hole solutions and non-spherical black hole horizons (both of which we will discuss in more detail in the context of five dimensions).

The classification results for the static case in Einstein-Maxwell gravity extend to higher dimensions in the obvious way (for non-extremal horizons). In the four-dimensional case assuming staticity is enough to determine that the DOC of the spacetime must be given by the Reissner–Nordström solution

[24, 25]. Exactly analogous results can be derived for $D > 4$, where now the metric must be given by the higher-dimensional generalisation of the Reissner–Nordström spacetime [49, 50, 51]:

$$
\begin{gathered}
g = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{D-2}^2, \qquad f(r) = 1 - \frac{2\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}}, \\
A = \frac{q}{Cr^{D-3}}\mathrm{d}t, \qquad C^2 = 2\frac{D-3}{D-2},
\end{gathered}
\tag{1.9}
$$

where $\mathrm{d}\Omega_{D-2}^2$ is the metric on the unit $D-2$-sphere, $r > 0$ and μ, q are parameters proportional to the mass and charge of the spacetime.

On the other hand, it is a notable feature of (electro)vacuum theories in higher dimensions that a naive generalisation of the no-hair theorem fails - for $D > 4$, mass and angular momenta are no longer sufficient to guarantee uniqueness of a solution. The first demonstration of this was in the discovery of the black ring solution by Emparan and Reall [52], a five-dimensional vacuum black hole with $S^2 \times S^1$ horizon topology. For the remainder of this section we will study this problem in more detail and consider what is known about the space of solutions. As we shall shortly see if we want want to make much progress in the AF case we will need restrict to $D = 5$.

1.2.1 Five-dimensional black holes, rod structure and integrability

The problem of extending classification results for stationary rotating black hole spacetimes to higher dimensions is a difficult one, and indeed there have only been successes in some special cases. One of the fundamental stumbling blocks here is the fact that the rigidity theorem only guarantees the existence of a single extra axial KVF, which is less and less restrictive as the number of dimensions D increases. An immediate consequence of this is that for $D \geq 5$ one cannot construct the Weyl-Papapetrou coordinates that we saw were fundamental in the proof of the no-hair theorem for rotating black holes. Therefore a natural class of solutions to consider are those for which these coordinates exist.

We consider spacetimes with $D-3$ commuting axial KVFs, m_i $(i = 1, \dots, D-3)$, which commute with the stationary KVF k (and are compatible with relevant matter fields). Note that in light of the results of the rigidity theorem it is reasonable to assume that this class of solutions is non-generic for $D > 4$, since extra symmetry has been assumed. An AF spacetime has at most $\lfloor (D-1)/2 \rfloor$ axial KVFs and so we see that AF is only compatible with these $D-3$ extra symmetries in the cases $D = 4, 5$ which we will now assume (note that it is possible to consider higher D if one considers Kaluza-Klein asymptotics instead). Also for simplicity we restrict to either vacuum gravity, Einstein-Maxwell gravity or (five-dimensional) minimal supergravity (MSG) (although some of the results we will present can be proved under slightly looser matter assumptions). We will consider MSG in more detail in Section 1.4 but for now we can think of it as Einstein-Maxwell gravity in five dimensions with a Chern-Simons term.

Assuming these new symmetries now gives enough additional structure to be able to define Weyl-Papapetrou coordinates in the DOC [53, 54]. Denoting the KVFs collectively as $\xi_A = (k, m_i)$ where $A = 0, \dots, D-3$, we can then define coordinates x^A such that $\partial_{x^A} = \xi_A$. Now consider the subbundle of the tangent bundle spanned by ξ_A. Using the fact that m_i vanish on the "axes" of symmetry and the compatibility of the matter with the symmetries, one can show that

$$
\xi_0 \wedge \dots \wedge \xi_{D-3} \wedge \mathrm{d}\xi_A = 0,
\tag{1.10}
$$

for all A, where by abuse of notation we've used ξ_A to stand in for its covector metric dual. Frobenius' theorem then implies that this subbundle is actually integrable (in four dimensions this is the circularity theorem previously mentioned [30, 31, 32]). The upshot of this is that one can use these x^A coordinates to split the metric into a $D-2$-dimensional and a 2-dimensional block.

We can then use g_{AB}, the metric with respect to x^A, to define the remaining coordinates. We introduce a new coordinate ρ in the DOC, away from the axes, through

$$\rho := \sqrt{-\det g_{AB}} > 0, \tag{1.11}$$

(which one can show to be well-defined [55]). This coordinate turns out to be harmonic upon using the Einstein equations (since the energy-momentum tensor is traceless) and so one can define its harmonic conjugate z according to $\mathrm{d}\rho = \star_2 \mathrm{d}z$ where $\star_2$ denotes the Hodge dual on the 2-space orthogonal to the space of KVFs. Then finally the metric (in the DOC) can be written in terms of these global coordinates as

$$g = g_{AB}(\rho, z)\mathrm{d}x^A \mathrm{d}x^B + e^{2\nu(\rho,z)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) \tag{1.12}$$

where, as with g_{AB}, the conformal factor $e^{2\nu}$ is independent of the Killing coordinates x^A.

These coordinates are a crucial tool in understanding the structure of D-dimensional spacetimes in this multi-axisymmetric class, just as they are in the 4d case. Notice that our symmetry assumptions mean that the isometry group must have $G = \mathbb{R} \times U(1)^{D-3}$ as a subgroup. Therefore one can define the orbit space $\hat{M} := \langle\langle M \rangle\rangle / G$, where we are quotienting the DOC out by the orbits of the KVFs. Considering a vacuum spacetime for simplicity (similar results hold for the Einstein-Maxwell and the five-dimensional MSG cases), the following theorem applies [56]:

Theorem 2. *Consider a $D = 4, 5$, AF, stationary vacuum spacetime (M, g) with $D - 2$ compatible commuting axial KVFs. Then the orbit space $\hat{M}$ is a manifold with boundaries and corners homeomorphic to the upper half plane given by the Weyl-Papapetrou coordinates (ρ, z). The $\rho = 0$ boundary is divided into intervals or "rods" on which g_{AB} has nullity 1, and points (which represent the corners) on which g_{AB} has nullity 2. These rods correspond to either (orbit spaces of) horizons or "axes" of symmetry where integer linear combinations of the axial KVFs vanish.*

Each axis rod therefore has an associated "rod vector" $v = v^i m_i$, giving the associated vanishing KVF, with coprime integer components v^i. For $D = 4$, the only choice for the rod vectors is $v = m$, the only axial KVF. For $D = 5$ one must also impose a so-called admissibility condition on the rod vectors of neighbouring axis rods as follows:

$$\det \begin{pmatrix} v_a^1 & v_a^2 \\ v_{a+1}^1 & v_{a+1}^2 \end{pmatrix} = \pm 1, \tag{1.13}$$

where the a and $a+1$ subscripts denote the rod vectors corresponding to the ath and $(a+1)$th rods - we will denote these by I_a and I_{a+1}. This condition is necessary to avoid potential orbifold singularities on these rods. Rods can either be infinite, semi-infinite or finite in which case they have a coordinate-independent length. The data encoded by the rods including their rod vectors and these rod lengths are collectively referred to as the *rod structure*.

For $D = 4$ the set of rod vectors simply tell us about the number of black hole horizons since regular solutions cannot have two horizon rods next to each other and all axis rod vectors are trivial. For $D = 5$ on the other hand, rod structures can be much more complicated since it is possible to have two adjacent axis rods with differing rod vectors, corresponding to non-trivial 2-cycles in the DOC. There are a couple of other specific ways that rod structures in five dimensions encode topology. First, the rod vectors for the semi-infinite rods on the right and left (i.e. as $z \to \pm\infty$) for an AF spacetime must agree with those for flat space. Up to certain choices this means that the left and right rods must be axis rods with rod vectors m_1 and m_2 (we consider concrete examples in the next section which will make this more explicit). Second, the rod vectors for the rods either side of a horizon rod tell us about the topology of that horizon [56]. Let p be equal to the determinant formed by these rod vectors, so if I_a is a horizon rod corresponding to a horizon with cross section B, then $p = \det(v_{a-1}, v_{a+1})$. If $p = \pm 1$ then $B \cong S^3$, a spherical horizon. However for $p \neq \pm 1$, we instead have $B \cong L(p,q)$ for some integer q where $L(p,q)$ is a lens space. This topological space can be described in the following way:

$$L(p,q) = \begin{cases} S^1 \times S^2, & p = 0 \\ S^3/\sim, & \text{otherwise} \end{cases} \tag{1.14}$$

where $\sim$ is given by $(w_1, w_2) \sim (e^{2\pi i/p} w_1, e^{2\pi i q/p} w_2)$ where $(w_1, w_2) \in \{(w_1, w_2) \in \mathbb{C}^2 : |w_1|^2 + |w_2|^2 = 1\} \cong S^3$. These exhaust all the possible horizon topologies, representing a refinement of the results of the horizon topology theorem discussed previously.

The final step in proving a uniqueness theorem for solutions in this class is to determine a divergence identity in line with the $D = 4$ method [34]. As we discussed, the crucial property allowing such an identity to be found is some kind of hidden symmetry in the theory being considered. More precisely we require that one can reformulate the theory as a harmonic map (also known as a sigma model) coupled to gravity with a symmetric target space X that has non-positive sectional curvature. If this holds for a theory then there exists a map $\Phi : \hat{M} \to X$ which must obey

$$\mathrm{d}(\rho \star_2 \mathrm{d}\Phi\Phi^{-1}) = 0, \tag{1.15}$$

where $\star_2$ is the Hodge dual defined on $\hat{M}$ and Φ^{-1} represents the inverse of Φ in X. The hidden symmetries of the theory can then be seen to arise from symmetries of the target space X. More explicitly one can rewrite X as a coset space G/H and these symmetries correspond to the action of G on the harmonic map equations above. For the vacuum theory in D dimensions the target space has an $SL(D-2)/SO(D-2)$ coset space structure [57, 58]. For MSG in five dimensions the corresponding coset space is $G_{2(2)}/SO(4)$ [59] where $G_{2(2)}$ is the non-compact real form of the group G_2. We will present the uniqueness results for these two cases in the next two sections. On the other hand it is not known how to reformulate Einstein-Maxwell theory in this way for $D \geq 5$, although we note that for $D = 4$ this can in fact be done with coset space $SU(2,1)/S(U(1) \times U(2))$ [60] - this is the basis of the four-dimensional electrovacuum uniqueness theorem. See [61] for a review of the details of these harmonic maps. Once we have this coset space reformulation, an expression known as Mazur's identity [62] applies, a divergence identity comparing two solutions to the harmonic map equations. Using Stokes' theorem one can show that if these two solutions are in sufficient agreement at "infinity" and on the axis (we will define this more precisely when we state the uniqueness theorems) then Mazur's identity implies that they must in fact be isometric. These uniqueness results can be generalised to target spaces which aren't necessarily symmetric spaces, although they must still have non-positive sectional curvature [63].

Finally it is worth remarking on the link between these hidden symmetries and integrability. In this context a theory of gravity being integrable means that the equations of the theory can be rewritten as the integrability conditions of a pair of linear PDEs. This is difficult to arrange in general, however once the equations have already been written in the form of a harmonic map (1.15), it becomes automatic. The way to do this is using the Belinski-Zakharov (BZ) equations [64, 65][2]

$$\partial_z \Psi = \frac{\rho V - \mu U}{\mu^2 + \rho^2} \Psi, \qquad \partial_\rho \Psi = \frac{\rho U + \mu V}{\mu^2 + \rho^2} \Psi, \tag{1.16}$$

where $\rho \mathrm{d}\Phi\Phi^{-1} = U \mathrm{d}\rho + V \mathrm{d}z$, Ψ is a complex matrix and $k = z + (\mu^2 - \rho^2)/(2\mu)$ is a spectral parameter which defines $\mu(k)$ on a 2-sheeted Riemann surface.

[2]Note that we use a slightly modified spectral parameter k as compared with the original BZ equations, as in [36].

One of the uses of this integrable structure is in applying the inverse scattering method to gravity [64, 65, 66]. This is a method adapted from classical integrability theory which allows one to take simple (possibly singular) seed solutions and transform them into much more complicated solutions using a particular soliton ansatz. Note that this is more powerful than just using the internal hidden symmetries of the theory and indeed as we shall see all known vacuum solutions in five dimensions can be constructed in this way. We shall use hidden symmetries in Einstein-Maxwell gravity in Chapter 2 and exploit this integrability structure in Chapters 3, 4 and 5.

1.3 Five-dimensional vacuum gravity

We now consider vacuum gravity in four and five dimensions. By following the procedure outlined in the previous section one can prove the following uniqueness theorem [56, 67]:

Theorem 3. *Consider the class of $D = 4, 5$ AF, stationary, vacuum spacetimes with 2 commuting axial KVFs that contain a non-degenerate[3] event horizon. There is at most one solution in this class for a given rod structure and set of horizon angular momenta.*

In $D = 4$, the horizon angular momentum associated with a horizon H is given by (1.6) with H replacing the S^2_∞ as the integration surface. For $D = 5$ the horizon angular momentum is instead given by

$$J_i = \frac{1}{16\pi} \int_H \star \mathrm{d} m_i. \tag{1.17}$$

For $D = 4$ and at most a single horizon component, the rod structure essentially just encodes the mass of the black hole (through the rod length of the horizon rod). Therefore combining this with the existence of the Kerr solution recovers the no-hair theorem (Theorem 1). For $D = 5$ this theorem provides an extension of this uniqueness results, where now additional metric and topological information in the form of the rod structure must be combined with the asymptotic charges to give a unique solution. We will shortly consider the known solutions in five dimensions and see explicitly that this extra information in the form of rod structure really is necessary to distinguish different spacetimes.

It is worth emphasising that this uniqueness theorem is non-constructive and is only really helpful *if* a particular solution exists; a separate analysis is needed to answer the existence problem. Earlier in this chapter we discussed a result by Weinstein [35] which partially answers this question in the four-dimensional case with possibly disconnected horizons. This result was extended to five dimensions by Khuri, Weinstein and Yamada [69]. In order to state this we must first define a technical condition on the rod structure required in five dimensions. This *compatiblity* condition states that if there are three consecutive axis rods I_{a-1}, I_a, I_{a+1}, then

$$v^1_{a-1} v^1_{a+1} \leq 0 \, , \tag{1.18}$$

whenever the admissibility condition (1.13) between the pairs I_{a-1}, I_a and I_a, I_{a+1} are obeyed with positive determinant. We also define an *unbalanced* solution, as a solution which is regular everywhere except for possible conical singularities on the axis rods. Then the five-dimensional existence result can be stated as:

[3] An analogous theorem can be established for degenerate horizons, i.e. for extreme black holes in this class [68]

Theorem 4. *Consider the class of solutions as in Theorem 3. Then for any admissible rod structure (obeying the compatibility condition (1.18) if $D = 5$) and set of horizon angular momenta, there exists exactly one unbalanced solution*[4] *in this class.*

This theorem does not settle the classification of *regular* solutions, though it does greatly simplify the problem reducing it to a regularity analysis near and on the axis.

We can illustrate this theorem by considering Weyl solutions in five dimensions. These are static solutions with $D-3$ orthogonal axial KVFs. If we were just to restrict to regular spacetimes then the only Weyl solution would be the five-dimensional Schwarzschild-Tangherlini solution ((1.9) with $D = 5, q = 0$) by the vacuum static uniqueness theorem. However if we allow for possible conical singularities then the space of solutions is much wider; for any given rod structure compatible with the admissibility condition (1.13) and the Weyl class, one can construct a unique solution in terms of harmonic functions on $\mathbb{R}^3$ [53]. These are precisely the solutions guaranteed by the theorem above. Although these solutions are not regular they are still useful as simple seed solutions for the inverse scattering method and we shall use them for something similar in Chapter 2 in the context of generating solutions in Einstein-Maxwell gravity.

Let us now consider the dimension of the moduli space of solutions in Theorem 4. Denote the moduli space of unbalanced solutions with h horizon rods and a finite axis rods by $\mathcal{M}^{h,a}_{\mathsf{sing}}$. The continuous parameters in Theorem 4 are given by $h + a$ rod lengths and $(D-3)h$ horizon angular momenta, and so one finds that

$$\dim \mathcal{M}^{h,a}_{\mathsf{sing}} = (D-2)h + a. \tag{1.19}$$

From experience with the known solutions one expects that removal of the conical singularities on each finite axis rod reduces the number of parameters by one, thus reducing the total by a. Hence, a natural conjecture, which agrees with the known solutions, is that provided regular solutions actually exist, the dimension of the moduli space of regular solutions $\mathcal{M}^{h,a}_{\mathsf{reg}}$ with h horizon rods and a finite axis rods is

$$\dim \mathcal{M}^{h,a}_{\mathsf{reg}} \stackrel{?}{=} (D-2)h. \tag{1.20}$$

1.3.1 Known solutions

We now give a list of the known stationary, biaxisymmetric, non-degenerate, AF solutions in the five-dimensional vacuum theory, noting that all of these solutions barring the first two were originally found using the inverse scattering method. Recall that given a horizon rod, the topology of the horizon B is determined by p, the determinant of the rod vectors associated to the rods either side. If $|p| = 1$ then $B \cong S^3$, if $p = 0$ then $B \cong S^2 \times S^1$, otherwise $B \cong L(p, q)$, a lens space (for some integer q).

(5d) Flat space: The metric for flat space (using Hopf coordinates on the S^3) can be written as

$$g = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}x^1)^2 + \cos^2\theta(\mathrm{d}x^2)^2), \tag{1.21}$$

where $r > 0$, $0 < \theta < \pi/2$ and the x^i have periods of 2π. We can write this in terms of Weyl-Papapetrou coordinates by using the definition of ρ (1.11) and the fact that the (ρ, z) part of the metric is conformally flat. Doing this one finds that

$$\rho = \frac{1}{2}r^2\sin(2\theta), \qquad z = \frac{1}{2}r^2\cos(2\theta), \tag{1.22}$$

[4] Technically there is also a potential problem over whether the metric is smooth and even in ρ up to the axis, however this is resolved for $D = 4$ [70] and it is believed that these results are applicable to the $D = 5$ case as well.

and so we can write the flat space metric as

$$g = \frac{\mu}{\rho^2 + \mu^2}(\mathrm{d}\rho^2 + \mathrm{d}z^2) - \mathrm{d}t^2 + \mu(\mathrm{d}x^1)^2 + \rho^2\mu^{-1}(\mathrm{d}x^2)^2, \\ \mu := \sqrt{\rho^2 + z^2} - z. \tag{1.23}$$

To analyse the rod structure we consider the axis $\rho = 0$. There are two rods: either $z < 0$ in which case $\partial_2 = 0$ or $z > 0$ in which case $\partial_1 = 0$. Note that there is a corner between these two rods at $z = 0$ where both these vectors vanish.

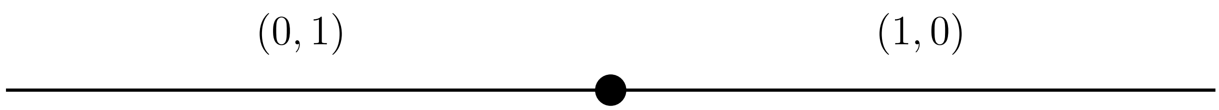

Figure 1.2: The rod structure for flat space in five dimensions. The vectors $(0, 1)$ and $(1, 0)$ give the rod vectors for the left and right rod in the m_i basis.

Figure 1.2 is a diagram representing this rod structure. All the rod structures we will consider have the same left and right semi-infinite rods as flat space, a reflection of the fact that we are considering AF spacetimes.

Myers-Perry: The analogue of the Kerr black hole (S^2 topology horizon) in D dimensions is known as the Myers-Perry spacetime (S^{D-2} topology horizon) and was originally derived using a Kerr-Schild ansatz [71]. The rod structure for the five-dimensional Myers-Perry solution is given in Figure 1.3. The determinant $|p| = 1$ and so we recover the fact that the horizon is topologically a 3-sphere.

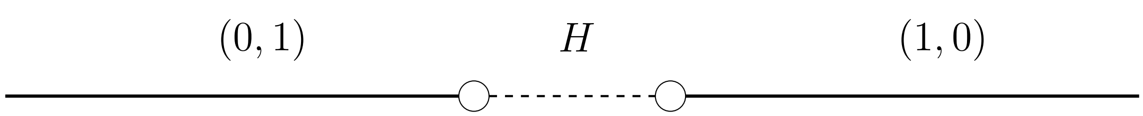

Figure 1.3: The rod structure for the five-dimensional Myers-Perry spacetime. H represents the orbit space of the horizon

Black ring: The black ring spacetime contains a black hole with $S^2 \times S^1$ horizon topology. This was the first known example of an AF, vacuum solution with non-spherical horizon topology. The singly spinning black ring was originally derived by Emparan and Reall [52] as a Wick-rotation of a C-type metric [72, 73] and the generic doubly spinning case was then derived by Pomeransky and Sen'kov [74, 75] using the inverse scattering method. We know that regular static black rings cannot exist due to the static uniqueness theorem and indeed there is a lower limit on the angular momentum of black rings - physically, one can think of this rotation as necessary to support the topology of the horizon. We give the rod structure in Figure 1.4 and the fact that the determinant $p = 0$ illustrates that the horizon has $S^2 \times S^1$ topology as stated.

These solutions provide the simplest counterexample to the generalisation of the no-hair theorem to higher dimensions - there exist black rings with the same mass and angular momentum as Myers-Perry black holes which are clearly not isometric. Moreover there is 2-fold non-uniqueness within the class of black rings themselves. So-called "fat" and "thin" black rings (which are not isometric) exist for the some overlapping region of M, J values, demonstrating that horizon topology is insufficient extra data in addition to the asymptotic charges to give uniqueness. Note these two classes are not required to be isometric from the uniqueness theorem stated above since they have different rod lengths for the finite axis rod.

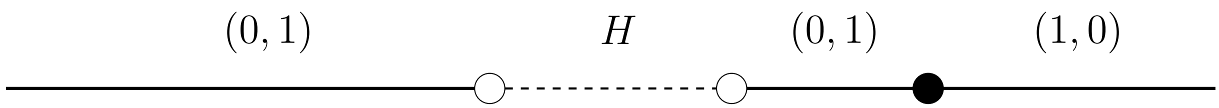

Figure 1.4: The rod structure for the black ring spacetime.

Black Saturn: The black Saturn [76] is a multi-black hole spacetime containing an S^3 black hole and an $S^2 \times S^1$ black ring. The known solution only has non-zero angular momentum in a single direction and so is expected to be non-generic. The rod structure is given in Figure 1.5.

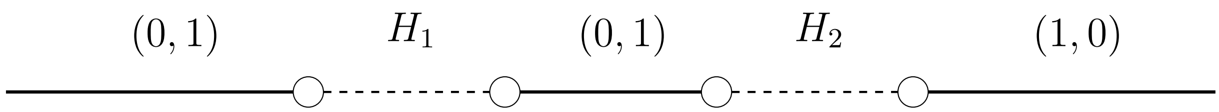

Figure 1.5: The rod structure for the black Saturn spacetime. H_1 corresponds to the black ring type black hole and H_2 corresponds to the S^3 black hole.

An interesting property of the black Saturn is that by counter balancing the rotation of the two black holes, one can arrange the total momentum to be zero. This means that this is not a unique property of the five-dimensional Schwarzschild solution as might otherwise be expected. Note on the other hand that for a single horizon $J = 0$ implies that the spacetime is static [68] and so must indeed be given by the five-dimensional Schwarzschild solution in line with the static uniqueness theorem.

Double black rings: There are two known double black ring solutions. Firstly there is the concentric di-ring spacetime consisting of the two rings rotating on the same plane [77, 78], with rod structure given in Figure 1.6. The other spacetime is the orthogonal di-ring spacetime consisting of the two rings balanced in orthogonal planes [79, 80], see Figure 1.7 for the rod structure. These solutions are both singly spinning and so are expected to be special cases of some more general doubly spinning solutions.

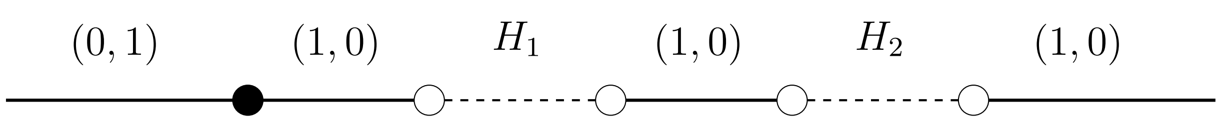

Figure 1.6: The rod structure for the concentric di-ring spacetime.

These spacetimes represent the extent of knowledge of exact solutions in the stationary vacuum class we are considering. Note that these are in fact all the known solutions in this class even without the $U(1)^2$ axial symmetry assumption. As mentioned before the rigidity theorem only guarantees a single $U(1)$ symmetry so these are expected to be non-generic solutions in the more general class. It is also interesting to note that the topologies of the black hole horizons in the spacetimes above are either S^3

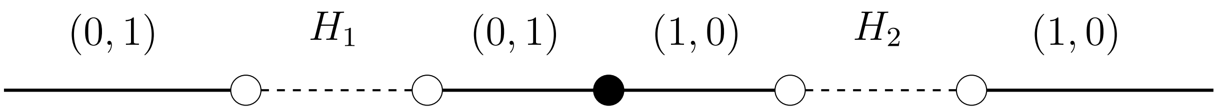

Figure 1.7: The rod structure for the orthogonal di-ring spacetime.

or $S^2 \times S^1$. Black holes with lens space horizon topologies have yet to be constructed in the vacuum theory, although as outlined previously, their existence still remains a possibility (such solutions can be constructed in MSG as we shall see shortly). We address the question of existence for a black lens with the simplest possible rod structure (see Figure 1.8) in Chapter 4.

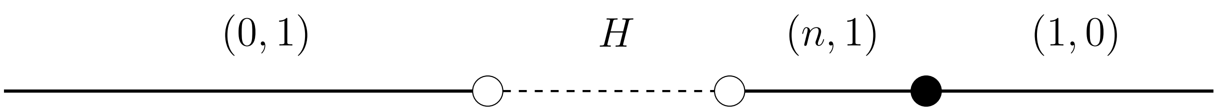

Figure 1.8: The rod structure for a black lens spacetime with horizon topology $L(n, 1)$ where $n \in \mathbb{Z}$.

Before we move on, it is interesting to consider the stability properties of some of these black hole solutions mentioned, although we will not consider these issues for the remainder of this thesis. The four-dimensional Kerr solution is thought to be stable against arbitrary perturbations (see [81, 82, 83] for recent progress), a result which seems to extend to the five-dimensional Myers-Perry solution (however note that higher-dimensional Myers-Perry solutions are in fact unstable [84, 85]). The black ring solution on the other hand is known to suffer a variety of instabilities; depending on the relative size of the S^2 and the S^1 of the ring, they possess Gregory-Laflamme type instabilities [86, 87], instabilities from radial perturbations [87, 88] or so-called "elastic" instabilities from non-axisymmetric perturbations [89]. Less is known about the instabilities of the black Saturn and di-ring solutions, although it is reasonable to expect that they too are unstable due to their black ring components, at least in the regime where there is minimal interaction between their constituent black holes.

1.4 Five-dimensional minimal supergravity

We now consider minimal $\mathcal{N} = 1$ supergravity in five dimensions, constructed in [90]. Note that this theory can be found through dimensional reduction of 11-dimensional supergravity. The action for the bosonic sector can be written as

$$S = \int R \star 1 - 2F \wedge \star F - \frac{8}{3\sqrt{3}} F \wedge F \wedge A, \tag{1.24}$$

with $F = \mathrm{d}A$. The first two terms in this action give the action for Einstein-Maxwell gravity, while the third is the five-dimensional Chern-Simons term. In the purely electric ($A_i = 0$) or magnetic ($A_0 = 0$) case, the Chern-Simons term is zero and so MSG is equivalent to Einstein-Maxwell gravity.

There is no general static classification known for AF, MSG solutions comparable to the one that exists for Einstein-Maxwell gravity, however certain similar results can be shown in special cases. First there is of course the obvious case when the Chern-Simons term is zero and so the Einstein-Maxwell uniqueness theorem holds. More interesting results are known in the case of supersymmetric solutions; all static solutions in this class must be in the (multi-centred) extreme Reissner–Nordström family [91]. On the other hand, if we just use the weaker assumption that the spacetime is just strictly stationary rather than static (i.e. the timelike KVF k is strictly timelike in the DOC), then any spacetime with a connected horizon that doesn't have an $S^2 \times S^1$ topology must be given by the BMPV solution - this is the supersymmetric limit of the charged Myers-Perry black hole, as we will see shortly. As mentioned previously, we note that the black hole horizon is non-rotating despite the spacetime not being static.

The results discussed above for non-rotating black holes did not require the $U(1)^2$ rotational symmetry to hold. If we now impose biaxisymmetry and consider rotating solutions, then strong classification results analogous to the vacuum case are known. This is ultimately because one can view the equations in this theory as arising from a harmonic map to a symmetric space [59], just as in the vacuum case. The uniqueness theorem in MSG extending Theorem 3 is given by [92, 93, 94, 95, 96, 97]:

Theorem 5. *Consider the class of five-dimensional AF, stationary, spacetimes in MSG with 2 commuting axial KVFs that contain a non-degenerate event horizon. There is at most one solution in this class for a given rod structure, set of horizon angular momenta and electric charges and set of dipole charges associated with each finite axis rod.*

The electric charge $\mathcal{Q}$ associated to a horizon H is given by

$$\mathcal{Q} = \frac{1}{4\pi}\int_H \star F - \frac{2}{\sqrt{3}} F \wedge A, \tag{1.25}$$

where the integrand is closed by the equations given by varying the MSG action (1.24) with respect to the gauge field A. The dipole charge $\mathcal{D}$ associated to axis rod I is given by

$$\mathcal{D} = \frac{1}{4\pi}\int_C F, \tag{1.26}$$

where C is the $2-$cycle corresponding to the orbit under the axial KVFs of I in M.

We stress that this gives a non-constructive uniqueness result, and so one must rely on other methods to determine whether there are actually any solutions for given uniqueness data. As in the vacuum case, an important existence result has been established for MSG which represents significant progress towards establishing a full classification of solutions in this class. We first recall that an unbalanced solution is a solution which is regular everywhere except for potential conical singularities on the axis. Then the theorem (which extends Theorem 4) can be stated as [97]:

Theorem 6. *Consider the class of solutions as in Theorem 5. Then for any admissible and compatible (1.18) rod structure, set of horizon angular momenta and electric charges and set of dipole charges associated with each finite axis rod, there exists exactly one unbalanced solution in this class.*

Let us now consider the dimension of the moduli space of solutions in Theorem 6. Consider the moduli space of unbalanced solutions $\mathcal{M}^{h,a}_{\mathsf{sing}}$ with h horizon rods and a finite axis rods. The continuous parameters in Theorem 6 are given by $h+a$ rod lengths, $2h$ horizon angular momenta, h horizon electric charges and a axis dipole charges and so one finds that

$$\dim \mathcal{M}^{h,a}_{\mathsf{sing}} = 4h + 2a. \tag{1.27}$$

As previously mentioned, experience with the known solutions suggests that removal of the conical singularities on each finite axis rod reduces the number of parameters by one, thus reducing the total by a. Hence, a natural conjecture, which agrees with the known solutions, is that provided regular solutions actually exist, the dimension of the moduli space of regular solutions $\mathcal{M}^{h,a}_{\mathsf{reg}}$ with h horizon rods and a finite axis rods is simply

$$\dim \mathcal{M}^{h,a}_{\mathsf{reg}} \stackrel{?}{=} 4h + a. \tag{1.28}$$

1.4.1 Known solutions

We now list of the known stationary, biaxisymmetric, AF solutions in five-dimensional MSG. There is of course some overlap with the vacuum solutions given in the previous section in the vacuum limit of these MSG solutions. It is also worth bearing in mind that the multi-black hole vacuum solutions can't be found as limits of any of the charged MSG solutions listed below (although multi-black hole supersymmetric do indeed exist).

Charged Myers-Perry: The charged Myers-Perry solution is an MSG solution with the same rod structure as the Myers-Perry solution (see Figure 1.3). From (1.28), one expects the most general family in this class to have 4 parameters and indeed such a family has been derived by Cvetič and Youm [98] (see also [99]). This has a limit to the extreme Reissner–Nordström solution when $J_i = 0$ and to the supersymmetric BMPV solution [20] when $J_1 = J_2$ and $M = (\sqrt{3}/2)|Q|$.

Charged black ring: The charged black ring is an MSG solution with the same rod structure as the black ring in the vacuum (see Figure 1.4). From (1.28), one expects the most general family in this class to have 5 parameters and such solutions have been constructed by Feldman and Pomeransky [100] [5]. Various special cases were previously constructed probing different lower dimensional sections of the full moduli space [101, 102, 103, 104], including a 3-parameter supersymmetric solution [105].

Supersymmetric solutions: Supersymmetric solutions in this context are solutions admitting a Killing spinor (which is also symplectic and Majorana). An important consequence of this is that the metric then must be given by a fibration over a 4d hyperkähler base space [106]. Combining this with global constraints coming from the stationary and $U(1)^2$ symmetry assumptions, one can show that this base space must in fact be of multi-centred Gibbons-Hawking form [107]. This allows for a constructive classification of these supersymmetric solutions (in the class we are considering) for arbitrary rod structures. Various black hole solutions were constructed before this classification was known: the BMPV [20] and black ring solution [105] mentioned above; $L(n, 1)$ black lenses [108, 109] (with rod structure given by Figure 1.8 up to unimportant choices); "bubbling" black hole spacetimes with non-trivial topology in the DOC (2-cycles) [110, 111]; various multi-black hole solutions [112, 113, 114].

The classification also allows for counting the number of continuous parameters of these supersymmetric solutions. The (conjectured) dimension of the moduli space of supersymmetric solutions $\mathcal{M}^{h,a}_{\text{susy}}$ based on considering the constraints appearing in the construction of solutions is given by [107]

$$\dim \mathcal{M}^{h,a}_{\text{susy}} \stackrel{?}{=} 2h + a. \tag{1.29}$$

Intuitively this makes sense, since for each horizon one must take the supersymmetric and extremal limit of the general solution, thus reducing the number of parameter by $2h$ in total (compare above with (1.28)). In any case this result agrees with all the known solutions.

Solitons: In this context soliton spacetimes are spacetimes with no event horizon (and therefore no horizon rods). Whilst for the vacuum theory it is well-known that smooth soliton solutions cannot exist (a simple proof follows from combining Stokes' theorem and the positive mass theorem), this result no longer holds in MSG. The only known non-supersymmetric soliton (see [115, 116]) has a single axis rod with rod structure given by Figure 1.9 (up to irrelevant choices). On the other hand, all supersymmetric

[5] This is actually a black ring solution in the more general $U(1)^3$ supergravity theory.

solitons have been constructed [117] and form part of the classification mentioned above [107], thereby proving that the number of axis rods a must be even for there to be supersymmetry. Note also that when the number of horizon rods $h = 0$, dim $\mathcal{M}^{0,a}_{\text{susy}} =$ dim $\mathcal{M}^{0,a}_{\text{reg}}$ (compare (1.28) and (1.29)). Therefore any solitons (that exist) with a odd must be non-supersymmetric and it is natural to expect that all solitons with a even are supersymmetric.

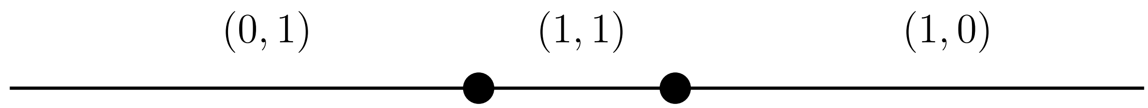

Figure 1.9: The rod structure for the known non-supersymmetric soliton

As with the known solutions in vacuum gravity there are some interesting gaps in the solutions here. For example, there are a wealth of solutions coming from the supersymmetric classification, almost all without charged non-supersymmetric counterparts. In a similar way to the vacuum we also see that there are no known (non-degenerate) black hole horizons with lens space topology. It is finally worth noting that all the solutions listed above have a $U(1)^2$ rotational symmetry. In contrast with the vacuum case, a family of solutions in MSG are known which do not posses this symmetry: these are the supersymmetric multi-BMPV black holes first constructed in [21]. In particular whilst the BMPV black hole has at least an $SU(2) \times U(1)$ rotational symmetry (of which $U(1)^2$ is a subgroup), the multi-BMPV solution has at most an $SO(3)$ symmetry (which doesn't have a $U(1)^2$ subgroup). It is worth bearing in mind however, that these multi-BMPV solutions are not smooth at the horizon, with the metric being at best C^1 and the Maxwell field C^0 [118].

1.5 Organisation

This thesis will be split into chapters as follows:

In Chapter 2 we begin by considering multi-axisymmetric, stationary solutions in D-dimensional Einstein-Maxwell gravity. We express the Einstein-Maxwell equations in two different ways by considering two possible reductions of the D-dimensional theory to a 3-dimensional base space. In the five-dimensional electrostatic case, $SL(2, \mathbb{R})$ symmetries become apparent which can then be used to either charge a black hole or immerse it in a background electric field depending on the choice of reduction. We use these transformations to generate regular solutions from neutral, static black Saturn and black lens seeds.

In Chapter 3 we consider AF, biaxisymmetric, stationary solutions in five-dimensional vacuum gravity. Taking advantage of the integrable structure we show that it is possible to derive complete metric information on the axis ($\rho = 0$) in terms of rational functions of z and various moduli. There are various consistency conditions on the moduli which lead to a system of polynomial equations and inequalities. We conjecture that these constraints fully classify the space of solutions in this theory.

In Chapter 4 we apply the method from the previous chapter to a number of examples. We rederive the moduli space of a few known solutions. In so doing we provide a proof that the known Pomeransky-Sen'kov black ring is indeed the most general solution for its rod structure. We also analyse a the rod structure corresponding to the simplest black lens. Through a combination of analytic and numerical methods we demonstrate that there is no regular solution corresponding to this rod structure.

In Chapter 5 we extend the method developed in Chapter 3 to AF, biaxisymmetric, stationary solutions in five-dimensional minimal supergravity, another theory admitting a description in terms of a harmonic map to a symmetric space. The results follow through in a similar way, although the equations on the moduli that one derives are significantly more complicated than in the vacuum setting.

Chapter 2

Electrostatic Solutions in a Melvin Background

In this chapter we consider five-dimensional electrostatic solutions to Einstein-Maxwell gravity with 2 commuting spacelike Killing fields. We take two different reductions to 3 dimensions, corresponding to reducing over either an $\mathbb{R} \times U(1)$ or a $U(1)^2$ subgroup of the isometry group. Using appropriately defined potentials we can write out the Einstein-Maxwell equations in both cases as PDEs for some axially symmetric functions on $\mathbb{R}^3$. By considering the isometries of the target space given by these equations we find a hidden $SL(2,\mathbb{R})$ symmetry group for each reduction.

The two different choices of reduction then give rise to two non-trivial 1-parameter families of transformations corresponding to either charging a black hole or immersing it in a background electric field. Using these transformations we charge a static black Saturn and a static $L(n,1)$ black lens spacetime and by tuning the strength of the external field, we cure the conical singularities to give new regular solutions. Notably the electrified black lens generated is the first example of a regular black lens in Einstein-Maxwell gravity with topologically trivial asymptotics[1].

2.1 Introduction

We now consider black hole solutions in Einstein-Maxwell gravity in D dimensions. Complete classification results are known in a couple of special cases: static, AF solutions must be isometric to the Reissner-Nordström solution [49, 50, 51] and in four dimensions the general stationary, axisymmetric, AF solution is known to be the Kerr-Newman solution (see Theorem 1). However considering either non-static or higher-dimensional spacetimes, the situation is less clear. In this chapter we will initially consider stationary, multi-axisymmetric, AF solutions in Einstein-Maxwell gravity. By considering a couple of natural reductions over the axial and stationary KVFs to 3 dimensions, we present a convenient description of the Einstein-Maxwell equations in this theory (this mirrors some of the work in [119]). It is notable that even with our restrictive assumptions, not a lot is known about this theory, in stark contrast to the corresponding sectors of vacuum gravity and minimal supergravity which are governed by powerful uniqueness theorems (see Theorems 3 and 5). The key to these uniqueness theorems is the fact that the reduction to 3 dimensions yields a gravitating harmonic map to a coset space - there is no such interpretation of the reduced equations in the Einstein-Maxwell case.

Along with these uniqueness theorems, another key consequence of this harmonic map description

[1] The results of this chapter have been written up in [1] with an additional appendix, not presented here, giving details of charging general Weyl solutions

is the existence of hidden symmetries. For example if the harmonic map is to a G/H coset space, then the isometries of this target space are given by G which in turn corresponds to symmetries of the original equations (see [61, 120] for reviews). The simplest example of these symmetries appears in stationary, axisymmetric, vacuum solutions in 4 dimensions where an Ernst system arises which has an $SL(2,\mathbb{R})$ symmetry [57]. To see this from the perspective of the harmonic map, one reduces to 3 dimensions by quotienting out the orbits of the $U(1)$ symmetry corresponding to the axisymmetric Killing vector. Similar results hold in various other theories of gravity: D-dimensional vacuum gravity has an $SL(D-2,\mathbb{R})$ symmetry [58]; 4-dimensional Einstein-Maxwell gravity has an $SU(2,1)$ symmetry [121]; 5-dimensional minimal supergravity has a $G_{2(2)}$ symmetry [59]; 11-dimensional supergravity has an $E_{8(8)}$ symmetry [122, 123]. Einstein-Maxwell gravity for $D > 4$ does not have this coset space harmonic map interpretation (as mentioned above) and therefore the hidden symmetries of the theory are obscured. However it is important to note that if we consider purely electric spacetimes in five dimensions then the Chern-Simons term vanishes (see (1.24)) and the theory becomes equivalent to minimal supergravity, thereby inheriting some symmetries (and also the uniqueness theorem) of that theory.

Although there are uniqueness results known in five-dimensional, AF vacuum gravity and minimal supergravity, it is still unknown which solutions actually exist (with the exception of supersymmetric solutions in minimal supergravity [107]). For example we know from general theorems that black hole horizons in five dimensions must either have S^3, $S^2 \times S^1$ or $L(p,q)$ (lens space) topology however only S^3 and $S^2 \times S^1$ horizon black holes are known (in the non-extreme case). Part of the motivation for studying Einstein-Maxwell theory in this chapter is as a toy model to understand solutions that haven't been constructed in these theories. To do this we will consider static solutions and then charge them using the hidden symmetries discussed above to give new electrostatic solutions which will preserve the rod structure of the original (see Section 1.2 for full definition of rod structure). These new charged solutions circumvent the static uniqueness theorem since they are no longer AF and are instead embedded in an external electric background. Whilst this breaks asymptotic flatness, it still preserves the asymptotic topology of the metric, i.e. the cross-sections are still topologically S^3 at infinity.

A large class of static vacuum solutions is given by Weyl solutions. These are five-dimensional solutions with 2 orthogonal axial KVFs, that can be trivially constructed out of axially symmetric harmonic functions on $\mathbb{R}^3$ [53]. Clearly, from the static uniqueness theorem, AF solutions in this class must be singular for all but the flat or Schwarzschild case, so they must be transformed somehow to make them regular. One way to do this is by adding rotation, which can be achieved using the inverse scattering method. This is a method based on integrability that takes a seed Weyl solution and uses it to generate a more general solution using a particular "soliton" ansatz [64, 65, 66]. A different approach is to add charge to these solutions to balance them. Again one can do this using inverse scattering [124], however there are other methods developed to do these charging transformations relying explicitly on hidden symmetries of the theory. In 11 dimensions one can charge solutions using the U-duality of supergravity (equivalently the $E_{8(8)}$ symmetry we discussed above), which is also inherited by supergravity theories in lower dimensions through dimensional reduction (see e.g. [104, 125, 126] for applications to minimal supergravity).

In this chapter we will develop charging transformations for biaxisymmetric, electrostatic solutions in five dimensions by using some $SL(2,\mathbb{R})$ hidden symmetry. We apply this transformation to the case of the black Saturn and a simple black lens. AF, charged black Saturns in Einstein-Maxwell gravity have been constructed previously, for example a singular static charged solution [127] and a regular rotating solution with a dipole charge [128]. However no AF black lens solutions have been constructed in Einstein-Maxwell gravity, and the new regular charged solution that we derive gives the first example of a solution that is even asymptotically topologically trivial. The topology of our solution at infinity is an important feature since it is clearly possible to construct any black lens solution if one relaxes this requirement, simply by quotienting a Schwarzschild solution by an appropriate discrete subgroup of $O(4)$.

This chapter is organised as follows: In Section 2.2 we discuss two different reductions from the D-dimensional theory to a 3-dimensional base space. This allows us to write the Einstein-Maxwell equations in a convenient way on this base space for both reductions. In Section 2.3 we specialise to the case of electrostatic solutions in five dimensions. Each of the two choices of reductions from the previous section lead to equations with different apparent symmetries. Exploiting these symmetries, we derive two different 1-parameter families of charging transformations. By combining them we discuss a 2-parameter family of transformations which charge a solution and then immerse it in a background electric field; In Section 2.4 we apply this combined transformation to the black Saturn and a $L(n,1)$ black lens solution and, by appropriately tuning the strength of the external electric field cure the conical singularities. We end with a discussion of these results and possible extensions in Section 2.5.

2.2 Background

We begin by considering a D-dimensional spacetime (M, g, F) in Einstein-Maxwell gravity with action (1.1). In addition we assume that the spacetime is AF, and possesses a stationary KVF k and $D-3$ compatible axial KVFs m_i $(i = 1, \ldots, D-3)$, a result of which is that the isometry group has an $G := \mathbb{R} \times U(1)^{D-3}$ subgroup. We saw in Section 1.2 that under appropriate assumptions, Weyl-Papapetrou coordinates can be chosen such that the metric can be written as

$$g = g_{AB}(\rho, z)\mathrm{d}x^A\mathrm{d}x^B + e^{2\nu(\rho,z)}(\mathrm{d}\rho^2 + \mathrm{d}z^2), \tag{2.1}$$

where $\partial_A = (k, m_i)$ for $(A = 0, \ldots D-3)$, $\rho^2 := -\det g_{AB}$ and $dz := -\star_2 d\rho$, with $\star_2$ the Hodge dual on the orbit space $\hat{M} := M/G$. Note that since ∂_A are Killing vector fields, the metric coefficients only depend on ρ and z.

The orbit space theorem (Theorem 2) states that the orbit space $\hat{M}$ is a simply connected manifold with boundaries and corners with the boundary given by the $\rho = 0$ axis in Weyl-Papapetrou coordinates. Furthermore the corners (rod points), occurring at specific values of z, divide this boundary up into intervals (rods) corresponding to either axes where integer linear combination of the KVFs vanish or horizon orbit spaces. The presence of finite axis rods is generically associated with conical singularities: for a given axis rod I with rod vector v, there is a conical singularity unless [54]

$$\lim_{\rho\to 0,\, z\in I} \frac{\rho^2 e^{2\nu}}{|v|^2} = 1. \tag{2.2}$$

Note that this assumes that the angles ϕ^i have a standard 2π period. We will solve some of these conditions explicitly in Section 2.4 when we consider black Saturn and black lens solutions.

Instead of looking at the 2-dimensional $\hat{M}$, to better understand the hidden symmetries of this theory we must instead reduce to a 3-dimensional base space which we will denote M_3. There are two obvious ways of doing this using the symmetries available, corresponding to either quotienting out the metric by $U(1)^{D-3}$ or $U(1)^{D-4} \times \mathbb{R}$. For the reduction by $U(1)^{D-3}$ one can write the metric as

$$g = e^{2\nu}(\mathrm{d}\rho^2 + \mathrm{d}z^2) - \gamma^{-1}\rho^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}\phi^i + w^i\mathrm{d}t)(\mathrm{d}\phi^j + w^j\mathrm{d}t), \tag{2.3}$$

where $\gamma = \det \gamma_{ij}$ and we've taken $x^A = (t, \phi^i)$ and so $\partial_0 = k$ and $\partial_i = m_i$.

Next we introduce some potentials as follows: First define the electric and magnetic potentials Φ and Ψ_i by

$$\mathrm{d}\Phi = \iota_1 \cdots \iota_{D-3} \star F, \qquad \mathrm{d}\Psi_i = \iota_i F, \tag{2.4}$$

where $\iota_i := \iota_{m_i}$. Φ and Ψ_i are well-defined (up to constants) through Maxwell's equations and the topological censorship theorem. Next we define the twist 1-forms Ω_i by

$$\Omega_i = \star(m_1 \wedge \cdots \wedge m_{D-3} \wedge \mathrm{d}m_i), \tag{2.5}$$

where by abuse of notation we've used m_i to stand in for their covector metric duals. Since $\iota_j \Omega_i = 0$, these can be viewed as 1-forms on M_3 or more explicitly

$$\Omega_i = |\gamma|^{-1/2} \star_3 \mathrm{d}m_i \tag{2.6}$$

where the m_i are viewed as functions on M_3 and $\star_3$ is the Hodge dual on this space. By further reducing down to $\hat{M}$ (and using the fact that $\iota_0 \Omega_i = 0$), Ω_i can be related to w^i through

$$\Omega_i = \rho^{-1} \gamma \gamma_{ij} \star_2 \mathrm{d}w^j, \tag{2.7}$$

where $\star_2$ is the Hodge dual on $\hat{M}$

Defining the Levi-Civita connection D on M_3, we can now write the Einstein-Maxwell equations for the Killing part of the metric and the potentials as[2]

$$\begin{aligned} \mathrm{D}^2\gamma_{ij} =& \gamma^{kl}\mathrm{D}\gamma_{ik} \cdot \mathrm{D}\gamma_{jl} - \gamma^{-1}\Omega_i \cdot \Omega_j - 4\mathrm{D}\Psi_i \cdot \mathrm{D}\Psi_j \\ &+ \frac{4}{D-2}\gamma_{ij}(\gamma^{kl}\mathrm{D}\Psi_k \cdot \mathrm{D}\Psi_l - \gamma^{-1}\mathrm{D}\Phi \cdot \mathrm{D}\Phi), \end{aligned} \tag{2.8}$$

$$\mathrm{D} \cdot \Omega_i = \gamma^{-1}\mathrm{D}\gamma \cdot \Omega_i + \gamma^{jk}\mathrm{D}\gamma_{ij} \cdot \Omega_k, \tag{2.9}$$

$$\mathrm{D}^2\Phi = \gamma^{-1}\mathrm{D}\gamma \cdot \mathrm{D}\Phi + \gamma^{ij}\mathrm{D}\Psi_i \cdot \Omega_j, \tag{2.10}$$

$$\mathrm{D}^2\Psi_i = \gamma^{jk}\mathrm{D}\gamma_{ij} \cdot \mathrm{D}\Psi_k - \gamma^{-1}\mathrm{D}\Phi \cdot \Omega_i, \tag{2.11}$$

and

$$\mathrm{d}\Omega_i = 4\mathrm{d}\Phi \wedge \mathrm{d}\Psi_i. \tag{2.12}$$

At first glance it may appear as though these equations involve the conformal factor $e^{2\nu}$ through the inner product and connection on M_3, however this turns out not to be the case. This is due to the fact that i) Ω_i and the functions we are considering are invariant under the action of the stationary KVF k and ii) the (ρ, z) part of the metric is conformally flat. Combining these pieces of information we find that the inner product on M_3 acts like the inner product on $\mathbb{R}^3$ in cylindrical polars, up to a conformal factor which can be scaled away in the equations above.

The equations for ν come from gravity on M_3 coupled to γ_{ij} and the various potentials. Reducing this to two dimensions gives a pair of PDEs

$$\frac{1}{\rho}\left(\nu_{,\rho} + \frac{1}{2}\gamma^{-1}\gamma_{,\rho}\right) = X_{\rho\rho} - X_{zz}, \qquad \frac{1}{\rho}\left(\nu_{,z} + \frac{1}{2}\gamma^{-1}\gamma_{,z}\right) = 2X_{\rho z}, \tag{2.13}$$

where

$$X_{pq} = \gamma^{-1}\Phi_{,p}\Phi_{,q} + \gamma^{ij}\Psi_{i,p}\Psi_{j,q} + \frac{1}{8}\gamma^{-2}\gamma_{,p}\gamma_{,q} + \frac{1}{8}\gamma^{ij}\gamma^{kl}\gamma_{ik,p}\gamma_{jl,q} + \frac{1}{4}\gamma^{-1}\gamma^{ij}(\iota_p\Omega_i)(\iota_q\Omega_j), \tag{2.14}$$

[2]These equations are identical to those appearing in [119], with a corrected factor of 2 on terms quadratic in the Maxwell potentials.

and p, q run over ρ and z. The integrability of these equations can be established using (2.8) to (2.12).

We have been considering the reduction over just the axial KVFs i.e. over $U(1)^{D-3}$, however there is another obvious reduction one can perform over the stationary KVF and all but one of the axial KVFs i.e. over $U(1)^{D-4} \times \mathbb{R}$. Without loss of generality we can choose the leftover KVF to correspond to ∂_{D-3}. Then we can write the metric as

$$g = e^{2\nu}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + h_{\mu\nu}(\mathrm{d}x^\mu + y^\mu \mathrm{d}\phi^{D-3})(\mathrm{d}x^\nu + y^\nu \mathrm{d}\phi^{D-3}) - h^{-1}\rho^2(\mathrm{d}\phi^{D-3})^2, \tag{2.15}$$

where $h = \det h_{\mu\nu}$ and $\mu, \nu = 0, \ldots, D-4$. We can define potentials in a similar way to the other reduction:

$$\begin{aligned} \mathrm{d}R &= \iota_0 \cdots \iota_{D-4} \star F, \qquad \mathrm{d}S_\mu = \iota_\mu F, \\ Z_\mu &= \star(k \wedge m_1 \wedge \cdots \wedge m_{D-4} \wedge \mathrm{d}e_\mu), \end{aligned} \tag{2.16}$$

where $e_\mu = (k, m_{i \neq D-3})$. The Einstein-Maxwell equations ((2.8) to (2.14)) then take the same form with $\{h_{\mu\nu}, R, S_\mu, Z_\mu\}$ replacing $\{\gamma_{ij}, \Phi, \Psi_i, \Omega_i\}$.

2.3 Charging transformations

We now set $D = 5$ and consider electrostatic solutions - we will shortly see how this condition can be written in terms of the potentials adapted to each reduction. We can then derive non-trivial transformations between solutions in this class by looking at the symmetries of the Einstein-Maxwell equations described in the previous section.

2.3.1 Charging black holes

We will start with the slightly less natural reduction over $\mathbb{R} \times U(1)$. Under this reduction staticity implies that

$$h_{01} = h_{10} = 0, \qquad Z_0 = 0, \qquad y^0 = 0 \tag{2.17}$$

and a pure electric spacetime must also satisfy

$$R = 0, \qquad S_{\mu \neq 0} = 0. \tag{2.18}$$

For convenience we define $S = S_0$, then the metric can be written

$$g = e^{2\nu}(\mathrm{d}\rho^2 + \mathrm{d}z^2) - e^{2V_0}\mathrm{d}t^2 + e^{2V_1}(\mathrm{d}\phi^1 + y\mathrm{d}\phi^2)^2 + e^{2V_2}(\mathrm{d}\phi^2)^2, \tag{2.19}$$

where $e^{2V_0} = h_{00}$, $e^{2V_1} = h_{11}$, $y = y^1$ and $V_0 + V_1 + V_2 = \ln \rho$.

The Einstein-Maxwell equations for V_0 and S come from (2.8) and (2.11) and are given by

$$\mathrm{D}^2 V_0 = \alpha^{-2} e^{-2V_0}(\mathrm{D}S)^2, \qquad \mathrm{D}^2 S = 2\mathrm{D}V_0 \cdot \mathrm{D}S \tag{2.20}$$

where $\alpha = \frac{\sqrt{3}}{2}$. Note that these two equations only depend on V_0 and S and so can be solved independently to the rest of the equations.

We now consider the target space T, defined by the equations for V_0 and S (2.20). The equations can be viewed as coming from the Lagrangian

$$\mathcal{L} = \mathrm{D}V_0^2 - \alpha^{-2} e^{-2V_0} \mathrm{D}S^2. \tag{2.21}$$

Therefore by defining coordinates $X^{\pm} = e^{V_0} \pm \alpha^{-1} S$, we can write the metric on T as

$$\mathrm{d}s^2 = 4\frac{dX^+ dX^-}{(X^+ + X^-)^2}, \tag{2.22}$$

which we recognise as $\mathbf{AdS}_2$ in lightcone coordinates. It has isometries given by

$$X^{\pm} \to aX^{\pm}, \qquad X^{\pm} \to X^{\pm} \pm b, \qquad X^{\pm} \to \frac{X^{\pm}}{1 \mp cX^{\pm}}, \tag{2.23}$$

for real constants a, b and c, with the KVFs corresponding to these transformations generating the Lie algebra $sl(2, \mathbb{R})$. These are hidden symmetries of the original equations (2.20). The dilation and translation transformations are both trivial, corresponding to rescaling t and gauge transforming S respectively. However, the third transformation is more interesting and can be used to generate a non-trivial 1-parameter family of new solutions given a starting seed solution.

It is convenient when performing the third transformation to simultaneously rescale t and gauge transform S in order to manifestly preserve the asymptotic conditions. Specifically we impose that if $e^{2V_0} \to 1$ and $S \to 0$ at asymptotic infinity for the seed metric then these conditions should hold for the final metric as well. Then the transformation can be written as

$$e^{2V_0} \to e^{2V_0} L^{-2}, \qquad S \to \frac{(1 - c\alpha^{-1}S)(S - \alpha c) + \alpha c e^{2V_0}}{(1 - c\alpha^{-1}S)^2 - c^2 e^{2V_0}}, \tag{2.24}$$

where

$$L = \frac{(1 - c\alpha^{-1}S)^2 - c^2 e^{2V_0}}{1 - c^2}. \tag{2.25}$$

The equations for the other metric components (2.8), (2.13) imply that they transform as

$$e^{2V_i} \to e^{2V_i} L \quad (i = 1, 2), \qquad y \to y, \qquad e^{2\nu} \to e^{2\nu} L. \tag{2.26}$$

Note that the condition $V_0 + V_1 + V_2 = \ln \rho$ is invariant under this transformation.

In order to preserve signature and avoid creating new singularities under this transformation, L must be positive which is satisfied if and only if $-1 < c < 1$. To show that this implies $L > 0$, it is sufficient to show that $X^{\pm} < 1$ [3], a result which is the content of lemma 2 in [51]. We also note that g_{ij} transforms with an overall factor of L meaning that a rod vector of the starting solution is a rod vector of the end solution and so the rod structure is partially preserved.

Since this transformation preserves asymptotic flatness whilst adding an electric field, it can be physically interpreted as adding electric charge into the bulk of the spacetime, or equivalently adding charge to black hole horizons that are present. In fact if one were to apply this to a higher dimensional Schwarzschild black hole, then the transformed metric would be a Reissner–Nordström black hole with charge proportional to c - this is essentially guaranteed by the static uniqueness theorem [50].

2.3.2 Immersing black holes in an electric background

Now we consider the reduction over $U(1)^2$. In this case staticity implies that

$$\Omega_i = 0, \qquad w^i = 0, \tag{2.27}$$

[3]We are using the fact that $e^{2V_0} \to 1$ and $S \to 0$ at asymptotic infinity and we also assume that the mass M and charge Q obey $M > |Q|$.

and a pure electric spacetime must also satisfy

$$\Psi_i = 0. \tag{2.28}$$

The metric can be written in the form

$$g = e^{2\nu}(\mathsf{d}\rho^2 + \mathsf{d}z^2) - \rho^2 e^{-2W}\mathsf{d}t^2 + e^{2W_1}(d\phi^1 + ud\phi^2)^2 + e^{2W_2}(d\phi^2)^2, \tag{2.29}$$

where $e^{2W_1} = \gamma_{11}$, $u = \gamma_{12}\gamma_{11}^{-1}$, $e^{2W_2} = \gamma_{22} - \gamma_{12}^2\gamma_{11}^{-1}$ and $W := W_1 + W_2$.

The Einstein-Maxwell equations for W and Φ (2.8), (2.10) are

$$\mathsf{D}^2 W = -\alpha^{-2}e^{-2W}(\mathsf{D}\Phi)^2, \qquad \mathsf{D}^2\Phi = 2\mathsf{D}W\cdot\mathsf{D}\Phi. \tag{2.30}$$

An almost identical analysis as in the previous section applies to these equations with the only difference being that in this case the target space $T \cong \mathbf{H}^2$, the hyperbolic plane. The isometries of T determine the transformations for W and Ψ as before. The non-trivial 1-parameter family of transformations is given by

$$\begin{gathered} e^{2W} \to e^{2W}M^{-2}, \qquad e^{2W_i} \to e^{2W_i}M^{-1}, \qquad u \to u \\ \Phi \to \left[\Phi(1 + k\alpha^{-1}\Phi) + \alpha k e^{2W}\right]M^{-1}, \qquad e^{2\nu} \to e^{2\nu}M^2 \end{gathered} \tag{2.31}$$

where

$$M = (1 + k\alpha^{-1}\Phi)^2 + k^2 e^{2W} \tag{2.32}$$

and k is a real parameter. Note that the condition $W - W_1 - W_2 = 0$ is invariant under this transformation.

Similarly to the previous transformation, M must be positive to preserve signature and avoid creating new singularities. $M > 0$ follows immediately from the definition and there are no restrictions on k. As before g_{ij} transforms with just an overall factor (M^{-1} in this case) and so the transformed solution has the same rod vectors as the seed.

An important difference between this transformation and the previous one is that now asymptotic flatness can no longer be preserved. In fact this transformation takes asymptotically flat spacetimes to asymptotically Melvin ones. These are spacetimes that asymptotically look like the 5d electric Melvin universe, a spacetime with metric

$$ds^2 = M^2\frac{\mu}{\rho^2+\mu^2}(\mathsf{d}\rho^2 + \mathsf{d}z^2) - M^2\mathsf{d}t^2 + M^{-1}\left(\mu(\mathsf{d}\phi^1)^2 + \frac{\rho^2}{\mu}(\mathsf{d}\phi^2)^2\right), \tag{2.33}$$

where

$$M = 1 + k^2\rho^2, \qquad \mu = \sqrt{\rho^2+z^2} - z, \tag{2.34}$$

and k determines the strength of the electric field. Therefore one can think of this transformation as taking a spacetime and immersing it in an electric background.

2.3.3 Combined transformation

We now consider applying these two transformations consecutively to neutral, static, AF black hole spacetimes. By convention we will take m_2 to be the rod vector for the left semi-infinite rod I_L (i.e. the rod with $z \to -\infty$) and m_1 for the right semi-infinite rod I_R (i.e. the rod with $z \to \infty$). As discussed in the previous section, the first transformation will give the black holes an electric charge and then the second will immerse them in a background electric field. We know (from the static uniqueness theorem) that these static seed solutions will generally have some kind of singularity. If these are conical in nature we will show how tuning the parameters of these transformations might allow one to cure these singularities to give regular solutions.

Consider a static seed metric as in (2.19), with $S = 0$, i.e. a neutral solution. If we charge this solution using the transformation associated to the $\mathbb{R} \times U(1)$ reduction (2.24), (2.26), (2.25), we find that

$$g = e^{2\nu} L(\mathsf{d}\rho^2 + \mathsf{d}z^2) - e^{2V_0} L^{-2}\mathsf{d}t^2 + e^{2V_1} L(\mathsf{d}\phi^1 + y\mathsf{d}\phi^2)^2 + e^{2V_2} L(\mathsf{d}\phi^2)^2, \tag{2.35}$$

$$S = \frac{\alpha c(e^{2V_0} - 1)}{1 - c^2 e^{2V_0}} \tag{2.36}$$

where

$$L = \frac{1 - c^2 e^{2V_0}}{1 - c^2}. \tag{2.37}$$

Next we need to convert these into the variables adapted to the $U(1)^2$ reduction. This is trivial for the metric components

$$e^{2W^{(0)}} = \rho^2 e^{-2V_0} L^2, \qquad e^{2W_1^{(0)}} = e^{2V_1} L, \qquad u = y, \qquad e^{2W_2^{(0)}} = e^{2V_2} L, \tag{2.38}$$

where we use (0) superscripts for this intermediate solution for later convenience. To do something similar for the Maxwell potential, we first work from the definition of S (2.16) and $\Phi^{(0)}$ (2.4) (and use the fact that F is purely electric) to find that

$$\mathsf{d}\Phi^{(0)} = \rho^{-1} e^{2W^{(0)}} \star_2 \mathsf{d}S. \tag{2.39}$$

Using the expression for $e^{2W^{(0)}}$ (2.38) and S (2.36) in terms of V_0 this simplifies to

$$\mathsf{d}\Phi^{(0)} = \frac{2\alpha c}{1 - c^2} \rho \star_2 \mathsf{d}V_0. \tag{2.40}$$

V_0 is an axially symmetric harmonic function on $\mathbb{R}^3$, which can be seen from (2.20) since the seed is neutral (or even just considering the above equation on M_3 and acting with the exterior derivative on both sides). We also know that V_0 must tend to 0 at asymptotic infinity (since the solution is AF) and be smooth everywhere except for on horizon rods where it should diverge as $\ln \rho$ (this is necessary for a smooth horizon). A candidate form for V_0 that satisfies these constraints can be written as

$$V_0 = \frac{1}{2} \sum_{k=1}^{n} s_k \ln \mu_k, \tag{2.41}$$

for constants s_k given by

$$s_k = \begin{cases} -1 & (z_k, z_{k+1}) \text{ is a horizon rod} \\ 1 & (z_{k-1}, z_k) \text{ is a horizon rod} \\ 0 & \text{otherwise} \end{cases}, \tag{2.42}$$

and

$$\mu_k = \sqrt{\rho^2 + (z - z_k)^2} - (z - z_k). \tag{2.43}$$

Note that $\ln \mu_k$ are axially symmetric harmonic functions and $(\ln \mu_{k+1} - \ln \mu_k)$ is smooth everywhere apart from $\rho = 0, z_k < z < z_{k+1}$ where it diverges as $\ln \rho$. Considering another function V_0' satisfying these constraints, it is simple to see that $V_0' - V_0$ is a smooth and bounded harmonic function and so must be constant everywhere. This demonstrates that the form of V_0 given above is unique up to a rescaling of the t coordinate.

Combining (2.41) with the identity

$$\rho \star_2 d \ln \mu_k = -d\bar{\mu}_k, \tag{2.44}$$

where

$$\bar{\mu}_k = \rho^2/\mu_k = \sqrt{\rho^2 + (z - z_k)^2} + (z - z_k), \tag{2.45}$$

and using the result in the equation for $\Phi^{(0)}$ in terms of V_0 (2.40), we finally get a solution for Φ^0

$$\Phi^{(0)} = -\frac{\alpha c}{1 - c^2} \sum_{k=1}^{n} s_k \bar{\mu}_k, \tag{2.46}$$

which is valid up to an arbitrary additive constant. Notice that we have chosen a gauge for $\Phi^{(0)}$ such that $\Phi^{(0)}|_{I_L} = 0$

Now we can use the transformation associated to the $U(1)^2$ reduction (2.31), (2.32) on this charged solution to give

$$g = e^{2\nu} L M^2 (\mathrm{d}\rho^2 + \mathrm{d}z^2) - e^{2V_0} L^{-2} M^2 \mathrm{d}t^2 + e^{2V_1} L M^{-1} (\mathrm{d}\phi^1 + y \mathrm{d}\phi^2)^2 + e^{2V_2} L M^{-1} (\mathrm{d}\phi^2)^2, \tag{2.47}$$

$$\Phi = \left[\Phi^{(0)}(1 + k\alpha^{-1}\Phi^{(0)}) + \alpha k e^{2W^{(0)}}\right] \tag{2.48}$$

where

$$M = (1 + k\alpha^{-1}\Phi^{(0)})^2 + k^2 e^{2W^{(0)}} \tag{2.49}$$

and $e^{2W^{(0)}}, \Phi^{(0)}$ are given in (2.38), (2.46).

Conical singularities

Lastly we discuss conical singularities. For a given rod I_a with rod vector v_a, there is a conical singularity unless the balance condition (2.2) is satisfied (taking ϕ^i to have period 2π). For the seed solution these constraints are only satisfied on the left and right semi-infinite rods. After the combined transformation the expression on the LHS of (2.2) will pick up a factor of $(M_0^a)^3$ where we define

$$M_0^a = \lim_{\rho \to 0,\, z \in I_a} M = (1 + k\alpha^{-1}\Phi^{(0)}|_{I_a})^2, \tag{2.50}$$

and we have used the fact that $e^{2W^{(0)}}$ vanishes on axis rods. Note that $\Phi^{(0)}$ is constant on axis rods, which can be seen from its definition (2.4) and the fact that $v_a = 0$.

Consider the left axis rod I_L. Then $\Phi^{(0)}|_{I_L} = 0$ from (2.46) and so $M_0^L = 1$ and there is no conical singularity in the transformed metric. However on the right rod I_R, from (2.42), (2.46) we see that

$$\Phi^{(0)}|_{I_R} = -\frac{2\alpha c}{1 - c^2} \sum_H \ell_H, \tag{2.51}$$

where the sum is over horizon rods H with associated rod length ℓ_H. Therefore the left hand side of (2.2) is given by

$$N^2 := (M_0^R)^3 = \left(1 - \frac{2ck}{1 - c^2} \sum_H \ell_H\right)^6, \tag{2.52}$$

for the transformed metric which is clearly not equal to 1 unless either $c, k = 0$ or there are no horizons (we return to these cases below). Therefore if we want to remove the conical singularity on I_R we must either relax the assumption that ϕ^1 has a period of 2π or equivalently rescale ϕ^1. We take the second option and rescale $\phi^1 \to N\phi^1$ (we take $N > 0$). This puts the metric into the new form

$$\begin{aligned} g = {} & e^{2\nu} L M^2 (\mathrm{d}\rho^2 + \mathrm{d}z^2) \\ & - e^{2V_0} L^{-2} M^2 N^{-2} \mathrm{d}t^2 + e^{2V_1} L M^{-1} (N \mathrm{d}\phi^1 + y \mathrm{d}\phi^2)^2 + e^{2V_2} L M^{-1} (\mathrm{d}\phi^2)^2, \end{aligned} \tag{2.53}$$

where we have also taken $t \to N^{-1} t$ in order to maintain the $W - W_1 - W_2 = 0$ condition. An immediate consequence of this is that ∂_1 transforms as $\partial_1 \to N^{-1}\partial_1$ under this coordinate change and so a rod vector should transform as well i.e. as $v = p\partial_1 + q\partial_2 \to N^{-1}p\partial_1 + q\partial_2$ for constants p and q. This is compatible with the earlier statements that these charging transformations shouldn't change rod vectors since all that is changing is the coordinates used to describe them. For the orbits of this rod vector to be closed we now have the requirement that qNp^{-1} must be rational (when $p \neq 0$). We will return to this condition when we discuss the black lens spacetime in the next section.

Analysis of the conical singularity condition for the finite axis rods is more difficult to do in general and we leave the discussion of this to the next section where we examine particular solutions. However there are some straightforward results in a couple of special cases. First consider the case $c = 0$. This implies that $\Phi^{(0)} = 0$ and so $N^2 = M_0^a = 1$ which in turn implies that the transformation doesn't affect the conical singularity condition (2.2). Similarly when $k = 0$, this condition is again unaffected by the transformation. We therefore see that both the charging and immersing transformations are needed to act non-trivially in order to have a chance at removing singularities. Lastly, consider a soliton solution, a spacetime with no black hole horizons. In this case $e^{2V_0} = 1$ (since $s_k = 0$ (2.42) on all rods) and so $S = 0, L = 1$, which means that the transformation is independent of c. By the arguments above this means that conical singularities cannot be cured and so this method cannot be used to construct regular solitons in an electric background.

2.4 Examples

2.4.1 Black Saturn

The neutral static black Saturn solution can be constructed from its rod structure (Figure 2.1) as a Weyl solution [53], with its metric given by

$$\begin{gathered} e^{2V_0} = \frac{\mu_1\mu_3}{\mu_4\mu_2}, \qquad e^{2V_1} = \mu_4, \qquad e^{2V_2} = \rho^2 \frac{\mu_2}{\mu_1\mu_3}, \\ e^{2\nu} = \mu_4 \frac{r_{12}^2 r_{23}^2 r_{14} r_{34}}{r_{13}^2 r_{24} \prod_{i=1}^4 r_{ii}^2}, \end{gathered} \tag{2.54}$$

where

$$r_{ij} = \rho^2 + \mu_i\mu_j, \tag{2.55}$$

μ_k is given by (2.43) and the rod points obey $z_1 < z_2 < z_3 < z_4$.

The metric physically corresponds to an S^3 black hole surrounded by a black ring in a flat background. One can isolate the central S^3 black hole by taking $z_2 \to z_1$, essentially removing the black ring horizon rod (note that this causes the dependence of the metric on z_1 to drop out). There is also a limit to the black ring by taking $z_4 \to z_3$ which removes the S^3 horizon. Since this is a static, AF solution to vacuum gravity (which is neither flat space nor a Schwarzschild black hole), it cannot be a smooth

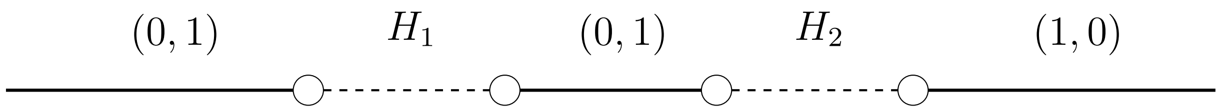

Figure 2.1: The rod structure for a black Saturn solution, where H_1 corresponds to a black ring and H_2 an S^3 black hole.

solution [50]. As expected this is because of a conical singularity on the finite axis rod $I_C = (z_2, z_3)$ which has rod vector $v_C = m_2$. Explicitly we find that for distinct z_k

$$\lim_{\rho\to 0,\, z\in I_C} \left(\frac{\rho^2 e^{2\nu}}{e^{2V_2}}\right) = \frac{(z_3-z_2)^2(z_4-z_1)}{(z_3-z_1)^2(z_4-z_2)}, \tag{2.56}$$

is always less than 1 and so the conical singularity cannot be removed through tuning z_k alone (see (2.2)).

We now charge and immerse this solution in the way described in the previous section. The equations for the metric components and Φ are trivially given from the general formalism (2.53), (2.48). Expression (2.56) picks up a factor of $(M_0^C)^3$ (2.50) which allows it to be solved in terms of k to give

$$k = \frac{1-c^2}{2c(z_2-z_1)}\left(1 \pm \left[\frac{z_3-z_1}{z_3-z_2}\left(\frac{z_4-z_2}{z_4-z_1}\right)^{1/2}\right]^{1/3}\right), \tag{2.57}$$

where we are assuming $k, c \neq 0$. This gives two disjoint families of regular solutions: if we take the upper sign then $k > \frac{1-c^2}{c(z_2-z_1)}$ and the charge of the black hole and the electric background have the same sign, on the other hand the lower sign implies that these electric charges have opposite signs since k has the opposite sign to c. Setting $z_4 = z_3$ gives a balanced static charged black ring immersed in an electric background (presumably matching the solution found in [125] for a choice of upper or lower sign).

2.4.2 Black lens

We next consider a neutral static $L(n,1)$ black lens, a black hole spacetime with $L(n,1)$ lens space horizon topology. We use the metric given in [129][4] which is written in (x,y) coordinates as

$$\begin{aligned} g = -\frac{1+\nu y}{1+\nu x}\mathrm{d}t^2 + \frac{2R^2(1+\nu x)}{(1-a^2)(x-y)^2 H(x,y)}\Bigg[&\frac{H(x,y)^2}{1-\nu}\left(\frac{\mathrm{d}x^2}{G(x)} - \frac{\mathrm{d}y^2}{G(y)}\right) \\ &+ (1-x^2)[(1-\nu-a^2(1+\nu y))\mathrm{d}\phi^2 - a\nu(1+y)\mathrm{d}\phi^1]^2 \\ &- (1-y^2)[(1-\nu-a^2(1+\nu x))\mathrm{d}\phi^1 - a\nu(1+x)\mathrm{d}\phi^2]^2\Bigg], \end{aligned} \tag{2.58}$$

where

$$G(\zeta) = (1-\zeta^2)(1+\nu\zeta), \qquad H(x,y) = (1-\nu)^2 - a^2(1+\nu x)(1+\nu y). \tag{2.59}$$

[4]The metric was originally derived in [130] but written in Weyl coordinates and not recognised as describing a black lens spacetime. We have also used ν and R in place of c and κ in [129] to avoid confusion with our charging parameters, and used (ϕ^1, ϕ^2) in place of their (ψ, ϕ).

The constants lie in the ranges $0 < \nu < 1, -1 < a < 1$ and $R > 0$ with the coordinates (x, y) constrained by $-1 \leq x \leq 1$ and $-1/\nu < y \leq -1$. The rod structure is given in Figure 2.2. The left axis rod corresponds to $x = -1$, the horizon rod corresponds to $y = -1/\nu$, the finite axis rod corresponds to $x = 1$ and the right axis rod corresponds to $y = -1$. The rod vector for the finite axis rod I_D is given by $v_D = \partial_1 + n\partial_2$ with n given by

$$n = \frac{2a\nu}{1 - \nu - a^2(1+\nu)}. \tag{2.60}$$

Requiring that the orbits of v_D are closed imposes that n is an integer, though we shall replace this condition with a slightly different one shortly when we discuss the transformed solution.

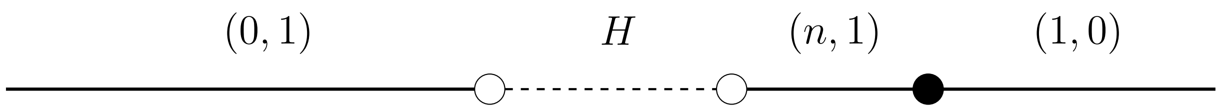

Figure 2.2: The rod structure for a simple $L(n, 1)$ black lens.

The metric has a limit to a black ring by taking $n \to 0$ (equivalently $a \to 0$). Similarly there is a limit to a Schwarzschild black hole by taking $n \to \infty$ (equivalently $a \to \pm\sqrt{\frac{1-\nu}{1+\nu}}$). Again, as with the black Saturn solution, there is a conical singularity associated with the finite axis rod I_D where $x = 1$.

The metric of the spacetime after performing both the transformations we've discussed (2.53), (2.48) can be written as

$$\begin{aligned} g = -\frac{1+\nu y}{1+\nu x} M^2 L^{-2} N^{-2} \mathrm{d}t^2 + \frac{2R^2(1+\nu x)}{(1-a^2)(x-y)^2 H(x,y)} M^{-1} L \Bigg[\frac{H(x,y)^2}{1-\nu} M^3 \left(\frac{\mathrm{d}x^2}{G(x)} - \frac{\mathrm{d}y^2}{G(y)} \right) \\ + (1-x^2)[(1-\nu-a^2(1+\nu y))\mathrm{d}\phi^2 - a\nu(1+y) N \mathrm{d}\phi^1]^2 \\ - (1-y^2)[(1-\nu-a^2(1+\nu x)) N \mathrm{d}\phi^1 - a\nu(1+x)\mathrm{d}\phi^2]^2 \Bigg], \end{aligned} \tag{2.61}$$

with L, M, N given by (2.25), (2.32), (2.52) and Φ given by (2.48). Note that

$$\rho^2 = -\frac{4R^2}{(x-y)^2} G(x) G(y) \tag{2.62}$$

from the definition of ρ in terms of the determinant of the Killing part of the metric. We also see that the rod vector for I_D is now written $v_D = \partial_1 + \bar{n}\partial_2$ for

$$\bar{n} = Nn = \left(1 - \frac{4ck\nu R^2}{1-c^2}\right)^3 \frac{2a\nu}{1-\nu-a^2(1+\nu)}. \tag{2.63}$$

This means that we should now take $\bar{n}$ to be an integer (and relax that requirement on n) to ensure that v_D has compact orbits, giving a $L(\bar{n}, 1)$ black lens.

Now we consider possible conical singularities on the axis rods. By construction there are no conical singularities on the semi-infinite axis rods as long as ϕ^i have periods 2π. In order to cure the conical singularity condition for the finite axis rod I_D, we use the fact that the conformal factor for the neutral seed is given by

$$
\begin{aligned}
e^{2\nu} = &- \frac{2(1+\nu x)(x-y)H(x,y)}{(1-\nu)(1-a^2)R^2} \\
&\left[(2+\nu(1+x)+\nu(1-x)y)(\nu+x+y+xy)(2-\nu(1-x-y-xy))\right]^{-1}.
\end{aligned} \tag{2.64}
$$

For the seed solution we find that

$$\lim_{x\to 1} \frac{\rho^2 e^{2\nu}}{|v_D|^2} = \frac{(a^2(1+\nu)-(1-\nu))^2}{(1-a^2)^2(1-\nu^2)}. \tag{2.65}$$

The right hand side of this expression is less than 1 for all allowed values of a, ν and therefore the balance condition (2.2) cannot be satisfied in the neutral case (as expected).

By the structure of the charging transformations we know that this expression will just be multiplied by an overall factor of $(M_0^D)^3$ (2.50) in the transformed solution. Therefore we can solve the balance condition (2.2) for the new solution in terms of k $(c, k \neq 0)$ to find

$$k = \frac{1-c^2}{4c\nu R^2}\left(1 \pm \left[\frac{(1-a^2)(1-\nu^2)^{1/2}}{|a^2(1+\nu)-(1-\nu)|}\right]^{1/3}\right), \tag{2.66}$$

where we have used (2.46), (2.42) and the fact that the horizon rod length is $z_2 - z_1 = 2\nu R^2$ (see [129]). Again this gives two distinct families of solutions corresponding to the charges of the transformations either having the same or opposite sign. Combining this with the expression for $\bar{n}$ we find

$$\bar{n} = s\frac{2a\nu(1-a^2)(1-\nu^2)^{1/2}}{((1-\nu)-a^2(1+\nu))^2}, \tag{2.67}$$

where s gives the sign of $((1-\nu)-a^2(1+\nu))$. Any value of $\bar{n}$ can be found for suitable a, ν, just as in the vacuum case with n.

2.5 Discussion

In this chapter we have considered 5d biaxisymmetric, electrostatic, black hole solutions. The Einstein-Maxwell equations take a different form depending on whether one reduces to a three-dimensional base space over $\mathbb{R} \times U(1)$ or $U(1)^2$. From these different formulations of the equations we have derived two distinct 1-parameter families of transformations. The first of these transformations preserves asymptotic flatness and can be interpreted as adding charge to black holes in a spacetime, for example acting on the Schwarzschild solution with this transformation gives the Reissner–Nordström solution (this must be the case because of the static uniqueness theorem [50]). The second transformation does not preserve asymptotic flatness but instead immerses the black hole in an external electric background known as the Melvin universe. Although the transformed solutions are no longer AF, they still preserve some of that structure, namely having an S^3 topology spatial cross-section at infinity.

We used both these transformations on a neutral static black Saturn and an $L(n,1)$ black lens. By tuning the charging parameters we were able to remove the conical singularities in these two solutions and so find regular black hole spacetimes. The regular black lens is of particular interest since it is the only known example which has trivial topology at infinity.

The other obvious type of reduction that we have not considered in this chapter are reductions over null KVFs. Different reductions allow new perspectives on the structure of the Einstein-Maxwell equations, making certain symmetries more manifest in some cases and it would be interesting to see what kind of symmetry this type of reduction would lead to. Another potentially useful extension of this work is to understand how the combined charging transformation works on the general class of Weyl solutions. Particularly this means being able to determine the rod structures for which the conical singularity conditions can be removed. As discussed, in the soliton case it is certainly not possible to balance solutions using these charging transformations, irrespective of any parameter tuning. However it remains a possibility that a new large class of regular charged (multi-)black hole solutions with non-trivial 2-cycles in the DOC could be found.

Chapter 3

Classification of Vacuum Solutions Using Integrability

This chapter looks at the classification of solutions in four and five-dimensional vacuum gravity. We consider AF, stationary, vacuum black hole spacetimes in four and five dimensions, that admit one and two commuting axial Killing fields respectively. The equations governing these spacetimes reduce to a harmonic map on the two-dimensional orbit space. This harmonic map can be rewritten as the integrability condition of a pair of linear PDEs. By taking advantage of this structure we derive a method which determines metric data on the $\rho = 0$ axis given a particular rod structure. This information can then further be leveraged to find the moduli space of solutions with this rod structure. We leave explicit examples of this method and the analysis of particular simple rod structures, including one corresponding to a black lens, to the following chapter.

This chapter draws heavily from sections 1-3 of [3].

3.1 Introduction

We now consider vacuum solutions with $D = 4, 5$ dimensions. We will be looking at the class of stationary, AF spacetimes with $D-3$ commuting axial KVFs that we originally discussed in Section 1.2. Crucially, the Einstein equations for these spacetimes reduce to a harmonic map on the two-dimensional orbit space. This leads to two important consequences. Firstly there is a uniqueness theorem governing these solutions, Theorem 3. This theorem dictates that each rod structure with a given angular momenta for each horizon component leads to a unique solution (if it exists). Secondly the Einstein equations are integrable, since the harmonic map can be written as the integrability condition of a pair of linear PDEs, these are the BZ equations (1.16). This feature will be fundamental to the results of this chapter.

As we've discussed in the introduction, in the vacuum $D = 4$ case with a connected horizon, Theorem 3 reduces to the classic black hole uniqueness theorem for the Kerr black hole: it says that any solution is uniquely parameterised by the horizon rod length ℓ_H and angular momentum J (there are no finite axis rods). The (non-extreme) Kerr solution realises all possible values of this data, $\ell_H > 0, J \in \mathbb{R}$, and hence the classification for this case is complete (in this case one can of course also use the M, J to label solutions as is traditionally done). As in the classic $D = 4$ case, the proof of Theorem 3 is non-constructive and involves a nonlinear divergence identity (Mazur identity) which characterises the 'difference' of two solutions to the corresponding harmonic map problem. Therefore this theorem does not address the crucial question of existence: *for what rod structures and horizon angular momenta do regular solutions actually exist?*

Indeed, the existence question is largely open even for $D = 4$. In this case the other possible rod structures correspond to black holes with multiple horizons, with finite axis rods separating the disjoint horizon rods. There is a general expectation that equilibrium configurations describing such solutions in the vacuum cannot exist due to their mutual gravitational attraction. In fact, by adapting existence results for harmonic maps to this problem, Weinstein has shown that a unique N-component black hole solution exists given any rod structure and horizon angular momenta, which is regular everywhere away from the axis [70, 131, 132]. However, such solutions may still suffer from conical singularities on the finite axis components (i.e. those not connected to infinity). Physically, these singularities are related to the force of attraction between the black holes and it is conjectured that for $N > 1$ such solutions always do possess conical singularities. Evidence that this force is always attractive has been obtained by studying various special cases [133, 132].

Candidate multi-black hole solutions, known as the multi-Kerr-NUT solutions, have been known for some time [134, 135, 65], although an analysis of the potential conical singularities has proven to be essentially intractable. Naturally, the $N = 2$ case corresponding to a double-black hole has been the most extensively studied. From the above theorem this solution depends on five-parameters (two horizon rod lengths, one axis rod length and the angular momentum of each horizon), that are related by the equilibrium condition (i.e. the condition for removal of the conical singularity on the finite axis rod). The study of the equilibrium condition for the double-Kerr-NUT solution has been the subject of much work, see e.g. [136, 137]. However, even if one can give a general proof that the equilibrium condition for the double-Kerr-NUT solution is never satisfied, this would still not give a proof of the non-existence of a regular double-black hole, since it is not a priori clear that it contains the general solution with these boundary conditions. Recently, this conjecture has been settled by Hennig and Neugebauer [39]: a regular double-black hole solution does not exist. The proof consists of two steps: (i) employing the inverse scattering method from integrability theory to prove that the general solution with such boundary conditions is contained in the known double-Kerr-NUT solution (this was already shown in earlier work by Varzugin [36] and Meinel and Neugebauer [38]); (ii) showing that the equilibrium conditions are incompatible with the area-angular momentum inequality for a marginally trapped surface [138, 139, 140].

The $D = 5$ case is more complicated for two principle reasons. Firstly, there are more horizon topologies compatible with biaxial symmetry: S^3, $S^2 \times S^1$ and lens spaces $L(p, q)$. Secondly, for every horizon topology (including multi-horizons) there can be an arbitrary number of finite axis rods on which different linear combinations of the two axial Killing fields vanish – these correspond to non-trivial 2-cycles in the domain of outer communication (DOC). Recently, a theorem which partially addresses the existence question in this context has been established by Khuri, Weinstein and Yamada [69] (Theorem 4 with $D = 5$). It is a five-dimensional analogue of Weinstein's theorem for $D = 4$ multi-black holes [35] using the theory of harmonic maps adapted to this setting. It essentially tells you that for given uniqueness moduli (rod structure and angular momenta), there exists a solution which is regular away from the axis. This greatly simplifies the problem of classification of regular solutions. In particular it reduces to an analysis of possible conical singularities at the inner axis rods (it has been shown that there are no conical singularities at the two semi-infinite axis rods [141]). Note that it also requires one to make the technical assumption that the rod structure satisfies some compatibility requirement (see (1.18)).

The purpose of this chapter is to use the spectral equation of BZ (3.54) to *systematically* investigate all possible solutions for any given rod structure. In particular, we explicitly integrate the BZ spectral equations along the axes and horizons, and around infinity. Then, using this we show that one can determine the metric everywhere on the axes and horizons for any given rod structure purely algebraically. Our main result can be summarised as follows (see Theorem 8 for a precise statement):

Theorem 7. *Consider a $D = 4, 5$ stationary vacuum spacetime as in Theorem 3. On every component of the axis and horizon, the general solution for the metric components and the associated Ernst or twist potentials are rational functions of z. These functions are explicitly determined in terms of the rod structure, horizon angular momenta, horizon angular velocities and certain gravitational fluxes, which are subject to a set of nonlinear algebraic constraints.*

Thus the solution depends on a number of continuous parameters which are geometrically defined: the rod lengths, the angular momenta and angular velocities of each horizon, and certain gravitational fluxes. The gravitational fluxes are invariants associated to each finite axis rod. In the spacetime the finite axis rods correspond to non-contractible $(D-3)$-cycles and the fluxes are integrals of certain closed $(D-3)$-forms constructed from the Killing fields. For every axis rod one can define an associated Ernst potential from the Killing fields which are nonzero on that rod. The change in Ernst potential across the associated rod is then precisely the gravitational flux through the corresponding 2-cycle. It is worth noting that similar gravitational fluxes arise in the recently found thermodynamic identities for $D = 5$ black holes in this class [142].

As mentioned in our theorem, the parameters in the general solution must obey certain nonlinear algebraic equations. These arise from integrating the BZ spectral equations along the z-axis and around the 'semi-circle' at infinity. Furthermore, imposing the metric is free of conical singularities on the axes and horizons typically imposes further constraints on the parameters. Thus we are able to address part (ii) of the regularity problem left open by Theorem 4. Hence, our method is particularly useful for ruling out regular solutions with a prescribed rod structure. For example, one can prove that a $D = 5$ solution with no horizon and one finite axis rod must be conically singular at the finite axis rod; this is of course guaranteed by the no-soliton theorem for vacuum solutions (even without biaxial symmetry), although it illustrates that our method is capable of showing that certain rod structures must lead to conically singular solutions.

Our method may be thought of as a higher-dimensional analogue of the $D = 4$ methods of Varzugin [36, 37] and Meinel and Neugebauer [38], which both lead to simple constructive uniqueness proofs for Kerr. In particular, Varzugin integrated the BZ spectral equations along the axis and horizons and used this to show that the N-black hole solution is contained in the $2N$-soliton solution of BZ [65]. We also integrate the BZ spectral equations along the boundaries, although our analysis of its solution differs, and we give a simple method to extract the spacetime metric, so even for $D = 4$ it offers an alternative approach. On the other hand, Meinel and Neugenbauer integrated a different spectral equation along the axis, whose integrability condition gives the Ernst equations, and used this to determine the Ernst potential on the axis. It would be interesting to investigate the precise relationship between these various methods.

This chapter is organised as follows. In Section 3.2 we recall well-known properties of stationary vacuum spacetimes with $D-3$ commuting axial Killing fields and introduce various Ernst potentials which will feature later (this section also serves to set our notation). In Section 3.3 we derive the general solution to the BZ spectral equations on the axes, horizons and around infinity, and use this to construct the general solution to the Einstein equations on the axes and horizons. In Sections 3.4 and 3.5 we specialise to $D = 4$ and $D = 5$ respectively and compute the asymptotic charges of the general solutions. In Section 3.6 we discuss our results and future work. We relegate various results to the Appendix.

3.2 Background

3.2.1 Einstein equations and rod structure

For this section we recall and expand on some of the results of Sections 1.2 and 1.3 and set up some notation for the rest of the chapter. Let $(M, \mathbf{g})$ be a D-dimensional stationary spacetime with $D-3$ commuting axial Killing vector fields that also commute with the stationary Killing field. We denote the stationary Killing field k and the remaining $D-3$ axial Killing fields $m_i, i = 1, \ldots, D-3$, and assume these generate an isometry group $G = \mathbb{R} \times U(1)^{D-3}$. We define coordinates (t, ϕ^i) adapted to the stationary and axial symmetries, so $k = \partial_t$ and $m_i = \partial_{\phi^i}$, and choose m_i to be generators with 2π-periodic orbits, i.e. the angles ϕ^i are 2π-periodic. We also assume that there is at least one point in spacetime that is a fixed point of the axial symmetry (as is the case for asymptotically flat spacetimes).

Under such assumptions the spacetime metric can be written in Weyl-Papapetrou coordinates [53, 54]

$$\mathbf{g} = g_{AB}(\rho, z)\mathrm{d}x^A\mathrm{d}x^B + e^{2\nu(\rho,z)}(\mathrm{d}\rho^2 + \mathrm{d}z^2), \tag{3.1}$$

where $A \in \{0, 1, \ldots, D-3\}$, ∂_A are the Killing fields and

$$\det g_{AB} = -\rho^2. \tag{3.2}$$

Then the vacuum Einstein equations reduce to

$$\partial_\rho U + \partial_z V = 0, \tag{3.3}$$

where

$$U = \rho \partial_\rho g g^{-1}, \qquad V = \rho \partial_z g g^{-1} \tag{3.4}$$

and the conformal factor, $e^{2\nu}$, is then determined by

$$\partial_\rho \nu = -\frac{1}{2\rho} + \frac{1}{8\rho}\mathrm{Tr}\,(U^2 - V^2), \qquad \partial_z \nu = \frac{1}{4\rho}\mathrm{Tr}\,UV\;. \tag{3.5}$$

Indeed, the integrability condition for (3.5) is (3.3).

One can also establish the orbit space theorem, Theorem 2. This shows that the orbit space M under the isometry group $\hat{M} = M/G$, is a 2d simply connected manifold with boundaries and corners, which may therefore be identified with the half-plane

$$\hat{M} = \{(\rho, z) \,|\, \rho > 0\}\;. \tag{3.6}$$

The boundary of the orbit space $\rho = 0$ corresponds to the z-axis and this splits into intervals, called rods, $(-\infty, z_1), (z_1, z_2), \ldots, (z_n, \infty)$, with $z_1 < z_2 < \cdots < z_n$, each of which corresponds to a connected component of the horizon orbit space $\hat{H} = H/U(1)^{D-3}$, or an axis with an associated vanishing "rod vector". The endpoints of the rods $z_a, a = 1, \ldots, n$, correspond to the corners of the orbit space, each of which corresponds to where an axis intersects a horizon, or for $D > 4$, a fixed point of the $U(1)^{D-3}$-action (i.e. $m_i = 0$ for all $i = 1, \ldots, D-3$, which occurs precisely where two axes intersect).

Let us denote the rods by I_a, for $a = 1, \ldots, n+1$, and the length of the finite rods $I_a = (z_{a-1}, z_a)$ by $\ell_a = z_a - z_{a-1}$ for $a = 2, \ldots, n$. Given any axis rod I_a the corresponding rod vector takes the form

$$v_a = v_a^i m_i \tag{3.7}$$

where $(v_a^i)_{i=1,\dots,D-3}$ are coprime integers. If $D = 5$, for any adjacent axis rods I_a and I_{a+1} separated by the corner $z = z_a$ the associated rod vectors must satisfy the condition

$$\det \begin{pmatrix} v_a^1 & v_a^2 \\ v_{a+1}^1 & v_{a+1}^2 \end{pmatrix} = \pm 1 \; . \tag{3.8}$$

Following [69], we will call any rod structure satisfying (3.8) *admissible*. We will denote the union of all axis rods by $\hat{A}$ and all horizon rods by $\hat{H}$. The collection of all this data

$$\{(\ell_a, v_a) \,|\, I_a \subset \hat{A}\} \cup \{\ell_a \,|\, I_a \subset \hat{H}\} \tag{3.9}$$

is known as the *rod structure*. We will often denote the semi-infinite axis rods by $I_L = I_1 = (-\infty, z_1)$ and $I_R = I_{n+1} = (z_n, \infty)$. For definiteness, in the $D = 5$ case we will choose the m_i such that $m_2 = 0$ on I_L and $m_1 = 0$ on I_R, i.e., the rod vectors $v_L = (0, 1)$ and $v_R = (1, 0)$ relative to the basis (m_1, m_2).

For $D = 5$ any finite axis rod I_a lifts to a 2-cycle in the spacetime. Explicitly this is given by the surface C_a obtained from the fibration of the nonzero $U(1)$ Killing field $u_a = u_a^i m_i$ over the closure of I_a (recall $v_a = 0$ on I_a). If the adjacent rods are both axis rods then u_a must vanish at the endpoints of I_a and C_a has the topology of S^2; if only one adjacent rod is an axis rod (and hence the other a horizon) then u_a only vanishes at the corresponding endpoint so C_a is topologically a 2-disc; finally if both adjacent rods are horizon rods then u_a does not vanish at either endpoint and C_a is topologically a cylinder.

Another important set of invariants for such solutions are the Komar angular momenta of each connected component of the horizon H_a defined by

$$J_i^a = \frac{1}{16\pi} \int_{H_a} \star \mathrm{d} m_i, \tag{3.10}$$

where we fix the orientation $\epsilon_{01\dots D-3\rho z} > 0$. From a standard argument, invoking Stokes' theorem and the Einstein equation, these are related to the total angular momenta of the spacetime $J_i = \sum_a J_i^a$. Due to the assumed symmetry these can be reduced to integrals over the horizon rods using

$$\int_{H_a} \star \alpha = (2\pi)^{D-3} \int_{I_a} \star (m_1 \wedge \dots \wedge m_{D-3} \wedge \alpha) \tag{3.11}$$

where α is any $U(1)^{D-3}$-invariant 2-form. This gives

$$J_i^a = \frac{1}{8} (2\pi)^{D-4} (\chi_i(z_a) - \chi_i(z_{a-1})) \; , \tag{3.12}$$

where χ_i are the twist potentials defined by

$$\mathrm{d}\chi_i = \star (m_1 \wedge \dots m_{D-3} \wedge \mathrm{d} m_i) \; . \tag{3.13}$$

The existence of globally defined twist potentials follows from the fact the vacuum Einstein equations imply the RHS of (3.13) is a closed 1-form and the fact that the DOC is simply connected using topological censorship [26, 42, 43]. Observe that the twist potentials are constant on any axis rod. Therefore, they can only vary across a horizon rod and the above shows that the change in twist potential across any horizon rod is precisely the angular momenta of the corresponding horizon in spacetime.

We now are now in a position to consider the uniqueness and existence theorems mentioned in Chapter 1 and the introduction in more detail. Theorem 3 guarantees that there is at most one solution for a given rod structure (3.9) and horizon angular momenta (3.12). However, as highlighted in the introduction, the main limitation of this theorem is that it does not address the crucial question of existence: for what rod structure and angular momenta do there exist regular black hole solutions? This is not an issue for $D = 4$ as the uniqueness theorem reduces to the classic no-hair theorem for the Kerr black holes (although for multi-black holes this is largely open, as explained in the Introduction).

However, for $D = 5$ the uniqueness theorem is less powerful as even for a connected horizon an arbitrary number of axis rods are allowed in principle. To this end, Theorem 4 has been established, which guarantees the existence of a solution for any admissible rod structure that obeys a certain technical compatibility condition (1.18). As explained in the Introduction, this theorem guarantees the solution is regular in the DOC away from the axes. Therefore it does not address regularity of the solution on the axes, which generically will possess conical singularities on the finite axis rods. It is instructive to consider certain special cases of Theorem 3 and 4.

First, consider the case of no horizon. Then, it is well known from the no-soliton theorem that the only regular solutions in this class of spacetimes is Minkowski spacetime (indeed, this result does not even assume biaxial symmetry). Hence, it must be that the only regular solution with the same rod structure as Minkowski spacetime is Minkowski spacetime itself. Furthermore, any solution with non-Minkowski rod structure must be singular on some component of the axis. For example, consider the Eguchi-Hanson soliton

$$\mathsf{d}s^2_{\mathsf{EH}} = -\mathsf{d}t^2 + \frac{\mathsf{d}R^2}{1-\frac{a^4}{R^4}} + \tfrac{1}{4}R^2\left(1-\frac{a^4}{R^4}\right)(\mathsf{d}\psi + \cos\theta \mathsf{d}\phi)^2 + \tfrac{1}{4}R^2(\mathsf{d}\theta^2 + \sin^2\theta \mathsf{d}\phi^2)\ , \tag{3.14}$$

where $R \geq a$. As is well known, if ψ is periodically identified with period 2π this gives a smooth metric with a bolt at $R = a$ which is asymptotically locally Euclidean with $S^3/\mathbb{Z}_2$ topology for large R. However, if instead we take (θ, ψ, ϕ) to be Euler angles on S^3, we get an asymptotically Minkowski spacetime, except now with a conical singularity at the bolt. This example then gives a non-trivial rod structure with one finite axis rod corresponding to the bolt $R = a$ separating the two semi-infinite rods. In particular, relative to the basis (m_1, m_2) introduced above, the rod vectors are $v_L = (0, 1)$, $v_B = (1, 1)$ and $v_R = (1, 0)$, where v_B is the rod vector on the bolt, thus giving an admissible rod structure (3.8). It is a one parameter family of solutions, where the parameter can be taken to be length of the axis rod, in line with the above theorems (since there is no horizon the only moduli are the rod lengths). One might wonder whether the more general Gibbons-Hawking solitons similarly give solutions with multiple axis rods in Theorem 4. In Appendix 3.A we show that in fact these do not possess an admissible rod structure (instead they possess orbifold singularities at the corners $z_2, \ldots, z_{n-1}$ and thus correspond to solutions of a different theorem in [69]).

Now consider black hole solutions with a single horizon. First, suppose that the angular momenta $J_i = 0$. Then it can be shown that the solution must be static [68] and hence by the static uniqueness theorem the solution must be the Schwarzschild black hole [49]. This implies that any regular solution in this class must have the same rod structure as Schwarzschild, i.e. one horizon rod separating the two semi-infinite axis rods. In other words, any solution with a single horizon, $J_i = 0$ and finite axis rods, must be conically singular on the axis rods. This shows that for single black holes, not all rod structures and angular momenta lead to regular solutions.

Next, consider the rod structure of the Myers-Perry solution: a single horizon rod and two semi-infinite axis rods (i.e. this is the same as that of Schwarzschild). Then, since the Myers-Perry solution realises all possible data $\ell_H > 0$ and $(J_1, J_2) \in \mathbb{R}^2$ it is the only solution in this class. A self-contained proof of this was given in earlier work [143]. This case is analogous to the Kerr solution in four dimensions.

For a black ring something more interesting happens. Consider the rod structure of the known black ring, i.e. one horizon rod and one finite axis rod (and two semi-infinite axis rods). In this case there are four parameters in the uniqueness theorem, namely the horizon and finite axis rod lengths ℓ_H, ℓ_A and the angular momenta J_i. However, the most general known regular black ring solution is the three parameter doubly spinning solution [74]. Thus, in this case the known regular solutions do not occupy all parts of the possible parameter space. A way to understand this is that generically one has a conical singularity at the finite axis rod and its removal imposes a constraint on the four available parameters thus leaving a three parameter subset. Nevertheless, this raises the question: are there other regular black rings which occupy different parts of the possible moduli space? A definitive answer requires constructing the most general solution with such a rod structure. We answer this question in the negative in the next chapter[1].

Remarkably, regular multi-black hole solutions do exist in five dimensions. The first such example constructed was the black Saturn, an equilibrium configuration of a spherical black hole surrounded by a black ring that is balanced by angular momentum [76]. This solution is a four parameter family corresponding to the horizon rod lengths and one angular momentum for each black hole (the rod length of the finite axis rod between the black holes is fixed by removing the associated conical singularity). There should be a more general six-parameter family where both the spherical black hole and black ring are doubly spinning which is yet to be constructed. Similarly, regular four-parameter multi-black rings have been constructed: di-rings are concentric rings rotating in the same plane [77], and bi-rings rotate in orthogonal planes [79, 80]. Again, these should be part of a more general six-parameter family of two doubly spinning black rings that remains to be constructed.

3.2.2 Geometry of axes and horizons

In this section we write down a general form for the metric near $\rho = 0$, i.e., near any axis or horizon, which will be useful for our purposes. The analysis of the geometry near an axis and near a horizon is very similar, although for clarity of presentation we will use different notations for the metric in these two cases. Most of the material in this section is well-known. In Appendix 3.B we also include a regularity analysis at the corners of the orbit space which is perhaps less well-known.

Axes

First consider an axis rod I_a. For simplicity of notation we temporarily drop the labelling of each rod. It is convenient to introduce an adapted basis for the $D-2$ commuting Killing fields $\tilde{E}_A = (e_\mu, v)$ where $\mu = 0, \ldots, D-4$ and $v = v^i m_i$ is the rod vector corresponding to I_a. For $D = 4$ we simply take $e_0 = k$. For $D = 5$ we take $e_\mu = (k, u)$ where u is an axial Killing field

$$u = u^i m_i, \qquad \text{such that} \qquad A = \begin{pmatrix} u^1 & u^2 \\ v^1 & v^2 \end{pmatrix} \in GL(2, \mathbb{Z}) \,, \tag{3.15}$$

i.e. (u, v) are 2π-periodic generators of the $U(1)^2$-action. It is worth emphasising that u is defined only up to an additive integer multiple of the rod vector v. Then, relative to the adapted basis the metric

[1] In fact, a four parameter family of 'unbalanced' doubly spinning back rings has been constructed [75], i.e., these suffer from a conical singularity at the axis rod. It is possible that these do fill out the whole moduli space, although as far as we know this has not been checked in the literature. If so, then by the uniqueness theorem this would have to be the general solution and hence the known three-parameter family of black rings would have to be the most general regular solution with this rod structure.

on the orbits of the isometry can be written as

$$\tilde{g} = \begin{pmatrix} h_{\mu\nu} - \rho^2 h^{-1} w_\mu w_\nu & h^{-1}\rho^2 w_\mu \\ h^{-1}\rho^2 w_\nu & -h^{-1}\rho^2 \end{pmatrix} . \tag{3.16}$$

Note that the normalisation (3.2) is automatically imposed in this basis. Here, $h_{\mu\nu}$ is an invertible $(D-3)\times(D-3)$ matrix and its determinant $h = \det h_{\mu\nu} < 0$. A regular axis requires $h_{\mu\nu}, w_\mu$ to be smooth functions of (ρ^2, z) and

$$\lim_{\rho\to 0,\ z\in I_a} \frac{\rho^2 e^{2\nu}}{|v|^2} = 1 . \tag{3.17}$$

This ensures the absence of a conical singularity at I_a [54].

The inverse metric in this adapted basis is

$$\tilde{g}^{-1} = \begin{pmatrix} h^{\mu\nu} & w^\mu \\ w^\nu & -h\rho^{-2} + w^\rho w_\rho \end{pmatrix} \tag{3.18}$$

where $h^{\mu\nu}$ is the inverse matrix of $h_{\mu\nu}$ and $w^\mu = h^{\mu\nu} w_\nu$. The requirement of a smooth axis implies the following limits exist

$$\mathring{U} = \lim_{\rho\to 0} U , \qquad \mathring{V} = \lim_{\rho\to 0} \frac{V}{\rho} , \tag{3.19}$$

where here and throughout we denote quantities evaluated in the limit $\rho \to 0$ by a circle above. Explicitly, relative to the adapted basis we find

$$\mathring{\tilde{U}} = \begin{pmatrix} 0 & -2w_\mu \\ 0 & 2 \end{pmatrix} , \qquad \mathring{\tilde{V}} = \begin{pmatrix} (\partial_z h_{\mu\rho}) h^{\rho\nu} & -h h_{\mu\nu}\partial_z (h^{-1} w^\nu) \\ 0 & -(h^{-1}\partial_z h) \end{pmatrix} \tag{3.20}$$

where here, and in what follows, all quantities on the RHS are understood to be evaluated at $\rho = 0$. Taking the $\rho \to 0$ limit of the second equation in (3.5) it follows that the conformal factor on the axis obeys

$$\partial_z \mathring{\nu} = -\frac{\partial_z h}{2h} \tag{3.21}$$

which integrates to

$$e^{2\mathring{\nu}} = -\frac{c^2}{h} \tag{3.22}$$

where $c > 0$ is a constant.

Collecting these results, we deduce that the metric induced on the axis component associated to I_a is

$$\mathbf{g}_a = -\frac{c_a^2 \mathrm{d}z^2}{h^a(z)} + h^a_{\mu\nu}(z)\mathrm{d}x^\mu \mathrm{d}x^\nu , \tag{3.23}$$

where x^μ are adapted coordinates so that $e_\mu = \partial_\mu$, $\mu = 0, 1$, and we have reinstated the rod labels. This is a $(D-2)$-dimensional smooth Lorentzian metric for $z \in I_a$. The condition for the absence of a conical singularity in the spacetime at I_a (3.17) is

$$c_a = 1 , \tag{3.24}$$

which is sometimes referred to as the equilibrium or balance condition.

For $D = 5$ one or both of the adjacent rods to I_a may be another axis rod (for $D = 4$ it must be the case that any adjacent rod is a horizon rod). If I_{a+1} is another axis rod then $u = \partial/\partial x^1 = 0$ at $z = z_a$ and the above metric will have a conical singularity at this endpoint unless

$$\frac{h^{a\prime}(z_a)^2}{h^a_{00}(z_a)} = -4c_a^2 \ , \tag{3.25}$$

in which case the metric extends smoothly at this point. Note that we used $h^{a\prime}(z_a) = h^a_{00}(z_a) h^a_{11}{}'(z_a)$ to simplify the above expression which in turn comes from $h^a_{\mu 1}(z_a) = 0$ and smoothness. Similarly, if I_{a-1} is an axis rod then $u = \partial/\partial x^1 = 0$ at $z = z_{a-1}$ and the above metric extends smoothly at this endpoint iff

$$\frac{h^{a\prime}(z_{a-1})^2}{h^a_{00}(z_{a-1})} = -4c_a^2 \ . \tag{3.26}$$

Therefore, if I_a is a finite axis rod and provided these regularity conditions are met, the axis metric extends to a smooth Lorentzian metric on $\mathbb{R} \times C_a$. The 2-cycle C_a is topologically S^2, a 2-disc or a 2-cylinder depending on if I_{a-1}, I_a are either both axis rods, one axis rod and one horizon, or both horizon rods, respectively. In Appendix 3.B we analyse the geometry where two axis rods meet and derive further relations that follow from the above regularity analysis. In particular, we find that for two axis rods I_a and I_{a+1} the function $|z - z_a| e^{2\mathring{\nu}}$ is continuous at $z = z_a$, a result that has been previously proven in [144].

Horizons

The analysis of the metric near a horizon is very similar. Consider a component of the horizon, H_a, with corresponding rod I_a (again, for simplicity of notation we will temporarily drop the labelling of each rod). The Killing field null on the horizon is

$$\xi = k + \Omega_i m_i \ , \tag{3.27}$$

where Ω_i are the angular velocities of the black hole. Now, working in an adapted basis for the $D-2$ commuting Killing fields, $\tilde{E}_A = (m_i, \xi)$, the metric can be written as

$$\tilde{g} = \begin{pmatrix} \gamma_{ij} - \rho^2 \gamma^{-1} \omega_i \omega_j & \gamma^{-1} \rho^2 \omega_i \\ \gamma^{-1} \rho^2 \omega_j & -\gamma^{-1} \rho^2 \end{pmatrix} \ , \tag{3.28}$$

where γ_{ij} is an an invertible $(D-3) \times (D-3)$ positive definite matrix with determinant $\gamma = \det \gamma_{ij}$ (again the normalisation (3.2) is automatically imposed in this basis). A regular non-degenerate horizon requires ω_i, γ_{ij} to be smooth functions of (ρ^2, z) and

$$\lim_{\rho \to 0,\, z \in I_a} \frac{\rho^2 e^{2\nu}}{|\xi|^2} = -\frac{1}{\kappa^2} \ , \tag{3.29}$$

where $\kappa \neq 0$ is the surface gravity [54].

The analysis of the metric induced on the horizon proceeds in an essentially identical fashion to the axis metric analysis above. The inverse metric in this adapted basis is

$$\tilde{g}^{-1} = \begin{pmatrix} \gamma^{ij} & \omega^i \\ \omega^j & -\gamma \rho^{-2} + \omega^i \omega_i \end{pmatrix} \tag{3.30}$$

where γ^{ij} is the inverse matrix of γ_{ij} and $\omega^i = \gamma^{ij}\omega_j$. The requirement of a smooth horizon then implies the limits (3.19) exist, which relative to the adapted basis are

$$\mathring{\tilde{U}} = \begin{pmatrix} 0 & -2\omega_i \\ 0 & 2 \end{pmatrix} , \qquad \mathring{\tilde{V}} = \begin{pmatrix} (\partial_z \gamma_{ik})\gamma^{kj} & -\gamma\gamma_{ij}\partial_z(\gamma^{-1}\omega^j) \\ 0 & -(\gamma^{-1}\partial_z\gamma) \end{pmatrix} . \tag{3.31}$$

Then the second equation in (3.5) integrates to

$$e^{2\mathring{\nu}} = \frac{\tilde{c}^2}{\gamma} \tag{3.32}$$

where $\tilde{c} > 0$ is a constant and imposing the smoothness condition (3.29) gives

$$\tilde{c} = \kappa^{-1} . \tag{3.33}$$

We deduce that the metric induced on the horizon component H_a associated to the rod I_a is

$$\mathbf{g}|_{H_a} = \frac{\mathrm{d}z^2}{\kappa_a^2 \gamma(z)} + \gamma_{ij}(z)\mathrm{d}\phi^i \mathrm{d}\phi^j , \tag{3.34}$$

where we have reinstated the rod labels. This is a $(D-2)$-dimensional smooth Riemannian metric for $z \in I_a$ (recall the axial Killing fields $m_i = \partial_{\phi_i}$).

Given the metric on a horizon H_a, one can determine the surface gravity as follows. In general there are conical singularities in the metric (3.34) at the endpoints $z = z_{a-1}, z_a$ and demanding that they are absent will fix κ_a. For $D = 4$ we have $m = \partial_\phi$ vanishing at each endpoint so the condition for no conical singularities is simply

$$\kappa_a = \frac{2}{\gamma'(z_{a-1})} = -\frac{2}{\gamma'(z_a)} . \tag{3.35}$$

In order to fix the sign we have used the fact that $\gamma'(z_{a-1}) > 0$ and $\gamma'(z_a) < 0$ (these follow from $\gamma > 0$ in the interior of I_a). Observe this gives two ways of calculating κ_a and hence in principle can provide a non-trivial constraint on the parameters of the solution. For $D = 5$ the adjacent rods I_{a-1} and I_{a+1} are axis rods with rod vectors v_{a-1} and v_{a+1}. In particular $v_{a-1} = 0$ at $z = z_{a-1}$ and $v_{a+1} = 0$ at $z = z_a$, so that the horizon metric has conical singularities at the endpoints of I_a. The horizon metric extends to a smooth metric at these end points iff the surface gravity

$$\kappa_a^2 = \frac{4}{\gamma'(z_{a-1})\gamma'_{ij}(z_{a-1})v^i_{a-1}v^j_{a-1}} = \frac{4}{\gamma'(z_a)\gamma'_{ij}(z_a)v^i_{a+1}v^j_{a+1}} . \tag{3.36}$$

Therefore, again, in principle this gives two independent expressions for κ_a and hence may provide a constraint on the parameters of the solution. In Appendix 3.B we obtain further relations for the surface gravity by studying the geometry near where an axis rod meets a horizon rod. Similarly to the analysis of a corner between two axes described in the previous section, we find that if an axis rod and horizon rod meet at $z = z_a$ then $|z - z_a|e^{2\mathring{\nu}}$ is continuous at $z = z_a$.

Using (3.34) one can also compute the area of a cross-section of the horizon

$$A = \int_{H_a} \kappa_a^{-1} \mathrm{d}z \mathrm{d}\phi^1 \cdots \mathrm{d}\phi^{D-3} = \frac{(2\pi)^{D-3}\ell_a}{\kappa_a} , \tag{3.37}$$

a relation which has been previously derived [67].

Standard basis

In order to compare the solutions on each rod it is useful to write them in a common basis of Killing fields. For definiteness we will take a basis adapted to the semi-infinite rod I_L, i.e. the standard basis $E_A = (k, m_1, \dots, m_{D-3})$. We can relate the adapted bases $\tilde{E}_A$ associated to each rod I_a to the standard basis by $\tilde{E}_A = (L_a^{-1})_A^{\ B} E_B$ where L_a is a change of basis matrix. The metric $\tilde{g}$ in the adapted basis $\tilde{E}_A$, relative to the standard basis is thus

$$g = L_a \tilde{g} L_a^T \; , \tag{3.38}$$

where $\tilde{g}$ is given by (3.16) or (3.28) for an axis rod or horizon rod respectively.

If I_a is a horizon rod then $\tilde{E}_A = (m_i, \xi_a)$ where ξ_a is the corotating Killing field (3.27) for the component of the horizon H_a, so

$$L_a = \begin{pmatrix} -\Omega_j^a & 1 \\ \delta_j^i & 0 \end{pmatrix} \; . \tag{3.39}$$

On the other hand, now suppose I_a is an axis rod. In 4d there is of course only one axial Killing field and so there is only one type of axis rod and hence the transformation matrices L_a are the identity matrix for all axis rods. In 5d we take the basis $\tilde{E}_A = (k, u_a, v_a)$, where (u_a, v_a) is a basis of $U(1)^2$ Killing fields such that v_a is the rod vector, which gives

$$L_a = \begin{pmatrix} 1 & 0 \\ 0 & A_a^{-1} \end{pmatrix} \; , \tag{3.40}$$

with A_a a $GL(2, \mathbb{Z})$ matrix given by (3.15). In particular, in 5d the right semi-infinite rod I_R has rod vector $v_R = m_1$ and choosing $u_R = m_2$ gives

$$L_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; . \tag{3.41}$$

It is worth noting that for any horizon and axis rods $\det L_a = \pm 1$. Therefore, using (3.38) we deduce that the normalisation (3.2) is also obeyed in the standard basis.

3.2.3 Ernst potentials and gravitational fluxes

We will need to introduce the following Ernst potentials b_μ^a associated to each axis rod I_a,

$$\mathrm{d}b_\mu^a = (-1)^{D-1} \tilde{\star}(e_0 \wedge \dots \wedge e_{D-4} \wedge \mathrm{d}e_\mu) \; , \tag{3.42}$$

where $\tilde{E}_A = (e_\mu, v_a)$ is the adapted basis defined above and we fix an orientation $\tilde{\epsilon}_{0 \cdots D-3 \rho z} > 0$. Therefore $\tilde{\star} = (\det L_a)\star$ where $\star$ is the Hodge dual with respect to the standard orientation (defined above) and L_a is the transformation matrix between the adapted basis and the standard basis. Closure of the 1-form on the RHS of (3.42) follows by the vacuum Einstein equations and simple connectedness ensures the potentials are globally defined. Explicitly, in Weyl coordinates we have

$$\partial_\rho b_\mu^a = \rho \tilde{g}^{D-3A} \tilde{g}_{A\mu,z} \; , \qquad \partial_z b_\mu^a = -\rho \tilde{g}^{D-3A} \tilde{g}_{A\mu,\rho} \; . \tag{3.43}$$

From the explicit form of the metric in the adapted basis (3.16) it follows that near each axis rod I_a

$$\partial_z b_\mu^a = 2 w_\mu + O(\rho) \; , \qquad \partial_\rho b_\mu^a = O(\rho) \tag{3.44}$$

as $\rho \to 0$.

The above Ernst potentials associated to each axis rod depend on the corresponding rod vector. For $D = 4$ there is only one type of axis rod and the corresponding Ernst potential is simply

$$\mathrm{d}b = - \star (k \wedge \mathrm{d}k) \; . \tag{3.45}$$

For $D = 5$, there are many possible axis rods, although there are two rods which appear in any asymptotically flat solution: the two semi-infinite axis rods I_L and I_R on which $m_2 = 0$ and $m_1 = 0$ respectively. The Ernst potentials (3.42) associated to I_L and I_R are

$$\mathrm{d}b^L_\mu = \star(k \wedge m_1 \wedge \mathrm{d}e^L_\mu), \qquad e^L_\mu = (k, m_1) \; , \tag{3.46}$$

$$\mathrm{d}b^R_\mu = - \star (k \wedge m_2 \wedge \mathrm{d}e^R_\mu), \qquad e^R_\mu = (k, m_2) \; . \tag{3.47}$$

where the sign in the latter arises from the transformation (3.41) between the adapted basis and the standard basis being orientation reversing, $\det L_R = -1$.

We will also need similar potentials associated to any horizon rod I_a. We define these analogously to the Ernst potentials (3.42). Thus, given our adapted basis for a horizon rod $\tilde{E}_A = (m_i, \xi)$, these potentials are precisely the usual twist potentials (3.13) (observe our choice of orientation in these two formulas is consistent). Therefore, similarly to the Ernst potentials, we find the twist potentials obey

$$\partial_\rho \chi_i = \rho \tilde{g}^{0A} \tilde{g}_{Ai,z} \; , \qquad \partial_z \chi_i = -\rho \tilde{g}^{0A} \tilde{g}_{Ai,\rho} \tag{3.48}$$

and using (3.28) we find that near a horizon rod I_a

$$\partial_z \chi_i = 2\omega_i + O(\rho) \; , \qquad \partial_\rho \chi_i = O(\rho) \tag{3.49}$$

as $\rho \to 0$.

As shown earlier, the change in twist potential over a horizon rod is related to the Komar angular momenta of the horizon (3.12). Similarly, one can relate the change in the Ernst potentials (3.42) across their associated axis rods I_a to certain gravitational fluxes. For $D = 4$ we can define the flux

$$\mathcal{G}[I_a] = - \int_{I_a} \star(k \wedge \mathrm{d}k) \tag{3.50}$$

for any finite axis rod. Since the integrand is closed by the vacuum Einstein equations these fluxes may be evaluated over any curve homotopic to I_a. Clearly, from (3.45) we deduce

$$\mathcal{G}[I_a] = b(z_a) - b(z_{a-1}), \tag{3.51}$$

which gives a geometric interpretation to the change in Ernst potential over an axis rod.

Similarly, for $D = 5$, given any finite axis rod I_a we may define gravitational fluxes on the corresponding 2-cycle C_a. Explicitly, for each 2-cycle C_a one can define a set of fluxes

$$\mathcal{G}_\mu[C_a] = \frac{1}{2\pi} \int_{C_a} \tilde{\star}(e_0 \wedge \mathrm{d}e_\mu), \tag{3.52}$$

where $e_\mu = (k, u_a)$, $\mu = 0, 1$, is our adaped basis of Killing fields on C_a (recall $\tilde{E}_A = (k, u_a, v_a)$ is the adapted basis of Killing fields in the full spacetime). The integrand is closed by the vacuum Einstein equations so one can evaluate these fluxes over any 2-surface homologous to C_a so it only depends on the homology class $[C_a]$. Thus these fluxes define gravitational topological charges. Due to the invariance under the Killing fields these integrals can be reduced to ones over the corresponding axis rods,[2]

$$\mathcal{G}_\mu[C_a] = \int_{I_a} \tilde{\star}(e_0 \wedge e_1 \wedge \mathrm{d}e_\mu) = b^a_\mu(z_a) - b^a_\mu(z_{a-1}) \;, \tag{3.53}$$

where we have used the definition of the Ernst potentials (3.42). Thus we see that the fluxes $\mathcal{G}_\mu[C_a]$ precisely correspond to the change in the Ernst potential $b^a_\mu(z)$ over the associated axis rod I_a giving it a geometric interpretation. A similar set of topological charges have appeared in recent identities that relate the thermodynamic variables to the topology of solutions in this class [142].

Finally, it is worth noting that one can also relate the changes in Ernst potentials $b^a_\mu(z)$ (3.42) over a horizon rod I_a to the standard thermodynamic quantities. We give these expressions in Appendix 3.C.

3.3 Integrability of Einstein equations

3.3.1 Belinski-Zakharov spectral equations

As we saw in Chapter 1, vacuum gravity is an integrable theory. More concretely as shown by Belinski and Zakharov (BZ), the Einstein equations (3.3) are the integrability conditions for the following auxiliary linear system [64, 65],

$$D_z\Psi = \frac{\rho V - \mu U}{\mu^2+\rho^2}\Psi, \qquad D_\rho\Psi = \frac{\rho U + \mu V}{\mu^2+\rho^2}\Psi \;, \tag{3.54}$$

where

$$D_z = \partial_z - \frac{2\mu^2}{\mu^2+\rho^2}\partial_\mu, \qquad D_\rho = \partial_\rho + \frac{2\mu\rho}{\mu^2+\rho^2}\partial_\mu \tag{3.55}$$

are commuting differential operators, μ is a complex 'spectral' parameter and Ψ is an invertible $(D-2)\times(D-2)$ complex matrix function of (ρ, z, μ).

We will work with a slightly different version of the BZ linear system [145, 146, 36]. This is obtained by a change of spectral parameter defined by the coordinate change $(\rho, z, \mu) \to (\rho, z, k)$ where

$$k = z + \frac{\mu^2 - \rho^2}{2\mu} \;, \tag{3.56}$$

which in particular implies $D_z \to \partial_z, D_\rho \to \partial_\rho$. This results in the linear system

$$\partial_z\Psi = \frac{\rho V - \mu U}{\mu^2+\rho^2}\Psi, \qquad \partial_\rho\Psi = \frac{\rho U + \mu V}{\mu^2+\rho^2}\Psi \;, \tag{3.57}$$

where $\mu = \mu(k)$ is defined implicitly by (3.56) and k is the new complex spectral parameter. We will assume Ψ is a smooth function of (ρ, z) and meromorphic in k (in a suitable domain). Henceforth we will work exclusively with this alternate form of the BZ linear system (3.57). It turns out to be more useful for our purposes since, as (3.56) shows, the spectral parameter k is defined on a two-sheeted Riemann surface.

Independently of (3.54), one can check directly from (3.57) that $\partial_z\partial_\rho\Psi = \partial_\rho\partial_z\Psi$ iff

$$\partial_\rho\left(\frac{V}{\rho}\right) - \partial_z\left(\frac{U}{\rho}\right) - \frac{1}{\rho^2}[U, V] = 0 \tag{3.58}$$

[2] Here we are using the identity $\int_{C_a}\tilde{\star}\alpha = -2\pi\int_{I_a}\tilde{\star}(u_a\wedge\alpha)$, valid for any $U(1)^2$-invariant 3-form α. This also shows that $\int_{C_a}\tilde{\star}(e_1\wedge \mathrm{d}e_\mu) = 0$ so that these quantities do not give rise to new charges.

and

$$\partial_\rho \mu = \frac{2\rho\mu}{\mu^2+\rho^2}, \qquad \partial_z \mu = -\frac{2\mu^2}{\mu^2+\rho^2} \tag{3.59}$$

and the Einstein equations (3.3) are satisfied. Equation (3.58) is in fact identically satisfied as it is the integrability condition for the existence of a matrix g such that (3.4), whereas the general solution to (3.59) is given by (3.56) where k is the integration constant. For some purposes it will be convenient to write the linear system in the equivalent form

$$(\rho\partial_\rho - \mu\partial_z)\Psi = U\Psi, \qquad (\mu\partial_\rho + \rho\partial_z)\Psi = V\Psi. \tag{3.60}$$

In particular, this form will be useful when evaluating on the boundary of the half-plane.

Although solving for Ψ in general is complicated, it is straightforward to solve for the general form of $\det\Psi$. Right multiplying (3.57) by Ψ^{-1} and taking the trace gives

$$\partial_\rho \det\Psi = \frac{2\rho\det\Psi}{\mu^2+\rho^2}, \qquad \partial_z \det\Psi = -\frac{2\mu\det\Psi}{\mu^2+\rho^2}\ , \tag{3.61}$$

where we have used Tr $U=2$ and Tr $V=0$. Comparing to (3.59) it follows that

$$\det\Psi = \mu f(k), \tag{3.62}$$

where $f(k)$ is an arbitrary function of k (i.e independent for ρ, z).

As we will take k to be a complex parameter we need to take care to treat the implicitly defined function μ in (3.56) properly. Locally, we may solve for μ to get

$$\mu = k - z \pm \sqrt{\rho^2+(k-z)^2}\ . \tag{3.63}$$

Thus there are branch points at $k=w$ and $k=\bar{w}$ where $w=z+i\rho$ and so we take the branch cut to be the finite line in the complex k-plane between these points. Hence we consider the linear system (3.57) on the two sheeted Riemann surface $\Sigma_w \subset \mathbb{C}^2$ defined by

$$y^2 = (k-w)(k-\bar{w})\ , \qquad (k,y)\in\mathbb{C}^2\ . \tag{3.64}$$

The square root function (3.63) is then defined by $\mu:\Sigma_w\to\mathbb{C}$ where

$$\mu(k,y) = k - z + y \tag{3.65}$$

We will denote y on the two sheets (i.e. the two square roots) by $y_\pm(k)$ and use k as a local coordinate on each sheet. For definiteness we define y_+ by having positive real part for Re $(k-w)>0$. We also define $\mu_\pm = \mu(k,y_\pm)$ and note the useful identity $\mu_+\mu_- = -\rho^2$.

We will also denote the corresponding Ψ on the two sheets by $\Psi_\pm$ and similarly for any other quantity on Σ_w. Since $\Psi_\pm$ corresponds to Ψ evaluated on the two sheets of the same Riemann surface we must require a continuity condition at the branch points:

$$\Psi_+(\rho,z,k) = \Psi_-(\rho,z,k) \qquad \text{at} \quad k = z\pm i\rho\ . \tag{3.66}$$

This condition will be important in our later analysis. Taking the determinant of this and comparing to (3.62) shows that $f_+(k)=f_-(k)$ (for Im $k\neq 0$, and by continuity, for all k except perhaps at isolated points) and so we drop the subscript on this quantity.

The spectral equations have an important involution symmetry which allow one to map solutions on one Riemann sheet to the other. The matrices defined by

$$\tilde{\Psi}_\pm = g\Psi_\mp^{T-1} , \tag{3.67}$$

where g is the metric associated to the Killing coordinates x^A, obey the same equations as $\Psi_\pm$, i.e.,

$$(\rho\partial_\rho - \mu_\pm\partial_z)\tilde{\Psi}_\pm = U\tilde{\Psi}_\pm, \qquad (\mu_\pm\partial_\rho + \rho\partial_z)\tilde{\Psi}_\pm = V\tilde{\Psi}_\pm . \tag{3.68}$$

It is easy to show that given two solutions $\Psi_\pm$ and $\tilde{\Psi}_\mp$ to the above equations their 'difference' $B_\pm = \tilde{\Psi}_\pm^{-1}\Psi_\pm$ must be independent of (ρ, z). Therefore, it follows from (3.67) that

$$\Psi_\pm = g\Psi_\mp^{T-1}B_\pm , \tag{3.69}$$

where $B_\pm = B_\pm(k)$ are invertible matrices. It immediately follows from this that $B_\pm = B_\mp^T$. Furthermore, for $\rho > 0$ we can write (3.69) as $B_\pm = \Psi_\mp^T g^{-1}\Psi_\pm$ and evaluating this at the branch points $k = z \pm i\rho$ and using the continuity condition (3.66) shows that $B_\pm(k)$ is symmetric (for Im $k \neq 0$, and by continuity, for all k except perhaps at isolated points). Putting all this together we deduce that $B_+ = B_-^T = B_-$ so we may drop the subscript on B. Thus this symmetry may be simply written as

$$\Psi_\pm = g\Psi_\mp^{T-1}B \tag{3.70}$$

where $B = B(k)$ is an invertible symmetric matrix. Taking the determinant shows

$$\det B(k) = f(k)^2 . \tag{3.71}$$

3.3.2 Spectral equations on semi-circle at infinity

We will consider asymptotically flat spacetimes in four and five dimensions. In both cases the asymptotic region corresponds to the semi-circle at infinity in the half-plane (3.6). Thus it is convenient to introduce polar coordinates (r, θ) on the half-plane where

$$\rho = r\sin\theta, \qquad z = r\cos\theta \tag{3.72}$$

and $0 \leq \theta \leq \pi$. In terms of the complex coordinate $w = z + i\rho$ we have $w = re^{i\theta}$. The semi-circle at infinity then simply corresponds to $r \to \infty$. More precisely, we introduce the contour $C_r = \{re^{i\theta} : 0 \leq \theta \leq \pi\}$ in the half-plane with anticlockwise orientation and consider large r.

Now, fix a sheet of Σ_w with local coordinate k and consider traversing C_r starting at $\theta = 0$. The branch points $w, \bar{w}$ trace out corresponding semi-circles in the upper and lower half of the complex k-plane with a moving branch cut between the upper and lower semi-circle. Any fixed value of k on the sheet must pass through the moving branch cut as we traverse C_r for large enough r (i.e. $r > |k|$). This occurs at an angle given by $\mathrm{Re}(k - w) = 0$, i.e.

$$\cos\theta_* = \frac{\text{Re}(k)}{r} \implies \theta_* = \frac{\pi}{2} - \frac{\text{Re}(k)}{r} + O(r^{-3}) . \tag{3.73}$$

Now, passing through the branch cut corresponds to changing sheet of Σ_w. Therefore, in effect, traversing C_r imposes a change of sheet as we pass through $\theta = \theta_*$. In particular, given a solution to the linear system $\Psi_\pm(r, \theta, k)$ on the two sheets, this implies the following continuity conditions on the semi-circle at infinity

$$\lim_{\epsilon\to 0^+} \Psi_\pm(r, \theta_* - \epsilon, k) = \lim_{\epsilon\to 0^+} \Psi_\mp(r, \theta_* + \epsilon, k) . \tag{3.74}$$

Notice this provides a relation between the Ψ_+ and Ψ_- fields at infinity.

The above considerations also affect the asymptotic expansion of quantities defined on each sheet along infinity. For instance, consider μ_+ on the $+$ sheet. Traversing C_r from $\theta = 0$, it is easy to see that the branch cut approaches a fixed k from the right (where $y_+(k)$ has negative real part) so

$$\mu_+(r,\theta,k) = (k-r)(1+\cos\theta) + O(r^{-1}) \qquad 0 \le \theta < \theta_* \ , \tag{3.75}$$

whereas traversing C_r from $\theta = \pi$, the branch cut approaches k from the left so

$$\mu_+(r,\theta,k) = (k+r)(1-\cos\theta) + O(r^{-1}) \qquad \theta_* < \theta \le \pi \ . \tag{3.76}$$

A similar argument for μ_- shows that

$$\mu_-(r,\theta,k) = \left\{ \begin{array}{ll} (k+r)(1-\cos\theta) + O(r^{-1}) & \quad 0 \le \theta < \theta_* \\ (k-r)(1+\cos\theta) + O(r^{-1}) & \quad \theta_* < \theta \le \pi \end{array} \right. . \tag{3.77}$$

Observe that the continuity conditions $\lim_{\epsilon\to 0^+} \mu_\pm(r,\theta_*-\epsilon,k) = \lim_{\epsilon\to 0^+} \mu_\mp(r,\theta_*+\epsilon,k)$ are indeed satisfied.

It is convenient to write our linear system (3.57) in polar coordinates, which gives,

$$\partial_r \Psi = Y_r \Psi, \qquad Y_r = \frac{r\sin^2\theta S - \mu T}{\mu^2 + r^2\sin^2\theta} \tag{3.78}$$

$$\partial_\theta \Psi = Y_\theta \Psi \ , \qquad Y_\theta = \frac{r\sin\theta(\mu S + rT)}{\mu^2 + r^2\sin^2\theta} \tag{3.79}$$

where $S = r\partial_r g g^{-1}$ and $T = \sin\theta\partial_\theta g g^{-1}$. We now consider the solution to the spectral equations in the limit $r \to \infty$.

The explicit solution depends on the dimension, although it has some common features which will be key in our analysis. Let $\bar{g}$ denote the Minkowski metric and $\bar{\Psi}$ a corresponding solution to the spectral equation (3.79). Now define the 'difference',

$$\Delta = \bar{\Psi}^{-1}\Psi \ , \tag{3.80}$$

between a Minkowski solution $\bar{\Psi}$ and a solution Ψ to (3.79) for any asymptotically flat metric g. Then, it easily follows that

$$\begin{aligned} (\partial_r\Delta)\Delta^{-1} &= \Upsilon_r, \qquad \Upsilon_r \equiv \bar{\Psi}^{-1}(Y_r - \bar{Y}_r)\bar{\Psi}, \\ (\partial_\theta\Delta)\Delta^{-1} &= \Upsilon_\theta \ , \qquad \Upsilon_\theta \equiv \bar{\Psi}^{-1}(Y_\theta - \bar{Y}_\theta)\bar{\Psi} \ . \end{aligned} \tag{3.81}$$

The matrices Υ depend on the explicit solution in Minkowski spacetime and the definition of asymptotic flatness, which for $D = 4,5$ will be given later. All that we need at this stage is that for both dimensions, all matrix entries of Υ_r and Υ_θ are $O(r^{-2})$ and $O(r^{-1})$ respectively, as $r \to \infty$. Thus, asymptotically, Δ must be only a function of k. In other words, the solution to the spectral equation for an asymptotically flat spacetime is asymptotic to that for Minkowski spacetime, as one would expect.

More precisely, consider the solution on the $+$ sheet of Σ_w

$$\Psi_+ = \bar{\Psi}_+\Delta_+ \ . \tag{3.82}$$

From the above it follows that

$$\Delta_+ = \left\{ \begin{array}{ll} N_R(k) + O(r^{-1}) & \quad 0 \le \theta < \theta_* \\ N_L(k) + O(r^{-1}) & \quad \theta_* < \theta \le \pi \end{array} \right. , \tag{3.83}$$

where $N_{R,L}(k)$ are invertible matrices and R, L denote the right and left segment (these in general are different since $\Upsilon_{r+}, \Upsilon_{\theta+}$ are discontinuous on C_r at $\theta = \theta_*$). Using the involution symmetry (3.70) we find that

$$\Psi_- = g \bar{\Psi}_+^{T-1} \Delta_+^{T-1} B \,, \tag{3.84}$$

where $B = B(k)$ is the matrix used in (3.70). Hence imposing the continuity conditions (3.74) we deduce that

$$\begin{aligned} C \;&\equiv\; N_R^{T-1}(k) B(k) N_L(k)^{-1} && (3.85)\\ &=\; \lim_{r\to\infty} \bar{\Psi}_+^T(r, \theta_*^-, k) g(r,\theta_*)^{-1} \bar{\Psi}_+(r, \theta_*^+, k) \,. && (3.86)\end{aligned}$$

The relation (3.86) allows one to compute C given the asymptotics of the Minkowski solution. It is worth remarking that although (3.74) consists of two continuity equations, the fact that B is a symmetric matrix (3.70) ensures that they are equivalent.

There is a certain freedom in the choice of $\bar{\Psi}_+$ corresponding to right-multiplication by a matrix function of k. Since the asymptotic expansion (3.76) for μ_+ to leading order is independent of k, we may choose $\bar{\Psi}_+(r, \theta, k)$ such that as $r \to \infty$ the leading term in each entry is independent of k. Making this choice, one then expects from (3.86) that C is independent of k and hence is a constant matrix (we will confirm this explicitly later).

3.3.3 General solution on the axes and horizons

We will now show that the linear system simplifies when evaluated on the boundary of the half-plane. Recall that smoothness of the axes and horizons requires the metric must be a smooth function of (ρ^2, z). Therefore we may assume Ψ is a smooth function of (ρ^2, z).

First we make a few general remarks. In order to evaluate limits to the boundary we will need the following useful relations

$$\mu_+ \sim 2(k - z), \qquad \mu_- \sim -\frac{\rho^2}{2(k-z)} \tag{3.87}$$

as $\rho \to 0$. Thus taking the limit of the determinant $\det \Psi_\pm$ and using (3.62) shows that Ψ_+ is generically a nonsingular matrix on the boundary whereas Ψ_- is singular. Therefore we will only consider Ψ_+ and use (3.70) to deduce Ψ_-.

We are now in a position to evaluate the limit of the linear system (3.60) for Ψ_+ as $\rho \to 0$. It is easy to see this system reduces to an ODE

$$(z - k)\partial_z \mathring{\Psi} = \tfrac{1}{2} \mathring{U} \mathring{\Psi}, \tag{3.88}$$

where we define $\mathring{\Psi}(z, k) = \lim_{\rho\to 0} \Psi_+(\rho, z, k)$ and the second equation vanishes identically due to our assumption that Ψ_+ is a smooth function of ρ^2. We will explicitly solve the linear system along the boundary $\rho = 0$.

First consider an axis rod I_a. In the corresponding adapted basis the metric is given by (3.16). The general solution to the linear system (3.88) on I_a in this basis can be written as

$$\tilde{X}_a(z,k)\tilde{M}_a(k), \qquad \tilde{X}_a(z,k) = \begin{pmatrix} -\delta_\mu^{\ \nu} & b^a_\mu(z) \\ 0 & 2(k - z) \end{pmatrix}, \quad z \in I_a \tag{3.89}$$

where we have used (3.20, 3.44) and $\tilde{M}_a(k)$ is an arbitrary integration matrix. The particular solution $\tilde{X}_a(z, k)$ satisfies

$$\partial_z \tilde{X}_a = -\mathring{\tilde{U}}_a. \tag{3.90}$$

We note there is a lot of freedom in the choice of particular solution $\tilde{X}_a(z,k)$. In particular, the integration constant for the Ernst potential $b^a_\mu(z)$ may be set to any value we like by right multiplying the particular solution by a constant upper triangular matrix with unit diagonals (which can then be absorbed into a redefinition of $\tilde{M}_a(k)$). For convenience we will choose the potentials to vanish at the lower endpoint of the finite rods

$$b^a_\mu(z_{a-1}) = 0 \tag{3.91}$$

for $a = 2, \dots, n$ and

$$\lim_{z\to-\infty} b^L_\mu(z) = 0 \;, \qquad \lim_{z\to\infty} b^R_\mu(z) = 0 \;. \tag{3.92}$$

The latter are consistent with the asymptotics $b^L_\mu \to 0$ and $b^R_\mu \to 0$ at infinity (even off axis).

In order to compare the solutions on each rod we will write them all relative to the standard basis. The metric near each axis rod I_a relative to the standard basis is given by (3.38), which implies $U = L_a \tilde{U} L_a^{-1}$. Hence, from (3.89), we deduce that the general solution to the linear system (3.88) on an axis rod I_a relative to the standard basis takes the form

$$\mathring{\Psi}_a(z,k) = X_a(z,k) M_a(k) \;, \quad z \in I_a \;, \tag{3.93}$$

where

$$X_a(z,k) = L_a \begin{pmatrix} -\delta_\mu{}^\nu & b^a_\mu(z) \\ 0 & 2(k-z) \end{pmatrix} L_a^{-1} \tag{3.94}$$

and $M_a(k)$ are arbitrary matrices. It is also worth recording that the metric on I_a relative to the standard basis (3.38) is simply

$$\mathring{g}(z) = L_a \begin{pmatrix} h^a_{\mu\nu}(z) & 0 \\ 0 & 0 \end{pmatrix} L_a^T \;. \tag{3.95}$$

Recall that in these formulas, if $D = 4$ the matrix L_a is the identity matrix, whereas if $D = 5$ it is given by (3.40).

Now consider a horizon rod I_a. An entirely analogous derivation of the solution can be given in this case using (3.31, 3.49). Thus we find the general solution to the linear system (3.88) on a horizon rod I_a relative to the standard basis can be again written as (3.93) where

$$X_a(z,k) = L_a \begin{pmatrix} -\delta_i{}^j & \chi^a_i(z) \\ 0 & 2(k-z) \end{pmatrix} L_a^{-1} \tag{3.96}$$

and $\chi^a_i(z) = \chi_i(z) - \chi_i(z_{a-1})$ (which corresponds to a choice of integration constant), the matrix L_a is given by (3.39) and $M_a(k)$ are arbitrary matrices. The metric on I_a relative to the standard basis (3.38) is simply

$$\mathring{g}(z) = L_a \begin{pmatrix} \gamma_{ij}(z) & 0 \\ 0 & 0 \end{pmatrix} L_a^T \;. \tag{3.97}$$

We now have the general solution to the linear system on all components of the boundary $\rho = 0$.

Before moving on it is worth noting that for both axis and horizon rods we have

$$\det X_a(z,k) = 2(-1)^{D-3}(k-z) \tag{3.98}$$

and combining this with (3.62) implies

$$\det M_a(k) = (-1)^{D-3} f(k) \;, \tag{3.99}$$

for all $a = 1, \dots, n+1$.

Clearly we must impose continuity of $\mathring{\Psi}(z,k)$ at $z = z_a$ for $a = 1,\dots,n$, where adjacent rods I_a and I_{a+1} touch, i.e.,

$$\mathring{\Psi}_a(z_a,k) = \mathring{\Psi}_{a+1}(z_a,k) \; . \tag{3.100}$$

Upon using the general solution this gives

$$M_a(k) = P_a(k) M_{a+1}(k) \tag{3.101}$$

where we have introduced the matrices

$$P_a(k) = X_a(z_a,k)^{-1} X_{a+1}(z_a,k) \; , \tag{3.102}$$

for each $a = 1,\dots,n$. Observe that from (3.98) it follows that $\det P_a(k) = 1$ automatically. Iterating we find

$$M_a(k) = Q_a(k) M_R(k), \tag{3.103}$$
$$Q_a(k) \equiv P_a(k) P_{a+1}(k) \cdots P_n(k) \tag{3.104}$$

for $a = 1,\dots,n+1$ with $Q_{n+1}(k)$ understood as the $(D-2)$-dimensional identity matrix. In particular

$$M_L(k) = Q_1(k) M_R(k) \; . \tag{3.105}$$

Note the fact $P_a(k)$ is unit determinant implies $\det Q_a(k) = 1$ automatically.

We may now match the solution on the semi-infinite axis rods to the solution for an asymptotically flat spacetime near infinity (3.82) and (3.83). Firstly, the solutions for Minkowski spacetime on the semi-infinite axes can be deduced from the above by setting $b_\mu^{L,R}(z) = 0$. A convenient choice, such that these solutions are independent of k to leading order as $|z| \to \infty$, is

$$\mathring{\bar{\Psi}}_L(z,k) = \begin{pmatrix} -\delta_\mu{}^\nu & 0 \\ 0 & 2(k-z) \end{pmatrix}, \qquad \mathring{\bar{\Psi}}_R(z,k) = L_R \begin{pmatrix} -\delta_\mu{}^\nu & 0 \\ 0 & 2(k-z) \end{pmatrix} L_R^{-1} \; . \tag{3.106}$$

Thus from (3.82) we get

$$\mathring{\Delta}_{+L} = M_L(k) + O(z^{-1}), \qquad \mathring{\Delta}_{+R} = M_R(k) + O(z^{-1}) \; , \tag{3.107}$$

where we have used (3.92) and further assumed the asympotic expansion for $b_\mu^{L,R}(z) = O(z^{-1})$ (this follows from the definition of asymptotic flatness as we will see later). Therefore, comparing to (3.83) we deduce that

$$N_R(k) = M_R(k) \; , \qquad N_L(k) = M_L(k) \; . \tag{3.108}$$

We may use this to eliminate the matrices $N_{L/R}$ in favour of $M_{L/R}$ and thus from (3.85) we obtain

$$M_L B^{-1} M_R^T = C^{-1} \; . \tag{3.109}$$

Recall that the choice of asymptotic solutions corresponds to a choice of matrix C (3.86). Later we will see that our choice (3.106) fixes C to be a dimension dependent constant matrix. In any case, taking the determinant of (3.109) and using (3.71) and (3.99) implies

$$\det C = 1 \tag{3.110}$$

independently of the dimension.

Using the invertible symmetric matrix $B = B(k)$ and the matrices $M_a = M_a(k)$ associated to each rod I_a, we now introduce an important set of symmetric matrices

$$F_a(k) = -M_a(k)B(k)^{-1}M_a(k)^T \tag{3.111}$$

for each rod I_a. The definition of F_a can be rewritten using (3.103), (3.105) and (3.109) to give

$$F_a = -Q_a Q_1^{-1} C^{-1} Q_a^T \tag{3.112}$$

or more explicitly in terms of $P_a(k)$ to give

$$\begin{aligned} F_L &= -C^{-1}P_n^T \cdots P_1^T, \\ F_a &= -P_{a-1}^{-1} \cdots P_1^{-1} C^{-1} P_n^T \cdots P_a^T, \\ F_R &= -P_n^{-1} \cdots P_1^{-1} C^{-1}, \end{aligned} \tag{3.113}$$

where a runs from 2 to n. In general the determinant of F_a is

$$\det F_a(k) = (-1)^{D-2} \tag{3.114}$$

as a consequence of $Q_a(k)$ being unit determinant and (3.110).

We are now ready to state the main result of this section.

Proposition 1. *The metric data on each rod satisfies the algebraic equation*

$$\mathring{g}(z) = X_a(z,z)F_a(z)\ , \qquad z \in I_a \tag{3.115}$$

where $F_a(z)$ is given by (3.111), whereas $\mathring{g}(z)$ and $X_a(z,z)$ are given by (3.95), (3.94) for an axis rod and (3.97), (3.96) for a horizon rod.

Proof. We impose continuity at the branch points (3.66) on the axis $\rho = 0$:

$$\lim_{k\to z} \Psi_+(0,z,k) = \lim_{k\to z} \Psi_-(0,z,k)\ . \tag{3.116}$$

Using (3.70) to write Ψ_- in terms of Ψ_+, the continuity condition (3.116) reads

$$\mathring{\Psi}(z,z) = \lim_{k\to z} \mathring{g}(z)\mathring{\Psi}(z,k)^{T-1}B(k)\ . \tag{3.117}$$

Evaluating on each rod and using the general solution (3.93), equation (3.111) and the elementary identity $\mathring{g}(z)X_a(z,k)^{T-1} = -\mathring{g}(z)$, gives (3.115) as claimed. □

We emphasise that, crucially, equation (3.115) does not depend on the arbitrary matrices $M_a(k)$ and hence provides a constraint on the spacetime geometry. In fact, (3.115) fully determines the functional form of the metric on each rod I_a. Indeed, both $\mathring{g}(z)$ and $X_a(z,z)$ for $z \in I_a$ are rank-$(D-3)$ so (3.115) gives $\frac{1}{2}(D-3)(D-2)+D-3$ algebraic equations for the $\frac{1}{2}(D-3)(D-2)+D-3$ unknowns, either $(h^a_{\mu\nu}(z), b^a_\mu(z))$ or $(\gamma_{ij}(z), \chi_i(z))$ (depending on if I_a is an axis or horizon rod).

3.3.4 Classification theorem

The explicit solution for the metric on each axis rod is summarised by the following theorem which is the main result of this chapter.

Theorem 8. *Consider a $D = 4$ or 5-dimensional vacuum spacetime as in Theorem 3.*

1. *The general solution $(h^a_{\mu\nu}(z), b^a_\mu(z))$ on any axis rod I_a is*

$$h^a_{\mu\nu}(z) = -\tilde{F}_{a\mu\nu}(z) + \frac{\tilde{F}_{a\mu N}(z)\tilde{F}_{aN\nu}(z)}{\tilde{F}_{aNN}(z)}, \qquad b^a_\mu(z) = \frac{\tilde{F}_{a\mu N}(z)}{\tilde{F}_{aNN}(z)}, \tag{3.118}$$

where $\mu = 0, \ldots, D-4$ and $N = D-3$ and the matrices $\tilde{F}_a(k)$ are defined by

$$F_a(k) = L_a \begin{pmatrix} \tilde{F}_{a\mu\nu}(k) & \tilde{F}_{a\mu N}(k) \\ \tilde{F}_{aN\nu}(k) & \tilde{F}_{aNN}(k) \end{pmatrix} L_a^T . \tag{3.119}$$

In particular, this implies

$$\det h^a_{\mu\nu}(z) = -\frac{1}{\tilde{F}_{aNN}(z)}. \tag{3.120}$$

2. *The general solution $(\gamma_{ij}(z), \chi^a_i(z))$ on any horizon rod I_a is*

$$\gamma_{ij}(z) = -\tilde{F}_{aij}(z) + \frac{\tilde{F}_{ai0}(z)\tilde{F}_{a0j}(z)}{\tilde{F}_{a00}(z)}, \qquad \chi^a_i(z) = \frac{\tilde{F}_{ai0}(z)}{\tilde{F}_{a00}(z)}, \tag{3.121}$$

where $i = 1, \ldots, D-3$ and $\tilde{F}_a(k)$ is defined by

$$F_a(k) = L_a \begin{pmatrix} \tilde{F}_{aij}(k) & \tilde{F}_{ai0}(k) \\ \tilde{F}_{a0j}(k) & \tilde{F}_{a00}(k) \end{pmatrix} L_a^T . \tag{3.122}$$

In particular,

$$\det \gamma_{ij}(z) = -\frac{1}{\tilde{F}_{a00}(z)}. \tag{3.123}$$

In both cases $F_a(k)$ are the matrices defined by (3.113). The solution depends on the 'moduli'

$$\{b^L_\mu(z_1), b^R_\mu(z_n)\} \cup \{(\ell_a, v_a, b^a_\mu(z_a) | I_{a\neq L,R} \subset \hat{A}\} \cup \{(\ell_a, \Omega^a_i, \chi^a_i(z_a) | I_a \subset \hat{H}\}, \tag{3.124}$$

where $\hat{A}$ and $\hat{H}$ are the union of axis and horizon rods respectively.

Proof. First consider an axis rod I_a and let us write $F_a(k)$ as (3.119). Then, using (3.95) and (3.94) reveals that (3.115) is equivalent to $h^a_{\mu\nu} = -\tilde{F}_{a\mu\nu} + b^a_\mu \tilde{F}_{aN\nu}$ and $\tilde{F}_{a\mu N} = F_{aNN} b^a_\mu$. We can solve this for $b^a_\mu = \tilde{F}_{a\mu N}/\tilde{F}_{aNN}$, since $\tilde{F}_{aNN} \neq 0$ for any $z \in I_a$; to see this latter condition simply note that if $\tilde{F}_{aNN} = 0$ then $\tilde{F}_{a\mu N} = 0$ which contradicts the fact $F_a(k)$ must be unimodular (3.114). Thus we find the unique solution on an axis rod I_a is (3.118). Then, recalling that $\det L_a = \pm 1$ for any rod, (3.114) implies that (3.120). A completely analogous analysis holds for any horizon rod I_a.

The matrices $F_a(k)$ are written in terms of the matrices $P_a(k)$ which in turn are defined by (3.102). From the explicit form for $X_a(z,k)$ on each axis rod (3.94) or horizon rod (3.96), it is clear that the set of matrices $P_a(k)$ depend on the parameters $z_a, v_a, b^a_\mu(z_a), b^R_\mu(z_n), \chi^a_i(z_a), \Omega^a_i$. However, due to the translation freedom in the choice of origin of the z-axis the solution can only depend on the constants z_a via the rod lengths $\ell_a = z_a - z_{a-1}$ and therefore the solution depends on (3.124). □

Remarks.

1. Alternate forms of the general solution can be obtained by replacing $F_a(k)$ with $F_a(k)^T$ for some $a \in \{1, \ldots, n+1\}$, although of course these are all equivalent since the $F_a(k)$ are symmetric.

2. The horizon moduli $\chi_i^a(z_a)$ are (up to a constant) the horizon angular momenta J_i^a (3.12). We will recover this result from an asymptotic analysis of the general solution later. On the other hand, the axis moduli $b_\mu^a(z_a)$ are equal to the gravitational fluxes (3.51) and (3.53).

3. From the explicit form of the matrices (3.113), (3.102), (3.94), (3.96) it is easy to see that the metric components and potentials on each rod are rational functions of z.

3.3.5 Moduli space of solutions

In order to complete this classification result one needs to develop a full understanding of the moduli space of unbalanced solutions. Note that in general, regularity of the axes would impose further constraints on the moduli from the conditions for the removal of conical singularities (3.24), (3.25), (3.26), (3.35), (3.36) (see also Appendix 3.B).

We split the discussion into two parts, first talking about the equations and then the inequalities that govern the moduli space.

Equation constraints:

There are a number of relations that arise from consistency conditions that the solution derived in Theorem 8 should satisfy. Since the general solution for the Ernst and twist potentials $\{b_\mu^a(z), \chi_i^a(z)\}$ on the finite rods (3.118), (3.121) depends on $\{b_\mu^a(z_a), \chi_i^a(z_a)\}$, there are potential constraints from the relations: $b_\mu^a(z)|_{z\to z_{a-1}} = 0$ (recall (3.91)) and $b_\mu^a(z)|_{z\to z_a} = b_\mu^a(z_a)$ and the corresponding constraints for horizon rods. In total these amount to $2(D-3)(n-1)$ conditions, $(D-3)(n-1)$ of which are automatically satisfied by our solution as the following shows (note we are not using the symmetry of the $F_a(k)$ matrices here).

Proposition 2. *For the general solution (3.118), (3.121), the following identities are satisfied for generic values of the moduli :*

$$\lim_{z\to z_{a-1}} b_\mu^a(z) = 0, \tag{3.125}$$
$$\lim_{z\to z_{a-1}} \chi_i^a(z) = 0, \tag{3.126}$$

if I_a is a finite axis rod or horizon rod respectively.

On the other hand, for the general solution with $F_a(k)$ replaced by $F_a(k)^T$ the following identities are satisfied for generic values of the moduli:

$$\lim_{z\to z_a} b_\mu^a(z) = b_\mu^a(z_a), \tag{3.127}$$
$$\lim_{z\to z_a} \chi_i^a(z) = \chi_i^a(z_a), \tag{3.128}$$

if I_a is a finite axis rod or horizon rod respectively.

Proof. First, using (3.113), we can write $F_a(k) = X_a(z_{a-1},k)^{-1}G_a(k)$, where $G_a(k)$ is a matrix with a finite limit as $k \to z_{a-1}$, for $a = 2,\ldots,n$. Then, if I_a is an axis rod, from (3.94) we get

$$\tilde{F}_a(k) = \begin{pmatrix} -\delta_\mu{}^\nu & 0 \\ 0 & \frac{1}{2(k-z_{a-1})} \end{pmatrix} \tilde{G}_a(k), \tag{3.129}$$

where $\tilde{F}_a(k)$ is defined in Theorem 8, and $G_a \equiv L_a \tilde{G}_a L_a^T$ is defined similarly. Using (3.118) implies the solution

$$b^a_\mu(z) = -\frac{2(z-z_{a-1})\tilde{G}_{a\mu N}(z)}{\tilde{G}_{aNN}(z)} \,. \tag{3.130}$$

Therefore, if $\lim_{k\to z_{a-1}} \tilde{G}_{aNN}(k) \neq 0$ for generic parameter values, the claim (3.125) follows. This is proved in Appendix 3.D. The analysis for a horizon rod is essentially identical.

Next, we can write $F_a(k)^T = X_a(z_a,k)^{-1}H_a(k)$, where $H_a(k)$ is a matrix with a finite limit as $k \to z_a$. Using (3.94) we find

$$\tilde{F}_a(k)^T = \begin{pmatrix} -\delta_\mu{}^\nu & \frac{b^a_\mu(z_a)}{2(k-z_a)} \\ 0 & \frac{1}{2(k-z_a)} \end{pmatrix} \tilde{H}_a(k), \tag{3.131}$$

where $H_a \equiv L_a \tilde{H}_a L_a^T$. Therefore the general solution (3.118) with $F_a(k)$ replaced with $F_a(k)^T$ gives

$$b^a_\mu(z) = b^a_\mu(z_a) - \frac{2(z-z_a)\tilde{H}_{a\mu N}(z)}{\tilde{H}_{aNN}(z)} \,, \tag{3.132}$$

which implies (3.127), since $\lim_{k\to z_a} \tilde{H}_{aNN}(k) \neq 0$ for generic parameter values (again, see Appendix 3.D). The analysis for a horizon rod is completely analogous. □

Remarks.

1. Conversely, for the general solution the conditions (3.127) and (3.128) generically provide non-trivial constraints on the moduli (3.124). Similarly, for the solution with $F_a(k)$ replaced by $F_a(k)^T$, the conditions (3.125) and (3.126) generically give non-trivial constraints. Thus, in either case these consistency relations on the finite rods generically provide $(D-3)(n-1)$ constraints on the moduli (3.124).

2. There are analogous relations that are satisfied automatically on the semi-infinite rods, i.e. for the solution (3.118) using F_R on I_R and F_L^T (rather than F_L) on I_L one finds that

$$\lim_{z\to z_n} b^R_\mu(z) = b^R_\mu(z_n) \,, \qquad \lim_{z\to z_1} b^L_\mu(z) = b^L_\mu(z_1) \,. \tag{3.133}$$

3. An interesting consequence of this Proposition is that if we now use the fact that the F_a matrices are symmetric, then both sets of consistency conditions (3.125, 3.126) and (3.127, 3.128) are satisfied and thus provide no further constraint on the moduli.

There are also consistency conditions on the solution from Theorem 8 on the semi-infinite rods as one approaches infinity. The most basic of these are simply that $b^L(z)$ and $b^R(z)$ go to zero as $|z| \to \infty$ as they should by definition (3.92). One additionally wishes to impose that the solution is asymptotically flat, as has been assumed. We will consider these conditions in more detail in the next two sections when we specialise to the $D=4$ and $D=5$ cases.

Inequality constraints:

For an axis rod I_a, we can combine the relation for $\det h^a_{\mu\nu}$ in terms of F_a (3.120) and the fact that $h^a_{\mu\nu}(z)$ is a smooth Lorentzian metric on I_a to give

$$\tilde{F}_{aNN}(z) > 0 \ , \qquad z \in I_a \ . \tag{3.134}$$

Similarly for a horizon rod I_a, using the formula for $\det \gamma_{ij}$ in terms of F_a (3.123) and the fact that γ_{ij} must be a smooth positive definite metric on I_a we have

$$\tilde{F}_{a00}(z) < 0 \ , \qquad z \in I_a \ . \tag{3.135}$$

In addition to these signature conditions there are a couple of constraints implicit in the use of rod structures. These are the fact that the rod lengths are positive for each finite rod

$$\ell_a > 0, \qquad I_{a\neq L,R} \tag{3.136}$$

and the fact

$$\det h^a(z) = 0, \quad z \in \partial I_a, \qquad \det \gamma(z) = 0, \quad z \in \partial \hat{H}. \tag{3.137}$$

This latter condition is simply a consequence of neighbouring rods having independent rod vectors. In fact combining this condition with the solutions for these determinants (3.120), (3.123) and using the fact that the F matrices are rational functions (ultimately a consequence of the form of X_a (3.94)), it follows that $\tilde{F}_{aNN}$ and $\tilde{F}_{a00}$ have simple poles at the endpoints of their respective rods. Then the signature conditions imply that

$$r_a^- := \mathrm{res}_{k=z_{a-1}} \tilde{F}_{a22}(k) > 0, \qquad r_a^+ := \mathrm{res}_{k=z_a} \tilde{F}_{a22}(k) < 0, \tag{3.138}$$

for axis rods, and

$$r_a^- := \mathrm{res}_{k=z_{a-1}} \tilde{F}_{a00}(k) < 0, \qquad r_a^+ := \mathrm{res}_{k=z_a} \tilde{F}_{a00}(k) > 0, \tag{3.139}$$

for horizon rods. These residue conditions will be useful in particular in the analysis of the black lens in the next chapter.

Counting arguments:

We are now ready to consider the moduli space of solutions with $n+1$ rods and h horizons (thus there are $n-1-h$ finite axis rods) that are potentially singular on the axis. The general solution on the z-axis we have found depends on a number of moduli (3.124): the rod structure, the change in Ernst and twist potentials across each axis and horizon rod, and the horizon angular velocities. Thus, the number of continuous parameters is given by $n-1+(n+1+h)(D-3)$. On the other hand, from the uniqueness and existence Theorems 3 and 4 we know that the solutions can be specified by the rod structure and the change in twist potentials across each horizon rod (recall by (3.12) these are equal to the horizon angular momenta $\{J_i^a\}$), which consists of $n-1+(D-3)h$ parameters (see (1.19)). Thus we expect $(D-3)(n+1)$ independent relations on the moduli (3.124); these may be thought of as determining $\{\Omega_i^a, b_\mu^a(z_a), b_\mu^L(z_1), b_\mu^R(z_n)\}$ in terms of the fundamental moduli $\{\ell_a, v_a, \chi_i^a(z_a)\}$ (although in practice these may not be the best parameters to express the solution with).

We find that these $(D-3)(n+1)$ constraints can be exactly accounted for by combining the non-trivial conditions on the Ernst/twist potentials at rod points (there are $(D-3)(n-1)$ of these as argued in Remark 1 under Proposition 2) with the $2(D-3)$ asymptotic conditions on b^L, b^R (3.92). This observation motivates the following conjecture:

Conjecture 1. *Consider a solution as in Theorem 8 (3.118), (3.121) with $F_a(k)$ replaced by $F_a(k)^T$ for $a = 1, \ldots, n$ and satisfying the moduli space inequalities above (3.134), (3.135), (3.136), (3.138), (3.139). Then using the consistency conditions on the Ernst and twist potentials (3.125), (3.126) and (3.92) one recovers the unique unbalanced solution described in Theorem 4.*

Remarks.

1. Here we use an alternate form of the solution in Theorem 8 with F_a being replaced by its transpose for all rods except I_R. This is motivated by Remark 2 under Proposition 2 which ensures that the non-trivial consistency conditions on b^L and b^R are both from the conditions at infinity (3.92). The choice of F_a or its transpose on finite rods is unimportant.

2. This conjecture holds for the simplest $D = 4, 5$ rod structures we consider i.e. those of flat space, the Kerr solution and the Myers-Perry solution. For the black ring and black lens rod structures in $D = 5$ that we consider in the next chapter we do not verify this conjecture since we do not fully analyse the space of unbalanced solutions in these cases.

3. Consider the special case where all the continuous moduli (3.124) are set to zero, except for the rod lengths ℓ_a. Also, for $D = 5$, suppose that any finite axis rods have rod vectors v_L or v_R. Then it is straightforward to see that b^a and χ^a are automatically zero since the F_a matrices are diagonal (the matrix C turns out to be diagonal for $D = 4, 5$, see (3.151, 3.179)). This satisfies the consistency conditions and we indeed have the unique unbalanced solution corresponding to this setup, a member of the class of (generalised) Weyl solutions. These are normally characterised by the requirement that the $D - 2$ commuting Killing fields are hypersurface-orthogonal [53] (so all Ernst/twist potentials must be constants which can be fixed to zero).

In light of this conjecture we will slightly abuse language in the succeeding sections in referring to the consistency conditions along with the various inequalities as the moduli space equations.

3.4 Four dimensions

3.4.1 General solution and physical parameters

In four spacetime dimensions the general solution on each components of the axis and horizon simplifies. It is therefore worth recording some of the key formulas and the solution again in this case. The main simplification arises because there is only one axial Killing field and hence the rod vector which vanishes on any axis rod is always $m = \partial_\phi$ (this of course includes the semi-infinite rods I_L and I_R).

Near any axis rod I_a, the metric (3.16) relative to the standard basis (k, m) is simply

$$g = \begin{pmatrix} h - h^{-1}\rho^2 w^2 & \rho^2 h^{-1} w \\ h^{-1}\rho^2 w & -h^{-1}\rho^2 \end{pmatrix}, \tag{3.140}$$

where $h < 0$. The general solution to the linear system (3.88) on the each axis rod can be written as (3.93) where

$$X_a(z, k) = \begin{pmatrix} -1 & b^a(z) \\ 0 & 2(k - z) \end{pmatrix} \tag{3.141}$$

and $b^a(z) = b(z) - b(z_{a-1})$ for $a = 2, \ldots, n$, $b^L(z) = b^R(z) = b(z)$, and $b(z)$ is the Ernst potential (3.45) fixed by imposing that $b \to 0$ at infinity.

On the other hand, near a horizon rod I_a the metric (3.28) relative to the standard basis is

$$g = L_a \begin{pmatrix} \gamma - \gamma^{-1}\rho^2\omega^2 & \rho^2\gamma^{-1}\omega \\ \gamma^{-1}\rho^2\omega & -\gamma^{-1}\rho^2 \end{pmatrix} L_a^T, \tag{3.142}$$

where $\gamma > 0$ and

$$L_a = \begin{pmatrix} -\Omega^a & 1 \\ 1 & 0 \end{pmatrix} . \tag{3.143}$$

The general solution to the linear system on I_a is (3.93) where

$$X_a(z,k) = L_a \begin{pmatrix} -1 & \chi^a(z) \\ 0 & 2(k-z) \end{pmatrix} L_a^{-1} \tag{3.144}$$

and $\chi^a(z) = \chi(z) - \chi(z_{a-1})$ is the twist potential defined by (3.13).

We now consider the general solution with rods $I_{a=1,\dots,n+1}$. This is given by Theorem 8 in terms of the matrices $F_a(k)$. In turn, the matrices $F_a(k)$ are constructed from the matrices $P_a(k)$ and a constant matrix C arising from the solution to the linear system at infinity using (3.113). To fix C we need to explicitly compute asymptotic solutions to the linear system (3.82), (3.83) which match on to the axis solution (3.106), (3.107). Then, from the definition (3.102) for matrices $P_a(k)$, we deduce that the general solution on the axis and horizons depends only on the following constants: the rod lengths $\ell_a = z_a - z_{a-1}$, the angular velocity of each horizon Ω^a, the jump in Ernst potentials $b(z_a) - b(z_{a-1})$ over each axis rod and jump in twist potentials $\chi(z_a) - \chi(z_{a-1})$ over each horizon rod.

We now turn to the computation of the constant matrix C. Firstly, Minkowski spacetime in polar coordinates (3.72) is given by

$$\bar{g} = \text{diag}(-1, r^2\sin^2\theta), \qquad \bar{\nu} = 0 \, , \tag{3.145}$$

which implies $\bar{S} = \text{diag}(0,2)$ and $\bar{T} = \text{diag}(0, 2\cos\theta)$, where S, T are defined in (3.79). The general solution to (3.79) in Minkowski space, which agrees with the axis solution (3.106), is

$$\bar{\Psi}_+ = \text{diag}(-1,\ \mu_+) \, . \tag{3.146}$$

Thus, using the asymptotic expansion for μ_+ in polar coordinates, given in Section 3.3.2, we find that

$$\bar{\Psi}_+(r,\theta,k) = \begin{cases} \text{diag}\left(-1,\ -r(1+\cos\theta) + O(1)\right) & 0 \le \theta < \theta_* \\ \text{diag}\left(-1,\ r(1-\cos\theta) + O(1)\right) & \theta_* < \theta \le \pi \end{cases} \tag{3.147}$$

as $r \to \infty$.

More generally, any four-dimensional asymptotically flat spacetimes in polar coordinates (3.72) must take the form

$$g = \begin{pmatrix} -1 + \frac{2M}{r} + O(r^{-2}) & -\frac{2J\sin^2\theta}{r}(1 + O(r^{-1})) \\ -\frac{2J\sin^2\theta}{r}(1+O(r^{-1})) & r^2\sin^2\theta(1+O(r^{-1})) \end{pmatrix} , \tag{3.148}$$

as $r \to \infty$, where M, J are the ADM mass and angular momentum. It follows that the corresponding matrices S, T in the linear system (3.79) are now given by

$$S - \bar{S} = \begin{pmatrix} O(r^{-1}) & O(r^{-3}) \\ O(r^{-1}) & O(r^{-1}) \end{pmatrix} , \qquad T - \bar{T} = \begin{pmatrix} O(r^{-2}) & O(r^{-3}) \\ O(r^{-2}) & O(r^{-1}) \end{pmatrix} , \tag{3.149}$$

which together with (3.147) imply that the RHS of equations (3.81) are

$$\Upsilon_{r+} = \begin{pmatrix} O(r^{-2}) & O(r^{-3}) \\ O(r^{-3}) & O(r^{-2}) \end{pmatrix}, \qquad \Upsilon_{\theta+} = \begin{pmatrix} O(r^{-1}) & O(r^{-2}) \\ O(r^{-2}) & O(r^{-1}) \end{pmatrix} \tag{3.150}$$

for all $0 \le \theta \le \pi$. This justifies the claim (3.83). Thus we may compute C from (3.86) using (3.147), which gives

$$C = -I_2 \; . \tag{3.151}$$

As a simple example, consider the rod structure of Minkowski spacetime, which is given by a single rod consisting of the whole z-axis. Thus the right and left semi-infinite axes are identified $I_L = I_R$ and there are no continuity conditions to be imposed. Then combining (3.109) with (3.117) gives

$$\mathring{g}(z) = X(z, z) \tag{3.152}$$

which using (3.141) is equivalent to

$$h(z) = -1, \qquad b(z) = 0 \; . \tag{3.153}$$

This is indeed the data for Minkowski spacetime (3.145). In itself this a non-trivial result: it shows that any asymptotically flat stationary and axisymmetric vacuum solution with the same rod structure as Minkowski spacetime is isometric to Minkowski spacetime on the axis. This of course follows from the well known no-soliton theorems.

Given a solution $(h(z), b(z))$ on I_L or I_R we can compute the mass and angular momentum. Comparing to (3.148) we find as $|z| \to \infty$

$$h(z) = -1 + \frac{2M}{|z|} + O(z^{-2}) \; , \qquad b(z) = -\frac{\text{sign}(z) 2J}{z^2} + O(z^{-3}) \tag{3.154}$$

where $b(z)$ is determined using (3.44) and we have fixed the integration constant so that it vanishes at infinity.

Finally, given the solution on a horizon rod, the surface gravity can be computed from (3.35), which in principle may impose a non-trivial constraint on the parameters.

3.4.2 Asymptotics of general solution

We now confirm our general solution (3.118) is asymptotically flat and compute the asymptotic charges. In particular, the metric and Ernst potential on I_L are given by the components of $F_L(k) = Q_1(k)^T$.

It is convenient to first write Q_1 in the form

$$Q_1(k) = X_L(z_1, k)^{-1} R(k) X_R(z_n, k), \tag{3.155}$$

where we have defined

$$R(k) = R_2(k) \dots R_n(k), \tag{3.156}$$

$$R_a(k) = X_a(z_{a-1}, k) X_a(z_a, k)^{-1} \tag{3.157}$$

for $a = 2, \dots, n$. Using this we can write F_L as

$$F_L(k) = \begin{pmatrix} {R_0}^0(k) - \frac{{R_1}^0(k) b(z_1)}{2(k-z_1)} & -\frac{{R_1}^0(k)}{2(k-z_1)} \\ \tilde{F}_{L10}(k) & \frac{{R_1}^0(k) b(z_n) + 2(k-z_n){R_1}^1(k)}{2(k-z_1)} \end{pmatrix} \; , \tag{3.158}$$

$$\tilde{F}_{L10}(k) = 2(k - z_n)\left(-{R_0}^1(k) + \frac{{R_1}^1(k) b(z_1)}{2(k - z_1)}\right) - b(z_n)\left({R_0}^0(k) - \frac{{R_1}^0(k) b(z_1)}{2(k - z_1)}\right)$$

and $R_A{}^B(k)$ denote the components of the matrix (3.156) in the standard basis. Hence using (3.118) we find that the solution on I_L is

$$h(z) = -\frac{2(z-z_1)}{R_1{}^0(z)b(z_n) + 2(z-z_n)R_1{}^1(z)}, \tag{3.159}$$

$$b(z) = -\frac{R_1{}^0(z)}{R_1{}^0(z)b(z_n) + 2(z-z_n)R_1{}^1(z)}\ , \tag{3.160}$$

where we have used the fact that $h(z) = -\tilde{F}_{L11}(z)^{-1}$ (using (3.120) in the $D=4$ case). We may now compute the asymptotics of the solution as $z \to -\infty$. To do this we first write R_a ($a = 2, \dots, n$) in the following convenient form

$$R_a(k) = I_{D-2} + \frac{S_a}{k - z_a}, \tag{3.161}$$

$$S_a \equiv L_a \begin{pmatrix} 0 & -\frac{1}{2}b^a_\mu(z_a) \\ 0 & \ell_a \end{pmatrix} L_a^{-1}, \tag{3.162}$$

where we have used our solution for X_a (3.94). Combining this with the definition of $R(k)$ (3.156), we can look at the asymptotic expansion around $k \to \infty$ to find

$$R(k) = I_{D-2} + \frac{S}{k} + O(k^{-2}), \qquad S = \sum_{a=2}^{n} S_a\ . \tag{3.163}$$

This then gives

$$h(z) = -1 - \frac{S_0{}^0}{z} + O(z^{-2}), \qquad b(z) = -\frac{S_1{}^0}{2z^2} + O(z^{-3}) \tag{3.164}$$

where $S_A{}^B$ denote the components of the matrix S defined in (3.163).

We can evaluate these relations more explicitly using (3.162). We find that

$$S_a = \begin{pmatrix} 0 & -\frac{1}{2}b^a(z_a) \\ 0 & \ell_a \end{pmatrix}\ , \qquad I_{a\neq L,R} \subset \hat{A}\ , \tag{3.165}$$

$$S_a = \begin{pmatrix} \ell_a + \frac{1}{2}\Omega^a\chi^a(z_a) & \Omega^a(\ell_a + \frac{1}{2}\Omega^a\chi^a(z_a)) \\ -\frac{1}{2}\chi^a(z_a) & -\frac{1}{2}\Omega^a\chi^a(z_a) \end{pmatrix}\ , \qquad I_a \subset \hat{H}\ . \tag{3.166}$$

Therefore, from the asymptotics of the general solution derived above we deduce

$$M = \sum_{I_a\subset\hat{H}} M_a\ , \qquad M_a = \tfrac{1}{2}(\ell_a + \tfrac{1}{2}\Omega^a\chi^a(z_a)), \tag{3.167}$$

$$J = \sum_{I_a\subset\hat{H}} J^a\ , \qquad J^a = \tfrac{1}{8}\chi^a(z_a)\ . \tag{3.168}$$

Observe that expressions for the angular momenta are the well-known relations (3.12). The expressions for the mass (3.167) together with (3.37) imply the Smarr relation (for multi-black holes).

On the other hand, suppose instead we use the alternate form of the solution where $F_L(k)$ is replaced by $F_L(k)^T$. Then the only change in the solution is that now $b(z) = \tilde{F}_{L10}(z)/\tilde{F}_{L11}(z)$. Working to first order in the expansion for $R(z)$ as in (3.163) allows us only to determine the $O(1)$ term,

$$b(z) = b(z_1) - 2S_0{}^1 - b(z_n) + O(z^{-1}). \tag{3.169}$$

Therefore $b(z) \to 0$ implies

$$b(z_n) - b(z_1) = -2S_0{}^1 = \sum_{\substack{I_a \subset \hat{A} \\ a \neq L,R}} b^a(z_a) - 4 \sum_{I_a \subset \hat{H}} \Omega^a M_a \, . \tag{3.170}$$

We provide an alternate derivation of this relation in Appendix 3.C (the same relation was also found in [36]).

We can similarly consider the asymptotics of the solution on I_R which is given by the matrix $F_R(k) = Q_1(k)^{-1}$. The computation is essentially the same as above and one finds the formulas (3.167) and (3.170).

3.5 Five dimensions

3.5.1 General solution and physical parameters

For $D = 5$ the general solution for the metric data $(h^a_{\mu\nu}(z), b^a_\mu(z))$ on any axis rod I_a takes the explicit form (3.118), with an analogous expression for the data $(\gamma^a_{ij}(z), \chi^a_i(z))$ on any horizon rod (3.121). The solution is given in terms of components of the matrices $F_a(k)$, which depend on the moduli (3.124) and a matrix C. The matrix C arises in the asymptotic solution (3.82), (3.83), in particular it relates the solution in the left and right segments (3.85). Therefore, to fully fix the general solution on the axis and horizon rods we need to find the solution to the linear system in Minkowski spacetime which matches onto our axis solution (3.106) and compute the corresponding matrix C using (3.86).

Five-dimensional Minkowski spacetime in polar coordinates (3.72) is

$$\bar{g} = \text{diag}\left(-1,\ r(1-\cos\theta),\ r(1+\cos\theta)\right), \qquad e^{2\bar{\nu}} = \frac{1}{2r}\,, \tag{3.171}$$

which gives

$$\bar{S} = \text{diag}(0,1,1), \qquad \bar{T} = \text{diag}(0, 1+\cos\theta, -(1-\cos\theta))\,. \tag{3.172}$$

For $r > |k|$, the solution to (3.79) on Minkowski space which agrees with the axis solution (3.106) is

$$\bar{\Psi}_+ = \text{diag}\left(-1,\ r(1-\cos\theta) - \mu_+,\ r(1+\cos\theta) + \mu_+\right) N(r,\theta,k)\,, \tag{3.173}$$

where the matrix

$$N(r,\theta,k) = \begin{cases} \text{diag}(1,\ -1,\ -2k)^{-1} & 0 \le \theta < \theta_* \\ \text{diag}(1,\ 2k,\ 1)^{-1} & \theta_* < \theta \le \pi \end{cases} \tag{3.174}$$

is needed to ensure the solution matches with the one on the axes. In particular, using the asymptotic expansions in Section 3.3.2 we find that

$$\bar{\Psi}_+(r,\theta,k) = \begin{cases} \text{diag}\left(-1,\ -2r + O(1),\ -\tfrac{1}{2}(1+\cos\theta) + O(r^{-1})\right) & 0 \le \theta < \theta_* \\ \text{diag}\left(-1,\ -\tfrac{1}{2}(1-\cos\theta) + O(r^{-1}),\ 2r + O(1)\right) & \theta_* < \theta \le \pi \end{cases}\,, \tag{3.175}$$

as $r \to \infty$.

Now, any five-dimensional asymptotically flat spacetimes in polar coordinates must take the form [54]

$$g = \begin{pmatrix} -1 + \frac{4M}{3\pi r} + O(r^{-2}) & -\frac{J_1(1-\cos\theta)}{\pi r}(1+O(r^{-1})) & -\frac{J_2(1+\cos\theta)}{\pi r}(1+O(r^{-1})) \\ -\frac{J_1(1-\cos\theta)}{\pi r}(1+O(r^{-1})) & r(1-\cos\theta)(1+O(r^{-1})) & \frac{\zeta\sin^2\theta}{r}(1+O(r^{-1})) \\ -\frac{J_2(1+\cos\theta)}{\pi r}(1+O(r^{-1})) & \frac{\zeta\sin^2\theta}{r}(1+O(r^{-1})) & r(1+\cos\theta)(1+O(r^{-1})) \end{pmatrix}, \tag{3.176}$$

as $r \to \infty$, where M, J_i are the ADM mass and angular momenta and ζ is a gauge invariant constant. From this one can show that S, T appearing in the linear system in polar coordinates (3.79) satisfy

$$S - \bar{S} = \begin{pmatrix} O(r^{-1}) & O(r^{-2}) & O(r^{-2}) \\ O(r^{-1}) & O(r^{-1}) & O(r^{-2}) \\ O(r^{-1}) & O(r^{-2}) & O(r^{-1}) \end{pmatrix}, \qquad T - \bar{T} = \begin{pmatrix} O(r^{-2}) & O(r^{-2}) & O(r^{-2}) \\ O(r^{-2}) & O(r^{-1}) & O(r^{-2}) \\ O(r^{-2}) & O(r^{-2}) & O(r^{-1}) \end{pmatrix}. \tag{3.177}$$

Then using (3.175) we find that the matrices Υ defined in (3.81) satisfy[3]

$$\Upsilon_{r+} = O(r^{-2}), \qquad \Upsilon_{\theta+} = O(r^{-1}), \tag{3.178}$$

for all $0 \leq \theta \leq \pi$, thus justifying (3.83). Finally, from (3.86) we find

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.179}$$

We therefore have fully fixed the general solution.

We now relate the parameters of the solution to the asymptotic quantities. Given a solution $(h^L_{\mu\nu}, b^L_\mu)$ on I_L we can compute the mass and angular momenta. From (3.176) we deduce that

$$\begin{aligned} h^L_{\mu\nu}(z) &= \begin{pmatrix} -1 - \frac{4M}{3\pi z} + O(z^{-2}) & \frac{2J_1}{\pi z} + O(z^{-2}) \\ \frac{2J_1}{\pi z} + O(z^{-2}) & -2z + O(1) \end{pmatrix}, \\ b^L_\mu(z) &= \begin{pmatrix} -\frac{2J_2}{\pi z} + O(z^{-2}) \\ \frac{4\zeta}{z} + O(z^{-2}) \end{pmatrix}, \end{aligned} \tag{3.180}$$

as $z \to -\infty$, where b^L_μ is determined using (3.44) and we have fixed the integration constant so that $b^L_\mu \to 0$ at infinity. Similarly, given a solution on I_R we can compute the asymptotic quantities again from (3.176) which in this case implies that

$$\begin{aligned} h^R_{\mu\nu}(z) &= \begin{pmatrix} -1 + \frac{4M}{3\pi z} + O(z^{-2}) & -\frac{2J_2}{\pi z} + O(z^{-2}) \\ -\frac{2J_2}{\pi z} + O(z^{-2}) & 2z + O(1) \end{pmatrix}, \\ b^R_\mu(z) &= \begin{pmatrix} -\frac{2J_1}{\pi z} + O(z^{-2}) \\ \frac{4\zeta}{z} + O(z^{-2}) \end{pmatrix}, \end{aligned} \tag{3.181}$$

as $z \to \infty$, again using (3.44) and fixing constants so $b^R_\mu \to 0$ at infinity.

On the other hand, given a solution on a horizon rod I_a we may compute the surface gravity from (3.36), which in principle may provide one constraint on the parameters. Similarly, given a solution on an axis rod I_a, smoothness requires that there are no conical singularities at any endpoint of the rod, the conditions for which are given by (3.25) and (3.26).

3.5.2 Asymptotics of general solution

We now confirm our general solution is asymptotically flat and deduce the asymptotic charges. First, consider the solution (3.118) on I_L which is derived from the components of $F_L(k) = -C^{-1}Q_1(k)^T$.

[3] In fact one obtains different fall-offs for $0 \leq \theta < \theta_*$ and $\theta_* < \theta \leq \pi$ where some components have faster fall-offs. We will not need these in our analysis.

Using (3.155) to write $Q_1(k)$ in terms of $R(k)$ defined in (3.156) and then using the asymptotic expansion (3.163) gives

$$h^L_{\mu\nu}(z) = \begin{pmatrix} -1 - \frac{{S_0}^0}{z} + O(z^{-2}) & -\frac{{S_1}^0}{z} + O(z^{-2}) \\ -2{S_0}^1 - b^R_0(z_n) + O(z^{-1}) & -2z + 2(z_n - {S_1}^1) + O(z^{-1}) \end{pmatrix}, \tag{3.182}$$

$$b^L_\mu(z) = \begin{pmatrix} \frac{{S_2}^0}{z} + O(z^{-2}) \\ 2{S_2}^1 + b^R_1(z_n) + O(z^{-1}) \end{pmatrix}, \tag{3.183}$$

as $z \to -\infty$. Thus comparing to the asymptotics (3.180) we deduce that

$$M = \frac{3\pi {S_0}^0}{4}, \qquad J_i = -\frac{\pi {S_i}^0}{2}, \qquad b^R_\mu(z_n) = -2\begin{pmatrix} {S_0}^1 \\ {S_2}^1 \end{pmatrix}. \tag{3.184}$$

Using (3.162) we can evaluate these expressions more explicitly. We find that

$$\begin{aligned} S_a &= \begin{pmatrix} 0 & -\frac{1}{2}v^1_a b^a_0(z_a) & -\frac{1}{2}v^2_a b^a_0(z_a) \\ 0 & -\frac{1}{2}\epsilon_a v^1_a(2u^2_a\ell_a + v^2_a b^a_1(z_a)) & -\frac{1}{2}\epsilon_a v^2_a(2u^2_a\ell_a + v^2_a b^a_1(z_a)) \\ 0 & \frac{1}{2}\epsilon_a v^1_a(2u^1_a\ell_a + v^1_a b^a_1(z_a)) & \frac{1}{2}\epsilon_a v^2_a(2u^1_a\ell_a + v^1_a b^a_1(z_a)) \end{pmatrix}, \quad I_{a\neq L,R} \subset \hat{A}, \\ S_a &= \begin{pmatrix} \ell_a + \frac{1}{2}\Omega^a_i\chi^a_i(z_a) & \Omega^a_1(\ell_a + \frac{1}{2}\Omega^a_i\chi^a_i(z_a)) & \Omega^a_2(\ell_a + \frac{1}{2}\Omega^a_i\chi^a_i(z_a)) \\ -\frac{1}{2}\chi^a_1(z_a) & -\frac{1}{2}\Omega^a_1\chi^a_1(z_a) & -\frac{1}{2}\Omega^a_2\chi^a_1(z_a) \\ -\frac{1}{2}\chi^a_2(z_a) & -\frac{1}{2}\Omega^a_1\chi^a_2(z_a) & -\frac{1}{2}\Omega^a_2\chi^a_2(z_a) \end{pmatrix}, \quad I_a \subset \hat{H}, \end{aligned} \tag{3.185}$$

where (u_a, v_a) is a basis of $U(1)^2$-Killing fields such that v_a is the rod vector and $\epsilon_a = (u^1_a v^2_a - u^2_a v^1_a)^{-1}$. Therefore, from the asymptotics of the general solution derived above we deduce

$$M = \sum_{I_a \subset \hat{H}} M_a, \qquad M_a = \tfrac{3\pi}{4}(\ell_a + \tfrac{1}{2}\Omega^a_i\chi^a_i(z_a)), \tag{3.186}$$

$$J_i = \sum_{I_a \subset \hat{H}} J^a_i, \qquad J^a_i = \tfrac{\pi}{4}\chi^a_i(z_a), \tag{3.187}$$

$$b^R_\mu(z_n) = \sum_{I_a \subset \hat{H}} \frac{4\Omega^a_1}{3\pi}\begin{pmatrix} -2M_a \\ 3J^a_2 \end{pmatrix} + \sum_{\substack{I_a \subset \hat{A} \\ a \neq L,R}} v^1_a \begin{pmatrix} b^a_0(z_a) \\ -\epsilon_a(2u^1_a\ell_a + v^1_a b^a_1(z_a)) \end{pmatrix}. \tag{3.188}$$

Note that again we reproduce the well-known relations between angular momenta and the change in twist potential across a horizon rod (3.12). Combining these with the above formulae for the mass, together with (3.37), gives the Smarr relation. Notice that in the absence of a black hole $M = 0$ and $J_i = 0$, in line with the no-soliton theorem.

If instead we use the alternate general solution with $F_L(k)$ replaced by $F_L(k)^T$ we find

$$b^L_\mu(z) = b^L_\mu(z_1) - 2\begin{pmatrix} {S_0}^2 \\ {S_1}^2 \end{pmatrix} + O(z^{-1}), \tag{3.189}$$

with $h^L_{\mu\nu}(z)$ given by the transpose of (3.182). Thus the asymptotics of b^L_μ now give

$$b^L_\mu(z_1) = 2\begin{pmatrix} {S_0}^2 \\ {S_1}^2 \end{pmatrix} = \sum_{I_a \subset \hat{H}} \frac{4\Omega^a_2}{3\pi}\begin{pmatrix} 2M_a \\ -3J^a_1 \end{pmatrix} - \sum_{\substack{I_a \subset \hat{A} \\ a \neq L,R}} v^2_a \begin{pmatrix} b^a_0(z_a) \\ \epsilon_a(2u^2_a\ell_a + v^2_a b^a_1(z_a)) \end{pmatrix}. \tag{3.190}$$

The $O(z^{-1})$ term, and hence J_2, requires a higher order calculation than the one given by (3.163). On the other hand, the asymptotics of $h^L_{\mu\nu}(z)$ give the mass (3.186), the angular momentum J_1 (3.187) and the Ernst potential $b^R_0(z_n)$ (3.188).

We may perform an analogous calculation on I_R. Using the general solution (3.118) and the explicit form $F_R(k) = -Q_1(k)^{-1}C^{-1}$, together with (3.155) and (3.163), the asymptotics yield the same expressions as above for $M, J_2, b^R_\mu(z_n), b^L_0(z_n)$; here J_1 requires the $O(z^{-1})$ term in $b^R_\mu(z)$ which needs a higher order calculation. If one instead considers the solution with $F_R(k)$ replaced by $F_R(k)^T$ one obtains $M, J_i, b^L_\mu(z_1)$ to this order.

We remark that if one uses the alternate solution as in Conjecture 1, then the above shows that the solution on I_L fixes $M, J_1, b^L_\mu(z_1)$ (and $b^R_0(z_n)$) and the solution on I_R fixes $M, J_2, b^R_\mu(z_n)$ (and $b^L_0(z_1)$), so taken together these give all the asymptotic quantities. Indeed, this partially motivates our conjecture. In Appendix 3.C we show how to derive these expression for $b^R_\mu(z_n), b^L_\mu(z_1)$ from general properties of the Ernst potentials.

In the case of a single black hole the above formulas simplify. In particular, if say $I_H = (z_1, z_2)$, the mass and angular momenta become

$$M = \tfrac{3\pi}{4}(\ell_H + \tfrac{1}{2}\Omega_i\chi_i(z_2)), \tag{3.191}$$

$$J_i = \tfrac{\pi}{4}\chi_i(z_2), \tag{3.192}$$

where $\ell_H = z_2 - z_1$ and we have chosen a gauge in which $\chi_i(z_1) = 0$ (for a single horizon one is always free to do this). In this case we find it convenient to work with the dimensionless parameters

$$j_i = J_i M^{-3/2}\left(\frac{27\pi}{32}\right)^{1/2}, \qquad \omega_i = \Omega_i M^{1/2}\left(\frac{8}{3\pi}\right)^{1/2}, \qquad \lambda_H = \frac{3\pi}{4M}\ell_H, \tag{3.193}$$

where we are of course now assuming $M > 0$. Then (3.191) gives

$$\lambda_H = 1 - \omega_i j_i \, . \tag{3.194}$$

For any finite axis rods I_a, $a = 2, \ldots, n$ we also define the associated dimensionless parameters

$$f^a_0 = b^a_0(z_a)\left(\frac{3\pi}{8M}\right)^{1/2}, \qquad f^a_1 = b^a_1(z_a)\left(\frac{3\pi}{8M}\right), \qquad \lambda_a = \frac{3\pi}{4M}\ell_a \, , \tag{3.195}$$

and f^L_μ, f^R_μ are similarly defined with $b^a_\mu(z_a)$ replaced by $b^L_\mu(z_1), b^R_\mu(z_n)$ respectively.

For the single black hole cases we study in the next chapter, we use another conjecture which extends the results of Conjecture 1, utilising a slightly more convenient set of equations to derive the unbalanced solutions. Given a single horizon rod, the condition (3.126) can be thought of as an equation for J_i, as follows. Since (3.128) is automatic we can write $\chi^a_i(z) = \chi^a_i(z_a) + (z - z_a)g_a(z)$ for some smooth function $g_a(z)$. Then evaluating at $z = z_{a-1}$ and using (3.12) we deduce that $J_i = \pi\ell_a g_a(z_{a-1})/4$ which gives a nonlinear equation for J_i (typically $g_a(z_{a-1})$ depends on all the moduli, including J_i). On the other hand, from the asymptotics (3.180) and (3.181), the $O(z^{-1})$ term in $b^L_0(z)$ and $b^R_0(z)$ gives J_2 and J_1 respectively. The above asymptotic analysis showed that, for the alternate solution, the computation of these $O(z^{-1})$ terms requires a higher order calculation, which in general will give different formulas for J_i than (3.187). Thus, one can take these asymptotic equations as new equations for J_i, instead of those from (3.126) described above. This motivates the following conjecture:

Conjecture 2. *Consider a solution as in Conjecture 1, further assuming that the consistency conditions on the Ernst potentials (3.125), (3.92) hold. Then the consistency conditions on the twist potentials (3.126) are equivalent to the $O(z^{-1})$ asymptotic conditions for b^L_0 (3.180) and b^R_0 (3.181), combined with the expression for the angular momentum (3.187).*

Remarks.

1. This conjecture holds when we consider the rod structure for the Myers-Perry solution. Whilst the backwards implication also holds for the black ring and black lens cases [4], we have been unable to show that the forward implication also works in these cases. The value of this conjecture is that it allows one to solve the relatively simpler asymptotic constraints on the Ernst potentials on $I_{L,R}$ rather than needing to consider the consistency constraints on the twist potentials (3.126).

2. This conjecture is automatic in the case of no black holes. On the other hand, for multi-black holes, this conjecture would need to be revisited, a matter which we will not consider this here.

3.6 Discussion

In this chapter we have considered the classification of $D = 4, 5$ asymptotically flat stationary vacuum black hole spacetimes that admit $D - 3$ commuting axial Killing fields. To do this we developed a method based on integrability of the Einstein equations for this class of spacetimes. In particular, we have presented a general solution for the metric and associated Ernst and twist potentials on each axis and horizon component, see Theorem 8. This solution depends on a number of geometrically defined moduli which obey a set of algebraic equations and inequalities. Generically the solutions possess conical singularities on the axes and (assuming Conjecture 1) correspond to the moduli space of solutions guaranteed to exist in Theorem 4. However, by imposing that the axis and horizon metric is free of conical singularities we obtain, at least in principle, the moduli space of *regular* black hole solutions in this class for any given rod structure (which may be empty depending on the rod structure).

In practice the equations which define the moduli spaces increase in complexity as one increases the number of rods. Therefore an analysis of the general solution remains out of reach. To this end, it would be interesting to better understand how the set of dependent equations that we have derived relate to each other and in particular to prove Conjecture 2, as this may lead to a more complete understanding of the moduli space equations. Nevertheless solving the moduli space equations is entirely possible for some of the simplest rod structures. We will discuss this in the next chapter.

By construction, we have obtained the general solution only on the boundary of the orbit space, i.e. on the axis and horizon rods. On the other hand, Theorem 4 shows that for given boundary data, there exists a unique unbalanced solution that is smooth everywhere away from the axes. Therefore an interesting problem is write down this full solution explicitly, given our boundary solution. Further methods from integrability theory are probably well suited to tackle this problem, since they have already been successful in this regard in four dimensions [147].

It would be interesting to develop our method to study the analogous classification problem for other types of boundary conditions. In particular, for $D = 5$ one can have asymptotically Kaluza-Klein (KK) or Taub-NUT (TN) vacuum solutions. This could be of interest, as in these cases, the space of regular solutions is richer since one can have regular soliton spacetimes (e.g. $\mathbb{R}\times$ Euclidean Schwarzschild and the KK monopole, for KK and TN asymptotics respectively). Presumably our analysis can be adapted to these cases, although clearly one would have to revisit the solution of the spectral equations near infinity.

Our method is based on the existence of an auxiliary linear system whose integrability condition is the vacuum Einstein equations for spacetimes in this symmetry class. It seems likely that this method could be employed in other theories of gravity which are integrable for spacetimes with $D - 2$

[4] Although in the case of the doubly spinning black lens, we have only been able to verify this numerically.

commuting Killing fields. For example, it is well-known that this is the case for $D = 4$ Einstein-Maxwell equations and an analogous inverse scattering method has been developed [38]. This was recently used to construct the general charged, rotating, double-black hole solution [40].

More generally, any theory which reduces to a two-dimensional sigma-model with coset target space is integrable in this sense. A notable example is $D = 5$ minimal supergravity (Einstein-Maxwell-Chern-Simons theory) [124]. This theory is particularly interesting to study as it is already known to contain a rich class of regular spacetimes with these symmetries. As we discussed in the introduction, besides the well-known charged versions of the Myers-Perry black holes and black rings, this theory also admits positive energy soliton solutions (a.k.a microstate geometries) [148], supersymmetric black lenses, and black holes with non-trivial topology in the DOC (2-cycles) [110, 108, 109, 111, 107, 149]. Recently a complete classification of supersymmetric spacetimes in this class was obtained revealing an infinite class of new black holes, black lenses and rings in spacetimes with non-trivial 2-cycles [107]. A method based on integrability as in this chapter provides a complementary perspective on the classification problem that also captures the much larger moduli space of non-supersymmetric solitons and black holes. This is precisely the problem we consider in Chapter 5.

Before we come to this however, we will first consider the applications of the methods developed in this chapter to some most basic rod structures, paying particular interest to the question of the existence of the simplest black lens spacetime.

3.A Rod structure of Gibbons-Hawking solitons

In Section 3.2.1 we showed that the Eguchi-Hanson soliton can be interpreted as an asymptotically Minkowski solution which is regular everywhere except for a conical singularity on its bolt. In particular, it gives a rod structure which satisfies the admissibility condition (3.8) and hence gives the corresponding solution that is guaranteed to exist in Theorem 4. It is natural to wonder whether the more general Gibbons-Hawking solitons can be similarly interpreted. In fact, we find that within this class of solutions, the only case which gives an admissible rod structure is the Eguchi-Hanson soliton.

The Gibbons-Hawking solitons are

$$\mathrm{d}s^2_{\mathsf{GH}} = -\mathrm{d}t^2 + H^{-1}(\mathrm{d}\tau + \chi_i \mathrm{d}x^i)^2 + H\mathrm{d}x^i\mathrm{d}x^i \, , \qquad H = \sum_{a=1}^{n} \frac{1}{|x - p_a|} \, , \tag{3.196}$$

where x^i are Cartesian coordinates on $\mathbb{R}^3$, $p_a \in \mathbb{R}^3$ are constants and χ is determined by $\mathrm{d}\chi = \star_3 \mathrm{d}H$. We assume $n > 1$ and note that for $n = 2$ this is the Eguchi-Hanson soliton (3.14) in different coordinates (for $n = 1$ this of course Minkowski spacetime). If we take the $p_a = (0, 0, z_a)$ collinear then the metric has biaxial symmetry and in cylindrical coordinates reads

$$\begin{aligned} &\mathrm{d}s^2_{\mathsf{GH}} = -\mathrm{d}t^2 + H^{-1}(\mathrm{d}\tau + \chi \mathrm{d}\phi)^2 + H\rho^2\mathrm{d}\phi^2 + H(\mathrm{d}\rho^2 + \mathrm{d}z^2) \, , \\ &H = \sum_{a=1}^{n} \frac{1}{\sqrt{\rho^2 + (z - z_a)^2}}, \qquad \chi = \sum_{a=1}^{n} \frac{z - z_a}{\sqrt{\rho^2 + (z - z_a)^2}} \, . \end{aligned} \tag{3.197}$$

Observe that this metric is also in Weyl coordinates. As is well-known, if (τ, ϕ) are identified as Euler angles on S^3 (i.e. such that the orbits of $\partial_\phi \pm \partial_\tau$ are independently 2π-periodic) this gives a smooth ALE metric with $S^3/\mathbb{Z}_n$ topology at infinity and any curve between the centres p_a corresponds to a 2-cycle (or bolt).

On the other hand, one can identify (τ, ϕ) such that the topology at infinity is S^3 resulting in an asymptotically Minkowski spacetime. Explicitly, as $r = |x| \to \infty$ we have $H \sim n/r$ and $\chi \sim n\cos\theta$, where (r, θ) are standard polar coordinates on $\mathbb{R}^3$, so

$$\mathrm{d}s^2_{\mathsf{GH}} \sim -\mathrm{d}t^2 + \mathrm{d}R^2 + \frac{1}{4}R^2\left[(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right] \; , \tag{3.198}$$

where we have defined coordinates $\psi = \tau/n$ and $R^2 = 4nr$. Thus identifying (θ, ψ, ϕ) to be Euler angles on S^3 gives an asymptotically Minkowski spacetime. In particular, the rod vectors with 2π-periodic orbits on the two semi-infinite axes $\theta = 0$ and $\theta = \pi$ are $v_R = \partial_\phi - \partial_\psi$ and $v_L = \partial_\phi + \partial_\psi$ respectively. Let us compute the rod structure for this asymptotically flat vacuum solution.

It is clear there are $n+1$ axis rods $I_1 = (-\infty, z_1)$, $I_a = (z_{a-1}, z_a)$ for $a = 2, \dots, n$ and $I_{n+1} = (z_n, \infty)$. The rod vector on each rod is a multiple of

$$\tilde{v}_a = \partial_\phi - \chi_a \partial_\tau \tag{3.199}$$

where

$$\chi_a \equiv \chi|_{I_a} = \sum_{b=1}^{n} \mathsf{sign}(z - z_b) = 2(a-1) - n \tag{3.200}$$

for $a = 1, \dots, n+1$. For $a = 1$ and $a = n+1$ this expression reduces to v_L and v_R respectively and hence is correctly normalised. With respect to the 2π-periodic basis (v_R, v_L) the rod vectors are

$$\tilde{v}_a = \left(\frac{a-1}{n}, 1 - \frac{a-1}{n}\right) \tag{3.201}$$

so $\tilde{v}_1 = v_1 = (0,1)$ and $\tilde{v}_{n+1} = v_{n+1} = (1,0)$ as previously noted. However, for $a = 2, \dots, n$ rod vectors must be rescaled to ensure they have integer entries with respect to a 2π-periodic basis. Thus for $a = 2, \dots, n$ the rod vectors are

$$v_a = \frac{1}{\mathsf{gcd}(a-1, n)}\,(a-1, n-a+1) \; , \tag{3.202}$$

where the prefactor is included to ensure the components are coprime and hence v_a has 2π-periodic orbits.

We will now examine whether this rod structure satisfies the admissibility condition (3.8). In general we have

$$v_2 = (1, n-1) \; , \qquad v_n = (n-1, 1) \; , \tag{3.203}$$

so $\det(v_1, v_2) = -1$ and $\det(v_n, v_{n+1}) = -1$ satisfy (3.8). Therefore, if $n = 2$, we have an admissible rod structure $v_1 = (0,1)$, $v_2 = (1,1)$, $v_3 = (1,0)$. This is the Eguchi-Hanson soliton discussed in the main text (3.14). However, for $n > 2$ and $a = 2, \dots, n-1$ we have

$$\det(v_a, v_{a+1}) = -\frac{n}{\mathsf{gcd}(a-1, n)\mathsf{gcd}(a, n)} \; , \tag{3.204}$$

which is never equal to ± 1 and hence the admissibility condition (3.8) is always violated for $n > 2$. Instead, for these cases the corners of the orbit spaces $z_2, \dots, z_{n-1}$ are orbifold singularities.

3.B Geometry near corners of orbit space

3.B.1 Intersection of axes

Here we consider the geometry of a $D = 5$ spacetime near a fixed point of the $U(1)^2$-action, i.e., we consider the geometry near a corner of the orbit space $z = z_a$ where two consecutive axis rods I_a and I_{a+1} meet.

Then, as shown in Section 3.2.2, smoothness of the metric on I_a at $z = z_a$ requires (3.25), whereas smoothness of the metric on I_{a+1} at $z = z_a$ requires (3.26) with a replaced by $a+1$, i.e., $h^{a+1\prime}(z_a)^2/h^{a+1}_{00}(z_a) = -4c^2_{a+1}$. On the other hand, for any axis rod

$$h^a_{00}(z) = \mathring{g}_{AB}k^A k^B \tag{3.205}$$

is simply the squared norm of the stationary Killing field k on the axis. Therefore, $h^a_{00}(z_a) = h^{a+1}_{00}(z_a)$ and hence eliminating the norm of k between the aforementioned regularity conditions we deduce that

$$c_a^{-1} h^{a\prime}(z_a) = -c^{-1}_{a+1} h^{a+1\prime}(z_a) \ , \tag{3.206}$$

where in order to fix the sign we have used the fact that $h^{a\prime}(z_a) > 0$ and $h^{a+1\prime}(z_a) < 0$ (these follow from $h^a < 0$ in the interior of I_a).

Finally, observe that using (3.22) the condition (3.206) is equivalent to continuity of $|z - z_a|e^{2\mathring{\nu}}$ at $z = z_a$. In fact this continuity condition for the conformal factor $e^{2\nu}$ has been previously proven in [144].

3.B.2 Intersection of horizon and axis

We now consider the geometry where a horizon rod I_a meets an axis rod I_{a+1}. In particular, the geometry on the axis corresponding to I_{a+1} (3.23) is a $(D-2)$-dimensional Lorentzian spacetime that must have a regular $(D-3)$-dimensional horizon as $z \to z_a$ corresponding to where the full horizon intersects the axis corresponding to I_{a+1}. We will now compute the surface gravity of this 'axis horizon' $z = z_a$, which must of course coincide with the surface gravity of the full horizon.

For $D = 5$, the Killing field null on the horizon ξ restricted to the axis rod I_{a+1} is $\xi = k + \Omega u_{a+1}$ where (k, u_{a+1}) is the adapted basis of I_{a+1} and Ω is a constant angular velocity. Therefore, the metric on this component of the axis (3.23) must be of the form

$$\begin{aligned}\mathbf{g}_{a+1} \;=\;& -\frac{c^2_{a+1}\mathrm{d}z^2}{h^{a+1}(z)} + (p_1(z-z_a) + O((z-z_a)^2))(\mathrm{d}x^0)^2 \\ &+\; O(z-z_a)\mathrm{d}x^0(\mathrm{d}x^1 - \Omega \mathrm{d}x^0) + (p_2 + O(z-z_a))(\mathrm{d}x^1 - \Omega\mathrm{d}x^0)^2 \ ,\end{aligned} \tag{3.207}$$

as $z \to z_a^+$, where we choose adapted coordinates such that $k = \partial/\partial x^0, u_{a+1} = \partial/\partial x^1$. The expansions of the metric components follow from smoothness, together with ξ being null on the axis horizon and u_{a+1} being tangent to the axis horizon. Here $p_1 < 0$, $p_2 > 0$ are constants related to the metric components ($p_1 = 0$ would correspond to an extremal horizon which we do not consider here). It follows that the determinant $h^{a+1}(z) = p_1 p_2 (z - z_a) + O((z-z_a)^2)$ and hence defining $\epsilon^2 = z - z_a$, the first two terms in (3.207) approach the Rindler metric

$$-\frac{4c^2_{a+1}}{p_1 p_2}\left(\mathrm{d}\epsilon^2 - \kappa_a^2 \epsilon^2 (\mathrm{d}x^0)^2\right) \ , \tag{3.208}$$

as $\epsilon \to 0$, with surface gravity

$$\kappa_a^2 = \frac{p_1^2 p_2}{4c_{a+1}^2} = \frac{h^{a+1\prime}(z_a)^2}{4c_{a+1}^2 h_{11}^{a+1}(z_a)} \ . \tag{3.209}$$

The second equality follows from the relations $p_2 = h_{11}^{a+1}(z_a)$ and $p_1 p_2 = h^{a+1\prime}(z_a)$. A similar analysis for $D = 4$ (which effectively can be obtained from dropping the dx^1 terms above) gives

$$\kappa_a^2 = \frac{h^{a+1\prime}(z_a)^2}{4c_{a+1}^2} \ . \tag{3.210}$$

This analysis confirms the axis geometry on I_{a+1} has a smooth non-degenerate horizon at $z = z_a$ with surface gravity (3.209) for $D = 5$ and (3.210) for $D = 4$.

On the other hand, as shown above, smoothness of the horizon metric at the corner $z = z_a$ leads to a different expression for κ_a. For $D = 4$ this is given by (3.35) and combining this with (3.210) implies

$$\kappa_a^2 \gamma'(z_a) = c_{a+1}^{-1} h^{a+1\prime}(z_a) \ , \tag{3.211}$$

where the signs are fixed from the fact that $\gamma'(z_a) < 0$ and $h^{a+1\prime}(z_a) < 0$. For $D = 5$, the expression for the surface gravity (3.36), written in coordinates $\hat{\phi}^i, i = 1, 2$, adapted to the horizon rod I_a so that $u_{a+1} = \partial_{\hat{1}}$ and $v_{a+1} = \partial_{\hat{2}}$, becomes

$$\kappa_a^{-2} = \frac{\gamma'(z_a)^2}{4\gamma_{\hat{1}\hat{1}}(z_a)} \ , \tag{3.212}$$

where we used $\gamma'(z_a) = \gamma_{\hat{1}\hat{1}}(z_a)\gamma'_{\hat{2}\hat{2}}(z_a)$. Next, note that

$$h_{11}^{a+1}(z) = \mathring{g}_{AB} u_{a+1}^A u_{a+1}^B, \qquad \gamma_{\hat{1}\hat{1}}(z) = \mathring{g}_{AB} u_{a+1}^A u_{a+1}^B \ , \tag{3.213}$$

on the rods I_{a+1} and I_a respectively, are both equal to the norm squared of u_{a+1}, so in particular $h_{11}^{a+1}(z_a) = \gamma_{\hat{1}\hat{1}}(z_a)$. Hence eliminating the norm of u_{a+1} between (3.209) and (3.212) we deduce that (3.211) also holds for $D = 5$. The analysis for a horizon rod I_a meeting an axis rod I_{a-1} is entirely analogous and similarly to (3.211) one can derive that

$$\kappa_a^2 \gamma'(z_{a-1}) = c_{a-1}^{-1} h^{a-1\prime}(z_{a-1}) \tag{3.214}$$

for $D = 4, 5$.

Finally, using (3.22) and (3.32) we see that (3.211) is equivalent to the continuity of $|z - z_a| e^{2\mathring{\nu}}$ at $z = z_a$ (with a similar condition at $z = z_{a-1}$ for (3.214)), just as in the case of a corner separating two axis rods.

3.C Ernst potential identities

Consider a component of the horizon H with corresponding rod I_a and we drop rod labels when convenient and unambiguous. First, recall the well-known identity

$$\int_H \star \mathrm{d}\xi = -2\kappa A \ , \tag{3.215}$$

where ξ is the horizon Killing field (3.27), κ is the surface gravity and A is the area of H. Therefore, using (3.11) we deduce that

$$\zeta(z_a) - \zeta(z_{a-1}) = -\frac{2\kappa A}{(2\pi)^{D-3}} \ , \tag{3.216}$$

where we have defined a new potential ζ by

$$\mathrm{d}\zeta = \star(m_1 \wedge \dots m_{D-3} \wedge \mathrm{d}\xi) \ . \tag{3.217}$$

Also, we will need the following fact: in coordinates adapted to the horizon rod (3.28) implies that the 1-form dual to the corotating Killing field is

$$\xi_A = \tilde{g}_{AD-3} = O(\rho^2) \tag{3.218}$$

near the horizon. Thus, in particular, $\xi = 0$ on the horizon (although $\mathrm{d}\xi \neq 0$ since ρ is not a good coordinate on the horizon).

For $D = 4$ we can write (3.45) in terms of the corotating Killing field

$$\mathrm{d}b = -\star(\xi \wedge \mathrm{d}\xi) + \Omega \star (\xi \wedge \mathrm{d}m) + \Omega \mathrm{d}\zeta - \Omega^2 \mathrm{d}\chi \ , \tag{3.219}$$

where we have used the definition of the twist potential (3.13) and (3.217). Evaluating this on the horizon we see that the first two terms must vanish due to (3.218). Thus we find that on the horizon

$$\mathrm{d}b = \Omega(\mathrm{d}\zeta - \Omega \mathrm{d}\chi) \tag{3.220}$$

and integrating this over the horizon rod I_a gives

$$b(z_a) - b(z_{a-1}) = -\Omega\left(\frac{\kappa A}{\pi} + 8\Omega J\right) = -4\Omega M \ , \tag{3.221}$$

where in the first equality we used (3.216) and (3.12) and in the final equality the standard Smarr relation for the Komar mass of the horizon $M = \frac{1}{8\pi}\int_H \star \mathrm{d}\xi$. This implies the identity (3.170).

For $D = 5$, one can show again using (3.218) that on the horizon

$$\mathrm{d}b^L_\mu = \begin{pmatrix} -\Omega_2\Omega_i \mathrm{d}\chi_i + \Omega_2 \mathrm{d}\zeta \\ \Omega_2 \mathrm{d}\chi_1 \end{pmatrix} \tag{3.222}$$

and hence integrating this over the horizon rod

$$b^L_\mu(z_a) - b^L_\mu(z_{a-1}) = \Omega_2 \begin{pmatrix} -\frac{4}{\pi}\left(\Omega_i J_i + \frac{\kappa A}{8\pi}\right) \\ \frac{4J_1}{\pi} \end{pmatrix} = \Omega_2 \begin{pmatrix} -\frac{8M}{3\pi} \\ \frac{4J_1}{\pi} \end{pmatrix} \ , \tag{3.223}$$

where in the first equality we used (3.12) and (3.216) and in the second the Smarr relation. Similarly, one finds that on the horizon

$$\mathrm{d}b^R_\mu = \begin{pmatrix} -\Omega_1\Omega_i \mathrm{d}\chi_i + \Omega_1 \mathrm{d}\zeta \\ \Omega_1 \mathrm{d}\chi_2 \end{pmatrix} \tag{3.224}$$

and hence

$$b^R_\mu(z_a) - b^R_\mu(z_{a-1}) = \Omega_1 \begin{pmatrix} -\frac{8M}{3\pi} \\ \frac{4J_2}{\pi} \end{pmatrix} \ . \tag{3.225}$$

In a similar manner, one can also evaluate the change in Ernst potential associated to any other axis rod over a horizon rod. Formulae for $b^L_\mu(z_a) - b^L_\mu(z_{a-1})$ and $b^R_\mu(z_a) - b^R_\mu(z_{a-1})$ across axis rods can also be derived, which combined with (3.223) and (3.225) imply the identities (3.190) and (3.188).

3.D Proof of Proposition 2

First we observe that for an axis rod $\tilde{G}_{aNN}(k) = v_a^T G_a(k) v_a$ where in the standard basis $v_a^T = (0, v_a^1, \ldots, v_a^{D-3})$ is the rod vector. Similarly, for a horizon rod we can write $\tilde{G}_{a00}(k) = v_a^T G_a(k) v_a$ where $v_a^T = (1, \Omega_1^a, \ldots, \Omega_{D-3}^a)$ denotes the horizon null vector. Similar statements hold for the matrices $H_a(k)$. Thus, to complete the proof of Proposition 2 we need to establish

$$\lim_{k \to z_{a-1}} v_a^T G_a(k) v_a \neq 0 \tag{3.226}$$

and

$$\lim_{k \to z_a} v_a^T H_a(k) v_a \neq 0, \tag{3.227}$$

for each finite rod I_a, for generic values of the parameters. We will only explicitly prove (3.226), though (3.227) can be proved in an almost identical fashion.

Writing out G_a explicitly in terms of the P_a matrices using the expression for F_a (3.113) gives

$$\begin{gathered} G_2(k) = -X_L(z_1, k) C^{-1} P_n(k)^T \cdots P_2(k)^T, \\ G_a(k) = -X_{a-1}(z_{a-1}, k) P_{a-2}(k)^{-1} \cdots P_1(k)^{-1} C^{-1} P_n(k)^T \cdots P_a(k)^T, \end{gathered} \tag{3.228}$$

where $a = 3, \ldots, n$. Consider a fixed, but arbitrary set of axis rod vectors v_a (this is of course only relevant for $D = 5$). Then, from the definition of the matrices $G_a(k)$ it is clear that the LHS of (3.226) is a rational function $\mathcal{R}_a(\vec{\varphi})$ where the vector $\vec{\varphi}$ denotes the continuous moduli in (3.124) (i.e. excluding the axis rod vectors). For the purposes of the proposition we need to prove $\mathcal{R}_a(\vec{\varphi}) \neq 0$ for *generic* values of the moduli $\vec{\varphi}$, i.e. the zero set of $\mathcal{R}_a$ is lower-dimensional. A simple strategy to prove this is to find an explicit value of the moduli φ_0 for which $\mathcal{R}_a(\varphi_0) \neq 0$, since when combined with analyticity of the numerator of $\mathcal{R}_a$, implies that the zero-set of $\mathcal{R}_a$ does not contain an open set. It is worth noting that for this argument the value φ_0 does not need to belong to the actual moduli space of solutions.

It is convenient to choose φ_0 for each rod I_a such that $P_b(z_{a-1}) = I_{D-3}$ for all $b \neq a-1$ and $1 \leq b \leq n$. This is achieved by setting $b_\mu^b(z_b) = 0$ or $\chi_i^b(z_b) = 0$, depending on whether I_b is an axis or horizon rod, and $z_b = z_{a-1} + 1/2$. The result of this is that for any finite rod I_a

$$\lim_{k \to z_{a-1}} v_a^T G_a(k) v_a \to -v_a^T X_{a-1}(z_{a-1}, z_{a-1}) C^{-1} v_a \tag{3.229}$$

under these parameter identifications. Therefore in order to prove (3.226) all that remains is to show that the right hand side of (3.229) is generically nonzero.

First consider $D = 4$. Using the explicit expression for C (3.151) one finds that

$$-v_a^T X_{a-1}(z_{a-1}, z_{a-1}) C^{-1} v_a = \begin{cases} -1 + \Omega^{a-1} \chi^{a-1}(z_{a-1}), & I_{a-1} \text{ horizon rod, } I_a \text{ axis rod,} \\ -1 + \Omega^a b^{a-1}(z_{a-1}), & I_{a-1} \text{ axis rod, } I_a \text{ horizon rod,} \end{cases} \tag{3.230}$$

which are indeed generically nonzero.

Now consider $D = 5$, in which case C is explicitly given by (3.179). If I_{a-1} is an axis rod and I_a is a horizon rod, we can also set $\Omega_i^a = 0$ which implies that the right hand side of (3.229) is simply given by -1. If I_{a-1} is a horizon rod and I_a is an axis rod then the right hand side of (3.229) is given by

$$[v_a^i \chi_i^{a-1}(z_{a-1})][\tilde{v}_a^T v_{a-1}] - v_a^T \tilde{v}_a, \tag{3.231}$$

where $\tilde{v}_a^T = \begin{pmatrix} 0 & -v_a^1 & v_a^2 \end{pmatrix}$, which is generically nonzero. Finally, if both I_{a-1} and I_a are axis rods then the right hand side of (3.229) is given by

$$(\det A_{a-1})^{-1} \det \begin{pmatrix} v_a^1 & v_a^2 \\ v_{a-1}^1 & v_{a-1}^2 \end{pmatrix} \tilde{v}_a^T (b_1^{a-1}(z_{a-1}) v_{a-1} - u_{a-1}), \tag{3.232}$$

where the matrix A_{a-1} and the axial Killing field u_{a-1} are introduced in (3.15). The first factor is nonzero since $A_{a-1} \in GL(2, \mathbb{Z})$, the second factor is nonzero since v_a and v_{a-1} must be linearly independent (in particular see (3.8)), and the third factor is generically nonzero since $\tilde{v}_a$ cannot be orthogonal to both v_{a-1} and u_{a-1}. This establishes the claim.

Chapter 4

Vacuum Solutions With Simple Rod Structures

In the previous chapter we derived the general solution on the axes and horizons for any given rod structure. This chapter complements that analysis by applying those methods to particular rod structures.

We start by briefly discussing the four dimensional case where we solve the moduli space equations for the Kerr rod structure, rederiving standard results. In five dimensions we do something similar for Minkowski space (no longer as trivial as the four dimensional case), the Eguchi-Hanson soliton and the Myers-Perry solution. The first new result comes from considering the rod structure for the black ring. By rederiving the known solution we prove that the Pomerasky-Sen'kov doubly spinning black ring [74] is indeed the most general solution with that rod structure, a fact which does not seem to have been addressed in the previous literature.

We end this chapter by considering the most basic open question in the existence problem in five dimensions: do regular black holes with lens space topology exist? A number of attempts at constructing such black lens solutions have resulted in singular spacetimes, the mildest being a conical singularity on the inner axis [150, 129, 151]. However, it has remained unclear whether these past works represent the most general solutions with their given rod structures, thereby leaving the question of existence unanswered. To address this question, we will analyse regularity of the general solution on the axes and horizon for the simplest rod structure corresponding to a $L(n,1)$ black lens (see Figure 4.5). As we will explain below, by a mix of analytic and numerical analysis, we will show that regular black lenses of this type do not in fact exist.

For all the solutions we consider in both four and five dimensions we find that it is convenient to use the alternate form of the solution $(h_{\mu\nu}(z), b_\mu(z))$ as in Conjecture 1 where F_L and F_H are replaced by their transpose. Unless otherwise specified this will be how $(h_{\mu\nu}(z), b_\mu(z))$ are constructed in the examples we consider.

This chapter draws heavily from [2] and sections 4 and 5 of [3].

4.1 Kerr solution

We start by considering a $D = 4$ case - the rod structure of the Kerr solution. Namely, we assume there are three rods $I_L = (-\infty, z_1), I_H = (z_1, z_2)$, $I_R = (z_2, \infty)$ where I_H is a horizon rod. The solution is given by Theorem 8 in terms of the matrices $F_a(k)$ given by (3.113), which in this case are simply

$$F_L(k) = P_2(k)^T P_1(k)^T \ , \qquad F_H(k) = P_1(k)^{-1} P_2(k)^T \ , \qquad F_R(k) = P_2(k)^{-1} P_1(k)^{-1} \ , \tag{4.1}$$

where the $P_a(k)$ are defined by (3.102).

First, let us consider $z < z_1$. We can compute the mass and angular momentum by comparing to the asymptotic expansions (3.154), which in fact also fixes $b(z_1), b(z_2)$. We find

$$M = \tfrac{1}{2}[\ell_H + \tfrac{1}{2}\Omega(\chi(z_2) - \chi(z_1))] , \tag{4.2}$$
$$b(z_1) = -b(z_2) = 2\Omega M , \tag{4.3}$$
$$J = \Omega M^2[4M - \tfrac{1}{2}\Omega(\chi(z_2) - \chi(z_1))] , \tag{4.4}$$

where $\ell_H = z_2 - z_1$ and we have written the latter quantities in terms of M.

On the other hand, from our general asymptotic analysis, (3.168) reduces to

$$\chi(z_2) - \chi(z_1) = 8J , \tag{4.5}$$

while the expressions (3.167) and (3.170) already follow from (4.2) and (4.3). We can use (4.5) to eliminate $\chi(z_2) - \chi(z_1)$. Then (4.2) gives[1]

$$\ell_H = 2(M - 2\Omega J) \tag{4.6}$$

and (4.4) can be solved for J

$$J = \frac{4\Omega M^3}{1 + 4\Omega^2 M^2} . \tag{4.7}$$

Substituting (4.7) back into (4.6) we find

$$\ell_H = \frac{2M(1 - 4\Omega^2 M^2)}{1 + 4\Omega^2 M^2} \tag{4.8}$$

and hence positivity of the horizon rod length $\ell_H > 0$ and of the mass $M > 0$ implies

$$|\Omega| < \frac{1}{2M} . \tag{4.9}$$

This determines the full moduli space of non-extreme Kerr black hole solutions. Indeed, the relation (4.7) now implies the well-known inequality

$$|J| < M^2 . \tag{4.10}$$

In terms of the physical quantities the solution simplifies a little. We find for $z < z_1$:

$$h(z) = \frac{-(z - z_1)(z - z_2)}{(z - z_2)^2 - 4\Omega J(z - z_2) + 4M\Omega J}, \tag{4.11}$$
$$b(z) = \frac{2J}{(z - z_2)^2 - 4\Omega J(z - z_2) + 4M\Omega J} . \tag{4.12}$$

It is worth noting that the relation for $b(z_1)$ in (4.3) is automatically satisfied by this solution (as it must be by Remark 2 below Proposition 2). Thus from the above analysis we see that the solution is naturally parameterised by (M, Ω)[2]. It is interesting to note that we have fully determined the moduli

[1] Combining this with (3.37) leads to the standard Smarr relation.

[2] Eq (4.7) can be solved for Ω, yielding $\Omega J = M - \sqrt{M^2 - \frac{J^2}{M^2}}$. Using this, the solution can be equivalently uniquely parameterised in terms of (M, J).

space (4.10) of non-extremal Kerr solutions by only analysing one semi-infinite axis (this was also found in [38]).

A similar analysis can be performed for the other semi-infinite axis $z > z_2$. One again finds (4.2)-(4.4) and the solution for $z > z_2$:

$$h(z) = \frac{-(z - z_1)(z - z_2)}{(z - z_1)^2 + 4\Omega J(z - z_1) + 4M\Omega J}, \tag{4.13}$$

$$b(z) = \frac{-2J}{(z - z_1)^2 + 4\Omega J(z - z_1) + 4M\Omega J}. \tag{4.14}$$

Again, the relation (4.3) is automatically satisfied by this $b(z_2)$ (as it must be). Thus, the analysis of this semi-infinite axis yields equivalent results.

Finally, consider the horizon rod $z_1 < z < z_2$. We find that (3.121) gives

$$\gamma(z) = \frac{-4(z - z_1)(z - z_2)}{1 + 4\Omega^2(z - (z_1 - M))(z - (z_2 + M))}, \tag{4.15}$$

$$\chi(z) = \frac{-8\Omega(z - z_1)^2(z - (z_2 + M))}{1 + 4\Omega^2(z - (z_1 - M))(z - (z_2 + M))}, \tag{4.16}$$

where we have used (4.3) and (4.2). The solution for $\chi(z)$ can be shown to automatically satisfy (4.5) as a consequence of the above relations (as guaranteed by Proposition 2). Furthermore, it can be checked that $\gamma'(z_1) = -\gamma'(z_2)$ automatically so (3.35) implies that the metric on the horizon has no conical singularities and the surface gravity simplifies to

$$\kappa = \frac{1 - 4\Omega^2 M^2}{4M} \ . \tag{4.17}$$

Notice that (4.9) is equivalent to the non-extremality condition $\kappa > 0$.

To summarise, we have fully determined the metric on the whole z-axis for any solution with the same rod structure as Kerr and computed all asymptotic and horizon physical quantites. We find this reproduces the full moduli space of non-extremal Kerr black holes, as it must from the no-hair theorem. It is interesting to note that our analysis does this without knowledge of the full spacetime metric.

4.2 Five-dimensional Minkowski space

Next we consider simple $D = 5$ rod structures. We begin with the rod structure of Minkowski spacetime as in Figure 4.1. We have two rods $I_L = (-\infty, z_1)$ and $I_R = (z_1, \infty)$. In this case the matrices which

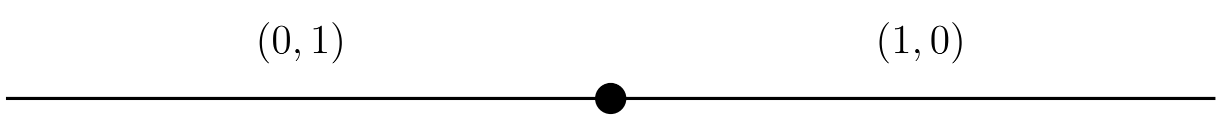

Figure 4.1: Rod structure for Minkowski spacetime.

give the general solution in Theorem 8 are $F_L(k) = -C^{-1}P_1(k)^T$ and $F_R(k) = -(CP_1(k))^{-1}$ where $P_1(k) = X_L(z_1, k)^{-1}X_R(z_1, k)$.

Let us first consider $z < z_1$. For I_L this gives

$$F_L(k)^T = \begin{pmatrix} 1 & b_0^R(z_1) - \frac{b_0^L(z_1)b_1^R(z_1)}{2(k-z_1)} & -\frac{b_0^L(z_1)}{2(k-z_1)} \\ 0 & 2(k-z_1) - \frac{b_1^L(z_1)b_1^L(z_1)}{2(k-z_1)} & -\frac{b_1^L(z_1)}{2(k-z_1)} \\ 0 & -\frac{b_1^R(z_1)}{2(k-z_1)} & -\frac{1}{2(k-z_1)} \end{pmatrix}, \tag{4.18}$$

and so

$$h^L_{\mu\nu}(z) = \begin{pmatrix} -1 & -b_0^R(z_1) \\ -b_0^R(z_1) + \frac{b_0^L(z_1)b_1^R(z_1)}{2(z-z_1)} & -2(z-z_1) \end{pmatrix}, \qquad b^L_\mu(z) = \begin{pmatrix} b_0^L(z_1) \\ b_1^L(z_1) \end{pmatrix}, \qquad z < z_1 \,. \tag{4.19}$$

Imposing our boundary condition $b^L_\mu(z) \to 0$ as $z \to -\infty$ then implies

$$b^L_\mu(z_1) = 0 \,, \tag{4.20}$$

which then immediately fixes $b^L_\mu(z) = 0$ for $z < z_1$.

The analysis for $z > z_1$ is analogous. One gets

$$F_R(k) = \begin{pmatrix} 1 & -b_0^L(z_1) + \frac{b_0^R(z_1)b_1^L(z_1)}{2(k-z_1)} & \frac{b_0^R(z_1)}{2(k-z_1)} \\ 0 & -2(k-z_1) + \frac{b_1^R(z_1)b_1^L(z_1)}{2(k-z_1)} & \frac{b_1^R(z_1)}{2(k-z_1)} \\ 0 & \frac{b_1^L(z_1)}{2(k-z_1)} & \frac{1}{2(k-z_1)} \end{pmatrix}, \tag{4.21}$$

and hence using the general solution (3.118) the metric data reads

$$h^R_{\mu\nu}(z) = \begin{pmatrix} -1 & b_0^L(z_1) \\ b_0^L(z_1) - \frac{b_1^L(z_1)b_0^R(z_1)}{2(z-z_1)} & 2(z-z_1) \end{pmatrix}, \qquad b^R_\mu(z) = \begin{pmatrix} b_0^R(z_1) \\ b_1^R(z_1) \end{pmatrix}, \qquad z > z_1 \,. \tag{4.22}$$

Imposing the boundary condition $b^R_\mu(z) \to 0$ as $z \to \infty$ implies

$$b^R_\mu(z_1) = 0 \tag{4.23}$$

and thus $b^R_\mu(z) = 0$ for $z < z_1$.

We have now fixed all non-trivial parameters. Notice that the asymptotic conditions for $h^L_{\mu\nu}, h^R_{\mu\nu}$ are both satisfied automatically with $M = J_1 = J_2 = \zeta = 0$. The final solution is simply

$$h^L_{\mu\nu}(z) = \begin{pmatrix} -1 & 0 \\ 0 & 2(z_1 - z) \end{pmatrix}, \qquad b^L_\mu(z) = 0, \qquad z < z_1 \tag{4.24}$$

$$h^R_{\mu\nu}(z) = \begin{pmatrix} -1 & 0 \\ 0 & 2(z - z_1) \end{pmatrix}, \qquad b^R_\mu(z) = 0, \qquad z > z_1 \,. \tag{4.25}$$

This of course is the metric data on axis for Minkowski spacetime (3.171). As in four dimensions this is a non-trivial result, showing that the only asymptotically flat spacetime in this symmetry class with the same rod structure as Minkowski spacetime is Minkowski spacetime itself. Of course, this is expected and follows from the more general no-soliton theorem for vacuum gravity.

4.3 Eguchi-Hanson soliton

Let us now attempt to construct a soliton solution, i.e. a non-trivial solution with no horizon. Of course, we know from the no-soliton theorem for asymptotically flat vacuum solutions that there can be no smooth solution in this case. Nevertheless, it is interesting to see how this emerges from our formalism.

The simplest rod structure without a horizon which is not flat space is given by three axis rods $I_L = (-\infty, z_1)$, $I_B = (z_1, z_2)$ and $I_R = (z_2, \infty)$ with rod vectors $(0,1)$, $v_B = (p,q)$ and $(1,0)$ respectively, where (p,q) are coprime integers. The finite axis rod I_B corresponds to a 2-cycle, or bolt, in the spacetime. The admissibility condition (3.8) between adjacent axis rods fixes $p = \pm 1$ and $q = \pm 1$ and without loss of generality we can fix $p = 1$ (since v_B is only defined up to a sign). We also fix $q = 1$ which can always be arranged since v_R is only defined up to a sign. Thus we take the rod vector for I_B to be $v_B = (1,1)$. The rod structure is depicted in Figure 4.2. We choose the other independent

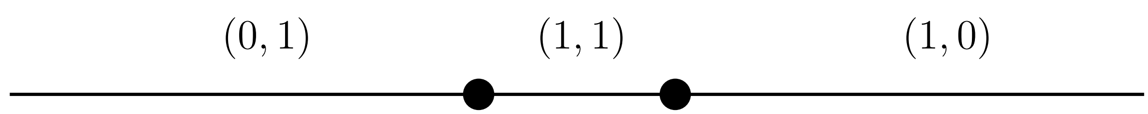

Figure 4.2: Rod structure for the simplest soliton spacetime.

axial vector to be $u_B = (1,0)$, so the change of basis matrix (3.40) is

$$L_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} . \tag{4.26}$$

The general solution in this case is determined by (3.118) where

$$\begin{gathered} F_L(k) = -C^{-1}P_2(k)^T P_1(k)^T, \qquad F_B(k) = -P_1(k)^{-1}C^{-1}P_2(k)^T, \\ F_R(k) = -P_2(k)^{-1}P_1(k)^{-1}C^{-1} \end{gathered} \tag{4.27}$$

and $P_a(k)$ are given by (3.102).

First, imposing that the general solution $(h^L_{\mu\nu}(z), b^L_\mu(z))$ on I_L obeys our boundary condition $b^L_\mu(z) \to 0$ as $z \to -\infty$ fixes the constants

$$b^L_\mu(z_1) = -b^B_\mu(z_2) \ , \tag{4.28}$$

with $b^L_\mu(z)|_{z\to z_1} = b^L_\mu(z_1)$ being automatically satisfied (as guaranteed by (3.133)). Next, imposing that $(h^R_{\mu\nu}(z), b^R_\mu(z))$ on I_R obeys $b^R_\mu(z) \to 0$ as $z \to \infty$ fixes

$$b^R_\mu(z_2) = \begin{pmatrix} b^B_0(z_2) \\ -b^B_1(z_2) + 2(z_1 - z_2) \end{pmatrix} \tag{4.29}$$

with $b^R_\mu(z)|_{z\to z_2} = b^R_\mu(z_2)$ being automatically satisfied (again, as guaranteed by (3.133)). These relations also follow from our general asymptotic analysis (3.190) and (3.188) respectively.

Finally, the solution $(h^B_{\mu\nu}(z), b^B_\mu(z))$ on I_B satisfies $b^B_\mu(z)|_{z\to z_2} = b^B_\mu(z_2)$ automatically (as guaranteed by Proposition 2) and $b^B_\mu(z)|_{z\to z_1} = 0$ fixes

$$b^B_\mu(z_2) = \begin{pmatrix} 0 \\ z_1 - z_2 \end{pmatrix} , \tag{4.30}$$

where we have used the above to simplify this expression. All parameters have been now fixed except for the axis rod length $\ell_B = z_2 - z_1$. The resulting solution is

$$h^L_{\mu\nu}(z) = \begin{pmatrix} -1 & 0 \\ 0 & -\frac{4(z-z_1)(z-z_2)}{2z-z_1-z_2} \end{pmatrix}, \qquad b^L_\mu(z) = \begin{pmatrix} 0 \\ -\frac{(z_2-z_1)^2}{2z-z_1-z_2} \end{pmatrix}, \qquad z \in I_L \tag{4.31}$$

$$h^B_{\mu\nu}(z) = \begin{pmatrix} -1 & 0 \\ 0 & -\frac{(z-z_1)(z-z_2)}{z_2-z_1} \end{pmatrix}, \qquad b^B_\mu(z) = \begin{pmatrix} 0 \\ \frac{(z-z_1)(z+z_1-2z_2)}{z_2-z_1} \end{pmatrix}, \quad z \in I_B \tag{4.32}$$

$$h^R_{\mu\nu}(z) = \begin{pmatrix} -1 & 0 \\ 0 & \frac{4(z-z_1)(z-z_2)}{2z-z_1-z_2} \end{pmatrix}, \qquad b^R_\mu(z) = \begin{pmatrix} 0 \\ -\frac{(z_2-z_1)^2}{2z-z_1-z_2} \end{pmatrix}, \qquad z \in I_R. \tag{4.33}$$

From the asymptotics $z \to \pm\infty$, we immediately deduce from (3.180) or (3.181) that

$$M = 0, \qquad J_1 = J_2 = 0, \qquad \zeta = -\tfrac{1}{8}(z_2 - z_1)^2 \,. \tag{4.34}$$

This corresponds to the unique unbalanced solution which is guaranteed to exist by Theorem 4.

We may now analyse regularity of the solution. The metric induced on the bolt (3.23) is

$$\mathbf{g}_B = -\mathrm{d}t^2 + \ell_B \left(\frac{c_B^2 \mathrm{d}y^2}{1-y^2} + \frac{1}{4}(1-y^2)(\mathrm{d}x^1)^2 \right), \qquad y = \frac{2z - z_1 - z_2}{z_2 - z_1}\,, \tag{4.35}$$

where (t, x^1) are coordinates such that $k = \partial_t, u = \partial_{x^1}$ and recall (k, u) is the adapted basis for I_B. Recall that $u = m_1$ and hence x^1 is a 2π-periodic angle. Therefore, it is clear that the spatial part of the metric on the bolt is a smooth round metric on S^2 iff $c_B = 1/2$ (indeed, one can check that the conditions for the removal of the conical singularity (3.25) and (3.26) at $z = z_1$ and $z = z_2$ are satisfied iff $c_B = 1/2$). Although this gives a smooth metric on the bolt, this shows that in this case the balance condition $c_B = 1$ (3.24) is violated so there must be a conical singularity at I_B. On the other hand, if we impose the balance condition $c_B = 1$, then inspecting the metric on the bolt shows that there must be conical singularities at the endpoints of I_B.

We have shown that any asymptotically flat solution with a single bolt must have a conical singularity. This is indeed consistent with the no-soliton theorem mentioned above. In fact, in this case it is easy to write down the full solution off axis. It is given by the Eguchi-Hanson soliton (3.14) where (θ, ψ, ϕ) are Euler angles on S^3. The rods I_L, I_B and I_R can be identified with $\theta = \pi, R = a$ and $\theta = 0$. It then follows that $v_L = \partial_\psi + \partial_\phi$ and $v_R = \partial_\psi - \partial_\phi$ are the 2π-periodic rod vectors on the semi-infinite axes, which implies $\phi^1 = (\psi - \phi)/2$ and $\phi^2 = (\psi + \phi)/2$. Weyl coordinates $(t, \phi^1, \phi^2, \rho, z)$ for this metric are

$$\rho = \tfrac{1}{2}\sqrt{R^4 - a^4}\sin\theta, \quad z = \tfrac{1}{2}(z_1 + z_2) + \tfrac{1}{2}R^2 \cos\theta \tag{4.36}$$

and the corresponding metric data is

$$\begin{aligned} g &= -\mathrm{d}t^2 + \tfrac{1}{4}R^2\left(1 - \frac{a^4}{R^4}\right)\left[(1-\cos\theta)\mathrm{d}\phi^1 + (1+\cos\theta)\mathrm{d}\phi^2\right]^2 + \tfrac{1}{4}R^2\sin^2\theta(\mathrm{d}\phi^1 - \mathrm{d}\phi^2)^2\,, \\ e^{2\nu} &= \frac{R^2}{R^4 - a^4\cos^2\theta}\,. \end{aligned} \tag{4.37}$$

Using $a^2 = \ell_B$, it is straightforward to show that g gives the same $(h^a_{\mu\nu}, b^a_\mu)$ on each rod as our general solution above (4.31)-(4.33). In addition $e^{2\nu}$ on the axes and the bolt agrees with our expressions (3.22) with $c_L = c_R = 1$ and $c_B = 1/2$.

4.4 Myers-Perry solution

We now consider the simplest rod structure of a single black hole with S^3 topology , i.e., the same rod structure as the Myers-Perry solution, see Figure 4.3. Thus we have three rods $I_L = (-\infty, z_1)$,

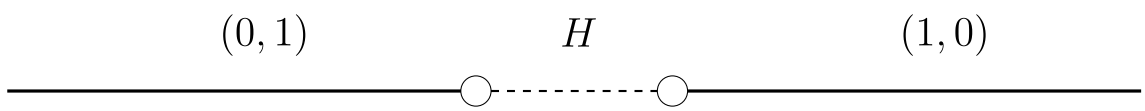

Figure 4.3: Rod structure for the Myers-Perry black hole.

$I_H = (z_1, z_2)$ and $I_R = (z_2, \infty)$ where I_H is a horizon rod.

The general solution can be obtained from Theorem 8 where the $F_a(z)$ are again given by (4.27) (although $X_2(z,k)$ now refers to the horizon rod). The solution depends on the parameters $(\ell_H, b^L_\mu(z_1), b^R_\mu(z_2), \chi_i(z_2), \Omega_i)$ where $\ell_H = z_2 - z_1$ and we choose a gauge in which $\chi_i(z_1) = 0$.

From the asymptotics for the solution on I_L and I_R given in (3.180) and (3.181) we find the mass M and angular momenta J_i are given by (3.191) and (3.192), the Ernst potentials are[3]

$$b^L_\mu(z_1) = \Omega_2 \begin{pmatrix} \frac{8}{3\pi} M \\ -\frac{4J_1}{\pi} \end{pmatrix}, \qquad b^R_\mu(z_2) = \Omega_1 \begin{pmatrix} -\frac{8}{3\pi} M \\ \frac{4J_2}{\pi} \end{pmatrix}, \tag{4.38}$$

and

$$J_1 = \tfrac{16}{9\pi} M \Omega_1 \left(M - \tfrac{3}{2}\Omega_2 J_2 \right) , \qquad J_2 = \tfrac{16}{9\pi} M\Omega_2 \left(M - \tfrac{3}{2}\Omega_1 J_1 \right) , \tag{4.39}$$

where we have eliminated ℓ_H and $\chi_i(z_2)$ in favour of M and J_i using (3.191) and (3.192). It is worth noting that the solutions on I_L and I_R automatically obey $b^L_\mu(z)|_{z\to z_1} = b^L_\mu(z_1)$ and $b^R_\mu(z)|_{z\to z_2} = b^R_\mu(z_2)$ and therefore no further constraints arise from these rods (as guaranteed by (3.133)). Observe that (4.39) are linear in J_i so we can straightforwardly solve these for J_i and therefore express all parameters in terms of the physical variables M, Ω_i.

It is convenient to use the dimensionless quantities (3.193). Then solving (4.39) gives

$$j_1 = \frac{\omega_1(1-\omega_2^2)}{1-\omega_1^2\omega_2^2}, \qquad j_2 = \frac{\omega_2(1-\omega_1^2)}{1-\omega_1^2\omega_2^2} \tag{4.40}$$

and $|\omega_1\omega_2| \neq 1$.[4] Thus as promised we can express all quantities in terms of M, ω_i. In particular, eliminating j_i we find that (3.194) becomes

$$\lambda_H = \frac{(1-\omega_1^2)(1-\omega_2^2)}{1-\omega_1^2\omega_2^2} \, . \tag{4.41}$$

To determine the precise moduli space, we will also need the invariants

$$\det h^L_{\mu\nu}(z) = -\frac{2(z-z_1)(z-z_2)}{\bar{z}_1 + z_1 - z}, \qquad \bar{z}_1 = \frac{4M}{3\pi}\frac{1-\omega_1^2}{1-\omega_1^2\omega_2^2} \, , \qquad z < z_1 \, , \tag{4.42}$$

$$\det h^R_{\mu\nu}(z) = -\frac{2(z-z_1)(z-z_2)}{z - z_2 + \bar{z}_2}, \qquad \bar{z}_2 = \frac{4M}{3\pi}\frac{1-\omega_2^2}{1-\omega_1^2\omega_2^2} \, , \qquad z > z_2 \, . \tag{4.43}$$

[3] Equation (4.38) also follows from our general asymptotic analysis (3.190) and (3.188). The same result can be established from general considerations using (3.223) and (3.225), together with the fact that $b^L_\mu(z) = 0$ on I_R and $b^R_\mu(z) = 0$ on I_L (from their definition (3.46, 3.47) the potentials b^L_μ, b^R_μ are constant on I_R, I_L respectively and vanish at infinity).

[4] If $|\omega_1\omega_2| = 1$ then (4.39) imply $\lambda_H = 0$ which contradicts our non-extremality assumption.

A smooth Lorentzian metric on I_L requires that the determinant $\det h^L_{\mu\nu}(z) < 0$ and is smooth for $z < z_1$ (see (3.134)) and from the above expression we see this is equivalent to $\bar{z}_1 > 0$. Similarly, the requirement that $\det h^R_{\mu\nu}(z)$ is smooth and negative on I_R is equivalent to $\bar{z}_2 > 0$. The inequalities $\lambda_H > 0, \bar{z}_1 > 0, \bar{z}_2 > 0$ are equivalent to

$$|\omega_i| < 1 \; . \tag{4.44}$$

This fully constrains the moduli space of solutions which is simply given by (4.44) and $M > 0$. One can show (4.44) implies

$$|j_1| + |j_2| < 1 \; , \tag{4.45}$$

which is a well-known inequality for the Myers-Perry black holes.

Now we turn to the solution $(\gamma_{ij}(z), \chi_i(z))$ on the horizon rod $z_1 < z < z_2$ which can be deduced from (3.121). Writing the parameters in terms of M, ω_i as above, we find that both $\chi_i(z)|_{z\to z_1} = 0$ and (3.192) are automatically satisfied (as they must be). Furthermore, using (3.36), we find that removal of the conical singularities of the horizon metric at the endpoints $z = z_1, z_2$ imposes no further constraints and fixes the surface gravity to be

$$\kappa = \sqrt{\frac{3\pi}{8M}(1-\omega_1^2)(1-\omega_2^2)} \; . \tag{4.46}$$

The horizon topology is of course S^3 with $m_2 = 0$ at $z = z_1$ and $m_1 = 0$ at $z = z_2$. Notice that the moduli space (4.44) is equivalent to the non-extremality condition $\kappa > 0$.

It is straightforward to check that the metric data for above solution agrees precisely with the Myers-Perry solution restricted to the z-axis, and the parameter region $|\omega_i| < 1$ we have derived agrees with the full moduli space of non-extremal Myers-Perry black holes (of course, this includes 5d Schwarzschild for $\omega_i = 0$). It is interesting to note that by combining (4.39) we obtain the thermodynamic identity recently obtained by integrating the sigma model equation over the boundary of the orbit space [142]. Thus our present method leads to a refinement of these identities.

4.5 Black ring

We now consider the rod structure of the black ring as depicted in Figure 4.4. Thus we have four rods

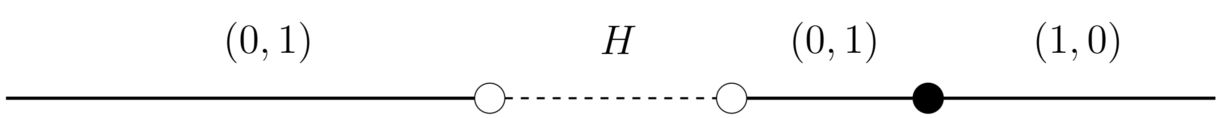

Figure 4.4: Rod structure for the black ring

$I_L = (-\infty, z_1)$, $I_H = (z_1, z_2)$, $I_D = (z_2, z_3)$ and $I_R = (z_3, \infty)$, where I_H is a horizon rod and I_D is an axis rod with rod vector $v_D = (0,1)$. The topology of the horizon is $S^2 \times S^1$ and the finite axis rod I_D lifts to a noncontractible 2-disc in spacetime. We use the adapted basis $\tilde{E}_A = (k, m_1, m_2)$ for I_D, i.e. $u_D = (1,0)$, so the change of basis matrix L_D (3.40) is simply the identity matrix.

The general solution is given by Theorem 8 and depends on the parameters

$$\{\ell_H, \ell_D, b^L_\mu(z_1), \chi_i(z_2), b^D_\mu(z_3), b^R_\mu(z_3), \Omega_i\} \tag{4.47}$$

where $\ell_H = z_2 - z_1, \ell_D = z_3 - z_2$ and we choose a gauge in which $\chi_i(z_1) = 0$.

From the asymptotics for the solution on I_L and I_R given in (3.180) and (3.181) we find the mass M and angular momenta J_i are given by (3.191) and (3.192) and the Ernst potentials are

$$b^L_\mu(z_1) + b^D_\mu(z_3) = \Omega_2 \begin{pmatrix} \frac{8}{3\pi}M \\ -\frac{4J_1}{\pi} \end{pmatrix}, \qquad b^R_\mu(z_3) = \Omega_1 \begin{pmatrix} -\frac{8}{3\pi}M \\ \frac{4J_2}{\pi} \end{pmatrix}, \tag{4.48}$$

where we have eliminated ℓ_H and $\chi_i(z_2)$ in favour of M and J_i using (3.191) and (3.192).

The asymptotics of the solution also give non-trivial equations for J_i which in terms of dimensionless variables introduced in (3.193) and (3.195) are given by

$$\begin{aligned} j_1 &= \omega_1(1 + \lambda_D + j_2(f^D_0 - \omega_2)) \, , \\ j_2 &= \omega_2 - \omega_1(j_1\omega_2 + f^D_1) + f^D_0(\omega_1 j_1 - 2) \, , \end{aligned} \tag{4.49}$$

where the previous relations have been used to eliminate variables in favour of $j_i, \omega_i, f^D_\mu, \lambda_D$. These equations correspond to (4.39) for the Myers-Perry solution.

Next consider I_D. $b^D_\mu(z)|_{z\to z_2} = 0$ gives the new constraints

$$f^D_\mu = \frac{j_2\lambda_D}{D}\begin{pmatrix} 1-\omega_1^2 \\ -\omega_1\lambda_D \end{pmatrix}, \quad D \equiv (1-\omega_1 j_1)^2 - j_2(f^D_0 + \omega_1(\omega_1 j_2 + f^D_1)). \tag{4.50}$$

These equations are well defined since $D > 0$. This follows from the relation $D = 2\lambda_H\lambda_D r^-_D$ together with (3.138) and positivity of the rod lengths.

Equations (4.49) and (4.50) are significantly more complicated than the corresponding parameter constraints for the Myers-Perry solution (4.40). Therefore it is instructive to first consider the S^1 singly spinning case.

4.5.1 Singly spinning black ring

The S^1 spinning black ring corresponds to setting $j_2 = 0$ in the above equations. In this case (4.50) simply gives that $f^D_\mu = 0$. Substituting this back into the equations for j_i (4.49) gives

$$\begin{aligned} j_1 &= \omega_1(1+\lambda_D) \\ \omega_2(1 - \omega_1 j_1) &= 0. \end{aligned} \tag{4.51}$$

The first of these two equations gives j_1 and the second implies that $\omega_2 = 0$ since $1 - \omega_1 j_1 = \lambda_H \neq 0$.

This gives the solution for the general unbalanced S^1 spinning black ring parameterised in terms of (M, ω_1, λ_D). The horizon rod length $\lambda_H = 1 - \omega_1^2(1+\lambda_D)$ and so $\lambda_H > 0$ gives the constraint

$$\omega_1^2 < \frac{1}{1+\lambda_D}, \tag{4.52}$$

which together with the conditions $M > 0$ and $\lambda_D > 0$ determines the moduli space of unbalanced solutions. It can then be checked that (3.134) and (3.135) are satisfied automatically and so impose no further constraints.

Next consider conical singularities on I_D. The balance condition (3.24) and regularity condition at $z = z_3$ (3.25) is equivalent to

$$(1-\omega_1^2)^2 - \lambda_D\omega_1^2(2-\omega_1^2) = 0 \tag{4.53}$$

which implies $\omega_1^2 > 0$ and

$$\lambda_D = \frac{(1-\omega_1^2)^2}{\omega_1^2(2-\omega_1^2)} \, . \tag{4.54}$$

Substituting this back in we obtain

$$\lambda_H = \frac{1-\omega_1^2}{2-\omega_1^2}, \tag{4.55}$$

which gives the moduli space of the balanced solution as

$$M > 0, \quad 0 < \omega_1^2 < 1. \tag{4.56}$$

In addition, the expression for j_1 (4.51) now takes the simple form

$$j_1 = \frac{1}{\omega_1(2-\omega_1^2)}. \tag{4.57}$$

Extremising this over the moduli space (4.56) gives the well-known inequality $|j_1| \geq \sqrt{27/32}$. Finally, condition (3.36) for the removal of conical singularities on I_H imposes no further constraints and fixes the surface gravity to be

$$\kappa = \sqrt{\frac{3\pi}{8M}}\frac{\sqrt{1-\omega_1^2}}{|\omega_1|}. \tag{4.58}$$

4.5.2 Doubly spinning black ring

Now we consider the doubly spinning solution corresponding to $j_2 \neq 0$. In this case it is no longer straightforward to solve (4.49) and (4.50) in terms of any of the variables already defined. Firstly, using (4.49) and (4.50), together with the balance condition (3.24) on I_D and the condition for removal of the conical singularity on I_D at $z = z_3$ (3.25), one can show that $\omega_2 = 0$ implies $j_2 = 0$ (here we are also assuming $\lambda_H, \lambda_D > 0$). Thus we deduce $\omega_2 \neq 0$.

It turns out it is convenient to define a new parameter t, using the denominator D defined in (4.50), by

$$t = \frac{\omega_2 D}{j_2 \lambda_D}. \tag{4.59}$$

Note that $t \neq 0$. This gives

$$f^D_\mu = \frac{\omega_2}{t}\begin{pmatrix} 1-\omega_1^2 \\ -\omega_1\lambda_D \end{pmatrix}. \tag{4.60}$$

Now we can solve (4.49) for j_i[5]

$$\begin{aligned} j_1 &= \frac{\omega_1(t^2(1+\lambda_D) - \omega_2^2(t-1+\omega_1^2)(t-2+\omega_1^2(2+\lambda_D))}{t^2 - \omega_1^2\omega_2^2(t-1+\omega_1^2)^2}\,, \\ j_2 &= \frac{t\omega_2(t-2+\omega_1^2)(1-\omega_1^2(1+\lambda_D))}{t^2 - \omega_1^2\omega_2^2(t-1+\omega_1^2)^2}\,, \end{aligned} \tag{4.61}$$

and then (4.59) for ω_2[6]

$$\omega_2^2 = \frac{t^2(1-\omega_1^2 - \lambda_D(t-2+2\omega_1^2))}{(t-2+2\omega_1^2)(1-\omega_1^2(1+\lambda_D))}. \tag{4.62}$$

This gives two branches of solutions corresponding to either $\omega_2 > 0$ or $\omega_2 < 0$. We have now solved for the generic[7] unbalanced doubly spinning black ring solution parameterised in terms of $(M, \omega_1, t, \lambda_D)$.

[5] Using (4.49) and (4.59) one can show that the denominator of (4.61) being zero is incompatible with $\lambda_H, \lambda_D > 0$ and the conditions for the removal of conical singularities (3.24) and (3.25).

[6] The denominator of (4.62) can never vanish since the denominator of (4.61) is nonzero and $j_2 \neq 0$.

[7] As explained above, a couple of possible special cases were ruled out using the balance condition.

Now consider the possible conical singularities on I_D. To remove this the balance condition (3.24) and the regularity condition (3.25) at $z = z_3$ must be satisfied, which in this case reduces to

$$\omega_1^4 \lambda_D + \omega_1^2 (1 + t\lambda_D) - 1 = 0. \tag{4.63}$$

Note that this implies that $\omega_1 \neq 0$. Solving this for λ_D one finds[8]

$$\lambda_D = \frac{1 - \omega_1^2}{\omega_1^2 (t + \omega_1^2)}. \tag{4.64}$$

The expressions (4.61), (4.62) and (3.194) can be simplified with this result and one finds

$$j_1 = \frac{1 + (t - 1 + 2\omega_1^2)(t - 1 + \omega_1^2)}{\omega_1 (t + \omega_1^2)^2}, \qquad j_2 = \frac{\omega_2 (t - 1 + \omega_1^2)(t - 2 + 2\omega_1^2)}{t(t + \omega_1^2)^2}, \tag{4.65}$$

$$\omega_2^2 = \frac{t^2 (\omega_1^4 - (t - 2)(1 - \omega_1^2))}{\omega_1^2 (t - 1 + \omega_1^2)(t - 2 + 2\omega_1^2)}, \tag{4.66}$$

$$\lambda_H = \frac{(1 - \omega_1^2)(t - 2 + \omega_1^2)}{\omega_1^2 (t + \omega_1^2)} = \lambda_D (t - 2 + \omega_1^2). \tag{4.67}$$

This gives the balanced doubly rotating solution, however one still needs to find the bounds on the parameters (M, ω_1, t). These turn out to be given by $M > 0$,

$$0 < 1 - \omega_1^2 < t - 1, \qquad t < ((1 - \omega_1^2) + (1 - \omega_1^2)^{-1}). \tag{4.68}$$

Positivity of the rod lengths $\lambda_H, \lambda_D > 0$ is equivalent to the first condition and the second condition then corresponds to $\omega_2^2 > 0$. The conditions (3.134) and (3.135) are then automatically satisfied and impose no further constraints.

Note that the limit curve given by $t \to ((1 - \omega_1^2) + (1 - \omega_1^2)^{-1})$ corresponds to the $\omega_2 \to 0$ (or equivalently $j_2 \to 0$) singly spinning limit. It turns out that taking this limit one recovers the results of the previous section on the S^1 spinning ring as one might expect. Therefore, although the original definition of t (4.59) only holds when $j_2 \neq 0$, this parameterisation can be extended to cover the singly spinning case as well.

Finally consider the horizon rod I_H. Using the parameters (M, ω_1, t), we find that both $\chi_i(z)|_{z \to z_1} = 0$ and (3.192) are automatically satisfied (as they must be). There are no further constraints from removing conical singularities at the endpoints of I_H since (3.36) is also satisfied automatically for a surface gravity given by

$$\kappa = \sqrt{\frac{3\pi}{8M(t - 1 + \omega_1^2)}} \frac{(1 - \omega_1^2)(t - 2 + \omega_1^2)(t + \omega_1^2)}{|\omega_1|(t - 2 + 2\omega_1^2)}. \tag{4.69}$$

From this one can explicitly see that the limit curve $\omega_1 \to \sqrt{2 - t}$, which is a boundary of the moduli space of solutions, corresponds to extremal solutions as one might expect. On the other hand although $\kappa = 0$ as $\omega_1 \to 1$, this corresponds to a singular solution since $\lambda_D \to 0$ in this limit.

We have now constructed the most general regular solution on the axes and horizon with the given rod structure. We will now show that our solution maps exactly to the Pomeransky-Sen'kov

[8] If $t + \omega_1^2 = 0$, using (4.63) and (4.62) one can show that the denominator of (4.61) is zero which is a contradiction. Therefore (4.64) is the unique solution of (4.63).

solution for the balanced doubly rotating black ring. Chen, Hong and Teo [75] present the solution for $\omega_1 > 0, \omega_2 > 0$ in terms of the parameters (χ, μ, ν), satisfying

$$\chi > 0, \qquad 0 < \nu < \mu < 1. \tag{4.70}$$

Note that we take $\nu \neq \mu$ since we are considering non-extremal solutions and $\nu \neq 0$ since we are considering $\omega_2 \neq 0$. To find an expression for t in terms of these variables, first use (4.67) to give

$$t = (2 - \omega_1^2) + \frac{\lambda_H}{\lambda_D}. \tag{4.71}$$

Using this, combined with the expressions for $M, \Omega_1, \ell_H, \ell_D$ from the known solution gives

$$M = \frac{3\pi\chi^2(\mu+\nu)}{(1-\mu)(1-\nu)}, \qquad \omega_1^2 = \frac{2(\mu+\nu)}{(1+\mu)(1+\nu)}, \qquad t = \frac{2(1+\mu^2)(1-\nu)}{(1-\mu^2)(1+\nu)}. \tag{4.72}$$

Inverting these relations for χ^2, μ and ν gives

$$\chi^2 = \frac{2M}{3\pi}\frac{(1-\omega_1^2)}{\omega_1^2}, \qquad \mu = \frac{x-(1-\omega_1^2)}{x+(1-\omega_1^2)}, \qquad \nu = \frac{1-x}{1+x}, \tag{4.73}$$

where

$$x = \sqrt{(1-\omega_1^2)(t-1+\omega_1^2)}. \tag{4.74}$$

A short calculation also demonstrates that these expressions give bijections between the subspaces defined by (4.70) and (4.68) restricted to $\omega_1 > 0, \omega_2 > 0$. One can also show that the metric data on the axis and horizon rods agrees precisely under this map. Therefore, we deduce that the Pomeransky-Sen'kov black ring is the most general regular solution within this class of rod structures (for $\omega_2 = 0$ see the singly spinning case above).

4.6 Black lens solution

An $L(p,q)$ lens space is a topological space which can be described in the following way:

$$L(p,q) = \begin{cases} S^1 \times S^2, & p = 0 \\ S^3/\sim, & \text{otherwise} \end{cases} \tag{4.75}$$

where $\sim$ is given by $(w_1, w_2) \sim (e^{2\pi i/p}w_1, e^{2\pi i q/p}w_2)$ where $(w_1, w_2) \in \{(w_1, w_2) \in \mathbb{C}^2 : |w_1|^2 + |w_2|^2 = 1\} \cong S^3$.

In this section we will look at a black hole with lens space horizon topology - a so-called black lens. We consider the rod structure for the simplest black lens as in Figure 4.5 below. Thus we have four

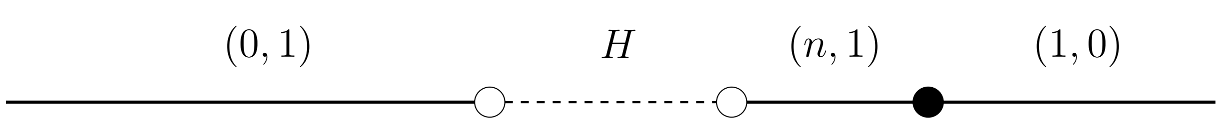

Figure 4.5: Rod structure for a black lens with $L(n,1)$ horizon topology.

rods: $I_L = (-\infty, z_1)$, $I_H = (z_1, z_2)$, $I_D = (z_2, z_3)$ and $I_R = (z_3, \infty)$, where I_H is a horizon rod and

I_D is an axis rod with rod vector to be $v_D = (n, 1)$ where $n \in \mathbb{Z}$. Indeed, up to irrelevant discrete choices this is the most general rod structure with one horizon and one finite axis rod that is compatible with asymptotic flatness and obeys the admissibility condition $\det(v_D, v_R) = \pm 1$ (this latter condition ensures the absence of orbifold singularities at $z = z_3$) [56]. The topology of the horizon is the lens space $L(n, 1)$ and the finite axis rod I_D lifts to a noncontractible 2-disc in the spacetime. For $n = 0$ this is the rod structure for the black ring solution (Figure 4.4). For $n = \pm 1$ the horizon topology $L(1, 1) \cong S^3$ is spherical, although the rod structure is distinct to that of the Myers-Perry black hole (Figure 4.3). We use the adapted basis $\tilde{E}_A = (k, m_1, v_D)$ for I_D, i.e. $u_D = (1, 0)$, so the change of basis matrix L_D (3.40) is given by

$$L_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -n & 1 \end{pmatrix}. \tag{4.76}$$

4.6.1 Moduli space equations

The general axis solution is given by Theorem 8 and depends on the same parameters as given for the black ring (4.47), where again we choose a gauge in which $\chi_i(z_1) = 0$.

From the asymptotics for the solution on I_L and I_R given in (3.180) and (3.181) we find the mass M and angular momenta J_i are given by (3.191) and (3.192) and the Ernst potentials are

$$f^L_\mu = \omega_2 \begin{pmatrix} 1 \\ -j_1 \end{pmatrix} - f^D_\mu, \qquad f^R_\mu = \omega_1 \begin{pmatrix} -1 \\ j_2 \end{pmatrix} + n \begin{pmatrix} f^D_0 \\ -\lambda_D - n f^D_1 \end{pmatrix}, \tag{4.77}$$

where we have eliminated ℓ_H and $\chi_i(z_2)$ in favour of M and J_i using (3.191) and (3.192), and introduced dimensionless variables according to (3.193) and (3.195). The asymptotics of the solution also give non-trivial equations for the angular momenta

$$\begin{aligned} j_1 &= \omega_1 - \omega_1 j_2(\omega_2 - f^D_0) + (\lambda_D + n f^D_1)(\omega_1 - n f^D_0), \\ j_2 &= -f^D_0 + (1 - \omega_1 j_1)(\omega_2 - f^D_0) - f^D_1(\omega_1 - n f^D_0). \end{aligned} \tag{4.78}$$

where the previous relations have been used to eliminate variables in favour of $j_i, \omega_i, f^D_\mu, \lambda_D$.

Next consider I_D. We find that $b^D_\mu(z)|_{z \to z_2} = 0$ gives the new constraints

$$f^D_\mu = \frac{\lambda_D}{D} \begin{pmatrix} j_2 + n j_1 - (n + j_2(\omega_1 - n\omega_2))(\omega_1 - n f^D_0) \\ -(n + j_2(\omega_1 - n\omega_2))\lambda_D \end{pmatrix} \tag{4.79}$$

where

$$\begin{aligned} D := &(1 - \omega_1 j_1)(1 - j_1(\omega_1 - n\omega_2)) - (j_2 + n j_1) f^D_0 \\ &+ (2n\lambda_D - \omega_1 j_2 + (n^2 - 1) f^D_1)(n + j_2(\omega_1 - n\omega_2)). \end{aligned} \tag{4.80}$$

These equations are well defined since $D > 0$. As with the black ring in the previous section this follows from the relation $D = 2\lambda_H \lambda_D r^-_D$ together with (3.138) and positivity of the rod lengths.

To summarise, we have shown that the solution is parameterised by the mass $M > 0$ and the seven dimensionless parameters $\lambda_D, f^D_\mu, j_i, \omega_i$ subject to the four algebraic equations (4.78), (4.79) and various inequalities (3.136), (3.134), (3.135), (3.138), (3.139). Generically this leaves a four-parameter family of solutions which we expect to correspond to the general unbalanced doubly spinning black lens (also a four-parameter family, proven to exist in [69]) by appealing to Conjecture 2.

Finally we restrict to regular solutions by demanding the absence of conical singularities as $\rho \to 0$ at the finite axis rod I_D and also at the corner $z = z_3$ where two axis rods I_D and I_R meet. This imposes conditions (3.24), (3.25), a consequence of which is the "continuity condition"[9]

$$h^{D\prime}(z_3) = -h^{R\prime}(z_3)\ . \tag{4.81}$$

Using (3.120), (3.138) one finds $h^{D\prime}(z_3) = -1/r_D^+$ and $h^{R\prime}(z_3) = -1/r_R^-$ so that the continuity condition is equivalent to

$$r_D^+ + r_R^- = 0\ . \tag{4.82}$$

Defining $C := -2\lambda_D(\lambda_H + \lambda_D)(r_D^+ + r_R^-)$, this becomes equivalent to

$$\begin{aligned} C = (1-\omega_1 j_1)(1 - j_1(\omega_1 - n\omega_2) + 2\lambda_D) - (j_2 + nj_1)f_0^D \\ + ((n^2-2)f_1^D - \omega_1 j_2)(n + j_2(\omega_1 - n\omega_2)) \\ + (n(n^2-2)f_1^D + n^2(1 - \omega_2 j_2 + \lambda_D) - 1)\lambda_D = 0. \end{aligned} \tag{4.83}$$

We find that (4.83) is a convenient way to impose the regularity conditions listed above, even though a priori it is only a consequence of the balance condition (3.24) and regularity at the corner $z = z_3$. Thus, by a slight abuse of terminology we will often refer to (4.83) simply as the balance condition.

Equations (4.78), (4.79) and (4.83) are substantially more difficult to solve than even the equivalent moduli space equations for the black ring considered in the previous section. For this reason it is convenient to first analyse various special cases that are tractable, such as the static limit and two distinct singly spinning limits, before considering the full doubly spinning solution.

4.6.2 Static black lens

For simplicity first suppose that $j_i = 0$ for $i = 1, 2$. We expect this case to be static (see staticity theorem [68]) and hence necessarily singular, in line with the static uniqueness theorem [49].

First consider (4.79), which in this case reduces to

$$f_\mu^D = -\frac{n\lambda_D}{D}\begin{pmatrix}\omega_1 - nf_0^D \\ \lambda_D\end{pmatrix}, \tag{4.84}$$

where

$$D = 1 + n(2n\lambda_D + (n^2-1)f_1^D). \tag{4.85}$$

Eliminating D between the two components of (4.84) gives $(\lambda_D + nf_1^D)(\omega_1 - nf_0^D) = \omega_1\lambda_D$ after some algebra. Using this in the first equation of (4.78) gives $\omega_1(1+\lambda_D) = 0$ and therefore $\omega_1 = 0$ since $\lambda_D > 0$. The $\mu = 0$ component of (4.84) then gives that either $f_0^D = 0$ or $D = n^2\lambda_D$, however the latter case can be shown to lead to a contradiction by combining it with $\mu = 1$ component of (4.84) and then with (4.85) and $\lambda_D > 0$. Now, the second equation of (4.78) gives $\omega_2 = 0$.

Next consider the $\mu = 1$ component of (4.84). To solve this it is convenient to define a new parameter t according to

$$t := \frac{D}{\lambda_D} - n^2, \tag{4.86}$$

which can be used to write f_1^D as

$$f_1^D = \frac{-n\lambda_D}{t + n^2}. \tag{4.87}$$

[9] So named since it is equivalent to the continuity of $|(z - z_3)|e^{2\nu}$. See appendix 3.B for further details.

Note that $t + n^2 \neq 0$ since, as we have already discussed, $D \neq 0$ is required by the positivity of the finite rod lengths. Using (4.87) in (4.86) with the explicit form of D and solving for λ_D in terms of t gives

$$\lambda_D = \frac{t + n^2}{t^2 - n^2}. \tag{4.88}$$

This expression is well-defined since if $t^2 = n^2$ that would imply that $t+n^2 = 0$, which as just mentioned is not allowed. Substituting this back into (4.87), one can write f_1^D purely in terms of t to find

$$f_1^D = \frac{-n}{t^2 - n^2}. \tag{4.89}$$

To summarise, we have shown that the asymptotic relations (4.77) and the conditions (4.78) and (4.79) imply $f_0^L = f_0^D = f_0^R = 0$ along with

$$\lambda_H = 1, \qquad f_1^L = -f_1^D, \qquad f_1^R = -n(\lambda_D + n f_1^D), \qquad \omega_i = 0\,, \tag{4.90}$$

with f_1^D and λ_D determined by a single parameter t as above. This is in line with our expectation that $j_i = 0$ implies the solution is static.

Finally we consider the constraints imposed by positivity of the rod lengths (3.136) and the signature conditions (3.134), (3.135). Obviously $\lambda_H > 0$ is trivially satisfied, whereas $\lambda_D > 0$ gives two possible branches:

$$\begin{aligned} &\text{Branch 1:} \qquad t > |n| \\ &\text{Branch 2:} \qquad -n^2 < t < -|n|, \quad n \neq 0, \pm 1. \end{aligned} \tag{4.91}$$

As discussed in the previous section the signature conditions imply the conditions (3.138), which for the upper endpoint of the finite axis rod I_D gives

$$r_D^+ = -\frac{1}{2}(t+1) < 0. \tag{4.92}$$

This is satisfied for Branch 1 (in which case $t > 0$) and is violated for Branch 2 (in which case $t < -2$). Thus we must discard Branch 2. Moreover, one can show that (3.134) and (3.135) are satisfied fully for Branch 1 demonstrating that this corresponds to the unique unbalanced static solution for this rod structure.

Finally, consider the balance condition (4.83), which simplifies to

$$C = \frac{t(1+t)(t^2 + n^2(1+t))}{(t^2 - n^2)^2} = 0. \tag{4.93}$$

This condition is clearly violated for any $n \in \mathbb{Z}$ since above we have shown $t > |n|$.[10] This shows that there are no regular static black lenses with this rod structure, as expected.

4.6.3 A singly spinning black lens

Next we consider the $j_2 = 0$ case of the black lens. The relations from the asymptotic conditions (4.77) and (4.78) reduce to

$$\begin{aligned} &\lambda_H = 1 - \omega_1 j_1, \\ f_\mu^L = \omega_2 \begin{pmatrix} 1 \\ -j_1 \end{pmatrix} - f_\mu^D, \qquad &f_\mu^R = \begin{pmatrix} n f_0^D - \omega_1 \\ -n(\lambda_D + n f_1^D) \end{pmatrix}, \end{aligned} \tag{4.94}$$

[10] In fact, for Branch 2 one can solve $C = 0$ for $|n| \geq 3$. However, in this case $r_D^\pm > 0$, $r_R^- < 0$, which implies that the invariants h^D and h^R are singular at an interior point of I_D and I_R respectively. This is consistent with [129].

and

$$
\begin{aligned}
j_1 &= \omega_1 + (\lambda_D + n f_1^D)(\omega_1 - n f_0^D), \\
0 &= -f_0^D + (1 - \omega_1 j_1)(\omega_2 - f_0^D) - f_1^D(\omega_1 - n f_0^D),
\end{aligned}
\tag{4.95}
$$

respectively. The consistency condition (4.79) becomes

$$
f_\mu^D = \frac{n\lambda_D}{D} \begin{pmatrix} j_1 - (\omega_1 - n f_0^D) \\ -\lambda_D \end{pmatrix}, \tag{4.96}
$$

where

$$
D = (1 - \omega_1 j_1)(1 - j_1(\omega_1 - n\omega_2)) - n j_1 f_0^D + n(2n\lambda_D + (n^2 - 1) f_1^D). \tag{4.97}
$$

We can solve these equations by introducing a new parameterisation as follows.

Define u by

$$
u := -\frac{j_1 - (\omega_1 - n f_0^D)}{\lambda_D}, \tag{4.98}
$$

which is well-defined since $\lambda_D > 0$. This allows us to write the $\mu = 0$ component of equation (4.96) as

$$
f_0^D = f_1^D u, \tag{4.99}
$$

and combining this with the definition of u we can solve for

$$
j_1 = \omega_1 - u(n f_1^D + \lambda_D). \tag{4.100}
$$

Substituting this into (4.95) implies[11]

$$
\omega_1 = (n f_1^D - 1)u, \qquad \omega_2 = f_1^D u\ , \tag{4.101}
$$

which then implies

$$
D = ((1 - u^2(1 + \lambda_D))^2 - n^2)(1 - n f_1^D) + n^2(1 + \lambda_D). \tag{4.102}
$$

Now define new parameters t and v according to

$$
t := \frac{D}{\lambda_D} - n^2, \qquad v := -u\sqrt{1 + \lambda_D}, \tag{4.103}
$$

so that just as for the static case we can write

$$
f_1^D = \frac{-n\lambda_D}{t + n^2}\ , \tag{4.104}
$$

where $t + n^2 \neq 0$ since $D \neq 0$. This allows one to solve the $\mu = 1$ component of (4.96) for λ_D to obtain

$$
\lambda_D = \frac{(t + n^2)(1 - v^2)^2}{t^2 - n^2(1 - v^2)^2}. \tag{4.105}
$$

This expression is well defined since $t^2 - n^2(1 - v^2)^2 = 0$ implies that $v^2 = 1$ which in turn implies $\lambda_H = -\lambda_D < 0$ from (4.94) which contradicts rod length positivity. Now we can express all remaining quantities in terms of t and v:

$$
\begin{aligned}
\lambda_H &= \frac{(t^2 - n^2(1 - v^2))(1 - v^2)}{t^2 - n^2(1 - v^2)^2}, \\
j_1 = v\sqrt{1 + \lambda_D}, \qquad \omega_i &= \frac{v}{(t^2 - n^2(1 - v^2)^2)\sqrt{1 + \lambda_D}} \begin{pmatrix} t^2 \\ n(1 - v^2)^2 \end{pmatrix}\ .
\end{aligned}
\tag{4.106}
$$

[11] In fact (4.95) admits a second solution, with $f_1^D = -\lambda_D/n$, however this implies that $r_R^- = 0$ which violates (3.138).

The conditions $\lambda_D > 0$ and $\lambda_H > 0$ constrain the precise moduli space of solutions. There are three possible branches

$$\begin{aligned} &\text{Branch 1:} \qquad v^2 < 1, \quad t > \sqrt{1-v^2}|n| \\ &\text{Branch 2:} \qquad v^2 < 1, \quad -n^2 < t < -\sqrt{1-v^2}|n|, \quad n \neq 0 \\ &\text{Branch 3:} \qquad v^2 > 1 + |n|, \quad -(v^2-1)|n| < t < -n^2, \quad n \neq 0. \end{aligned} \tag{4.107}$$

To determine which branch actually corresponds to the unbalanced black lens consider the signature conditions (3.134) and (3.135). As in the static case it is helpful to consider the simpler conditions (3.138) on the finite axis rod I_D which are

$$r_D^- = \frac{1}{2\lambda_H}(t+n^2) > 0, \qquad r_D^+ = -\frac{1}{2}\left(\frac{t}{1-v^2}+1\right) < 0\,. \tag{4.108}$$

The first condition rules out Branch 3, whilst the second condition implies that $t/(1-v^2) > -1$, which rules out Branch 2 (since $t/(1-v^2) < -1$ in that case). On the other hand, Branch 1 satisfies both these conditions and furthermore it can be shown that the full conditions (3.134) and (3.135) are also satisfied in this case. Therefore Branch 1 gives the unique unbalanced solution for a singly spinning black lens for this rod structure. This is as one might expect since the limit $v \to 0$ of this branch gives the unbalanced static solution found in the previous section.

It is interesting to consider the moduli space of solutions in terms of physical parameters. In particular, we find that j_1 for the unbalanced singly spinning black lens ($n \neq 0$) (4.106) is bounded above:

$$|j_1| < \sqrt{1+\frac{1}{|n|}}\,. \tag{4.109}$$

Solutions with j_1 arbitrarily close to this upper bound can be found near the corner in (t,v)-space given by $t = |n|\sqrt{1-v^2}$ and $v = 0$. To see this, first note that from (4.106) it is easy to show that $|j_1|$ for fixed v is a monotonically decreasing function of t over the domain defined by Branch 1 and therefore is bounded by its value at $t = |n|\sqrt{1-v^2}$. Then one finds that $|j_1(t = |n|\sqrt{1-v^2}, v)|$ is monotonically decreasing in $|v|$ and is thus bounded by its value at $v = 0$ which is given by the RHS of (4.109). We remark that the curve $t = |n|\sqrt{1-v^2}$ corresponds to a component of the extremal locus $\lambda_H = 0$ and therefore the upper bound on j_1 arises from the extremality bound. This result is in clear contrast to the singly spinning black rings ($n = 0$) for which j_1 can be arbitrarily large for both the unbalanced and balanced solutions (note that this is consistent with the $n \to 0$ limit of (4.109)).

Finally, the balance condition (4.83) reduces to

$$C = \frac{t(1-v^2)^2\left[(n^2-1)t^2 + (t+1-v^2)(t^2+n^2(1-v^2)) + t(1-v^2)(t+n^2)\right]}{(t^2-n^2(1-v^2)^2)^2} = 0. \tag{4.110}$$

For $n = 0$ this can be solved to give $t = 2v^2 - 1$ and $\frac{1}{2} < v^2 < 1$, which is the regular S^1 spinning black ring. For $n \neq 0$ the expression after the first equality is strictly positive in the domain defined by Branch 1, so the balance condition cannot be satisfied for any values of t and v in the moduli space of unbalanced solutions. Curiously, $C = 0$ can be satisfied for $n \neq 0$ in the domain defined by Branch 2 or 3, although in either case $r_D^- r_D^+ > 0$ so the invariant $h^D(z)$ must change sign on I_D and is thus singular at an interior point of this axis rod (this follows from (3.120) and the fact that the only possible singularities of $F_a(z)$ on I_a are at its endpoints)[12]. This proves that there are no regular singly

[12]Similarly for Branch 2, one has $r_R^- < 0$ so $h^R(z)$ must change sign on I_R and thus posses a singularity on this axis.

spinning black lenses for this rod structure with $n \neq 0$. This explains why the previously constructed singly spinning black lens solutions must have conical singularities that cannot be removed or naked singularities [129, 151].

4.6.4 A distinct singly spinning black lens

Another natural singly spinning black lens is given by $j_2 + nj_1 = 0$. Solving for $j_2 = -nj_1$, the relations from the asymptotic conditions (4.77), (4.78) now give

$$\begin{gathered} \lambda_H = 1 - j_1(\omega_1 - n\omega_2) \\ f^L_\mu = \omega_2 \begin{pmatrix} 1 \\ -j_1 \end{pmatrix} - f^D_\mu, \qquad f^R_\mu = -\omega_1 \begin{pmatrix} 1 \\ nj_1 \end{pmatrix} + n \begin{pmatrix} f^D_0 \\ -\lambda_D - nf^D_1 \end{pmatrix} \end{gathered} \tag{4.111}$$

and the equations

$$\begin{aligned} j_1 &= \omega_1 + n\omega_1 j_1(\omega_2 - f^D_0) + (\lambda_D + nf^D_1)(\omega_1 - nf^D_0) \\ -nj_1 &= -f^D_0 + (1 - \omega_1 j_1)(\omega_2 - f^D_0) - f^D_1(\omega_1 - nf^D_0). \end{aligned} \tag{4.112}$$

Condition (4.79) becomes

$$f^D_\mu = \frac{-n\lambda_D}{P} \begin{pmatrix} \omega_1 - nf^D_0 \\ \lambda_D \end{pmatrix}, \tag{4.113}$$

where

$$P := \frac{D}{1 - j_1(\omega_1 - n\omega_2)} = 1 + 2n^2\lambda_D + (n^2 - 1)(nf^D_1 + \omega_1 j_1). \tag{4.114}$$

Note that (4.113) is well-defined since $P \neq 0$ as a consequence of $1 - j_1(\omega_1 - n\omega_2) = \lambda_H > 0$ and $D \neq 0$.

$L(1,1)$ horizon

First, consider the special case $n = \pm 1$. Then the equations (4.112) and (4.113) can be solved straightforwardly in terms of λ_D and ω_1 to give

$$\begin{gathered} j_1 = \frac{\omega_1(1+\lambda_D)}{1+\omega_1^2(1+2\lambda_D)}, \quad \omega_2 = \mp \frac{\omega_1(2(1+\lambda_D)^2 - 1)}{1+\lambda_D}, \\ f^D_0 = \mp \frac{\omega_1 \lambda_D}{1+\lambda_D}, \quad f^D_1 = \mp \frac{\lambda_D^2}{1+2\lambda_D}, \end{gathered} \tag{4.115}$$

which imply that

$$\lambda_H = \frac{1 - \omega_1^2(1+\lambda_D)(1+2\lambda_D)}{1+\omega_1^2(1+2\lambda_D)}. \tag{4.116}$$

The inequalities $\lambda_H, \lambda_D > 0$ are thus equivalent to the region

$$\lambda_D > 0, \qquad \omega_1^2 < \omega_{\text{ext}}^2 := \frac{1}{(1+\lambda_D)(1+2\lambda_D)}, \tag{4.117}$$

where the upper bound for ω_1 is equivalent to the extremality bound $\lambda_H = 0$ (hence the notation). The signature conditions (3.134), (3.135) now turn out to be satisfied automatically. Hence we have fully determined the moduli space in this case.

This case also turns out to have an upper angular momentum bound given by

$$|j_1| < \frac{1}{\sqrt{2}}\,. \tag{4.118}$$

This arises from the following facts: (i) $j_1(\omega_1, \lambda_D)$ is monotonic in ω_1 over the domain (4.117) and is thus bounded by its value at the extremality bound; (ii) $|j_1(\omega_{\text{ext}}, \lambda_D)|$ is monotonic in λ_D and hence bounded by its value as $\lambda_D \to \infty$ which is the RHS of (4.118). Observe that in this case we get a more stringent bound for j_1 than in the singly spinning case $j_2 = 0$ which from (4.109) gives $|j_1| < \sqrt{2}$.

Next, the balance condition (4.83) in this case can be written as

$$C = \frac{\lambda_H(1+2\lambda_D)}{1+\lambda_D} + \frac{\lambda_D(1+3\lambda_D(1+\lambda_D)(2+\lambda_D))}{(1+\lambda_D)(1+2\lambda_D)(1+\omega_1^2(1+2\lambda_D))} = 0\,, \tag{4.119}$$

which is clearly incompatible with $\lambda_H > 0, \lambda_D > 0$ and so there are no regular solutions in this class.

$L(n,1)$ horizon

We now return to the general case $n^2 > 1$. Since $P \neq n^2\lambda_D$[13] we can define a new nonzero parameter

$$t := \frac{n\lambda_D}{P - n^2\lambda_D}. \tag{4.120}$$

Rearranging this for P gives

$$P = \frac{n\lambda_D(1+nt)}{t}\,, \tag{4.121}$$

and using this expression for P in equations (4.113) one can solve for f^D_μ to obtain

$$f^D_\mu = -t\begin{pmatrix}\omega_1 \\ \frac{\lambda_D}{1+nt}\end{pmatrix}. \tag{4.122}$$

Note that since $P \neq 0$ we must have $1 + nt \neq 0$ and so the expression for f^D_1 is well defined. Next, substituting into the definition (4.114) gives an expression linear in j_1 which is solved to give

$$j_1 = -\frac{t(1+nt) - n\lambda_D(1-t^2)}{t\omega_1(n^2-1)(1+nt)}\,, \tag{4.123}$$

where we assume $\omega_1 \neq 0$ ($\omega_1 = 0$ leads to the static black lens which we have already analysed separately in section 4.6.2)[14]. Now consider equations (4.112): the first equation can be solved for ω_2 and then the second can be solved for ω_1^2 resulting in

$$\omega_2 = \frac{t(1+nt) - n\lambda_D(1-t^2) - t\omega_1^2((1+nt)(1+n(t-n)(1+\lambda_D)) - (n^2-1)\lambda_D)}{n\omega_1(t(1+nt) - n\lambda_D(1-t^2))}, \tag{4.124}$$

$$\omega_1^2 = -\frac{\lambda_D(1-t^2)(t(1+nt) - n\lambda_D(1-t^2))}{t(1+\lambda_D)(1+nt)^2(t(n+t) - \lambda_D(1-t^2))}\,, \tag{4.125}$$

where the choice of sign for ω_1 is not fixed. It is easy to see that the denominator for ω_2 is nonzero in view of our assumptions $n^2 > 1, \omega_1 \neq 0$ and $j_1 \neq 0$ together with (4.123) (if $j_1 = 0$ then $j_2 = 0$

[13] If $P = n^2\lambda_D$ then (4.113) gives $\omega_1 = 0$ and $nf^D_1 = -\lambda_D$ and (4.114) reduces to $\lambda_D = -1$ which violates $\lambda_D > 0$.

[14] To see this, note that eliminating P in (4.113) gives $(\omega_1 - nf^D_0)f^D_1 = \lambda_D f^D_0$ which implies that $(\lambda_D + nf^D_1)(\omega_1 - nf^D_0) = \omega_1\lambda_D$. Substituting this into the first equation of (4.112) gives $j_1 = 0$ and hence $j_2 = 0$.

which is the static case). It is also easy to show that the denominator of ω_1^2 is nonzero under these assumptions[15] so both expressions are indeed well-defined.

Equations (4.112) and (4.113) have now been solved in terms of t and λ_D with an additional choice of sign from ω_1. In these variables λ_H is given by

$$\lambda_H = \frac{(1+nt)(t^2(n+t)(1+\lambda_D) - n(1-t^2)\lambda_D^2)}{t\lambda_D(n^2-1)(1-t^2)}. \tag{4.126}$$

The moduli space is defined by $\lambda_H, \lambda_D, \omega_1^2 > 0$ along with the conditions (3.134), (3.135). As for the special cases considered previously it is convenient to consider these latter conditions in the weaker form (3.138) and in particular

$$r_D^+ = -\frac{n+t}{2t} < 0. \tag{4.127}$$

By combining this with $\lambda_H, \lambda_D, \omega_1^2 > 0$, one can show that t and λ_D must satisfy

$$\begin{gathered} \frac{t(1+nt)}{n(1-t^2)} < \lambda_D < \lambda_D^{\text{ext}} := \frac{t^2(n+t) + t\sqrt{t^2(n+t)^2 + 4n(1-t^2)(t+n))}}{2n(1-t^2)}\,, \\ nt > 0, \qquad 1-t^2 > 0. \end{gathered} \tag{4.128}$$

These conditions turn out to be sufficient to imply the full conditions (3.134) and (3.135) and so describe the full space of unbalanced solutions in this case. The upper bound on λ_D corresponds to extremal solutions with $\lambda_H = 0$ and the lower bound corresponds to the static limit of Section 4.6.2 (though note that a different parameter t is used there).

One again finds a bound on the angular momentum which in this case is given by,

$$|j_1| < \sqrt{\frac{1}{|n|(1+|n|)}}\,. \tag{4.129}$$

This follows from the fact that (4.123) for fixed t is monotonically increasing in λ_D in the region (4.128), so it is bounded by its value at $\lambda_D = \lambda_D^{\text{ext}}$. Next one can show that $|j_1(t, \lambda_D^{\text{ext}})|$ is monotonically increasing in $|t|$ so it is bounded by its value at $|t| = 1$ which gives (4.129). Note that (4.129) agrees with the $n = \pm 1$ special case we found above (4.118). Thus for general n we again have a more stringent bound for j_1 than in the $j_2 = 0$ singly spinning case (4.109).

Finally consider the balance condition (4.83). This can be written as

$$C = \frac{\lambda_D}{n^2-1} + \frac{\lambda_H\lambda_D(n+2t)}{t(1+nt)} + \frac{t(n+t)((n^2-2)+nt)(1+\lambda_D)}{(1-t^2)(n^2-1)} = 0\,, \tag{4.130}$$

which cannot be satisfied for any t and λ_D in the moduli space (4.128) since each of the three terms after the first equality is manifestly positive. This proves that for $n^2 > 1$ there are no regular singly spinning solutions of this kind.

4.6.5 Myers-Perry limit

Next we consider the question of what happens to these $L(n,1)$ black lens solutions as n becomes arbitrarily large. The rod vector on the finite axis rod $v_D = (n,1)$ clearly diverges in this limit, however,

[15] If $t(n+t) - \lambda_D(1-t^2) = 0$ one can solve for $\lambda_D = \frac{t(n+t)}{1-t^2}$ (since $n^2 \neq 1$ means that $1-t^2 \neq 0$). Then (4.112) gives $t = -n$ (since $\omega_1, j_1 \neq 0$) and hence $\lambda_D = 0$, which is not allowed.

since any multiple of the rod vector also vanishes on I_D we can rescale it so that it has a finite limit to obtain $n^{-1}v_D \to (1,0) = v_R$ as $n \to \infty$ (of course this will spoil the periodicity of v_D for finite n, although not in the limit $n \to \infty$). Therefore, in this limit I_D becomes part of I_R and hence the rod structure as $n \to \infty$ reduces to that of the Myers-Perry black hole. This suggests that the $n \to \infty$ limit of the unbalanced $L(n,1)$ black lens solution should be the Myers-Perry solution. This turns out to be the case and can be seen to emerge from our moduli space equations as follows.

Recall the moduli space of unbalanced solutions is given by four equations (4.78) and (4.79) for seven parameters $f^D_\mu, j_i, \omega_i, \lambda_D$.[16] We take ω_i, λ_D as the independent parameters and solve for f^D_μ, j_i by assuming they are finite (possibly vanishing) in the $n \to \infty$ limit and that they admit an expansion in n^{-1}. Then, expanding (4.79) in n^{-1} we find that the leading order terms immediately give $f^D_0 = O(n^{-1})$ and $f^D_1 = -\lambda_D/n + O(n^{-2})$. Equations (4.78) and (4.77) then imply that

$$\begin{gathered} j_1 = \frac{\omega_1(1-\omega_2^2)}{1-\omega_1^2\omega_2^2} + O(n^{-1}) \,, \qquad j_2 = \frac{\omega_2(1-\omega_1^2)}{1-\omega_1^2\omega_2^2} + O(n^{-1}) \\ \lambda_H = \frac{(1-\omega_1^2)(1-\omega_2^2)}{1-\omega_1^2\omega_2^2} + O(n^{-1}). \end{gathered} \tag{4.131}$$

The leading order terms here are precisely the expressions for the Myers-Perry solution [3]. Next we consider the bounds on the moduli for these unbalanced solutions. In particular from (3.138) we obtain

$$r_H^- = -\frac{3\pi}{16M}(1-\omega_1^2) + O(n^{-1}) < 0, \qquad r_H^+ = \frac{3\pi}{16M}(1-\omega_2^2) + O(n^{-1}) > 0 \,, \tag{4.132}$$

which immediately gives $|\omega_i| < 1$ to leading order and is sufficient to imply the remaining inequalities that define the moduli space $\lambda_H > 0$, (3.134) and (3.135). These are also the moduli space conditions for Myers-Perry solutions and in particular imply that $|j_1| + |j_2| < 1$ to leading order.

Now we consider the balance condition (4.83) in this limit. Using the expansions of f^D_μ and j_i found above, the leading term in the large n expansion of (4.83) fixes the n^{-2} term in f^D_1:

$$f^D_1 = -\frac{\lambda_D}{n} + \frac{\omega_1\omega_2\lambda_D(1-\omega_1^2)}{n^2((1-\omega_2^2)+\lambda_D(1-\omega_1^2\omega_2^2))} + O(n^{-3}). \tag{4.133}$$

Note that the conditions $|\omega_i| < 1$ ensure the denominator is nonzero. Next, one can solve the $\mu = 0$ component of (4.79) for the n^{-1} term in the expansion of f^D_0 provided $\omega_1\omega_2 \neq 0$ (this involves the n^{-1} terms in the expansion of j_1 and j_2). Finally, using these expansions the first equation of (4.78) becomes

$$\frac{\omega_2(1-\omega_1^2)(\lambda_D+\lambda_H)}{n((1-\omega_2^2)+\lambda_D(1-\omega_1^2\omega_2^2))} + O(n^{-2}) = 0 \tag{4.134}$$

for $\omega_1\omega_2 \neq 0$. The coefficient of the leading term is nonzero since $|\omega_i| < 1, \lambda_D, \lambda_H > 0$, resulting in a contradiction for large enough n. For $\omega_1\omega_2 = 0$ one has to repeat the above analysis, although once again one finds that no regular solutions are possible in this limit. This demonstrates that no regular solutions exist in the large n limit.

It is worth noting that there is another, more obvious, limit of the unbalanced black lens which reduces to the Myers-Perry solution. This is $\lambda_D \to 0$ with all other parameters held fixed, which corresponds to simply shrinking the finite axis rod I_D away, resulting in the same rod structure as the Myers-Perry solution. One can see this explicitly by solving our moduli space equations in this limit, which imply

$$f^D_0 = \omega_2\lambda_D + O(\lambda_D^2), \qquad f^D_1 = O(\lambda_D^2) \tag{4.135}$$

[16] For simplicity, in this section we are assuming the validity of the conjecture stated at the end of Section 4.6.1.

and $j_i, \lambda_H, r_H^{\pm}$ are given by the above expressions with $O(n^{-1})$ replaced by $O(\lambda_D)$. Thus to leading order $|\omega_i| < 1$ and the j_i are given by the Myers-Perry expressions. Then, the balance condition (4.83) in this limit is

$$C = \frac{(1-\omega_1^2)^2}{1-\omega_1^2\omega_2^2} + O(\lambda_D) = 0 \,, \tag{4.136}$$

which clearly cannot be satisfied for small enough λ_D since the first term is positive. Thus no regular solutions exist in this limit either.

4.6.6 Doubly spinning black lens

We now study the general moduli space equations for the black lens (4.78), (4.79) together with the balance condition (4.83). On general grounds one would expect the moduli space of regular solutions to fill out a 2-dimensional subset of the (j_1, j_2) plane (or be empty). To see this, first note that from an existence theorem we know there must be a 4-dimensional moduli space of unbalanced black lens solutions parameterised by $(J_1, J_2) \in \mathbb{R}^2$ and $\ell_H > 0, \ell_D > 0$ [69]. Therefore, since we expect the balance condition on the finite axis rod to reduce the number of parameters by one, it is reasonable to expect that the moduli space of regular solutions is 3-dimensional (if it is non-empty). For a given mass $M > 0$ this is then equivalent to a 2-dimensional subset of the (j_1, j_2)-plane. For reference, for the Myers-Perry S^3 black holes this region is simply $|j_1| + |j_2| < 1$, whereas for the black ring it is an unbounded region [15] (we plot this in our figures below).

The moduli space equations and balance condition possess the following discrete symmetries:[17]

$$g_1 : (n, j_1, \omega_1, j_2, \omega_2, f_0^D, f_1^D, \lambda_D) \to (-n, -j_1, -\omega_1, j_2, \omega_2, f_0^D, -f_1^D, \lambda_D), \tag{4.137}$$
$$g_2 : (n, j_1, \omega_1, j_2, \omega_2, f_0^D, f_1^D, \lambda_D) \to (-n, j_1, \omega_1, -j_2, -\omega_2, -f_0^D, -f_1^D, \lambda_D). \tag{4.138}$$

The g_1-symmetry originates from the orientation-reversing symmetry $m_1 \to -m_1, n \to -n, v_R \to -v_R$, whereas the g_2-symmetry arises from $m_2 \to -m_2, n \to -n, v_D \to -v_D, v_L \to -v_L$. Using these we may restrict to the region $j_1 \geq 0$ and $n > 0$, which we will henceforth (we exclude the $n = 0$ case as that corresponds to the black ring).

Unfortunately, we have been unable to solve the moduli space equations and balance condition analytically in general. Nevertheless, we have verified numerically that for a large sample of solutions to these equations at least one of the inequalities $\lambda_H > 0, \lambda_D > 0$, (3.138), (3.139) is violated. We give more details on our numerical checks below. First, it is instructive to consider the moduli space of solutions to the moduli space equations (4.78), (4.79) without imposing the balance condition. As explained above this should correspond to a 4-dimensional space, or, in terms of our dimensionless variables, a 3-dimensional space. We have numerically solved these equations and plotted the projection of this space to the (j_1, j_2) plane for $n = 1, 2, 3, 10$ in Figure 4.6 and $n = 100$ in Figure 4.7. Specifically, these plots were obtained by numerically solving the moduli space equations at values of j_1 and j_2 centred around the origin (on a square grid of spacing 0.02) and values of λ_D from 0 to 10^3 with greatest density of sampling in the interval $0 < \lambda_D < 1$. Then the plotted points correspond to solutions of these equations which also satisfy $\lambda_H > 0$, (3.134) and (3.135).

These plots suggest that the moduli space for unbalanced $L(n, 1)$ black lenses is a bounded region in the (j_1, j_2)-plane somewhat akin to that of the Myers-Perry solution. As n increases this region approaches that of the Myers-Perry solution, indeed, the $n = 100$ solution already closely approximates the Myers-Perry moduli space, in line with our large n analysis in Section 4.6.5. For general n, the upper bound for j_1 is clearly determined by that of the $j_2 = 0$ solution and is consistent with the analytic

[17] It is easy to see this is the case for (4.78), (4.79).

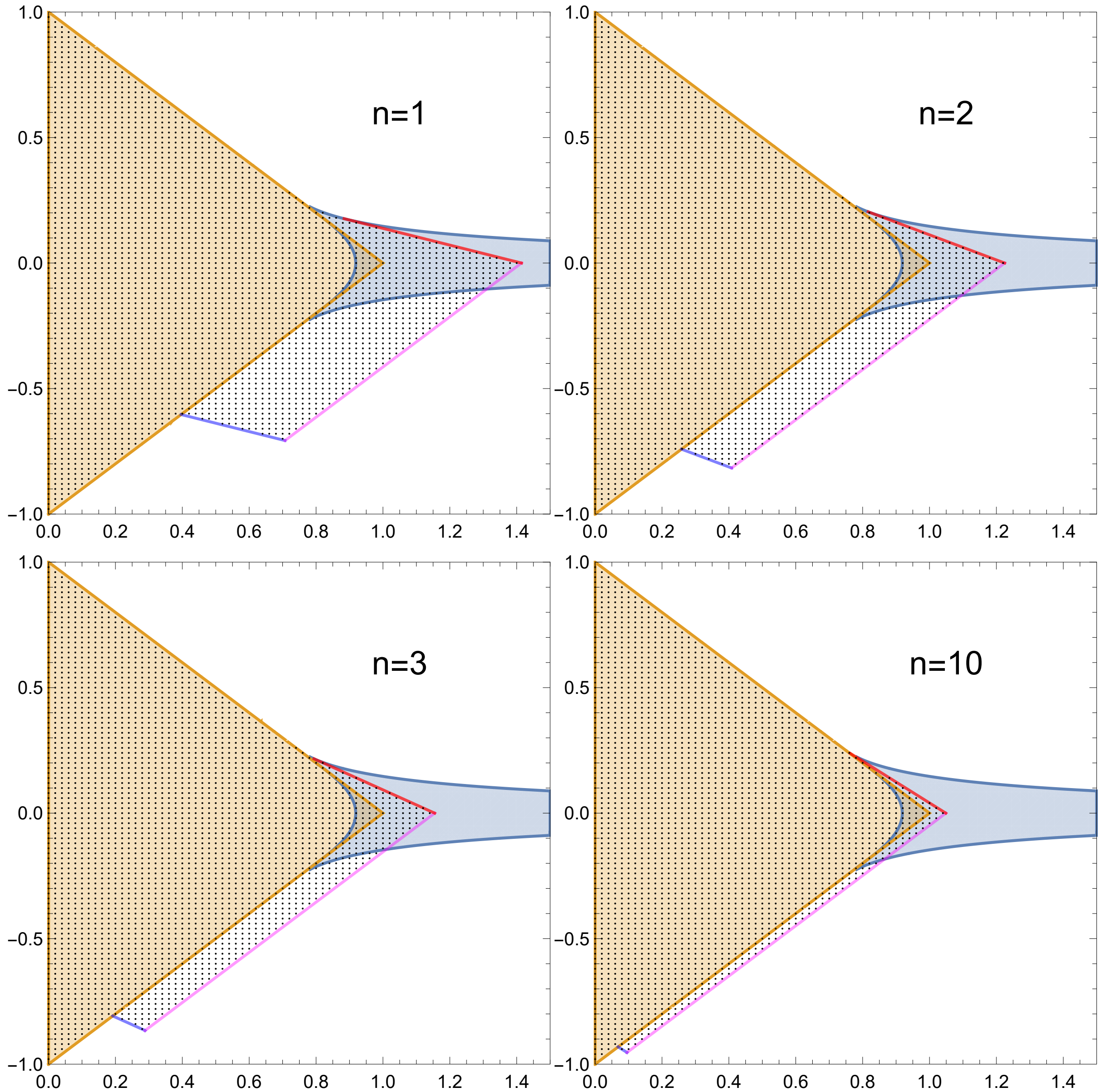

Figure 4.6: The unbalanced $L(n,1)$ black lens solution projected to the (j_1, j_2) plane (black dots). The shaded regions correspond to the Myers-Perry black hole (orange) and balanced black ring (blue). The line segments for $j_2 < 0$ are (4.139) (blue, pink) and for $j_2 > 0$ is (4.141) (red).

bound (4.109) we found in this special case. The bounds for j_2 are clearly not symmetric about the j_1-axis and for $j_2 < 0$ they are consistent with the analytic bound we found for the $j_2 = -nj_1$ case (4.129). Note that the lack of symmetry about the j_1-axis is a consequence of the rod structure and that we have fixed the discrete g_2-symmetry discussed above.

In more detail, the plots suggest that the boundary of the moduli space consists of several segments. For $j_2 < 0$ there appear to be three boundary segments which, based on the numerics and the analytic upper bounds (4.109), (4.129), we conjecture are:

$$j_2 = \begin{cases} j_1 - 1 & 0 < j_1 < q_n \\ \frac{n}{n+2}(-j_1 - s_n) & q_n < j_1 < r_n \\ j_1 - s_n & r_n < j_1 < s_n \end{cases} , \tag{4.139}$$

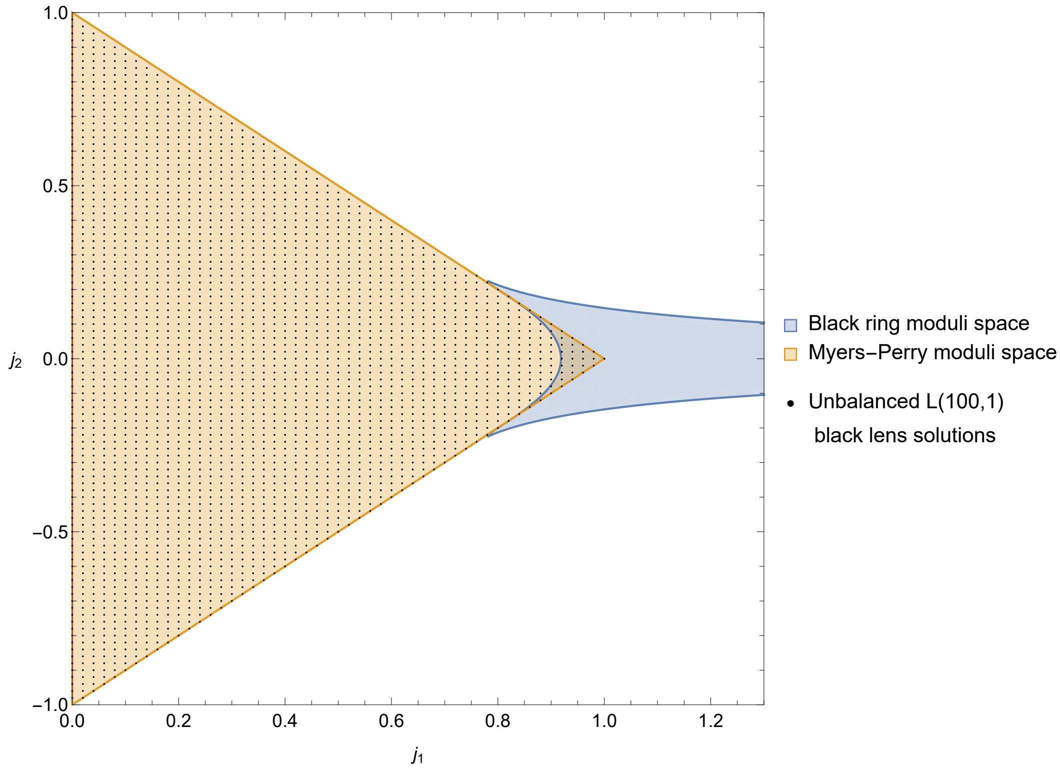

Figure 4.7: The unbalanced $L(100,1)$ black lens solution projected to the (j_1, j_2) plane. This closely approximates the Myers-Perry moduli space, consistent with the analysis in Section 4.6.5.

where s_n, r_n denotes the j_1 upper bound in (4.109), (4.129) respectively, and

$$q_n := \frac{1 - \frac{1}{2}n(s_n - 1)}{n+1} \,. \tag{4.140}$$

In particular, for $n = 1, 2, 3, 10$ we find $q_n \approx 0.396, 0.258, 0.192, 0.069$ with $q_n \to 0$ as $n \to \infty$, which are consistent with the numerics. The first line segment coincides with part of the Myers-Perry boundary; the second line segment starts at the Myers-Perry boundary and ends at the upper limit of the $j_2 = -nj_1$ solution (4.129) (blue line in the plots); the third line segment ends at the upper limit of the $j_2 = 0$ solution (4.109) (pink line in the plots).

For $j_2 > 0$ the numerics indicate that there are also three boundary segments, although from the plots displayed here only two are apparent. The first is a line segment given by part of the Myers-Perry boundary $j_2 = 1 - j_1$ for $0 < j_1 < u_n < 3/4$; the third line segment is[18] (red line on the plots)

$$j_2 = \frac{n}{n+2}(-j_1 + s_n) \,, \qquad \frac{3}{4} < v_n < j_1 < s_n \,.^{19} \tag{4.141}$$

The second segment is a curve which joins these two lines although it is not visible on the plots (we will not study this here). Note that $j_1 = 3/4$ corresponds to where the balanced black ring and the Myers-Perry moduli space boundaries meet.

[18] The blue and red lines are parallel, intersect the j_2 axis at equal and opposite values of j_2, and the red line ends at $(s_n, 0)$ and the blue line ends at $(r_n, -nr_n)$. In fact, one can check that the blue, red and pink lines correspond to extremal $\lambda_H = 0$, $\lambda_D \to \infty$, solutions to the moduli space equations.

[19] The value of v_n seems to be close to the intersection of the upper curve for the black ring moduli space with the line defined in (4.141). This is given by $v_n = \frac{2+3n}{4\sqrt{n+n^2}}$ and is what has been plotted for the endpoint of that line.

It is interesting to note that for both the $j_2 = 0$ and $j_2 = -nj_1$ singly spinning special cases that we studied analytically one has the bound

$$|j_1| + |j_2| < \sqrt{1 + \frac{1}{|n|}}\,. \tag{4.142}$$

It is tempting to conjecture that this bound is satisfied for all doubly spinning unbalanced black lenses (although the moduli space does not fill out the whole of this region as can be seen from the plots). Indeed inspecting the plots, this inequality seems to be saturated for the part of the moduli space boundary between $j_2 = 0$ and $j_2 = -nj_1$ illustrated by the pink line segments in Figure 4.6, which correspond to extremal $\lambda_H = 0$, $\lambda_D \to \infty$ solutions.

We now return to the question of regular black lenses, i.e., solutions in the moduli space of unbalanced black lenses which also obey the balance condition (4.83). For all the points sampled in Figures 4.6 and 4.7 we find that C defined in (4.83) is positive. We have also performed further searches by numerically solving (4.78), (4.79) and (4.83) for $n = 1, 2, 3, 10, 100$, sampling (j_1, j_2) in a square grid in the region $-1 \leq j_2 \leq 1, 0 \leq j_1 \leq 1.5$ with a spacing of 0.015. For all points we find at least one of the inequalities $\lambda_H > 0, \lambda_D > 0$, (3.138), (3.139) is violated. We therefore conclude that regular black lenses do not exist in this class.

4.7 Discussion

In the previous chapter we looked at the problem of classifying regular, asymptotically flat, stationary, vacuum black holes in D dimensions with $D-3$ commuting axial Killing fields. The method we derived allowed us to construct metric data on the axis along with the corresponding moduli space equations. In this chapter we have applied this method to various special cases in which it is possible to fully solve the moduli space equations. In particular, for rod structures corresponding to the Kerr black hole, the Myers-Perry black holes and the known doubly spinning black rings, we find that the resulting moduli space of regular solutions coincide precisely with that of the known solutions. Thus our analysis, together with Theorem 4, provides a proof of uniqueness of these solutions within their class of rod structures (of course, for the Kerr case we recover the classic no-hair theorem). These proofs are constructive in the sense that we also obtain the metric and associated Ernst or twist potentials on the axes and horizon.

We have also presented evidence that black holes with lens space $L(n, 1)$ topology do not exist, for the simplest possible rod structure (see Figure 4.5). Our evidence is based on an analytic proof that the conical singularities on the inner axis rod cannot be removed for two different singly spinning cases ($J_2 = 0$ and $J_2 + nJ_1 = 0$), together with numerical evidence for the generic doubly spinning case. In particular, based on the examples studied in this chapter, we conjecture that any solution to the black lens moduli space equations (4.78), (4.79) will give $C > 0$ and hence violate the balance condition (4.83). Of course, it would be desirable to provide a fully analytic proof of this, ideally by solving the moduli space equations together with the balance (regularity) condition. Perhaps this could be achieved by finding an alternate parameterisation of the moduli space.

It is interesting to compare our results to analogous supersymmetric solutions in five-dimensional minimal supergravity. In that case, regular $L(n, 1)$ black lenses and S^3-black holes with 2-cycles in the DOC are known [110, 108, 109, 111], demonstrating that at least in the presence of supersymmetry, non-trivial rod structures can be realised. In fact, a complete classification of asymptotically flat, supersymmetric and biaxisymmetric black hole and soliton solutions has been obtained, revealing an infinite class of new black hole/ring/lens solutions with 2-cycles in the DOC [107, 149]. A natural physical explanation for the existence of such configurations is that the presence of a Maxwell field allows magnetic flux to 'support' the 2-cycles. Indeed, even in the absence of a black hole, this theory possesses supersymmetric soliton solutions, i.e., asymptotically flat, stationary, everywhere regular spacetimes with positive energy, which posses 2-cycles supported by magnetic flux [117]. Furthermore, some of the aforementioned supersymmetric S^3-black hole spacetimes with non-trivial topology can be interpreted as black holes sitting in such soliton spacetimes [111, 149].

The supersymmetric classification reveals that the only regular supersymmetric solutions with the rod structure studied in the present chapter (see Figure 4.5) is the original $L(2,1)$ black lens for $|n|=2$ [108] and the black ring for $n=0$ [105] (the supersymmetric $L(n,1)$ black lenses for $|n|>2$ necessarily posses extra finite axis rods [109, 107]). Therefore, at least for $|n|=2$, one expects that regular near-supersymmetric black lenses should exist.[20] In contrast, here we have found that regular $|n|=2$ vacuum black lenses do not exist. A simple physical interpretation of this is that rotation alone is not sufficient to support non-trivial topology, although the presence of a sufficiently strong magnetic flux is. Indeed, this is consistent with the non-existence of vacuum soliton spacetimes. Perhaps this suggests that the same goes for more complicated rod structures with a single horizon. If so then the black ring would be an exceptional case for which rotation alone is sufficient to support non-trivial topology.

In the next chapter we consider the case of minimal supergravity in more detail and in particular we adapt the method from the previous chapter into this setting, essentially reproducing the same results.

[20] Existence of soliton and non-extremal black hole solutions, with potential conical singularities on the inner axis rods, has been recently proven in this theory [97].

Chapter 5

Constructing Minimal Supergravity Solutions Using Integrability

5.1 Introduction

In this chapter we will consider AF, stationary and biaxisymmetric solutions in five-dimensional minimal supergravity. Supersymmetric solutions in this class are fully classified [107], and as we have shown in Chapter 3, it is possible to write down equations determining the moduli space of any vacuum solutions, at least in principle. However this leaves generic charged black holes relatively poorly understood. It is reasonable to expect that given any particular solution in minimal supergravity, there should be a family of generic charged solutions containing it. This implies that there are a wide range of unknown regular, non-supersymmetric solutions arbitrarily close to the known families of supersymmetric solutions. For example we know that supersymmetric black lens and "bubbling" black hole spacetimes exist [108, 109, 110, 111], however their non-supersymmetric counterparts are yet to be constructed.

Our aim in this chapter is to develop a similar classification result in this theory as we have done in vacuum gravity in Chapter 3. That classification theory was possible because of the fact that the vacuum Einstein equations are integrable, ultimately a consequence of the equations reducing to a gravitating 2-dimensional harmonic map with a coset target space $SL(3,\mathbb{R})/SO(3)$. In the minimal supergravity case there is a similar harmonic map, this time with target space $G_{2(2)}/SO(4)$. As with the vacuum theory this leads to a pair of linear PDEs whose integrability condition implies the equations of motion of the theory (BZ pair). We shall see that by integrating these PDEs on the axes and horizons we are able to derive various metric and gauge field data in these regions, together with constraints on the moduli space of solutions, just as we did in the vacuum case.

5.2 Background

Consider five-dimensional Minimal supergravity (M, g, F) with action given by

$$S = \int R \star 1 - 2F \wedge \star F - \frac{8}{3\sqrt{3}} F \wedge F \wedge A, \tag{5.1}$$

with where A is defined (at least locally) via $F = \mathrm{d}A$. Varying this action with respect to the metric g and the gauge potential A gives the following equations of motion

$$R_{AB} = 2F_A{}^C F_{BC} - \frac{1}{3} F^2 g_{AB}, \tag{5.2}$$

$$\mathsf{d}\star F+\frac{2}{\sqrt{3}}F\wedge F=0,\qquad \mathsf{d}F=0,\tag{5.3}$$

where $A,B=0,\ldots,4$. We now restrict to the class of solutions with a stationary vector k and 2 commuting axial KVFs m_i, $i=1,2$. As we've seen previously, this means that the metric can then be written in Weyl-Papapetrou coordinates (ρ,z,x^A)

$$\mathbf{g}=g_{AB}(\rho,z)\mathsf{d}x^A\mathsf{d}x^B+e^{2\nu(\rho,z)}(\mathsf{d}\rho^2+\mathsf{d}z^2),\tag{5.4}$$

where $x^A=(t,\phi^i)$ for $\partial_t=k$, $\partial_{\phi_i}=m_i$ and $\det g_{AB}=-\rho^2$. Note also that all elements of F other than $F_{pA}=-F_{Ap}$ $(p=\rho,z)$ vanish (see e.g. [92]).

The equations in this sector of MSG are equivalent to a gravitating harmonic map defined on the orbit space $\hat{M}:=M/(\mathbb{R}\times U(1)^2)$ with coset target space $G/H:=G_{2(2)}/(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))$, where $G_{2(2)}$ is the non-compact (split) real form of the group G_2. The equations for the Killing part of the metric and gauge potential can be written as

$$\mathsf{d}\star_2 J=0,\qquad J=\rho\,\mathsf{d}\mathcal{M}\mathcal{M}^{-1},\tag{5.5}$$

where $\star_2$ is the Hodge dual on $\hat{M}$, or equivalently

$$\begin{gathered}\partial_\rho U+\partial_z V=0,\\ U=\rho\partial_\rho\mathcal{M}\mathcal{M}^{-1},\qquad V=\rho\partial_z\mathcal{M}\mathcal{M}^{-1},\end{gathered}\tag{5.6}$$

for some $\mathcal{M}\in G/H$ satisfying $\det\mathcal{M}=1$. We will give this coset representative $\mathcal{M}$ explicitly in the next section.

These equations are of the same form as the vacuum Einstein equations for g, except with $\det\mathcal{M}=1$ instead of $\det g=-\rho^2$ (compare (5.6) with (3.3),(3.4)), and indeed the equation for ν is again given by (3.5). Therefore the supergravity equations are integrable in the same sense as the vacuum Einstein equations in that they can be written as the integrability condition of an auxiliary linear system. We will cover this in more detail in Section 5.3.

In this chapter we will perform a general analysis of this linear system for AF solutions, in particular on the boundary of the orbit space, following closely the analysis in Chapter 3.

5.2.1 Coset representative

The expression for $\mathcal{M}$ comes from reducing the spacetime $(M,\mathbf{g},F)$ over the two axial KVFs i.e. over a $U(1)^2$ subgroup of the isometry group. From this perspective g can be rewritten as

$$g=\begin{pmatrix}-\rho^2\gamma^{-1}+\omega_i\omega^i & \omega_i\\ \omega_i & \gamma_{ij}\end{pmatrix}\tag{5.7}$$

where $\gamma=\det\gamma_{ij}$ and $\omega_i=\gamma_{ij}\omega^j$. We can describe the Maxwell field F using ψ_i and μ defined by

$$\begin{aligned}\mathsf{d}\psi_i&=-\frac{2}{\sqrt{3}}\iota_i F,\\ \mathsf{d}\mu&=-\epsilon^{ij}\psi_i\mathsf{d}\psi_j-\frac{2}{\sqrt{3}}\iota_1\iota_2\star F,\end{aligned}\tag{5.8}$$

where $\iota_i=\iota_{m_i}$ and $\epsilon^{12}=1$. The 1-forms on the RHS of these equations are closed by equation (5.3) and can be used to fully reconstruct F (see e.g. [152]). It will also be necessary to define the twist potential

$$\mathsf{d}\chi_i=\Omega_i+\psi_i(3\mathsf{d}\mu+\epsilon^{jk}\psi_j\mathsf{d}\psi_k)\tag{5.9}$$

where the twist 1-forms Ω_i are given by

$$\Omega_i = \star(m_1 \wedge m_2 \wedge \mathsf{d}m_i) = -\gamma\rho^{-1}\gamma_{ij} \star_2 \mathsf{d}\omega^j. \tag{5.10}$$

The equation for χ_i is integrable using (5.2). These potentials are only defined up to the gauge transformations

$$\begin{aligned} \psi_i &\to \psi_i + \psi_i^{(0)}, \\ \mu &\to \mu - \epsilon^{ij}\psi_i^{(0)}\psi_j + \mu^{(0)} \\ \chi_i &\to \chi_i + 3\psi_i^{(0)}\mu - \epsilon^{jk}\psi_j^{(0)}\psi_k(\psi_i + 2\psi_i^{(0)}) + \chi_i^{(0)} \end{aligned} \tag{5.11}$$

for constants $\psi_i^{(0)}, \mu^{(0)}, \chi_i^{(0)}$.

We will construct $\mathcal{M}$ using a 14-dimensional fundamental representation of the Lie algebra $\mathfrak{g}_2$, written in terms of a basis of 7×7 matrices denoted by $\{\mathbf{m}_i{}^j, \mathbf{n}^i, \mathbf{l}_i, \mathbf{p}_i, \mathbf{q}, \mathbf{r}^i, \mathbf{t}\}$[1]. To write things in a convenient way we also introduce the homomorphisms $\iota : SL(3,\mathbb{R}) \to G_{2(2)}$ and $\iota_0 : GL(2,\mathbb{R}) \to G_{2(2)}$ (we slightly abuse notation here and throughout this chapter by referring to $G_{2(2)}$ and $\mathfrak{g}_2$ when strictly speaking we mean their representations). These are given by

$$\iota[A] = \begin{pmatrix} A & 0 & 0 \\ 0 & A^{T-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \iota_0[B] = \iota\left[\begin{pmatrix} B & 0 \\ 0 & (\det B)^{-1} \end{pmatrix},\right] \tag{5.12}$$

where $A \in SL(3,\mathbb{R})$ and $B \in GL(2,\mathbb{R})$. The images of ι and ι_0 are the subgroups given by exponentiating the subalgebras $\langle \mathbf{m}_i{}^j, \mathbf{n}^i, \mathbf{l}_i \rangle \cong \mathfrak{sl}_3$ and $\langle \mathbf{m}_i{}^j \rangle \cong \mathfrak{gl}_2$ respectively. Finally we can define

$$\mathcal{M} = S^T \mathcal{M}_0 S, \qquad S = \exp(\psi_i \mathbf{r}^i)\exp(\mu\mathbf{q})\exp(\chi_i \mathbf{n}^i), \tag{5.13}$$

where

$$\mathcal{M}_0 = \iota_0(\gamma_{ij}). \tag{5.14}$$

$\mathcal{M}$ is dependent on the gauges chosen for the potentials. Under the gauge transformations (5.11), S transforms as

$$S \to SS^{(0)}, \qquad S^{(0)} = \exp(\psi_i^{(0)}\mathbf{r}^i)\exp(\mu^{(0)}\mathbf{q})\exp(\chi_i^{(0)}\mathbf{n}^i). \tag{5.15}$$

This means that $\mathcal{M}$ transforms as $\mathcal{M} \to (S^{(0)})^T \mathcal{M} S^{(0)}$ leaving the harmonic map equations (5.6) unchanged, as expected.

It is helpful to see how a harmonic map formulation of vacuum gravity comes out in the vacuum limit. Taking ψ_i and μ to vanish implies that $\mathcal{M}$ can be written as

$$\mathcal{M} = \exp(\chi_i \mathbf{n}^i)^T \mathcal{M}_0 \exp(\chi_i \mathbf{n}^i) = \iota[\Phi], \tag{5.16}$$

where

$$\Phi = \begin{pmatrix} \gamma_{ij} + \gamma^{-1}\chi_i\chi_j & -\gamma^{-1}\chi_i \\ -\gamma^{-1}\chi_j & -\gamma^{-1} \end{pmatrix}. \tag{5.17}$$

This is the well-known coset representative for five-dimensional vacuum gravity [58]; we see that the MSG coset representative contains two copies of this in the vacuum limit. Note that in the vacuum case in Chapter 3 we did not use Φ and instead just used the fact that g satisfies the harmonic map equations (albeit with $\det g = -\rho^2$).

[1]We use the notation of [153] which also gives the explicit form of these basis elements along with their commutation relations.

5.2.2 Adapted basis for an axis rod

We can also write the metric and potentials in coordinates adapted to the non-vanishing KVFs on particular axis rods. We consider an axis rod I and define a basis for the KVFs adapted to this rod (e_μ, v), $\mu = 0, 1$ with $e_\mu = (k, u)$ where $v = v^i m_i$ is the rod vector and $u = u^i m_i$ is an independent vector defined such that [2]

$$A = \begin{pmatrix} u^1 & u^2 \\ v^1 & v^2 \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{5.18}$$

Then g can be written in this adapted basis as

$$\tilde{g} = \begin{pmatrix} h_{\mu\nu} - \rho^2 h^{-1} w_\mu w_\nu & \rho^2 h^{-1} w_\mu \\ \rho^2 h^{-1} w_\nu & -\rho^2 h^{-1} \end{pmatrix}. \tag{5.19}$$

In addition one can define analogous potentials to (5.8) and (5.9) with respect to this adapted basis. These are ϕ_μ, ν and b_μ, defined by

$$\begin{gathered} \mathrm{d}\phi_\mu = -\frac{2}{\sqrt{3}} \iota_{e_\mu} F, \qquad \mathrm{d}\nu = -\epsilon^{\sigma\tau} \phi_\sigma \mathrm{d}\phi_\tau - \frac{2}{\sqrt{3}} \iota_{e_0} \iota_{e_1} \star F \\ \mathrm{d}b_\mu = \Theta_\mu + \phi_\mu (3\mathrm{d}\nu + \epsilon^{\sigma\tau} \phi_\sigma \mathrm{d}\phi_\tau), \end{gathered} \tag{5.20}$$

where

$$\Theta_\mu = \star(e_0 \wedge e_1 \wedge \mathrm{d}e_\mu). \tag{5.21}$$

We can relate the metric in this adapted basis to that in the standard basis (5.7) through

$$[\gamma_{ij}] = A^{-1} \begin{pmatrix} h_{11} - \rho^2 h^{-1} (w_1)^2 & \rho^2 h^{-1} w_1 \\ \rho^2 h^{-1} w_1 & -\rho^2 h^{-1} \end{pmatrix} (A^{-1})^T, \qquad \gamma = -\rho^2 h_{11} h^{-1}, \tag{5.22}$$

and

$$\omega^i = A_j{}^i \tilde{\omega}^j, \qquad [\tilde{\omega}^i] = \begin{pmatrix} h_{01} h_{11}^{-1} \\ h_{01} h_{11}^{-1} w_1 - w_0 \end{pmatrix}. \tag{5.23}$$

Similarly, we can relate the Maxwell potentials in this adapted basis to those in the standard basis (5.8) by using

$$\begin{aligned} u^i \mathrm{d}\psi_i &= \mathrm{d}\phi_1, \\ v^i \mathrm{d}\psi_i &= \frac{\rho^2 h^{-1}}{1 - \rho^2 h w_\mu w^\mu} [w^\nu \mathrm{d}\phi_\nu + \rho^{-1} \star_2 (\mathrm{d}\nu + \epsilon^{\sigma\tau} \phi_\sigma \mathrm{d}\phi_\tau)], \\ \frac{2}{\sqrt{3}} \iota_1 \iota_2 \star F &= \gamma \rho^{-1} \star_2 (\mathrm{d}\phi_0 - \omega^i \mathrm{d}\psi_i), \end{aligned} \tag{5.24}$$

where $w^\mu = h^{\mu\nu} w_\nu$.

Finally, it is also helpful to record some results about how the basis elements of the Lie algebra $\mathfrak{g}_2$ interact with the change of basis matrix A[3]:

$$\begin{gathered} L\mathbf{r}^i L^{-1} = (A^{-1})_i{}^j \mathbf{r}^j, \qquad L\mathbf{q}L^{-1} = \mathbf{q}, \qquad L\mathbf{n}^i L^{-1} = (A^{-1})_i{}^j \mathbf{n}^j, \\ L\mathbf{p}_i L^{-1} = A_j{}^i \mathbf{p}_j, \qquad L\mathbf{t}L^{-1} = \mathbf{t}, \qquad L\mathbf{l}_i L^{-1} = A_j{}^i \mathbf{l}_j \\ L\mathbf{m}_i{}^j L^{-1} = A_k{}^i \mathbf{m}_k{}^l (A^{-1})_j{}^l. \end{gathered} \tag{5.25}$$

[2] In Chapter 3, we also allow for $\det A = -1$ (3.15), however here we restrict to the case $\det A = 1$ without loss of generality.

[3] We note that some of the indices in these expressions are contracted with other indices of the same type. This is because the $\mathfrak{g}_2$ basis elements do not transform as their i, j labels might suggest. It is nonetheless convenient to give them these indices as a way to organise the matrices and write down the coset representative $\mathcal{M}$ (5.16).

The 7×7 transformation matrix L is given by $L = \iota_0[A]$. Note that these relations essentially just encode information about the commutation relations between all the basis elements and $\mathbf{m}_i{}^j$.

5.3 BZ equations

As we discussed, the fact that the equations in this theory (5.6) take a similar form to the vacuum equations means that they can be reformulated as the integrability condition of a certain auxiliary linear pair of PDEs. These are the BZ equations [64, 65]

$$(\rho\partial_\rho - \mu\partial_z)\Psi = U\Psi, \qquad (\mu\partial_\rho + \rho\partial_z)\Psi = V\Psi, \tag{5.26}$$

where U, V are matrices defined in (5.6), Ψ is a complex matrix and $k = z + (\mu^2 - \rho^2)/(2\mu)$ is a spectral parameter which defines $\mu(k)$ on a 2-sheeted Riemann surface.

There are a few general statements that we can make without using the exact form of $\mathcal{M}$, which closely follow the results in the vacuum case. We define the $\pm$ sheets of the Riemann surface as in the vacuum case, using k as a local coordinate. This definition is equivalent to the fact that

$$\mu_+ \sim 2(k-z), \qquad \mu_- \sim -\rho^2/(2(k-z)), \tag{5.27}$$

as $\rho \to 0$. Next we see that there is a continuity condition between these sheets at the branch points $k = z \pm i\rho$

$$\Psi_+ = \Psi_-, \qquad k = z \pm i\rho. \tag{5.28}$$

The BZ system also possesses an involution symmetry

$$\Psi_- = \mathcal{M}\Psi_+^{T-1}B(k), \tag{5.29}$$

where $B(k)$ is an invertible symmetric matrix. The main value in this expression is that it allows us to mostly ignore the linear system (5.26) on the $-$ sheet; given a solution to Ψ_+ we can then trivially find a solution for Ψ_- using the above symmetry.

Taking the trace of the BZ system and using the fact that $\det \mathcal{M} = 1$ shows that $\det \Psi$ is independent of the Weyl coordinates and hence

$$\det \Psi = f(k), \tag{5.30}$$

for some f. Evaluating this on the $\pm$ sheets and imposing continuity at the branch points implies $f_+ = f_-$. Note that generically both $\Psi_\pm$ are invertible. Thus taking the trace of the involution transformation implies

$$\det B = f(k)^2. \tag{5.31}$$

5.4 Linear system on the axis

Now we consider the limit of the BZ equations (5.26) on the axis $\rho = 0$. We shall shortly see that $\mathring{U} = \lim_{\rho\to 0} U$ and $\mathring{V} = \lim_{\rho\to 0} V/\rho$ exist on all components of the axis and horizons. Thus we can evaluate the linear system as $\rho \to 0$ using (5.27) and find that the only equation not involving ρ-derivatives of Ψ_+ is

$$2(z-k)\partial_z\mathring{\Psi}_+ = \mathring{U}\mathring{\Psi}_+, \tag{5.32}$$

where $\mathring{\Psi}_\pm = \lim_{\rho\to 0}\Psi_\pm$. In this section we will integrate this ODE along the z-axis and find $\mathring{\Psi}_+$ on the axes and horizons.

5.4.1 Horizon limit

On a horizon rod I_H, $\mathcal{M}$ is smooth in ρ^2 near the axis with a non-singular limit (since γ_{ij} is finite and non-singular). This means that $\mathring{U} = 0$ and $\mathring{V}$ is finite on a horizon rod I_a and so one can solve (5.32) trivially to find

$$\mathring{\Psi}^a_+ = M_a(k), \qquad \text{on } I_a, \tag{5.33}$$

for some arbitrary matrix M_a.

5.4.2 Axis rod limit

On axis rods, S (5.13) is still smooth in ρ^2 near the axis with a non-singular limit $\mathring{S}$. However $\mathcal{M}_0$ no longer has a finite limit, so it is less clear that $\mathring{U}$ and $\mathring{V}$ exist. We now demonstrate that they are still well-defined.

Finiteness of $\mathring{U}$ and $\mathring{V}$

From the definition of $\mathcal{M}$ in terms of S and $\mathcal{M}_0$ one finds

$$J = \rho\, \mathsf{d}\mathcal{M}\mathcal{M}^{-1} = S^T D S^{T-1}, \tag{5.34}$$

where

$$D := \rho\left[(\mathsf{d}SS^{-1})^T + \mathsf{d}\mathcal{M}_0\mathcal{M}_0^{-1} + \mathcal{M}_0\mathsf{d}SS^{-1}\mathcal{M}_0^{-1}\right]. \tag{5.35}$$

Using the commutation relations for $\mathfrak{g}_2$ and the definitions of the potentials μ (5.8) and χ_i (5.9), one can show that

$$\begin{aligned} \mathsf{d}SS^{-1} &= \mathsf{d}\psi_i \mathbf{r}^i - \frac{2}{\sqrt{3}}\left(\iota_1\iota_2 \star F\right)\mathbf{q} + \Omega_i \mathbf{n}^i, \\ \mathsf{d}\mathcal{M}_0\mathcal{M}_0^{-1} &= \mathsf{d}\gamma_{ik}\gamma^{kj}\mathbf{m}_i{}^j, \\ \mathcal{M}_0\mathsf{d}SS^{-1}\mathcal{M}_0^{-1} &= \mathsf{d}\psi_i\gamma^{ij}\mathbf{r}^j - \frac{2}{\sqrt{3}}\gamma^{-1}\left(\iota_1\iota_2\star F\right)\mathbf{q} - \rho^{-1}\star_2 \mathsf{d}\omega^i\mathbf{n}^i. \end{aligned} \tag{5.36}$$

At this point it is helpful to start using a basis adapted to the axis rod we are considering. g can then be written as (5.19) and we can introduce various potentials as in (5.20). Similarly to the vacuum case we will assume these variables adapted to the axis rod are smooth in ρ^2 up to that rod. To relate these quantities to those defined in the standard basis above we have the expressions (5.24) which we can take the $\rho \to 0$ limits of to give

$$\begin{aligned} u^i\mathsf{d}\psi_i &= \partial_z\phi_1\mathsf{d}z + O(\rho), \\ v^i\mathsf{d}\psi_i &= \rho h^{-1}(\partial_z\nu + \epsilon^{\sigma\tau}\phi_\sigma\partial_z\phi_\tau)\mathsf{d}\rho + O(\rho^2), \\ \frac{2}{\sqrt{3}}\iota_1\iota_2\star F &= \gamma\rho^{-1}(\partial_z\phi_0 - \tilde{\omega}^1\partial_z\phi_1)d\rho + O(\rho^2). \end{aligned} \tag{5.37}$$

Finally we use the fact that ω^i must be smooth in ρ^2 up to the axis rod from (5.23) to write

$$\rho^{-1}\star_2 \mathsf{d}\omega^i = \rho^{-1}\partial_z\omega^i\mathsf{d}\rho + O(1), \tag{5.38}$$

which in turn implies that $\Omega_i = O(\rho)$ using (5.10).

Combining these relations with the terms in (5.36), we see that the z component of each 1-form is at least $O(1)$ and the ρ component $O(\rho^{-1})$. These 1-forms sum to give D (5.35) along with a factor of ρ. Since S has a finite axis limit, this means that $\mathring{U}$ and $\mathring{V}$ are well-defined as claimed.

Determining $\mathring{U}$

In order to write down $\mathring{U}$ explicitly we need to consider the ρ components of these 1-forms in more detail. Using (5.37) in (5.36), we see that

$$\begin{aligned}
\partial_\rho S S^{-1} &= O(\rho), \\
\partial_\rho \mathcal{M}_0 \mathcal{M}_0^{-1} &= \rho^{-1} t_i v^j \mathbf{m}_i{}^j + O(\rho), \\
\mathcal{M}_0 \partial_\rho S S^{-1} \mathcal{M}_0^{-1} &= -\rho^{-1}(\partial_z \nu + \epsilon^{\sigma\tau}\phi_\sigma \partial_z \phi_\tau) v^i \mathbf{r}^i \\
&\qquad - \rho^{-1}(\partial_z \phi_0 - \tilde{\omega}^1 \partial_z \phi_1)\mathbf{q} - \rho^{-1}\partial_z \omega^i \mathbf{n}^i + O(\rho).
\end{aligned} \tag{5.39}$$

where

$$t = A^{-1}\tilde{t}, \qquad \tilde{t} = \begin{pmatrix} -2w_1 \\ 2 \end{pmatrix}. \tag{5.40}$$

This implies that

$$\mathring{D}_\rho = t_i v^j \mathbf{m}_i{}^j - (\partial_z \nu + \epsilon^{\sigma\tau}\phi_\sigma \partial_z \phi_\tau) v^i \mathbf{r}^i - (\partial_z \phi_0 - \tilde{\omega}^1 \partial_z \phi_1)\mathbf{q} - \partial_z \omega^i \mathbf{n}^i, \tag{5.41}$$

where we define $\mathring{D}_\rho := \lim_{\rho\to 0} D_\rho$. Note that the various metric components and potentials we use here are strictly speaking their $\rho \to 0$ axis limits.

Next we need to find $\mathring{S}$. To do this we solve for the potentials in the standard basis (5.8), (5.9) on axis. Using (5.37) and the fact that $(\Omega_i)_z = 0$ on axis, we see that

$$\psi_i(z) = (A^{-1})_i{}^1 \phi_1(z), \qquad \nu(z) = 0, \qquad \chi(z) = 0, \tag{5.42}$$

up to gauge transformations (5.11). Combining this with the transformation relations for the $\mathfrak{g}_2$ basis elements (5.25), we find that

$$\mathring{S} = L^T e^{\phi_1 \mathbf{r}^1} L^{T\,-1} S^{(0)}, \tag{5.43}$$

where we have allowed for the possibility of a more general set of gauges for the potentials by including a $S^{(0)}$ factor (5.15). Note that for a globally defined set of potentials this expression defines $S^{(0)}$ for this particular axis rod. From the definition of D and U We know that $\mathring{U} = \mathring{S}^T \mathring{D}_\rho (\mathring{S}^T)^{-1}$ and so defining $\tilde{Y} = e^{\phi_1 \mathbf{p}_1} L \mathring{D}_\rho L^{-1} e^{-\phi_1 \mathbf{p}_1}$, we can write

$$\mathring{U} = (S^{(0)})^T L^{-1} \tilde{Y} L (S^{(0)})^{T\,-1}, \tag{5.44}$$

where we have used the fact that $\mathbf{p}_i = (\mathbf{r}^i)^T$.

In order to evaluate $\tilde{Y}$, we first need the fact that

$$\partial_z b_\mu = 2w_\mu + \phi_\mu(3\partial_z \nu + \epsilon^{\sigma\tau}\phi_\sigma \partial_z \phi_\tau), \tag{5.45}$$

which comes from the definition of b_μ (5.20) after using the result that $\Theta_\mu = 2w_\mu \mathrm{d}z + O(\rho)$. Then using (5.41) one can express $\tilde{Y}$ as

$$\begin{aligned}
\tilde{Y} = \partial_z\big[C\mathbf{m}_1{}^2 + 2z\mathbf{m}_2{}^2 + R\mathbf{r}^2 + Q\mathbf{q} - \tilde{\omega}^i \mathbf{n}^i\big], \\
C := (\phi_1^2 Q - b_1), \qquad R := (\phi_1 Q - \nu) \\
Q := -(\phi_0 - \phi_1 \tilde{\omega}^1),
\end{aligned} \tag{5.46}$$

where we have used the transformation rules (5.25) and commutation relations of the $\mathfrak{g}_2$ basis elements.

It is convenient at this point to fix the gauges of the adapted basis potentials (ϕ_μ, ν, b_μ). We will do this in line with the gauge choices for b^a_μ in Chapter 3. Namely for finite axis rods we set these potentials to vanish at the lower rod point and to vanish on the semi-infinite axis rods at infinity. Explicitly we impose the conditions

$$\begin{aligned} \phi^a_\mu(z_{a-1}) &= 0, \qquad \lim_{z\to-\infty} \phi^L_\mu(z) = 0, \qquad \lim_{z\to\infty} \phi^R_\mu(z) = 0, \\ \nu^a(z_{a-1}) &= 0, \qquad \lim_{z\to-\infty} \nu^L(z) = 0, \qquad \lim_{z\to\infty} \nu^R(z) = 0, \\ b^a_\mu(z_{a-1}) &= 0, \qquad \lim_{z\to-\infty} b^L_\mu(z) = 0, \qquad \lim_{z\to\infty} b^R_\mu(z) = 0, \end{aligned} \tag{5.47}$$

where we have temporarily reinstated rod labels a which run over finite axis rods $I_a = (z_{a-1}, z_a)$. Note that because of the gauge conditions on ϕ_1 in particular, we have the following expressions for $S^{(0)}$, derived from (5.43),

$$S^{(0)}_a = \mathring{S}(z_{a-1}), \qquad S^{(0)}_L = \lim_{z\to-\infty} \mathring{S}(z), \qquad S^{(0)}_R = \lim_{z\to\infty} \mathring{S}(z). \tag{5.48}$$

Solving for $\mathring{\Psi}_+$

Now that we know $\mathring{U}$, we can solve for $\mathring{\Psi}_+$ using (5.32). We begin by writing $\mathring{\Psi}_+$ as

$$\mathring{\Psi}_+ = (S^{(0)})^T X, \qquad X = L^{-1}\tilde{X}L. \tag{5.49}$$

Then the equation for $\mathring{\Psi}_+$ (5.32) can be written in terms of $\tilde{X}, \tilde{Y}$ as

$$\partial_z \tilde{X}\tilde{X}^{-1} = \frac{\tilde{Y}}{2(z-k)}, \tag{5.50}$$

where $\tilde{Y}$ is given in (5.46).

Using (5.46), we find a solution for $\tilde{X} \in G_{2(2)}$ of the form

$$\tilde{X} = \tilde{X}_0 \exp\left(R\mathbf{r}^2\right) \exp\left(Q\mathbf{q}\right) \exp\left(N_i \mathbf{n}^i\right). \tag{5.51}$$

for

$$\tilde{X}_0 = \iota_0\left[\begin{pmatrix} 1 & C \\ 0 & 2(z-k) \end{pmatrix}\right], \tag{5.52}$$

$$\begin{aligned} N_1 &= -\tilde{\omega}^1, \\ N_2 &= Q(\phi_1 - 3\nu Q) - (b_0 - b_1\tilde{\omega}^1) + 2(z-k)\tilde{\omega}^2. \end{aligned} \tag{5.53}$$

To demonstrate that does solve equation (5.50), one must also use the fact that $w_0 = w_1\tilde{\omega}^1 - \tilde{\omega}^2$, a consequence of (5.23).

Therefore the most general solution for $\mathring{\Psi}_+(z,k)$ is given by

$$\mathring{\Psi}^a_+ = (S^{(0)}_a)^T X_a M_a \tag{5.54}$$

where X_a is given by (5.49), (5.51), M_a is some arbitrary matrix function of k and we have reinstated the label a for axis rod I_a.

5.5 Solving for the metric and potentials

5.5.1 Continuity conditions

At each rod point we impose that $\mathring{\Psi}_+$ must be continuous in z. For $z = z_a$, this condition is given by

$$\mathring{\Psi}^a_+(z_a, k) = \mathring{\Psi}^{a+1}_+(z_a, k). \tag{5.55}$$

Using the solutions on the axis and horizon rods (5.54), (5.33), one can then relate M_a and M_{a+1}

$$M_a M_{a+1}^{-1} = \begin{cases} [(S_a^{(0)})^T X_a(z_a)]^{-1} & I_a \text{ axis}, I_{a+1} \text{ horizon} \\ (S_{a+1}^{(0)})^T X_{a+1}(z_a) & I_a \text{ horizon}, I_{a+1} \text{ axis} \\ [(S_a^{(0)})^T X_a(z_a)]^{-1}[(S_{a+1}^{(0)})^T X_{a+1}(z_a)] & I_a, I_{a+1} \text{ axes} \end{cases}. \tag{5.56}$$

Using these expressions one can calculate $M_a M_b^{-1}$ for arbitrary rods I_a and I_b, which will necessarily be an element of $G_{2(2)}$.

5.5.2 Asymptotics

The asymptotic analysis for this theory will closely follow the analysis in Chapter 3 for the vacuum theory with matrices appropriately redefined in terms of $\mathcal{M}$ rather than g.

First consider Minkowski space. In this case

$$\begin{gathered} \bar{\gamma}_{ij} = \operatorname{diag}(\mu, \rho^2\mu^{-1}), \qquad \bar{\omega}^i = 0, \qquad \bar{F} = 0, \\ \mu = \sqrt{\rho^2 + z^2} - z, \end{gathered} \tag{5.57}$$

This implies that we can take $\psi_i = \mu = \chi_i = 0$ (μ here is the potential (5.8), not to be confused with the function defined above) and so $\bar{S} = 1$ which implies that $\overline{\mathcal{M}} = \overline{\mathcal{M}}_0 = \iota(\bar{\gamma})$. Using this and (5.26), one can solve for $\bar{\Psi}_+(r, \theta, k)$ to find

$$\bar{\Psi}_+ = \iota_0[\operatorname{diag}(\mu - \mu_+, \rho^2\mu^{-1} + \mu_+))]\bar{N}(k) \tag{5.58}$$

where $\bar{N}$ is an arbitrary matrix function of k which we take to be invertible.

Now consider an AF solution. To do this analysis it is convenient to introduce polar coordinates (r, θ) where

$$\rho = r\sin\theta, \qquad z = r\cos\theta, \tag{5.59}$$

and $0 \leq \theta \leq \pi$. Then AF solutions obey

$$\gamma_{ij} = \bar{\gamma}_{ij} + \begin{pmatrix} O(1) & O(r^{-1}) \\ O(r^{-1}) & O(1) \end{pmatrix}, \qquad \omega^i = O(r^{-2}), \qquad F = O(r^{-1}) \tag{5.60}$$

as $r \to \infty$. This implies that $\mu = \chi_i = O(1)$ as $r \to \infty$ and we can choose a gauge for ψ_i such that $\psi_i = O(r^{-1})$[4].

Next we consider solving the BZ system (5.26) for these AF spacetimes. The equations can be written in polar coordinates as

$$\begin{aligned} \partial_r \Psi = Y_r \Psi, \qquad & Y_r = \frac{r\sin^2\theta S - \mu T}{\mu^2 + r^2\sin^2\theta}, \\ \partial_\theta \Psi = Y_\theta \Psi\ , \qquad & Y_\theta = \frac{r\sin\theta(\mu S + rT)}{\mu^2 + r^2\sin^2\theta}, \end{aligned} \tag{5.61}$$

[4] See section IV of [92] for further details

where $S = r\partial_r \mathcal{M}\mathcal{M}^{-1}$ and $T = \sin\theta\partial_\theta\mathcal{M}\mathcal{M}^{-1}$. Consider a solution for Ψ of the form

$$\Psi = \bar{\Psi}\Delta, \tag{5.62}$$

where Δ is a new matrix function of (k, r, θ). From (5.61) it follows that

$$\begin{aligned} (\partial_r\Delta)\Delta^{-1} &= \Upsilon_r, \qquad & \Upsilon_r &\equiv \bar{\Psi}^{-1}(Y_r - \bar{Y}_r)\bar{\Psi}, \\ (\partial_\theta\Delta)\Delta^{-1} &= \Upsilon_\theta\ , \qquad & \Upsilon_\theta &\equiv \bar{\Psi}^{-1}(Y_\theta - \bar{Y}_\theta)\bar{\Psi}. \end{aligned} \tag{5.63}$$

One can calculate these Υ matrices for the $+$ sheet using the solution for Minkowski space $\bar{\Psi}$ (5.58) and the expansion of an AF metric (5.60). Using this one can show that $\Upsilon_{r+} = O(r^{-1}), \Upsilon_{\theta+} = O(r^{-2})$ which implies that asymptotically Δ_+ must be only a function of k. This can be written as

$$\Delta_+ = \begin{cases} N_R + O(r^{-1}), & 0 \leq \theta < \theta_* \\ N_L + O(r^{-1}), & \theta_* < \theta \leq \pi \end{cases}, \tag{5.64}$$

where $N_{R,L}$ are matrix functions of k which we take to be invertible and θ_* is defined by $\cos\theta_* = \frac{\text{Re}(k)}{r}$. These N matrices are in general different because Υ_+ is discontinuous across $\theta = \theta_*$; instead one should impose the continuity condition

$$\lim_{\epsilon\to 0^+} \Psi_\pm(r, \theta_* - \epsilon, k) = \lim_{\epsilon\to 0^+} \Psi_\mp(r, \theta_* + \epsilon, k), \tag{5.65}$$

since as one follows the contour at infinity, k moves through the branch cut of the Riemann surface at $\theta = \theta_*$ and the sheet of Ψ swaps. Now combining this condition with the involution symmetry (5.29), one finds that

$$\begin{aligned} N_R(k)^{T-1}B(k)N_L(k)^{-1} &= \lim_{r\to\infty} \bar{\Psi}_+^T(r, \theta_*^-, k)\mathcal{M}(r, \theta_*)^{-1}\bar{\Psi}_+(r, \theta_*^+, k) \\ &= \bar{N}^T\iota_0[\,\text{diag}\,(-2k, 2k)]\bar{N}, \end{aligned} \tag{5.66}$$

where in going to the second line we have used the form of $\bar{\Psi}$ (5.58) and the asymptotics of $\mathcal{M}$ for an AF spacetime.

Now that we have an asymptotic solution for Ψ_+ for an AF spacetime, we want to relate this to our axis solutions $\mathring{\Psi}_+$ (5.54). As $|z| \to \infty$, $\mathring{\Psi}_+^{R,L}$ should approach the solution for an AF spacetime (5.62), in other words we impose

$$\begin{aligned} \lim_{r\to\infty} \mathring{\Psi}_+^R(r, k)^{-1}\bar{\Psi}_+(r, 0, k)N_R(k) &= 1 \\ \lim_{r\to\infty} \mathring{\Psi}_+^L(-r, k)^{-1}\bar{\Psi}_+(r, \pi, k)N_L(k) &= 1. \end{aligned} \tag{5.67}$$

In order to evaluate this limit we need the $|z| \to \infty$ behaviour of $\mathring{\Psi}_+^{R,L}$. We have already noted that as $r \to \infty$, μ and χ_i are $O(1)$ and we have chosen a gauge such that ψ_i vanishes. In addition, (ϕ_μ, ν, b_μ) on the left and right semi-infinite rods as $|z| \to \infty$ (5.47) and the asymptotic conditions imply that $\tilde{\omega}^i = O(|z|^{-2})$. Using these asymptotic behaviours we find

$$\begin{aligned} M_R &= \iota_0[\,\text{diag}\,(1, 2k)]\bar{N}N_R \\ M_L &= \iota_0[\,\text{diag}\,(-2k, -1)]\bar{N}N_L, \end{aligned} \tag{5.68}$$

where we have taken $u_R = (0, -1)$, i.e. (5.18)

$$A_R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.69}$$

Combining this with (5.66) gives

$$C := M_L B^{-1} M_R^T = \iota_0[\,\mathrm{diag}\,(1,-1)] \tag{5.70}$$

providing a relation between M_L and M_R, which doesn't involve the matrices $\bar{N}, N_R$ or N_L. Note that this precise form for C depends on the form of our particular axis solutions $\tilde{X}_R$ and $\tilde{X}_L$.

Finally one can define F_a matrices according to[5]

$$F_a := M_a B^{-1} M_a^T = [M_L M_a^{-1}]^{-1} C [M_a M_R^{-1}]^T. \tag{5.71}$$

The first equality in this expression dictates that F_a must be symmetric for each a. The second equality gives an expression for F_a that can be evaluated explicitly using the expressions for $M_a M_b^{-1}$ found previously (5.56). Furthermore since $M_a M_b^{-1}, C \in G_{2(2)}$ it follows that $F_a \in G_{2(2)}$ and in particular $\det F_a = 1$.

5.5.3 Branch point compatibility conditions

Horizon Rods

The branch point compatibility condition (5.28) on horizon rods is given by

$$\mathring{\Psi}_+(z,z) = \mathring{\Psi}_-(z,z). \tag{5.72}$$

The involution condition (5.29) can then be used to express $\mathring{\Psi}_-$ in terms of $\mathring{\Psi}_+$, and so the above equation can be written as

$$\mathring{\Psi}_+(z,z) = \mathring{\mathcal{M}}(z)\mathring{\Psi}_+(z,z)^{T-1}B(z), \tag{5.73}$$

where $\mathring{\mathcal{M}} = \lim_{\rho\to 0}\mathcal{M}$. Using the fact that $\mathring{\Psi}_+^a(z,k) = M_a(k)$ on horizon rods and rearranging gives

$$\mathring{\mathcal{M}}(z) = F_a(z), \tag{5.74}$$

where F_a is given in (5.71).

Axis Rods

$\mathcal{M}$, and so Ψ_-, is not well-defined on axis rods which means that slightly more care must be paid to this case in order to recover metric data. Instead of considering Ψ_- on axis we will consider $E_-\Psi_-$ where

$$E = \iota_0\left[\begin{pmatrix}1 & 0\\ 0 & \mu^{-1}\end{pmatrix}\right] L S^{T-1}. \tag{5.75}$$

$E_-\Psi_-$ then has a well-defined axis limit. To demonstrate this consider the involution condition (5.29) with this factor of E_- included

$$E_-\Psi_- = (E_-\mathcal{M})\Psi_+^{T-1}B. \tag{5.76}$$

Clearly the axis limit of $E_-\Psi_-$ is well-defined if the axis limit of $E_-\mathcal{M}$ is well-defined. We can calculate this explicitly to find

$$\hat{\mathcal{M}} := \lim_{\rho\to 0}(E_-\mathcal{M}) = \iota_0\left[\begin{pmatrix}h_{11} & 0\\ 2(z-k)w_1 h^{-1} & -2(z-k)h^{-1}\end{pmatrix}\right] L^{T-1}\mathring{S}, \tag{5.77}$$

[5] These F_a are defined without a minus sign in contrast to (3.111).

where we have used the fact that $\mu_- \to -\rho^2/(2(k-z))$ as $\rho \to 0$ and the form of the metric in the basis adapted to axis rod we are considering (5.19).

Now note that $E_+\Psi_+ = E_-\Psi_-$ at $k = z \pm i\rho$ (a small variation of identity (5.28)). So using this and taking first the $\rho \to 0$ limit of (5.76) and then the $k \to z$ limit one finds that

$$\lim_{k\to z}\left[\mathring{E}_+\mathring{\Psi}_+\right] = \lim_{k\to z}\left[\hat{\mathcal{M}}\mathring{\Psi}_+^{T-1}B\right]. \tag{5.78}$$

There follows a slightly convoluted calculation, using the axis solution (5.54), the commutation relations of the algebra $\mathfrak{g}_2$ and a relation for $\tilde{\omega}^1$ (5.23). However in the end one finds a rather simple relation between F_a and the metric and potentials on an axis rod I_a

$$\tilde{F}_a = B^T\iota_0[h^a]B, \qquad B = \exp\left(\phi^a_\mu \mathbf{r}^{\mu+1}\right)\exp\left(\nu^a\mathbf{q}\right)\exp\left(b^a_\mu\mathbf{n}^{\mu+1}\right) \tag{5.79}$$

where $\tilde{F}_a := (KL_a)F_a(KL_a)^T$,

$$K = \iota\left[\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\right], \tag{5.80}$$

and we have reinstated rod labels.

5.6 Classification theorem and moduli space of solutions

We can summarise the results of the previous sections in the following theorem, analogous to Theorem 8 (we suppress the rod labels for legibility):

Theorem 9. *Consider a spacetime in 5-dimensional minimal supergravity as in Theorem 5.*

1. *The general solution $(h_{\mu\nu}(z), \phi_\mu(z), \nu(z), b_\mu(z))$ on an axis rod is given by solving (5.79) where $\tilde{F} = (KL)F(KL)^T$. This solution can be written explicitly in terms of the components of $\tilde{F}$ as*

$$\begin{gathered} h^{\mu\nu} = \tilde{F}^{4+\mu\;4+\nu} - \frac{1}{\tilde{F}^{33}}\tilde{F}^{4+\mu\;3}\tilde{F}^{3\;4+\nu}, \\ \phi_\mu = \frac{1}{\tilde{F}^{33}}\begin{pmatrix}-\tilde{F}^{35}\\ \tilde{F}^{34}\end{pmatrix}, \qquad \nu = -\frac{\tilde{F}^{37}}{\sqrt{2}\tilde{F}^{33}}, \\ b_\mu = \frac{1}{\sqrt{2}(\tilde{F}^{33})^2}\begin{pmatrix}\tilde{F}^{35}\tilde{F}^{37} - \sqrt{2}\tilde{F}^{31}\tilde{F}^{33}\\ -\tilde{F}^{34}\tilde{F}^{37} - \sqrt{2}\tilde{F}^{32}\tilde{F}^{33}\end{pmatrix}, \end{gathered} \tag{5.81}$$

   In particular, this implies

$$\det h_{\mu\nu} = \frac{1}{\tilde{F}^{33}}. \tag{5.82}$$

2. *The general solution $(\gamma_{ij}(z), \psi_i(z), \mu(z), \chi_i(z))$ on a horizon rod is given by solving (5.74). This solution can be written explicitly in terms of the components of F as*

$$\begin{gathered} \gamma^{ij} = F^{3+i\;3+j} - \frac{1}{F^{33}}F^{3+i\;3}F^{3\;3+j}, \\ \psi_i = \frac{1}{F^{33}}\begin{pmatrix}-F^{35}\\ F^{34}\end{pmatrix}, \qquad \mu = -\frac{F^{37}}{\sqrt{2}F^{33}}, \\ \chi_i = \frac{1}{\sqrt{2}(F^{33})^2}\begin{pmatrix}F^{35}F^{37} - \sqrt{2}F^{31}F^{33}\\ -F^{34}F^{37} - \sqrt{2}F^{32}F^{33}\end{pmatrix}, \end{gathered} \tag{5.83}$$

In particular, this implies

$$\det \gamma_{ij} = \frac{1}{F^{33}}. \tag{5.84}$$

In both cases $F(k)$ are the matrices defined by (5.71) and the components of F and $\tilde{F}$ are F^{MN} and $\tilde{F}^{MN}$ respectively with $M, N = 1, \dots 7$. The solution depends on the following 'moduli': rod lengths for each finite rod; rod vectors for each finite axis rod; $\tilde{\omega}^i$ evaluated at the endpoints of each axis rod; for each finite axis rod the potentials $(\phi_\mu(z), \nu(z), b_\mu(z))$ evaluated at each upper endpoint and similarly for the single endpoints of the semi-infinite axis rods.

Proof. The equations (5.79), (5.74) follow from the analysis in the previous section. Multiplying the exponentials out and using the explicit basis representatives of $\mathfrak{g}_2$ one can determine these particular relations in terms of the components of F and $\tilde{F}$. Note that we use the fact that F_{33} and $\tilde{F}_{33}$ can't vanish on the relevant rods since that would correspond to a singularity of the metric, with the determinant of either γ_{ij} or $h_{\mu\nu}$ diverging.

The dependence of the solution on the specific moduli mentioned above comes directly from the construction of F (5.71) in terms of the axis solution to Ψ_+ , (5.56), (5.54). There are only constants associated to the potentials for a single endpoint per axis rod because we fixed a gauge (5.47). □

Remarks.

1. Alternate forms of the general solution can be obtained by replacing $F_a(k)$ with $F_a(k)^T$ for some $a \in \{1, \dots, n+1\}$ which are equivalent since the $F_a(k)$ are symmetric.

2. $\mathbf{q}$ and any linear combination of either $\mathbf{r}^i$ or $\mathbf{n}^i$ are nilpotent. This means that matrix exponentials of any terms of these types remain polynomial in the coefficients of the basis elements. This implies from (5.79), (5.74) that the metric components and potentials on each rod are rational functions of z.

3. In general, regularity of the axes imposes further constraints on these moduli from the conditions for the removal of conical singularities. We will not discuss these further here though they were dealt with at length in Chapter 3 and the analysis is identical in this case.

We now illustrate this theorem by applying the method described to the simplest possible rod structure, that of flat space.

5.6.1 Example: flat space

Flat space in five dimensions has two rods: the left rod $I_L = (-\infty, z_1)$ with rod vector $v_L = (0, 1)$ and the right rod $I_R = (z_1, \infty)$ with rod vector $v_R = (1, 0)$. This rod structure is illustrated in Figure 5.1.

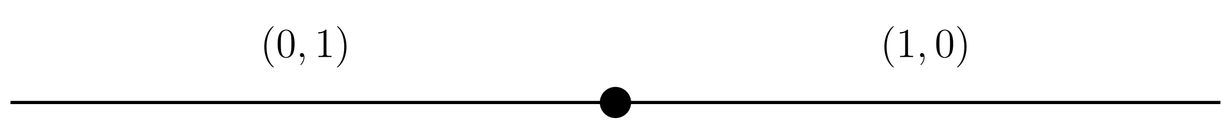

Figure 5.1: The Rod structure for five-dimensional flat space.

Using the rod point continuity conditions (5.56) and the definition of F_a (5.71), we find that

$$\begin{aligned} &F_L = C[M_L M_R^{-1}]^T, \qquad F_R = [M_L M_R^{-1}]^{-1} C \\ &M_L M_R^{-1} = X_L(z_1)^{-1} (S_R^{(0)})^{T-1} (S_L^{(0)})^T X_R(z_1) \end{aligned} \tag{5.85}$$

We are free to impose the following conditions on the gauge potentials ψ_i, μ, χ_i [92]

$$
\begin{gathered}
\lim_{|z|\to\infty} \psi_i(z) = 0, \\
\lim_{z\to-\infty} \mu(z) = 0, \qquad \lim_{z\to-\infty} \chi_i(z) = 0.
\end{gathered} \tag{5.86}
$$

This means that at infinity on I_L these potentials vanish, and so from (5.48) we find that $S_L^{(0)} = 1$. We also know that μ and χ_i are constant along axis rods (5.42) and continuous at rod points. Therefore they must vanish at infinity on I_R along with ψ_i, implying that $S_R^{(0)} = 1$.

Using this result in (5.85) and after lengthy manipulations we can write $\tilde{F}_L$ as a product in the following form

$$
\begin{aligned}
\tilde{F}_L &= \mathcal{L}^T \iota_0[f] \mathcal{R}, \\
\mathcal{L} &= \exp\left(A_i \mathbf{r}^i\right) \exp\left(B\mathbf{q}\right) \exp\left(C_i \mathbf{n}^i\right), \\
\mathcal{R} &= \exp\left(I_i \mathbf{r}^i\right) \exp\left(J\mathbf{q}\right) \exp\left(K_i \mathbf{n}^i\right),
\end{aligned} \tag{5.87}
$$

for particular functions A_i, B, C_i, I_i, J, K_i and a 2×2 matrix f (which we shall not present here). Using this we can immediately solve for the metric and potentials on the axis rods from (5.79), up to the symmetry of $\tilde{F}_L$. Imposing this symmetry (i.e. $\mathcal{L} = \mathcal{R}, f^T = f$) together with consistency conditions coming from the metric being asymptotically flat (3.180), and the potentials being in a particular gauge (5.47), one can solve for the moduli of the solution. These are given by

$$
\phi_\mu^{L,R}(z_1) = 0, \qquad \nu^{L,R}(z_1) = 0, \qquad b_\mu^{L,R}(z_1) = 0, \qquad \tilde{\omega}^i_{L,R}(z_1) = 0, \tag{5.88}
$$

where we have also used (5.23) and (5.45). The metric and potentials on I_L can also be solved explicitly to give

$$
\begin{gathered}
\phi_\mu^L(z) = 0, \qquad \nu^L(z) = 0, \qquad b_\mu^L(z) = 0, \qquad \tilde{\omega}^i_L(z) = 0 \\
h^L_{\mu\nu}(z) = \operatorname{diag}\left(-1, -2(z - z_1)\right),
\end{gathered} \tag{5.89}
$$

which is indeed the solution for flat space on I_L. We can find the solution for flat space on I_R in the same way by considering $\tilde{F}_R$.

This proves the non-trivial result that the only AF, stationary, biaxisymmetric solution in five-dimensional minimal supergravity with the same rod structure as flat space is flat space itself. This result can be proved through other methods, for example by using the fact that the spacetime associated with this rod structure cannot have any 2-cycles. Combining this with the Smarr relation in five-dimensional minimal supergravity (see e.g. [154]) we see that the mass must vanish and so by the positive mass theorem the solution is flat space.

5.6.2 Vacuum limit

As we discussed in Section (5.2.1), the harmonic map for minimal supergravity reduces in the vacuum limit (taking all the Maxwell potentials to be constant) to two copies of a vacuum harmonic map. Equivalently the $G_{2(2)}$ matrices we are dealing with reduce down to a direct sum of two $SL(3,\mathbb{R})$ matrices in the image of the map ι (5.12). This means that our results of the last few sections that allow us to solve for the metric and potentials on axis can be applied to the vacuum case in an appropriate limit. Note that we do not expect this to reproduce the formalism of Chapter 3 since this vacuum harmonic map is not the same as the one previously considered.

To start with, we can consider the vacuum limit of the solution for $\mathring{\Psi}_+$. The horizon rod case is trivial (5.33). In the axis limit case (5.54) one finds that

$$\tilde{X} \to \iota_0 \left[\begin{pmatrix} 1 & -b_1 \\ 0 & 2(z-k) \end{pmatrix}\right] \exp(-\tilde{\omega}^1 \mathbf{n}_1 + (2(z-k)\tilde{\omega}^2 - (b_0 - b_1\tilde{\omega}^1))\mathbf{n}_2), \tag{5.90}$$

$$S^{(0)} \to \exp(\chi_i^{(0)} \mathbf{n}_i). \tag{5.91}$$

Note that both of these matrices are now in the image of the map ι, as stated above. This means that now F and $\tilde{F}$ are in the image of ι as well and so the results of Theorem 9 can be somewhat refined. For an axis rod we find

$$h^{\mu\nu} \to \tilde{F}^{4+\mu\ 4+\nu}, \qquad b_\mu \to -\begin{pmatrix} \tilde{F}^{31} \\ \tilde{F}^{32} \end{pmatrix}, \tag{5.92}$$

and for a horizon

$$\gamma^{ij} \to F^{3+i\ 3+j}, \qquad \chi_i \to -\begin{pmatrix} F^{31} \\ F^{32} \end{pmatrix}. \tag{5.93}$$

This allows one to start solving for axis metric data in the vacuum theory, given a particular rod structure. Using this we have, for example, derived the solution for the doubly rotating Myers-Perry black hole. However, this formalism is in practice less useful than the one developed in Chapter 3 since the constraints on the moduli in this version are more complicated. This ultimately seems to be because of the fact that the harmonic map in the previous chapter was fully covariant with respect to all the KVFs whereas this one is only covariant with respect to the axial KVFs. As a result the moduli that the solutions are written in terms of can be rather unnatural choices (for example $\tilde{\omega}^i$ evaluated at rod points), which makes solving moduli space equations significantly more challenging.

5.7 Discussion

In this chapter we have considered the classification of stationary, AF and biaxisymmetric solutions to five-dimensional minimal supergravity. This theory of gravity is integrable, meaning that the equations governing it can be recovered as the integrability conditions of a pair of linear PDEs (BZ equations). Following a very similar method to the one presented in Chapter 3, we were able to use these equations to derive a general expression for the metric and various potentials on each axis and horizon component, see Theorem 9. These solutions are written in terms of a set of moduli governing the spacetime which can be constrained using a variety of consistency conditions on the metric and potentials. We applied these methods to the rod structure corresponding to flat space and by solving for the metric data and moduli were able to demonstrate that flat space is the unique spacetime with this rod structure.

There is clearly much work still to be done using the formalism developed in this chapter. It would be interesting to see how easily one could derive (and consequently provide uniqueness theorems for) other known solutions such as the 1-bubble soliton [115, 116] or the Cvetič-Youm black hole [98]. Extending this for more complicated rod structures would probably start to present some serious calculational difficulties, since even with the flat space rod structure things are already far from trivial. If some of these difficulties were overcome however, there are a large class of supersymmetric solutions without known non-supersymmetric limits which would be interesting to investigate. Of particular interest would be to consider the rod structure for the simplest $L(n,1)$ black lens (see Figure 4.5). A supersymmetric solution realising this rod structure is known [108], however as we demonstrated in Chapter 4, it has no corresponding regular vacuum solution. Using the methods developed in this chapter would provide a chance to bridge this gap by constructing a charged non-supersymmetric black lens.

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