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# Methods for the analysis of oscillatory integrals and Bochner-Riesz operators 

Reuben Wheeler

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## Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.
(Reuben Wheeler)

To Rachel and Byron

## Lay summary

Think of the signal emitted from speakers when we play a piece of music. This signal may be altered, for example by removing the bass frequencies. Such an alteration is an example of a (Fourier) multiplier operator applied to the signal. This multiplier removes the bass frequencies with a multiplication by 0 , the remaining frequencies are multiplied by 1 to be preserved. Bochner-Riesz operators are also multiplier operators, rather than a sharp multiplier which jumps from 0 to 1 , they have additional regularity. We consider when these operators are well behaved. For example, in what sense is the returned signal to be understood as piece of music? Is the sound evenly distributed through the piece's duration? Are there deafening peaks? This behaviour is expressed in terms of certain operator bounds. The study of BochnerRiesz operators is a long-standing problem in harmonic analysis dating back to the 1930s and has strong connections to other fundamental areas of research in the field, namely Fourier restriction, the Kakeya problem, and local smoothing for the wave equation. One of the major themes in this thesis concerns how we might extend the definition of Bochner-Riesz operators to particular curved or flat surfaces. We classify curved surfaces defining Bochner-Riesz operators which display behaviour like the classical Bochner-Riesz operators. For certain surfaces with flat components, we show that the behaviour of the corresponding operators differs substantially from what might be expected according to standard tests. This extends work of Mockenhoupt, using restriction estimates to obtain operator bounds. Carrying out the standard test on $L^{p}$ boundedness requires the adaptation of a nuanced technique of Arkhipov, Chubarikov, and Karatsuba, which was originally brought to bear on Tarry's problem in number theory, for bounding oscillatory integrals.

Let us consider an oscillating sum. We start with 1 . If we add to this number $-\frac{1}{2}$, one may verify we are left with $\frac{1}{2}$. If we now add to this $\frac{1}{4}$, we obtain $\frac{3}{4}$. Taking $-\frac{1}{8}$ from this we have $\frac{5}{8}$. We may continue this procedure through as many steps as we like, even ask what happens as we approach an infinite number of steps in the procedure. One of the things we can observe is that the positive and negative terms in the sum display some cancellation. What we have outlined is an example of an oscillating sum. For other sums, the oscillation can be more complicated. We might ask, 'which are the most significant parts of the oscillating sum?' or 'what bounds can we obtain for the sum?' Oscillating sums (or, more generally, oscillatory integrals) are important objects of study in harmonic analysis, for instance they appear in our analysis of Bochner-Riesz operators. We make use of a variety of methods for understanding the effects of oscillation on such sums. In particular, we extend a proof of Hickman and Wright which bounds oscillatory sums by splitting up different sized pieces of the sum and presenting a novel categorisation of the small number of large pieces. We also make use of nuanced bounds of Arkhipov, Chubarikov, and Karatsuba, who are able to abound oscillatory sums by considering them in a suitable average sense. Finally, we also make use of bounds of Phong and Stein, who obtain bounds oscillatory sums which are expressed in terms of clusters of polynomial roots. With our later results on the structure of polynomial roots, we are able to recover the bounds of Hickman and Wright by an application of the estimates of Phong and Stein.

Finding the roots of polynomials is a task familiar to many. We are taught methods to find the roots of linear and quadratic equations of the form $3 x+5=0$ and $x^{2}+2 x-3=0$. For higher order equations with more terms, for example the unwieldy equation $x^{10}+700 x^{4}-$ $200 x^{3}-100 x^{2}+10 x-1=0$, the situation is much more complicated. Nevertheless, it is possible to figure out roughly where the roots can be found and how close different roots can get to each other. For example, just by looking at the previous polynomial equation featuring $x^{10}$, our results reveal that at most 5 roots can get close to each other. More specifically, our results
characterise the different types of structure that can occur when the powers of $x$ are fixed and the coefficients can vary. This analysis extends work of Kowalski and Wright characterising the structure of 'small' and 'large' collections of roots, similar to how we might think of the distinction between the sun and the space that lies beyond its surface. Our work uncovers a richer picture, showing how roots appear in multiple strata, akin layers of the sun and the orbits of planets.

## Abstract

For a smooth surface $\Gamma$ of arbitrary codimension, one can consider the $L^{p}$ mapping properties of the Bochner-Riesz multiplier

$$
m_{\Gamma, \alpha}(\zeta)=\operatorname{dist}(\zeta, \Gamma)^{\alpha} \phi(\zeta)
$$

where $\alpha>0$ and $\phi$ is an appropriate smooth cutoff function. Even for the sphere $\Gamma=\mathbb{S}^{N-1}$, the exact $L^{p}$ boundedness range remains a central open problem in Euclidean harmonic analysis. We consider the $L^{p}$ integrability of the Bochner-Riesz convolution kernel for a particular class of surfaces (of any codimension). For a subclass of these surfaces the range of $L^{p}$ integrability of the kernels differs substantially from the $L^{p}$ boundedness range of the corresponding Bochner-Riesz multiplier operator. Extending work of Mockenhoupt, we then establish a range of operator bounds, which are sharp in the $\alpha$ exponent, under the assumption of an appropriate $L^{2}$ restriction estimate. Hickman and Wright established sharp oscillatory integral estimates, associated with a particular class of surfaces, and derived restriction estimates. We extend this work to certain curves of standard type and corresponding surfaces of revolution. These surfaces are discussed as an explicit class for which we have $L^{p} \rightarrow L^{p}$ boundedness of the corresponding Bochner-Riesz operators.

Understanding the structure of the roots of real polynomials is important in obtaining stable bounds for oscillatory integrals with polynomial phases. For real polynomials with exponents in some fixed set,

$$
\Psi(t)=x+y_{1} t^{k_{1}}+\ldots+y_{L} t^{k_{L}}
$$

we analyse the different possible root structures that can occur as the coefficients vary. We first establish a stratification of roots into tiers containing roots of comparable sizes. We then show that at most $L$ non-zero roots can cluster about a point. Supposing additional restrictions on the coefficients, we derive structural refinements. These structural results extend work of Kowalski and Wright and provide a characteristic picture of root structure at coarse scales. As an application, these results are used to recover the sharp oscillatory integral estimates of Hickman and Wright, using bounds for oscillatory integrals of Phong and Stein.

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## Publications

Some of the work contained herein has been adapted from work previously published in the Journal of Fourier Analysis and its Applications, [Whe20]. This work has been adapted throughout Part I, but is primarily concentrated in Chapter 4. This work of the author is available via open access and is published under a creative commons license. The results in [Whe20] concern certain radial polynomial surfaces. In this thesis we extend the results to radial surfaces of standard type.

Part III contains unpublished work of the author, which is to be submitted and also made available on a pre-print server.

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## Introduction

In this thesis, we explore aspects of two central areas in Harmonic Analysis: Bochner-Riesz operators and oscillatory integrals.

Bochner-Riesz operators are classical Fourier multiplier operators associated with the surface $\Gamma=S^{N-1}$. With $m_{\mathrm{BR}}(\zeta):=\left(1-|\zeta|^{2}\right)_{+}^{\alpha}$, the operator $T_{m_{\mathrm{BR}}, \alpha}$ is defined by

$$
\widehat{T_{m_{\mathrm{BR}, \alpha}}} f=m_{\mathrm{BR}, \alpha}(\zeta) \hat{f}(\zeta) .
$$

These operators have been studied since the 1930's, [Boc35], and arise concerning fundamental questions about the summability of Fourier series and integrals. These questions have played an important role in the development of the field of harmonic analysis and Bochner-Riesz operators display deep connections to other fundamental areas of research, namely Fourier restriction, local smoothing for the wave equation, and the Kakeya problem. The Bochner-Riesz conjecture, concerning the $L^{p} \rightarrow L^{p}$ boundedness of these operators, remains a central open problem. In Part I, we consider Bochner-Riesz operators associated with smooth surfaces, $\Gamma$, which may be of co-dimension greater than 1 . Intimately connected with the study of BochnerRiesz operators is the Fourier restriction problem, where surface curvature plays a critical role. Despite the connection between the research of Bochner-Riesz operators and the restriction problem, our study reveals some interesting differences in the implications of curvature in the two areas; in Bochner-Riesz flatness at isolated points has no clear effect on the $L^{p}$ boundedness of the operator but flatness of the whole surface significantly alters the $L^{p}$ boundedness of the operator, whilst in restriction flatness at isolated points impacts the $L^{p}$ boundedness of the operator and flatness of the whole surface ensures only trivial restriction estimates are possible.

Harmonic analysis is riven with oscillatory integrals, a typical oscillatory integral can be written

$$
\int e^{i \Phi(\xi)} \phi(\xi) d \xi
$$

We refer to $\Phi$ as the phase and $\phi$ as the amplitude. Most evident is the Fourier transform of a function, which is an oscillatory integral with a linear phase. Another interesting kind of oscillatory integral appears out of the extension operator, which is dual to the restriction operator. In this thesis, various oscillatory integrals will play an essential role in our analysis and we use a variety of methods for their study. Classical techniques for bounding oscillatory integrals include the non-stationary phase lemma, the method of stationary phase, and van der Corput's lemma. We make liberal use of each of these, including a lesser used variant of van der Corput's lemma found in [ACK08]. There is a more geometric sibling of van der Corput's lemma due to Phong and Stein, [PS97], which expresses bounds on oscillatory integrals with phase $\Phi$ in terms of certain cluster estimates involving the roots of $\Phi^{\prime}$-i.e. the stationary points. Combined with these tools, there are some more nuanced methods we appeal to. In Section 4.2, we adapt a method of Arkhipov, Chubarikov, and Karatsuba, taken from [ACK79], to determine if our Bochner-Riesz kernel $K_{\Gamma, \alpha} \in L^{p}$. Whilst using the aforementioned variant of van der Corput's estimate, this technique relies on controlling oscillatory integrals in an average sense. A refined method for bounding the Fourier transform of particular surface supported measures, decomposing the integral on dyadic scales and obtaining suitable control at exceptional scales where the phase degenerates, due to Hickman and Wright, [HW20], is extended in Part II.

In Part III, we carry out a structural analysis of the roots of real polynomials with exponents
$0=k_{0}<k_{1}<\ldots<k_{L}$ taken from some fixed set,

$$
\Psi(t)=x+\sum_{j=1}^{L} y_{j} t^{k_{j}}
$$

Locating roots of polynomials has a long history in mathematics, pre-dating all of the techniques discussed above. There can, however, be no explicit formualae for the solution of polynomials of degree at least 5 . Nevertheless, numerical techniques provide algorithms for the increasingly precise approximation of roots. Our interest in locating polynomial roots is motivated in part by the applications to oscillatory integrals, via the estimates of Phong and Stein, and previous work of Kowalski and Wright [KW12]. For our applications, understanding the overall structure and approximate location of roots is sufficient. In particular, to recover the oscillatory integral bound of Hickman and Wright, we will show that the roots of $\Psi$ are stratified into at most $L$ separated tiers and, furthermore, for a suitable small parameter $\epsilon>0$ and any non-zero root, $w$, of $\Psi$,

$$
B(w, \epsilon|w|)
$$

can contain at most $L$ other roots of $\Psi$, counted with multiplicity. Finally, in Chapter 9, we recover the bounds of Hickman and Wright via our results on polynomial root structure and the oscillatory integral bounds of Phong and Stein.

## Preliminaries

We here recall some classical results that we will require use of in this document. First, we recall the non-stationary phase lemma.
Lemma 0.0.1. Suppose that $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right), \Phi \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and $|\nabla \Phi(\xi)| \gtrsim 1$ for $\xi \in \operatorname{supp} \phi$, then, with

$$
I(\lambda):=\int e^{i \lambda \Phi(\xi)} \phi(\xi) d \xi
$$

we have that

$$
|I(\lambda)| \lesssim_{M}|\lambda|^{-M}\left(1+\|\Phi\|_{C^{M+1}(\operatorname{supp} \phi)}\right)^{M}\|\phi\|_{C^{M}(\operatorname{supp} \phi)}|\operatorname{supp} \phi|
$$

for all $M \in \mathbb{N}$.
Let us also recall the classical Riesz-Thorin interpolation theorem, as found in [SW71].
Theorem 0.0.2. Let $T$ be a linear operator for which

$$
\begin{aligned}
\|T f\|_{q_{0}} & \leq C_{0}\|f\|_{p_{0}} \\
\|T f\|_{q_{1}} & \leq C_{1}\|f\|_{p_{1}}
\end{aligned}
$$

for some $p_{0} \leq p_{1}, 1 \leq p_{j}, q_{j} \leq \infty$. Then, for $\theta \in[0,1]$, there exists a constant $C_{\theta}=C_{0}^{1-\theta} C_{1}^{\theta}$ for which

$$
\|T f\|_{q_{\theta}} \leq C_{\theta}\|f\|_{p_{\theta}}
$$

for all $f \in L^{1}$ and $\frac{1}{p_{\theta}}=\frac{1-\theta}{p_{0}}+\frac{\theta}{p_{1}}, \frac{1}{q_{\theta}}=\frac{1-\theta}{q_{0}}+\frac{\theta}{q_{1}}$.
We also make use of the weak-type Young's inequality; see, for instance, [Tao].
Theorem 0.0.3. Suppose that $h_{1} \in L^{s, \infty}\left(\mathbb{R}^{L}\right)$ with $s \in(1, \infty)$. Then, for $p, r \in(1, \infty)$ with $\frac{1}{p}+\frac{1}{s}=1+\frac{1}{r}$ and $h_{2} \in L^{p}\left(\mathbb{R}^{L}\right), h_{2} * h_{1} \in L^{r}\left(\mathbb{R}^{L}\right)$ with

$$
\left\|h_{2} * h_{1}\right\|_{L^{r}} \leq\left\|h_{1}\right\|_{L^{s, \infty}}\left\|h_{2}\right\|_{L^{p}} .
$$

## A generalised van der Corput estimate

We will make use of van der Corput's lemma throughout this document.
Lemma 0.0.4. Suppose that we have $\Phi \in C^{\infty}$ and, for some $k \geq 1,\left|\Phi^{(k)}(t)\right|^{\frac{1}{k}} \geq \kappa$ for $t \in[a, b]$, with the additional hypothesis that $\Phi^{\prime}$ monotonic in the case that $k=1$. Then we have the bound

$$
\left|\int_{a}^{b} e^{2 \pi i \Phi(t)} d t\right| \lesssim \min \left\{(b-a), \kappa^{-1}\right\} .
$$

There is also the following simple corollary of the above, which we also refer to as van der Corput's lemma.

Lemma 0.0.5. Suppose that we have $\Phi \in C^{\infty}$ and, for some $k \geq 1,\left|\Phi^{(k)}(t)\right|^{\frac{1}{k}} \geq \kappa$ for $t \in[a, b]$, with the additional hypothesis that $\Phi^{\prime}$ monotonic in the case that $k=1$. Then, for $\phi \in C_{c}^{\infty}$ with $\operatorname{supp} \phi \subset(a, b)$, we have the bound

$$
\left|\int_{a}^{b} e^{2 \pi i \Phi(t)} \phi(t) d t\right| \lesssim \min \left\{(b-a), \kappa^{-1}\right\}\left\|\phi^{\prime}\right\|_{1} .
$$

In particular

$$
\left|\int_{a}^{b} e^{2 \pi i \Phi(t)} \phi(t) d t\right| \lesssim \min \left\{(b-a), \kappa^{-1}\right\}\left\|\phi^{\prime}\right\|_{\infty}(b-a) .
$$

The versions of the lemma we utilise do not require the exact specification of which derivative of the phase is bounded below in size. To avoid this specification, we require that the phase satisfies suitable monotonicity properties. A model is a polynomial, which is monotone, with monotone derivatives, on finitely many intervals partitioning its domain. The following result is adapted from [ACK08].

Lemma 0.0.6. Let $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that, for some $k \geq 1$,

$$
\begin{equation*}
\inf _{t \in[a, b]}\left|\Phi^{(k)}(t)\right| \geq c \sup _{t \in[a, b]}\left|\Phi^{(k)}(t)\right|>0 \tag{0.0.1}
\end{equation*}
$$

for some $c>0$, with $\Phi^{\prime}$ monotone on a bounded number of intervals in the case the $k=1$. Given the bound $\inf _{t \in[a, b]} \sum_{j=1}^{k}\left|\frac{\Phi^{(j)}(t)}{j!}\right|^{\frac{1}{j}} \geq \kappa$ we have that

$$
\left|\int_{a}^{b} e^{2 \pi i \Phi(t)} d t\right| \lesssim_{k} \min \left\{\kappa^{-1},(b-a)\right\}
$$

Remark 0.0.7. Lemma 0.0.6 is stated in greater generality in [ACK08], with more specific monotonicity conditions on $\Phi^{\prime}$. However, the assumption (0.0.1), which provides the required monotonicity, is sufficient for our present purposes.

Proof. There is nothing to prove in the case that $k=1$, since we can apply the classical van der Corput lemma on each of the intervals where $\Phi^{\prime}$ is monotone. Henceforth, we suppose that $k \geq 2$.

Firstly, we use the fact that $\Phi^{(k)}(t)$ is bounded away from 0 on $[a, b]$. As a consequence of this, we see that $\Phi^{(k-1)}$ is strictly monotonic on $[a, b]$ and, in particular, there are at most two intervals on which $\Phi^{(k-1)}(t)$ is of constant sign. We consider each of these intervals separately.

Continuing as before, if $\Phi^{(j+1)}$ is of constant sign on $\left(a^{\prime}, b^{\prime}\right)$ then $\Phi^{(j)}$ is strictly monotonic on $\left(a^{\prime}, b^{\prime}\right)$ and there are at most 2 connected subintervals of $\left(a^{\prime}, b^{\prime}\right)$ on which $\Phi^{(j)}(t)$ has constant sign. This recursive procedure ensures that we can (up to finitely many excluded points) cover $[a, b]$ by at most $2^{k}$ open intervals $J$ upon which $\Phi^{(j)}(t)$ is of constant sign for each $j$. For each of these intervals $J=(\tilde{a}, \tilde{b})$ and each derivative $j$, we can see by smoothness of $\Phi$ and monotonicity of the derivatives, that $\inf _{J}\left|\Phi^{(j)}(t)\right|=\min \left\{\left|\Phi^{(j)}(\tilde{a})\right|,\left|\Phi^{(j)}(\tilde{b})\right|\right\}$. We know that
$\sum_{j=1}^{k}\left|\frac{\Phi^{(j)}(t)}{j!}\right|^{\frac{1}{j}} \geq \kappa$ so that $\min _{t \in\{\tilde{a}, \tilde{b}\}} \max _{1 \leq j \leq k}\left|\frac{\Phi^{(j)}(t)}{j!}\right|^{\frac{1}{j}} \gtrsim \kappa$. For the $j_{0}$ which realises this expression, we then know that $\inf _{t \in J}\left|\Phi^{\left(j_{0}\right)}(t)\right|^{\frac{1}{j_{0}}} \gtrsim \kappa$ and we can apply the classical van der Corput's estimate to bound the integral over $J \subset(a, b)$. Summing the resulting estimates over all subintervals $J$, of which there are finitely many, gives the result.

Corresponding with the second van der Corput estimate, we have the following corollary of Lemma 0.0.6.

Lemma 0.0.8. Let $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function with $\inf _{t \in[a, b]}\left|\Phi^{(k)}(t)\right| \geq c \sup _{t \in[a, b]}\left|\Phi^{(k)}(t)\right|>$ 0 , for some $c>0$, with $\Phi^{\prime}$ monotone on a bounded number of intervals if $k=1$, and $\phi$ be a smooth function with $\operatorname{supp} \phi \subset(a, b)$. Given the bound $\inf _{t \in[a, b]} \sum_{j=1}^{k}\left|\frac{\Phi^{(j)}(t)}{j!}\right|^{\frac{1}{j}} \geq \kappa$, we have that

$$
\left|\int_{a}^{b} e^{2 \pi i \Phi(t)} \phi(t) d t\right| \lesssim_{k} \min \left\{(b-a), \kappa^{-1}\right\}\left\|\phi^{\prime}\right\|_{L^{1}}
$$

In particular, we have that

$$
\left|\int_{a}^{b} e^{2 \pi i \Phi(t)} \phi(t) d t\right| \lesssim \min \left\{(b-a), \kappa^{-1}\right\}(b-a)\left\|\phi^{\prime}\right\|_{L^{\infty}} .
$$

Proof. By the fundamental theorem of calculus, we see that

$$
\begin{gathered}
\int_{a}^{b} e^{2 \pi i \Phi(t)} \phi(t) d t=\int_{a}^{b} e^{2 \pi i \Phi(t)} \int_{a}^{t} \phi^{\prime}(s) d s d t \\
=\int_{a}^{b} \int_{s}^{b} e^{2 \pi i \Phi(t)} d t \phi^{\prime}(s) d s
\end{gathered}
$$

so then

$$
\begin{gathered}
\left|\int_{a}^{b} e^{2 \pi i \Phi(t)} \phi(t) d t\right| \leq \sup _{a \leq s \leq b}\left|\int_{s}^{b} e^{2 \pi i \Phi(t)} d t\right| \int_{a}^{b}\left|\phi^{\prime}(s)\right| d s \\
\lesssim_{k} \min \left\{(b-a), \kappa^{-1}\right\} \int_{a}^{b}\left|\phi^{\prime}(s)\right| d s
\end{gathered}
$$

## Notation

For $1 \leq p \leq \infty$, we denote by $L^{p}$ the space of $L^{p}$ integrable functions on $\mathbb{R}^{N}$. More generally, a function $f \in L^{p}(\Gamma, \mu)$ if

$$
\|f\|_{L^{p}}=\left(\int_{\Gamma}|f(x)|^{p} d \mu(x)\right)^{\frac{1}{p}}<\infty
$$

with the understanding that, in the case of $p=\infty$, the integral is to be understood as an essential supremum. We also denote by $L^{p, \infty}=L^{p, \infty}\left(\mathbb{R}^{N}\right)$ the space of weak- $L^{p}$ functions: $f \in L^{p, \infty}\left(\mathbb{R}^{N}\right)$ if

$$
\|f\|_{L^{p, \infty}}=\sup _{\lambda>0} \lambda\left|\left\{x \in \mathbb{R}^{N} ;|f(x)|>\lambda\right\}\right|^{\frac{1}{p}}<\infty .
$$

We denote by $B^{m}(x, r)$ the Euclidean ball of radius $r$ centred at $x$ in $\mathbb{R}^{m}$, or simply $B(x, r)$ if the dimension is clear. We use the notation

$$
\mathcal{F}(f)(\xi)=\hat{f}(\xi)=\int e^{-2 \pi i \xi \cdot x} f(x) d x
$$

for the Fourier transform and likewise for the inverse Fourier transform we write

$$
\mathcal{F}^{-1}(g)(x)=\check{g}(x)=\int e^{2 \pi i x \cdot \xi} g(\xi) d \xi
$$

We denote by $\mathscr{S}=\mathscr{S}\left(\mathbb{R}^{n}\right)$ the class of Schwartz functions, a function $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{C}$ is an element of $\mathscr{S}$ if it is infinitely differentiable and

$$
\left|\partial^{\alpha} \phi(x)\right| \leq C_{\alpha, M}|x|^{-M}
$$

for all multi-indices $\alpha \in \mathbb{N}_{0}^{n}$ and $M \in \mathbb{N}_{0}$. We denote by $\mathscr{S}^{\prime}$ the space of tempered distributions. These are the bounded linear functionals on $\mathscr{S}$. For $\Lambda \in \mathscr{S}^{\prime}$, we define its Fourier transform as a distribution by

$$
\hat{\Lambda}(\varphi)=\Lambda(\hat{\varphi})
$$

We also denote by $\hat{\sigma}_{S^{m-1}}$ the Fourier transform of the surface measure of the sphere $S^{m-1}$ :

$$
\hat{\sigma}_{S^{m-1}}(x)=\int_{S^{m-1}} e^{2 \pi i x \cdot \omega} d \sigma(\omega)
$$

Similarly, for a Borel measure, $\mu$, on $\mathbb{R}^{N}$ of bounded variation, we define its Fourier transform by

$$
\hat{\mu}(x)=\int e^{2 \pi i x \cdot \omega} d \mu(\omega)
$$

For the Lebesgue measure on $\mathbb{R}^{n}$, we sometimes denote this by $\mu_{\mathbb{R}^{n}}$. We deviate from convention in this way as typically throughout this document, $\lambda$ will be used to denote a real factor in the phase of an oscillatory integral. Often, the Lebesgue measure will be written classically as a density $d x$.

Throughout this document $C$ will be used to denote a constant, its value may change from line to line. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ if there exists some implicit constant $C$ such that $X \leq C Y$. When we wish to highlight the dependence of the implied constant $C$ on
some other parameter, say $C=C(M)$, we will use the notation $X \lesssim_{M} Y$. We use the notation $X \ll Y$ or $Y \gg X$ if there exists some suitable large constant $D$ such that $D X \leq Y$.

For $p \in(1, \infty)$ we denote by $\mathcal{M}_{p}$ the space of Borel measurable functions $m$ for which the multiplier operator defined a priori by

$$
f \mapsto \mathcal{F}^{-1}(m \cdot \hat{f})
$$

is bounded from $L^{p}$ to $L^{p}$, we denote by $\|m\|_{\mathcal{M}_{p}}$ the norm of the operator, $T_{m}$, thus defined.
We do not use vector notation and, at different points 0 will be used to denote zero as a scalar, a zero vector, or the zero matrix. Each of these uses should be clear from context and the use will typically be highlighted with a set containment reference.

We will write the co-dimension of surfaces appearing throughout Part I as $L=\tilde{L}+L^{\prime}$. Typically, in what follows $x, y$, and $z$ will denote points in $\mathbb{R}^{n}, \mathbb{R}^{\tilde{L}}$, and $\mathbb{R}^{L^{\prime}}$, respectively. Since we may have that $L^{\prime}=0$ but integrate with respect to $d z$ we follow the convention that, when $L^{\prime}=0, \mathbb{R}^{L^{\prime}}=\mathbb{R}^{0}=\{0\}$ and $d z$ is the counting measure. Similarly for $S^{0}$, which we regard as the set $\{-1,1\}$ equipped with the counting measure.

## Part I

## Bochner-Riesz operators

## Chapter 1

## Introduction

In Part I, we will be considering Fourier multiplier operators in $N$ dimensions. These will be defined with respect to surfaces of dimension $n$ and co-dimension $L$, with $N=n+L$.

The disc multiplier operator $S_{R}$ is defined by

$$
S_{R} f(x)=\int_{B_{R}} e^{2 \pi i x \cdot \xi} \hat{f}(\xi) d \xi
$$

or more simply by

$$
\widehat{S_{R} f}=\hat{f}(\xi) \mathbb{1}_{B_{R}}
$$

where $B_{R}$ is the unit ball of radius $R$ centred at the origin. This multiplier arises in the question of whether the integral

$$
\int_{|\xi|<R} \hat{f}(\xi) e^{2 \pi i \xi \cdot x} d \xi
$$

converges to the function $f$ as $R \rightarrow \infty$, in some appropriate sense, for example in $L^{p}$.
As we establish in Section 2.1, when one studies Fourier multiplier operators $T_{m}$, with a compactly supported multiplier $m, \widehat{T_{m}(f)}(\xi)=m(\xi) \hat{f}(\xi)$, a necessary condition for $T_{m}$ to be bounded on $L^{p}\left(\mathbb{R}^{N}\right)$ is that the kernel $K=\check{m} \in L^{p}$.

Since $\mathbb{1}_{B_{R}} \in L^{\infty}$, it is a consequence of Plancherel's theorem that $S_{R} \in \mathcal{M}_{2}\left(\mathbb{R}^{N}\right)$. Furthermore, corresponding kernel, $\check{\mathbb{1}}_{B_{R}}$ is an element of $L^{p}$ for

$$
\frac{L+n}{L+\frac{n}{2}}<p
$$

By the above necessary condition, and the fact real valued multipliers define operators which are self-dual, we see that $T_{\mathbb{1}_{B_{R}}}$ can define a bounded operator on $L^{p}$ only for

$$
\frac{L+n}{L+\frac{n}{2}}<p<\frac{L+n}{\frac{n}{2}}
$$

where the left and right terms in the inequality are dual exponents. Historically, it was conjectured that the disc multiplier defined a bounded operator on $L^{p}$ for

$$
\begin{equation*}
\frac{L+n}{L+\frac{n}{2}}<p<\frac{L+n}{\frac{n}{2}} \tag{1.0.1}
\end{equation*}
$$

The disc multiplier conjecture is true in one dimension; the multiplier is an element of $\mathcal{M}_{p}(\mathbb{R})$ for all $1<p<\infty$. To verify this one may write $\mathbb{1}_{(-R, R)}=\mathbb{1}_{(-\infty, 0)}(\cdot-R)-\mathbb{1}_{(-\infty, 0]}(\cdot+R)$. Each of these two terms in the sum for the multiplier defines an operator which is a modulation followed by a Hilbert transform, these are bounded on $L^{p}(\mathbb{R})$, [Rie28].

It is a famous result of Fefferman that the disc multiplier is not an element of $\mathcal{M}_{p}\left(\mathbb{R}^{N}\right)$ for $N \geq 2$ and $p \neq 2$ [Fef71], disproving the disc multiplier conjecture in dimensions $N \geq 2$.

In a remarkable paper of Heo, Nazarov, and Seeger, [HNS11], they show that the natural
necessary condition $\check{m} \in L^{p}$ is also sufficient for $T_{m}$ to be bounded on $L^{p}$ in the range $1<$ $p<\frac{2(N-1)}{N+1}$, whenever $m$ is a compactly supported radial multiplier. The radial multiplier conjecture states that for radial multipliers $m, \check{m} \in L^{p}$ implies $T_{m}$ is bounded on $L^{p}$ holds in the range $1<p<\frac{2 N}{N+1}$, i.e. below the critical exponent from the (disproved) disc multiplier conjecture.

Further canonical examples of compactly supported radial multipliers are the classical Bochner-Riesz multipliers. The Bochner-Riesz multipliers can be viewed as regularised versions of the disc multiplier, they are given by

$$
m_{R}^{\alpha}(\xi)=\left(1-\frac{|\xi|^{2}}{R^{2}}\right)_{+}^{\alpha} .
$$

These multipliers are $\alpha$-Hölder, although they are still singular at the boundary $\partial B_{R}$. The corresponding convolution kernel $K_{\alpha}=\check{m}_{\alpha}$ is well known to lie in $L^{p}$ for

$$
p>p_{\alpha, N}=\frac{2 N}{N+1+2 \alpha}
$$

see [Her54].
The Bochner-Riesz conjecture states that, for $\alpha>0$, the Bochner-Riesz operator is bounded on $L^{p}$ if and only if

$$
\alpha>(n+1)\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2} .
$$

For $\alpha>0$, this may be equivalently stated as

$$
\frac{L+n}{L+\frac{n}{2}+\alpha}<p<\frac{L+n}{\frac{n}{2}-\alpha} .
$$

The radial multiplier conjecture is a significant generalisation of the Bochner-Riesz conjecture. However, the result in [HNS11] gives no new improvements on the Bochner-Riesz conjecture.

It can easily be established the classical Bochner-Riesz multiplier operator and the multiplier operator with multiplier $(1-|\zeta|)_{+}^{\alpha}$ are mutually bounded. ${ }^{1}$ Furthermore, observe that, for $\alpha>0$, the multiplier $(1-|\zeta|)_{+}^{\alpha}$ has the same regularity as $|1-|\zeta||^{\alpha}=d\left(\zeta, S^{n}\right)^{\alpha}$. The latter are the multipliers we generalise.

We let $\Gamma$ denote a smooth surface in $\mathbb{R}^{N}$ of codimension $L$. The generalised Bochner-Riesz operator we will consider is given by, for $\zeta \in \mathbb{R}^{n+L}$,

$$
\begin{equation*}
m_{\Gamma, \alpha}(\zeta, \Gamma)=d(\zeta, \Gamma)^{\alpha} \phi_{\Gamma}(\zeta), \tag{1.0.2}
\end{equation*}
$$

for some $\phi_{\Gamma} \in C_{c}\left(\mathbb{R}^{N}\right)$ with $\operatorname{supp} \phi_{\Gamma} \cap \Gamma$ non-empty. We wish to determine for which $p$ the multiplier $m_{\alpha, \Gamma} \in \mathcal{M}_{p}\left(\mathbb{R}^{N}\right)$.

Our approach in Part I takes multiple directions. First, in Chapter 3, we work to establish that $m_{\Gamma, \alpha} \in \mathcal{M}_{p}$ for smooth surfaces $\Gamma$, provided suitable $L^{2}$ restriction estimates hold. This extends work of Mockenhoupt, [Moc90]. Extending a method of Hickman and Wright, we also provide a specific class of surfaces to which restriction estimates hold. These surfaces are the (symmetric) curves of standard type found in [SW78] and associated surfaces of revolution. A model for these surfaces, and the case considered by Hickman and Wright, are given by

$$
\begin{equation*}
\Gamma=\left\{\xi,|\xi|^{d_{1}}, \ldots,|\xi|^{d_{L}} ;|\xi|<1\right\}, \tag{1.0.3}
\end{equation*}
$$

for some choice of even $2 \leq d_{1}<d_{2}<\ldots<d_{L}$. As a corollary of this restriction estimate and the previous result that restriction implies Bochner-Reisz, we obtain $L^{p} \rightarrow L^{p}$ bounds for the Bochner-Riesz operator associated with these surfaces.

For the aforementioned class of surfaces, modelled by (1.0.3), in Chapter 4, we investigate the sharpness of the above results. We determine precisely for which $p$ the kernel $\check{m}_{\Gamma, \alpha} \in L^{p}$. In the first instance, the work of Section 4.1 relies on classical stationary phase analysis in a

[^0]restricted region of $\mathbb{R}^{N}$; we derive pointwise lower bounds $\check{m}_{\Gamma, \alpha}$ to determine the necessary condition for $\check{m}_{\Gamma, \alpha} \in L^{p}$,
$$
\check{m}_{\Gamma, \alpha} \in L^{p} \Longrightarrow p>\frac{L+n}{L+\alpha+\frac{n}{2}} .
$$

To complete the analysis of where $\check{m}_{\Gamma, \alpha} \in L^{p}$, in Section 4.2 we seek upper estimates on $\left\|\check{m}_{\Gamma, \alpha}\right\|_{L^{p}}$ and this requires more modern methods. In particular, we are able to make use of a method of Arkhipov, Chubarikov, and Karatsuba, to bound oscillatory integrals in an $L^{p}$ average sense; this method was introduced in [ACK79] and allows us to establish the sufficient condition for $\check{m}_{\Gamma, \alpha} \in L^{p}$,

$$
p>\frac{L+n}{L+\alpha+\frac{n}{2}} \Longrightarrow \check{m}_{\Gamma, \alpha} \in L^{p} .
$$

In Section 4.3, using our analysis of where $\check{m}_{\Psi, \alpha} \in L^{p}$, we discuss the sharpness of our operator bounds. For a particular class of surfaces we consider, the the range of $\alpha$ such that the BochnerRiesz kernel $K_{\Gamma, \alpha}=\check{m}_{\Psi, \alpha} \in L^{p}$ differs from the range of $\alpha$ for which $m_{\Psi, \alpha} \in \mathcal{M}_{p}$. To conclude Part I, we consider this case, and present a test which properly captures the critical $\alpha$-exponent. These examples are instances of smooth surfaces $\Gamma$ which are contained in a proper subspace of $\mathbb{R}^{N}$. Model surfaces are given by

$$
\begin{equation*}
\Gamma=\left\{\left(\xi, \frac{|\xi|^{d_{1}}}{d_{1}!}, \ldots, \frac{|\xi|^{d_{\tilde{L}}}}{d_{\tilde{L}}!}, 0\right) \in \mathbb{R}^{n} \times \mathbb{R}^{\tilde{L}} \times \mathbb{R}^{L^{\prime}} ;|\xi|<1\right\} \tag{1.0.4}
\end{equation*}
$$

where $2 \leq d_{1}<d_{2}<\ldots<d_{\tilde{L}}$ are even and $1 \leq \tilde{L}<L$. In particular, we show the following.
Proposition. Let $\Gamma$ be given by (1.0.4) such that $\tilde{L}<L$, $d_{1}<d_{2},<\ldots,<d_{\tilde{L}}$, and $d_{1} \geq$ $n(\tilde{L}+1)$. Define $D=d_{1}+d_{2}+\ldots+d_{\tilde{L}}$. We set $\frac{q^{\prime}}{2}=1+\frac{D}{n}$. For $1<p \leq q$ or $q^{\prime} \leq p<\infty$, $T_{m_{\Gamma, \alpha}}$ is bounded on $L^{p}$ if and only if $\frac{\tilde{L}+n}{\tilde{L}+\alpha+\frac{n}{2}}<p<\frac{\tilde{L}+n}{\tilde{L}-\alpha+\frac{n}{2}}$.

To conclude the introduction, we now define the surfaces of standard type that we will pay particular attention to throughout Part I. The surfaces described at (1.0.3) and (1.0.4) should consistently be kept in mind as models.

Definition 1.0.1. The class of surfaces $\mathcal{S}_{0}$ is given as follows. This class contains surfaces

$$
\Gamma=\{(\xi, \Psi(|\xi|)) ;|\xi|<4 \delta\},
$$

where $\delta$ is some suitable small parameter and $\Psi=\left(\psi_{1}, \ldots, \psi_{L}\right):(-4 \delta, 4 \delta) \rightarrow \mathbb{R}$ is a smooth function with components satisfying the following. There exist $1 \leq \tilde{L} \leq L$ and even indices $d_{1}<d_{2}<\ldots<d_{\tilde{L}}$ for which

$$
\psi_{j}(r)=\frac{r^{d_{j}}}{d_{j}!}+\varepsilon_{j}(r)
$$

where $\varepsilon_{j}(r)$ is a higher order remainder term according with Taylor's theorem, for which we additionally assume ${ }^{2}$ that

$$
\begin{equation*}
\varepsilon_{j}^{\left(d_{i}\right)}(0)=0, \quad \text { for } \quad i>j . \tag{1.0.5}
\end{equation*}
$$

Additionally, for $\tilde{L}<j \leq L$, we set

$$
\psi_{j}(r)=0 .
$$

More specifically, the control we have on the higher order remainder terms $\varepsilon_{j}(r)$ is given as follows, provided $\delta$ is small enough. We have that $\varepsilon_{j}:(-4 \delta, 4 \delta) \rightarrow \mathbb{R}$ is a smooth function such that, for $0 \leq i \leq d_{j}$,

$$
\left|\varepsilon_{j}^{(i)}(r)\right| \ll|r|^{d_{j}-i}
$$

and, for $i>j$,

$$
\left|\varepsilon_{j}^{\left(d_{i}\right)}(r)\right| \ll 1
$$

[^1]Corresponding to the surfaces in the class $\mathcal{S}_{0}$ are those surfaces which are essentially linear transformations of these, which we now define.

Definition 1.0.2. We define the class of radial surfaces $\mathcal{S}$, which are of dimension $n$ and codimension $L$, as follows. Each element of $\mathcal{S}$ can be expressed as

$$
\begin{equation*}
\Gamma=\{(\xi, \Psi(|\xi|)) ;|\xi|<4 \delta\} \subset \mathbb{R}^{n} \times \mathbb{R}^{L} \tag{1.0.6}
\end{equation*}
$$

where $\delta$ is some suitably small parameter and $\Psi(r)=\left(\psi_{1}(r), \psi_{2}(r), \ldots, \psi_{L}(r)\right)$ is a smooth symmetric function on $\mathbb{R}$ which satisfies the following. First we require that $\Psi(0)=\Psi^{\prime}(0)=0$. We also require that, for

$$
\begin{align*}
& \tilde{L}:=\operatorname{dim} \operatorname{span}\left\{\psi_{j} ; j=1, \ldots, L\right\} \\
\text { we have that } & \tilde{L} \geq 1  \tag{1.0.7}\\
\text { and } & \tilde{L}=\sup _{M \geq 2} \operatorname{dim} \operatorname{span}\left\{\psi_{j}^{M} ; j=1, \ldots, L\right\}
\end{align*}
$$

where $\psi_{j}^{M}(r):=\sum_{l=0}^{M} \frac{\psi_{j}^{(l)}(0)}{l!} r^{l}$. We define $d_{\tilde{L}}$ to be the smallest $M$ such that

$$
\tilde{L}=\operatorname{dim} \operatorname{span}\left\{\psi_{j}^{M} ; j=1, \ldots, L\right\} .
$$

In other words, the class $\mathcal{S}$ of surfaces that we consider are those radial surfaces $\Gamma$ for which there exist Taylor approximants of finite order which characterise any linear dependence of the graphing functions $\psi_{1}, \psi_{2}, \ldots, \psi_{L}$. For example, they include the surfaces $\mathcal{S}_{0}$ if we relax the condition (1.0.5). The surfaces thus parametrised include certain curves of standard type defined in [SW78].

## Chapter 2

## A preliminary study

In this chapter, we make a preliminary study of the Bochner-Riesz multipliers, $m_{\Gamma, \alpha}$, which we defined previously; we derive a related graphical Bochner-Riesz multiplier, $m_{\Psi, \alpha}$, and give an explicit expression of the corresponding convolution kernel $\check{m}_{\Psi, \alpha}$. First, in Section 2.1, we introduce some classical lemmas for multiplier operators. Then, in Section 2.2, we outline the procedure that shows the mutual boundedness of the Bochner-Riesz multipliers $m_{\Gamma, \alpha}$ and the graphical Bochner-Riesz multpliers $m_{\Psi, \alpha}$. Finally, we express the kernel, $K_{\Psi, \alpha}=\check{m}_{\Psi, \alpha}$, and establish some preliminary estimates on its size.

### 2.1 Multiplier lemmas

Lemma 2.1.1. Real valued multipliers $m$ define operators $T_{m}$ which are self dual.
Proof. Using Fubini's theorem, for any $f, g \in \mathscr{S}$ we see that

$$
\begin{aligned}
&\left\langle T_{m} f, g\right\rangle=\int\left(\int e^{2 \pi i x \cdot \xi} \hat{f}(\xi) m(\xi) d \xi\right) \bar{g}(x) d x=\int \hat{f}(\xi) m(\xi) \overline{\hat{g}}(\xi) d \xi \\
&=\int\left(\int e^{-2 \pi i x \cdot \xi} f(x) d x\right) m(\xi) \overline{\hat{g}}(\xi) d \xi=\int f(x)\left(\int e^{-2 \pi i x \cdot \xi} m(\xi) \overline{\hat{g}}(\xi) d \xi\right) d x \\
&=\int f(x) \overline{\left(\int e^{2 \pi i x \cdot \xi} m(\xi) \hat{g}(\xi) d \xi\right)} d x=\left\langle f, T_{m} g\right\rangle
\end{aligned}
$$

Therefore $T_{m}^{*}=T_{m}$.
We will also make use of de Leeuw's Theorem, which we now state. A short proof can be found in [Jod71].

Theorem 2.1.2. Suppose that $m \in \mathcal{M}_{p}\left(\mathbb{R}^{n}\right)$ and $k<n$ in $\mathbb{R}^{n}$. Then, with $m_{x}$ defined by $m_{x}(y)=m(x, y)$ for $x \in \mathbb{R}^{k}, y \in \mathbb{R}^{n-k}$, $m_{x} \in \mathcal{M}_{p}\left(\mathbb{R}^{n-k}\right)$ for almost every $x$. In particular, $m_{x} \in \mathcal{M}_{p}$ for those $x$ such that $(x, y)$ is a Lebesgue point of $m$ for almost every $y$.

As discussed in the introduction, we have the following important test for the $L^{p} \rightarrow L^{p}$ boundedness of a compactly supported multiplier operator.

Lemma 2.1.3. For a Fourier multiplier operator, $T_{m}$, with a compactly supported multiplier $m$, a necessary condition for $T_{m}$ to be bounded on $L^{p}\left(\mathbb{R}^{N}\right)$ is that the kernel $K=\check{m} \in L^{p}$.

Proof. Take a Schwartz function $\varphi$ with $\hat{\varphi}(\xi)=1$ for $\xi \in \operatorname{supp} m$ so that $T_{m} \varphi=\check{m}$. If $T_{m}$ is bounded on $L^{p}$, then

$$
\|\check{m}\|_{L^{p}} \lesssim\|\varphi\|_{p}<\infty .
$$

Lemma 2.1.4. Boundedness of multipliers is preserved under invertible affine transformations. Suppose that $m \in \mathcal{M}_{p}$ and define $\tilde{m}(\xi)=m(T(\xi))$ for some invertible affine transformation $T$, then $\tilde{m} \in \mathcal{M}_{p}$ with

$$
\|\tilde{m}\|_{\mathcal{M}_{p}}=\|m\|_{\mathcal{M}_{p}}
$$

Proof. For $a \in \mathbb{R}^{n}$, we define the modulation operator $M_{a}$ by $M_{a}(f)(x)=e^{2 \pi i a \cdot x} f(x)$. Modulation preserves $L^{p}$ norms. For $R \in G L\left(\mathbb{R}^{n}\right)$, we define the rescaling operators $N_{R}$ by $N_{R}(f)=|\operatorname{det} R| f(R x)$.

Let $m \in \mathcal{M}_{p}$. We write $T(\xi)=R \xi+a$, for some matrix $R \in G L\left(\mathbb{R}^{n}\right)$. We let $\tilde{m}(\xi)=$ $m \circ T(\xi)$. Observe that $\mathcal{F}\left(M_{a} \circ N_{R^{t}}(f)\right)(\eta)=\hat{f}\left(R^{-1}(\eta-a)\right)$.

Take $f \in \mathscr{S}$

$$
\begin{gathered}
\mathcal{F}^{-1}(\tilde{m} \hat{f})(x)=\int e^{2 \pi i x \cdot \xi} m(T(\xi)) \hat{f}(\xi) d \xi \\
=\int e^{2 \pi i x \cdot R^{-1}(\eta-a)} m(\eta) \hat{f}\left(R^{-1}(\eta-a)\right)|\operatorname{det}(R)|^{-1} d \eta \\
=\int e^{2 \pi i\left(R^{-1}\right)^{t} x \cdot(\eta-a)} m(\eta) \mathcal{F}\left(M_{a} \circ N_{R^{t}}(f)\right)(\eta) d \eta \\
=e^{2 \pi i\left(R^{-1}\right)^{t} x \cdot(-a)} T_{m}\left(M_{a} \circ N_{R^{t}}(f)\right)\left(\left(R^{-1}\right)^{t} x\right)\left|\operatorname{det}\left(R^{t}\right)^{-1}\right| \\
=\left(N_{\left(R^{t}\right)^{-1}} \circ M_{-a}\left(T_{m}\left(M_{a} \circ N_{R^{t}}(f)\right)\right)\right)(x) .
\end{gathered}
$$

From this we establish that

$$
\begin{gathered}
\|\tilde{m}\|_{\mathcal{M}_{p}}=\left\|T_{\tilde{m}}\right\|_{L^{p} \rightarrow L^{p}} \\
\leq\left\|N_{\left(R^{t}\right)^{-1}}\right\|\left\|M_{-a}\right\|\left\|T_{m}\right\|\left\|M_{a}\right\|\left\|N_{R^{t}}\right\| \\
=\left\|T_{m}\right\|_{L^{p} \rightarrow L^{p}}=\|m\|_{\mathcal{M}_{p}}
\end{gathered}
$$

since $\left\|N_{\left(R^{t}\right)^{-1}}\right\|_{L^{p} \rightarrow L^{p}}\left\|N_{R^{t}}\right\|_{L^{p} \rightarrow L^{p}}=1$. Likewise, $\left\|T_{m}\right\|_{L^{p} \rightarrow L^{p}} \leq\left\|T_{\tilde{m}}\right\|_{L^{p} \rightarrow L^{p}}$.
We now turn to a simple but important lemma.
Lemma 2.1.5. Boundedness of multipliers is preserved by multiplication by $C_{c}^{\infty}$ functions: if $m \in \mathcal{M}_{p}$ and $\psi \in C_{c}^{\infty}$, then $\psi m \in \mathcal{M}_{p}$ and

$$
\|\psi m\|_{\mathcal{M}_{p}} \leq\|m\|_{\mathcal{M}_{p}}\|\check{\psi}\|_{1} .
$$

Proof. Let $m \in \mathcal{M}_{p}$, and take $f \in \mathscr{S}$. We see that $\mathcal{F}^{-1}(\psi \hat{f})=\check{\psi} * f \in \mathscr{S}$ so that,

$$
\left\|\mathcal{F}^{-1}(m \psi \hat{f})\right\|_{p} \leq\|m\|_{\mathcal{M}_{p}}\|\check{\psi} * f\|_{p} \leq\|m\|_{\mathcal{M}_{p}}\|\check{\psi}\|_{1}\|f\|_{p}
$$

by Young's inequality.

### 2.2 Multiplier reductions

Certain reductions are necessary to carry out effective calculations with the Bochner-Riesz multipliers. We present these in this section, working to establish Lemma 2.2.3, which reduces the study of the Bochner-Riesz multipliers given by Definition 2.2.1 to the graphical BochnerRiesz multipliers given by Definition 2.2.2.

In this section we will be working in $\mathbb{R}^{N}=\mathbb{R}^{n} \times \mathbb{R}^{L}$, for $\zeta \in \mathbb{R}^{N}$, we write $\zeta=(\xi, \eta)$, where $\xi \in \mathbb{R}^{n}$ and $\eta \in \mathbb{R}^{L}$.

Definition 2.2.1. For a smooth surface $\Gamma$ and some $\alpha>0$,

$$
m_{\Gamma, \alpha}(\zeta)=d(\zeta, \Gamma)^{\alpha} \phi_{\Gamma}(\zeta)
$$

is called the Bochner-Riesz multiplier with exponent $\alpha>0$. Here $\phi_{\Gamma}$ is some appropriate bump function whose support intersects $\Gamma$.

Definition 2.2.2. Suppose

$$
\Gamma_{\Psi}=\{(\xi, \Psi(\xi)) ;|\xi|<4 \delta\}
$$

is a smooth surface expressible near the origin as the graph of smooth $\Psi$ with $\Psi(0)=0$ and $\nabla \Psi(0)=0$. The (graphical) Bochner-Riesz multiplier is given by

$$
m_{\Psi, \alpha}(\zeta)=m_{\Psi, \alpha}(\xi, \eta)=\phi(\xi)|\eta-\Psi(\xi)|^{\alpha} \chi(\eta-\Psi(\xi))
$$

where $\phi$ and $\chi$ are smooth cutoff functions with $\phi(0)=\chi(0)=1$, $\operatorname{supp} \phi \subset B^{n}(0, \delta)$, and $\operatorname{supp} \chi \subset B^{L}(0, \delta)$, for some suitable small $\delta$.

Provided $\delta$ is chosen sufficiently small, we have the following.
Lemma 2.2.3. The Bochner-Riesz multiplier $m_{\Gamma, \alpha}$ given by Definition 2.2.1 is an element of $\mathcal{M}_{p}$ if and only if the multipliers $m_{\Psi, \alpha}$ given by Definition 2.2.2 are elements of $\mathcal{M}_{p}$, for all $\Psi$ which, up to a translation and change of coordinates, locally graph $\Gamma \cap \operatorname{supp} \phi_{\Gamma}$ and are such that

$$
\nabla \Psi(0)=0, \Psi(0)=0
$$

Here $\phi$ and $\chi$ are smooth, compactly supported, and radial bump functions with $\operatorname{supp} \phi \subset$ $B^{n}(0, \delta), \operatorname{supp} \chi \subset B^{L}(0, \delta)$, which can be taken uniformly over our choice of graphing function $\Psi$. Additionally, if we consider $\delta$ as a small variable parameter, which is implicitly related to $\phi$ in Definition 2.2.2, we may consider $\phi$ with

$$
\|\phi\|_{C^{2}} \lesssim \delta^{-2}
$$

Before proving Lemma 2.2.3, we state a lemma which is central to its proof.
Lemma 2.2.4. For

$$
\Gamma=\{(\xi, \Psi(\xi)) ;|\xi|<4 \delta\}
$$

where $\Psi(0)=0$ and $\nabla \Psi(0)=0$, the map

$$
\zeta \mapsto \frac{d(\zeta, \Gamma)^{\alpha}}{|\eta-\Psi(\xi)|^{\alpha}}
$$

has a positive smooth extension to $\Gamma \cap B(0, \delta)$, provided $\delta$ is chosen sufficiently small. Furthermore, since the function is positive, the map

$$
\zeta \mapsto \frac{|\eta-\Psi(\xi)|^{\alpha}}{d(\zeta, \Gamma)^{\alpha}}
$$

is also smooth.
Let us see how Lemma 2.2.3 follows before proving this result.
Proof of Lemma 2.2.3. Let us first suppose that $m_{\Gamma, \alpha} \in \mathcal{M}_{p}$. Firstly, we require that, for all $\zeta \in \Gamma \cap \operatorname{supp} \phi_{\Gamma}, B(\zeta, \delta) \cap \Gamma$, after a rotation and translation, which we denote by $T$, is expressible as a segment about the origin of the graph $\{(\xi, \Psi(\xi)) ;|\xi|<4 \delta\}$ with $\xi \in \mathbb{R}^{n}, \Psi \in C_{c}^{\infty}\left(\mathbb{R}^{n} ; \mathbb{R}^{L}\right)$, $\Psi(0)=0, \nabla \Psi(0)=0$. We can do this since $\Gamma$ is a smooth embedded manifold. Now, for some $\zeta_{1} \in \Gamma \cap \operatorname{supp} \phi_{\Gamma}$, we set $m_{1}(\zeta)=\chi_{1}(\zeta) m_{\Gamma, \alpha}$, where $\chi_{1} \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$ is a cutoff function with supp $\chi_{1} \subset B\left(\zeta_{1}, 2 \delta\right)$. More specifically, if we denote $p_{\xi}$ and $p_{\eta}$ projection onto the $\xi$ and $\eta$ coordinates, respectively, we choose $\chi_{1}$ so that $\chi_{1}(T \zeta)=\phi\left(p_{\xi}(T \zeta)\right) \chi\left(p_{\eta}(T \zeta)-\Psi\left(p_{\xi}(T \zeta)\right)\right)$, with $\phi$ and $\chi$ as in Definition 2.2.2. If $T_{m_{\Gamma, \alpha}}$ is bounded on $L^{p}$, then $T_{m_{1}}$ is bounded on $L^{p}$, by Lemma 2.1.5 and Lemma 2.1.4. Furthermore, by Lemma 2.1.4, if we make the above change of coordinates corresponding to the graphical expression of $B\left(\zeta_{1}, 4 \delta\right) \cap \Gamma$, to the multiplier, $m_{1}$, its operator norm is preserved. The multiplier $m_{1}=m \cdot \chi_{1}$ is not yet amenable to simple calculations, due to the appearance of the factor $d(\zeta, \Gamma)^{\alpha}$. We can make further improvements using Lemma 2.1.5. According with our above discussion of Lemma 2.1.4, we henceforth suppose that $\Gamma$ is expressed as the graph of $\Psi$.

We note that, for $\zeta \notin \Gamma$

$$
d(\zeta, \Gamma)^{\alpha} \chi_{1}(\zeta) \phi_{\Gamma}(\zeta)=\frac{d(\zeta, \Gamma)^{\alpha}}{|\eta-\Psi(\xi)|^{\alpha}}|\eta-\Psi(\xi)|^{\alpha} \chi_{1}(\zeta)
$$

so that, if we can show that, in the support of $\chi_{1}$,

$$
\begin{equation*}
\zeta \mapsto \frac{d(\zeta, \Gamma)^{\alpha}}{|\eta-\Psi(\xi)|^{\alpha}} \tag{2.2.1}
\end{equation*}
$$

defines a smooth function, then this suffices to establish that $n(\zeta)=n(\xi, \eta)=|\eta-\Psi(\xi)|^{\alpha} \chi_{1}(\zeta)$, which is precisely the multiplier specified in Definition 2.2 .2 , is an $L^{p}$ multiplier. We know, from Lemma 2.2.4, that (2.2.1) is smooth. This concludes the first half of the proof.

It remains to show that $m_{\Psi, \alpha} \in \mathcal{M}_{p}$ for all $\Psi$ which locally graph $\Gamma$ implies that $m_{\Gamma, \alpha} \in \mathcal{M}_{p}$. So, let us suppose that $m_{\Psi, \alpha} \in \mathcal{M}_{p}$ for all $\Psi$ which locally graph $\Gamma$. We begin with a partition of unity $\left\{\chi_{j}\right\}_{j \in \mathcal{J}}$, where $\operatorname{supp} \chi_{j} \subset B\left(\zeta_{j}, \delta / 2\right)$ for some choice of $\zeta_{j} \in \Gamma \cap \operatorname{supp} \phi_{\Gamma}$. As previously, corresponding to each element of this partition is a linear map, $T_{j}$, and a graphing function $\Psi_{j}$. We find that

$$
\left\|m_{\Gamma, \alpha} \chi_{j}\right\|_{\mathcal{M}_{p}}=\left\|m_{\Gamma, \alpha}\left(T_{j}^{-1} \cdot\right) \chi_{j}\left(T_{j}^{-1} \cdot\right)\right\|_{\mathcal{M}_{p}}
$$

We now make use of Lemma 2.1.5. First, note the the above map, (2.2.1), and its reciprocal are smooth, by Lemma 2.2.4. We also see that

$$
\chi_{j}\left(T_{j}^{-1} \cdot\right)=\phi(\xi(\cdot)) \chi\left(\eta-\Psi_{j}(\xi(\cdot))\right) \chi_{j}\left(T_{j}^{-1} \cdot\right)
$$

Bringing this all together with Lemma 2.1.5, we find that

$$
\begin{gathered}
\left\|m_{\Gamma, \alpha}\right\|_{\mathcal{M}_{p}} \leq \sum_{j \in \mathcal{J}}\left\|m_{\Gamma, \alpha} \chi_{j}\right\|_{\mathcal{M}_{p}}=\sum_{j}\left\|m_{\Gamma, \alpha}\left(T_{j}^{-1} \cdot\right) \chi_{j}\left(T_{j}^{-1} \cdot\right)\right\|_{\mathcal{M}_{p}} \\
\lesssim \lesssim \sum_{j}\left\||\eta(\cdot)-\Psi(\xi(\cdot))|^{\alpha} \chi_{j}\left(T_{j}^{-1} \cdot\right)\right\|_{\mathcal{M}_{p}} \lesssim \sum_{j}\left\|m_{\Psi_{j}, \alpha}\right\|_{\mathcal{M}_{p}}
\end{gathered}
$$

proving our claim.
We did not previously specify our choice of $\phi$. However, we may choose $\phi$ with

$$
\begin{equation*}
\phi(\xi)=\phi_{0}\left(\delta^{-1} \xi\right)^{2} \tag{2.2.2}
\end{equation*}
$$

where $\phi_{0}$ is a compactly supported bump function such that $\operatorname{supp} \phi_{0} \subset B^{n}(0,1)$ and $\phi_{0}(\xi)=1$ for $|\phi| \leq \frac{1}{2}$. Thus we have that

$$
\|\phi\|_{C^{2}} \lesssim \delta^{-2}
$$

Now we turn to the proof of the critical lemma, that the map (2.2.1) is smooth.
Proof of Lemma 2.2.4. One can use the fundamental theorem of calculus to bound differences of the form to obtain bounds of the form $\left|f(\xi)-f\left(\xi_{0}^{\prime}\right)\right| \lesssim\|f\|_{C^{1}}\left|\xi_{0}-\xi_{0}^{\prime}\right|$. We do so throughout this proof.

We denote by $\tilde{B}$ the ball $B^{n}(0, \delta)$ and note that for $\zeta=(\xi, \eta) \in B, \xi \in \tilde{B}$. Throughout this proof $\epsilon>0$ is a suitable small constant. Provided we take $\delta$ sufficiently small, any choice of $\epsilon$ can be made. Note that, for $\zeta$ such that $\phi(\xi) \chi(\eta-\Psi(\xi)) \neq 0$, we have, by the fundamental theorem of calculus, that $|\eta| \leq \delta+|\Psi(\xi)| \leq \delta+C\|\Psi\|_{C^{1}(\operatorname{supp} \phi)} \delta \leq \epsilon$, provided we choose $\delta$ small enough.

Throughout this proof, we understand $\nabla \Psi(\xi)$ to be an $n \times L$ matrix whose $j$ th column is given by $\nabla \psi_{j}(\xi)$. It is a simple matter to verify the following bound. For any given $\epsilon>0$, provided we choose $\delta$ sufficiently small, for $\omega \in \tilde{B}$

$$
\begin{equation*}
\|\nabla \Psi(\omega)\|_{\mathbb{R}^{L} \rightarrow \mathbb{R}^{n}} \leq \epsilon \quad \text { and } \quad,\|\nabla \Psi(\omega)\|_{\mathbb{R}^{n} \rightarrow \mathbb{R}^{L}} \leq \epsilon, \tag{2.2.3}
\end{equation*}
$$

where the relevant linear operators are defined naturally by the left and right action of the matrix, respectively. More precisely, we have that

$$
\sup _{\xi \in \tilde{B}}\|\nabla \Psi(\xi)\|_{\mathbb{R}^{L} \rightarrow \mathbb{R}^{n}} \leq \epsilon
$$

and likewise for the other norm. We also have that

$$
\begin{equation*}
\|\Psi\|_{C^{2}(\tilde{B})} \lesssim 1 \tag{2.2.4}
\end{equation*}
$$

Finally, by making a sufficiently small choice of $\delta$,

$$
\begin{equation*}
|\Psi(\xi)| \leq|\Psi(0)|+C \delta\|\Psi\|_{C^{1}(\tilde{B})} \leq \epsilon \text { for } \xi \in B^{n}(0, \delta) \tag{2.2.5}
\end{equation*}
$$

Away from $\Gamma$, smoothness of the map

$$
\zeta \mapsto \frac{d(\zeta, \Gamma)^{\alpha}}{|\eta-\Psi(\xi)|^{\alpha}}
$$

is directly apparent since the function is the quotient of two smooth functions, where the denominator is non-zero. We must show it has a smooth extension to $\Gamma$. In fact, it suffices to prove that the function $S$, with

$$
S(\zeta)=\frac{d(\zeta, \Gamma)^{2}}{|\eta-\Psi(\xi)|^{2}},
$$

has a smooth extension to $\Gamma$ such that $S(\zeta)>0$. The desired result follows, since we then have that $S^{\frac{\alpha}{2}}>0$ is also smooth by the chain rule and likewise for $S^{-\frac{\alpha}{2}}$.

Recall the notation $\zeta=(\xi, \eta)$. Let us denote by $F$ the function $(\zeta, \omega) \mapsto d\left(\zeta,(\omega, \Psi(\omega))^{2}=\right.$ $|\xi-\omega|^{2}+|\eta-\Psi(\omega)|^{2}$. We first establish the existence of a unique map $\xi_{0}: B \rightarrow \mathbb{R}^{n}$ such that for each $\zeta \in B d(\zeta, \Gamma)=d\left(\zeta,\left(\xi_{0}(\zeta), \Psi\left(\xi_{0}(\zeta)\right)\right)\right)$; the function $\xi_{0}$ characterises the nearest point on $\Gamma$ to a given $\zeta$. Since $\Gamma$ is smooth, we see that those points $\omega$ for which $F(\zeta, \omega)=d(\zeta, \Gamma)^{2}$ are stationary points of $F(\zeta, \cdot)$, as they give local minima. If $\xi_{0}$ is a stationary point of $F(\zeta, \cdot)$, then

$$
\begin{equation*}
\nabla_{\omega} F(\zeta, \cdot)\left(\xi_{0}\right)=2\left(\xi_{0}-\xi\right)^{t}+2 \nabla \Psi\left(\xi_{0}\right)\left(\eta-\Psi\left(\xi_{0}\right)\right)^{t}=0 \tag{2.2.6}
\end{equation*}
$$

In order to show that $\xi_{0}$ is uniquely defined, suppose that $\xi_{0}^{\prime}$ is also a stationary point of $F(\zeta, \cdot)$. Then

$$
\left[\left(\xi_{0}-\xi\right)^{t}+\nabla \Psi\left(\xi_{0}\right)\left(\eta-\Psi\left(\xi_{0}\right)\right)^{t}\right]-\left[\left(\xi_{0}^{\prime}-\xi\right)^{t}+\nabla \Psi\left(\xi_{0}^{\prime}\right)\left(\eta-\Psi\left(\xi_{0}^{\prime}\right)\right)^{t}\right]=0
$$

In particular, using (2.2.3), (2.2.4), (2.2.5), and the fundamental theorem of calculus, we may find that

$$
\begin{gather*}
\left|\xi_{0}-\xi_{0}^{\prime}\right| \leq\left|\left(\nabla \Psi\left(\xi_{0}\right)-\nabla \Psi\left(\xi_{0}^{\prime}\right)\right) \eta^{t}\right|+\left|\nabla \Psi\left(\xi_{0}^{\prime}\right) \Psi\left(\xi_{0}^{\prime}\right)^{t}-\nabla \Psi\left(\xi_{0}\right) \Psi\left(\xi_{0}\right)^{t}\right| \\
=\left|\left(\nabla \Psi\left(\xi_{0}\right)-\nabla \Psi\left(\xi_{0}^{\prime}\right)\right) \eta^{t}\right| \\
+\left|\nabla \Psi\left(\xi_{0}^{\prime}\right)\left(\left(\Psi\left(\xi_{0}^{\prime}\right)-\Psi\left(\xi_{0}\right)\right)+\Psi\left(\xi_{0}\right)\right)^{t}-\left(\left(\nabla \Psi\left(\xi_{0}\right)-\nabla \Psi\left(\xi_{0}^{\prime}\right)\right)+\nabla \Psi\left(\xi_{0}^{\prime}\right)\right) \Psi\left(\xi_{0}\right)^{t}\right| \\
\leq\left|\left(\nabla \Psi\left(\xi_{0}\right)-\nabla \Psi\left(\xi_{0}^{\prime}\right)\right) \eta^{t}\right|+\left|\nabla \Psi\left(\xi_{0}^{\prime}\right)\left(\Psi\left(\xi_{0}^{\prime}\right)-\Psi\left(\xi_{0}\right)\right)^{t}\right|+\left|\left(\nabla \Psi\left(\xi_{0}^{\prime}\right)-\nabla \Psi\left(\xi_{0}\right)\right) \Psi\left(\xi_{0}\right)^{t}\right| \\
\lesssim \epsilon\|\Psi\|_{C^{2}(\tilde{B})}\left|\xi_{0}-\xi_{0}^{\prime}\right|, \\
\lesssim \epsilon\left|\xi_{0}-\xi_{0}^{\prime}\right| . \tag{2.2.7}
\end{gather*}
$$

Recall that $\epsilon$ parameter, taken from the bounds (2.2.3) and (2.2.4), can be made arbitrarily small by our choice of $\delta$. Thus, for $\delta$ sufficiently small, the inequality (2.2.7) can only hold for $\xi_{0}-\xi_{0}^{\prime}=0$. That is, there exists a unique $\xi_{0}=\xi_{0}(\zeta)$ such that $d(\zeta, \Gamma)=d\left(\zeta,\left(\xi_{0}, \Psi\left(\xi_{0}\right)\right)\right.$.

We now establish that the map $\zeta \mapsto \xi_{0}(\zeta)$ is smooth. This follows by the implicit function theorem. Let us denote by $G$ the vector valued function $\frac{1}{2} \nabla_{\omega} F$ :

$$
G(\zeta, \omega)=(\xi-\omega)^{t}+\nabla \Psi(\omega)(\eta-\Psi(\omega))^{t} .
$$

Because $\xi_{0}$ is a stationary point of $F(\zeta, \cdot)$, the function $\xi_{0}(\zeta)$ is given implicitly by the equation $G\left(\zeta, \xi_{0}\right)=0$. It can be seen that $G$ is smooth, and

$$
D_{\omega} G(\zeta, \cdot)(\omega)=-I-D_{\omega}\left(\nabla \Psi \Psi^{t}\right)(\omega)+D_{\omega}\left(\nabla \Psi \eta^{t}\right)(\omega) .
$$

We now use the fact that $|\eta|<\epsilon$. As a consequence of the control on $\Psi$, (2.2.3), (2.2.4), and (2.2.5), it can be seen that, for $\omega \in \tilde{B}$

$$
\left\|D_{\omega} G(\zeta, \cdot)(\omega)\right\|_{\mathbb{R}^{n} \rightarrow \mathbb{R}^{n}} \geq 1-\left\|D_{\omega}\left(\nabla \Psi \Psi^{t}\right)(\omega)\right\|_{\mathbb{R}^{n} \rightarrow \mathbb{R}^{n}}-\epsilon C\|\Psi\|_{C^{2}(\tilde{B})} \geq \frac{1}{2}
$$

provided $\delta$ is chosen sufficiently small. Therefore, by the implicit function theorem $\xi_{0}$ is a smooth function of $\zeta$, since $\nabla \Psi(0)=0$ and $\Psi(0)=0$, and, for $\omega \in \tilde{B}$,

$$
\left\|D_{\omega} G(\zeta, \cdot)(\omega)\right\|_{\mathbb{R}^{n} \rightarrow \mathbb{R}^{n}} \geq \frac{1}{2}
$$

Now we complete the proof that $S$ has a smooth extension to $\Gamma$, where

$$
S(\zeta)=\frac{d(\zeta, \Gamma)^{2}}{|\eta-\Psi(\xi)|^{2}}=\frac{\left|\xi-\xi_{0}(\zeta)\right|^{2}+\left|\eta-\Psi\left(\xi_{0}(\zeta)\right)\right|^{2}}{|\eta-\Psi(\xi)|^{2}}
$$

We use the implicit formula defining $\xi_{0}$ to show this. By definition,

$$
F\left(\zeta, \xi_{0}(\zeta)\right)=\left|\xi-\xi_{0}(\zeta)\right|^{2}+\left|\eta-\Psi\left(\xi_{0}(\zeta)\right)\right|^{2}
$$

and the implicit formula, (2.2.6), tells us that

$$
2\left(\xi-\xi_{0}(\zeta)^{t}\right)=2 \nabla \Psi\left(\xi_{0}(\zeta)\right)\left(\eta-\Psi\left(\xi_{0}(\zeta)\right)^{t}\right.
$$

Thus, if we can find a smooth function $G_{0}>0$ such that

$$
\left|\eta-\Psi\left(\xi_{0}(\zeta)\right)\right|=G_{0}(\zeta)|\eta-\Psi(\xi)|
$$

then, using the implicit formula (2.2.6),

$$
F\left(\zeta, \xi_{0}(\zeta)\right)=\left(1+\left|\nabla \Psi\left(\xi_{0}(\zeta)\right)\right|^{2}\right) G_{0}(\zeta)^{2}|\eta-\Psi(\xi)|^{2}
$$

and we can see that $\zeta \mapsto S(\zeta)=F\left(\zeta, \xi_{0}(\zeta)\right) /|\eta-\Psi(\xi)|^{2}$ is smooth and bounded away from 0 , provided $\left|\nabla \Psi\left(\xi_{0}(\zeta)\right)\right|<1$, which is possible if we choose $\delta$ small enough.

To show that $\left|\eta-\Psi\left(\xi_{0}(\zeta)\right)\right|=G_{0}(\zeta)|\eta-\Psi(\xi)|$, we expand the left hand term using the implicit formula for $\xi_{0}$, (2.2.6). We can replace instances of $\left(\xi-\xi_{0}(\zeta)\right)$ with $\left(\eta-\Psi\left(\xi_{0}(\zeta)\right) \nabla \Psi\left(\xi_{0}(\zeta)\right)^{t}\right.$. We find that

$$
\begin{gathered}
H(\zeta):=\left(\eta-\Psi\left(\xi_{0}(\zeta)\right)\right) \\
=[\eta-\Psi(\xi)]+\left[\Psi(\xi)-\Psi\left(\xi_{0}(\zeta)\right)\right] \\
=[\eta-\Psi(\xi)]+\left[\int_{0}^{1}\left(\xi-\xi_{0}(\zeta)\right) \nabla \Psi\left(\xi_{0}(\zeta)+s\left(\xi-\xi_{0}(\zeta)\right)\right) d s\right] \\
=[\eta-\Psi(\xi)]+\left[\int_{0}^{1}\left(\eta-\Psi\left(\xi_{0}(\zeta)\right) \nabla \Psi\left(\xi_{0}(\zeta)\right)^{t} \nabla \Psi\left(\xi_{0}(\zeta)+s\left(\xi-\xi_{0}(\zeta)\right)\right) d s\right]\right. \\
=[\eta-\Psi(\xi)]+H(\zeta)\left[\int_{0}^{1} \nabla \Psi\left(\xi_{0}(\zeta)\right)^{t} \nabla \Psi\left(\xi_{0}(\zeta)+s\left(\xi-\xi_{0}(\zeta)\right)\right) d s\right] .
\end{gathered}
$$

We can rearrange the equation to see that

$$
H(\zeta)\left(I-\int_{0}^{1} \nabla \Psi\left(\xi_{0}(\zeta)\right)^{t} \nabla \Psi\left(\xi_{0}(\zeta)+s\left(\xi-\xi_{0}(\zeta)\right)\right) d s\right)=[\eta-\Psi(\xi)]
$$

Note that, for $\xi \in \tilde{B}$,

$$
\left\|\int_{0}^{1} \nabla \Psi\left(\xi_{0}(\zeta)\right)^{t} \nabla \Psi\left(\xi_{0}(\zeta)+s\left(\xi-\xi_{0}(\zeta)\right)\right) d s\right\| \lesssim \delta \ll 1
$$

so that $\left(I-\int_{0}^{1} \nabla \Psi\left(\xi_{0}(\zeta)\right)^{t} \nabla \Psi\left(\xi_{0}(\zeta)+s\left(\xi-\xi_{0}(\zeta)\right)\right) d s\right)$ is invertible with a smooth inverse. We then see that

$$
\begin{equation*}
H(\zeta)=(\eta-\Psi(\xi))\left(I-\int_{0}^{1} \nabla \Psi\left(\xi_{0}(\zeta)\right)^{t} \nabla \Psi\left(\xi_{0}(\zeta)+s\left(\xi-\xi_{0}(\zeta)\right)\right) d s\right)^{-1} \tag{2.2.8}
\end{equation*}
$$

We sought to find a smooth function $G$ with $|H(\zeta)|=\left|\eta-\Psi\left(\xi_{0}\right)\right|=G(\zeta)|\eta-\Psi(\xi)|$. From the above, we know that

$$
G(\zeta)=\left|\frac{(\eta-\Psi(\xi))}{|\eta-\Psi(\xi)|}\left(I-\int_{0}^{1} \nabla \Psi\left(\xi_{0}(\zeta)\right)^{t} \nabla \Psi\left(\xi_{0}(\zeta)+s\left(\xi-\xi_{0}(\zeta)\right)\right) d s\right)^{-1}\right|
$$

which is smooth for $\zeta \in B$ because $H$ is smooth, $|\cdot|$ is smooth away from 0 , and

$$
\left|\frac{(\eta-\Psi(\xi))}{|\eta-\Psi(\xi)|}\right|=1
$$

### 2.3 Bochner-Riesz kernels

Recall the reductions of Section 2.2. We consider the graphical Bochner-Riesz operators with multipliers given by

$$
m_{\Psi, \alpha}(\zeta)=m_{\Psi, \alpha}(\xi, \eta)=\phi(\xi)|\eta-\Psi(\xi)|^{\alpha} \chi(\eta-\Psi(\xi)) .
$$

Working with this multiplier, whose expression is more amenable to direct calculation, we can now express the corresponding convolution kernel. We then establish a relationship between the kernels for surfaces in the class $\mathcal{S}$ and surfaces in the class $\mathcal{S}_{0}$, which were defined at Definitions 1.0.1 and 1.0.2. This will reduce a large part of our analysis to the class $\mathcal{S}_{0}$.

Multiplier operators are expressible as convolution operators. We turn our sights to the convolution kernel of $T_{m}, K_{\Psi, \alpha}=\check{m}_{\Psi, \alpha}$, as given by Definition 2.2.2. Observe that

$$
\begin{gather*}
K_{\Psi, \alpha}(x, y, z)=\int_{\mathbb{R}^{D}} \int_{\mathbb{R}^{n}} e^{2 \pi i(x \cdot \xi+(y, z) \cdot \eta)} m_{\Psi, \alpha}(\xi, \eta) d \xi d \eta \\
=\int_{\mathbb{R}^{\boldsymbol{L}}} \int_{\mathbb{R}^{n}} e^{2 \pi i(x \cdot \xi+(y, z) \cdot \eta)} \phi(\xi)|\eta-\Psi(\xi)|^{\alpha} \chi(\eta-\Psi(\xi)) d \xi d \eta \\
=\int_{\mathbb{R}^{\text {L }}} \int_{\mathbb{R}^{n}} e^{2 \pi i(x \cdot \xi+(y, z) \cdot(\eta+\Psi(\xi)))} \phi(\xi)|\eta|^{\alpha} \chi(\eta) d \xi d \eta \\
=A_{\alpha}(y, z)\left(\int_{\mathbb{R}^{n}} e^{2 \pi i(x \cdot \xi+(y, z) \cdot \Psi(\xi))} \phi(\xi) d \xi\right)  \tag{2.3.1}\\
=A_{\alpha}(y, z) k(x, y, z),
\end{gather*}
$$

where

$$
k(x, y, z)=\int_{\mathbb{R}^{n}} e^{2 \pi i(x \cdot \xi+(y, z) \cdot \Psi(\xi))} \phi(\xi) d \xi
$$

and

$$
A_{\alpha}(y, z)=\int_{\mathbb{R}^{L}} e^{2 \pi i(y, z) \cdot \eta}|\eta|^{\alpha} \chi(\eta) d \eta
$$

Note that if $\Psi(\xi)$ is a graphing function for one of the surfaces in $\mathcal{S}_{0}$, its final $L^{\prime}$ coordinates
are 0 . In this case, $k(x, y, z)$ is simply a function of $(x, y)$. For surfaces in $\mathcal{S}$ such that $\tilde{L}<L$, we would prefer to be able to work in this framework and, later in this section, we outline how a linear change of variables puts us in this setting. Where we are considering the class $\mathcal{S}_{0}$, $k(x, y, z)$ is independent of $z$, so we will instead write

$$
k(x, y)=\int_{\mathbb{R}^{n}} e^{2 \pi i\left(x \cdot \xi+y \cdot \Psi_{\tilde{L}}(\xi)\right)} \phi(\xi) d \xi
$$

Obtaining pointwise control $k$ is a delicate matter. Indeed, in establishing that $K_{\Psi, \alpha} \in L^{p}$ for specified $p$, we do not use pointwise control, instead adapting the method from [ACK08] to control the measure of regions for which the kernel has a specified size. Obtaining pointwise control on $A_{\alpha}$ is more routine, and we here state and prove bounds that we will use through this document.
Lemma 2.3.1. For large $|(y, z)|, \alpha \neq 0$ with $\alpha \notin 2 \mathbb{N}$,

$$
\begin{equation*}
\left|A_{\alpha}((y, z))\right| \sim|(y, z)|^{-L-\alpha} . \tag{2.3.2}
\end{equation*}
$$

For large $|(y, z)|$, and all $\alpha>0$,

$$
\begin{equation*}
\left|A_{\alpha}((y, z))\right| \lesssim|(y, z)|^{-L-\alpha} \tag{2.3.3}
\end{equation*}
$$

Proof. For ease of notation, we suppose that $L^{\prime}=0$ so that we seek bounds on

$$
A(y):=\int e^{2 \pi i y \cdot \eta}|\eta|^{\alpha} \chi(\eta) d \eta
$$

One can easily see that the analysis also works in the $L^{\prime}>0$ case. Before we proceed, let us recall that for the distribution, $\Lambda_{\alpha}$, with

$$
\Lambda_{\alpha}(\varphi)=\int_{\mathbb{R}^{L}}|\eta|^{\alpha} \varphi(\eta) d \eta
$$

we have that $\widehat{\Lambda_{\alpha}}$, regarded as a tempered distribution, agrees with the function $c_{\alpha}|y|^{-L-\alpha}$ away from the origin, i.e.

$$
\widehat{\Lambda_{\alpha}}(\varphi)=\int_{\mathbb{R}^{L}} \varphi(y) c_{\alpha} \frac{1}{|y|^{L+\alpha}} d y
$$

for $\varphi$ with $0 \notin \operatorname{supp} \varphi$, with $c_{\alpha} \neq 0$ if $\alpha \notin 2 \mathbb{N}$; see p. 363 of [IMG64]. To make use of this fact, we introduce a cutoff function $a_{0}$, with $\operatorname{supp} a_{0} \subset B(0, \epsilon)$ with $a_{0}(y)=1$ for $y \in B(0, \epsilon / 2)$. We consider the tempered distribution $\chi \Lambda_{\alpha}$, with

$$
\chi \Lambda_{\alpha}(f)=\int|\eta|^{\alpha} \chi(\eta) f(\eta) d \eta
$$

We see that $\chi \Lambda_{\alpha}$ can be regarded as the function $\eta \mapsto|\eta|^{\alpha} \chi(\eta)$. Therefore, if we regard $\widehat{\chi \Lambda_{\alpha}}$ as a function, we see that

$$
A_{\alpha}=\widehat{\chi \Lambda_{\alpha}} .
$$

Now we write

$$
\begin{gather*}
\widehat{\chi \Lambda_{\alpha}} \\
=\hat{\chi} * \hat{\Lambda}_{\alpha} \\
=\hat{\chi} *\left(\hat{\Lambda}_{\alpha}\left(a_{0}+\left(1-a_{0}\right)\right)\right) \\
=\hat{\chi} *\left(a_{0} \hat{\Lambda}_{\alpha}\right)+\hat{\chi} *\left(\left(1-a_{0}\right) \hat{\Lambda}_{\alpha}\right) \\
=E_{1}+\hat{\chi} *\left(\left(1-a_{0}\right) \hat{\Lambda}_{\alpha}\right), \tag{2.3.4}
\end{gather*}
$$

where

$$
\begin{equation*}
E_{1}=\hat{\chi} *\left(a_{0} \hat{\Lambda}_{\alpha}\right) \tag{2.3.5}
\end{equation*}
$$

We later establish $E_{1}$ as an error term. The term $\left(1-a_{0}\right) \hat{\Lambda}_{\alpha}$ from (2.3.4) is supported away from the origin, and we know know that $\hat{\Lambda}_{\alpha}$ coincides with the function $c_{\alpha}|y|^{-L-\alpha}$ away from the origin. Thus, we see that

$$
\begin{gathered}
\hat{\chi} *\left(\left(1-a_{0}\right) \hat{\Lambda}_{\alpha}\right)(y) \\
=\int_{\mathbb{R}^{L}} \hat{\chi}(y-w)\left(1-a_{0}(w)\right) c_{\alpha} \frac{1}{|w|^{L+\alpha}} d w \\
=\int_{|y-w| \leq \epsilon|y|} \hat{\chi}(y-w) c_{\alpha} \frac{1}{|w|^{L+\alpha}} d w+E_{2}(y),
\end{gathered}
$$

where

$$
\begin{equation*}
E_{2}(y)=\int_{|y-w|>\epsilon|y|} \hat{\chi}(y-w)\left(1-a_{0}(w)\right) c_{\alpha} \frac{1}{|w|^{L+\alpha}} d w . \tag{2.3.6}
\end{equation*}
$$

We later establish $E_{2}$ as an error term. A further splitting shows that

$$
\begin{gathered}
\int_{|y-w| \leq \epsilon|y|} \hat{\chi}(y-w) c_{\alpha} \frac{1}{|w|^{L+\alpha}} d w \\
=c_{\alpha} \frac{1}{|y|^{L+\alpha}} \int_{|y-w| \leq \epsilon|y|} \hat{\chi}(y-w) d w \\
+E_{3}(y),
\end{gathered}
$$

where

$$
\begin{equation*}
E_{3}(y)=c_{\alpha} \int_{|y-w| \leq \epsilon|y|} \hat{\chi}(y-w)\left(\frac{1}{|w|^{L+\alpha}}-\frac{1}{|y|^{L+\alpha}}\right) d w . \tag{2.3.7}
\end{equation*}
$$

We will show that $E_{3}$ is an error term. Finally, we write

$$
\begin{gathered}
c_{\alpha} \frac{1}{|y|^{L+\alpha}} \int_{|y-w| \leq \epsilon|y|} \hat{\chi}(y-w) d w \\
=c_{\alpha} \frac{1}{|y|^{L+\alpha}} \int \hat{\chi}(y-w) d w \\
+E_{4}(y),
\end{gathered}
$$

where

$$
\begin{equation*}
E_{4}(y)=c_{\alpha} \frac{1}{|y|^{L+\alpha}} \int_{|y-w|>\epsilon|y|} \hat{\chi}(y-w) d w . \tag{2.3.8}
\end{equation*}
$$

Note that

$$
\begin{gathered}
c_{\alpha} \frac{1}{|y|^{L+\alpha}} \int \hat{\chi}(y-w) d w \\
\quad=c_{\alpha} \frac{1}{|y|^{L+\alpha}} \chi(0) .
\end{gathered}
$$

The above presents a decomposition

$$
A_{\alpha}(y)=c_{\alpha} \frac{1}{|y|^{L+\alpha}}+E_{1}(y)+E_{2}(y)+E_{3}(y)+E_{4}(y) .
$$

To complete the proof, we show that, for $1 \leq j \leq 4$,

$$
\left|E_{j}(y)\right| \leq \frac{\left|c_{\alpha}\right|}{5} \frac{1}{|y|^{L+\alpha}}
$$

Let us first consider $E_{3}$, (2.3.7). We recall that

$$
E_{3}(y)=c_{\alpha} \int_{|y-w| \leq \epsilon|y|} \hat{\chi}(y-w)\left(\frac{1}{|w|^{L+\alpha}}-\frac{1}{|y|^{L+\alpha}}\right) d w
$$

Let us set $g(r)=r^{-L-\alpha}$. We can see that

$$
\begin{gathered}
\frac{1}{|w|^{L+\alpha}}-\frac{1}{|y|^{L+\alpha}} \\
=\int_{0}^{1}(|w|-|y|) g^{\prime}(|y|+r(|w|-|y|)) d r .
\end{gathered}
$$

As such, we find, using the Schwartz decay of $\hat{\chi}$, that

$$
\begin{gathered}
\left|E_{3}(y)\right| \lesssim m \int_{|y-w| \leq \epsilon|y|}(1+|y-w|)^{-m} \int_{0}^{1}| | w|-|y||| | y|+r(|w|-|y|)|^{-L-\alpha-1} d r d w \\
\lesssim \epsilon|y| \int_{|y-w| \leq \epsilon|y|} \frac{1}{|y|^{L+\alpha+1}}(1+|y-w|)^{-m} d w \\
\lesssim \epsilon \frac{1}{|y|^{L+\alpha}},
\end{gathered}
$$

so that

$$
\left|E_{3}(y)\right| \leq \frac{\left|c_{\alpha}\right|}{5|y|^{L+\alpha}}
$$

provided we choose $\epsilon$ small enough.
Let us now consider $E_{1}$, here we must use the distributional definition of convolution and the Fourier transform. We set $\hat{\chi}_{y}=\hat{\chi}(y-\cdot)=\check{\chi}(\cdot-y)$. Considering $a_{0} \hat{\Lambda}_{\alpha}$ as a distribution, we find that we can express (2.3.5) as

$$
\begin{gathered}
E_{1}(y)=\hat{\chi} *\left(a_{0} \hat{\Lambda}_{\alpha}\right) \\
=\left(a_{0} \hat{\Lambda}_{\alpha}\right)\left(\hat{\chi}_{y}\right) \\
=\hat{\Lambda}_{\alpha}\left(a_{0} \hat{\chi}_{y}\right) \\
=\Lambda_{\alpha}\left(\mathcal{F}\left(a_{0} \check{\chi}(\cdot-y)\right)\right) \\
=\int|w|^{\alpha} \int \hat{a}_{0}(w-z) e^{2 \pi i z \cdot y} \chi(z) d z d w .
\end{gathered}
$$

We can see that

$$
\frac{1}{(2 \pi i|y|)^{2}} \Delta_{z} e^{2 \pi i z \cdot y}=e^{2 \pi i z \cdot y}
$$

We use this to integrate by parts and find that

$$
\begin{gathered}
\int \hat{a}_{0}(w-z) e^{2 \pi i z \cdot y} \chi(z) d z \\
=\frac{1}{(2 \pi i|y|)^{2 m}} \int \hat{a}_{0}(w-z) \Delta_{z}^{m}\left(e^{2 \pi i z \cdot y}\right) \chi(z) d z \\
=\frac{1}{(2 \pi i|y|)^{2 m}} \int \sum_{\beta \in \mathcal{I}}(-1)^{\left|\beta_{1}\right|} \partial_{z}^{\beta_{1}} \hat{a}_{0}(w-z) e^{2 \pi i z \cdot y} \partial_{z}^{\beta_{2}} \chi(z) d z,
\end{gathered}
$$

where the finite sum is taken over a finite set of multi-indices $\mathcal{I}$ with elements $\left(\beta_{1}, \beta_{2}\right) \in \mathbb{N}^{L} \times \mathbb{N}^{L}$.

One can see that, with $f_{1}:=\partial_{z}^{\beta_{1}} \hat{a}_{0} \in \mathscr{S}$ and $f_{2}:=\partial_{z}^{\beta_{2}} \chi \in \mathscr{S}$ so that $\operatorname{supp} f_{2} \subset \operatorname{supp} \chi$, we have

$$
\begin{gathered}
\int\left|f_{1}(w-z) f_{2}(z)\right| d z \\
\lesssim \int_{\operatorname{supp} \chi}(1+|w-z|)^{-L-1-\lceil\alpha\rceil} d z \\
\lesssim(1+|w|)^{-L-1-\lceil\alpha\rceil}
\end{gathered}
$$

Thus, we find that

$$
\begin{gathered}
\left|E_{1}(y)\right| \\
=\int|w|^{\alpha}\left|\int \hat{a}_{0}(w-z) e^{2 \pi i z \cdot y} \chi(z) d z\right| d w \\
\lesssim m \frac{1}{|y|^{2 m}} \int|w|^{\alpha}(1+|w|)^{-L-1-\lceil\alpha\rceil} d w \\
\ll \frac{1}{|y|^{L+\alpha}},
\end{gathered}
$$

provided we choose $2 m>L+\alpha$ and $|y| \gg 1$ sufficiently large. In particular, we may find that

$$
\left|E_{1}(y)\right| \leq \frac{\left|c_{\alpha}\right|}{5|y|^{L+\alpha}}
$$

Let us now consider $E_{2}$, (2.3.6). We have that

$$
\begin{aligned}
\left|E_{2}(y)\right|= & \left|\int_{|y-w|>\epsilon|y|} \hat{\chi}(y-w)\left(1-a_{0}(w)\right) c_{\alpha} \frac{1}{|w|^{L+\alpha}} d w\right| \\
& \left.\lesssim m\left|\int_{|y-w|>\epsilon|y|}\right| y-\left.w\right|^{-m} \frac{1}{|w|^{L+\alpha}} d w \right\rvert\, \\
& \lesssim\left|\int_{|y-w|>\epsilon|y|}(\epsilon|y|)^{-m} \frac{1}{|w|^{L+\alpha}} d w\right| \\
& \lesssim(\epsilon|y|)^{-m} \\
& \ll|y|^{-L-\alpha}
\end{aligned}
$$

so that

$$
\left|E_{2}(y)\right| \leq \frac{\left|c_{\alpha}\right|}{5|y|^{L+\alpha}}
$$

provided we choose $m>L+\alpha$ and $y$ sufficiently large, dependent on $\epsilon$.
Let us now consider $E_{4}$, (2.3.8). We see that, by choosing $m \geq L+1$,

$$
\begin{gathered}
\lesssim_{m} \frac{1}{|y|^{L+\alpha}} \int_{|y-w|>\epsilon|y|}(1+|y-w|)^{-m} d w \\
\lesssim_{\epsilon} \frac{1}{|y|^{L+\alpha+1}}
\end{gathered}
$$

In particular, we have that

$$
\left|E_{4}(y)\right| \leq \frac{\left|c_{\alpha}\right|}{5|y|^{L+\alpha}}
$$

provided we choose $y$ large enough depending on $\epsilon$.
This completes the proof.

The two main aspects of our analysis in Part I concern whether $m_{\Psi, \alpha} \in \mathcal{M}_{p}$ and whether $\check{m}_{\Psi, \alpha} \in L^{p}$. In particular, we consider this question for surfaces in the class $\mathcal{S}$ (Definition 1.0.2). However, in practice, it is useful to restrict our attention to the class $\mathcal{S}_{0}$ and we will use the following lemma to do so.

Lemma 2.3.2. For functions $\Psi$ graphing $\Gamma \in \mathcal{S}$, i.e. satisfying the spanning assumption (1.0.7), there exists an invertible matrix $R$ and a function $\widetilde{\Psi}=\Psi R=\left(\tilde{\psi}_{1}, \ldots, \tilde{\psi}_{\tilde{L}}, 0, \ldots, 0\right)$ which graphs $\widetilde{\Gamma} \in \mathcal{S}_{0}$.

Proof. Recall that $d_{\tilde{L}}$ was defined to be the smallest $M$ such that

$$
\tilde{L}=\operatorname{dim} \operatorname{span}\left\{\psi_{j}^{M} ; j=1, \ldots, L\right\}
$$

holds. Here $\psi_{j}^{M}(t):=\sum_{l=0}^{M} \frac{\psi_{j}^{(l)}(0)}{l!} t^{l}$ is a Taylor approximation of $\psi_{j}$. Consider the $d_{\tilde{L}} \times L$ matrix $M_{\Psi}$ whose columns are given by the vectors

$$
\left(\psi_{j}^{\prime}(0), \ldots, \psi_{j}^{\left(d_{\tilde{L}}\right)}(0)\right)^{t}
$$

for $1 \leq j \leq L$. Using the spanning assumption $\tilde{L}=\operatorname{dim} \operatorname{span}\left\{\psi_{1}^{d_{\tilde{L}}}, \ldots, \psi_{L}^{d_{\tilde{L}}}\right\}$, from (1.0.7), we find the following. Applying a sequence of elementary column elimination operations, we see that there exists an $L \times L$ invertible matrix $R$, with

$$
M_{\Psi} R=M_{\tilde{\Psi}}
$$

where $M_{\tilde{\Psi}}$ is in reduced column echelon form. Furthermore, $M_{\tilde{\Psi}}$ is the matrix corresponding to function $\tilde{\Psi}=\left(\psi_{1}, \ldots, \psi_{L}\right) R=\left(\tilde{\psi}_{1}, \ldots, \tilde{\psi}_{L}\right)$, i.e. the columns of $M_{\tilde{\Psi}}$ are given by

$$
\left(\tilde{\psi}_{j}^{\prime}(0), \ldots, \tilde{\psi}_{j}^{\left(d_{\tilde{L}}\right)}(0)\right)^{t}
$$

We see that the functions $\tilde{\psi}_{\tilde{L}+1}, \tilde{\psi}_{\tilde{L}+2}, \ldots, \tilde{\psi}_{L}$ are 0 , which is again a consequence of the spanning assumption 1.0.7. For the remaining $\tilde{\psi}_{j}$, since the corresponding matrix is in reduced column echelon form with rank $\tilde{L}$, we have that there exist $d_{1}<d_{2}<\ldots<d_{\tilde{L}}$ such that $\tilde{\psi}_{j}^{\left(d_{j}\right)}(0)=1$, for $0 \leq l \leq d_{\tilde{L}}$ with $l<d_{j}$, and, for $j^{\prime}>j, \tilde{\psi}_{j}^{\left(d_{j^{\prime}}\right)}(0)=0$. It is then a consequence of Taylor's theorem that $\tilde{\Psi}$ describes an element $\widetilde{\Gamma} \in \mathcal{S}_{0}$.

## Chapter 3

## Fourier restriction

It is well known that the Bochner-Riesz conjecture is connected to another fundamental area of Euclidean harmonic analysis: the Fourier restriction problem. In this context, the problem concerns the precise $L^{p}\left(\mathbb{R}^{N}\right) \rightarrow L^{q}\left(\Gamma, \sigma_{\Gamma}\right)$ mapping properties of the restriction operator $R f=$ $\left.\hat{f}\right|_{\Gamma}$, where $\sigma_{\Gamma}$ denotes surface measure on $\Gamma$. Even in the original setting of $\Gamma=S^{N-1}$, as proposed by Stein in the mid 1960's, the Fourier restriction problem is unresolved for $N \geq 3$. In this chapter, we introduce the Fourier restriction problem and its relation to the BochnerRiesz conjecture. In particular, in Section 3.3, we show that sharp Bochner-Riesz estimates follow from restriction estimates.

Progress with the Bochner-Riesz conjecture has historically paralleled progress with the restriction conjecture. Tao established that the Bochner-Riesz conjecture implies the restriction conjecture on the sphere, [Tao99]; the table contained therein also outlines some of the parallel progress in these two areas.

It is well known that sharp $L^{p}$ estimates for Bochner-Riesz multiplier operators, $T_{m_{\Psi, \alpha}}$, follow from $L^{2}$ Fourier restriction estimates for $\Gamma$;

$$
\begin{equation*}
\left(\int|\hat{f}(\zeta)|^{2} \phi_{\Gamma}(\zeta) d \sigma_{\Gamma}\right)^{\frac{1}{2}} \leq C\|f\|_{L^{q}\left(\mathbb{R}^{N}\right)} \tag{3.0.1}
\end{equation*}
$$

The implication that restriction implies sharp Bochner-Riesz estimates and surrounding ideas date back to Fefferman's thesis, where the analysis is for the sphere $\Gamma=S^{N-1}$; see [Fef70].

In practice, we will work with local restriction estimates at small scales. We use a function $\Psi$ which, after a rotation and translation graphs a small section of $\Gamma$ chosen such that $\Psi(0)=0$ and $\nabla \Psi(0)=0$. The relavant local restriction estimate is

$$
\begin{equation*}
\left(\int|\hat{f}(\xi, \Psi(\xi))|^{2} \phi(\xi) d \xi\right)^{\frac{1}{2}} \leq C\|f\|_{L^{q}\left(\mathbb{R}^{N}\right)} \tag{3.0.2}
\end{equation*}
$$

Recall that we defined $m_{\Gamma, \alpha}$ only for $\alpha>0$. For varieties $\Gamma$ of arbitrary codimension we have the following result of G. Mockenhoupt, [Moc90].

Theorem 3.0.1. Suppose that $\Gamma \subset \mathbb{R}^{N}$ is a smooth surface and the restriction inequality (3.0.1) holds. Then, for $1<p \leq q$ or $q^{\prime} \leq p<\infty$, the Bochner-Riesz multiplier $m_{\Gamma, \alpha}$ defined in (1.0.2) with $\alpha>0$ defines a multiplier operator $T_{m_{\Gamma, \alpha}}$ which is bounded on $L^{p}$ for

$$
\alpha>(n+L)\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{L}{2} .
$$

We extend this result to the case of smooth surfaces $\Gamma \subset \mathbb{R}^{N}$ which are contained in a proper affine subspace of $\mathbb{R}^{N}$. In this case, there may only be a restriction estimate in a suitable subspace. Suppose that $\Gamma \subset P$ for some affine subspace $P$ of dimension $\tilde{N}$. We
consider $\widetilde{\Gamma} \subset \mathbb{R}^{\tilde{N}}$ to be the corresponding embedding and consider the $L^{2}$ restriction inequality

$$
\begin{equation*}
\left(\int|\hat{f}(\zeta)|^{2} \phi_{\widetilde{\Gamma}}(\zeta) d \sigma_{\widetilde{\Gamma}}\right)^{\frac{1}{2}} \leq C\|f\|_{L^{q}\left(\mathbb{R}^{\tilde{N}}\right)} \tag{3.0.3}
\end{equation*}
$$

where $\sigma_{\widetilde{\Gamma}}$ is the surface measure on $\widetilde{\Gamma} \subset \mathbb{R}^{\tilde{N}}$ and $\phi_{\widetilde{\Gamma}}=\left.\phi_{\Gamma}\right|_{P}$ is the restriction of $\phi_{\Gamma}$ to $P$.
Theorem 3.0.2. Let $\Gamma \subset \mathbb{R}^{N}$ be a smooth surface such that $\Gamma \subset P$ for some proper affine subspace $P$, which is of dimension $\tilde{N}$. Let $\widetilde{\Gamma}$ be the corresponding embedding of $\Gamma$ into $\mathbb{R}^{\tilde{N}}$. Suppose that the $L^{q}\left(\mathbb{R}^{\tilde{N}}\right) \rightarrow L^{2}\left(\widetilde{\Gamma}, \sigma_{\widetilde{\Gamma}}\right)$ restriction inequality (3.0.3) holds. Then, for $1<p \leq q$ or $q^{\prime} \leq p<\infty$, the Bochner-Riesz multiplier $m_{\Gamma, \alpha}$ given by (1.0.2) with $\alpha>0$ defines a multiplier operator $T_{m_{\Gamma, \alpha}}$ which is bounded on $L^{p}$ if

$$
\alpha>(n+\tilde{L})\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{\tilde{L}}{2} .
$$

Remark 3.0.3. In the case that $L^{\prime}=0, \tilde{L}=1$ the condition

$$
\alpha>\max \left\{\left|\frac{\tilde{N}}{p}-\frac{\tilde{N}}{2}\right|-\frac{\tilde{L}}{2}, 0\right\}
$$

is the classical necessary condition in the Bochner-Riesz problem.
The following $L^{2}$ restriction estimate was established by J. Hickman and J. Wright in the algebraic case, [HW20]. Their proof has been extended here to cover the symmetric curves of standard type and their surfaces of revolution. We provide a proof in Section 3.2.

Theorem 3.0.4. With $\Gamma \in \mathcal{S}_{0}$, graphed by $\Psi$ with $d_{1}$ such that $d_{1} \geq n(\tilde{L}+1)$, then the restriction inequality (3.0.3) holds if $\frac{q^{\prime}}{2} \geq 1+\frac{D}{n}$, where $D=\sum_{j=1}^{\tilde{L}} d_{j}$.

One expects this proposition to hold without the extra condition $d_{1} \geq n(\tilde{L}+1)$. For the curves described by the equation (1.0.4) when $n=1$ and $\tilde{L}=L$, this is indeed the case; see [DM87], which gives the corresponding restriction estimate.

### 3.1 A brief introduction to Fourier restriction

The restriction phenomenon is well studied in harmonic analysis. The question is whether, for $f \in L^{p}\left(\mathbb{R}^{n}\right)$, the Fourier transform of $f$ can meaningfully be restricted to a proper subset, $\Gamma \subset \mathbb{R}^{n}$.

When studying the restriction phenomenon we seek a priori inequalities of the form

$$
\|\hat{f}\|_{L^{q}(\Gamma, \mu)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{n}\right)}, \quad \text { for } \quad f \in \mathscr{S},
$$

for some choice of Borel measure $\mu$. The restriction operator can then be defined as the continuous extension of this operator.

For restriction to smooth hypersurfaces, it turns out that curvature plays a pivotal role. An extreme example of the role of curvature in restriction estimates is that of a smooth surface with a segment supported in an affine hyperplane. Testing extension estimates, which are dual to restriction estimates, and using the fact the extension operator remains constant along fibres normal to the surface shows that no non-trivial restriction estimates hold for these surfaces. We present this analysis at Example 3.1.3. Similarly, for algebraic surfaces which are not totally flat, the nature of the degeneracy of the surface plays a role in what restriction estimates are possible. ${ }^{1}$

Definition 3.1.1. The extension operator, $E$, is defined a priori, i.e. on the class of Schwartz functions, by $g \mapsto \widehat{g \sigma}, g \in \mathscr{S}\left(\mathbb{R}^{n}\right)$.

[^2]We now express the duality between restriction and extension, with a third dual formulation of restriction estimates. For a proof, see, for instance, [SM13].
Proposition 3.1.2. In the following, $\sigma$ is some measure of bounded variation supported in $\mathbb{R}^{n}$. The following three estimates are equivalent,

$$
\begin{gather*}
\|\hat{f}\|_{L^{2}(\sigma)} \leq A\|f\|_{L^{p}\left(\mathbb{R}^{n}\right)}, f \in \mathscr{S}\left(\mathbb{R}^{n}\right)  \tag{3.1.1}\\
\|\widehat{g \sigma}\|_{L^{p^{\prime}}\left(\mathbb{R}^{n}\right)} \leq A\|g\|_{L^{2}(\sigma)}, g \in \mathscr{S}\left(\mathbb{R}^{n}\right),  \tag{3.1.2}\\
\|h * \widehat{\sigma}\|_{L^{p^{\prime}}\left(\mathbb{R}^{n}\right)} \leq A^{2}\|h\|_{L^{p}\left(\mathbb{R}^{n}\right)}, h \in \mathscr{S}\left(\mathbb{R}^{n}\right) . \tag{3.1.3}
\end{gather*}
$$

A well known case of the restriction phenomenon is the famous Tomas restriction theorem [Tom75], which is an $L^{2}$ restriction inequality for $S^{n-1}$ equipped with its surface measure:

$$
\|\hat{f}\|_{L^{2}\left(S^{N-1}\right)} \leq\|f\|_{L^{p}\left(\mathbb{R}^{N}\right)}
$$

for $1 \leq p<\frac{2 N+2}{N+3}$. The inequality was extended to include the endpoint $p=\frac{2 N+2}{N+3}$ by Stein, with a proof using analytic interpolation. Critical these proofs is the non-vanishing Gaussian curvature of the sphere, which leads to a certain order of decay in the Fourier transform of the surface measure. In particular, the range of valid $L^{p}$ exponents is determined exactly by the order of decay.

As an extreme example, we now show how flat surfaces may satisfy no restriction estimates.
Example 3.1.3. For smooth surfaces $\Gamma \subset P$, where $P$ is some affine hyperplane, no non-trivial restriction estimate to $\Gamma$ holds. In particular, the only $L^{2}$ restriction estimate that holds to $\Gamma$, with respect to (localised) surface measure, is the $L^{1} \rightarrow L^{2}\left(\sigma_{\Gamma}\right)$ restriction estimate. To see this, we consider the corresponding extension estimate (see Proposition 3.1.2). We choose non-zero $f \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$ with $\operatorname{supp} f \subset P$ such that, after a rotation and translation, $\Gamma$ is expressible as the graph of a function $\Psi(\xi)=(\widetilde{\Psi}(\xi), 0, \ldots, 0)$ for $|\xi|<\delta$. The corresponding extension estimate we consider is

$$
\|E f\|_{L^{q}\left(\mathbb{R}^{N}\right)} \lesssim\|f\|_{L^{2}\left(\mathbb{R}^{N}, \sigma_{\Gamma}\right)}
$$

where

$$
E f(x, y, z)=\int_{|\xi|<\delta} e^{2 \pi i(x, y, z) \cdot(\xi, \Psi(\xi))} f(\xi)\left(\sqrt{1+|\nabla \Psi(\xi)|^{2}}\right) d \xi
$$

Now we observe that, for all $z^{\prime}$ and $z$,

$$
E f(x, y, z)=E f\left(x, y, z^{\prime}\right)
$$

since $(x, y, z) \cdot(\xi, \widetilde{\Psi}(\xi), 0, \ldots, 0)=\left(x, y, z^{\prime}\right) \cdot(\xi, \widetilde{\Psi}(\xi), 0, \ldots, 0)$. Thus we have that the extension operator (applied to $f$ ) is constant along fibres normal to the surface. As such, the only way that $\|E f\|_{L^{q}\left(\mathbb{R}^{N}\right)}<\infty$ is that $E f=0$ or $q=\infty$. In the case that $q<\infty$, we thus find that $E$ is the zero operator, which is a contradiction.

Another example for the role of curvature in restriction is given by the model surfaces discussed in the introduction. Let

$$
\Gamma=\left\{\left(\xi,|\xi|^{d_{1}}, \ldots,|\xi|^{d_{\tilde{L}}}, 0\right) ;|\xi|<4 \delta\right\} \subset \mathbb{R}^{n} \times \mathbb{R}^{\tilde{L}} \times \mathbb{R}^{L^{\prime}}
$$

whose graphing function $\Psi$ is given by (1.0.4) with $d_{1}<d_{2}<\ldots<d_{\tilde{L}}$ even. Then a necessary condition for the restriction inequality (3.0.3) to hold is

$$
\begin{equation*}
\frac{q^{\prime}}{2} \geq 1+\frac{D}{n} \tag{3.1.4}
\end{equation*}
$$

where $D=\sum_{j=1}^{\tilde{L}} d_{j}$. This follows from a standard Knapp example, which we now provide.
Example 3.1.4. To test the restriction inequality, we consider a one parameter family of $1 \geq g_{\epsilon} \geq 0$ with $g_{\epsilon} \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ and $\operatorname{supp} g_{\epsilon} \subset B(0, \epsilon)$ with $g_{\epsilon}(\xi)=1$ for $|\xi| \leq \epsilon / 2$. We consider
the extension formulation of the estimate (see Proposition 3.1.2) and test whether the inequality

$$
\begin{equation*}
\left\|E g_{\epsilon}\right\|_{L^{q^{\prime}}\left(\mathbb{R}^{\tilde{N}}\right)} \leq C\left\|g_{\epsilon}\right\|_{L^{2}(\phi(\xi) d \xi)} \tag{3.1.5}
\end{equation*}
$$

holds. One can easily see that, in the region where $\left|y_{j}\right| \ll \epsilon^{-d_{j}}$ for each $j$ and $|x| \ll \epsilon^{-1}$, $\left|E g_{\epsilon}(x, y)\right| \sim 1$. One can thus see, since the phase is near vanishing for $(x, y)$ in the region just mentioned, that

$$
\epsilon^{-\frac{1}{q^{\prime}}\left(n+\sum_{j=1}^{\tilde{L}} d_{j}\right)} \lesssim\left\|E g_{\epsilon}\right\|_{L^{q^{\prime}}}
$$

We also have that

$$
\left\|g_{\epsilon}\right\|_{L^{2}(\phi(\xi) d \xi)} \sim \epsilon^{-\frac{n}{2}}
$$

Considering the inequality (3.1.5) for $g_{\epsilon}$ as $\epsilon \rightarrow 0$, we obtain a necessary condition

$$
\frac{q^{\prime}}{2} \geq 1+\frac{D}{n}
$$

as desired.
Regarding the Fourier restriction problem for (a neighbourhood of the origin of) the moment curve,

$$
\left(t, t^{2}, t^{3}, \ldots, t^{d}\right)
$$

there is the initial result of Drury, [Dru85], which tells us that the restriction inequality (3.0.1) holds for $q^{\prime} \geq d(d+1)$. In the field, it was not known that this result was sharp until the discovery of an earlier publication, [ACK79], where the necessary condition $q^{\prime} \geq d(d+1)$ had been established.

### 3.2 An $L^{2}$ restriction theorem for radial surfaces of standard type

In this section, we present a proof of Theorem 3.0.4, the statement of which is recalled below. The proof is due to Jonathan Hickman and Jim Wright, [HW20]. We work along the lines of the classical fractional integration proof of restriction and make use of the weak-type Young's inequality, Theorem 0.0.3

Let us recall the statement of Theorem 3.0.4, which is an $L^{2}$ restriction result for surfaces $\Gamma \in \mathcal{S}_{0}$ (Definition 1.0.1).
Theorem. With $\Gamma \in \mathcal{S}_{0}$, graphed by $\Psi$ with such that $d_{1} \geq n(\tilde{L}+1)$, then (3.0.3) holds if $\frac{q^{\prime}}{2} \geq 1+\frac{D}{n}$, where $D=\sum_{j=1}^{\tilde{L}} d_{j}$.
Proof. To apply the weak-type Young's inequality it will suffice to verify the following decay on the measure $\mu$

$$
\begin{equation*}
|\hat{\mu}(x, y)| \leq C \min _{j=1, \ldots, L}\left|y_{j}\right|^{-\frac{n}{d_{j}}} \tag{3.2.1}
\end{equation*}
$$

For now, let us suppose that the estimate (3.2.1) holds and see how the result follows. We later return to (3.2.1), which we obtain as a corollary of Theorem 5.0.5. We now work to establish the estimate

$$
\|\hat{\mu} * f\|_{L^{p^{\prime}}} \leq C\|f\|_{L^{p}}
$$

which, by Proposition 3.1.2, is equivalent to $L^{p} \rightarrow L^{2}$ Fourier restriction. Indeed, let us consider the convolution

$$
\hat{\mu} * f(x, y)=\int_{\mathbb{R}^{n}} K_{y-z} * f_{z}(x) d z
$$

where $K_{y}(x)=\hat{\mu}(x, y)$ and $f_{z}(x)=f(x, z)$. The bounds

$$
\begin{gathered}
\left\|K_{y} * g\right\|_{L^{2}\left(\mathbb{R}^{n}\right)} \leq C\|g\|_{L^{2}\left(\mathbb{R}^{n}\right)} \\
\text { and }\left\|K_{y} * g\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq C \min _{j=1, \ldots, L}\left|y_{j}\right|^{-\frac{n}{d_{j}}}\|g\|_{L^{1}\left(\mathbb{R}^{n}\right)}
\end{gathered}
$$

are easily verified. The first follows because $g \mapsto K_{y} * g$ is a Fourier multiplier operator, with multiplier $m(\xi)=\phi(\xi) e^{2 \pi i y \cdot \Psi(\xi)}$, whose multiplier is uniformly bounded in $y$. The second is obtained by a direct application of Young's inequality and the $L^{\infty}$ bound

$$
\left\|K_{y}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq C \min _{j=1, \ldots, L}\left|y_{j}\right|^{-\frac{n}{d_{j}}}
$$

For $p \in[1,2]$ we may interpolate, using Riesz-Thorin interpolation (Theorem 0.0.2), between the $L^{1} \rightarrow L^{\infty}$ and $L^{2} \rightarrow L^{2}$ bounds. For $p \in[1,2]$, by choosing $\theta \in[0,1]$ with $\frac{1}{p}=\frac{1-\theta}{1}+\frac{\theta}{2}$, that is $1-\theta=\frac{2}{p}-1$, we see that

$$
\left\|K_{y} * g\right\|_{L^{p^{\prime}}\left(\mathbb{R}^{n}\right)} \leq C\left(\min _{j=1, \ldots, L}\left|y_{j}\right|^{-\frac{n}{d_{j}}}\right)^{\frac{2-p}{p}}\|g\|_{L^{p}}
$$

Now, one can easily verify that $\left(\min _{j=1, \ldots, L}\left|y_{j}\right|^{-\frac{n}{d_{j}}}\right)^{\frac{2-p}{p}} \in L^{r, \infty}\left(\mathbb{R}^{\tilde{L}}\right)$ for $r=\frac{p}{2-p} \frac{D}{n}$. Given this, we may apply Minkowski's inequality and the weak-type Young's inequality to see that

$$
\begin{gathered}
\|\hat{\mu} * f\|_{L^{p^{\prime}}}=\| \| \int_{\mathbb{R}^{n}} K_{y-z} * f_{z}(x) d z\left\|_{L_{x}^{p^{\prime}}\left(\mathbb{R}^{n}\right)}\right\|_{L_{y}^{p^{\prime}}\left(\mathbb{R}^{\tilde{L}}\right)} \\
\leq\left\|\int_{\mathbb{R}^{n}}\right\| K_{y-z} * f_{z}(x)\left\|_{L_{x}^{p^{\prime}}} d z\right\|_{L_{y}^{p^{\prime}}} \\
\leq C\left\|\int_{\mathbb{R}^{n}}\left|\left(\min _{j=1, \ldots, L}\left|y_{j}-z_{j}\right|^{-\frac{n}{d_{j}}}\right)^{\frac{2-p}{p}}\right|^{-\frac{n}{D}\left(\frac{2-p}{p}\right)}\right\| f_{z}\left\|_{L^{p}} d z\right\|_{L_{y}^{p^{\prime}}} \\
\leq C\|f\|_{L^{p}\left(\mathbb{R}^{N}\right)},
\end{gathered}
$$

provided $1+\frac{1}{p^{\prime}}=\frac{1}{p}+\frac{n}{D}\left(\frac{2-p}{p}\right)$, i.e. $\frac{2}{p^{\prime}}=1+\frac{D}{n}$. By interpolation with the trivial $L^{1} \rightarrow L^{\infty}$ estimate, $\|f * \hat{\mu}\|_{\infty} \leq\|f\|_{1}\|\hat{\mu}\|_{\infty} \leq\|f\|_{1} \mu\left(\mathbb{R}^{N}\right)$, we have that $f \mapsto f * \hat{\mu}$ is bounded from $L^{p}$ to $L^{p^{\prime}}$ for all $\frac{2}{p^{\prime}} \geq 1+\frac{D}{n}$.

To prove the key decay estimate (3.2.1) we perform polar integration to see that

$$
\hat{\mu}(x, y)=\iint e^{2 \pi i\left(r x \cdot \omega+\sum_{j=1}^{\tilde{L}} y_{j} \psi_{j}(r)\right)} \phi(r \omega) r^{n-1} d r d \sigma(\omega) .
$$

The essential estimate is given by Theorem $5.0 .5^{2}$, which gives the bound

$$
\begin{equation*}
\left|\int e^{2 \pi i\left(r x \cdot \omega+\sum_{j=1}^{\tilde{L}} y_{j} \psi_{j}(r)\right)} \phi_{0}(r) r^{n-1} d r\right| \leq C \min _{1 \leq j \leq L}\left|y_{j}\right|^{-\frac{n}{d_{j}}} . \tag{3.2.2}
\end{equation*}
$$

Integrating over the sphere, we then find that

$$
|\hat{\mu}(x, y)| \leq C^{\prime} \min _{j=1, \ldots, L}\left\{\frac{1}{\left|y_{j}\right|^{\frac{n}{d_{j}}}}\right\}
$$

which is all that we require.

### 3.3 Restriction implies Bochner-Riesz

In this section, we work to establish Theorem 3.0.2. This result concerns the $L^{p}$-boundedness of a Bochner-Riesz operators defined relative to certain smooth surfaces. Under the assumption

[^3]that an appropriate restriction estimate holds with respect to a smooth surface $\Gamma$, we establish boundedness of the corresponding operator.

The surfaces $\Gamma$ we consider may be contained in a proper affine subspace $P \subset \mathbb{R}^{N}$ of dimension $\tilde{N}<N$. In this case, a rotation and translation are all that is required to ensure that $\Gamma \subset \mathbb{R}^{\tilde{N}} \times\{0\} \subset \mathbb{R}^{\tilde{N}} \times \mathbb{R}^{L^{\prime}}$. Henceforth, we may suppose without loss of generality that $\Gamma$ is expressed in this way.

We approach the study of the Bochner-Riesz operator defined with respect to these surfaces locally, considering segments $\Gamma_{\Psi}$ which, after a translation and change of coordinates, give a section of $\Gamma$ as a graph. We may additionally choose the graphing function $\Psi=\left(\Psi_{\tilde{L}}, 0\right) \in$ $\mathbb{R}^{\tilde{L}} \times \mathbb{R}^{L^{\prime}}$ such that $\Psi_{\tilde{L}}(0)=0$ and $\nabla \Psi_{\tilde{L}}(0)=0$. This notation represents a slight change from previous sections, as we here need to work without reference to the 0 components, and our $\Psi_{\tilde{L}}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{\tilde{L}}$ corresponds to the first $\tilde{L}$ non-vanishing coordinates.

In this section we prove Theorem 3.0.2, whose statement we now recall. The theorem is a corollary of Theorem 3.3.2, the graphical multiplier reduction, Lemma 2.2.3, and Lemma 3.3.1, which is a restriction reduction.

Theorem. Let $\Gamma \subset \mathbb{R}^{N}$ be a smooth surface such that $\Gamma \subset P$ for some proper affine subspace $P$, which is of dimension $\tilde{N}$. Let $\widetilde{\Gamma}$ be the corresponding projection of $\Gamma$ onto $\mathbb{R}^{\tilde{N}}$. Suppose that the $L^{q}\left(\mathbb{R}^{\tilde{N}}\right) \rightarrow L^{2}\left(\widetilde{\Gamma}, \sigma_{\widetilde{\Gamma}}\right)$ restriction inequality (3.0.3) holds. Then, for $1<p \leq q$ or $q^{\prime} \leq p<\infty$, the Bochner-Riesz multiplier $m_{\Gamma, \alpha}$ given by (1.0.2) with $\alpha>0$ defines a multiplier operator $T_{m_{\Gamma, \alpha}}$ which is bounded on $L^{p}$ if

$$
\alpha>(n+\tilde{L})\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{\tilde{L}}{2}
$$

Before turning to Theorem 3.3.2, let us show how the local restriction to fine segments of the surface, are related to the restriction estimate appearing in Theorem 3.0.2.
Lemma 3.3.1. The $L^{p}\left(\mathbb{R}^{\tilde{N}}\right) \rightarrow L^{2}\left(\widetilde{\Gamma}, \phi_{\widetilde{\Gamma}}(\zeta) d \sigma_{\widetilde{\Gamma}}\right)$ restriction inequality

$$
\left.\left.\left|\int\right| \hat{f}(\zeta)\right|^{2} \phi_{\widetilde{\Gamma}}(\zeta) d \sigma_{\widetilde{\Gamma}}(\zeta)\right|^{\frac{1}{2}} \leq\left.\left.\left|\int_{\mathbb{R}^{\tilde{N}}}\right| f(\zeta)\right|^{p} d \zeta\right|^{\frac{1}{p}}
$$

holds if and only if the local restriction inequalities

$$
\left.\left.\left|\int\right| \hat{f}\left(\xi, \Psi_{\tilde{L}}(\xi)\right)\right|^{2} \phi(\xi) d \xi\right|^{\frac{1}{2}} \leq\left.\left.\left|\int_{\mathbb{R}^{\tilde{N}}}\right| f(\zeta)\right|^{p} d \zeta\right|^{\frac{1}{p}}
$$

where $\phi$ is given as in Lemma 2.2.3 for all $\Psi_{\tilde{L}}$ which, after a rotation and translation, graphs a segment of $\widetilde{\Gamma} \cap \operatorname{supp} \phi_{\widetilde{\Gamma}}(\cdot, 0)$.

Proof. Let us first suppose that we have the local restriction inequalities. Recall that we chose $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ with $\operatorname{supp} \phi \subset B^{n}(0, \delta)$ for some suitable small $\delta$. We now take a partition of unity $\left\{\chi_{j} ; j \in \mathcal{J}\right\}$, where each $\chi_{j}(\zeta)=1$ for $\zeta \in B\left(\zeta_{j}, \delta / 4\right)$ and supp $\chi_{j} \subset B\left(\zeta_{j}, \delta / 2\right)$ for finitely many $\zeta_{j} \in \widetilde{\Gamma} \cap \operatorname{supp} \phi_{\widetilde{\Gamma}}$, and $\sum \chi_{j}(\zeta)=1$ for $\zeta \in \widetilde{\Gamma} \cap \operatorname{supp} \phi_{\widetilde{\Gamma}}$. We see that, for $\zeta \in \widetilde{\Gamma}$,

$$
\hat{f}(\zeta)=\sum \hat{f}_{j}(\zeta),
$$

where $f_{j}=f * \check{\chi}_{j}$. As discussed in Section 2.2 to each $\chi_{j}$ is a linear map, $T_{j}$, which is the combination of a rotation and translation, such that $T_{j} B\left(\zeta_{j}, 4 \delta\right) \cap \widetilde{\Gamma}$ can be expressed as the graph of a function $\Psi_{\tilde{L},(j)}$ with $\Psi_{\tilde{L},(j)}(0)=0$ and $\nabla \Psi_{\tilde{L},(j)}(0)=0$. For $\xi \in \operatorname{supp} \phi$, we see that $\left|\nabla \Psi_{\tilde{L},(j)}(\xi)\right|<\frac{1}{2}$. We see that

$$
\left.\left.\left|\int\right| \sum_{j} \hat{f}_{j}(\zeta)\right|^{2} \phi_{\widetilde{\Gamma}}(\zeta) d \sigma_{\widetilde{\Gamma}}(\zeta)\right|^{\frac{1}{2}}
$$

$$
\begin{gathered}
\left.\left.\lesssim \sum_{j}\left|\int\right| \hat{f}_{j}(\zeta)\right|^{2} \phi_{\widetilde{\Gamma}}(\zeta) d \sigma_{\widetilde{\Gamma}}(\zeta)\right|^{\frac{1}{2}} \\
\left.\left.\lesssim \sum_{j}\left|\int\right| \hat{f}_{j}\left(T_{j}^{-1} \zeta\right)\right|^{2} \phi_{\widetilde{\Gamma}}\left(T_{j}^{-1} \zeta\right) d \sigma_{\widetilde{\Gamma}_{j}}(\zeta)\right|^{\frac{1}{2}} \\
\left.\lesssim \sum_{j}\left|\int\right| \hat{f}_{j}\left(T_{j}^{-1}\left(\xi, \Psi_{\tilde{L},(j)}(\xi)\right)\right)\right|^{2} \phi_{\widetilde{\Gamma}}\left(\left.T_{j}^{-1}\left(\xi, \Psi_{\tilde{L},(j)}(\xi)\right) \sqrt{1+\left|\nabla \Psi_{\tilde{L},(j)}(\xi)\right|^{2}} d \xi\right|^{\frac{1}{2}}\right. \\
\left.\left.\lesssim \sum_{j}\left|\int\right| \hat{f}\left(T_{j}^{-1}\left(\xi, \Psi_{\tilde{L},(j)}(\xi)\right)\right)\right|^{2} \phi(\xi) d \xi\right|^{\frac{1}{2}} \\
\lesssim \sum_{j}\left\|\mathcal{F}^{-1}\left(\hat{f}\left(T_{j}^{-1} \cdot\right)\right)\right\|_{L^{p}\left(\mathbb{R}^{\tilde{N}}\right)} \\
\sim\|f\|_{L^{p}\left(\mathbb{R}^{\tilde{N}}\right)} .
\end{gathered}
$$

The reverse implication is simpler, after a rotation and translation $T$, we apply Hölder's inequality to see that

$$
\begin{gathered}
\left(\int\left|\hat{f}\left(\xi, \Psi_{\tilde{L}}(\xi)\right)\right|^{2} \phi(\xi) d \xi\right)^{\frac{1}{2}} \\
\lesssim\left(\int\left|\hat{f}\left(\xi, \Psi_{\tilde{L}}(\xi)\right)\right|^{2} \phi_{\widetilde{\Gamma}}\left(T^{-1}\left(\xi, \Psi_{\tilde{L}}(\xi)\right) \sqrt{1+\left|\nabla \Psi_{\tilde{L}}(\xi)\right|^{2}} d \xi\right)^{\frac{1}{2}}\right. \\
\lesssim\left(\int|\hat{f}(T(\zeta))|^{2} \phi_{\widetilde{\Gamma}}(\zeta) d \sigma_{\tilde{\Gamma}}\right)^{\frac{1}{2}} \\
\lesssim\|f\|_{L^{p}\left(\mathbb{R}^{\tilde{N}}\right)} .
\end{gathered}
$$

Theorem 3.3.2. Provided $\delta$ is chosen sufficiently small, we have the following. Let

$$
\Gamma_{\Psi}=\left\{\left(\xi, \Psi_{\tilde{L}}(\xi), 0\right) ;|\xi|<4 \delta \in \mathbb{R}^{n}\right\} \subset \mathbb{R}^{n} \times \mathbb{R}^{\tilde{L}} \times \mathbb{R}^{L^{\prime}}
$$

and

$$
\widetilde{\Gamma}_{\Psi_{\tilde{L}}}=\left\{\left(\xi, \Psi_{\tilde{L}}(\xi)\right) ; \xi \in \mathbb{R}^{n},|\xi|<4 \delta\right\} \subset \mathbb{R}^{n} \times \mathbb{R}^{\tilde{L}}
$$

be the corresponding projection onto $\mathbb{R}^{\tilde{N}}$. Suppose that, for $\tilde{p}=\max \left\{p, p^{\prime}\right\}$, the $L^{2} \rightarrow L^{\tilde{p}}$ extension inequality

$$
\left(\iint|E g(x, y)|^{\tilde{p}} d x d y\right)^{\frac{1}{p}} \leq C\left(\int|g(\xi)|^{2} \phi(\xi) d \xi\right)^{\frac{1}{2}}
$$

extension holds, where

$$
E g(x, y)=\int e^{2 \pi i(x, y) \cdot\left(\xi, \Psi_{\dot{\mathcal{L}}}(\xi)\right)} g(\xi) \phi(\xi) d \xi .
$$

Then the Bochner-Riesz multiplier $m=m_{\Psi, \alpha}$ with

$$
m(\zeta)=|\eta-\Psi(\xi)|^{\alpha} \phi(\xi) \chi(\eta-\Psi(\xi))
$$

is an element of $\mathcal{M}_{p}$ for

$$
\alpha>\max \left\{\left|\frac{\tilde{N}}{p}-\frac{\tilde{N}}{2}\right|-\frac{\tilde{L}}{2}, 0\right\} .
$$

Remark 3.3.3. Critical to the behaviour of the operator that we are considering is the oscillation of the kernel, for large $|y|$ this oscillation is captured by the $k(x, y)$ term. We must
carefully analyse the effect of this oscillation. In particular, we will utilise extension estimates for an $(x, y)$ region where the mass of the kernel is concentrated.

Proof. Provided $\delta$ is chosen small enough, we have that

$$
\begin{equation*}
\left\|\Psi_{\tilde{L}}\right\|_{C^{1}(\operatorname{supp} \phi)} \leq \frac{1}{2 L} \tag{3.3.1}
\end{equation*}
$$

We first prove the result for $p \geq 2$. The full range of estimates is then a consequence of duality, Lemma 2.1.1.

By a kernel decomposition, we work to isolate the main piece of the operator in what follows. We start with a dyadic partition of the kernel, in positive dyadic scales of $|(y, z)|$. Let us recall that $K(x, y, z)=A(y, z) k(x, y)$, where

$$
k(x, y)=\int e^{2 \pi i\left(x \cdot \xi+y \cdot \Psi_{\tilde{L}}(\xi)\right)} \phi(\xi) d \xi
$$

and

$$
A(y, z)=\iint e^{2 \pi i(y, z) \cdot(\eta, \lambda)}|(\eta, \lambda)|^{\alpha} \chi(\eta, \lambda) d \eta d \lambda
$$

We let $\psi_{0} \in C_{c}^{\infty}\left(\mathbb{R}^{L}\right)$ with $\psi_{0}(y, z)=1$ for $|(y, z)| \leq \frac{1}{2}, \psi_{0}(y, z)=0$ for $|(y, z)| \geq 1$, and, for $l \geq 1$, define

$$
\psi_{l}(y, z):=\psi_{0}\left(2^{-l}(y, z)\right)-\psi_{0}\left(2^{-l+1}(y, z)\right)
$$

so that $\sum_{l \geq 0} \psi_{l}(y, z)=1$ for all $(y, z) \in \mathbb{R}^{L}$. We can write, for $l \geq 1, \psi_{l}(y, z)=\psi\left(2^{-l}(y, z)\right)$, where

$$
\psi(y, z)=\psi_{0}(y, z)-\psi_{0}(2(y, z)) .
$$

It is seen that $\operatorname{supp} \psi \subset B(0,1) \backslash B\left(0, \frac{1}{4}\right)$.
We write

$$
K(x, y, z)=\sum_{l \geq 0} \psi_{l}(y, z) A(y, z) k(x, y)=\sum_{l \geq 0} K_{l}(x, y, z) .
$$

We define the operators $T_{l}$ by

$$
T_{l} f=f * K_{l} .
$$

We first consider the operator $T_{0}$. We set $\widetilde{\psi_{0}}(y, z)=A(y, z) \psi_{0}(y, z)$. We will insert a cutoff function, $\chi_{0}$, in the first argument of the kernel $K_{0}$ :

$$
\begin{aligned}
& K_{0}(x, y, z)=\widetilde{\psi_{0}}(y, z) k(x, y) \\
& =K_{0}^{1}(x, y, z)+K_{0}^{2}(x, y, z)
\end{aligned}
$$

where

$$
\begin{gathered}
K_{0}^{1}(x, y, z)=\widetilde{\psi_{0}}(y, z)\left(\chi_{0}(x) \int e^{2 \pi i\left(x \cdot \xi+y \cdot \Psi_{\tilde{L}}(\xi)\right)} \phi(\xi) d \xi\right), \\
K_{0}^{2}(x, y, z)=\widetilde{\psi_{0}}(y, z)\left(\left(1-\chi_{0}(x)\right) \int e^{\left.2 \pi i\left(x \cdot \xi+y \cdot \Psi_{\tilde{L}}(\xi)\right)\right)} \phi(\xi) d \xi\right) .
\end{gathered}
$$

For $(x, y, z) \in \operatorname{supp} K_{0}^{2}$, provided $\chi_{0}$ is chosen appropriately, the phase appearing in the integral expansion of $k(x, y)$ has no critical points in the integrand's support. We may choose smooth $\chi_{0}$ with

$$
\begin{equation*}
1 \geq \chi_{0} \geq 0 \text { with } \chi_{0}(x)=1 \text { for }|x| \leq 1 \text { and } \chi_{0}(x)=0 \text { for }|x| \geq 2 . \tag{3.3.2}
\end{equation*}
$$

For $|x| \geq|y|$, which is true for the $x$ and $y$ in the support of $K_{0}^{2}$, where we know $1 \leq|x|$ and $|y| \leq 1$, we show that

$$
\begin{equation*}
|k(x, y)|=\left|\int e^{2 \pi i\left(x \cdot \xi+y \cdot \Psi_{\tilde{L}}(\xi)\right)} \phi(\xi) d \xi\right| \leq C_{M, \Psi_{\tilde{L}}}(1+|x|)^{-M} . \tag{3.3.3}
\end{equation*}
$$

This follows by an application of the non-stationary phase lemma. Indeed, setting $\Phi(\xi)=$
$\frac{x}{|x|} \cdot \xi+\frac{y}{|x|} \cdot \Psi_{\tilde{L}}(\xi)$ we have

$$
|k(x, y)|=\left|\int e^{2 \pi i|x| \Phi(\xi)} \phi(\xi) d \xi\right|
$$

with $\left|\nabla \Phi_{x, y}(\xi)\right| \geq\left|\frac{x}{|x|}\right|-\left|\frac{y}{|x|}\right|\left|\nabla \Psi_{\tilde{L}}(\xi)\right| \geq 1-L\left\|\Psi_{\tilde{L}}\right\|_{C^{1}(\operatorname{supp} \phi)} \geq \frac{1}{2}$, using (3.3.1), for $\xi \in$ $\operatorname{supp} \phi$ for those $x, y$ in the support of $K_{0}^{2}$. For $(x, y, z) \in \operatorname{supp} K_{0}^{2}$, we also have the control $\|\Phi\|_{C^{M+1}(\operatorname{supp} \phi)} \lesssim_{M} 1+\left\|\Psi_{\tilde{L}}\right\|_{C^{M+1}(\operatorname{supp} \phi)}$ and $\|\phi\|_{C^{M}} \lesssim_{M} 1$ so that, by an application of the non-stationary phase lemma, Lemma 0.0 .1 , together with the trivial estimate $|k(x, y)| \leq C$, we have (3.3.3).

We define the operators $T_{0}^{1}$ and $T_{0}^{2}$ by

$$
T_{0}^{1} f(x, y)=f * K_{0}^{1}(x, y), T_{0}^{2} f(x, y, z)=f * K_{0}^{2}(x, y, z)
$$

By choosing $M=N+1$, we then see that

$$
\begin{equation*}
\left\|K_{0}^{2}\right\|_{1} \leq C \tag{3.3.4}
\end{equation*}
$$

Therefore, by Young's inequality, $\left\|T_{0}^{2} f\right\|_{p} \leq C\|f\|_{p}$.
The operator $f \mapsto K_{0}^{1} * f$ is also given by convolution with an $L^{1}$ kernel $K_{0}^{1}$. Indeed,

$$
\begin{equation*}
\left\|K_{0}^{1}\right\|_{1} \leq C \tag{3.3.5}
\end{equation*}
$$

since $K_{0}^{1}$ is bounded and compactly supported. By Young's inequality it follows that $\left\|T_{0}^{1} f\right\|_{p} \leq$ $\left\|K_{0}^{1}\right\|_{1}\|f\|_{p}$. By the triangle inequality we then have that $\left\|T_{0} f\right\|_{p} \leq C\|f\|_{p}$.

We now consider the operators $T_{l}$, defined by convolution with the kernels $K_{l}(x, y, z)=$ $A(y, z) \psi_{l}(y, z) k(x, y)$. We know that $|A(y, z)| \leq C|(y, z)|^{-L-\alpha}$, so we write

$$
A(y, z) \psi_{l}(y, z)=2^{-l(L+\alpha)}\left(2^{l(L+\alpha)} \psi_{l}(y, z) A(y, z)\right)
$$

and define $\widetilde{\psi}_{l}$ by

$$
\widetilde{\psi}_{l}(y, z)=2^{l(L+\alpha)} \psi_{l}(y, z) A(y, z)
$$

We then have the $L^{\infty}$ bound

$$
\left\|\widetilde{\psi}_{l}\right\|_{L^{\infty}} \leq C 2^{l(L+\alpha)} \frac{1}{2^{(l+1)(L+\alpha)}}=\frac{C}{2^{L+\alpha}}
$$

which is uniform in $l$. We wish to obtain operator bounds on $f \mapsto f * K_{l}$. To do so we will find the operator bound for convolution with the kernel determined by $k(x, y, z) \widetilde{\psi}_{l}(y, z)$ and multiply the resultant bounds by $2^{-l(L+\alpha)}$.

We first carry out a rescaling, which will allow for a uniform treatment of operator pieces. To find an appropriate rescaling we briefly suppose that $A(y, z)=\frac{c}{\left.(y, z)\right|^{L+\alpha}}$. In truth, $A$ is not homogoneous, but assuming it to be so will help us to identify the right rescaling. Under this assumption we have that $\widetilde{\psi}_{l}(y, z)=\frac{c 2^{l(L+\alpha)}}{|(y, z)|^{l(L+\alpha)}} \psi_{l}(y, z)$. Most importantly, $\widetilde{\psi}_{l}(y, z)=$ $\psi_{\text {hom }}\left(2^{-l}(y, z)\right)$, where $\psi_{h o m}(y, z):=\frac{c}{|(y, z)|^{L+\alpha}} \psi(y, z)$.

We look to find operator bounds for the operators $T_{l}$ which are given by

$$
\begin{gathered}
T_{l} f(x, y, z)=\iiint \widetilde{\psi}_{l}(y-u, z-v) k(x-t, y-u) f(t, u, v) d t d d u d v \\
\quad=\iint \psi_{h o m}\left(2^{-l}(y-u, z-v)\right) k(x-t, y-u) f(t, u, v) d t d u d v
\end{gathered}
$$

Changing variables we find that

$$
T_{l} f(x, y, z)
$$

$$
=\iiint \psi_{h o m}\left(2^{-l} y-u, 2^{-l} z-v\right) k\left(2^{l}\left(2^{-l} x-t\right), 2^{l}\left(2^{-l} y-u\right)\right) f\left(2^{l} t, 2^{l} u, 2^{l} v\right) 2^{l(n+L)} d t d u d v
$$

$$
=2^{l(n+L)} \widetilde{T}_{l} f_{2^{l}}\left(2^{-l} x, 2^{-l} y, 2^{-l} z\right)
$$

where

$$
\widetilde{T}_{l} f(x, y, z)=\iiint \psi_{h o m}(y-u, z-v) k\left(2^{l}(x-t), 2^{l}(y-u)\right) f(t, u, v) d t d u d v
$$

and $f_{2^{l}}(t, u, v)=f\left(2^{l} t, 2^{l} u, 2^{l} v\right)$.
Since $\frac{\left\|f_{2 l}\right\|_{L^{p}}}{\|f\|_{L^{p}}}=2^{-\frac{i N}{p}}$, we also see that

$$
\begin{gathered}
\left\|T_{l}\right\|_{L^{p} \rightarrow L^{p}}=\sup _{\|f\|_{L^{p}} \leq 1}\left\|T_{l} f\right\|_{L^{p}}=2^{l N} 2^{\frac{l N}{p}} \sup _{\|f\|_{L^{p} \leq 1} \leq 1}\left\|\widetilde{T}_{l} f_{2^{l}}\right\|_{L^{p}} \\
=2^{l N} \sup _{\left\|f_{2^{l}}\right\|_{L^{p}} \leq 2^{-\frac{l N}{p}}}\left\|\widetilde{T}_{l}\left(2^{\frac{l N}{p}} f_{2^{l}}\right)\right\|_{L^{p}}=2^{l N}\left\|\widetilde{T}_{l}\right\|_{L^{p} \rightarrow L^{p}}
\end{gathered}
$$

This reduces the problem to the study of the operators $\widetilde{T}_{l}$ with kernels $\widetilde{K}_{l}(t, u, v)=\psi_{h o m}(u, v) k\left(2^{l} t, 2^{l} u\right)$.
We assumed that $A(y, z)=\frac{c}{|(y, z)|^{L+\alpha}}$. In truth, the relation obtained from such homogeneous $A, \widetilde{\psi}_{l}(y, z)={\underset{\sim}{h}}_{h o m}\left(2^{-l}(y, z)\right)$, does not hold. We indicate the required modifications. We define $\psi_{l, h}(y, z):=\widetilde{\psi}_{l}\left(2^{l}(y, z)\right)$ so that $\widetilde{\psi}_{l}(y, z)=\psi_{l, h}\left(2^{-l}(y, z)\right)$. It is seen that

$$
\begin{equation*}
\operatorname{supp} \psi_{l, h} \subset \operatorname{supp} \psi \text { and }\left\|\psi_{l, h}\right\|_{L^{\infty}} \leq C \tag{3.3.6}
\end{equation*}
$$

hold uniformly in $l$. The above calculations may be repeated with the operator $\widetilde{T}_{l}$ now rightly given by

$$
\begin{gather*}
\widetilde{T}_{l} f(x, y, z)=\widetilde{K}_{l} * f(x, y, z) \\
=\iiint \psi_{l, h}(y-u, z-v) k\left(2^{l}(x-t), 2^{l}(y-u)\right) f(t, u, v) d t d u d v \tag{3.3.7}
\end{gather*}
$$

The bounds we now seek are those

$$
\begin{equation*}
\left\|\widetilde{T}_{l}\right\|_{L^{p} \rightarrow L^{p}} \lesssim C_{l} \tag{3.3.8}
\end{equation*}
$$

such that $2^{-l(L+\alpha)} 2^{l N} C_{l}$ is summable over $l \in \mathbb{N}$.
As in our treatment of $T_{0}$, we begin with a kernel decomposition. We write

$$
\widetilde{K_{l}}(x, y, z)=\widetilde{K_{l}^{1}}(x, y, z)+\widetilde{K_{l}^{2}}(x, y, z)
$$

where

$$
\begin{align*}
\widetilde{K_{l}^{1}}(x, y, z) & :=\psi_{l, h}(y, z) \chi_{0}(x) k\left(2^{l} x, 2^{l} y\right) \\
\text { and } \widetilde{K_{l}^{2}}(x, y, z) & :=\psi_{l, h}(y, z)\left(1-\chi_{0}(x)\right) k\left(2^{l} x, 2^{l} y\right) \tag{3.3.9}
\end{align*}
$$

Here $\chi_{0}$ was defined at (3.3.2). So we now define the operators $\widetilde{T_{l}^{1}}$ and $\widetilde{T_{l}^{2}}$ by

$$
\widetilde{T_{l}^{1}} f(x, y, z)=\widetilde{K_{l}^{1}} * f(x, y, z), \widetilde{T_{l}^{2}} f(x, y, z)=\widetilde{K_{l}^{2}} * f(x, y, z)
$$

To analyse the operator piece corresponding with the kernel $\widetilde{K_{l}^{2}}$, we again make use of the non-stationary phase lemma. Indeed, we expand

$$
k\left(2^{l} x, 2^{l} y\right)=\int e^{2 \pi i 2^{l}|x| \Phi_{x, y}(\xi)} \phi(\xi) d \xi
$$

where

$$
\Phi_{x, y}(\xi)=\frac{x}{|x|} \cdot \xi+\frac{y}{|x|} \cdot \Psi_{\tilde{L}}(\xi)
$$

This phase factor, $\Phi_{x, y}$, is exactly the same as we saw in the analysis of (3.3.3), where we used the non-stationary phase lemma. We also know that $|x| \geq 1$ and $|y| \leq 1$ for $(x, y, z) \in \operatorname{supp} \widetilde{K_{l}^{2}}$.

For these, $(x, y, z)$, we also have the control $\left\|\Phi_{x, y}\right\|_{C^{M+1}(\operatorname{supp} \phi)} \lesssim_{M} 1+\left\|\Psi_{\tilde{L}}\right\|_{C^{M+1}(\operatorname{supp} \phi)}$ and $\|\phi\|_{C^{M}(\operatorname{supp} \phi)} \lesssim_{M} 1$. We then apply the non-stationary phase lemma, Lemma 0.0 .1 , which gives the bound

$$
\left|k\left(2^{l} x, 2^{l} y\right)\right| \lesssim_{M} \frac{1}{\left.\left(1+\left|2^{l} x\right|\right)^{M}\right)}
$$

Recall from (3.3.6) that we have a uniform bound on $\left\|\psi_{l, h}\right\|_{L^{\infty}} \leq C$ and we also know $\operatorname{supp} \psi_{l, h}=\operatorname{supp} \psi$ is compact so that

$$
\begin{equation*}
\left\|\widetilde{K_{l}^{2}}\right\|_{L^{1}} \lesssim 2^{-l M} \tag{3.3.10}
\end{equation*}
$$

provided $M>n$, where the constant is uniform in $l$. We choose $M=N+1>\tilde{N}$. For such $M$ we also have that $M>\frac{\tilde{N}}{p}+\frac{n}{2}$. By Young's Inequality, this gives

$$
\begin{equation*}
\left\|\widetilde{T_{l}^{2}} f\right\|_{L^{p}} \lesssim 2^{-l M}\|f\|_{L^{p}} \leq 2^{-l\left(\frac{\tilde{N}}{p}+\frac{n}{2}\right)}\|f\|_{L^{p}} \tag{3.3.11}
\end{equation*}
$$

The operator $\widetilde{T_{l}^{1}}$ is local. Indeed, we see that $\operatorname{supp} \widetilde{K_{l}^{1}} \subset \operatorname{supp} \chi_{0} \times \operatorname{supp} \widetilde{\psi_{0}} \subset[-2,2]^{n} \times$ $[-1,1]^{L}$. Let us now take a standard tiling of $\mathbb{R}^{N}$-up to a set of measure 0 -by translates of an open hypercube of width 100 , we denote this tiling by $\mathcal{Q}$. We also refer to the family of fattened cubes $\mathcal{Q}^{*}$, where each element of $\mathcal{Q}^{*}$ has width 101 and corresponds to an element of $\mathcal{Q}$ with the same centre. The family $\mathcal{Q}^{*}$ forms a cover of $\mathbb{R}^{N}$ and there exists a corresponding partition of unity: there exist functions $\chi_{Q} \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$ such that $\operatorname{supp} \chi_{Q} \subset Q^{*}$ and $\sum_{Q \in \mathcal{Q}} \chi_{Q}=1$. For each $Q \in \mathcal{Q}$, we set $f_{Q}=f \cdot \chi_{Q}$. We have that $\operatorname{supp} \widetilde{K_{l}^{1}} * f_{Q} \subset Q^{*}+[-2,2]^{n} \times[-1,1]^{L} \subset \widetilde{Q}$, where $\widetilde{Q}$ is the cube of width 105 which has the same centre as $Q$. The distance between the centres of non-adjacent cubes $Q$ is at least 200. As such, if $Q_{1}, Q_{2} \in \mathcal{Q}$ are not adjacent, then the functions $T_{0}^{1}\left(f_{Q_{1}}\right)$ and $T_{0}^{1}\left(f_{Q_{2}}\right)$ have disjoint support, contained in $\widetilde{Q}_{1}$ and $\widetilde{Q}_{2}$, respectively.

It follows that

$$
\begin{gather*}
\left\|\widetilde{K_{l}^{1}} * f\right\|_{p}^{p}=\left\|\widetilde{K_{l}^{1}} *\left(\sum_{Q_{j} \in \mathcal{Q}} f_{Q_{j}}\right)\right\|_{p}^{p} \\
=\left\|\sum_{Q_{j} \in \mathcal{Q}} \chi_{\widetilde{Q}_{j}} \widetilde{K_{l}^{1}} *\left(f_{Q_{j}}\right)\right\|_{p}^{p}  \tag{3.3.12}\\
\lesssim_{N} \sum_{Q_{j} \in \mathcal{Q}}\left\|\widetilde{K_{l}^{1}} *\left(f_{Q_{j}}\right)\right\|_{p}^{p},
\end{gather*}
$$

since finitely many cubes $\widetilde{Q_{j}}$ overlap: those which correspond with adjacent $Q_{j}$. In particular, to show that $\left\|\widetilde{T_{l}^{1}} f\right\|_{p}^{p} \lesssim C_{l}^{p}\|f\|_{p}^{p}$ it is sufficient to prove the inequality

$$
\left\|\widetilde{T_{l}^{1}} f_{Q_{j}}\right\|_{L^{p}\left(\widetilde{Q_{j}}\right)}^{p} \leq C_{l}^{p}\left\|f_{Q_{j}}\right\|_{p}^{p}
$$

for functions $f_{Q_{j}}$ with supp $f_{Q_{j}} \subset Q_{j}^{*}\left(\subset \widetilde{Q_{j}}\right)$ coming from our partition $\mathcal{Q}^{*}$. For ease of notation, henceforth in our analysis of $\widetilde{T_{l}^{1}}$ we consider functions $f=f_{Q}$, supported in some $Q^{*} \in \mathcal{Q}^{*}$. At stages in the proof when the support becomes relevant we will replace $f$ with $f_{Q}$ to make this apparent.

The operator $\widetilde{T_{l}^{1}}$, which is local, corresponds (via our prior rescaling) with the most significant portion of the kernel $K$ at scale $|(y, z)| \sim 2^{l}$. In our analysis of this part of the operator, we must consider the effect of the oscillation of the kernel carefully.

The appearance of the $x$ variable away from the argument of $k$ in the expression for $\widetilde{K_{l}^{1}}(x-$ $t, y-u, z-v),(3.3 .9)$, would present a technical difficulty preventing the application of extension estimates if we were to simply proceed with the proof. However, we have already established that this piece of the operator is local and now require only local estimates. To avoid the
aforementioned technical difficulty, we write

$$
\begin{gathered}
\widetilde{K_{l}^{1}}(x-t, y-u, z-v)=\psi_{l, h}(y-u, z-v) \chi_{0}(x-t) k\left(2^{l}(x-t), 2^{l}(y-u)\right) \\
=\widetilde{K_{l}^{11}}(x-t, y-u, z-v)+\widetilde{K_{l}^{12}}(x-t, y-u, z-v)
\end{gathered}
$$

where

$$
\begin{aligned}
\widetilde{K_{l}^{11}}(x, y, z) & :=\psi_{l, h}(y, z) k\left(2^{l} x, 2^{l} y\right) \\
\text { and } \widetilde{K_{l}^{12}}(x, y, z) & :=\psi_{l, h}(y, z)\left(\chi_{0}(x)-1\right) k\left(2^{l} x, 2^{l} y\right) .
\end{aligned}
$$

We define $\widetilde{T_{l}^{11}} f(x, y, z)=f * K_{l}^{11}(x, y, z)$ and $\widetilde{T_{l}^{12}} f(x, y, z)=f * K_{l}^{12}(x, y, z)$. Note that $\widetilde{K_{l}^{12}}=-\widetilde{K_{l}^{2}}$ and we previously established the $L^{1}$ bound (3.3.10) using the non-stationary phase lemma, which may be equivalently stated

$$
\left\|\widetilde{K_{l}^{12}}\right\|_{1} \lesssim 2^{-l M}
$$

with $M=N+1>\frac{\tilde{N}}{p}+\frac{n}{2}$ chosen as previously. By Young's inequality, this shows that

$$
\begin{equation*}
\left\|\widetilde{T_{l}^{12}} f_{Q_{j}}\right\|_{L^{p}\left(\widetilde{Q_{j}}\right)} \lesssim 2^{-l M}\left\|f_{Q_{j}}\right\|_{p} . \tag{3.3.13}
\end{equation*}
$$

For any choice $Q \in \mathcal{Q}$, we will apply Fubini's Theorem to analyse the operator $f_{Q} \mapsto$ $\left(f_{Q}\right) * \widetilde{K_{l}^{11}}$. We know that $f_{Q}$ is supported in the cube $Q^{*}$. We here write, $f=f_{Q}$, emphasising $Q$ wherever it becomes relevant,

$$
\begin{gathered}
\widetilde{K_{l}^{11}} * f(x, y, z) \\
=\int k\left(2^{l}(x-t), 2^{l}(y-u)\right) \psi_{l, h}(y-u, z-v) f(t, u, v) d \mu_{\mathbb{R}^{N}}(t, u, v) \\
=\int\left(\int e^{2 \pi i\left(2^{l}(x-t) \cdot \xi+2^{l}(y-u) \cdot \Psi_{\tilde{L}}(\xi)\right)} \phi(\xi) d \xi\right) \psi_{l, h}(y-u, z-v) f(t, u, v) d \mu_{\mathbb{R}^{N}}(t, u, v) \\
=\int\left(\int e^{2 \pi i\left(2^{l} x \cdot \xi+2^{l}(y-u) \cdot \Psi_{\tilde{L}}(\xi)\right)}\left(\int e^{-2 \pi i\left(2^{l} t \cdot \xi\right)} f(t, u, v) d t\right) \phi(\xi) d \xi\right) \psi_{l, h}(y-u, z-v) d \mu_{\mathbb{R}^{L}}(u, v) \\
=\int\left(\int e^{\left.2 \pi i\left(2^{l} x \cdot \xi+2^{l}(y-u) \cdot \Psi_{\tilde{L}}(\xi)\right) \widehat{f_{u, v}}\left(2^{l} \xi\right) \phi(\xi) d \xi\right) \psi_{l, h}(y-u, z-v) d \mu_{\mathbb{R}^{L}}(u, v),} .\right.
\end{gathered}
$$

where $f_{u, v}(t)=f(t, u, v)$.
Let us define $f_{u, v}^{l}(t)=2^{-l n} f_{u, v}\left(2^{-l} t\right)$ so that $\mathcal{F}\left(f_{u, v}^{l}\right)(\xi)=\widehat{f_{u, v}}\left(2^{l} \xi\right)$. We then see that

$$
\widetilde{K_{l}^{11}} * f(x, y, z)=\iint E\left(\widehat{f_{u, v}^{l}}\right)\left(2^{l} x, 2^{l}(y-u)\right) \psi_{l, h}(y-u, z-v) d u d v,
$$

where, according with Lemma 3.3.1, the extension operator is given by

$$
E g(x, y)=\int e^{2 \pi i\left(x \cdot \xi+y \cdot \Psi_{\tilde{L}}(\xi)\right)} g(\xi) \phi(\xi) d \xi
$$

The expression $\left\|\widetilde{T_{l}^{11}} f\right\|_{L^{p}(\widetilde{Q})}^{p}$ may now be bounded. We denote by $P_{z}$ the orthogonal projection onto the $(x, y)$-plane. We use Minkowski's inequality, then the bounded extension of $\tilde{Q}$ in the $z$-coordinate, then repeatedly apply Hölder's inequality, together with an $L^{2} \rightarrow L^{p}$ extension estimate and Plancherel's identity to find that

$$
\begin{gathered}
\left\|\widetilde{T_{l}^{11}} f\right\|_{L^{p}(\tilde{Q})}^{p} \\
\left.=\int_{\tilde{Q}} \mid \int E\left(\widehat{f_{u, v}^{l}}\right)\left(2^{l} x, 2^{l}(y-u)\right) \psi_{l, h}(y-u, z-v)\right)\left.d \mu_{\mathbb{R}^{L}}(u, v)\right|^{p} d \mu_{\mathbb{R}^{N}}(x, y, z)
\end{gathered}
$$

$$
\begin{align*}
& \lesssim\left(\int\left(\int_{\tilde{Q}}\left|E\left(\widehat{f_{u, v}^{l}}\right)\left(2^{l} x, 2^{l}(y-u)\right)\right|^{p} d \mu_{\mathbb{R}^{N}}(x, y, z)\right)^{\frac{1}{p}} d \mu_{\mathbb{R}^{L}}(u, v)\right)^{p} \\
& \lesssim\left(\int\left(\int_{P_{z}(\tilde{Q})}\left|E\left(\widehat{f_{u, v}^{l}}\right)\left(2^{l} x, 2^{l}(y-u)\right)\right|^{p} d \mu_{\mathbb{R}^{\tilde{N}}}(x, y)\right)^{\frac{1}{p}} d \mu_{\mathbb{R}^{L}(u, v)}\right)^{p} \\
& \lesssim 2^{-l \tilde{N}}\left(\int\left(\int_{2^{l} P_{z}(\tilde{Q})}\left|E\left(\widehat{f_{u, v}^{l}}\right)\left(x, y-2^{l} u\right)\right|^{p} d \mu_{\mathbb{R}^{\tilde{N}}}(x, y)\right)^{\frac{1}{p}} d \mu_{\mathbb{R}^{L}(u, v)}\right)^{p} \\
& \lesssim 2^{-l \tilde{N}}\left(\int\left(\int\left|\widehat{f_{u, v}^{l}}(\xi)\right|^{2} \phi(\xi) d \xi\right)^{\frac{1}{2}} d \mu_{\mathbb{R}^{L}}(u, v)\right)^{p} \\
& \lesssim 2^{-l \tilde{N}}\left(\int\left(\int\left|\widehat{f_{u, v}^{l}}(\xi)\right|^{2} d \xi\right)^{\frac{1}{2}} d \mu_{\mathbb{R}^{L}(u, v)}\right)^{p} \\
& =2^{-l \tilde{N}}\left(\int\left(\int\left|2^{-l n} f\left(2^{-l} t, u, v\right)\right|^{2} d t\right)^{\frac{1}{2}} d \mu_{\mathbb{R}^{L}}(u, v)\right)^{p} \\
& =2^{-l \tilde{N}_{2}} 2^{-\frac{l n p}{2}}\left(\int\left(\int\left|f_{Q}(t, u, v)\right|^{2} d t\right)^{\frac{1}{2}} d \mu_{\mathbb{R}^{L}}(u, v)\right)^{p} \\
& \lesssim 2^{-l \tilde{N}} 2^{-\frac{\ln p}{2}}\left(\int\left|f_{Q}(t, u, v)\right|^{2} d \mu_{\mathbb{R}^{N}}(t, u, v)\right)^{\frac{p}{2}} \\
& \lesssim 2^{-l \tilde{N}} 2^{-\frac{l n p}{2}} \int|f(t, u, v)|^{p} d \mu_{\mathbb{R}^{N}}(t, u, v) . \tag{3.3.14}
\end{align*}
$$

Combining the bounds (3.3.13) and (3.3.14), since we chose $M=N+1>\frac{\tilde{N}}{p}+\frac{n}{2}$, we find that

$$
\left\|\widetilde{T_{l}^{1}}\right\|_{L^{p}\left(Q^{*}\right) \rightarrow L^{p}(\widetilde{Q})} \leq\left\|\widetilde{T_{l}^{11}}\right\|_{L^{p}\left(Q^{*}\right) \rightarrow L^{p}(\widetilde{Q})}+\left\|\widetilde{T_{l}^{12}}\right\|_{L^{p}\left(Q^{*}\right) \rightarrow L^{p}(\widetilde{Q})} \lesssim 2^{-\frac{l \tilde{N}}{p}} 2^{-\frac{l n}{2}}
$$

Since $\widetilde{T_{l}^{1}}$ is local, we thus have that

$$
\left\|\widetilde{T_{l}^{1}}\right\|_{L^{p} \rightarrow L^{p}} \lesssim 2^{-\frac{l \tilde{N}}{p}} 2^{-\frac{l n}{2}} .
$$

In (3.3.11) we chose $M=N+1>\frac{\tilde{N}}{p}+\frac{n}{2}$ so we may now find, for $l \in \mathbb{N}$, that

$$
\left\|\widetilde{T_{l}}\right\|_{L^{p} \rightarrow L^{p}} \leq\left\|\widetilde{T_{l}^{1}}\right\|_{L^{p} \rightarrow L^{p}}+\left\|\widetilde{T_{l}^{2}}\right\|_{L^{p} \rightarrow L^{p}} \lesssim 2^{-\frac{L \tilde{\tilde{N}}}{p}} 2^{-\frac{l n}{2}} .
$$

Thus we have the bound $C_{l}$ we were seeking for (3.3.8). By the rescaling, we see that

$$
\begin{gathered}
\left\|T_{l}\right\|_{L^{p} \rightarrow L^{p}} \lesssim 2^{-l(L+\alpha)} 2^{l N}\left(\left\|\widetilde{T_{l}^{1}}\right\|_{L^{p} \rightarrow L^{p}}+\left\|\widetilde{T_{l}^{2}}\right\|_{L^{p} \rightarrow L^{p}}\right) \\
\quad 2^{-l(L+\alpha)} 2^{l N} 2^{-l \bar{N}} 2^{-\frac{l n}{2}} \\
=2^{-l\left(\tilde{L}+\alpha-\tilde{N}\left(1-\frac{1}{p}\right)+\frac{n}{2}\right)} .
\end{gathered}
$$

This expression is summable in $l$ precisely when $\tilde{L}+\alpha-\tilde{N}\left(1-\frac{1}{p}\right)+\frac{n}{2}=\tilde{L}+\alpha-\tilde{N}\left(1-\frac{1}{p}\right)+\frac{\tilde{N}-\tilde{L}}{2}=$ $\frac{\tilde{L}}{2}+\alpha+\frac{\tilde{N}}{p}-\frac{\tilde{N}}{2}>0$, i.e. for $\alpha>\frac{\tilde{N}}{2}-\frac{\tilde{N}}{p}-\frac{\tilde{L}}{2}$.

By Lemma 2.1.1, $T_{m_{\Psi, \alpha}}$ is self dual. For $p>2$ we thus have that $m_{\Psi, \alpha} \in \mathcal{M}_{p}$ for $\alpha>$ $\tilde{N}\left(1-\frac{1}{p}\right)-\frac{\tilde{N}}{2}-\frac{\tilde{L}}{2}=\frac{\tilde{N}}{2}-\frac{\tilde{N}}{p}-\frac{\tilde{L}}{2}$. This concludes the proof that, provided we have the appropriate $L^{2} \rightarrow L^{\tilde{p}}$ extension estimate and $\alpha>\max \left\{\left|\frac{\tilde{N}}{2}-\frac{\tilde{N}}{p}\right|-\frac{\tilde{L}}{2}, 0\right\}$, that $m_{\Psi, \alpha} \in \mathcal{M}_{p}$.

## Chapter 4

## A test for $L^{p} \rightarrow L^{p}$ boundedness

To determine the sharpness of the above results on the $p$ and $\alpha$ for which $m_{\Gamma, \alpha} \in \mathcal{M}_{p}$, we carry out a natural test for the $L^{p} \rightarrow L^{p}$ boundedness of the corresponding operator. In this section, for the class of surfaces $\mathcal{S}$ defined in the introduction (Definition 1.0.1), we determine precisely the range of $p$ exponents for which $K_{\Psi, \alpha} \in L^{p}$.

To recall, a model for the surfaces we consider in this context are those considered in [Whe20]. These are given as graphs by

$$
\left\{\left(\xi,|\xi|^{d_{1}},|\xi|^{d_{2}}, \ldots,|\xi|^{d_{L}}\right) \in \mathbb{R}^{n} \times \mathbb{R}^{L} ;|\xi|<\delta\right\}
$$

and

$$
\left\{\left(\xi,|\xi|^{d_{1}},|\xi|^{d_{2}}, \ldots,|\xi|^{d_{\tilde{L}}}, 0\right) \in \mathbb{R}^{n} \times \mathbb{R}^{\tilde{L}} \times \mathbb{R}^{L^{\prime}} ;|\xi|<\delta\right\}
$$

where $2 \leq d_{1}<d_{2}<\ldots<d_{L}$ are even and $\tilde{L} \geq 1$. The more general surfaces we consider can essentially be thought of as (linear transformations of) smooth perturbations of these model surfaces.

Theorem 4.0.1. We consider $\Gamma \in \mathcal{S}$. For $\alpha>0$ with $\alpha \notin 2 \mathbb{N}$, the convolution kernel $K_{\Psi, \alpha}=$ $\check{m}_{\Psi, \alpha} \in L^{p}\left(\mathbb{R}^{N}\right)$ if and only if $p>\frac{L+n}{L+\alpha+\frac{n}{2}}$.

The proofs of necessity and sufficiency in Theorem 4.0.1 are different. The kernel $K_{\Psi, \alpha}$ is given as an oscillatory integral with an explicit phase. To prove that

$$
\begin{equation*}
K_{\Psi, \alpha} \in L^{p} \Longrightarrow p>\frac{L+n}{L+\alpha+\frac{n}{2}} \tag{4.0.1}
\end{equation*}
$$

we restrict our attention to a region where the critical point of the phase is non-degenerate and use stationary phase techniques to obtain pointwise estimates on the size of the kernel. To obtain the sufficient condition

$$
\begin{equation*}
p>\frac{L+n}{L+\alpha+\frac{n}{2}} \Longrightarrow K_{\Psi, \alpha} \in L^{p} \tag{4.0.2}
\end{equation*}
$$

we adapt methods developed in work by V. Chubarikov, G. I. Arkhipov, and A. Karatsuba, [ACK79], which were used to settle a problem arising from Tarry's problem in Number Theory. Rather than pointwise control on the size of the kernel, we form a dyadic partition of space according to its size and estimate the measure of elements of the partition.
Lemma 4.0.2. To prove Theorem 4.0.1 for some $\tilde{L}$, it suffices to consider only those surfaces which are in the class $\mathcal{S}_{0}$, given at Definition 1.0.1.

In the following proof of Lemma 4.0.2, we use the atypical notation $(x, y) \in \mathbb{R}^{n} \times \mathbb{R}^{L}$. It is this proof and the arguments contained therein that justify the notation we typically use of $(x, y, z) \in \mathbb{R}^{n} \times \mathbb{R}^{\tilde{L}} \times \mathbb{R}^{L^{\prime}}$. With the notation $(x, y) \in \mathbb{R}^{n} \times \mathbb{R}^{L}$, we write the product expression
for the kernel, (2.3.1), as

$$
K_{\Psi, \alpha}(x, y)=A_{\alpha}(y) \int_{\mathbb{R}^{n}} e^{2 \pi i(x \cdot \xi+y \cdot \Psi(\xi))} \phi(\xi) d \xi
$$

Proof of Lemma 4.0.2. We have that $K_{\Psi, \alpha}(x, y)=A_{\alpha}(y) k(x, y)$, where

$$
A_{\alpha}(y)=\int_{\mathbb{R}^{L}} e^{2 \pi i y \cdot \eta}|\eta|^{\alpha} \chi(\eta) d \eta \text { and } k(x, y)=\int_{\mathbb{R}^{n}} e^{2 \pi i(x \cdot \xi+y \cdot \Psi(\xi))} \phi(\xi) d \xi .
$$

With $R$ as in Lemma 2.3.2, we set $\tilde{y}=y\left(R^{-1}\right)^{*}$ and $\widetilde{\Psi}=\Psi R$ to see that

$$
k(x, y)=\tilde{k}(x, \tilde{y})
$$

where

$$
\tilde{k}(x, y)=\int_{\mathbb{R}^{n}} e^{2 \pi i\left(x \cdot t+\sum_{j=1}^{\tilde{L}} y_{j} \tilde{\psi}_{j}(t)\right)} \phi(\xi) d \xi
$$

since

$$
x \cdot t+y \cdot \Psi(t)=x \cdot t+y \cdot \widetilde{\Psi}(t) R^{-1}=x \cdot t+\left(y\left(R^{-1}\right)^{*}\right) \cdot \widetilde{\Psi}(t) .
$$

We denote by $S_{0}$ the $(x, y)$-region with suitably large $|y| \sim\left|y\left(R^{-1}\right)^{*}\right|$, such that we can use the comparison

$$
\begin{equation*}
|A(\tilde{y})| \sim|\tilde{y}|^{-L-\alpha} \sim\left|\tilde{y} R^{*}\right|^{-L-\alpha}=|y|^{-L-\alpha} \sim|A(y)| . \tag{4.0.3}
\end{equation*}
$$

Let $\widetilde{S_{0}}$ be the corresponding region under the change of variables $(x, y) \mapsto\left(x, y\left(R^{-1}\right)^{*}\right)$. We can make the change of variables $\tilde{y}=y\left(R^{-1}\right)^{*}$ in the $L^{p}$ integration of $K_{\Psi, \alpha}(x, y)=k(x, y) A(y)$. The change of variables has a constant Jacobian. Using the comparison (4.0.3), it is simply verified that

$$
K_{\Psi, \alpha} \in L^{p}\left(S_{0}\right) \Longleftrightarrow K_{\widetilde{\Psi}, \alpha}\left(\widetilde{S_{0}}\right)
$$

where $\widetilde{\Gamma}$ is expressed as a graph by

$$
\widetilde{\Gamma}=\left\{\left(\xi, \tilde{\psi}_{1}(\xi), \tilde{\psi}_{2}(\xi), \ldots, \tilde{\psi}_{\tilde{L}}(\xi), 0, \ldots, 0\right) ;|\xi|<\delta\right\} .
$$

It remains to consider the mutual $L^{p}$ boundedness of the kernels on the complements, $S_{0}^{c}$ and $\widetilde{S}_{0}^{c}$, where we have $|y| \lesssim 1$ : with $S_{1}:=S_{0}^{c}$ and $\widetilde{S_{1}}:=\widetilde{S}_{0}^{c}$ its image under the change of variables $\tilde{y}=y\left(R^{-1}\right)^{*}$ in the $y$-coordinate: we must show $K_{\Psi, \alpha} \in L^{p}\left(S_{1}\right) \Longleftrightarrow K_{\widetilde{\Psi}, \alpha}\left(\widetilde{S_{1}}\right)$. In fact, we routinely show that $K_{\Psi, \alpha} \in L^{p}\left(S_{1}\right)$ and $K_{\widetilde{\Psi}, \alpha} \in L^{p}\left(\widetilde{S_{1}}\right)$ for all $1 \leq p \leq \infty$ using an $L^{\infty}$ estimate on $K_{\Psi, \alpha}$ and the method of non-stationary phase. We show this for $K_{\Psi, \alpha}$, the analysis may be repeated for $K_{\widetilde{\Psi}, \alpha}$. Let us set $S_{10}$ to be the region where $|y| \lesssim 1$ and $|x| \lesssim 1$ and $S_{11}$ to be the region where $|y| \lesssim 1$ and $|x| \gg 1$. Since $K_{\Psi, \alpha}$ is bounded and $\left|S_{10}\right| \lesssim 1$, we see that $K_{\Psi, \alpha} \in L^{p}\left(S_{10}\right)$ for all $1 \leq p \leq \infty$. For $(x, y) \in S_{11}$, we work by the method of nonstationary phase. Set $\Phi(t)=\frac{1}{|x|}\left(x \cdot t+\sum_{j=1}^{\tilde{L}} y_{j} \psi_{j}(t)\right)$. One can easily see that, for $t \in \operatorname{supp} \phi$, $|\nabla \Phi(t)| \gtrsim 1$ and $\|\Phi\|_{C^{n+2}(\operatorname{supp} \phi)} \lesssim 1$. Therefore, by the non-stationary phase lemma, Lemma 0.0.1, we have that

$$
\left|\int e^{2 \pi i|x| \Phi(t)} \phi(t) d t\right| \lesssim \frac{1}{|x|^{n+1}} .
$$

Integrating $\frac{1}{|x|^{n+1}} \times 1$ over $S_{11}$ shows that $K_{\Psi, \alpha} \in L^{p}\left(S_{11}\right)$.
To prove Theorem 4.0.1, in Section 4.1 we prove the implication (4.0.1) for surfaces $\Gamma \in \mathcal{S}_{0}$ and in Section 4.2 we prove the implication (4.0.2) for surfaces $\Gamma \in \mathcal{S}_{0}$. By Lemma 4.0.2, this will give 4.0.1.

### 4.1 A necessary condition for $K \in L^{p}$; the implication (4.0.1)

Recall from Section 2.3 the Bochner-Riesz kernels we are considering,

$$
\begin{equation*}
K_{\Psi, \alpha}(x, y, z)=A_{\alpha}(y, z) \int_{\mathbb{R}^{n}} e^{2 \pi i\left(x \cdot \xi+y \cdot \Psi_{\tilde{L}}(|\xi|)\right)} \phi(\xi) d \xi . \tag{4.1.1}
\end{equation*}
$$

Recall also Lemma 2.3.1, which tells us that, provided $\alpha \notin 2 \mathbb{N}$, for large ( $y, z$ ),

$$
\begin{equation*}
|A(y, z)| \sim \frac{1}{|(y, z)|^{L+\alpha}} \tag{4.1.2}
\end{equation*}
$$

Here, according with the reductions of Section 2.2 and the statement of Lemma 2.2.3, the function $\phi$ is chosen with supp $\phi \subset[-\delta, \delta]^{n}$ and we also have that $\|\phi\|_{C^{2}} \lesssim \delta^{-2}$ for some suitable small parameter $\delta$.

Proposition 4.1.1. We consider surfaces $\Gamma \in \mathcal{S}_{0}$. Provided the implicit $\delta$ parameter is taken sufficiently small and $\alpha \notin 2 \mathbb{N}$, with $K_{\Psi, \alpha}$ given as in (4.1.1),

$$
\begin{equation*}
K_{\Psi, \alpha} \in L^{p} \Longrightarrow p>\frac{L+n}{L+\alpha+\frac{n}{2}} \tag{4.1.3}
\end{equation*}
$$

Proposition 4.1.1 is a direct corollary of the following lemmas, corresponding to the $n=1$ case and $n \geq 1$ case.

Lemma 4.1.2. Provided $\operatorname{supp} \phi \subset[-\delta, \delta]$ with sufficiently small $\delta$ in the definition of the Bochner-Riesz multiplier, we have the following. We define the region

$$
\begin{gather*}
R=\left\{(x, y) \in \mathbb{R}^{1+L} ; \frac{\delta_{1}}{2} \leq \frac{|x|}{\left|y_{1}\right|} \leq \delta_{1}, \frac{\left|y_{i}\right|}{\left|y_{1}\right|} \leq \delta_{1} \text { for } i \geq 2,\right. \\
\left.y_{1} \geq C_{L r g}, y_{i} \geq 1 \text { for } i \geq 2, x \leq-1\right\}, \tag{4.1.4}
\end{gather*}
$$

where the parameter $\delta_{1}$ is some sufficiently small constant and the parameter $C_{L r g}$ is some sufficiently large constant. For $(x, y) \in R$,

$$
\begin{equation*}
|k(x, y)| \gtrsim|x|^{-\frac{1}{2}} \tag{4.1.5}
\end{equation*}
$$

Lemma 4.1.3. The region $R$ is given as

$$
\begin{gather*}
R=\left\{(x, y) \in \mathbb{R}^{n+L} ; \frac{\delta_{1}}{2} \leq \frac{|x|}{\left|y_{1}\right|} \leq \delta_{1}, \frac{\left|y_{i}\right|}{\left|y_{1}\right|} \leq \delta_{1} \text { for } i \geq 2,\right. \\
\left.y_{1} \geq C_{L r g}, y_{i} \geq 1 \text { for } i \geq 2,|x| \geq 1\right\}, \tag{4.1.6}
\end{gather*}
$$

the parameter $\delta_{1}$ is some sufficiently small constant, the parameter $C_{L r g}$ is some sufficiently large constant. For $(x, y) \in R$,

$$
\begin{equation*}
|k(x, y)| \gtrsim|x|^{-\frac{n}{2}} . \tag{4.1.7}
\end{equation*}
$$

The analysis is simpler in the $n=1$ case, as reflected in the proof of Lemma 4.1.2. For $n \geq 1$, we extend the argument, though there are additional technicalities present in the proof of Lemma 4.1.3.

Before proceeding with the proofs of Lemma 4.1.2 and Lemma 4.1.3, let us show how Proposition 4.1.1 follows.

Proof of Proposition 4.1.1. Recall the product expression (4.1.1) and our size estimate on $A(y)$ for large $y,(4.1 .2)$. Together with the estimate (4.1.7), we can use this to estimate $\left\|K_{\Psi, \alpha}\right\|_{L^{p}\left(R \times \mathbb{R}^{L^{\prime}}\right)}^{p}$.

We see that

$$
\begin{gathered}
\int_{R \times \mathbb{R}^{L^{\prime}}}\left|K_{\Psi, \alpha}(x, y)\right|^{p} d \mu_{\mathbb{R}^{N}}(x, y, z) \gtrsim \int_{R} \frac{1}{|y|^{p(L+\alpha)}} \frac{1}{|x|^{p \frac{n}{2}}} \mu_{\mathbb{R}^{\tilde{N}}}(x, y) \\
\sim \int_{D}^{\infty} r^{n+L-1} r^{-p\left(L+\alpha+\frac{n}{2}\right)} d r
\end{gathered}
$$

which is finite if and only if $p\left(L+\alpha+\frac{n}{2}\right)>n+L$.
Proof of Lemma 4.1.2. As we will see, our choice of parameters is such that the phase in the integral defining $k(x, y)$ has a unique and isolated critical point. We work by the method of stationary phase. We choose $\gamma$ such that

$$
\gamma|x|=\frac{\gamma^{d_{1}} y_{1}}{\left(d_{1}-1\right)!} \quad \text { and define } \quad \lambda:=2 \pi \gamma|x|
$$

With $\Phi(\xi)=2 \pi(x \xi+y \cdot \Psi(\xi))$ we see that

$$
\Phi(\gamma \xi)=\lambda \Phi_{0}(\xi)
$$

where

$$
\Phi_{0}(\xi):=\left(-\xi+\frac{\xi^{d_{1}}}{d_{1}}\right)+\frac{1}{\lambda}\left(y_{1} \varepsilon_{1}(\gamma \xi)+\sum_{j=2}^{\tilde{L}} y_{j} \psi_{j}(\gamma \xi)\right) .
$$

The principal term in $\Phi_{0}$ is the left bracketed term. The principal term suggests that $\Phi_{0}$ has a unique positive critical point $\xi_{0}$ which is close to 1 . We now set the scaffold for our stationary phase analysis, upon which we later fix our proof construction.

We have that

$$
\begin{gathered}
k(x, y)=\int e^{i \Phi(\xi)} \phi(\xi) d \xi=\gamma \int e^{i \lambda \Phi_{0}\left(\gamma^{-1} \xi\right)} \phi\left(\gamma\left(\gamma^{-1} \xi\right)\right) \gamma^{-1} d \xi \\
=\gamma \int e^{i \lambda \Phi_{0}(\xi)} \phi(\gamma \xi) d \xi
\end{gathered}
$$

Using a cutoff function $a$ to localise about $\xi_{0}$, we see that

$$
k(x, y)=\gamma e^{i \lambda \Phi_{0}\left(\xi_{0}\right)} \int e^{i \lambda\left(\Phi_{0}(\xi)-\Phi_{0}\left(\xi_{0}\right)\right)} \phi(\gamma \xi) a\left(\xi-\xi_{0}\right) d \xi+E_{1}(x, y)
$$

where our first error term $E_{1}$ is defined by

$$
\begin{equation*}
E_{1}(x, y):=\gamma \int e^{i \lambda \Phi_{0}(\xi)} \phi(\gamma \xi)\left(1-a\left(\xi-\xi_{0}\right)\right) d \xi \tag{4.1.8}
\end{equation*}
$$

The critical point $\xi_{0}$ is isolated and, as such, is non-degenerate. For $\xi$ near $\xi_{0}$ we expect that

$$
\Phi(\xi)-\Phi\left(\xi_{0}\right) \approx \frac{1}{2!} \Phi^{(2)}\left(\xi_{0}\right)\left(\xi-\xi_{0}\right)^{2}
$$

Informed by this, we will make a change of variables $\xi \mapsto \tau(\xi)$, where $\tau(\xi)$ is such that

$$
\begin{equation*}
\Phi(\xi)-\Phi\left(\xi_{0}\right)=\frac{1}{2!} \Phi^{(2)}\left(\xi_{0}\right) \tau(\xi)^{2} \tag{4.1.9}
\end{equation*}
$$

According with this change of variables, corresponding to each $\tau$ will be a unique $\xi_{\tau}=\xi(\tau)$. Note that $\xi(0)=\xi_{0}$. With $J(\tau)$ denoting the appropriate Jacobian factor, we find that

$$
\begin{gathered}
k(x, y) \\
=\gamma e^{i \lambda \Phi_{0}\left(\xi_{0}\right)} \int e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\xi_{0}\right) \tau^{2}} \phi\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J(\tau) d \tau+E_{1}(x, y)
\end{gathered}
$$

$$
=k_{p}(x, y)+E_{1}(x, y)+E_{2}(x, y)
$$

where the principal part is given by

$$
\begin{equation*}
k_{p}(x, y):=\gamma e^{i \lambda \Phi_{0}\left(\xi_{0}\right)} \int e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\xi_{0}\right) \tau^{2}} \phi\left(\gamma \xi_{0}\right) a(0) J(0) d \tau \tag{4.1.10}
\end{equation*}
$$

and our second error term is given by

$$
\begin{gather*}
E_{2}(x, y) \\
:=\gamma e^{i \lambda \Phi_{0}\left(\xi_{0}\right)} \int e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\xi_{0}\right) \tau^{2}}\left[\phi\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}\right) J(\tau)-\phi\left(\gamma \xi_{0}\right) a(0) J(0)\right] d \tau . \tag{4.1.11}
\end{gather*}
$$

Looking to the principal part of $k$, (4.1.10), we will see that $\phi\left(\gamma \xi_{0}\right) a(0)=1$. By setting $\tilde{\tau}=\sqrt{\lambda \Phi^{(2)}\left(\xi_{0}\right)} \tau$, we then find

$$
\left|k_{p}(x, y)\right|=\frac{\gamma J(0)}{\sqrt{\lambda \Phi^{(2)}\left(\xi_{0}\right)}}\left|\int e^{i \frac{1}{2!} \tilde{\tau}^{2}} d \tilde{\tau}\right| \sim_{\delta_{1}} \frac{1}{|x|^{\frac{1}{2}}}
$$

Thus, if we can show that, for $j \in\{1,2\}$,

$$
\begin{equation*}
\left|E_{j}(x, y)\right| \ll \frac{1}{|x|^{\frac{1}{2}}}, \tag{4.1.12}
\end{equation*}
$$

with a suitable constant, then the result follows as $k(x, y)=k_{p}(x, y)+E_{1}(x, y)+E_{2}(x, y)$.
Before establishing the desired control, (4.1.12), on the error terms let us more precisely consider what appears above. We did not specify the control on the error terms in the phase or establish that there is a unique critical point of $\Phi_{0}$. Also, we did not show that the implicit change of variables $\xi \mapsto \tau$ in (4.1.9) was well defined or give the explicit Jacobian factor $J(\tau)$. Let us now do so. The sequencing of the following analysis reflects the sequential dependence of parameters $\delta_{2}$ (later to appear), $\delta, \delta_{1}$, and, finally, $C_{\mathrm{Lrg}}$.

Observe that $\gamma$, which we defined at the start of the proof, is such that $\gamma=\left(-\frac{\left(d_{1}-1\right)!x}{y_{1}}\right)^{\frac{1}{d_{1}-1}} \sim$ $\delta_{1}^{-\frac{1}{d_{1}-1}} \sim_{\delta_{1}} 1$. For the phase error from $\Phi_{0}$,

$$
\begin{equation*}
\theta_{0}(\xi):=\frac{1}{\lambda}\left(y_{1} \varepsilon_{1}(\gamma \xi)+\sum_{j=2}^{\tilde{L}} y_{j} \psi_{j}(\gamma \xi)\right) \tag{4.1.13}
\end{equation*}
$$

we now observe that, for $\xi \in \operatorname{supp} \phi\left(\gamma^{-1}\right)$ and $l \in\{0,1,2\}$,

$$
\begin{equation*}
\left|\theta_{0}^{(l)}(\xi)\right| \leq \epsilon_{\delta}|\xi|^{d_{1}-l} \tag{4.1.14}
\end{equation*}
$$

where $\epsilon_{\delta}$ can be made arbitrarily small provided $\delta$ is chosen sufficiently small. Recall that $\lambda=2 \pi y_{1} \gamma^{d_{1}} /\left(d_{1}-1\right)!,\left|y_{j}\right| \leq \delta_{1}\left|y_{1}\right|$, and note $\gamma \lesssim 1$. For $2 \leq j \leq \tilde{L}, l \in\{0,1,2\}$, and $\xi \in \operatorname{supp} \phi\left(\gamma^{-1}.\right) \subset\left[-\gamma^{-1} \delta, \gamma^{-1} \delta\right]$, we see that, by Taylor's theorem,

$$
\begin{align*}
& \left|\frac{1}{\lambda} y_{j} \gamma^{l} \psi_{j}^{(l)}(\gamma \xi)\right| \leq \gamma^{-d_{1}} \gamma^{l} \delta_{1}\left|\psi_{j}^{(l)}(\gamma \xi)\right| \\
& \quad \lesssim \gamma^{d_{j}-d_{1}}|\xi|^{d_{j}-l} \\
& \ll|\xi|^{d_{1}-l} \tag{4.1.15}
\end{align*}
$$

provided $\delta \ll 1$ is chosen sufficiently small. Similarly, by Taylor's theorem, for $\xi \in \operatorname{supp} \phi\left(\gamma^{-1}\right)$ and $l \in\{0,1,2\}$, we have that

$$
\begin{align*}
\frac{y_{1}}{\lambda}\left|\gamma^{l} \varepsilon_{1}^{(l)}(\gamma \xi)\right|= & \frac{y_{1}}{\lambda}\left|\gamma^{l} \psi_{1}^{(l)}(\gamma \xi)-\gamma^{l} \frac{(\gamma \xi)^{d_{1}-l}}{\left(d_{1}-l\right)!}\right| \\
& \ll|\xi|^{d_{1}-l} \tag{4.1.16}
\end{align*}
$$

provided we take $\delta \ll 1$ sufficiently small, as discussed in Definition 1.0.1 for the class $\mathcal{S}_{0}$. Summing (4.1.15) and (4.1.16) gives the desired bound (4.1.14). We also have the following constant bounds on $\theta_{0}$. For $l \in\{0,1,2,3,4\}$ and $\xi \in \operatorname{supp} \phi\left(\gamma^{-1}\right.$.),

$$
\begin{equation*}
\left|\theta_{0}^{(l)}(\xi)\right| \lesssim C_{1}, \tag{4.1.17}
\end{equation*}
$$

for some fixed $C_{1}>0$. The proof of (4.1.17) is analogous to the proof of (4.1.14) and is omitted. With this control on the phase error, we can make the above analysis rigorous.

Firstly, we observe that $\Phi_{0}^{\prime}$ has a unique critical point, $\xi_{0}$, because $\left|\theta_{0}^{(2)}(\xi)\right| \leq \epsilon_{\delta}|\xi|^{d_{1}-2}$ and, in particular, since $d_{1}$ is even, $\Phi^{(2)}(\xi)=\left(d_{1}-1\right) \xi^{d_{1}-2}+\theta_{0}^{(2)}(\xi)>0$ for non-zero $\xi \in \operatorname{supp} \phi(\gamma \cdot)$ and $\Phi_{0}^{\prime}(0)=-1<0$. Furthermore, we can verify that

$$
\begin{equation*}
\frac{3}{4} \leq \xi_{0} \leq \frac{5}{4} \tag{4.1.18}
\end{equation*}
$$

Indeed, for now, let us claim that, for $|\xi-1| \geq \frac{1}{4}$,

$$
\left|-1+\xi^{d_{1}-1}\right| \gtrsim|\xi|^{d_{1}-1} .
$$

Given the claim, provided the $\delta$ we later fix is chosen sufficently small, we see that, for $|\xi-1| \geq \frac{1}{4}$, with $\xi \neq 0$

$$
\left|\Phi^{\prime}(\xi)\right| \gtrsim\left|-1+\xi^{d_{1}-1}\right|-\left|\theta_{0}^{\prime}(\xi)\right| \gtrsim|\xi|^{d_{1}-1}>0
$$

by (4.1.14). It remains to prove our claim. To see this, let us consider $\xi \geq \frac{5}{4}$. We observe that $1 \leq\left(\frac{4}{5} \xi\right)^{d_{1}-1}$ and write

$$
\begin{gathered}
\xi^{d_{1}-1}-1 \\
\geq \xi^{d_{1}-1}-\left(\frac{4}{5} \xi\right)^{d_{1}-1} \\
\gtrsim \xi^{d_{1}-1}
\end{gathered}
$$

Likewise, for $0 \leq \xi \leq \frac{3}{4}, 1 \geq\left(\frac{4}{3} \xi\right)^{d_{1}-1}$ so that

$$
\begin{aligned}
& 1-\xi^{d_{1}-1} \\
& \gtrsim \xi^{d_{1}-1} .
\end{aligned}
$$

For $\xi \leq 0$,

$$
1-\xi^{d_{1}-1} \gtrsim-\xi^{d_{1}-1} .
$$

We now consider the change of variables we claimed implicitly at (4.1.9). Observe that

$$
\begin{gathered}
\Phi_{0}(\xi)-\Phi_{0}\left(\xi_{0}\right)=\int_{0}^{1}\left(\xi-\xi_{0}\right) \Phi_{0}^{\prime}\left(\xi_{0}+s\left(\xi-\xi_{0}\right)\right) d s \\
=-\int_{0}^{1} \frac{d}{d s}(1-s)\left(\xi-\xi_{0}\right) \Phi_{0}^{\prime}\left(\xi_{0}+s\left(\xi-\xi_{0}\right)\right) d s \\
=\frac{1}{2!} \Phi_{0}^{(2)}\left(\xi_{0}\right)\left(\xi-\xi_{0}\right)^{2} G\left(\xi-\xi_{0}\right)
\end{gathered}
$$

where

$$
G\left(\xi-\xi_{0}\right)=\int_{0}^{1} \frac{2(1-s) \Phi_{0}^{(2)}\left(\xi_{0}+s\left(\xi-\xi_{0}\right)\right)}{\Phi_{0}^{(2)}\left(\xi_{0}\right)} d s
$$

One can see that, for $\xi \in \operatorname{supp} a\left(\cdot-\xi_{0}\right), G(\xi)>0$. The implicit change of variables (4.1.9) is thus given explictly by $\xi \mapsto \tau(\xi)=\left(\xi-\xi_{0}\right) \sqrt{ } G\left(\xi-\xi_{0}\right)$. We see that

$$
\frac{d \tau}{d \xi}(\xi)=\sqrt{G\left(\xi-\xi_{0}\right)}+\left(\xi-\xi_{0}\right) \frac{G^{\prime}\left(\xi-\xi_{0}\right)}{2 \sqrt{G\left(\xi-\xi_{0}\right)}}
$$

We will make our choice of cutoff function $a$ with sufficiently small support to ensure that, for $\xi \in \operatorname{supp} a\left(\cdot-\xi_{0}\right)$,

$$
\begin{equation*}
\frac{d \tau}{d \xi}(\xi) \sim 1 \tag{4.1.19}
\end{equation*}
$$

We see that the Jacobian factor is given by

$$
J(\tau)=\frac{d \xi}{d \tau}(\tau)=\frac{\sqrt{G\left(\xi_{\tau}-\xi_{0}\right)}}{G\left(\xi_{\tau}-\xi_{0}\right)+\left(\xi_{\tau}-\xi_{0}\right) \frac{G^{\prime}\left(\xi_{\tau}-\xi_{0}\right)}{2}}
$$

Note that

$$
J(0)=\frac{\sqrt{G(0)}}{G(0)+0}=1
$$

For $\xi \in \operatorname{supp} a$, we require explicit control on $G$ and its derivatives. We choose a preliminary bound $\delta \leq \delta_{0}$ so that the constant $\epsilon_{\delta}<1 / 2$, where $\epsilon_{\delta}$ is the error parameter in (4.1.14), for all $\delta \leq \delta_{0}$. We later fix $\delta$. We also suppose the following preliminary structure for our cutoff function $a \in C_{c}^{\infty}(\mathbb{R}): \operatorname{supp} a \subset\left[-\frac{1}{4}, \frac{1}{4}\right]$.

First, we bound $G(\tilde{\xi})$. Recall that

$$
\Phi_{0}(\xi)=\left(-\xi+\frac{\xi^{d_{1}}}{d_{1}}\right)+\theta_{0}(\xi)
$$

For $\tilde{\xi} \in \operatorname{supp} a$ and $s \in[0,1], \frac{1}{2} \leq\left|\xi_{0}+s \tilde{\xi}\right| \leq \frac{3}{2}$. In particular, for these $s,\left|\xi_{0}+s \tilde{\xi}\right| \sim\left|\xi_{0}\right|$. We can see that, for $\tilde{\xi} \in \operatorname{supp} a$

$$
\begin{gathered}
|G(\tilde{\xi})|=\left|\int_{0}^{1} \frac{2(1-s) \Phi_{0}^{(2)}\left(\xi_{0}+s \tilde{\xi}\right)}{\Phi_{0}^{(2)}\left(\xi_{0}\right)} d s\right| \\
\leq\left|\int_{0}^{1} \frac{2(1-s)\left(\left(d_{1}-1\right)\left|\xi_{0}+s \tilde{\xi}\right|^{d_{1}-2}+\left|\theta_{0}^{(2)}\left(\xi_{0}+s \tilde{\xi}\right)\right|\right)}{\left(d_{1}-1\right)\left|\xi_{0}\right|^{d_{1}-2}-\left|\theta_{0}^{(2)}\left(\xi_{0}\right)\right|} d s\right| \\
\leq\left|\int_{0}^{1} \frac{3(1-s)\left(d_{1}-1\right)\left|\xi_{0}+s \tilde{s}\right|^{d_{1}-2}}{\left(d_{1}-1\right)\left|\xi_{0}\right|^{d_{1}-2} / 2} d s\right| \\
\sim\left|\int_{0}^{1} \frac{3(1-s)\left|\xi_{0}\right|^{d_{1}-2}}{\left|\xi_{0}\right|^{d_{1}-2}} d s\right| \\
\sim 1 .
\end{gathered}
$$

Similarly,

$$
\begin{gathered}
|G(\tilde{\xi})| \geq\left|\int_{0}^{1} \frac{2(1-s)\left(\left(d_{1}-1\right)\left|\xi_{0}+s \tilde{\xi}\right|^{d_{1}-2}-\left|\theta_{0}^{(2)}\left(\xi_{0}+s \tilde{\xi}\right)\right|\right)}{\left(d_{1}-1\right)\left|\xi_{0}\right|^{d_{1}-2}+\left|\theta_{0}^{(2)}\left(\xi_{0}\right)\right|} d s\right| \\
\geq\left|\int_{0}^{1} \frac{2(1-s) \frac{1}{2}\left(d_{1}-1\right)\left|\xi_{0}+s \tilde{\xi}\right|^{d_{1}-2}}{3\left|\xi_{0}\right|^{d_{1}-2} / 2} d s\right| \\
\sim 1 .
\end{gathered}
$$

We will also make use of upper bounds on the size of $G^{\prime}, G^{\prime \prime}$, and $G^{\prime \prime \prime}$. We can see that, for $\tilde{\xi} \in \operatorname{supp} a$,

$$
\left|G^{\prime}(\tilde{\xi})\right| \leq\left|\int_{0}^{1} \frac{2(1-s)\left(\left(d_{1}-1\right)\left(d_{1}-2\right)\left|\xi_{0}+s \tilde{\xi}\right|^{d_{1}-3}+\left|\theta_{0}^{(3)}\left(\xi_{0}+s \tilde{\xi}\right)\right|\right)}{\left(d_{1}-1\right)\left|\xi_{0}\right|^{d_{1}-2}-\left|\theta_{0}^{(2)}\left(\xi_{0}\right)\right|} d s\right|
$$

$$
\begin{gathered}
\leq\left|\int_{0}^{1} \frac{2(1-s)\left(\left(d_{1}-1\right)\left(d_{1}-2\right)\left|\xi_{0}+s \tilde{\xi}\right|^{d_{1}-3}+\left|\theta_{0}^{(3)}\left(\xi_{0}+s \tilde{\xi}\right)\right|\right)}{\left(d_{1}-1\right)\left|\xi_{0}\right|^{d_{1}-2}-\left|\theta_{0}^{(2)}\left(\xi_{0}\right)\right|} d s\right| \\
\leq\left|\int_{0}^{1} \frac{2(1-s)\left(\frac{3}{2}\left(d_{1}-1\right)\left(d_{1}-2\right)\left|\xi_{0}+s \tilde{\xi}\right|^{d_{1}-3}\right)}{\frac{1}{2}\left(d_{1}-1\right)\left|\xi_{0}\right|^{d_{1}-2}} d s\right| \\
\sim\left|\int_{0}^{1} \frac{2(1-s)\left(\xi_{0}^{d_{1}-3}\right)}{\left|\xi_{0}\right|^{d_{1}-2}} d s\right| \\
\sim 1 .
\end{gathered}
$$

Likewise for $G^{\prime \prime}$ and $G^{\prime \prime \prime}$. We write these bounds explicitly: for $\tilde{\xi} \in \operatorname{supp} a$,

$$
\begin{equation*}
0<c_{0-} \leq|G(\tilde{\xi})| \leq c_{0+},\left|G^{\prime}(\tilde{\xi})\right| \leq c_{1},\left|G^{\prime \prime}(\tilde{\xi})\right| \leq c_{2},\left|G^{\prime \prime \prime}(\tilde{\xi})\right| \leq c_{3} \tag{4.1.20}
\end{equation*}
$$

We can now verify (4.1.19). It is here we will fix our choice of cutoff function $a$, with $\operatorname{supp} a \subset\left[-\delta_{2}, \delta_{2}\right]$, where $0<\delta_{2}<\frac{1}{4}$ is taken sufficiently small so that $\delta_{2} \frac{c_{1}}{2 \sqrt{c_{0-}}} \leq \sqrt{c_{0+}}$ and $\delta_{2} \frac{c_{1}}{2 \sqrt{c_{0+}}} \leq \sqrt{c_{0-}} / 2$. We see that, for $\xi \in \operatorname{supp} a\left(\cdot-\xi_{0}\right)$,

$$
\begin{aligned}
\left|\frac{d \tau}{d \xi}(\xi)\right| & =\left|\sqrt{G\left(\xi-\xi_{0}\right)}+\left(\xi-\xi_{0}\right) \frac{G^{\prime}\left(\xi-\xi_{0}\right)}{2 \sqrt{G\left(\xi-\xi_{0}\right)}}\right| \\
& \leq \sqrt{c_{0+}}+\delta_{2} \frac{c_{1}}{2 \sqrt{c_{0-}}} \leq 2 \sqrt{c_{0+}}
\end{aligned}
$$

and

$$
\begin{gathered}
\left|\frac{d \tau}{d \xi}(\xi)\right|=\left|\sqrt{G\left(\xi-\xi_{0}\right)}+\left(\xi-\xi_{0}\right) \frac{G^{\prime}\left(\xi-\xi_{0}\right)}{2 \sqrt{G\left(\xi-\xi_{0}\right)}}\right| \\
\geq \sqrt{c_{0-}}-\delta_{2} \frac{c_{1}}{2 \sqrt{c_{0+}}} \\
\geq \sqrt{c_{0-}} / 2 .
\end{gathered}
$$

We must track more than the support properties of $a$, we later use the fact that

$$
\begin{equation*}
\|a\|_{C^{2}} \lesssim \delta_{2}^{-2} \tag{4.1.21}
\end{equation*}
$$

which we can see if we define

$$
a(\xi)=a_{0}\left(\delta_{2}^{-1} \xi\right)
$$

for some fixed $a_{0} \in C_{c}^{\infty}(\mathbb{R})$ with $\operatorname{supp} a_{0} \subset[-1,1]$ and $a_{0}(\xi)=1$ for $\xi \in\left[-\frac{1}{2}, \frac{1}{2}\right]$. By this definition, we also see that $\operatorname{supp}(1-a(\cdot)) \subset\left(-\infty, \frac{\delta_{2}}{2}\right] \cup\left[\frac{\delta_{2}}{2}, \infty\right)$.

The next part of our analysis concerns the appropriate choice of $\delta$. We previously supposed $\delta$ was subject to the condition $\epsilon_{\delta}<\frac{1}{2}$ when we obtained bounds for $G$ and its derivatives and also that it was sufficiently small for (4.1.18) to hold. Now, we have fixed $\delta_{2}$ the $\delta$ we choose must be sufficiently small to ensure $\left|\xi_{0}-1\right| \leq \delta_{2} / 4$, with some additional conditions that we will set out momentarily. Recall that we previously showed, at (4.1.18), that $\left|\xi_{0}-1\right| \leq \frac{1}{4}$. For $\frac{1}{4} \geq|\tilde{\xi}| \geq \delta_{2} / 4$, we can write

$$
\Phi^{\prime}(1+\tilde{\xi})=\Phi^{\prime}(1)+\tilde{\xi} \int_{0}^{1} \Phi^{\prime \prime}(1+s \tilde{\xi}) d s
$$

We can see by (4.1.14) that $\left|\Phi^{\prime}(1)\right| \leq \epsilon_{\delta}$. Likewise, for $\frac{1}{4} \geq|\tilde{\xi}| \geq \delta_{2} / 4$ and $s \in[0,1]$,

$$
\left.\left(d_{1}-1\right)\left(\frac{3}{4}\right)^{d_{1}-2}-\epsilon_{\delta}\left(\frac{5}{4}\right)^{d_{1}-2} \leq \inf _{|1-\xi| \in\left[\delta_{2} / 4, \frac{1}{4}\right]}\left|\left(d_{1}-1\right) \xi\right|^{d_{1}-2}-\sup _{|1-\xi| \in\left[\delta_{2} / 4, \frac{1}{4}\right]} \right\rvert\, \theta_{0}^{\prime \prime}(\xi) \leq \Phi^{\prime \prime}(1+s \tilde{\xi})
$$

so that

$$
\int_{0}^{1} \Phi^{\prime \prime}(1+s \tilde{\xi}) d s \geq c>0
$$

provided our later choice of $\delta$ sufficiently small depending on $d_{1}$. Therefore, for $\frac{1}{4} \geq|\tilde{\xi}| \geq \delta_{2} / 4$,

$$
\begin{gathered}
\left|\Phi^{\prime}(1+\tilde{\xi})\right| \\
\geq|\tilde{\xi}|\left|\int_{0}^{1} \Phi^{\prime \prime}(1+s \tilde{\xi}) d s\right|-\left|\Phi^{\prime}(1)\right| \\
\geq c \delta_{2} / 4-\epsilon_{\delta}
\end{gathered}
$$

We now fix choose $\delta$ sufficiently small so this final term is $>c \delta_{2} / 8$, which shows that $\xi_{0} \in$ $\left(1-\delta_{2} / 4,1+\delta_{2} / 4\right)$ and, further, for $\xi \in\left[\frac{3}{4}, \frac{5}{4}\right] \backslash\left(1-\delta_{2} / 4,1+\delta_{2} / 4\right)$,

$$
\left|\Phi^{\prime}(\xi)\right|>c \delta_{2} / 8
$$

By monotonicity of $\Phi^{\prime}$, we then have that, for all $\xi \in \operatorname{supp} \phi(\gamma \cdot) \backslash\left(1-\delta_{2} / 4,1+\delta_{2} / 4\right)$,

$$
\left|\Phi_{0}^{\prime}(\xi)\right| \gtrsim \delta_{2}
$$

If $\left|\xi-\xi_{0}\right|>\delta_{2} / 2$, then $|\xi-1|>\delta_{2} / 4$, since $\left|\xi_{0}-1\right|<\delta_{2} / 4$. Therefore, for $\xi \in \operatorname{supp} \phi(\gamma \cdot) \cap$ $\operatorname{supp}\left(1-a\left(\cdot-\xi_{0}\right)\right)$,

$$
\begin{equation*}
\left|\Phi_{0}^{\prime}(\xi)\right| \gtrsim \delta_{2} \tag{4.1.22}
\end{equation*}
$$

To recall, we used preliminary bounds on $\delta_{2}$ and $\delta$ to control $G, G^{\prime}$, and $G^{\prime \prime}$. We then fixed $\delta_{2}$ sufficiently small so that the change of variables was well controlled by (4.1.19). We then fixed $\delta$ sufficiently small to ensure that $\xi_{0}$ was close to 1 and the size of $\Phi_{0}^{\prime}$ was controlled away from $\xi_{0}$. We must now choose $\delta_{1}$ sufficiently small to ensure that $F(0)=1$, this gives us our principal term (4.1.10).

To obtain the principle term (4.1.10), we require that $\phi\left(\gamma \xi_{0}\right)=1$ and also that $a(0)=1$. Because $\gamma=\left(-\frac{\left(d_{1}-1\right)!x}{y_{1}}\right)^{\frac{1}{d_{1}-1}} \sim \delta_{1}^{\frac{1}{d_{1}-1}}$ and $\xi_{0} \sim 1$, the first of these conditions is ensured by now fixing $\delta_{1}$ sufficiently small. The second condition is true for any choice of smooth cutoff $a$ about the origin.

Having captured the principal term, $k_{p}$, we turn our consideration to the error terms. First, for $E_{1}$, which we gave at (4.1.8), we can use the bound on (4.1.22) on $\Phi_{0}^{\prime}$ and the non-stationary phase lemma, Lemma 0.0.1, to see that

$$
\begin{equation*}
\left|E_{1}(x, y)\right| \lesssim \delta_{2} \lambda^{-1} \lesssim \delta_{1}|x|^{-1} . \tag{4.1.23}
\end{equation*}
$$

Let us now analyse the error $E_{2}$ coming from the change of variables, (4.1.11). Setting $F(\tau)=\phi\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J(\tau)$, we can write

$$
E_{2}(x, y)=\gamma e^{i \lambda \Phi_{0}\left(\xi_{0}\right)} \int e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\xi_{0}\right) \tau^{2}}(F(\tau)-F(0)) d \tau
$$

We find that

$$
\begin{gathered}
\left|E_{2}(x, y)\right|=\gamma\left|\int e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\xi_{0}\right) \tau^{2}} \int_{0}^{1} \tau F^{\prime}(s \tau) d s d \tau\right| \\
=\gamma\left|\int_{0}^{1} \int e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\xi_{0}\right) \tau^{2}} \tau F^{\prime}(s \tau) d \tau d s\right| \\
=\gamma\left|\int_{0}^{1} \int \frac{1}{i \lambda \Phi^{(2)}\left(\xi_{0}\right)} \frac{d}{d \tau}\left(e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\xi_{0}\right) \tau^{2}}\right) F^{\prime}(s \tau) d \tau d s\right| \\
\leq \gamma \frac{1}{\lambda \Phi^{(2)}\left(\xi_{0}\right)} \int_{0}^{1} \int\left|s F^{\prime \prime}(s \tau)\right| d \tau d s
\end{gathered}
$$

$$
\begin{equation*}
\lesssim \delta_{1} \frac{1}{|x|}\left\|F^{\prime \prime}\right\|_{L^{\infty}}\left|\operatorname{supp} F^{\prime \prime}(\cdot)\right| \tag{4.1.24}
\end{equation*}
$$

Recall from (4.1.19) that, for $\xi \in \operatorname{supp} a\left(\cdot-\xi_{0}\right)$,

$$
\frac{d \tau}{d \xi}(\xi) \sim 1
$$

Thus we see that $J(\tau)=\frac{d \xi}{d \tau} \sim 1$ for $\tau \in \operatorname{supp} a(\xi(\cdot))$ and also $\operatorname{supp} a(\xi(\cdot)) \subset\left[-C \delta_{2}, C \delta_{2}\right] \subset$ $[-1,1]$. Using the fact that $J(\tau)=\frac{d \xi}{d \tau}$, we consider the derivatives of $F(\tau)=\phi\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\right.$ $\left.\xi_{0}\right) J(\tau)$. We have that

$$
\begin{aligned}
F^{\prime}(\tau) & =\gamma J(\tau) \phi^{\prime}\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J(\tau)+\phi\left(\gamma \xi_{\tau}\right) J(\tau) a^{\prime}\left(\xi_{\tau}-\xi_{0}\right) J(\tau)+\phi\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J^{\prime}(\tau) \\
& =\gamma \phi^{\prime}\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J(\tau)^{2}+\phi\left(\gamma \xi_{\tau}\right) a^{\prime}\left(\xi_{\tau}-\xi_{0}\right) J(\tau)^{2}+\phi\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J^{\prime}(\tau)
\end{aligned}
$$

Furthermore,

$$
F^{\prime \prime}(\tau)
$$

$$
\begin{aligned}
= & \gamma^{2} \phi^{\prime \prime}\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J(\tau)^{3}+\gamma \phi^{\prime}\left(\gamma \xi_{\tau}\right) a^{\prime}\left(\xi_{\tau}-\xi_{0}\right) J(\tau)^{3}+\gamma \phi^{\prime}\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) 2 J^{\prime}(\tau) J(\tau) \\
& +\gamma \phi^{\prime}\left(\gamma \xi_{\tau}\right) a^{\prime}\left(\xi_{\tau}-\xi_{0}\right) J(\tau)^{3}+\phi\left(\gamma \xi_{\tau}\right) a^{\prime \prime}\left(\xi_{\tau}-\xi_{0}\right) J(\tau)^{3}+\phi\left(\gamma \xi_{\tau}\right) a^{\prime}\left(\xi_{\tau}-\xi_{0}\right) 2 J^{\prime}(\tau) J(\tau) \\
& +\gamma \phi\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J^{\prime}(\tau) J(\tau)+\phi\left(\gamma \xi_{\tau}\right) a^{\prime}\left(\xi_{\tau}-\xi_{0}\right) J^{\prime}(\tau) J(\tau)+\phi\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J^{\prime \prime}(\tau)
\end{aligned}
$$

Thus, if we can find a suitable $L^{\infty}$ bound for all the terms appearing in this sum, then we can obtain an $L^{\infty}$ bound for $F^{\prime \prime}$. Our choice of $\phi$ in Section 2.2, at (2.2.2), ensured that

$$
\|\phi\|_{C^{2}} \lesssim \delta^{-2}
$$

We also have that

$$
\|a\|_{C^{2}} \lesssim \delta_{2}^{-2}
$$

as we previously discussed at (4.1.21). Recall that

$$
J(\tau)=\frac{\sqrt{G\left(\xi_{\tau}-\xi_{0}\right)}}{G\left(\xi_{\tau}-\xi_{0}\right)+\left(\xi_{\tau}-\xi_{0}\right) \frac{G^{\prime}\left(\xi_{\tau}-\xi_{0}\right)}{2}} .
$$

We already established that $|J(\tau)| \lesssim 1$. We now see

$$
\begin{gathered}
J^{\prime}(\tau) \\
=\frac{1}{2} \frac{G^{\prime}\left(\xi_{\tau}-\xi_{0}\right)}{\sqrt{G\left(\xi_{\tau}-\xi_{0}\right)}\left(G\left(\xi_{\tau}-\xi_{0}\right)+\left(\xi_{\tau}-\xi_{0}\right) \frac{G^{\prime}\left(\xi_{\tau}-\xi_{0}\right)}{2}\right)} J(\tau) \\
-\frac{\left(G^{\prime}\left(\xi_{\tau}-\xi_{0}\right)+\frac{G^{\prime}\left(\xi_{\tau}-\xi_{0}\right)}{2}+\left(\xi_{\tau}-\xi_{0}\right) \frac{G^{\prime \prime}\left(\xi_{\tau}-\xi_{0}\right)}{2}\right) \sqrt{G\left(\xi_{\tau}-\xi_{0}\right)}}{\left(G\left(\xi_{\tau}-\xi_{0}\right)+\left(\xi_{\tau}-\xi_{0}\right) \frac{G^{\prime}\left(\xi_{\tau}-\xi_{0}\right)}{2}\right)} J(\tau),
\end{gathered}
$$

so that, from our bounds on $G,(4.1 .20)$, and our choice of $\delta_{2} \frac{c_{1}}{2 \sqrt{c_{0+}}} \leq \sqrt{c_{0-}} / 2$

$$
\begin{gathered}
\left|J^{\prime}(\tau)\right| \\
\lesssim \frac{c_{1}}{\sqrt{c_{0-}}\left(c_{0-}-\delta_{2} \frac{c_{1}}{2}\right)} J(\tau) \\
+\frac{\left(c_{1}+\frac{c_{1}}{2}+\delta_{2} \frac{c_{2}}{2}\right)}{\left(\sqrt{c_{0-}}-\delta_{2} \frac{c_{1}}{2}\right)} J(\tau) \\
\lesssim 1 .
\end{gathered}
$$

The situation is analogous for $J^{\prime \prime}$. We thus find that

$$
\begin{equation*}
\left\|F^{\prime \prime}\right\|_{L^{\infty}(a)} \lesssim \delta, \delta_{2} 1 . \tag{4.1.25}
\end{equation*}
$$

To summarise, we have shown that

$$
k(x, y)=k_{p}(x, y)+E_{1}(x, y)+E_{2}(x, y)
$$

where

$$
\begin{aligned}
& \left|k_{p}(x, y)\right| \sim_{\delta_{1}} \frac{1}{|x|^{\frac{1}{2}}} \\
& \left|E_{1}(x, y)\right| \lesssim \delta_{2} \frac{1}{|x|}
\end{aligned}
$$

as we established at (4.1.23), and

$$
\left|E_{2}(x, y)\right| \lesssim \delta, \delta_{1}, \delta_{2} \frac{1}{|x|}
$$

which we obtain from (4.1.24) and (4.1.25). The desired result follows by taking $C_{\mathrm{Lrg}}$ large enough so that $\left|E_{1}(x, y)\right| \ll|x|^{-\frac{1}{2}} \sim\left|k_{p}(x, y)\right|$ and $\left|E_{2}(x, y)\right| \ll|x|^{-\frac{1}{2}}$, which is possible since $|x| \gtrsim \delta_{1}\left|y_{1}\right| \gtrsim \delta_{1} C_{\text {Lrg }}$.

Proof of Lemma 4.1.3. The proof is similar to the $n=1$ case. However, since we will be carrying out stationary phase analysis with respect to one dimensional oscillatory integrals, we must first use polar integration. In order that the resulting oscillatory integrals can be expressed with explicit phases, we use the asymptotic expansion for the surface measure of the sphere. As such, we will need to analyse additional error terms.

First, we write

$$
\begin{gathered}
k(x, y)=\int e^{2 \pi i\left(x \cdot \xi+y \cdot \Psi_{\tilde{L}}(|\xi|)\right)} \phi(\xi) d \xi \\
\int_{0}^{\infty} \int_{S^{n-1}} e^{2 \pi i\left(x \cdot(r \omega)+y \cdot \Psi_{\tilde{L}}(r)\right)} \phi(r \omega) d \sigma(\omega) r^{n-1} d r \\
\int_{0}^{\infty} e^{2 \pi i \sum_{j=1}^{\tilde{L}} y_{j} \psi_{j}(r)} \int_{S^{n-1}} e^{2 \pi i r x \cdot \omega} d \sigma(\omega) \phi_{0}(r) r^{n-1} d r \\
\int_{0}^{\infty} e^{2 \pi i \sum_{i=1}^{\tilde{L}} y_{j} \psi_{j}(r)} \hat{\sigma}(2 \pi r x) \phi_{0}(r) r^{n-1} d r,
\end{gathered}
$$

where $\phi_{0}$ is such that $\phi_{0}(|\xi|)=\phi(\xi)$. We now rescale, choosing $\gamma$ such that $\gamma|x|=\frac{\gamma^{d_{1}} y_{1}}{\left(d_{1}-1\right)!}$ and setting $\gamma \rho=r$, we see that

$$
\gamma^{n} \int_{0}^{\infty} e^{2 \pi i \sum_{j=1}^{\tilde{L}} y_{j} \psi_{j}(\gamma \rho)} \hat{\sigma}(2 \pi \gamma \rho x) \phi_{0}(\gamma \rho) \rho^{n-1} d \rho .
$$

We use the asymptotic expression for the Fourier transform of the surface measure of the sphere

$$
\hat{\sigma}(2 \pi r x)=\frac{1}{|r x|^{\frac{n-1}{2}}}\left(c_{+} e^{2 \pi i r|x|}+c_{-} e^{-2 \pi i r|x|}+E_{\sigma}(r x)\right),
$$

for $|r x| \gtrsim 1$, where the $c_{-}$and $c_{+}$are non-zero constants and $\left|E_{\sigma}(r x)\right| \leq C|r x|^{-1}$; see, for instance, [Ste93]. We will use a cutoff function $b$, with small support about the origin, to split the region of integration about $\left.|\rho| x\right|^{\frac{1}{2}} \mid \sim \delta_{3}$, for some small $\delta_{3}$ later to be fixed. More specifically, we take $b(r)=b_{0}\left(\delta_{3}^{-1} r\right)$ for some fixed $b_{0} \in C_{c}^{\infty}$ with $b_{0}(r)=1$ for $|r| \leq \frac{1}{2}$ and $b_{0}(r)=1$ for $|r| \geq 1$. This will ensure we can apply asymptotic expansion of $\hat{\sigma}$ and also provide an error term coming from integration near the origin. In particular, we find that

$$
k(x, y)=\gamma^{n} \int_{0}^{\infty} e^{2 \pi i \sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)} \hat{\sigma}(2 \pi \gamma \rho x) \phi_{0}(\gamma \rho) \rho^{n-1} b\left(\rho|x|^{\frac{1}{2}}\right) d \rho
$$

$$
\begin{aligned}
& +\gamma^{n} \int_{0}^{\infty} e^{2 \pi i \sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)} \hat{\sigma}(2 \pi \gamma \rho x) \phi_{0}(\gamma \rho) \rho^{n-1}\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) d \rho \\
& \left.\quad=\gamma^{n} \int_{0}^{\infty} e^{2 \pi i \sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)} \hat{\sigma}(2 \pi \gamma \rho x) \phi_{0}(\gamma \rho) \rho^{n-1} b\left(\rho|x|^{\frac{1}{2}}\right)\right) d \rho \\
& +\frac{\gamma^{n}}{|x|^{\frac{n-1}{2}}} \int e^{2 \pi i\left(\sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)\right)} \phi_{0}(\gamma \rho) \rho^{\frac{n-1}{2}} E_{\sigma}(\gamma \rho x)\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) d \rho \\
& +\frac{\gamma^{n-\frac{n-1}{2}}}{|x|^{\frac{n-1}{2}}} c_{+} \int_{0}^{\infty} e^{2 \pi i\left(\gamma \rho|x|+\sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)\right)} \phi_{0}(\gamma \rho) \rho^{\frac{n-1}{2}}\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) d \rho \\
& +\frac{\gamma^{n-\frac{n-1}{2}}}{|x|^{\frac{n-1}{2}}} c_{-} \int e^{2 \pi i\left(-\gamma \rho|x|+\sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)\right)} \phi_{0}(\gamma \rho) \rho^{\frac{n-1}{2}}\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) d \rho .
\end{aligned}
$$

So far, the final term is the principal term. Indeed, let us set

$$
\begin{gather*}
E_{0}(x, y):=\gamma^{n} \int_{0}^{\infty} e^{2 \pi i \sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)} \hat{\sigma}(2 \pi \gamma \rho x) \phi_{0}(\gamma \rho) \rho^{n-1} b\left(\rho|x|^{\frac{1}{2}}\right) d \rho,  \tag{4.1.26}\\
E_{-1}(x, y):=\frac{\gamma^{n}}{|x|^{\frac{n-1}{2}}} \int_{0}^{\infty} e^{2 \pi i\left(\sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)\right)} \phi_{0}(\gamma \rho) \rho^{\frac{n-1}{2}} E_{\sigma}(\gamma \rho x)\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) d \rho, \tag{4.1.27}
\end{gather*}
$$

and

$$
\begin{equation*}
E_{-2}(x, y):=\frac{\gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} c_{+} \int_{0}^{\infty} e^{2 \pi i\left(\gamma \rho|x|+\sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)\right)} \phi_{0}(\gamma \rho) \rho^{\frac{n-1}{2}}\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) d \rho \tag{4.1.28}
\end{equation*}
$$

Also, we set

$$
k_{0}(x, y):=\frac{\gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} c_{-} \int e^{2 \pi i\left(-\gamma \rho|x|+\sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)\right)} \phi_{0}(\gamma \rho) \rho^{\frac{n-1}{2}}\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) d \rho
$$

We know that

$$
k(x, y)=k_{0}(x, y)+E_{0}(x, y)+E_{-1}(x, y)+E_{-2}(x, y)
$$

Working as in the one dimensional case, by splitting $k_{0}(x, y)=k_{p}(x, y)+E_{1}(x, y)+E_{2}(x, y)$, we show that

$$
\left|k_{0}(x, y)\right| \sim \frac{1}{|x|^{\frac{n}{2}}}
$$

Finally, we will show that, for $l \in\{-2,-1,0\}$,

$$
\left|E_{l}(x, y)\right| \ll \frac{1}{|x|^{\frac{n}{2}}}
$$

We set $\lambda=2 \pi \rho|x|$ and

$$
\Phi_{0}(\rho)=-\rho+\frac{\rho^{d_{1}}}{d_{1}}+\frac{1}{\lambda}\left(\psi_{1}(\gamma \rho)-\frac{(\gamma \rho)^{d_{1}}}{d_{1}!}\right)+\frac{1}{\lambda} \sum_{j=2}^{L} y_{j} \psi_{j}(\gamma \rho)
$$

To reflect the structure of the above proof, let us set

$$
\begin{equation*}
\phi_{x}(\gamma \rho)=\phi_{0}(\gamma \rho) \rho^{\frac{n-1}{2}}\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) . \tag{4.1.29}
\end{equation*}
$$

We have that

$$
k_{0}(x, y)=\frac{c_{-} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} \int e^{i \lambda \Phi_{0}(\rho)} \phi_{x}(\gamma \rho) d \rho
$$

The phase $\lambda \Phi_{0}$ is exactly the same we considered in the one dimensional case. As such, much of the above analysis can be carried through. In particular, we will carry out a decomposition
with respect to the unique critical point, $\rho_{0}$, of $\Phi_{0}$. As before, we have that

$$
\begin{equation*}
\frac{3}{4} \leq \rho_{0} \leq \frac{5}{4} \tag{4.1.30}
\end{equation*}
$$

We may find that $k_{0}(x, y)=k_{p}(x, y)+E_{1}(x, y)+E_{2}(x, y)$, where the error terms $E_{1}$ and $E_{2}$ are defined as we did in the one dimensional case; the original definitions for $E_{1}$ and $E_{2}$ were given at (4.1.8) and (4.1.11), respectively. These terms are now given by

$$
\begin{align*}
k_{p}(x, y) & :=\frac{c_{-} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} e^{i \lambda \Phi_{0}\left(\rho_{0}\right)} \int e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\rho_{0}\right) \tau^{2}} \phi_{x}\left(\gamma\left(\rho_{0}\right)\right) a(0) J(0) d \tau  \tag{4.1.31}\\
& E_{1}(x, y):=\frac{c_{-} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} \int e^{i \lambda \Phi_{0}(\rho)} \phi_{x}(\gamma(\rho))\left(1-a\left(\rho-\rho_{0}\right)\right) d \rho \tag{4.1.32}
\end{align*}
$$

and

$$
\begin{equation*}
:=\frac{c_{-} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} e^{i \lambda \Phi_{0}\left(\rho_{0}\right)} \int e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\rho_{0}\right) \tau^{2}}\left[\phi_{x}\left(\gamma\left(\rho_{\tau}\right)\right) a\left(\rho_{\tau}-\rho_{0}\right) J(\tau)-\phi_{x}\left(\gamma\left(\rho_{0}\right)\right) a(0) J(0)\right] d \tau . \tag{4.1.33}
\end{equation*}
$$

In our proof of Lemma 4.1.2, much of the analysis does not refer to properties of the amplitude $\phi$, but the key things we used were $\operatorname{supp} \phi \subset[-\delta, \delta],\|\phi\|_{C^{2}} \lesssim \delta^{-2}$ and $\phi\left(\gamma \xi_{0}\right) \sim 1$ (actually, we used that $\left.\phi\left(\gamma \xi_{0}\right)=1\right)$. For $\xi \in \operatorname{supp}\left(1-a\left(\cdot-\rho_{0}\right)\right)$, we established at (4.1.22) that

$$
\begin{equation*}
\left|\Phi_{0}^{\prime}(\rho)\right| \gtrsim \delta_{2} \tag{4.1.34}
\end{equation*}
$$

only using the fact that $\operatorname{supp} \phi \subset[-\delta, \delta]$. We previously used this along with the non-stationary phase lemma to obtain the bound (4.1.23) on $E_{1}$ and here the amplitude $\phi$ made an (implicit) contribution to the analysis in the lemma's application. Presently, we must be more exact and integrate by parts explicitly to understand the contribution of $\phi_{x}$ to $E_{1}$. The exact same method is used to bound $E_{-2}$, and we refer the reader to our later proof of the bound (4.1.40). The proof of (4.1.40) can be repeated to provide the same bound for $E_{1}$, though the implicit constant will have a $\delta_{2}$ dependence coming from (4.1.34). In particular, we can show that

$$
\begin{equation*}
\left|E_{1}(x, y)\right| \lesssim \delta_{2}, \delta_{3}, \delta_{1} \frac{1}{|x|^{\frac{n+1}{2}}} . \tag{4.1.35}
\end{equation*}
$$

Regarding the error term $E_{2}$ and the contribution of $\phi_{x}$, (4.1.29), the bounds we work with correspond in the previous proof to (4.1.24). With the same change of variables $\rho \mapsto \tau$ we made in the previous proof, presently we define

$$
F(\tau):=\phi_{x}\left(\gamma\left(\rho_{\tau}\right)\right) a\left(\rho_{\tau}-\rho_{0}\right) J(\tau)
$$

where $J(\tau)=\frac{d \rho}{d \tau}(\tau)$. In the one dimensional proof, $F(\tau)$ was given by $\phi\left(\gamma \xi_{\tau}\right) a\left(\xi_{\tau}-\xi_{0}\right) J(\tau)$. In obtaining the bound (4.1.24), the information we used about $\phi$ was, in essence, the support condition $\operatorname{supp} \phi \subset[-\delta, \delta]$. To properly establish $E_{2}$ as an error term, we derived a bound with $\left\|F^{\prime \prime}\right\|_{L^{\infty}} \lesssim \delta, \delta_{1}, \delta_{2} 1$ as a factor in (4.1.24). Here we saw a contribution of $\|\phi\|_{C^{2}} \lesssim_{\delta} 1$ to the bound on $\left\|F^{\prime \prime}\right\|_{L^{\infty}}$. Due to the presence of the $\left(1-b\left(|x|^{\frac{1}{2}} \cdot\right)\right)$ factor, we no longer have this global $C^{2}$-norm control on (4.1.29),

$$
\phi_{x}(\gamma \rho)=\phi_{0}(\gamma \rho) \rho^{\frac{n-1}{2}}\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) .
$$

Indeed, one can verify that we only have

$$
\begin{equation*}
\left\|\phi_{x}\right\|_{C^{2}} \lesssim \delta, \delta_{1}, \delta_{2}, \delta_{3}|x| . \tag{4.1.36}
\end{equation*}
$$

However, despite this we can still show that

$$
\left\|F^{\prime \prime}\right\|_{L^{\infty}} \lesssim \delta, \delta_{1}, \delta_{2} 1
$$

Indeed, note that if $b^{\prime}\left(\rho|x|^{\frac{1}{2}}\right) \neq 0$, then $\frac{\delta_{3}}{2}|x|^{-\frac{1}{2}} \leq|\rho| \leq \delta_{3}|x|^{-\frac{1}{2}}<\frac{1}{4}$. However, $\operatorname{supp} a\left(\cdot-\rho_{0}\right) \subset$ $\left[\frac{1}{2}, \frac{3}{2}\right]$, by (4.1.30) and the fact $a$ has small support. From this it follows that

$$
\begin{aligned}
\left\|\phi_{x}\right\|_{C^{2}\left(\operatorname{supp} a\left(\cdot-\rho_{0}\right)\right)} & \lesssim\left(1+\left\|\phi_{0}\right\|_{C^{2}}\right)\|b\|_{\infty} \\
& \lesssim \delta^{-2}
\end{aligned}
$$

which is sufficient for the desired $L^{\infty}$ bound on $F^{\prime \prime}$. Thus we have that

$$
\begin{equation*}
\left|E_{2}(x, y)\right| \lesssim \delta, \delta_{1}, \delta_{2}, \delta_{3} \frac{1}{|x|^{\frac{n+1}{2}}} . \tag{4.1.37}
\end{equation*}
$$

Let us consider the error term $E_{0}$ coming from the region of integration close to the origin, (4.1.26). It is here that our choice of the $\delta_{3}$ parameter will be important. This parameter appears defining $b(r):=b_{0}\left(\delta_{3}^{-1} r\right)$, for some cutoff function $b_{0} \in C_{c}^{\infty}$ with $b_{0}(r)=1$ for $|r| \leq \frac{1}{2}$ and $b_{0}(r)=1$ for $|r| \geq 1$. We see that

$$
\begin{gather*}
\left|E_{0}(x, y)\right|=\left|\gamma^{n} \int_{0}^{\infty} e^{2 \pi i \sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)} \hat{\sigma}(2 \pi \gamma \rho x) \phi_{0}(\gamma \rho) \rho^{n-1} b\left(\rho|x|^{\frac{1}{2}}\right) d \rho\right| \\
\lesssim\left|\gamma^{n} \int_{0}^{\delta_{3}|x|^{-\frac{1}{2}}} \rho^{n-1} d \rho\right| \\
\lesssim \delta_{1} \frac{\delta_{3}^{n}}{|x|^{\frac{n}{2}}} . \tag{4.1.38}
\end{gather*}
$$

We now consider the error term $E_{-1}$ coming from the asymptotic remainder, (4.1.27). Using the fact that, $\left|E_{\sigma}(r x)\right| \leq C|r x|^{-1}$, we see that

$$
\begin{gather*}
\left|E_{-1}(x, y)\right|=\frac{\gamma^{n}}{|x|^{\frac{n-1}{2}}}\left|\int e^{2 \pi i\left(\sum_{j=1}^{L} y_{j} \psi_{j}(\gamma \rho)\right)} \phi_{0}(\gamma \rho) \rho^{\frac{n-1}{2}} E_{\sigma}(\gamma \rho x)\left(1-b\left(\rho|x|^{\frac{1}{2}}\right)\right) d \rho\right| \\
\left.\left.\lesssim \frac{\gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}}\left|\int_{\frac{\delta_{3}}{2}|x|^{-1}}^{\delta \gamma^{-1}} \rho^{\frac{n-1}{2}}\right| \rho x\right|^{-1} d \rho \right\rvert\, \\
\lesssim \frac{\gamma^{\frac{n+1}{2}}}{|x|^{\frac{n+1}{2}}}\left(\delta \gamma^{-1}\right)^{\frac{n-1}{2}} \\
\sim_{\delta_{1}, \delta} \frac{1}{|x|^{\frac{n+1}{2}}} \tag{4.1.39}
\end{gather*}
$$

We also consider the error term $E_{-2}$, (4.1.28), which is defined by an integral with nonstationary phase. We let $\lambda=2 \pi \gamma|x|$ as previously and write the phase $\Phi_{+}(\rho)=\rho+\frac{\rho^{d_{1}}}{d_{1}}+\theta_{0}(\rho)$, where

$$
\theta_{0}(\rho):=\frac{1}{\lambda}\left(y_{1}\left(\psi_{1}(\gamma \rho)-\frac{(\gamma \rho)^{d_{1}}}{d_{1}!}\right)+\sum_{j=1}^{\tilde{L}} y_{j} \psi_{j}(\gamma \rho)\right)
$$

The phase error $\theta_{0}$ is defined exactly as in the one dimensional case, (4.1.13), and we have the same control (4.1.14). In particular, for $\rho \in \operatorname{supp} \phi_{x}$, it is easy to verify that $\left|\Phi_{+}^{\prime}(\rho)\right| \gtrsim 1$. We can use integration by parts to express

$$
\begin{gathered}
E_{-2}(x, y)=\frac{c_{+} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} \int_{0}^{\infty} \frac{1}{i \lambda \Phi_{+}^{\prime}(\rho)} \frac{d}{d \rho}\left(e^{i \lambda \Phi_{+}(\rho)}\right) \phi_{x}(\gamma \rho) d \rho \\
=\frac{c_{+} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} \int_{0}^{\infty} e^{i \lambda \Phi_{+}(\rho)} \frac{d}{d \rho}\left(\frac{1}{i \lambda \Phi_{+}^{\prime}(\rho)} \phi_{x}(\gamma \rho)\right) d \rho
\end{gathered}
$$

$$
\begin{aligned}
& =\frac{c_{+} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} \int_{0}^{\infty} \frac{1}{i \lambda \Phi_{+}^{\prime}(\rho)} \frac{d}{d \rho}\left(e^{i \lambda \Phi_{+}(\rho)}\right) \frac{d}{d \rho}\left(\frac{1}{i \lambda \Phi_{+}^{\prime}(\rho)} \phi_{x}(\gamma \rho)\right) d \rho \\
& =\frac{c_{+} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} \int_{0}^{\infty} e^{i \lambda \Phi_{+}(\rho)} \frac{d}{d \rho}\left(\frac{1}{i \lambda \Phi_{+}^{\prime}(\rho)} \frac{d}{d \rho}\left(\frac{1}{i \lambda \Phi_{+}^{\prime}(\rho)} \phi_{x}(\gamma \rho)\right)\right) d \rho
\end{aligned}
$$

Using the fact that $\left|\Phi_{+}^{\prime}(\rho)\right| \gtrsim 1,\left\|\phi_{x}(\gamma \cdot)\right\|_{C^{2}} \lesssim \delta, \delta_{1}, \delta_{2}, \delta_{3}|x|$, and $\left\|\Phi_{+}\right\|_{C^{3}\left(\operatorname{supp} \phi_{x}\right)} \lesssim 1$, we thus find

$$
\begin{align*}
\left|E_{-2}(x, y)\right| \lesssim \delta, \delta_{1}, \delta_{2}, \delta_{3} & \frac{c_{+} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}}\left|\operatorname{supp} \phi_{x}\right| \frac{1}{\lambda^{2}}|x| \\
\quad \delta_{3}, \delta_{1} & \frac{1}{|x|^{\frac{n+1}{2}}} . \tag{4.1.40}
\end{align*}
$$

To summarise, we have decomposed

$$
k(x, y)=k_{p}(x, y)+E_{1}(x, y)+E_{2}(x, y)+E_{0}(x, y)+E_{-1}(x, y)+E_{-2}(x, y)
$$

Recall the principal term $k_{p}$, which we defined at (4.1.31), is given by

$$
k_{p}(x, y)=\frac{c_{-} \gamma^{\frac{n+1}{2}}}{|x|^{\frac{n-1}{2}}} e^{i \lambda \Phi_{0}\left(\rho_{0}\right)} \int e^{i \lambda \frac{1}{2!} \Phi^{(2)}\left(\rho_{0}\right) \tau^{2}} \phi_{x}\left(\gamma\left(\rho_{0}\right)\right) a(0) J(0) d \tau .
$$

We see for $\phi_{x}$, (4.1.29), that

$$
\begin{gathered}
\phi_{x}\left(\gamma\left(\rho_{0}\right)\right) a(0) J(0)=\phi_{0}(\gamma \rho) \rho_{0}^{\frac{n-1}{2}}\left(1-b\left(\rho_{0}|x|^{\frac{1}{2}}\right)\right) \\
=\phi_{0}(\gamma \rho) \rho_{0}^{\frac{n-1}{2}} \sim 1,
\end{gathered}
$$

as we will show that

$$
\begin{equation*}
\delta_{3}<\rho_{0}|x|^{\frac{1}{2}} \tag{4.1.41}
\end{equation*}
$$

Therefore

$$
\left|k_{p}(x, y)\right| \sim_{\delta_{1}} \frac{1}{|x|^{\frac{n}{2}}} .
$$

We now fix $\delta_{3}$ sufficiently small, according with (4.1.38), to ensure that

$$
\left|E_{0}(x, y)\right| \leq \frac{\left|k_{p}(x, y)\right|}{10}
$$

Finally, summing the bounds (4.1.35), (4.1.37), (4.1.39), and (4.1.40), we see that

$$
\begin{aligned}
& \mid E_{-2}(x, y)+ E_{-1}(x, y)+E_{1}(x, y)+E_{2}(x, y) \mid \\
& \lesssim \delta, \delta_{1}, \delta_{2}, \delta_{3} \\
&|x|^{\frac{n+1}{2}}
\end{aligned}
$$

We can thus ensure, provided we take $C_{\text {Lrg }}$ sufficiently large, that

$$
\left|E_{-2}(x, y)+E_{-1}(x, y)+E_{1}(x, y)+E_{2}(x, y)\right| \leq \frac{4}{10}\left|k_{p}(x, y)\right|
$$

because $\left|k_{p}(x, y)\right| \sim|x|^{-\frac{n}{2}}$ and $|x|^{\frac{1}{2}} \gtrsim \delta_{1}\left|y_{1}\right|^{\frac{1}{2}} \geq C_{\mathrm{Lrg}}^{\frac{1}{2}}$. A large choice of $C_{\mathrm{Lrg}}$ will also ensure that (4.1.41) holds, since $\rho_{0}|x|^{\frac{1}{2}} \gtrsim \delta_{1} C_{\mathrm{Lrg}}^{\frac{1}{2}}$, which can be made sufficiently large. This completes the proof.

### 4.2 A sufficient condition for $K \in L^{p}$; the implication (4.0.2)

In this section, for surfaces $\Gamma \in \mathcal{S}$, we work to establish sufficient conditions for the BochnerRiesz kernel, given at (4.1.1), to be an element of $L^{p}$. By Lemma 4.0.2, without loss of generality, in this section, we may restrict our attention to the surfaces in class $\mathcal{S}_{0}$.

Proposition 4.2.1. We consider $\Gamma \in \mathcal{S}_{0}$ and the corresponding kernel $K_{\Psi, \alpha}$. If $p>\frac{L+n}{L+\alpha+\frac{n}{2}}$, then the kernel

$$
K_{\Psi, \alpha} \in L^{p}\left(\mathbb{R}^{N}\right)
$$

One of the main results that we make use of in this section is a version of van der Corput's lemma we proved in the preliminary part of this document, Lemma 0.0.8. Let us recall its statement.

Lemma. Let $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function for which there exists $k \in \mathbb{N}$ with

$$
\inf _{t \in[a, b]}\left|\Phi^{(k)}(t)\right| \geq c \sup _{t \in[a, b]}\left|\Phi^{(k)}(t)\right|>0,
$$

for some $c>0$, and $\phi$ be a smooth function with $\operatorname{supp} \phi \subset(a, b)$. Additionally, we assume that $\Phi^{\prime}$ is monotonic on boundedly many intervals in the case that $k=1$. Given the bound

$$
\inf _{t \in[a, b]} \sum_{j=1}^{k}\left|\frac{\Phi^{(j)}(t)}{j!}\right|^{\frac{1}{j}} \geq \kappa,
$$

we have that

$$
\left|\int_{a}^{b} e^{2 \pi i \Phi(t)} \phi(t) d t\right| \lesssim_{k} \min \left\{(b-a), \kappa^{-1}\right\}\left\|\phi^{\prime}\right\|_{L^{1}}
$$

In particular, we have that

$$
\left|\int_{a}^{b} e^{2 \pi i \Phi(t)} \phi(t) d t\right| \lesssim \min \left\{(b-a), \kappa^{-1}\right\}(b-a)\left\|\phi^{\prime}\right\|_{L^{\infty}} .
$$

For the surfaces in $\mathcal{S}_{0}$ from Definition 1.0.1, it is easily verified, provided $\delta$ is chosen sufficiently small, that, for $t \in I=[-\delta, \delta]$,

$$
\text { and, for } 1 \leq j \leq \tilde{L} \text { with } j \neq l, \quad \begin{array}{r}
\frac{1}{2} \leq\left|\psi_{l}^{\left(d_{l}\right)}(t)\right| \leq 2,  \tag{4.2.1}\\
\left|\psi_{j}^{\left(d_{l}\right)}(t)\right| \leq \epsilon,
\end{array}
$$

for some small parameter $\epsilon$ to be made explicit in the course of our analysis. The parameter $\delta$ is some small constant to be made explicit in the course of the proof.

The following is a model for our main result.
Proposition 4.2.2. We consider the algebraic variety

$$
\Gamma=\left\{\left(\xi, \frac{\xi^{d_{1}}}{d_{1}!}, \frac{\xi^{d_{2}}}{d_{2}!}, \ldots, \frac{\xi^{d_{\tilde{L}}}}{d_{\tilde{L}}!}, 0, \ldots, 0\right) ;-\delta<\xi<\delta\right\}
$$

where $2 \leq d_{1}<d_{2}<\ldots<d_{\tilde{L}}$ are even. If $p>\frac{L+1}{L+\alpha+\frac{1}{2}}$, then the (graphical) Bochner-Reisz kernel

$$
K_{\Psi, \alpha}(x, y, z)=k(x, y) A(y, z) \in L^{p}\left(\mathbb{R}^{N}, d x d y d z\right)
$$

In higher dimensions, for $n \geq 1$, we have the more general Proposition 4.2.1. We state and prove the simpler Proposition 4.2.2 first as the core ideas are set in clearer relief in this setting.

The specific application of van der Corput's lemma which we make use of to prove Proposition 4.2.2 is clarified by the following lemma.

Lemma 4.2.3. With $\Phi_{x, y}(t):=x t+\sum_{j=1}^{\tilde{L}} y_{j} t^{d_{j}} / d_{j}!$, and

$$
k(x, y)=\int e^{2 \pi i \Phi_{x, y}(t)} \phi(t) d t
$$

we have the following. If $H(x, y) \geq \kappa>0$, where

$$
H(x, y)=\inf _{t \in[-\delta, \delta]} \sum_{j=1}^{d_{\tilde{L}}}\left|\frac{\Phi_{x, y}^{(j)}(t)}{j!}\right|^{\frac{1}{j}}
$$

then

$$
|k(x, y)| \lesssim \kappa^{-1}
$$

in the region $|y| \gtrsim|x|,|y| \gg 1$.
In higher dimensions, the lemma corresponding to Lemma 4.2.3 has a more complicated statement and proof. Indeed, we will require the following to prove Proposition 4.2.1, where we must make use of a new cutoff function $\eta$.

Lemma 4.2.4. With $\Phi(r)=\Phi_{x, y, \pm}(r):= \pm|x| r+\sum_{j=1}^{\tilde{L}} y_{j} \psi_{j}(r)$. We define

$$
H(x, y)=\inf _{r \in[0, \delta]} \sum_{j=1}^{d_{\tilde{L}}}\left|\frac{\Phi^{(j)}(r)}{j!}\right|^{\frac{1}{j}}
$$

If $H(x, y) \geq \kappa$, then we have that

$$
\left|\int_{0}^{\infty} e^{2 \pi i \Phi(r)} \frac{1}{|r x|^{\frac{n-1}{2}}} \eta(|x| r) \phi_{0}(r) r^{n-1} d r\right| \lesssim \kappa^{-1} \frac{1}{|x|^{\frac{n-1}{2}}} .
$$

We require the following lower bound on $H$, which is evident in the polynomial case.
Lemma 4.2.5. For $|y| \gtrsim 1$, we have that $H(x, y) \gtrsim|y|^{\frac{1}{d_{\tilde{L}}}}$.
Proof. We prove this by relating $H$ to the quantity

$$
\widetilde{H^{1}}(x, y)=\max _{j=1,2, \ldots, d_{\tilde{L}}} \inf _{t \in I}\left|\Phi_{x, y}^{(j)}(t)\right| .
$$

We show that this, which is clearly homogeneous, is bounded strictly away from 0 on the sphere. We first show that it is non-vanishing on the sphere.

The non-vanishing comes by considering the $d_{j}$ derivatives. If

$$
\widetilde{H^{1}}(x, y)=\max _{j=1,2, \ldots, d_{\tilde{L}}} \inf _{t \in I}\left|\Phi_{x, y}^{(j)}(t)\right|=0,
$$

then, in particular,

$$
\begin{gathered}
\inf _{t \in I}\left|\Phi_{x, y}^{\left(d_{\tilde{L}}\right)}(t)\right| \\
=\inf _{t \in I}\left|\sum_{l<L} y_{l} \psi_{l}^{\left(d_{\tilde{L}}\right)}(t)+y_{L} \psi_{L}^{\left(d_{\tilde{L}}\right)}(t)\right|=0 .
\end{gathered}
$$

Therefore, using our observation from (4.2.1) that $\left|\psi_{l}^{\left(d_{\tilde{L}}\right)}(t)\right| \leq \epsilon$ for $l<L$ and $\frac{1}{2} \leq\left|\psi_{L}^{\left(d_{\tilde{L}}\right)}(t)\right| \leq$ 2 , we find that $\frac{1}{2}\left|y_{L}\right| \leq(L-1) \epsilon|y|$.

Next, we see that

$$
\inf _{t \in I}\left|\sum_{l<L-1} y_{l} \psi_{l}^{\left(d_{L-1}\right)}(t)+y_{L} \psi_{L}^{\left(d_{L-1}\right)}(t)+y_{L-1} \psi_{L-1}^{\left(d_{L-1}\right)}(t)\right|=0
$$

so that, since $\frac{1}{2} \leq\left|\psi_{L-1}^{\left(d_{L-1}\right)}(t)\right| \leq 2,\left|\psi_{l}^{\left(d_{L-1}\right)}(t)\right| \leq \epsilon$ for $l<L-1$, and $\left|\psi_{L}^{\left(d_{L-1}\right)}(t)\right| \leq 2$, we find that

$$
\frac{1}{2}\left|y_{L-1}\right| \leq((L-2)+4(L-1)) \epsilon|y| .
$$

Continuing in this fashion, we find for each $1 \leq l \leq L$ that

$$
\left|y_{l}\right| \lesssim \epsilon|y|,
$$

so that $|y| \lesssim \epsilon|y|$. As a consequence, provided $\epsilon$ is chosen small enough, we must have that $y=0$. Further, we find that $x=0$, since $\inf _{t \in I}\left|\Phi^{\prime}(t)\right|=|x|=0$.

To show that $\inf _{(x, y) \in S^{N-1}} \widetilde{H^{1}}(x, y)>0$ we suppose by way of contradiction that there exists a sequence $z_{n} \in S^{N-1}$ with $\widetilde{H^{1}}\left(z_{n}\right) \rightarrow 0$. Passing to a subsequence if necessary we can assume that $z_{n} \rightarrow z \in S^{N-1}$. We see that, for $1 \leq j \leq d_{\tilde{L}}$, by choosing $t_{n}$ such that $\inf _{t \in I}\left|\Phi_{z_{n}}^{(j)}(t)\right|=\left|\Phi_{z_{n}}^{(j)}\left(t_{n}\right)\right|$, we have

$$
\begin{aligned}
\widetilde{H^{1}}\left(z_{n}\right) \geq \inf _{t \in I}\left|\Phi_{z_{n}}^{(j)}(t)\right| & =\left|\Phi_{z_{n}}^{(j)}\left(t_{n}\right)\right| \geq\left|\Phi_{z}^{(j)}\left(t_{n}\right)\right|-\left|\Phi_{z}^{(j)}\left(t_{n}\right)-\Phi_{z_{n}}^{(j)}\left(t_{n}\right)\right| \\
& \geq \inf _{t \in I}\left|\Phi_{z}^{(j)}(t)\right|-C\left|z_{n}-z\right|
\end{aligned}
$$

Taking the maximum over $j$ shows that

$$
\widetilde{H^{1}}\left(z_{n}\right) \geq \widetilde{H^{1}}(z)-C\left|z_{n}-z\right|
$$

We know that $\widetilde{H^{1}}(z) \neq 0$ and $\left|z_{n}-z\right| \rightarrow 0$ so taking $n \rightarrow \infty$ we obtain a contradiction, $0>0$. Therefore, we can conclude $\inf _{(x, y) \in S^{N-1}} \widetilde{H^{1}}(x, y)>0$. As a consequence of this and homogeneity, we have that $\widetilde{H^{1}}(x, y) \gtrsim|(x, y)| \geq|y|$.

Now relating $H$ to the homogenous $\widetilde{H^{1}}$, which is non-vanishing on the sphere, shows that $H(x, y) \gtrsim|y|^{\frac{1}{d_{\tilde{L}}}}$. Indeed, if $\widetilde{H^{1}}(x, y) \gtrsim 1$, which we have if $|y| \gtrsim 1$, then there exists $t^{*}=$ $t^{*}(x, y) \in I$ and a suitable $j_{0}$ so that

$$
\begin{gathered}
H(x, y)=H_{t^{*}}(x, y)=\sum_{j=1}^{d_{\tilde{L}}}\left|\frac{\Phi^{(j)}\left(t^{*}\right)}{j!}\right|^{\frac{1}{j}} \\
\geq \max _{j=1, \ldots d_{\tilde{L}}}\left|\frac{\Phi^{(j)}\left(t^{*}\right)}{j!}\right|^{\frac{1}{j}} \geq\left|\frac{\Phi^{\left(j_{0}\right)}\left(t^{*}\right)}{j_{0}!}\right|^{\frac{1}{j_{0}}} \geq\left(\inf _{t \in I}\left|\frac{\Phi^{\left(j_{0}\right)}(t)}{j_{0}!}\right|\right)^{\frac{1}{j_{0}}} \\
=\left(\widetilde{H^{1}}(x, y)\right)^{\frac{1}{j_{0}}} \gtrsim\left(\widetilde{H^{1}}(x, y)\right)^{\frac{1}{d_{\tilde{L}}}} \gtrsim|y|^{\frac{1}{d_{\widetilde{L}}}} .
\end{gathered}
$$

Now let us establish the oscillatory integral estimate we make use of in the model one dimensional case, with a polynomial phase.

Proof of Lemma 4.2.3. Suppose that $H(x, y) \geq \kappa>0$. We must have that $(x, y) \neq 0$. Let the index $0 \leq m \leq L$ be chosen maximally so that $y_{m} \neq 0$. Then we know that $\left|\Phi^{\left(d_{m}\right)}(\xi)\right|=\left|y_{m}\right| \neq$ 0 so that the van der Corput estimate, Lemma 0.0.8, can be applied. In particular, we find that

$$
\left|\int e^{2 \pi i \Phi_{x, y}(t)} \phi(t) d t\right| \lesssim \kappa^{-1} .
$$

With Lemma 4.2.3 proved, we can prove the model case, Proposition 4.2.2.

Proof of Proposition 4.2.2. We prove the result in the case that $p<2$, the full result follows since we also have that $K_{\Psi, \alpha} \in L^{\infty}$. Throughout, we will use the bounds on $A_{\alpha},\left|A_{\alpha}(y, z)\right| \sim$ $(1+|(y, z)|)^{-L-\alpha}$ from Lemma 2.3.1.

The main region for our analysis is

$$
R:=\left\{(x, y) \in \mathbb{R}^{\tilde{N}} ;|x| \lesssim|y| \text { and }|y| \gg 1 .\right\} .
$$

In the region

$$
R_{0}:=\left\{(x, y) \in \mathbb{R}^{\tilde{N}} ;|x| \gg|y| \text { and }|x| \gg 1\right\}
$$

we can use the non-stationary phase lemma, Lemma 0.0.1. We find that, for $(x, y) \in R_{0}$,

$$
\left\|\frac{1}{|x|} \Phi(\cdot)\right\|_{C^{n+2}(\operatorname{supp} \phi)} \lesssim 1,
$$

so that

$$
\left|\int e^{2 \pi i \Phi(t)} \phi(t) d t\right| \lesssim \frac{1}{|x|^{n+1}} .
$$

In particular, using the fact that $\left|A_{\alpha}(y, z)\right| \lesssim(1+|(y, z)|)^{-L-\alpha}$, we can see that

$$
\left\|K_{\Psi, \alpha}\right\|_{L^{p}\left(R_{0} \times \mathbb{R}^{L^{\prime}}\right)} \lesssim \int_{R_{0} \times \mathbb{R}^{L^{\prime}}} \frac{1}{|x|^{p(n+1)}} \frac{1}{(1+|(y, z)|)^{p(L+\alpha)}} d \mu_{\mathbb{R}^{N}}(x, y, z)<\infty .
$$

For the region

$$
R_{-1}:=\left\{(x, y) \in \mathbb{R}^{\tilde{N}} ;|(x, y)| \lesssim 1\right\}
$$

we can easily verify that

$$
\left\|K_{\Psi, \alpha}\right\|_{L^{p}\left(R_{-1} \times \mathbb{R}^{L^{\prime}}\right)}<\infty .
$$

To estimate

$$
\left\|K_{\Psi, \alpha}\right\|_{L^{p}\left(R \times \mathbb{R}^{L^{\prime}}\right)}
$$

we partition on dyadic scales $|(y, z)| \sim 2^{m}$ and $H(x, y) \sim 2^{l}$. We know that, in the region $R$, $|y| \gg 1$ and we also know from Lemma 4.2 .5 that $H(x, y) \gtrsim|y|^{\frac{1}{d_{\tilde{L}}}}$. Therefore, we know that the only dyadic scales we must consider are non-negative. Using the van der Corput estimate, Lemma 4.2.3, we find that

$$
\begin{gathered}
\int_{R \times \mathbb{R}^{L^{\prime}}}\left|K_{\Psi, \alpha}(x, y, z)\right|^{p} d \mu_{\mathbb{R}^{N}}(x, y, z) \\
\lesssim \sum_{m \geq 0} \sum_{l \geq 0} \int_{H(x, y) \sim 2^{l},|(y, z)| \sim 2^{m},|x| \lesssim|y|} \frac{1}{(1+|(y, z)|)^{p(L+\alpha)}}|k(x, y)|^{p} d \mu_{\mathbb{R}^{N}}(x, y, z) \\
\lesssim \sum_{m \geq 0} \sum_{l \geq 0} 2^{-p m(L+\alpha)} 2^{-l p} \int_{H(x, y) \sim 2^{l},|(y, z)| \sim 2^{m},|x| \lesssim|y|} d \mu_{\mathbb{R}^{N}}(x, y, z) \\
\lesssim \sum_{m \geq 0} \sum_{l \geq 0} 2^{-p m(L+\alpha)} 2^{-l p} 2^{m L^{\prime}} J_{l, m},
\end{gathered}
$$

where

$$
J_{l, m}=\int_{H(x, y) \sim 2^{l},|x| \lesssim|y| \lesssim 2^{m}} d \mu_{\mathbb{R}^{\tilde{N}}}(x, y) .
$$

We use two estimates on the size of $J_{l, m}$. The first estimate is the trivial estimate

$$
\begin{equation*}
J_{l, m} \lesssim 2^{m(\tilde{L}+1)} \tag{4.2.2}
\end{equation*}
$$

Where $H(x, y)=\inf _{t \in I} \sum_{j=1}^{d_{\tilde{L}}}\left|\Phi^{(j)}(t) / j!\right|^{\frac{1}{j}} \sim 2^{l}$, we will take a collection of $\lesssim 2^{l}$ sample
points $\mathcal{T}_{l} \subset[0, \delta]$. For one of these sample points $t_{j}(x, y)$, we will show that

$$
\left|\Phi^{\prime}\left(t_{j}(x, y)\right)\right| \lesssim 2^{l}
$$

From this it follows that

$$
\left\{(x, y) ; H(x, y) \sim 2^{l},|x| \lesssim|y| \lesssim 2^{m}\right\} \subset \bigcup_{t_{j} \in \mathcal{T}_{l}}\left\{(x, y) ;\left|\Phi^{\prime}\left(t_{j}\right)\right| \lesssim 2^{l},|x| \lesssim|y| \lesssim 2^{m}\right\}
$$

where the right hand side is expressed as a finite union of $\lesssim 2^{l}$ sets. For a fixed $t_{j}$, since $\Phi^{\prime}\left(t_{j}\right)=x+\sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}^{\prime}\left(t_{j}\right)$, we can make a change of $x \mapsto \tilde{x}=\Phi^{\prime}\left(t_{j}\right)$, which has unit Jacobian, to see that

$$
\begin{gather*}
J_{l, m} \lesssim \sum_{t_{j} \in \mathcal{T}_{l}} \int_{\left|\Phi^{\prime}\left(t_{j}\right)\right| \lesssim 2^{l},|y| \lesssim 2^{m}} d \mu_{\mathbb{R}^{\tilde{N}}}(x, y) \\
\lesssim \sum_{t_{j} \in \mathcal{T}_{l}} 2^{m \tilde{L}} \int_{|\tilde{x}| \lesssim 2^{l}} d \tilde{x} \\
\lesssim 2^{2 l} 2^{m \tilde{L}} . \tag{4.2.3}
\end{gather*}
$$

The bounds (4.2.2) and (4.2.3) are comparable precisely when $2 l=m$. Splitting the summation at this point, we find that

$$
\begin{aligned}
& \int_{R \times \mathbb{R}^{L^{\prime}}}\left|K_{\Psi, \alpha}(x, y, z)\right| d \mu_{\mathbb{R}^{N}}(x, y, z) \\
& \lesssim \sum_{m \geq 0} \sum_{l \geq \frac{m}{2}} 2^{-p m(L+\alpha)} 2^{-l p} 2^{m L^{\prime}} J_{l, m} \\
& +\sum_{m \geq 0} \sum_{\frac{m}{2}>l \geq 0} 2^{-p m(L+\alpha)} 2^{-l p} 2^{m L^{\prime}} J_{l, m} \\
& \lesssim \sum_{m \geq 0} \sum_{l \geq \frac{m}{2}} 2^{-p m(L+\alpha)} 2^{-l p} 2^{m L^{\prime}} 2^{m(\tilde{L}+1)} \\
& +\sum_{m \geq 0} \sum_{\frac{m}{2}>l \geq 0} 2^{-p m(L+\alpha)} 2^{-l p} 2^{m L^{\prime}} 2^{2 l} 2^{m \tilde{L}} \\
& \quad \lesssim \sum_{m \geq 0} 2^{m(L+1)} 2^{-p m\left(L+\alpha+\frac{1}{2}\right)} .
\end{aligned}
$$

This last expression is finite if and only if

$$
p>\frac{L+1}{L+\alpha+\frac{1}{2}}
$$

It remains to discuss the sample points: we must define $\mathcal{T}_{l}$ with $\left|\mathcal{T}_{l}\right| \lesssim 2^{l}$ such that, where $H(x, y) \sim 2^{l}$, there exists $t_{j}(x, y) \in \mathcal{T}_{l}$ with $\left|\Phi^{\prime}\left(t_{j}\right)\right| \lesssim 2^{l}$. This is in fact a simple consequence of Taylor's theorem. We first choose $t^{*}=t^{*}(x, y)$ such that

$$
2^{l} \sim H(x, y)=H_{t^{*}}(x, y)=\sum_{i=1}^{d_{\tilde{L}}}\left|\frac{\Phi^{(i)}\left(t^{*}\right)}{i!}\right|^{\frac{1}{i}}
$$

Thus, we find that, for $0 \leq i \leq d_{\tilde{L}}-1$,

$$
\left|\Phi^{(i+1)}\left(t^{*}\right)\right| \lesssim 2^{(i+1) l}
$$

In particular, expanding the polynomial $\Phi^{\prime}$ as a Taylor series about $t^{*}$, we see that

$$
\Phi^{\prime}\left(t^{*}+h\right)=\sum_{i=0}^{d_{\tilde{L}-1}} \frac{\Phi^{(i+1)}\left(t^{*}\right)}{i!} h^{i}
$$

so that, if $|h| \lesssim 2^{-l}$, then

$$
\left|\Phi^{\prime}\left(t^{*}+h\right)\right| \lesssim \sum_{i=0}^{d_{\tilde{L}}-1} 2^{l(i+1)} 2^{-l i} \sim 2^{l}
$$

If we then set $t_{j}(x, y)=2^{-l}\left[2^{l} t^{*}(x, y)\right]$, we have $\left|t_{j}(x, y)-t^{*}(x, y)\right| \lesssim 2^{-l}$, so that

$$
\left|\Phi^{\prime}\left(t_{j}(x, y)\right)\right| \lesssim 2^{l} .
$$

Therefore, if we take as our sample points the set

$$
\mathcal{T}_{l}=\left\{0,2^{-l}, \ldots, 2^{-l}\left[2^{l} \delta\right]-2^{-l}, 2^{-l}\left[-2^{l} \delta\right]\right\}
$$

we are done.
Proceeding from the above arguments, we may now turn our sights to the higher dimensional case, proving Lemma 4.2.4 and Proposition 4.2.1 in turn.

Proof of Lemma 4.2.4. The proof requires the division of $(x, y)$ space so that we can apply the van de Corput estimate, Lemma 0.0.8. Henceforth, until stated otherwise, we denote by $\Phi=\Phi_{x, y, \pm}$ the phase appearing in the integral expansion we are interested in. To recall,

$$
\Phi(r)=\Phi_{x, y, \pm}(r):= \pm|x| r+\sum_{j=1}^{\tilde{L}} y_{j} \psi_{j}(r) .
$$

Likewise we denote by $H(x, y)$ the quantity

$$
H(x, y)=\inf _{r \in[0, \delta]} \sum_{j=1}^{d_{\tilde{L}}}\left|\frac{\Phi^{(j)}(r)}{j!}\right|,
$$

where the phase $\Phi, d_{\tilde{L}}$, and $\delta$ depend implicitly on the surface $\Gamma$ under consideration.
The main regions are those defined as follows. Let

$$
\begin{gathered}
R_{1}=\left\{(x, y)| | x|\lesssim| y_{1}\left|;\left|y_{1}\right| \gg\right| y_{j} \mid \text { for } j>1 ;\left|y_{1}\right| \gg 1\right\}, \\
R_{2}=\left\{(x, y)| | x\left|,\left|y_{1}\right| \lesssim\right| y_{2}\left|;\left|y_{2}\right| \gg\right| y_{j} \mid \text { for } j>2 ;\left|y_{2}\right| \gg 1\right\}, \\
\vdots \\
R_{L}=\left\{(x, y)| | x\left|,\left|y_{1}\right|, \ldots,\left|y_{L-1}\right| \lesssim\right| y_{L}\left|;\left|y_{L}\right| \gg 1\right\} .\right.
\end{gathered}
$$

We are free to choose the constants in the $\gg$ inequalities as we wish, with the remaining conditions on all of the $\lesssim$ inequalities chosen so that the regions $R_{l}$ cover the whole of the region where $|y| \gtrsim|x|$ and $|y| \gg 1$.

We now show that for each one of these regions the van de Corput estimate can be applied. Although we introduced the quantity $H(x, y)=\inf _{r \in I} \sum_{j=1}^{d_{\tilde{\tilde{L}}}}\left|\frac{\Phi_{x, y}^{(j)}(r)}{j!}\right|$, we must also work with reference to $H^{d_{l}}(x, y)=\inf _{r \in I} \sum_{j=1}^{d_{l}}\left|\frac{\Phi_{x, y y}^{(j)}(r)}{j!}\right|$, for $1 \leq l \leq \tilde{L}$. We show that $H(x, y) \sim H^{d_{l}}(x, y)$ in the regions $R_{l}$, for $1 \leq l \leq \tilde{L}$.

Now we verify that, in the region $R_{l}$,

$$
\begin{equation*}
\inf _{r \in I}\left|\Phi^{\left(d_{l}\right)}(r)\right| \geq c \sup _{r \in I}\left|\Phi^{\left(d_{l}\right)}(r)\right| . \tag{4.2.4}
\end{equation*}
$$

Noting that the even $d_{l} \geq 2$, it is seen that

$$
\Phi^{\left(d_{l}\right)}(r)=\sum_{j=1}^{L} y_{j} \psi_{j}^{\left(d_{l}\right)}(r),
$$

since $\left|y_{j}\right| \ll\left|y_{l}\right|$ for $j>l$ and $\left|\psi_{j}^{\left(d_{l}\right)}(r)\right| \leq \epsilon$ for $j<l$. We can then see, using the bound (4.2.1) on $\left|\psi_{l}^{\left(d_{l}\right)}(r)\right|$, that

$$
\begin{equation*}
\left|\Phi^{\left(d_{l}\right)}(r)\right| \geq \frac{1}{2}\left|y_{l}\right|-(L-l) \epsilon|y|-\sum_{j>L} 2\left|y_{j}\right| \geq \frac{1}{4}\left|y_{l}\right|, \tag{4.2.5}
\end{equation*}
$$

provided $\epsilon$ is chosen small enough, since $\left|y_{j}\right| \ll\left|y_{l}\right|$ for $j>l$. Trivially, in $R_{l}$, we have

$$
\begin{equation*}
\left|\Phi^{\left(d_{l}\right)}(r)\right| \lesssim\left|y_{l}\right| . \tag{4.2.6}
\end{equation*}
$$

The desired inequality (4.2.4) follows from (4.2.5) and (4.2.6).
In $R_{l}$ we also see that, for $j>d_{l}$,

$$
\left|\frac{\Phi^{(j)}(r)}{j!}\right|^{\frac{1}{j}} \lesssim\left|\frac{y_{l}}{j!}\right|^{\frac{1}{j}} \lesssim\left|\frac{y_{l}}{d_{l}!}\right|^{\frac{1}{d_{l}}} \sim \inf _{r \in I}\left|\frac{\Phi^{\left(d_{l}\right)}(r)}{d_{l}!}\right|^{\frac{1}{d_{l}}} \lesssim H(x, y) .
$$

It follows that $H^{d_{l}}(x, y) \sim H(x, y)$.
Thus in each of the regions $R_{l}$ for $l \geq 1$ we can apply our van de Corput estimate, Lemma 0.0 .8 , with $k=d_{l}$ and, further, we have that $H(x, y) \sim H^{d_{l}}(x, y)$.

For $(x, y) \in R_{l}$, when we apply Lemma 0.0 .8 we see that, if $H(x, y) \geq \kappa$, then

$$
\begin{gathered}
\left|\int_{0}^{\infty} e^{2 \pi i\left( \pm r|x|+\sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}(r)\right)} \frac{1}{|r x|^{\frac{n-1}{2}}} \eta(|x| r) \phi_{0}(r) r^{n-1} d r\right| \\
\lesssim_{d}\left\|\tilde{\phi}^{\prime}\right\|_{L^{1}} \min \left\{1, \kappa^{-1}\right\},
\end{gathered}
$$

where $\tilde{\phi}(r)=\frac{1}{|r x|^{\frac{n-1}{2}}} \eta(|x| r) \phi_{0}(r) r^{n-1}=\frac{1}{|x|^{\frac{n-1}{2}}} \eta(|x| r) \phi_{0}(r) r^{\frac{n-1}{2}}$. We see that

$$
\begin{gathered}
\tilde{\phi}^{\prime}(r)=\frac{1}{|x|^{\frac{n-1}{2}}}\left(|x| \eta^{\prime}(|x| r) \phi_{0}(r) r^{\frac{n-1}{2}}\right. \\
\left.+\eta(|x| r) \phi_{0}^{\prime}(r) r^{\frac{n-1}{2}}+\frac{n-1}{2} \eta(|x| r) \phi_{0}(r) r^{\frac{n-3}{2}}\right)
\end{gathered}
$$

so that

$$
\begin{gathered}
\left\|\tilde{\phi}^{\prime}\right\|_{L^{1}} \leq \frac{1}{|x|^{\frac{n-1}{2}}} \int| | x\left|\eta^{\prime}(|x| r) \phi_{0}(r) r^{\frac{n-1}{2}}\right| d r \\
+\frac{1}{|x|^{\frac{n-1}{2}}} \int\left|\eta(|x| r) \phi_{0}^{\prime}(r) r^{\frac{n-1}{2}}\right| d r \\
\quad+\frac{1}{|x|^{\frac{n-1}{2}}} \int\left|\frac{n-1}{2} \eta(|x| r) \phi_{0}(r) r^{\frac{n-3}{2}}\right| d r \\
\lesssim \frac{1}{|x|^{\frac{n-1}{2}}}\left(|x||x|^{-\frac{n-1}{2}}|x|^{-1}+1+\left(1+|x|^{-\frac{n-1}{2}}\right)\right) \sim \frac{1}{|x|^{\frac{n-1}{2}}} .
\end{gathered}
$$

This completes the proof since the union of $R_{l}$ is the desired region, where $|x| \lesssim|y|$ and
$|y| \gg 1$.
Proof of Proposition 4.2.1. We prove that

$$
k(x, y)(1+|(y, z)|)^{-L-\alpha} \in L^{p}\left(\mathbb{R}^{N}, d x d y d z\right)
$$

which, combined with bounds on $A_{\alpha},(2.3 .2)$ and $\left|A_{\alpha}\right| \leq C$, gives the desired result. We show the result for $p \leq 2$, the full range of $p$ follows because $L^{p} \cap L^{\infty} \subset L^{q}$ for all $q \in[2, \infty]$.

We first consider the regions where the kernel $k(x, y)(1+|(y, z)|)^{-(L+\alpha)}$ contributes what can be considered error terms in the $L^{p}$ integration. The first of these is the region $R_{0}$, where $|x| \gg|y|$. The next region is $R_{-1}$, where $|x| \lesssim \min \{1,|y|\}$. The remaining region, where $1 \lesssim|x| \lesssim|y|$ and $|y| \gg 1$, is the main region $R$. Trivially, we always have the estimate $|k(x, y)| \leq C$.

In the region $R_{0}$, we consider the phase function

$$
\widetilde{\Phi_{x, y}}(t)=\frac{1}{|x|}(x \cdot t+y \cdot \Psi(t)),
$$

for which $\left\|\widetilde{\Phi_{x, y}}\right\|_{C^{n+2}(\overline{B(0, \delta)})} \lesssim 1$ and $\left|\nabla \widetilde{\Phi_{x, y}}(t)\right| \gtrsim 1$, for $|t| \leq \delta$. Therefore, by the nonstationary phase lemma, Lemma 0.0.1,

$$
|k(x, y)|=\left|\int e^{2 \pi i|x| \widetilde{\Phi_{x, y}}(t)} \phi(t) d t\right| \lesssim \frac{1}{|x|^{n+1}} .
$$

Integrating the resulting estimate we see

$$
\begin{gathered}
\int_{R_{0} \times \mathbb{R}^{\prime}}|k(x, y)|^{p}(1+|(y, z)|)^{-p(L+\alpha)} d x d y d z \\
\lesssim \\
\int_{R_{0} \times \mathbb{R}^{\prime}} \frac{1}{(1+|x|)^{(n+1) p}} \frac{1}{(1+|(y, z)|)^{p(L+\alpha)}} d \mu_{\mathbb{R}^{N}}(x, y, z),
\end{gathered}
$$

which is finite.
Next, we see for $R_{-1}$ that

$$
\begin{aligned}
& \int_{R_{-1} \times \mathbb{R}^{L^{\prime}}}|k(x, y)|^{p}(1+|(y, z)|)^{-p(L+\alpha)} d \mu_{\mathbb{R}^{N}}(x, y, z) \\
& \lesssim \int_{|x| \lesssim 1} 1 d x \iint(1+|(y, z)|)^{-p(L+\alpha)} d y d z<\infty .
\end{aligned}
$$

So far, we have established that

$$
k(x, y) A(y, z) \in L^{p}\left(R_{0}, d x d y d z\right) \quad \text { and } \quad k(x, y) A(y, z) \in L^{p}\left(R_{-1}, d x d y d z\right),
$$

for all $1 \leq p \leq \infty$. It remains for us to prove that

$$
k(x, y) A(y, z) \in L^{p}(R, d x d y d z) .
$$

Using polar integration and writing $\phi(t)=\phi_{0}(|t|)$ we expand

$$
\begin{gathered}
k(x, y)=\int e^{2 \pi i\left(x \cdot t+y \cdot \Psi_{\tilde{L}}(|t|)\right)} \phi(t) d t \\
=\iint e^{2 \pi i\left(x \cdot r \omega+y \cdot \Psi_{\tilde{L}}(r)\right)} \phi_{0}(r) d \sigma(\omega) r^{n-1} d r \\
=\int e^{2 \pi i y \cdot \Psi_{\tilde{L}}(r)} \hat{\sigma}(r x) \phi_{0}(r) r^{n-1} d r .
\end{gathered}
$$

For $n=1$ we have that $\hat{\sigma}(r x)=e^{2 \pi i r|x|}+e^{-2 \pi i r|x|}$. For $n>1$ we use the asymptotic
expansion for $\hat{\sigma}$

$$
\hat{\sigma}(r x)=\frac{1}{|r x|^{\frac{n-1}{2}}}\left(c_{+} e^{2 \pi i r|x|}+c_{-} e^{-2 \pi i r|x|}+E_{\sigma}(r x)\right),
$$

for $|r x| \gtrsim 1$, where the $c_{-}$and $c_{+}$are non-zero constants and $\left|E_{\sigma}(r x)\right| \leq C|r x|^{-1}$; see, for example, Chapter 8 of [Ste93]. To make use of this we introduce a cut-off function, $\eta$. We choose $\eta$ such that $\eta(r)=0$ for $r \leq \frac{1}{2}$ and $\eta(r)=1$ for $r \geq 1$. We see

$$
\begin{gathered}
k(x, y) \\
=E_{1}(x, y)+I(x, y),
\end{gathered}
$$

where

$$
E_{1}(x, y)=\int_{0}^{\infty} e^{2 \pi i \sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}(r)} \hat{\sigma}(r x)(1-\eta(|x| r)) \phi_{0}(r) r^{n-1} d r
$$

and

$$
I(x, y)=\int_{0}^{\infty} e^{2 \pi i \sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}(r)} \hat{\sigma}(r x) \eta(|x| r) \phi_{0}(r) r^{n-1} d r
$$

Let us look first to $I(x, y)$. Using the asymptotic expansion, we see that

$$
\begin{gathered}
I(x, y) \\
=I_{+}(x, y)+I_{-}(x, y)+E_{2}(x, y)
\end{gathered}
$$

where

$$
\begin{aligned}
& I_{+}(x, y)=c_{+} \int_{0}^{\infty} e^{2 \pi i\left(r|x|+\sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}(r)\right)} \frac{1}{|r x|^{\frac{n-1}{2}}} \eta(|x| r) \phi_{0}(r) r^{n-1} d r \\
& I_{-}(x, y)=c_{-} \int_{0}^{\infty} e^{2 \pi i\left(-r|x|+\sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}(r)\right)} \frac{1}{|r x|^{\frac{n-1}{2}}} \eta(|x| r) \phi_{0}(r) r^{n-1} d r, \\
& E_{2}(x, y)=\int_{0}^{\infty} e^{2 \pi i\left(\sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}(r)\right)} \frac{1}{|r x|^{\frac{n-1}{2}}} E_{\sigma}(r x) \eta(|x| r) \phi_{0}(r) r^{n-1} d r .
\end{aligned}
$$

The main terms are $I_{+}$and $I_{-}$. The phases we consider are thus

$$
\Phi_{x, y,-}(r)=-r|x|+\sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}(r)
$$

and

$$
\Phi_{x, y,+}(r)=r|x|+\sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}(r)
$$

We estimate the contributions of the terms $I_{+}(x, y)$ and $I_{-}(x, y)$ separately. We apply the van der Corput estimate, Lemma 4.2.4, to bound the oscillatory integrals. Since the analysis is the same in either case, we only present the argument for $I_{+}(x, y)$, with the relevant phase denoted by $\Phi=\Phi_{x, y}=\Phi_{x, y,+}$. We use the quantity $H(x, y)=\inf _{r \leq \delta} \sum_{j=1}^{d_{\tilde{L}}}\left|\frac{\Phi_{x, y}^{(j)}(r)}{j!}\right|^{\frac{1}{j}}$.

To carry out the $L^{p}$ integration of $I_{+}(x, y)(1+|(y, z)|)^{-(L+\alpha)}$, we perform a dyadic partition on scales $H(x, y) \sim 2^{l}$ and $|y| \sim 2^{m}$. By Lemma 4.2 .5 we have $H(x, y) \gtrsim|y|^{\frac{1}{d} \tilde{L}}$. It follows that, for $(x, y, z) \in R$, the indices $l, m \geq 0$, provided $|y| \gg 1$ with a suitable constant in the main region $R$. When $H(x, y) \sim 2^{l}$, we have as a consequence of Lemma 4.2.4 that $\left|I_{+}(x, y)\right|^{p} \lesssim \frac{1}{|x|^{p \frac{n-1}{2}}} 2^{-l p}$. Thus we see

$$
\int\left|I_{+}(x, y)\right|^{p} \frac{1}{(1+|(y, z)|)^{p(L+\alpha)}} d \mu_{\mathbb{R}^{N}}(x, y, z)
$$

$$
\begin{gathered}
\lesssim \sum_{m \geq 0} \sum_{l \geq 0} 2^{-l p} \int_{2^{l-1}<H(x, y) \leq 2^{l} ; \max \{|x|,|z|\} \lesssim|y| \sim 2^{m}} \frac{1}{|x|^{p \frac{n-1}{2}}} \frac{1}{(1+|y|)^{p(L+\alpha)}} \mu_{\mathbb{R}^{N}}(x, y, z) \\
+\sum_{m \geq 0} \sum_{l \geq 0} 2^{-l p} \int_{2^{l-1}<H(x, y) \leq 2^{l} ;|x| \lesssim|y| \sim 2^{m} \leqq|z|} \frac{1}{|x|^{p^{\frac{n-1}{2}}} \frac{1}{(1+|z|)^{p(L+\alpha)}} \mu_{\mathbb{R}^{N}}(x, y, z)} \\
\quad \lesssim \sum_{m \geq 0} \sum_{l \geq 0} 2^{-l p} 2^{-p m(L+\alpha)} 2^{L^{\prime} m} \int_{2^{l-1}<H(x, y) \leq 2^{l},|x| \lesssim|y| \sim 2^{m}} \frac{1}{|x|^{p^{\frac{n-1}{2}}} \mu_{\mathbb{R}^{\tilde{N}}}(x, y)} \\
=\sum_{m \geq 0} \sum_{l \geq 0} 2^{-l p} 2^{-m p(L+\alpha)} 2^{L^{\prime} m} J_{l, m},
\end{gathered}
$$

where

$$
J_{l, m}=\iint_{2^{l-1}<H(x, y) \leq 2^{l} ;|x| \lesssim|y| \sim 2^{m}} \frac{1}{|x|^{p^{\frac{n-1}{2}}}} d x d y
$$

We split the sum into two and write

$$
\sum_{m \geq 0} \sum_{l \geq 0} 2^{-l p} 2^{-m p(L+\alpha)} 2^{L^{\prime} m} J_{l, m}=S_{1}+S_{2}
$$

where

$$
S_{1}=\sum_{l \geq 0} 2^{-p l} \sum_{m \geq 2 l, 0} 2^{-p m(L+\alpha)} 2^{L^{\prime} m} J_{l, m}
$$

and

$$
S_{2}=\sum_{l \geq 0} 2^{-p l} \sum_{2 l>m \geq 0} 2^{-p m(L+\alpha)} 2^{L^{\prime} m} J_{l, m} .
$$

We provide two different estimates on the size of $J_{l, m}$. The first estimate is the trivial estimate

$$
\begin{gather*}
J_{l, m} \leq \iint_{|y| \lesssim 2^{m},|x| \lesssim 2^{m}} \frac{1}{|x|^{p \frac{n-1}{2}}} d x d y \\
\leq 2^{m L} 2^{m n-p\left(\frac{n-1}{2}\right) m} \tag{4.2.7}
\end{gather*}
$$

The second estimate requires a little more care and it is here we work by the method used in [ACK79]. We now work via polar integration.

Since $H(x, y) \sim 2^{l}$ we will later show that there exists some

$$
r_{j} \in \mathcal{T}_{l}:=\left\{0, \frac{1}{2^{l}}, \ldots, \frac{1}{2^{l}}\left[2^{l} \delta\right]-\frac{1}{2^{l}}, \frac{1}{2^{l}}\left[2^{l} \delta\right]\right\}
$$

such that $\left|\Phi^{\prime}\left(r_{j}\right)\right| \lesssim 2^{l}$. Indeed, as before, we choose $r_{j}(x, y)=\frac{1}{2^{l}}\left[2^{l} r^{*}(x, y)\right]$, where

$$
\sum_{i=1}^{d_{\tilde{L}}}\left|\Phi^{(i)}\left(r^{*}(x, y)\right)\right|^{\frac{1}{i}}=\inf _{r \in[0, \delta]} \sum_{i=1}^{d_{\tilde{L}}}\left|\Phi^{(i)}(r)\right|^{\frac{1}{i}}
$$

We claim that

$$
\begin{equation*}
\left|\Phi^{\prime}\left(r_{j}(x, y)\right)\right| \lesssim 2^{l} . \tag{4.2.8}
\end{equation*}
$$

Let us for now assume that (4.2.8) holds and show how the result follows. We later return to the proof of this claim. As a consequence, we see that

$$
\begin{aligned}
& \left\{(x, y) ; H(x, y) \sim 2^{l},|y| \sim 2^{m},|s| \lesssim|y|\right\} \\
\subset & \bigcup_{r_{j} \in \mathcal{T}_{l}}\left\{(s \omega, y) ;\left|\Phi^{\prime}\left(r_{j}\right)\right| \lesssim 2^{l},|y| \sim 2^{m},|s| \lesssim|y|\right\} .
\end{aligned}
$$

For each $j$, we will make the change of variables $s \mapsto \tilde{s}=\Phi^{\prime}\left(r_{j}\right)=|x|+\sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}^{\prime}(s)$, where $s=|x|$ in the following polar integration. The change of variables has unit Jacobian. We
denote by $e_{1}$ the vector $(1,0, \ldots, 0) \in \mathbb{R}^{n}$ and see that

$$
\begin{gathered}
J_{l, m}=\iiint_{2^{l-1}<H(s \omega, y) \leq 2^{l},|y| \sim 2^{m}, s \lesssim|y|} \frac{1}{s^{p \frac{n-1}{2}}} d \sigma(\omega) s^{n-1} d s d y \\
\lesssim \sum_{j=0}^{\left[2^{l} \delta\right]} \iiint_{\left|\Phi_{s \omega, y}^{\prime}\left(r_{j}\right)\right| \lesssim 2^{l},|y| \lesssim 2^{m}, s^{p^{\frac{n-1}{2}}} d \sigma(\omega) s^{n-1} d s d y} \quad \sum_{j=0} \iint_{\left|\Phi_{s s_{1}, y}^{\prime}\left(r_{j}\right)\right| \lesssim 2^{l},|y| \lesssim 2^{m},} s^{n-1-p \frac{n-1}{2}} d s d y \\
\sim \sum_{j=0}^{\left[2^{l} \delta\right]} \iint_{|\tilde{s}| \lesssim 2^{l},|y| \lesssim 2^{m}}\left|\tilde{s}-\sum_{i=1} y_{i} \psi_{i}^{\prime}\left(r_{j}\right)\right|^{n-1-p \frac{n-1}{2}} d \tilde{s} d y \\
\lesssim \sum_{j=0}^{\left[2^{l} \delta\right]} \iint_{|\tilde{s}| \lesssim 2^{l},|y| \lesssim 2^{m}}\left(|\tilde{s}|^{n-1-p \frac{n-1}{2}}+\left|\sum_{i=1}^{\tilde{L}} y_{i} \psi_{i}^{\prime}\left(r_{j}\right)\right|^{n-1-p \frac{n-1}{2}}\right) d \tilde{s} d y \\
\\
\lesssim 2^{l} \int_{|y| \lesssim 2^{m}}\left(2^{l\left(n-p \frac{n-1}{2}\right)}+2^{l} 2^{m\left(n-1-p \frac{n-1}{2}\right)}\right) d y \\
\\
\lesssim 2^{2 l} 2^{\tilde{L} m}\left(2^{l\left(n-1-p \frac{n-1}{2}\right)}+2^{m\left(n-1-p \frac{n-1}{2}\right)}\right)
\end{gathered}
$$

We use this last estimate in the case that $2 l \leq m$, in which case we see that

$$
\begin{equation*}
J_{l, m} \lesssim 2^{2 l} 2^{\tilde{L} m} 2^{m\left(n-1-p \frac{n-1}{2}\right)} \tag{4.2.9}
\end{equation*}
$$

We use this bound to see that

$$
\begin{gathered}
S_{1}=\sum_{l \geq 0} 2^{-p l} \sum_{m \geq 2 l, 0} 2^{-p m(L+\alpha)} 2^{L^{\prime} m} J_{l, m} \\
\lesssim \\
\sum_{m \geq 0} 2^{-p m(L+\alpha)} 2^{L m} 2^{m\left(n-1-p \frac{n-1}{2}\right)} \sum_{0 \leq l \leq \frac{m}{2}} 2^{2 l} 2^{-p l} \\
\lesssim \\
\sum_{m \geq 0} 2^{-p m(L+\alpha)} 2^{L m} 2^{m\left(n-1-p \frac{n-1}{2}\right)} 2^{m} 2^{-p \frac{m}{2}},
\end{gathered}
$$

which is finite if $p>\frac{L+n}{L+\alpha+\frac{n}{2}}$.
Using the trivial bound (4.2.7), we see that

$$
\begin{aligned}
& S_{2}=\sum_{l \geq 0} 2^{-p l} \sum_{2 l>m \geq 0} 2^{-p m(L+\alpha)} 2^{L^{\prime} m} J_{l, m} \\
= & \sum_{m \geq 0} 2^{-p m(L+\alpha)} 2^{m L} 2^{m n-p m\left(\frac{n-1}{2}\right)} \sum_{2 l>m \geq 0} 2^{-p l} \\
& \lesssim \sum_{m \geq 0} 2^{-p m(L+\alpha)} 2^{m L} 2^{m n-p m\left(\frac{n-1}{2}\right)} 2^{-p \frac{m}{2}},
\end{aligned}
$$

which is finite if $p>\frac{L+n}{L+\alpha+\frac{n}{2}}$.
The analysis may be repeated for the contribution of $I_{-}(x, y)$.
It remains to consider the contribution of the error terms. For $E_{2}$, we have that

$$
\left|E_{2}(x, y)\right|=\left|\int_{0}^{\infty} e^{2 \pi i\left(\sum_{i=1}^{L} y_{i} \psi_{i}(r)\right)} \frac{1}{|r x|^{\frac{n-1}{2}}} E_{\sigma}(r x) \eta(|x| r) \phi_{0}(r) r^{n-1} d r\right|
$$

$$
\leq C \frac{1}{|x|^{\frac{n-1}{2}}} \frac{1}{|x|} \int_{\frac{1}{2}|x|^{-1}}^{1} r^{-1} r^{\frac{n-1}{2}} d r \sim \frac{1}{|x|^{\frac{n+1}{2}}} .
$$

For $E_{1}$, we find

$$
\begin{gathered}
\left|E_{1}(x)\right|=\left|\int_{0}^{\infty} e^{2 \pi i \sum_{i=1}^{L} y_{i} \psi_{i}(r)} \hat{\sigma}(r x)(1-\eta(|x| r)) \phi_{0}(r) r^{n-1} d r\right| \\
\lesssim \int_{0}^{|x|^{-1}} r^{n-1} d r \sim \frac{1}{|x|^{n}}
\end{gathered}
$$

Since we know in either case that $|x| \gtrsim 1$, we see $\left|E_{i}(x, y)\right| \lesssim \frac{1}{|x|^{\frac{n+1}{2}}}$ for each $i$.
Of course, we also have the bound $\left|E_{i}(x, y)\right| \leq C$ for each $i$. Performing polar integration, we find that

$$
\begin{gathered}
\left\|E_{i}(x, y)(1+|(y, z)|)^{-L-\alpha}\right\|_{L^{p}(R, d x d y d z)}^{p} \\
\lesssim \iiint(1+|(y, z)|)^{-p(L+\alpha)}(1+|x|)^{-p \frac{n+1}{2}} d x d y d z \\
\sim \int_{1}^{\infty} r^{(n+L-1)-p\left(L+\alpha+\frac{n+1}{2}\right)} d r
\end{gathered}
$$

which is finite if $p>\frac{n+L}{L+\alpha+\frac{n+1}{2}}$. In particular, this is true if $p>\frac{n+L}{L+\alpha+\frac{n}{2}}$. To conclude, in the region $R$, we have made the decomposition

$$
\begin{gathered}
k(x, y) A(y, z) \\
=I_{+}(x, y) A(y, z)+I_{-}(x, y) A(y, z)+E_{1}(x, y) A(y, z)+E_{2}(x, y) A(y, z)
\end{gathered}
$$

and we proved that each of the summands is in $L^{p}(R, d x d y d z)$ for $p>\frac{L+n}{L+\alpha+\frac{n}{2}}$. Applying the triangle inequality establishes the required result.

Let us finally return to the proof of (4.2.8). Recall that we chose $r^{*}$ with

$$
\sum_{i=1}^{d_{\tilde{L}}}\left|\Phi^{(i)}\left(r^{*}\right)\right|^{\frac{1}{i}}=\inf _{r \in[0, \delta]} \sum_{i=1}^{d_{\tilde{L}}}\left|\Phi^{(i)}(r)\right|^{\frac{1}{i}}
$$

Recall also that we took $r_{j}=\frac{1}{2^{l}}\left[2^{l} r^{*}\right]$, for which $\left|r_{j}-r^{*}\right| \leq 2^{-l}$. Now observe that, for $1 \leq i \leq d_{\tilde{L}}$,

$$
\left|\Phi^{(i)}\left(r^{*}\right)\right| \lesssim 2^{l i}
$$

Before we can apply Taylor's theorem, we must leverage this bound to adequately control the remainder term. We work with reference to the same regions $R_{a}$ which we introduced in the proof of Lemma 4.2.4. To recall, these were

$$
\begin{gathered}
R_{1}=\left\{(x, y)| | x|\lesssim| y_{1}\left|;\left|y_{1}\right| \gg\right| y_{j} \mid \text { for } j>1 ;\left|y_{1}\right| \gg 1\right\}, \\
R_{2}=\left\{(x, y)| | x\left|,\left|y_{1}\right| \lesssim\right| y_{2}\left|;\left|y_{2}\right| \gg\right| y_{j} \mid \text { for } j>2 ;\left|y_{2}\right| \gg 1\right\}, \\
\vdots \\
R_{L}=\left\{(x, y)| | x\left|,\left|y_{1}\right|, \ldots,\left|y_{L-1}\right| \lesssim\right| y_{L}\left|;\left|y_{L}\right| \gg 1\right\} .\right.
\end{gathered}
$$

Let us fix an $a$ and consider $(x, y) \in R_{a}$, then we can write

$$
\Phi^{\left(d_{a}\right)}(r)=y_{a}+\sum_{i=1}^{\tilde{L}} y_{i} \varepsilon_{i}^{\left(d_{a}\right)}(r)+\sum_{i=a+1}^{\tilde{L}} y_{i} \frac{r^{d_{i}-d_{a}}}{\left(d_{i}-d_{a}\right)!} .
$$

We know by definition of $\mathcal{S}_{0}$ (Definition 1.0.1) that, provided $\delta$ has been chosen sufficiently small $\left|\varepsilon_{i}^{\left(d_{a}\right)}(r)\right| \ll 1$ for $0 \leq r \leq \delta$. We also know that $\left|y_{b}\right| \lesssim\left|y_{a}\right|$ and, for $b>a,\left|y_{b}\right| \ll\left|y_{a}\right|$.

Therefore, for $0 \leq r \leq \delta$,

$$
\left|\Phi^{\left(d_{a}\right)}(r)\right| \sim\left|y_{a}\right|
$$

and we know, since $H(x, y) \sim 2^{l}$, that

$$
\left|y_{a}\right| \sim\left|\Phi^{\left(d_{a}\right)}\left(r^{*}\right)\right| \lesssim 2^{l d_{a}}
$$

Using this, we can establish suitable control on the error term coming from Taylor's theorem. In particular, we find that

$$
\Phi^{\prime}\left(r^{*}+h\right)=\sum_{i=0}^{d_{a}-1} \Phi^{(i+1)}\left(r^{*}\right) h^{i}+\theta(h),
$$

where

$$
|\theta(h)| \lesssim h^{d_{a}} \sup _{\xi \in\left[r^{*}-h, r^{*}+h\right]}\left|\Phi^{\left(d_{a}+1\right)}(r)\right| \lesssim h^{d_{a}}\left|y_{a}\right| \lesssim h^{d_{a}} 2^{l d_{a}} .
$$

Therefore, if we choose $r_{j}=2^{-l}\left[2^{l} r^{*}\right]$, since $\left|r_{j}-r^{*}\right| \lesssim 2^{-l}$ and $H(x, y) \sim 2^{l}$,

$$
\left|\Phi^{\prime}\left(r_{j}\right)\right| \lesssim\left(\sum_{i=0}^{d_{\tilde{a}-1}} 2^{l(i+1)} 2^{-l i}\right)+2^{-l d_{a}} 2^{l d_{a}} \sim 2^{l}
$$

### 4.3 Sharp Bochner-Riesz estimates

To conclude this part, we observe that there is a class of surfaces for which the range of $p$ such that the Bochner-Riesz kernel $K_{\Psi, \alpha}=\check{m}_{\Psi, \alpha} \in L^{p}$ differs from the $L^{p}$ boundedness range for $T_{m_{\Psi, \alpha}}$. These examples are instances of smooth surfaces $\Gamma$ that are contained in a proper affine subspace, $P \subset \mathbb{R}^{N}$. In this case it is well known that the Fourier restriction inequality (3.0.1) can not hold for any $1<q$; see Example 3.1.3 in Section 3.1. Nevertheless, a restriction inequality may hold within the ambient subspace. For the surfaces in $\mathcal{S}$ in Definition 1.0.2 such that $\tilde{L}<L$, this is indeed the case.

In fact, the sharp result we have is slightly stronger than how it was stated in the introduction, as we established a restriction estimate for surfaces in $\mathcal{S}_{0}$ and also precisely determined the range of $p$ for which $K_{\Psi, \alpha} \in L^{p}$.

Proposition 4.3.1. Let $\Gamma \in \mathcal{S}_{0}$ with graphing function $\Psi$ such that $\tilde{L}<L, d_{1}<d_{2},<\ldots,<$ $d_{\tilde{L}}$, and $d_{1} \geq n(\tilde{L}+1)$. We set $\frac{q^{\prime}}{2}=1+\frac{D}{n}$. For $1<p \leq q$ or $q^{\prime} \leq p<\infty, T_{m_{\Psi, \alpha}}$ is bounded on $L^{p}$ if and only if $\frac{\tilde{L}+n}{\tilde{L}+\alpha+\frac{n}{2}}<p<\frac{\tilde{L}+n}{\tilde{L}-\alpha+\frac{n}{2}}$.

It is easily seen for $\alpha<\frac{n}{2}$ that $\frac{\tilde{L}+n}{\tilde{L}+\alpha+\frac{n}{2}} \geq \frac{L+n}{L+\alpha+\frac{n}{2}}$ if $\tilde{L} \leq L$, with strict inequality if $\tilde{L} \neq L$. Indeed, with $\alpha<\frac{n}{2}$

$$
\begin{gathered}
\frac{\tilde{L}+n}{\tilde{L}+\alpha+\frac{n}{2}}-\frac{\tilde{L}+L^{\prime}+n}{\tilde{L}+L^{\prime}+\alpha+\frac{n}{2}} \\
=\frac{1}{\left(\tilde{L}+\alpha+\frac{n}{2}\right)\left(L+\alpha+\frac{n}{2}\right)}\left((\tilde{L}+n)\left(\tilde{L}+L^{\prime}+\alpha+\frac{n}{2}\right)-\left(\tilde{L}+L^{\prime}+n\right)\left(\tilde{L}+\alpha+\frac{n}{2}\right)\right) \\
=\frac{1}{\left(\tilde{L}+\alpha+\frac{n}{2}\right)\left(L+\alpha+\frac{n}{2}\right)}\left((\tilde{L}+n) L^{\prime}-L^{\prime}\left(\tilde{L}+\alpha+\frac{n}{2}\right)\right) \geq 0,
\end{gathered}
$$

with strict inequality if $L^{\prime} \geq 1$. Hence, for the smooth surfaces in Theorem 4.0 .1 with $\tilde{L}<L$, the $L^{p}$ boundedness range for $T_{m_{\Psi, \alpha}}$ differs from the $L^{p}$ integrability range of $K_{\Psi, \alpha}$.

It is useful to consider the situation in terms of the reciprocal, $\frac{1}{p}$, of the $L^{p}$ exponents. We have a line of critical indices $\left(\frac{1}{p}, \alpha\right)$ coming from the $K_{\Psi, \alpha} \in L^{p}$ test. We also have a distinct line of critical indices $\left(\frac{1}{p}, \alpha\right)$ coming from Theorem 3.0.2. The situation is rendered in Figure 4.1. In particular, for $2<p<\infty$, the sketch indicates the range of exponents

$$
\left(\frac{1}{p}, \alpha\right)
$$

for which the Bochner-Riesz operator is bounded. Above the critical line connecting

$$
\left(0, \frac{n}{2}\right) \quad \text { and } \quad\left(\frac{1}{2},-\frac{\tilde{L}}{2}\right)
$$

provided we have an $L^{2} \rightarrow L^{p}$ extension estimate, the Bochner-Riesz operator is bounded. Below this critical line, the operator is unbounded. However, the critical line for the $K_{\Psi, \alpha} \in L^{p}$ test is the line connecting

$$
\left(0, \frac{n}{2}\right) \quad \text { and } \quad\left(\frac{1}{2},-\frac{L}{2}\right)
$$

which is below the critical line for the operator. Marked on the figure is the vertical line extending from the inverse of the critical exponent for $L^{2} \rightarrow L^{p}$ extension.

Proposition 4.3.1 is obtained as a corollary of Theorem 3.0.4 (which is the restriction estimate), Theorem 3.0.2, and Theorem 4.0.1. Let us recall Theorem 3.0.2.

Theorem. Let $\Gamma \subset \mathbb{R}^{N}$ be a smooth surface such that $\Gamma \subset P$ for some proper affine subspace $P$, which is of dimension $\tilde{N}$. Let $\widetilde{\Gamma}$ be the corresponding projection of $\Gamma$ onto $\mathbb{R}^{\tilde{N}}$. Suppose that the $L^{q}\left(\mathbb{R}^{\tilde{N}}\right) \rightarrow L^{2}\left(\widetilde{\Gamma}, \sigma_{\widetilde{\Gamma}}\right)$ restriction inequality (3.0.3) holds. Then, for $1<p \leq q$ or


Figure 4.1: Indices $\left(\frac{1}{p}, \alpha\right)$ for which $m_{\Psi, \alpha} \in \mathcal{M}_{p}$ or $K_{\Psi, \alpha} \in L^{p}$
$q^{\prime} \leq p<\infty$, the Bochner-Riesz multiplier $m_{\Gamma, \alpha}$ given by (1.0.2) with $\alpha>0$ defines a multiplier operator $m_{\Gamma, \alpha} \in \mathcal{M}_{p}$ if

$$
\alpha>(n+\tilde{L})\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{\tilde{L}}{2} .
$$

Let us also recall the statement of Theorem 4.0.1.
Theorem. Consider $\Gamma$ as given as a neighbourhood about the origin of the graph given by (1.0.6). Then the convolution kernel $K_{\Psi, \alpha}=\check{m}_{\Psi, \alpha} \in L^{p}\left(\mathbb{R}^{N}\right)$ if and only if $p>\frac{L+n}{L+\alpha+\frac{n}{2}}$.

Proof of Proposition 4.3.1. By Theorem 3.0.4, we see that the $L^{q} \rightarrow L^{2}$ restriction inequality (3.0.3) also holds as an $L^{p} \rightarrow L^{2}$ inequality for $1<p \leq q$. Therefore, we can apply Theorem 3.0.2 to see that, where $1<p \leq q$ or $q^{\prime} \leq p<\infty, T_{m}$ is bounded on $L^{p}$ for $\frac{\tilde{L}+n}{\tilde{L}+\alpha+\frac{n}{2}}<p<$ $\frac{\tilde{L}+n}{\tilde{L}-\alpha+\frac{n}{2}}$.

To prove the converse, instead of considering the multiplier

$$
m(\xi, \eta, \lambda)=|(\eta, \lambda)-\Psi(\xi)|^{\alpha} \chi((\eta, \lambda)-\Psi(\xi)) \phi(\xi),
$$

we restrict this continuous multiplier to $V=\mathbb{R}^{n+\tilde{L}}$, corresponding to $\lambda=0$. de Leeuw's theorem, [dL65], tells us that if $m \in \mathcal{M}_{p}\left(\mathbb{R}^{N}\right)$, then $\left.m\right|_{V} \in \mathcal{M}_{p}\left(\mathbb{R}^{n+\tilde{L}}\right)$. This is the BochnerRiesz multiplier on $\mathbb{R}^{n+\tilde{L}}$ corresponding to the surface $\widetilde{\Gamma}=\left\{(\xi, \widetilde{\Psi}(\xi)) ; \xi \in \mathbb{R}^{n}\right\}$. We apply Theorem 4.0.1 to see that the corresponding kernel is in $L^{p}$ only for $p>\frac{\tilde{L}+n}{\tilde{L}+\alpha+\frac{n}{2}}$. By duality, we must also have $p<\frac{\tilde{L}+n}{\tilde{L}-\alpha+\frac{n}{2}}$ for $T_{m}$ to be bounded on $L^{p}$.

## Part II

## Uniform oscillatory integral estimates

## Chapter 5

## Uniform oscillatory integral estimates

In this chapter, we work to extend work of J. Hickman and J. Wright regarding bounds for certain oscillatory integrals. These oscillatory integrals appeared previously in Section 3.2, arising in our derivation of $L^{2}$ restriction estimates for the surfaces in the class $\mathcal{S}_{0}$ (Definition 1.0.1). They arose as the Fourier transform of the surface measures for $\Gamma \in \mathcal{S}_{0}$. As we have seen with the application to restriction estimates, obtaining uniform estimates on such oscillatory integrals has important applications. Such estimates can also be used to obtain sublevel set estimates for the phase; see, for instance [KW12]. We will later return to some of the estimates we here consider: in Part III they are recovered in the polynomial case, by an analysis of polynomial root structure.

We now consider the one dimensional oscillatory integral

$$
I(x, y)=\int_{0}^{\infty} e^{2 \pi i\left(r x+\sum_{j=1}^{L} y_{j} \psi_{j}(r)\right)} \phi_{0}(r) r^{n-1} d r
$$

where $\Phi_{x, y}(r)=\Phi(r)=x r+\sum_{j=1}^{L} y_{j} \psi_{j}(r)$, where the $\psi_{j}$ correspond with the class $\mathcal{S}_{0}$ in Definition 1.0.1. Here $(x, y) \in \mathbb{R} \times \mathbb{R}^{L}$. We recall that

$$
\psi_{j}(r)=\frac{r^{d_{j}}}{d_{j}!}+\varepsilon_{j}(r)
$$

where $\varepsilon_{j}(r)$ is a higher order remainder term.
Under suitable restrictions, we work to establish uniform estimate

$$
\begin{equation*}
|I(x, y)| \leq C \min _{j=1, \ldots, L}\left|y_{j}\right|^{-\frac{n}{d_{j}}} \tag{5.0.1}
\end{equation*}
$$

Note that we also have the inequality $|I(x, y)| \lesssim 1$, since $\phi_{0}$ has compact support, although we make no use of this. One can also see that (5.0.1) is trivial in the case that $y=0$, if we interpret $0^{-1}=\infty$.

Remark 5.0.1. The estimate 5.0 .1 is sharp in the following sense. For each $1 \leq j \leq L$, we consider those $(x, y)$ such that, $y_{j} \neq 0, x=0$ and $y_{j^{\prime}}=0$ for $j^{\prime} \neq j$. It is a simple matter to verify that in this region, provided $\left|y_{j}\right| \gtrsim 1$,

$$
|I(x, y)| \sim\left|y_{j}\right|^{-\frac{n}{d_{j}}}
$$

This is essentially a special case of our later Proposition 9.0.1.
The following is a result of Hickman and Wright, in the polynomial case. We have extended their proof to cover the (radial) curves of standard type. We also later extend their method to establish some stronger uniform bounds, at the cost of stronger restrictions on the region of $(x, y) \in \mathbb{R} \times \mathbb{R}^{L}$ for which the corresponding estimate holds.

Theorem 5.0.2. The estimate (5.0.1) holds uniformly over $(x, y) \in \mathbb{R} \times \mathbb{R}^{L}$, provided $d_{1} \geq$ $n(L+1)$.

Proof. Suppose that $d_{1} \geq n(L+1)$. Note that, provided $\delta$ is chosen sufficiently small defining $\phi$, (2.2.2), using Taylor's theorem to bound the higher order term, $\varepsilon_{j}$, we can write each

$$
\begin{align*}
\psi_{j}^{(\iota)}(r) & =\frac{r^{d_{j}-\iota}}{\left(d_{j}-\iota\right)!}+\varepsilon_{j}^{(\iota)}(r) \\
\text { where }\left|\varepsilon_{j}^{(\iota)}(r)\right| & \ll\left|\frac{r^{d_{j}-\iota}}{\left(d_{j}-\iota\right)!}\right| \text { for } 0 \leq \iota \leq d_{j} \text { and } r \in \operatorname{supp} \phi_{0} \subset[-\delta, \delta] \tag{5.0.2}
\end{align*}
$$

To begin with, we rescale. We set $\sigma=\min _{1 \leq j \leq L}\left|y_{j}\right|^{-\frac{1}{d_{j}}}$ and $\tilde{\psi}_{j}(s)=\sigma^{-d_{j}} \psi_{j}(\sigma s)$. Making the change of variables $\sigma s=r$ and using the definition of the $\tilde{\psi}_{j}$, we find that

$$
\begin{aligned}
I(x, y) & =\int_{0}^{\infty} e^{2 \pi i\left(r x+\sum_{j=1}^{L} y_{j} \psi_{j}(r)\right)} \phi_{0}(r) r^{n-1} d r \\
& =\sigma^{n} \int_{0}^{\infty} e^{i \Phi_{w, z}(s)} \phi_{0}(\sigma s) s^{n-1} d s
\end{aligned}
$$

where

$$
\Phi_{w, z}(s)=w s+\sum_{j=1}^{L} z_{j} \sigma^{-d_{j}} \psi_{j}(\sigma s)
$$

with $w=2 \pi \sigma x$ and $z_{j}=2 \pi \sigma^{d_{j}} y_{j}$, for $1 \leq j \leq L$. We now seek to prove that, with

$$
\begin{gathered}
J(x, y):=\int_{0}^{\infty} e^{i \Phi_{w, z}(s)} \phi_{0}(\sigma s) s^{n-1} d s \\
|J(x, y)| \lesssim 1
\end{gathered}
$$

uniformly over $(x, y) \in \mathbb{R}^{N}$, which gives the desired result. Note that $\max _{1 \leq j \leq L}\left|z_{j}\right| \sim 1$.
Let us now decompose the integral over dyadic scales. We choose a bump function $\chi_{0}$ with $\chi_{0}(s)=1$ for $|s| \leq \frac{1}{2}$ and $\chi_{0}(s)=0$ for $|s| \geq 1$. For $l \geq 1$, we set $\chi_{l}(s)=\chi_{0}\left(2^{-l} s\right)-\chi_{0}\left(2^{-l+1} s\right)$. We can see that $\sum_{l \geq 0} \chi_{l}(s)=1$ and also that $\chi_{l}(s)=\chi\left(2^{-l} s\right)$, where $\chi(s)=\chi_{0}(s / 2)-\chi_{0}(s)$. Also, supp $\chi \subset[-2,-1 / 2] \cup[1 / 2,2]$. We see that

$$
J(x, y)=\sum_{l \geq 0} J_{l}(x, y)
$$

where

$$
\begin{aligned}
& J_{l}(x, y):=\int_{0}^{\infty} e^{i \Phi_{w, z}(s)} \phi_{0}(\sigma s) s^{n-1} \chi_{l}(s) d s \\
& =2^{l n} \int_{0}^{\infty} e^{i \Phi_{w, z}\left(2^{l} t\right)} \phi_{0}\left(2^{l} \sigma t\right) t^{n-1} \chi(t) d t
\end{aligned}
$$

For $l=0$, we can use the trivial bound

$$
\left|J_{0}(x, y)\right| \lesssim 1
$$

To bound the oscillatory integrals $J_{l}(x, y)$ for $l \geq 1$, we must get a handle on the phase. We set

$$
\begin{gathered}
\Phi_{l}(t):=\Phi_{w, z}\left(2^{l} t\right)=2^{l} w t+\sum_{j=1}^{L} 2^{l d_{j}} z_{j}\left(2^{-l d_{j}} \sigma^{-d_{j}} \psi_{j}\left(\sigma 2^{l} t\right)\right) \\
=2^{l} w t+\sum_{j=1}^{L} 2^{l d_{j}} z_{j} \tilde{\psi}_{j, l}(t)
\end{gathered}
$$

where

$$
\tilde{\psi}_{j, l}(t)=2^{-l d_{j}} \sigma^{-d_{j}} \psi_{j}\left(\sigma 2^{l} t\right)
$$

We can also write

$$
\tilde{\psi}_{j, l}(t)=\frac{t^{d_{j}}}{d_{j}!}+\tilde{\varepsilon}_{j, l}(t),
$$

where

$$
\tilde{\varepsilon}_{j, l}(t)=2^{-l d_{j}} \sigma^{-d_{j}} \varepsilon_{j}\left(\sigma 2^{l} t\right) .
$$

It is a simple matter to verify from (5.0.2) that

$$
\begin{equation*}
\left|\tilde{\varepsilon}_{j, l}^{(\iota)}(s)\right| \ll\left|\frac{s^{d_{j}-\iota}}{\left(d_{j}-\iota\right)!}\right| \text { for } 0 \leq \iota \leq d_{j} \text { and } s \in \operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi . \tag{5.0.3}
\end{equation*}
$$

As a consequence, for $t \in \operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi$, and $0 \leq \iota \leq L+1$,

$$
\left|\tilde{\psi}_{j, l}^{(\iota)}(t)\right| \sim\left|t^{d_{j}-\iota}\right| \sim 1 .
$$

We can also see that

$$
\begin{equation*}
\left\|\Phi_{l}^{\prime \prime}\right\|_{C^{n}\left(\operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi\right)} \lesssim 1, \tag{5.0.4}
\end{equation*}
$$

uniformly in $l$. We now work to establish that

$$
\sum_{l \geq 0}\left|J_{l}(x, y)\right| \lesssim 1
$$

We will find that, for all but finitely many exceptional indices $l$, the oscillatory integral $J_{l}$ has a well controlled non-stationary phase and we have strong bounds on $J_{l}$. For the exceptional indices, bounding $J_{l}$ requires more care.

Let us first define the set of exceptional indices $\mathcal{L}^{\prime}$. We include $l=0$ and, for $l \geq 1$, we also include in $\mathcal{L}^{\prime}$ those indices $l$ for which

$$
\inf _{t \in \operatorname{supp} \chi}\left|\Phi_{l}^{\prime}(t)\right| \lesssim 2^{l} .
$$

We find that there are finitely many exceptional indices for which such a relationship holds, with the number of exceptional indices uniformly bounded over $(x, y) \in \mathbb{R}^{N}$. Indeed, let us first choose $t^{*}$ such that

$$
\min _{t \in \operatorname{supp} \chi}\left|\Phi_{l}^{\prime}(t)\right|=\left|\Phi_{l}^{\prime}\left(t^{*}\right)\right| .
$$

Before continuing, let us introduce some new notation. We may henceforth denote $w$ by $z_{0}$ and 1 by $d_{0}$, although we still write $z=\left(z_{1}, z_{2}, \ldots, z_{L}\right)$. Likewise, we set $\tilde{\psi}_{0, l}(s)=2^{l} s$ so that $\Phi(s)=\sum_{j=0}^{L} z_{j} \tilde{\psi}_{j, l}(s)$. Now fixing $1 \leq j^{*} \leq L$ such that

$$
\max _{1 \leq j \leq L}\left|2^{l d_{j}} z_{j} \tilde{\psi}_{j, l}^{\prime}\left(t^{*}\right)\right|=\left|2^{l d_{j^{*}}} z_{j^{*}} \tilde{\psi}_{j^{*}, l}^{\prime}\left(t^{*}\right)\right|,
$$

we evidently have that

$$
\left|2^{l d_{j^{*}}} z_{j^{*}} \tilde{\psi}_{j^{*}, l}^{\prime}\left(t^{*}\right)\right| \gtrsim 2^{l d_{1}},
$$

since $\max _{1 \leq j \leq L}\left|z_{j}\right| \sim 1$ and $\tilde{\psi}_{j^{*}, l}^{\prime}\left(t^{*}\right) \sim 1$. Therefore, in order to have that

$$
\inf _{t \in \operatorname{supp} \chi}\left|\Phi_{l}^{\prime}(t)\right| \lesssim 2^{l},
$$

there must exist $0 \leq j^{\prime} \leq L$ with $j^{\prime} \neq j^{*}$ such that

$$
\left|2^{l d_{j^{\prime}}} z_{j^{\prime}} \tilde{\psi}_{j^{\prime}, l}^{\prime}\left(t^{*}\right)\right| \gtrsim 2^{l d_{1}} .
$$

In particular, we find that

$$
\begin{equation*}
2^{l d_{j^{\prime}}}\left|z_{j^{\prime}}\right| \sim 2^{l d_{j^{*}}}\left|z_{j^{*}}\right|>0 \tag{5.0.5}
\end{equation*}
$$

For fixed $(w, z)=\left(z_{0}, z_{1}, \ldots, z_{L}\right)$, there are at most finitely many indices where such a comparison may hold. Specifically, for a suitable fixed number $a$ the only integers $l$ for which the relation (5.0.5) might hold are those

$$
l \in\left[\frac{1}{d_{j^{\prime}}-d_{j^{*}}} \log _{2}\left(\frac{\left|z_{j^{*}}\right|}{\left|z_{j^{\prime}}\right|}\right)-\frac{a}{2}, \frac{1}{d_{j^{\prime}}-d_{j^{*}}} \log _{2}\left(\frac{\left|z_{j^{*}}\right|}{\left|z_{j^{\prime}}\right|}\right)+\frac{a}{2}\right] .
$$

We now establish the estimate

$$
\left|J_{l}(x, y)\right| \lesssim 1
$$

which we will use to estimate the contribution of the terms indexed by the exceptional $\mathcal{L}^{\prime}$ to our estimate on $|J(x, y)|$. This bound is trivial for $l=0$. For $l \geq 1$ we proceed as follows. Recall from (5.0.3) that, for $t \in \operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi$

$$
\tilde{\psi}_{j, l}^{(\iota)}(t)=\frac{t^{d_{j}-\iota}}{\left(d_{j}-\iota\right)!}+\tilde{\varepsilon}_{j, l}^{(\iota)}(t)
$$

where $\left|\tilde{\varepsilon}_{j, l}^{(\iota)}(t)\right| \ll\left|\frac{t^{d_{j}-\iota}}{\left(d_{j}-\iota\right)!}\right|$. One can see that, for $2 \leq \iota \leq L+1$,

$$
\begin{aligned}
& \Phi_{l}^{(\iota)}(t)=\sum_{j=1}^{L} 2^{l d_{j}} z_{j} \tilde{\psi}_{j, l}^{(\iota)}(t) \\
= & \sum_{j=1}^{L} 2^{l d_{j}} z_{j} \frac{t^{d_{j}-\iota}}{\left(d_{j}-\iota\right)!}+\theta_{l}^{(\iota)}(t),
\end{aligned}
$$

where $\theta_{l}(t)=\sum_{j=1}^{L} 2^{l d_{j}} z_{j} \tilde{\varepsilon}_{j, l}^{(l)}(t)$. Note that, for $t \in \operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi$

$$
\left|t^{\iota} \theta_{l}^{(\iota)}(t)\right| \sim\left|\theta_{l}^{(\iota)}(t)\right| \ll \max _{1 \leq j \leq L} 2^{l d_{j}}\left|z_{j}\right| .
$$

For $2 \leq \iota \leq L+1$, we write

$$
t^{\iota} \Phi_{l}^{(\iota)}(t)=\sum_{j=1}^{L} 2^{l d_{j}} z_{j} \frac{t^{d_{j}}}{\left(d_{j}-\iota\right)!}+t^{\iota} \theta_{l}^{(\iota)}(t) .
$$

In matrix form, we have that

$$
\left(\begin{array}{c}
t^{2} \Phi_{l}^{(2)}(t) \\
t^{3} \Phi_{l}^{(3)}(t) \\
\vdots \\
t^{L+1} \Phi_{l}^{(L+1)}(t)
\end{array}\right)=M\left(\begin{array}{c}
2^{l d_{1}} z_{1} \frac{t^{d_{1}}}{\left(d_{1}-1\right)!} \\
2^{l d_{2}} z_{2} \frac{t^{d_{2}}}{\left(d_{2}-1\right)!} \\
\vdots \\
2^{l d_{L}} z_{L} \frac{t^{d} L}{\left(d_{L}-1\right)!}
\end{array}\right)+\left(\begin{array}{c}
t^{2} \theta_{l}^{(2)}(t) \\
t^{3} \theta_{l}^{(3)}(t) \\
\vdots \\
t^{L+1} \theta_{l}^{(L+1)}(t)
\end{array}\right),
$$

where

$$
M(b, j)=\left(d_{j}-1\right)\left(d_{j}-2\right) \ldots\left(d_{j}-b\right) .
$$

We claim that $M$ is invertible. Given this claim, we see that, for $t \in \operatorname{supp} \chi$, since $|t| \sim 1$,

$$
\max _{2 \leq \iota \leq L+1}\left|\Phi^{(\iota)}(t)\right| \sim \max _{2 \leq \iota \leq L+1}\left|t^{\iota} \Phi^{(\iota)}(t)\right| \sim\left|\left(\begin{array}{c}
t^{2} \Phi_{l}^{(2)}(t) \\
t^{3} \Phi_{l}^{(3)}(t) \\
\vdots \\
t^{L+1} \Phi_{l}^{(L+1)}(t)
\end{array}\right)\right|
$$

$$
\gtrsim \operatorname{det} M\left|\left(\begin{array}{c}
2^{l d_{1}} z_{1} \frac{t^{d_{1}}}{\left(d_{1-1}\right)!} \\
2^{l d_{2}} z_{2} \frac{t^{d_{2}}}{\left(d_{2}-1\right)!} \\
\vdots \\
2^{l d_{L}} z_{L} \frac{t^{d_{L}}}{\left(d_{L}-1\right)!}
\end{array}\right)\right|-\left|\left(\begin{array}{c}
t^{2} \theta_{l}^{(2)}(t) \\
t^{3} \theta_{l}^{(3)}(t) \\
\vdots \\
t^{L+1} \theta_{l}^{(L+1)}(t)
\end{array}\right)\right|
$$

We know that, for $t \in \operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi$,

$$
\left|\left(\begin{array}{c}
t^{2} \theta_{l}^{(2)}(t) \\
t^{3} \theta_{l}^{(3)}(t) \\
\vdots \\
t^{L+1} \theta_{l}^{(L+1)}(t)
\end{array}\right)\right| \ll \max _{1 \leq j \leq L} 2^{l d_{j}}\left|z_{j}\right| \sim \operatorname{det} M\left|\left(\begin{array}{c}
2^{l d_{1}} z_{1} \frac{t^{d_{1}}}{\left(d_{1}-1\right)!} \\
2^{l d_{2}} z_{2} \frac{t^{d_{2}}}{\left(d_{2}-1\right)!} \\
\vdots \\
2^{l d_{L}} z_{L} \frac{t^{d_{L}}}{\left(d_{L}-1\right)!}
\end{array}\right)\right|
$$

We also know that $\max _{1 \leq j \leq L}\left|z_{j}\right| \sim 1$ so that $\max _{1 \leq j \leq L} 2^{l d_{j}}\left|z_{j}\right| \gtrsim 2^{l d_{1}}$. Therefore, for $t \in$ $\operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi$,

$$
\begin{gathered}
\max _{2 \leq \iota \leq L+1}\left|\Phi^{(\iota)}(t)\right| \sim \max _{1 \leq j \leq L} 2^{l d_{j}}\left|z_{j} t^{d_{j}}\right| \\
\sim \max _{1 \leq j \leq L} 2^{l d_{j}}\left|z_{j}\right| \gtrsim 2^{l d_{1}}
\end{gathered}
$$

We can now apply the van der Corput estimate, Lemma $0.0 .8,{ }^{1}$ to find that

$$
\begin{gathered}
\left|J_{l}(x, y)\right| \\
\lesssim 2^{l n} 2^{-\frac{l d_{1}}{L+1}} \int\left(2^{l} \sigma\left|\phi_{0}^{\prime}\left(2^{l} \sigma t\right) t^{n-1} \chi(t)\right|+\left|\phi_{0}\left(2^{l} \sigma t\right)(n-1) t^{n-2} \chi(t)\right|\right) d t \\
+2^{l n} 2^{-\frac{l d_{1}}{L+1}} \int\left|\phi_{0}\left(2^{l} \sigma t\right) t^{n-1} \chi^{\prime}(t)\right| d t \\
\lesssim 2^{l n} 2^{-\frac{l d_{1}}{L+1}}
\end{gathered}
$$

since if $\operatorname{supp} \phi_{0}^{\prime}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi \neq \emptyset$, then $2^{l} \sigma \sim 1$. By assumption, $d_{1} \geq n(L+1)$, so that upon taking the finite sum,

$$
\sum_{l \in \mathcal{L}^{\prime}}\left|J_{l}(x, y)\right| \lesssim 1
$$

It remains to bound those terms with non-stationary phases: $J_{l}(x, y)$ for $l \notin \mathcal{L}^{\prime}$. Here we expect to have stronger control on $J_{l}(x, y)$. We integrate by parts. Specifically, we define the operator $T_{l}$ by

$$
T_{l} g(t)=\frac{1}{i \Phi_{l}^{\prime}(t)} \frac{d g}{d t}(t)
$$

so that $T_{l}^{*} h(t)=-\frac{d}{d t}\left(\frac{1}{i \Phi_{l}^{\prime}(\cdot)} h\right)(t)$. Then

$$
\begin{aligned}
& \left|J_{l}(x, y)\right|=2^{l n}\left|\int e^{i \Phi_{l}(t)} \phi_{0}\left(2^{l} \sigma t\right) t^{n-1} \chi(t) d t\right| \\
= & 2^{l n}\left|\int T_{l}^{n+1}\left(e^{i \Phi_{l}(\cdot)}\right)(t) \phi_{0}\left(2^{l} \sigma t\right) t^{n-1} \chi(t) d t\right| \\
= & 2^{l n}\left|\int e^{i \Phi_{l}(t)}\left(T_{l}^{*}\right)^{n+1}\left(\phi_{0}\left(2^{l} \sigma(\cdot)\right)(\cdot)^{n-1} \chi(\cdot)\right)(t) d t\right| \\
\leq & 2^{l n} \int\left|\left(T_{l}^{*}\right)^{n+1}\left(\phi_{0}\left(2^{l} \sigma(\cdot)\right)(\cdot)^{n-1} \chi(\cdot)\right)(t)\right| d t
\end{aligned}
$$

[^4]\[

$$
\begin{gathered}
\lesssim 2^{l n} \frac{1}{2^{l(n+1)}}\left(1+\left\|\Phi_{l}^{\prime \prime}\right\|_{\left.C^{n}\left(\operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi\right)\right)^{n+1}\left(1+\left\|\phi_{0}\left(2^{l} \sigma(\cdot)\right)\right\|_{C^{n+1}(\operatorname{supp} \chi)}\right)}^{\lesssim 2^{-l},}\right.
\end{gathered}
$$
\]

since $\left|\Phi_{l}^{\prime}(t)\right| \mid \gtrsim s^{l}$ in the region of integration, $\left\|\Phi_{l}^{\prime \prime}\right\|_{C^{n}\left(\operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi\right)} \lesssim 1$ from (5.0.4) and, if $\operatorname{supp} \phi_{0}^{\prime}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi \neq \emptyset$, then $2^{l} \sigma \sim 1$.

Bringing these bounds together, we have that

$$
\begin{gathered}
|I(x, y)|=\sigma^{n}|J(x, y)| \\
\leq \sigma^{n}\left(\sum_{l \in \mathcal{L}^{\prime}}\left|J_{l}(x, y)\right|+\sum_{l \geq 0, l \notin \mathcal{L}^{\prime}}\left|J_{l}(x, y)\right|\right) \\
\lesssim \sigma^{n}\left(1+\sum_{l \geq 0} 2^{-l}\right) \lesssim \sigma^{n}
\end{gathered}
$$

which is what we require.
It remains to show that $M$ is invertible, where

$$
M(b, j)=\left(d_{j}-1\right)\left(d_{j}-2\right) \ldots\left(d_{j}-b\right)
$$

We see that the $j$ th column of $M$ is the evaluation of the vector valued function

$$
v(t)=\left(\begin{array}{c}
(t-1) \\
(t-1)(t-2) \\
\vdots \\
(t-1) \ldots(t-L)
\end{array}\right)
$$

at the point $t=d_{j}$. Considering the Maclaurin expansion of $(t-1)^{-1} v(t)$, we can also write

$$
v(t)=\frac{t-1}{t} \widetilde{M}\left(\begin{array}{c}
t / 1! \\
t^{2} / 2! \\
\vdots \\
t^{L} / L!
\end{array}\right)
$$

where $\widetilde{M}$ is a lower triangular matrix with non-zero diagonal entries. As a triangular matrix with non-zero diagonal entries, the matrix $\widetilde{M}$ is invertible. With

$$
w(t)=\left(\begin{array}{c}
t / 1! \\
t^{2} / 2! \\
\vdots \\
t^{L} / L!
\end{array}\right)
$$

we see that

$$
\begin{gathered}
\operatorname{dim} \operatorname{span}\left\{v\left(k_{1}\right), v\left(k_{2}\right), \ldots, v\left(k_{L}\right)\right\} \\
=\operatorname{dim} \operatorname{span}\left\{\frac{k_{1}-1}{k_{1}} \widetilde{M} w\left(k_{1}\right), \frac{k_{1}-2}{k_{2}} \widetilde{M} w\left(k_{2}\right), \ldots, \frac{k_{L}-1}{k_{L}} \widetilde{M} w\left(k_{L}\right)\right\} \\
=\operatorname{dim} \operatorname{span}\left\{\widetilde{M} w\left(k_{1}\right), \widetilde{M} w\left(k_{2}\right), \ldots, \widetilde{M} w\left(k_{L}\right)\right\} \\
=\operatorname{dim} \operatorname{span}\left\{w\left(k_{1}\right), w\left(k_{2}\right), \ldots, w\left(k_{L}\right)\right\} \\
=L
\end{gathered}
$$

since points on the moment curve are in general position. Therefore, $M$ is invertible.
If we pay careful attention to the proof, then we see that it relies in determining which of
the summands defining the phase,

$$
\Phi(t)=\sum_{j=0}^{L} y_{j} \psi_{j}(t),
$$

are substantial in a given region $|t| \sim 2^{-l} \sigma$ for $l \geq 1$. We previously considered uniform oscillatory integral estimates over all $(x, y) \in \mathbb{R}^{N}$. If we introduce further restrictions on the region of $\mathbb{R}^{N}$ under consideration, then we can ensure that some of the phase's summands are insubstantial in the above proof. In particular, we are able to obtain the following refinement of Theorem 5.0.2.

Proposition 5.0.3. For $1 \leq m \leq L$ we define, with $d_{0}=1$,

$$
R_{m}:=\left\{\left(y_{0}, y\right) \in \mathbb{R} \times \mathbb{R}^{L} ;\left|y_{j}\right|^{\frac{1}{d_{j}}} \leq\left|y_{m}\right|^{\frac{1}{d_{m}}} \text { for all } 0 \leq j \leq L\right\}
$$

The estimate (5.0.1) holds uniformly over $\left(y_{0}, y\right) \in R_{m}$, provided $d_{m} \geq n(L-m+1)$.
Remark 5.0.4. Note that the regions $R_{m}$ do not cover $\mathbb{R}^{N}$, as they exclude the region where $|x|=\left|y_{0}\right|>\left|y_{j}\right|^{\frac{1}{d_{j}}}$ for $1 \leq j \leq L$.

Similarly, we have the following generalisation of Theorem 5.0.2.
Theorem 5.0.5. For $1 \leq m \leq L$ we define

$$
S_{m}:=\left\{y \in \mathbb{R}^{L} ;\left|y_{j}\right|^{\frac{1}{d_{j}}} \leq\left|y_{m}\right|^{\frac{1}{d_{m}}} \text { for all } 1 \leq j \leq L\right\}
$$

For $1 \leq m \leq L$, the estimate (5.0.1) holds uniformly over $(x, y) \in \mathbb{R} \times S_{m}$, provided $d_{m} \geq n(L-m+2)$. In particular, (5.0.1) holds over all non-zero $(x, y) \in \mathbb{R}^{1} \times \mathbb{R}^{L}$ provided $d_{1} \geq n(L+1)$.

We prove Proposition 5.0.3, noting the essential difference with our previous proof of Theorem 5.0.2. The modifications to these proofs then necessary to prove Theorem 5.0.5 will then be made apparent to the reader, and we provide only an outline.

Proof of Proposition 5.0.3. Let us consider $(x, y) \in R_{m}$. We proceed as above. Set $\sigma=$ $\left|y_{m}\right|^{-\frac{1}{d_{m}}}$ and, for $0 \leq j \leq L, \tilde{\psi}_{j}(s)=\sigma^{-d_{j}} \psi_{j}(\sigma s)$. We set $w=z_{0}=\sigma x, d_{0}=1$, and $z_{j}=\sigma^{d_{j}} y_{j}$ and then see that $\max _{0 \leq j \leq L}\left|z_{j}\right| \leq\left|z_{m}\right|=1$.

With $\chi_{l}$ and $\Phi_{w, z}$ defined as previously, we find that

$$
I(x, y)=\sigma^{n} \sum_{l \geq 0} J_{l}(x, y)
$$

where

$$
\begin{aligned}
& J_{l}(x, y):=\int_{0}^{\infty} e^{i \Phi_{w, z}(s)} \phi_{0}(\sigma s) s^{n-1} \chi_{l}(s) d s \\
& =2^{l n} \int_{0}^{\infty} e^{i \Phi_{w, z}\left(2^{l} t\right)} \phi_{0}\left(2^{l} \sigma t\right) t^{n-1} \chi(t) d t
\end{aligned}
$$

To bound the oscillatory integrals $J_{l}(x, y)$, we must get a handle on the phase, $\Phi_{w, z}\left(2^{l} \cdot\right)$. We set

$$
\begin{gathered}
\Phi_{l}(t):=\Phi_{w, z}\left(2^{l} t\right)=2^{l} w t+\sum_{j=1}^{L} 2^{l d_{j}} z_{j}\left(2^{-l d_{j}} \tilde{\psi}_{j}\left(2^{l} t\right)\right) \\
=2^{l} w t+\sum_{j=1}^{L} 2^{l d_{j}} z_{j} \tilde{\psi}_{j, l}(t)
\end{gathered}
$$

where

$$
\tilde{\psi}_{j, l}(t)=2^{-l d_{j}} \sigma^{-d_{j}} \psi_{j}\left(2^{l} \sigma t\right)
$$

$$
=\frac{t^{d_{j}}}{d_{j}!}+\tilde{\varepsilon}_{j, l}(t)
$$

where $\tilde{\varepsilon}_{j, l}(t)=2^{-l d_{j}} \sigma^{-d_{j}} \varepsilon_{j}\left(2^{l} \sigma t\right)$. Now we will see the essential difference from the above proof. We here define

$$
\begin{equation*}
\theta_{l}(t):=\sum_{j=0}^{m-1} 2^{l d_{j}} z_{j} \frac{t^{d_{j}}}{d_{j}!}+\sum_{j=1}^{L} 2^{l d_{j}} z_{j} \tilde{\varepsilon}_{j, l}(t) \tag{5.0.6}
\end{equation*}
$$

In our analysis of the exceptional indices, $\theta_{l}$ will be treated as an error term in the phase. We consider the $\iota$ derivative of the phase, where $1 \leq \iota \leq L-m+1$. One can see that, for $t \in \operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right)$,

$$
\left|\sum_{\substack{j=0 \\ d_{j} \geq \iota}}^{m-1} 2^{l d_{j}} z_{j} \frac{t^{d_{j}-\iota}}{\left(d_{j}-\iota\right)!}\right| \lesssim 2^{l d_{m-1}} .
$$

Therefore, for $l \geq l_{0}$ for some sufficiently large $l_{0}$, we have that, for $t \in \operatorname{supp} \chi \cap \operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right)$,

$$
\left|\sum_{\substack{j=0 \\ d_{j} \geq \iota}}^{m-1} 2^{l d_{j}} z_{j} \frac{t^{d_{j}-\iota}}{\left(d_{j}-\iota\right)!}\right| \ll 2^{l d_{m}} \lesssim \max _{1 \leq j \leq L} 2^{l d_{j}}\left|z_{j}\right|
$$

since $2^{l d_{m-1}} \ll 2^{l d_{m}}$. As previously, provided $\delta$ is chosen sufficiently small, we also have that

$$
\left|\sum_{j=1}^{L} 2^{l d_{j}} z_{j} \tilde{\varepsilon}_{j, l}^{(\iota)}(t)\right| \ll \max _{1 \leq j \leq L} 2^{l d_{j}}\left|z_{j}\right| .
$$

Thus we see that, for $t \in \operatorname{supp} \chi \cap \operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right)$ and $l \geq l_{0}$,

$$
\begin{equation*}
\left|t^{\iota} \theta_{l}^{(\iota)}(t)\right| \sim\left|\theta_{l}^{(\iota)}(t)\right| \ll \max _{1 \leq j \leq L} 2^{l d_{j}}\left|z_{j}\right|=\max _{1 \leq j \leq L-m+1} 2^{l d_{m+j-1}}\left|z_{m+j-1}\right|, \tag{5.0.7}
\end{equation*}
$$

since $\left|z_{m}\right|=1 \geq \max _{0 \leq j \leq L}\left|z_{j}\right|$. As such, another distinction we make in this proof is that we also include in the exceptional set $\mathcal{L}^{\prime}$ those $0 \leq l \leq l_{0}$ for a suitable $l_{0}$.

For $0 \leq l \leq l_{0}$, one can easily see that

$$
\left|J_{l}(x, y)\right| \lesssim 2^{l_{0} n} \lesssim l_{0} 1
$$

For those $l>l_{0}$, the elements of $\mathcal{L}^{\prime}$ are defined as previously to be chosen to be those indices $l$ such that

$$
\inf _{t \in \operatorname{supp} \chi \cap \operatorname{supp} \phi_{0}\left(2^{l} \sigma^{\cdot}\right)}\left|\Phi_{l}^{\prime}(t)\right| \leq 2^{l} .
$$

For $l \in \mathcal{L}^{\prime}$ with $l>l_{0}$, we find as before that, for $1 \leq \iota \leq L-m+1$,

$$
\begin{aligned}
& \Phi_{l}^{(\iota)}(t)=\sum_{j=0}^{L} 2^{l d_{j}} z_{j} \tilde{\psi}_{j, l}^{(\iota)}(t) \\
= & \sum_{j=m}^{L} 2^{l d_{j}} z_{j} \frac{t^{d_{j}-\iota}}{\left(d_{j}-\iota\right)!}+\theta_{l}^{(\iota)}(t),
\end{aligned}
$$

where $\theta_{l}(t)$ is given by (5.0.6). We write, for $1 \leq \iota \leq L-m+1$,

$$
t^{\iota} \Phi_{l}^{(\iota)}(t)=\sum_{j=m}^{L} 2^{l d_{j}} z_{j} \frac{t^{d_{j}}}{\left(d_{j}-\iota\right)!}+t^{\iota} \theta_{l}^{(\iota)}(t)
$$

We can express this in matrix form. We have that

$$
\left(\begin{array}{c}
t^{1} \Phi_{l}^{(1)}(t) \\
t^{2} \Phi_{l}^{(2)}(t) \\
\vdots \\
t^{L-m+1} \Phi_{l}^{(L-m+1)}(t)
\end{array}\right)=M\left(\begin{array}{c}
2^{l d_{m}} z_{m} \frac{t^{d_{m}}}{\left(d_{m}-m\right)!} \\
2^{l d_{m+1}} z_{2} \frac{t^{d_{m+1}}}{\left(d_{m+1}-1\right)!} \\
\vdots \\
2^{l d_{L}} z_{L} \frac{t^{d_{L}}}{\left(d_{L}-1\right)!}
\end{array}\right)+\left(\begin{array}{c}
t^{1} \theta_{l}^{(1)}(t) \\
t^{2} \theta_{l}^{(2)}(t) \\
\vdots \\
t^{L-m+1} \theta_{l}^{(L-m+1)}(t)
\end{array}\right),
$$

where

$$
M(b, j)=\left(d_{m+j-1}-1\right)\left(d_{m+j-1}-2\right) \ldots\left(d_{m+j-1}-b\right)
$$

As previously, we can verify that $M$ is invertible. Thus, we find that, for $t \in \operatorname{supp} \chi$, since $|t| \sim 1$,

$$
\begin{aligned}
& \max _{1 \leq \iota \leq L-m+1}\left|\Phi^{(\iota)}(t)\right| \sim \max _{1 \leq \iota \leq L-m+1}\left|t^{\iota} \Phi^{(\iota)}(t)\right| \sim\left|\left(\begin{array}{c}
t^{1} \Phi_{l}^{(1)}(t) \\
t^{2} \Phi_{l}^{(2)}(t) \\
\vdots \\
t^{L-m+1} \Phi_{l}^{(L-m+1)}(t)
\end{array}\right)\right| \\
& \quad \gtrsim \operatorname{det} M\left|\left(\begin{array}{c}
2^{l d_{m}} z_{m} \frac{t^{d_{m}}}{\left(d_{m}-1\right)!} \\
2^{l d_{m+1}} z_{m+1} \frac{t^{d_{m+1}}}{\left(d_{m+1}-1\right)!} \\
\vdots \\
2^{l d_{L}} z_{L} \frac{t^{d_{L}}}{\left(d_{L}-1\right)!}
\end{array}\right)\right|-\left|\left(\begin{array}{c}
t^{1} \theta_{l}^{(1)}(t) \\
t^{2} \theta_{l}^{(2)}(t) \\
\vdots \\
t^{L-m+1} \theta_{l}^{(L-m+1)}(t)
\end{array}\right)\right|
\end{aligned}
$$

Using the bound (5.0.7), we see that, for $t \in \operatorname{supp} \phi_{0}\left(2^{l} \sigma \cdot\right) \cap \operatorname{supp} \chi$,

$$
\begin{aligned}
&\left|\left(\begin{array}{c}
t^{1} \theta_{l}^{(1)}(t) \\
t^{2} \theta_{l}^{(2)}(t) \\
\vdots \\
t^{L-m+1} \theta_{l}^{(L-m+1)}(t)
\end{array}\right)\right| \ll \max _{1 \leq j \leq L-m+1} 2^{l d_{m+j-1}}\left|z_{m+j-1}\right| \\
& \sim \operatorname{det} M\left|\left(\begin{array}{c}
2^{l d_{m}} z_{m} \frac{t^{d_{m}}}{\left(d_{m}-1\right)!} \\
2^{l d_{m+1}} z_{m+1} \frac{t^{d_{m+1}}}{\left(d_{m+1}-1\right)!} \\
\vdots \\
2^{l d_{L}} z_{L} \frac{t^{d_{L}}}{\left(d_{L}-1\right)!}
\end{array}\right)\right| \sim \max _{1 \leq j \leq L-m+1} 2^{l d_{m+j-1}\left|z_{m+j-1}\right|}
\end{aligned}
$$

It follows that

$$
\inf _{t \in \operatorname{supp} \chi \text { กsupp } \phi_{0}\left(\sigma 2^{l .}\right)} \max _{1 \leq \iota \leq L-m+1}\left|\Phi_{l}^{(j)}(t)\right| \gtrsim \max _{1 \leq j \leq L-m+1} 2^{l d_{m+j-1}}\left|z_{m+j-1}\right| \gtrsim 2^{l d_{m}}
$$

Therefore, by van der Corput's lemma, we see that for the exceptional $\mathcal{L}^{\prime}$ with $l \geq l_{0}$,

$$
\left|J_{l}(x, y)\right| \lesssim 2^{l n} 2^{-\frac{l d_{m}}{L-m+1}} \lesssim 1,
$$

provided $d_{m} \geq n(L-m+1)$. As before, we also have summable bounds on the remaining $J_{l}$ with $l \notin \mathcal{L}^{\prime}$.

Proof of Theorem 5.0.5. The situation is wholly analogous to the above proofs, so we simply provide an outline. To prove the required bound in the region $S_{m}$, one may work as follows. Rescale the integral and carry out a dyadic decomposition of the integration, $J=\sum_{l>0} J_{l}$. The heart of the analysis takes place on (finitely many) exceptional dyadic pieces, $J_{l}$. For each of these, one defines the error term $\theta_{l}$ for the phase, analogously to the proof of Proposition 5.0.3. To bound the exceptional $\left|J_{l}\right| \lesssim 2^{l n} 2^{-\frac{l d_{m}}{L-m+2}}$, we work exactly as in our proof of Theorem
5.0.2: we can use van der Corput's lemma, with reference to derivatives of the phase of order $\iota$, where $2 \leq \iota \leq L-m+2$. The remaining dyadic pieces, which have non-stationary phase, can be bounded using integration by parts. Provided $n(L-m+2) \leq d_{m}$, we obtain the desired bound.

Remark 5.0.6. Other refinements of Theorem 5.0.2 are possible. For example, if $\left|y_{j}\right| \ll\left|y_{m}\right|$ for $j>m$, then, in the support of $\phi_{0}$, the corresponding terms $y_{j} r^{k_{j}}$ are dominated by $y_{m} r^{k_{m}}$ so that those $y_{j} r^{k_{j}}$ with $j>m$ can not substantially contribute to any cancellation in the phase. Further consideration of this and other possible refinements are left to the reflective reader.

## Part III

## The structure of polynomial roots

## Chapter 6

## Introduction

In this part, we make a structural analysis of the roots of real polynomials. We show how roots are stratified in separated tiers and provide further analysis to bound how many roots can cluster about a point. Though our analysis is carried out over $\mathbb{R}$, we expect that the proofs contained herein are valid over other fields. We consider real polynomials, in particular, because understanding the root structure of real polynomials has applications in obtaining uniform oscillatory integral estimates.

This analysis is inspired by work of Kowalski and Wright, [KW12], and an unpublished oscillatory integral estimate of Hickman and Wright [HW20], Theorem 5.0.2, which we reformulate for this part at Theorem 6.0.5. It takes the perspective of root clusters, in the spirit of the famous oscillatory integral estimates of Phong and Stein, [PS97].

Consider the polynomial

$$
\Psi_{1}(t)=x+y_{1} t+y_{2} t^{2}+\ldots y_{L} t^{L}
$$

We know, by the fundamental theorem of algebra, that $\Psi_{1}$ has $L$ roots, counted with multiplicity. In particular, for some small $\epsilon>0$, we know that, for any non-zero root $w$, at most $L$ roots are contained in $B(w, \epsilon|w|)$, which is, of course, trivial. Now consider the polynomial

$$
\Psi_{k}(t)=\Psi_{1}\left(t^{k}\right) .
$$

Corresponding to each root of $w$ of $\Psi_{1}$, we see that there are $k$ roots of $\Psi_{k}$, these are the $k$ th roots of $w$. As before, we also have that, for some small $\epsilon>0$, and for any non-zero root $w$, at most $L$ roots are contained in $B(w, \epsilon|w|)$. See Figures 6.1 and 6.2 for a sketch of the roots of $\Psi_{1}$ and $\Psi_{k}$, respectively, in the case that $L=4, k=5$, and

$$
\Psi_{1}(t)=y_{L}\left(t-h_{1}\right)\left(t-h_{2}\right)\left(t-h_{3}\right)\left(t+h_{4}\right)
$$

More specifically, these figures are sketches of root structure in the case where $h_{1} \gg h_{2} \approx h_{3} \gg$ $h_{4}>0$. The dotted black circle corresponds to roots with modulus $h_{1}$. The solid blue lines correspond to roots with modulus close to $h_{2}$. The dashed red line corresponds with roots of modulus $h_{4}$. However, for any choice $h_{j}$ one can see we have the following. For some suitable small $\epsilon>0$ and for any non-zero root $w^{\prime}$ of $\Psi_{k}$, there are at most $L=4$ roots contained in $B\left(w^{\prime}, \epsilon\left|w^{\prime}\right|\right)$. This particular example reflects polynomial root structure more generally, as outlined in the Theorem 6.0.1.

Theorem 6.0.1. We fix a set of exponents $0=k_{0}<k_{1}<k_{2}<\ldots<k_{L}$ and consider real polynomials whose exponents are drawn from this set. For any real polynomial $\Psi(t)=$ $x+y_{1} t^{k_{1}}+y_{2} t^{k_{2}}+\ldots+y_{L} t^{k_{L}}$ with $y_{L} \neq 0$, we have the following structure.

The roots of $\Psi$ are stratified into $s$ tiers of roots $\mathcal{T}_{1}, \mathcal{T}_{2}, \ldots, \mathcal{T}_{s}$, where $1 \leq s \leq L$. The tiers are separated in the sense that, if we take $w_{i} \in \mathcal{T}_{i}$, then

$$
\left|w_{1}\right| \gg\left|w_{2}\right| \gg \ldots \gg\left|w_{s}\right| .
$$

At most $L$ non-zero roots can cluster about a root: there is some suitable small parameter


Figure 6.1: The roots of $\Psi_{1}$.


Figure 6.2: The roots of $\Psi_{k}$.
$\epsilon$ such that, for all $0 \neq w \in \mathcal{T}_{i}$, there are at most $L$ roots of $\Psi$ in $B(w, \epsilon|w|)$, counted with multiplicity.

Let us recall the following structural result of Kowalski and Wright, Theorem 1.6 of [KW12]. Note that this result found applications in bounding oscillatory integrals and discrete exponential sums and for bounding the measure of sublevel sets of polynomials.

Theorem 6.0.2. Let

$$
\Psi(t)=a_{k_{L}} t^{k_{L}}+a_{k_{L}-1} t^{k_{L}-1}+\ldots+a_{0}=a_{d} \prod_{j}\left(t-z_{j}\right)
$$

be a complex polynomial with $\max _{l}\left|a_{l}\right|=1$. Suppose that the coefficients satisfy $0<\gamma \leq\left|a_{k_{L}-k}\right|$ and $\left|a_{k_{L}-j}\right| \leq \delta_{j}(\gamma), 0 \leq j \leq k-1$ for some $0 \leq k \leq k_{L}$, where $\delta_{j}$ is some suitably small constant for each $0 \leq j \leq k-1$. Then there are exactly $k$ large roots $z_{1}, z_{2}, \ldots, z_{k}$ and the remaining roots are bounded. In particular, with ordered roots $\left|z_{1}\right| \geq\left|z_{2}\right| \geq \ldots \geq\left|z_{k_{L}}\right|$, we have that

$$
\left(\frac{\gamma}{\max _{j} \delta_{j}}\right)^{\frac{1}{k_{L}}} \lesssim\left|z_{1}\right|, \ldots,\left|z_{k}\right| \text { and }\left|z_{k+1}\right|, \ldots,\left|z_{k_{L}}\right| \lesssim 1
$$

Beyond Theorem 6.0.1, analogous to Theorem 6.0.2, we are able to obtain the following refined structural result.

Theorem 6.0.3. We fix a set of exponents $0=k_{0}<k_{1}<k_{2}<\ldots<k_{L}$ and consider certain polynomials whose exponents are drawn from this set. We consider real polynomials

$$
\Psi(t)=x+y_{1} t^{k_{1}}+y_{2} t^{k_{2}}+\ldots+y_{L} t^{k_{L}}
$$

with $y_{L} \neq 0$ and $\max _{1 \leq j \leq L}\left|y_{j}\right|=1$. Let $\gamma \in(0,1]$. We suppose, additionally, that there exists $m$ such that

$$
\begin{align*}
& \left|y_{m}\right|^{\frac{1}{k_{m}}} \geq \gamma \\
& \text { and, for } n>m,\left|y_{n}\right|^{\frac{1}{k_{n}}} \leq \delta, \tag{6.0.1}
\end{align*}
$$

where $\delta=\delta(\gamma)>0$ is some suitably small constant. We have the following refined structure for the roots of $\Psi$.

The roots of $\Psi$ may be stratified into $s=s(1)+s(2)$ tiers of roots $\mathcal{T}_{1}, \mathcal{T}_{2}, \ldots, \mathcal{T}_{s}$, where $1 \leq s \leq L$, which are ordered and separated in the sense that, if we take $w_{i} \in \mathcal{T}_{i}$, then

$$
\left|w_{1}\right| \gg\left|w_{2}\right| \gg \ldots \gg\left|w_{s}\right| .
$$

Furthermore, the tiers have the following additional structure.
We refer to each tier $\mathcal{T}_{r}$ with $1 \leq r \leq s(1)$ as a large tier. For all roots $w$ in a large tier $\mathcal{T}_{r}$, $|w| \gtrsim_{\gamma} 1$. At most $L-m+1$ large roots can cluster about a point: there is some suitable small parameter $\epsilon$ such that, for all $w \in \mathcal{T}_{r}$ with $1 \leq r \leq s(1)$, there are at most $L-m+1$ roots of $\Psi$ in $B(w, \epsilon|w|)$.

In the case that $s(2) \geq 1$, we have the following improvement. We refer to any $\mathcal{T}_{r}$ with $s(1)<r \leq s(1)+s(2)$ as a small tier. For any root $w$ in a small tier, $|w| \lesssim_{\gamma}$ 1. For any non-zero $w$ taken from a small tier, $w \in \mathcal{T}_{r}$ with $s(1)<r \leq s(1)+s(2), B(w, \epsilon|w|)$ can contain at most $m$ roots. For any root $w$ in a large tier, $B(w, \epsilon|w|)$ can contain at most $L-m$ roots.
Remark 6.0.4. Other explicit refinements of the structure theorem, which are amenable to direct calculation, are possible. These refinements can be achieved according with our tier height estimation procedure, Lemma 8.1.1.

As mentioned above, we use these structural results to obtain bounds on oscillatory integrals with polynomial phases. We consider polynomial phases

$$
\begin{array}{r}
\Phi(t)=x t+\frac{y_{1}}{k_{1}+1} t^{k_{1}+1}+\frac{y_{2}}{k_{2}+1} t^{k_{2}+1}+\ldots+\frac{y_{L}}{k_{L}+1} t^{k_{L}+1}  \tag{6.0.2}\\
\text { with } 1<k_{1}<k_{2}<\ldots<k_{L}
\end{array}
$$

with $y_{L} \neq 0$. Given our main structure result, Theorem 6.0.3, we can make use of bounds for oscillatory integrals due to Phong and Stein, Theorem 6.0.6. We are thus able to provide an alternative proof of the following oscillatory integral estimate due to Hickman and Wright, [HW20], which was considered in a slightly different formulation at Theorem 5.0.2.
Theorem 6.0.5. For oscillatory integrals

$$
I(x, y)=\int_{\mathbb{R}} e^{i \Phi(t)} d t
$$

with phases $\Phi$ given by (6.0.2), we have that

$$
|I(x, y)| \lesssim \min _{j=1,2, \ldots, L}\left|y_{j}\right|^{-\frac{1}{k_{j}+1}}
$$

provided $k_{1} \geq L$.
A cluster, $\mathcal{C}$, is a non-empty subcollection of roots of $\Phi^{\prime}$. We make use of the following result of Phong and Stein, [PS97].

Theorem 6.0.6. Suppose that $\Phi$, (6.0.2), is a real polynomial such that $y_{L} \neq 0$ and $\Phi^{\prime}$ has roots $z_{1}, z_{2}, \ldots, z_{k}$, counted with multiplicity. Then we have the oscillatory integral estimate

$$
|I(x, y)|=\left|\int_{\mathbb{R}} e^{i \Phi(t)} d t\right| \leq C_{k} \max _{j} \min _{\mathcal{C} \ni z_{j}} \frac{1}{\left(\left|y_{L}\right| \prod_{l \notin \mathcal{C}}\left|z_{j}-z_{l}\right|\right)^{\frac{1}{|C|+1}}}
$$

where the maximum is taken over roots $z_{j}$ and the minimum over clusters of roots, $\mathcal{C} \subset$ $\left\{z_{1}, z_{2}, \ldots, z_{k}\right\}$, such that $z_{j} \in \mathcal{C}$.

In fact, our refined structural statement, Theorem 6.0.3 can be applied to obtain a refined oscillatory integral estimate. Hickman and Wright established the previous estimate, Theorem 6.0.5 and, as we saw in Part II, their method of proof is sufficient to establish the below result. The following is close to being a reformulation of Theorem 5.0.5.
Theorem 6.0.7. For oscillatory integrals

$$
I(x, y)=\int_{\mathbb{R}} e^{i \Phi(t)} d t
$$

with phases $\Phi$ given by (6.0.2) and amplitude $\phi \in C_{c}^{\infty}(\mathbb{R})$, we have that

$$
\begin{equation*}
|I(x, y)| \lesssim \min _{j=1,2, \ldots, L}\left|y_{j}\right|^{-\frac{1}{k_{j}+1}} \tag{6.0.3}
\end{equation*}
$$

provided $k_{1} \geq L$. More generally, if $\max _{j}\left|y_{j}\right|^{\frac{1}{k_{j}}}=\left|y_{m}\right|^{\frac{1}{k_{m}}}$, then

$$
|I(x, y)| \lesssim \min _{j=1,2, \ldots, L}\left|y_{j}\right|^{-\frac{1}{k_{j}+1}}
$$

provided $k_{m} \geq L-m+1$.
The structural analysis of polynomial roots in this part is expected to hold with respect to polynomials over fields other than $\mathbb{R}$. This corresponds with the arguments in [KW12], which are presented for non-Archimedean fields but also hold over $\mathbb{R}$. Wright has developed a framework for the study of oscillatory integrals over fields other than $\mathbb{R}$ and proved analogues of the Phong-Stein cluster bound for such oscillatory integrals. Using bounds for oscillatory integrals over $\mathbb{C}$ from [Wri20], one could obtain a complex analogue of Theorem 6.0.7; the proof would be the same as the proof of Theorem 6.0.7, as the root structure we make use of depends only on the size of the coefficients.

Theorem 6.0.8. We consider complex oscillatory integrals

$$
I(x, y)=\int_{\mathbb{C}} e(\Phi(t)) \phi(t) d t
$$

with $\phi \in C_{c}^{\infty}(\mathbb{C})$ and $e(z)=e^{i(\Re(z)+\Im(z))}$. The phases $\Phi$ we consider are given by

$$
\Phi(z)=x z+\frac{y_{1}}{k_{1}+1} z^{k_{1}+1}+\ldots+\frac{y_{L}}{k_{L}+1} z^{k_{L}+1}
$$

with $x, y_{1}, \ldots, y_{L} \in \mathbb{C}$ and $y_{L} \neq 0$. We have that

$$
\begin{equation*}
|I(x, y)| \lesssim \min _{j=1,2, \ldots, L}\left|y_{j}\right|^{-\frac{2}{k_{j}+1}} \tag{6.0.4}
\end{equation*}
$$

provided $k_{1} \geq L$. More generally, if $\max _{j}\left|y_{j}\right|^{\frac{1}{k_{j}}}=\left|y_{m}\right|^{\frac{1}{k_{m}}}$, then

$$
|I(x, y)| \lesssim \min _{j=1,2, \ldots, L}\left|y_{j}\right|^{-\frac{2}{k_{j}+1}}
$$

provided $k_{m} \geq L-m+1$.
Remark 6.0.9. Note that we have the exponents $\frac{2}{k_{j}+1}$ appearing on the right hand side of (6.0.4), in distinction to the real case. This is a natural feature of complex oscillatory integrals, as examples in [Wri20] show.

The primary structural result, Theorem 6.0.1, is not framed for explicit computation. For example, it makes reference to the relative size of non-zero roots rather than their actual size. For applications, explicit size estimates are desirable. Indeed, we outline a procedure for estimating the size of roots in Lemma 8.1.1 and it is this procedure which allows us to strengthen the statement of Theorem 6.0.1 to the refined structural result, Theorem 6.0.3.

There is one final component of our structural investigations. Our analysis for tier stratification suggests the consideration of a factorised polynomial expression, with distinct factors corresponding to distinct root tiers. For monic polynomials, $\Psi$, we analyse such a factorisation in Section 8.2. We find that the factorisation is quantifiably close to the original polynomial $\Psi$. Furthermore, the root structure of $\Psi$ is essentially preserved by its rough factorisation.
Theorem. We consider monic polynomials

$$
x+y_{1} t^{k_{1}}+\ldots+y_{L-1} t^{k_{L-1}}+t^{k_{L}}
$$

There exists a polynomial $\widetilde{\Psi}=\prod_{l=1}^{s} \widetilde{\Psi}_{l}(t)$ which roughly factorises $\Psi$ in the following sense.
The polynomial $\widetilde{\Psi}_{l}$ has roots, $\widetilde{\mathcal{T}}_{l}$, which are all of mutually comparable magnitude. Furthermore, for any choice of roots $w_{j} \in \widetilde{\mathcal{T}}_{j}$, we have that

$$
\left|w_{1}\right| \gg\left|w_{2}\right| \gg \ldots \gg\left|w_{s}\right| .
$$

There exists a covering, $N(\mathcal{R})$, of the roots, $\mathcal{R} \subset \mathbb{C}$, of $\Psi$ which satisfies the following. Each connected component of $N(\mathcal{R})$, which we call a cell, is given by a ball. Each cell containing non-zero roots contains at most $L$ roots. For a cell $B$ containing exactly $m$ roots of $\Psi, B$ contains exactly $m$ roots of $\widetilde{\Psi}$.

For $t \notin N(\mathcal{R})$,

$$
\begin{equation*}
|\Psi(t)-\widetilde{\Psi}(t)| \ll|\Psi(t)| . \tag{6.0.5}
\end{equation*}
$$

Remark 6.0.10. The constant in (6.0.5) can be made arbitrarily small provided the tier regime, which we define in Section 7.1, is specified with sufficiently strong separation between tiers.

### 6.1 Overview

The critical observation that forms the foundation of Part III is that many of the symmetric functions of the roots of $\Psi$ are vanishing. Those that are far from vanishing determine the size of the roots. Additionally, they tell us how many roots might cluster about a point. We denote the roots of $\Psi$ by $\mathcal{R}$, they may appear with multiplicity. For at most $L$ non-zero critical indices $j \in\left\{D_{1}, D_{2}, \ldots, D_{L}\right\}$, we have that

$$
S_{j}(\mathcal{R})=\sum_{\substack{|\mathcal{S}|=j \\ \mathcal{S} \subset \mathcal{R}}} \prod_{z \in \mathcal{S}}(-z) \neq 0
$$

In particular, these indices are given by $D_{1}=k_{L}-k_{L-1}, D_{2}=k_{L}-k_{L-2}, \ldots, D_{L}=k_{L}-k_{0}=$ $k_{L}$. Throughout, we work with reference to these symmetric functions, as well as those which are vanishing.

In Section 6.2, we introduce notation. In Section 6.3, we present a model example for root structure and its relation to the polynomial coefficients (corresponding to the symmetric functions).

The core of our structural analysis is developed in Chapter 7. Indeed, Sections 7.1, 7.2, and 7.3 contain the proof of the main structure Theorem 6.0.1. This analysis pays no great heed to the polynomial coefficients, $x, y_{1}, \ldots, y_{L}$. Indeed, the structural results we obtain in Chapter 7 depend implicitly on the coefficients.

In Section 7.1, we use the vanishing of certain $S_{j}$ to provide a stratification of the roots into tiers and formulate the symmetric equations with respect to these tiers.

In Section 7.2 we characterise how clustering might occur within a given tier $\mathcal{T}$. Here, we analyse the simultaneous (near) vanishing of particular $S_{j}(\mathcal{T})$ and determine when this is inconsistent with many roots being close together. For this part of the analysis, we form a series expansion of particular $S_{j}(\mathcal{T})$ in terms of highlighted roots, which we suppose are close together. Section 7.3 contains the derivation of the distinguished root expansion that we use to prove Theorem 7.2.1, which concerns the root structure in a single tier.

In Chapter 8, we build on the tools developed in Chapter 7 to reveal more explicit root structure in specific instances. In Section 8.1, we outline an algorithm for estimating the heights of root tiers. This height estimation procedure can then be applied to polynomials satisfying the hypotheses of the refined structure Theorem 6.0.3. The result we thus obtain feeds directly into our previous work to give Theorem 6.0.3 as a corollary. Section 8.2 provides an explicit rough factorisation, $\widetilde{\Psi}=\prod_{l=1}^{s} \widetilde{\Psi}_{l}(t)$, of monic polynomials $\Psi$. We show that the roots of $\Psi$ and the roots of $\widetilde{\Psi}$ almost coincide.

Chapter 9 contains a proof of Theorem 6.0.7 and presents some more refined oscillatory integral estimates. We also give examples for the sharpness of some of the oscillatory integral estimates, these examples are also examples of the sharpness of the structure theorems.

### 6.2 Notation

We here introduce some of the notation that we will be using throughout Part III. We use $\varepsilon$ throughout, often with indexing subscripts, for error terms that appear in the analysis, where we can have suitable control on their size. Any $\varepsilon$ that appears will be the sum of (signed) products of $k$ roots, for some $k$, with size bounds adapted to an appropriate scale.

To account for repeated roots, it should be understood that we are working with multi-sets. For example, $\{1,1,1\}$ should be considered a 3 -element (multi-)set. One would more formally write this multi-set as $\left\{1^{(1)}, 1^{(2)}, 1^{(3)}\right\}$, indexing set elements by their multiplicity, so we can properly speak about distinct set elements. Another example is the fundamental theorem of algebra, which may be expressed as follows. If $\Psi$ is a degree $k_{L}$ polynomial, then the (multi-)set of roots of $\Psi$, which we denote by $\mathcal{R}$, contains $k_{L}$ elements.

We use $\mathcal{R}$ to denote the roots of $\Psi$. We also use $\mathcal{C}, \mathcal{T}$, and $\mathcal{S}$ to denote appropriate subcollections of roots. We use $\mathcal{K}$ and $\mathcal{D}$ to denote sets of integer indices, these will be specified but should be thought of as the exponents of terms in $\Psi$ and the differences between these exponents.

Throughout, we fix some ordering of the roots of $\Psi(t)=x+y_{1} t^{k_{1}}+y_{2} t^{k_{2}}+\ldots+y_{L} t^{k_{L}}$ :

$$
\begin{equation*}
\left|z_{1}\right| \geq\left|z_{2}\right| \geq \ldots \geq\left|z_{k_{L}}\right| \tag{6.2.1}
\end{equation*}
$$

Later, we will use the notation $w_{1}, w_{2}, \ldots, w_{k_{L}}$ when we wish to take an arbitrary enumeration of the roots of $\Psi$.

Definition 6.2.1. Throughout, we consider elementary symmetric functions of (a subset of) roots of $\Psi$,

$$
S_{j}(\mathcal{A})=\sum_{\mathcal{S} \subset \mathcal{A},|\mathcal{S}|=j} \prod_{z \in \mathcal{S}}(-z)
$$

where $\mathcal{A}$ is some subset of the roots of $\Psi$.

### 6.3 A model example for root structure

Working by example, we now give an indication of the possible root structure of a particular $\Psi$ and see how this relates to the coefficients. We consider $\Psi$ with

$$
\Psi(t)=y_{2} \prod_{j=1}^{2}\left(t^{k_{1}}-\alpha_{j}^{k_{1}}\right) .
$$

Note that all real polynomials $x+y_{1} t^{k_{1}}+y_{2} t^{2 k_{1}}$ can be expressed in this way. The roots of the polynomial $\Psi$ are easily recognised: they appear as the $k_{1}$ th roots of $\alpha_{1}^{k_{1}}$ and $\alpha_{2}^{k_{1}}$. There are some qualitatively different scenarios for the structure of these roots. These depend on the relative size of $\alpha_{1}$ and $\alpha_{2}$. They also depend on the cancellation between $\alpha_{1}^{k_{1}}$ and $\alpha_{2}^{k_{1}}$. Without loss of generality, suppose that $\left|\alpha_{1}\right| \geq\left|\alpha_{2}\right|$. In the case of positive $\alpha_{1}$ and $\alpha_{2}$ with $k_{1}=9$, the roots of $\Psi$ are sketched in Figure 6.3. Throughout the remainder of this section, $\epsilon$ is some suitably small fixed parameter.

Let us first consider the case where $\boldsymbol{\alpha}_{\boldsymbol{2}}=\mathbf{0}$ and $\boldsymbol{\alpha}_{\boldsymbol{1}} \neq \mathbf{0}$. Here, the are two tiers of roots. There are $k_{1}$ repeated 0 roots and if we divide out the corresponding factor $t^{k_{1}}$ from $\Psi$ we are left with the polynomial $y_{2}\left(t^{k_{1}}-\alpha_{1}^{k_{1}}\right)$, from which the location of the non-zero roots can be observed directly as the $k_{1}$ th roots of $-S_{k_{1}}(\mathcal{R})=-\frac{y_{1}}{y_{2}}=\alpha_{1}^{k_{1}}$. The non-zero roots are in $\mathcal{T}_{1}$ and the tier $\mathcal{T}_{2}$ consists of all zero roots. In this case, for roots $w \in \mathcal{T}_{1}, \boldsymbol{B}(\boldsymbol{w}, \boldsymbol{\epsilon}|\boldsymbol{w}|)$ contains only the root $\boldsymbol{w}$. Later on, being able to decouple equations for large and small roots in a similar fashion will critically allow us to analyse the structure of roots in distinct tiers.

The case that $\boldsymbol{\alpha}_{1}^{k_{1}}$ is close to $-\boldsymbol{\alpha}_{2}^{k_{1}}$, in particular, $\left|\alpha_{1}^{k_{1}}+\alpha_{2}^{k_{1}}\right| \ll\left|\alpha_{1}^{k_{1}}\right|$. The roots of $\Psi$ appear in one tier and they are close to being the $2 k_{1}$ th roots of $-\alpha_{1}^{k_{1}} \alpha_{2}^{k_{1}}$ : there are $2 k_{1}=k_{2}$ roots $w_{j}$ with $\left|w_{j}\right| \sim\left|\alpha_{1}\right|$ such roots are in tier $\mathcal{T}_{1}$. Within $\mathcal{T}_{1}$ the roots are separated in the sense that, for roots $w \in \mathcal{T}_{1}, \boldsymbol{B}(\boldsymbol{w}, \boldsymbol{\epsilon}|\boldsymbol{w}|)$ contains only the root $\boldsymbol{w}$. Note that the coefficients of $t^{0}, t^{k_{1}}$, and $t^{2 k_{1}}$ in $\Psi$ reflect this behaviour in the fact that $\left|\frac{x}{y_{2}}\right|^{\frac{1}{2 k_{1}}}=$ $\left|S_{2 k_{1}}(\mathcal{R})\right|^{\frac{1}{2 k_{1}}}=\left|\alpha_{1}^{k_{1}} \alpha_{2}^{k_{1}}\right|^{\frac{1}{2 k_{1}}} \gg\left|\alpha_{1}^{k_{1}}+\alpha_{2}^{k_{1}}\right|^{\frac{1}{k_{1}}}=\left|S_{k_{1}}(\mathcal{R})\right|^{\frac{1}{k_{1}}}=\left|\frac{y_{1}}{y_{2}}\right|^{\frac{1}{k_{1}}}$.

The case where $\left|\boldsymbol{\alpha}_{\boldsymbol{1}}\right|$ is comparable to $\left|\boldsymbol{\alpha}_{2}\right|$ but the $t^{k_{1}}$ coefficient of $\Psi,-y_{2}\left(\alpha_{1}^{k_{1}}+\alpha_{2}^{k_{1}}\right)$, does not display significant cancellation, i.e. $\left|\alpha_{1}^{k_{1}}+\alpha_{2}^{k_{1}}\right| \sim\left|\alpha_{1}^{k_{1}}\right|$. This scenario has one tier


Figure 6.3: The roots of $\Psi$ for $k_{1}=9$ and positive $\alpha_{j}$.
of roots, $\mathcal{T}_{1}=\left\{z_{1}, z_{2}, \ldots, z_{k_{2}}\right\}$. Here, all the roots are of comparable size and $\boldsymbol{B}(\boldsymbol{w}, \boldsymbol{\epsilon}|\boldsymbol{w}|)$ contains at most two roots for any root $\boldsymbol{w} \in \mathcal{T}_{\mathbf{1}}$. Two roots which might get close are those roots which share the same argument if $\alpha_{1}^{k_{1}}$ and $\alpha_{2}^{k_{1}}$ are real valued with the same sign. The coefficients of $t^{0}, t^{k_{1}}$, and $t^{2 k_{1}}$ in $\Psi$ reflect this behaviour in the fact that

$$
\left|\frac{x}{y_{2}}\right|^{\frac{1}{2 k_{1}}}=\left|S_{2 k_{1}}(\mathcal{R})\right|^{\frac{1}{2 k_{1}}}=\left|\alpha_{1}^{k_{1}} \alpha_{2}^{k_{1}}\right|^{\frac{1}{2 k_{1}}} \sim\left|\alpha_{1}^{k_{1}}+\alpha_{2}^{k_{1}}\right|^{\frac{1}{k_{1}}}=\left|\frac{y_{1}}{y_{2}}\right|^{\frac{1}{k_{1}}}=\left|S_{k_{1}}(\mathcal{R})\right|^{\frac{1}{k_{1}}} .
$$

Finally, we consider the case where $\mathbf{0} \neq\left|\boldsymbol{\alpha}_{\mathbf{2}}\right| \ll\left|\boldsymbol{\alpha}_{\mathbf{1}}\right|$. Here roots of $\Psi$ appear in two tiers: there are $k_{1}$ roots $w_{j, 2}$ with $\left|w_{j, 2}\right|=\left|\alpha_{2}\right|$-such roots are in tier $\mathcal{T}_{2}$-and the remaining $k_{1}$ roots $w_{l, 1}$ satisfy $\left|w_{l, 1}\right|=\left|\alpha_{1}\right|$-such roots are in tier $\mathcal{T}_{1}$. Within each tier the roots are separated in the sense that, $\boldsymbol{B}(\boldsymbol{w}, \boldsymbol{\epsilon}|\boldsymbol{w}|)$ contains only one root for any root $\boldsymbol{w} \in \boldsymbol{T}_{\boldsymbol{j}}$. We also have separation between tiers: given roots $w_{1}, \in \mathcal{T}_{1}$ and $w_{2} \in \mathcal{T}_{2},\left|w_{1}\right| \gg\left|w_{2}\right|$. Note that the coefficients of $t^{0}, t^{k_{1}}$, and $t^{2 k_{1}}$ in $\Psi$ reflect this behaviour in the fact that

$$
\left|S_{2 k_{1}}(\mathcal{R})\right|^{\frac{1}{2 k_{1}}}=\left|\frac{x}{y_{2}}\right|^{\frac{1}{2 k_{1}}}=\left|\alpha_{1} \alpha_{2}\right|^{\frac{1}{2}}<\left|\frac{y_{1}}{y_{2}}\right|^{\frac{1}{k_{1}}}=\left|S_{k_{1}}(\mathcal{R})\right|^{\frac{1}{k_{1}}}=\left|\alpha_{1}^{k_{1}}+\alpha_{2}^{k_{1}}\right|^{\frac{1}{k_{1}}} \sim\left|\alpha_{1}\right|
$$

and

$$
\begin{gathered}
\left|S_{k_{1}}\left(\mathcal{T}_{2}\right)\right|^{\frac{1}{k_{1}}}=\left|\alpha_{2}\right| \sim\left|\frac{\alpha_{1}^{k_{1}} \alpha_{2}^{k_{1}}}{\alpha_{1}^{k_{1}}+\alpha_{2}^{k_{1}}}\right|^{\frac{1}{k_{1}}}=\left|\frac{x}{y_{1}}\right|^{\frac{1}{k_{1}}}=\left|\frac{S_{2 k_{1}}(\mathcal{R})}{S_{k_{1}}(\mathcal{R})}\right|^{\frac{1}{k_{1}}} \\
\ll\left|\frac{y_{1}}{y_{2}}\right|^{\frac{1}{k_{1}}}=\left|\alpha_{1}^{k_{1}}+\alpha_{2}^{k_{1}}\right|^{\frac{1}{k_{1}}} \sim\left|\alpha_{1}\right|=\left|S_{k_{1}}\left(\mathcal{T}_{1}\right)\right|^{\frac{1}{k_{1}}}
\end{gathered}
$$

These last equations correspond to the separation of certain height estimates. Such height estimates will not form a part of our initial structural analysis, but we eventually consider them more explicitly in Section 8.1.

## Chapter 7

## Implicit root structure

In this chapter, we work to uncover some of the root structure of real polynomials

$$
\Psi(t)=x+y_{1} t^{k_{1}}+\ldots+y_{L} t^{k_{L}}
$$

with exponents taken from a fixed set $\left\{0, k_{1}, \ldots, k_{L}\right\}$. Throughout, we denote the set of roots of $\Psi$ by $\mathcal{R}$. Essentially, as there are at most $L+1$ non-vanishing coefficients, we will find that at most $L$ roots can coalesce about a point. Throughout this chapter, our analysis will be carried out with respect to particular reference heights, $h_{1}, h_{2}, \ldots, h_{L}$, which depend implicitly on a given polynomial $\Psi$. We postpone the discussion of more explicit tools for locating roots to Chapter 8.

### 7.1 Root tier stratification

In this section, we work to stratify the roots into tiers. The stratification of roots is suggested by the example in Section 6.3. With this in mind, we define certain reference heights $h_{j}$ for the roots. Before proceeding, let us introduce some useful indexing notation.

Definition 7.1.1. We set

$$
\begin{array}{llll}
d_{0}(\mathcal{R})=0, & d_{1}(\mathcal{R})=k_{L}-k_{L-1}, & \ldots & d_{L}(\mathcal{R})=k_{1}-k_{0}, \quad \text { and } \\
D_{0}(\mathcal{R})=d_{0}(\mathcal{R}), & D_{1}(\mathcal{R})=d_{0}(\mathcal{R})+d_{1}(\mathcal{R}), & \ldots & D_{L}(\mathcal{R})=d_{1}(\mathcal{R})+\ldots+d_{L}(\mathcal{R}) . \tag{7.1.1}
\end{array}
$$

We also set $\mathcal{D}(\mathcal{R})=\left\{0, D_{1}(\mathcal{R}), \ldots, D_{L}(\mathcal{R})\right\}$.
Note that $D_{j}(\mathcal{R})=k_{L}-k_{L-j}$, although we expressed it slightly differently in the definition to emphasise that it is the sum of consecutive $d_{i}$.

Throughout this section, we will be working with reference to all roots, $\mathcal{R}$, and so we suppress the argument of $D_{j}=D_{j}(\mathcal{R})$ and $d_{j}=d_{j}(\mathcal{R})$. Similarly, in this section, we set $\mathcal{D}=\mathcal{D}(\mathcal{R})$. We also write $d(j)=d_{j}$ and $D(j)=D_{j}$.

Definition 7.1.2. According with the root ordering, (6.2.1), we define the reference heights $h_{1}=\left|z_{1}\right|, h_{2}=\left|z_{D(1)+1}\right|, h_{3}=\left|z_{D(2)+1}\right|, \ldots, h_{L}=\left|z_{D(L-1)+1}\right|$.

The following Lemma 7.1.3 shows exactly why our Definition 7.1.2 is a sensible one.
Lemma 7.1.3. Suppose that, for some $1 \leq j \leq k_{L}, S_{j}(\mathcal{R})=0$. Then $\left|z_{j+1}\right| \sim\left|z_{j}\right|$.
As a consequence, since $S_{j}(\mathcal{R})=0$ for $j \notin \mathcal{D}=\left\{0, D_{1}, D_{2}, \ldots, D_{L}\right\}$, we have that

$$
\begin{array}{lllll}
h_{1} & =\left|z_{1}\right| & \sim\left|z_{2}\right| & \sim \ldots & \sim\left|z_{D(1)}\right|, \\
h_{2} & =\left|z_{D(1)+1}\right| & \sim\left|z_{D(1)+2}\right| & \sim \ldots & \sim\left|z_{D(2)}\right|,  \tag{7.1.2}\\
& & & \vdots \\
h_{L} & =\left|z_{D(L-1)+1}\right| & \sim\left|z_{D(L-1)+2}\right| & \sim \ldots & \sim\left|z_{D(L)}\right| .
\end{array}
$$

Proof. Suppose that $S_{j}(\mathcal{R})=0$ and consider the corresponding root $z_{j}$. There are two cases to consider. In the case that $\left|z_{j}\right|=0$, the desired comparison follows directly from the inequality $\left|z_{j+1}\right| \leq\left|z_{j}\right|=0$.

It remains to consider the case that $z_{j} \neq 0$. The largest root outwith $\left\{z_{1}, z_{2}, \ldots, z_{j}\right\}$ is $z_{j+1}$. Additionally, the largest $j-1$ roots within $\left\{z_{1}, z_{2}, \ldots, z_{j}\right\}$ are $z_{1}, z_{2}, \ldots, z_{j-1}$. Thus we find that

$$
0=\left|S_{j}\right| \geq\left|\left(-z_{1}\right)\left(-z_{2}\right) \ldots\left(-z_{j}\right)\right|-\binom{k_{L}}{j}\left|z_{1} \ldots z_{j-1} z_{j+1}\right|
$$

Rearranging and dividing through by $\left|\left(-z_{1}\right)\left(-z_{2}\right) \ldots\left(-z_{j}\right)\right|$ gives the desired inequality.
Remark 7.1.4. The constants of comparison we obtain in (7.1.2) can be chosen to depend on the indices $k_{1}, k_{2}, \ldots, k_{L}$.

In the statement of Theorem 6.0.1, we said that tiers of roots are well separated. It is thus natural, and in accordance with Lemma 7.1.3, to define the tiers of roots relative to the separation of the reference heights.

Definition 7.1.5. If we have that

$$
\begin{gathered}
h_{1} \sim h_{2} \sim \ldots \sim h_{l(1)} \\
>h_{l(1)+1} \sim \ldots \sim h_{l(1)+l(2)} \\
\vdots \\
\gg h_{l(1)+\ldots+l(s-1)+1} \sim \ldots \sim h_{l(1)+\ldots+l(s)}=h_{L},
\end{gathered}
$$

then we define the tiers as follows. First, let $l(0)=0$ and $L(i)=l(0)+l(1)+\ldots+l(i)$. Then we set

$$
\mathcal{T}_{i}=\left\{z_{D(L(i-1))+1}, z_{D(L(i-1))+2} \ldots, z_{D(L(i))}\right\} .
$$

We also define the reference height for the tiers by

$$
h\left(\mathcal{T}_{r}\right)=h_{L(r-1)+1} .
$$

This definition establishes the first part of our structure theorem, Theorem 6.0.1. Indeed, by Lemma 7.1.3, we have that, for any choice of $w_{i} \in \mathcal{T}_{i}$

$$
\left|w_{1}\right| \gg\left|w_{2}\right| \gg \ldots \gg\left|w_{s}\right| .
$$

Remark 7.1.6. The choice of separation constants defining the tier regime must be suitably strong. Having a well separated tier regime will ensure we have suitable control on error terms. In this section, up to an error term, we derive explicit equations for the symmetric functions of roots in a given tier. The error terms must be small enough to feed into our later Theorem 7.2.1. For the rough factorisation of monic $\Psi$, Theorem 8.2.2, the definition of a tier regime requires much stronger separation, as we will see in Section 8.2.

Analogous to Definition 7.1.1, it will be useful to have distinguished indices for each tier. These distinguished indices will later allow us to pick out critical symmetric functions of roots within each tier.

Definition 7.1.7. In the tier $\mathcal{T}_{r}$, we set

$$
\begin{align*}
& d_{0}\left(\mathcal{T}_{r}\right)=0, d_{1}\left(\mathcal{T}_{r}\right)=d_{L(r-1)+1}, d_{2}\left(\mathcal{T}_{r}\right)=d_{L(r-1)+2}, \ldots d_{l(r)}\left(\mathcal{T}_{r}\right)=d_{L(r)}, \text { and }  \tag{7.1.3}\\
& D_{0}\left(\mathcal{T}_{r}\right)=0, D_{1}\left(\mathcal{T}_{r}\right)=d_{0}\left(\mathcal{T}_{r}\right)+d_{1}\left(\mathcal{T}_{r}\right), \ldots D_{l(r)}\left(\mathcal{T}_{r}\right)=d_{1}\left(\mathcal{T}_{r}\right)+\ldots+d_{l(r)}\left(\mathcal{T}_{r}\right)
\end{align*}
$$

There is an appropriate $L\left(\mathcal{T}_{r}\right)=l(r)=L(r)-L(r-1)$. Note that $D_{l(r)}\left(\mathcal{T}_{r}\right)=\left|\mathcal{T}_{r}\right|=D(L(r))-$ $D(L(r-1))$, and we also denote this by $D\left(\mathcal{T}_{r}\right)$.

If $\mathcal{T}=\mathcal{T}_{r}$, then the distinguished indices for the symmetric functions in $\mathcal{T}$ are given by $\mathcal{D}(\mathcal{T})=\left\{0, D_{1}(\mathcal{T}), \ldots, D_{L(\mathcal{T})}(\mathcal{T})\right\}$.

We now present the main result of this section, this lemma characterises the critical symmetric functions with respect to roots within each tier.
Lemma 7.1.8. According with Definition 7.1.5, fix a $\operatorname{tier} \mathcal{T}=\mathcal{T}_{r}$.
For the positive critical indices $j \in \mathcal{D}(\mathcal{T})$,

$$
S_{j}(\mathcal{T})=c_{j}(\mathcal{T})+\varepsilon_{j}(\mathcal{T})
$$

where $\left|\varepsilon_{j}(\mathcal{T})\right| \ll h(\mathcal{T})^{j}$ and

$$
c_{j}(\mathcal{T})=\frac{S_{D(L(r-1))+j}(\mathcal{R})}{S_{D(L(r-1))}\left(\mathcal{T}_{1} \cup \mathcal{T}_{1} \cup \ldots \mathcal{T}_{r-1}\right)}
$$

For $j \notin \mathcal{D}(\mathcal{T})$, with $1 \leq j \leq|\mathcal{T}|$, we have that

$$
S_{j}(\mathcal{T})=\varepsilon_{j}(\mathcal{T})
$$

where $\left|\varepsilon_{j}(\mathcal{T})\right| \ll h(\mathcal{T})^{j}$.
Proof. We consider the symmetric functions of order $D(L(r-1))+j$ for $1 \leq j \leq D(\mathcal{T})$. For ease of notation, we set $D=D(L(r-1))$ and we denote the reference height $h(\mathcal{T})=h_{L(r-1)+1}$ by $h$.

Splitting the sum defining symmetric functions according to the size of the summands, we have that

$$
\begin{equation*}
S_{D+j}(\mathcal{R})=S_{D}\left(\mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \ldots \cup \mathcal{T}_{r-1}\right) S_{j}(\mathcal{T})+\sum_{\left|\mathcal{S}^{\prime}\right|=D+j} \prod_{z \in \mathcal{S}^{\prime}}(-z) \tag{7.1.4}
\end{equation*}
$$

where the sum in $\mathcal{S}^{\prime} \subset \mathcal{R}$ is taken over $\mathcal{S}^{\prime} \cap\left(\mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \ldots \mathcal{T}_{r-1} \cup \mathcal{T}_{r}\right)^{c} \neq \emptyset$. Any $\mathcal{S}^{\prime}$ appearing in this proof should be understood as subject to these restrictions. Note that

$$
\left|S_{D}\left(\mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \ldots \cup \mathcal{T}_{r-1}\right)\right|=\left|z_{1} z_{2} \ldots z_{D}\right|
$$

Every product appearing in

$$
\sum_{\left|\mathcal{S}^{\prime}\right|=D+j} \prod_{z \in \mathcal{S}^{\prime}}(-z)
$$

contains at least one root, $z$, outwith $\mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \ldots \cup \mathcal{T}_{r}$ and, for any such $z,|z| \ll h$. The largest $D+j-1$ roots are $z_{1}, z_{2}, \ldots, z_{D+j-1}$. Thus we see that

$$
\left|\sum_{\left|\mathcal{S}^{\prime}\right|=D+j} \prod_{z \in \mathcal{S}^{\prime}}(-z)\right| \ll\left|z_{1} z_{2} \ldots z_{D+j-1} h\right| .
$$

Normalising and rearranging (7.1.4) we thus see that

$$
S_{j}(\mathcal{T})=S_{D}\left(\mathcal{T}_{1} \cup \mathcal{T}_{1} \cup \ldots \cup \mathcal{T}_{r-1}\right)^{-1} S_{D+j}(\mathcal{R})+\varepsilon_{j}(\mathcal{T})
$$

where

$$
\varepsilon_{j}(\mathcal{T})=-S_{D}\left(\mathcal{T}_{1} \cup \mathcal{T}_{1} \cup \ldots \cup \mathcal{T}_{r-1}\right)^{-1} \sum_{\left|\mathcal{S}^{\prime}\right|=D+j} \prod_{z \in \mathcal{S}^{\prime}}(-z)
$$

To conclude, we observe that

$$
\left|\varepsilon_{j}(\mathcal{T})\right| \ll\left|z_{D+1} z_{D+2} \ldots z_{D+j-1} h\right| \sim h^{j}
$$

as required.

### 7.2 Root structure within tiers

Recall how we defined the tiers of roots in Definition 7.1.5, which immediately gives the separation between tiers and part of Theorem 6.0.1. In the regime with tiers $\mathcal{T}_{1}, \mathcal{T}_{2}, \ldots, \mathcal{T}_{s}$, we have
that

$$
h\left(\mathcal{T}_{1}\right) \gg h\left(\mathcal{T}_{2}\right) \gg \ldots \gg h\left(\mathcal{T}_{s}\right)
$$

where $h\left(\mathcal{T}_{r}\right)=\max _{w \in \mathcal{T}_{r}}|w|$. To complete the proof of Theorem 6.0.1, we must establish the structure of roots within a tier. In this section, we prove Theorem 7.2.1, which, combined with Lemma 7.1.8, gives the structure Theorem 6.0.1.

If there are any zero roots, these will all fall in the smallest tier and they necessarily coincide. What remains is to consider tiers containing non-zero roots. To this end, let us fix $\mathcal{T}=\mathcal{T}_{r}$ for some $r$ such that $h(\mathcal{T})=h_{L(r-1)+1}>0$. For ease of notation, throughout this section, we denote by $h$ the reference height for roots in $\mathcal{T}_{r}, h=h\left(\mathcal{T}_{r}\right)=h_{L(r-1)+1}$. With care, we can leverage the statement of Lemma 7.1 .8 to determine how roots in $\mathcal{T}$ may cluster about a point.

Recall from Lemma 7.1.8 that, for $j \notin \mathcal{D}(\mathcal{T}), S_{j}(\mathcal{T})$ is near vanishing. The following theorem tells us what kind kind of clustering can occur according to how many symmetric functions of $\mathcal{T}$ are far from vanishing. In particular, we consider what happens when the symmetric functions, $S_{j}(\mathcal{T})$, are near vanishing away from some (unspecified) set of exceptional indices $\widetilde{\mathcal{D}}(\mathcal{T}) \subset \mathcal{D}(\mathcal{T})$.

Theorem 7.2.1. Suppose that, for $0 \leq j \leq D(\mathcal{T})$,

$$
S_{j}(\mathcal{T})=\varepsilon_{j}(\mathcal{T}), \text { for } j \notin \widetilde{\mathcal{D}}(\mathcal{T})
$$

where $\left|\varepsilon_{j}(\mathcal{T})\right| \ll h^{j}$ and $\widetilde{\mathcal{D}}(\mathcal{T}) \subset \mathcal{D}(\mathcal{T})$. Then at most $\tilde{L}(\mathcal{T}):=|\widetilde{\mathcal{D}}(\mathcal{T})|-1 \leq|\mathcal{D}(\mathcal{T})|-1=L(\mathcal{T})$ roots in $\mathcal{T}$ can cluster about a point. More precisely, there is is some suitably small $\epsilon>0$ such that, for any root $w \in \mathcal{T}$, we have $|w| \sim h>0$ and $B(w, \epsilon h)$ contains at most $\tilde{L}(\mathcal{T})$ roots $w_{i} \in \mathcal{T}$.

We have as a corollary of Theorem 7.2.1 and Lemma 7.1.8 the following structure theorem, of which Theorem 6.0.1 is a special case.

Theorem 7.2.2. We fix a set of exponents $0=k_{0}<k_{1}<k_{2}<\ldots<k_{L}$ and consider real polynomials whose exponents are drawn from this set. For any real polynomial $\Psi(t)=$ $x+y_{1} t^{k_{1}}+y_{2} t^{k_{2}}+\ldots+y_{L} t^{k_{L}}$ with $y_{L} \neq 0$, we have the following.

We suppose that the polynomial coefficients $(x, y)$ are such that we are in the tier regime indexed by $\left(l_{1}, l_{2}, \ldots, l_{s}\right)$ in Definition 7.1.5.

The roots of $\Psi$ are stratified into $s$ tiers of roots $\mathcal{T}_{1}, \mathcal{T}_{2}, \ldots, \mathcal{T}_{s}$, where $1 \leq s \leq L$. The tiers are separated in the sense that, if we take $w_{i} \in \mathcal{T}_{i}$, then

$$
\left|w_{1}\right| \gg\left|w_{2}\right| \gg \ldots \gg\left|w_{s}\right| .
$$

At most $L\left(\mathcal{T}_{r}\right)=l_{r} \leq L$ non-zero roots can cluster about a root: there is some suitable small parameter $\epsilon$ such that, for all $0 \neq w \in \mathcal{T}_{r}$, there are at most $L\left(\mathcal{T}_{r}\right)=l_{r}$ roots of $\Psi$ in $B(w, \epsilon|w|)$.

We have expressed Theorem 7.2.1 in a slightly more general form than is required to prove Theorem 6.0.1. As a black box, Theorem 7.2.1 can give improvements to our main structure Theorem 6.0.1. Indeed, if it is applied with reference to our later Proposition 8.1.2, we can obtain 6.0.3. It is for this reason we have framed the theorem in terms of critical indices $\widetilde{\mathcal{D}}$, rather than $\mathcal{D}$, since some of the symmetric functions indexed by $\mathcal{D}$ might still be close to vanishing. Nevertheless, all of the tildes appearing in this section can safely be ignored on first reading under the assumption that there is only one tier as all of the essential ideas are contained in this case.

In this section, roots in $\mathcal{T}$ are not ordered in terms of size: $w_{1}, w_{2}, \ldots, w_{D(\mathcal{T})}$ is some enumeration of the roots in $\mathcal{T}$. Our analysis works by highlighting some of these roots, $\kappa \subset \mathcal{T}$, which we will assume are close together. We expand the symmetric functions in terms of these highlighted roots. We will recover some structural statements about the roots from the highlighted expansions. We denote the excluded roots by $e=\mathcal{T} \backslash 反$.

First, we state the following lemma, which is verified at a glance.
Lemma 7.2.3. Let $\{\subset \mathcal{T}$ with $|\hbar|=m$. Then

$$
S_{D(\mathcal{T})-m}(\mathcal{T})=\sum_{l=0}^{\min \{m, D(\mathcal{T})-m\}} S_{l}(\hbar) S_{D(\mathcal{T})-m-l}(e)
$$

Since we will be investigating highlighted roots that are close together, we distinguish one of these roots to further refine the expansion. By a suitable recursive procedure, which is outlined in Section 7.3, we can apply Lemma 7.2.3 to obtain the following.

Lemma 7.2.4. Set

$$
a_{m}(j)=(-1)^{j-1}\binom{m-1+j}{m-1}
$$

Suppose we have $m$ highlighted roots $\boldsymbol{\kappa}=\left\{w_{1}, w_{2}, \ldots, w_{m}\right\} \subset \mathcal{T}$. Then,

$$
\begin{aligned}
& S_{D(\mathcal{T})-m}(\mathcal{T})=\left(-w_{m+1}\right)\left(-w_{m+2}\right) \ldots\left(-w_{D(\mathcal{T})}\right) \\
& +\sum_{j=1}^{D(\mathcal{T})-m} a_{m}(j)\left(-w_{1}\right)^{j} S_{D(\mathcal{T})-m-j}(\mathcal{T})+\varepsilon_{D(\mathcal{T})-m}
\end{aligned}
$$

where

$$
\left|\varepsilon_{D(\mathcal{T})-m}\right| \lesssim \max _{w, w^{\prime} \in \hbar}\left|w-w^{\prime}\right| h^{D(\mathcal{T})-m-1}
$$

if $m \geq 2$ and $\varepsilon_{D(\mathcal{T})-m}=0$ if $m=1$.
Remark 7.2.5. It may appear that there is an error in the statement of Lemma 7.2.4, due to the apparent double counting of certain expressions. However, these are accounted for in the error term. The reason we desire such a series expansion is because those expressions $S_{D(\mathcal{T})-m-j}(\mathcal{T})$ are much more explicit than, for example, $S_{D(\mathcal{T})-m-j}(e)$. Indeed, we can relate $S_{D(\mathcal{T})-m-j}(\mathcal{T})$ to the coefficients of our original polynomial via Lemma 7.1.8 and we have no such tools for $S_{D(\mathcal{T})-m-j}(e)$ or $S_{j}(\kappa)$.

Recall Definition 7.1.7, the definition of distinguished indices for a tier, $\mathcal{D}(\mathcal{T})$. It will also be necessary to count backwards from $D(\mathcal{T})$ to 0 and pick out corresponding critical symmetric functions of roots in the tier. We make the following further definition of distinguished exponents for a given tier $\mathcal{T}=\mathcal{T}_{r}$.

Definition 7.2.6. We set $k_{0}(\mathcal{T})=0$,

$$
\begin{equation*}
k_{1}(\mathcal{T})=d_{L(r)}, k_{2}(\mathcal{T})=d_{L(r)}+d_{L(r)-1}, \ldots k_{L(\mathcal{T})}(\mathcal{T})=d_{L(r)}+\ldots+d_{L(r-1)+1} \tag{7.2.1}
\end{equation*}
$$

Note that $k_{L(\mathcal{T})}(\mathcal{T})=D(\mathcal{T})$. We denote by $\mathcal{K}(\mathcal{T})$ these exponents. Note that

$$
\mathcal{K}(\mathcal{T})=D(\mathcal{T})-\mathcal{D}(\mathcal{T})=\left\{D(\mathcal{T}), D(\mathcal{T})-D_{1}(\mathcal{T}), \ldots, 0\right\}
$$

According with Theorem 7.2.1, if we have $\tilde{L}(\mathcal{T})+1$ distinguished indices $0=\tilde{D}_{0}(\mathcal{T})<$ $\tilde{D}_{1}(\mathcal{T})<\ldots<\tilde{D}_{\tilde{L}(\mathcal{T})}(\mathcal{T})=D(\mathcal{T})$, given as $\widetilde{\mathcal{D}}(\mathcal{T}) \subset \mathcal{D}(\mathcal{T})$, then, corresponding with the above, we set $\widetilde{\mathcal{K}}(\mathcal{T})=D(\mathcal{T})-\widetilde{\mathcal{D}}(\mathcal{T})$. Naturally, we enumerate $\widetilde{\mathcal{K}}(\mathcal{T})$ by $0=\tilde{k}_{0}<\tilde{k}_{1}<\ldots<\tilde{k}_{\tilde{L}(\mathcal{T})}$.

With this notation and our series expansion tools, we are now ready to carry out the analysis of how roots can cluster within $\mathcal{T}$.
Proof of Theorem 7.2.1. In this proof, we use the notation $D=D(\mathcal{T})=|\mathcal{T}|, \tilde{L}=\tilde{L}(\mathcal{T})=$ $|\widetilde{\mathcal{D}}(\mathcal{T})|-1, \tilde{D}(j)=\tilde{D}_{j}(\mathcal{T}), \tilde{k}_{j}=\tilde{k}_{j}(\mathcal{T})$.

We suppose, by way of contradiction, that, given some small $\epsilon>0$, there exist roots $w_{1}, w_{2}, \ldots, w_{\tilde{L}+1} \in B\left(w_{1}, \epsilon h\right) \cap \mathcal{T}$. Working with the symmetric functions $S_{D-1}(\mathcal{T}), S_{D-2}(\mathcal{T})$, $\ldots, S_{D-(\tilde{L}+1)}(\mathcal{T})$ we will derive a system of equations which has no solution.

We can apply Lemma 7.2 .4 to obtain the following. For highlighted roots $\mathcal{K} \subset\left\{w_{1}, w_{2}, \ldots, w_{\tilde{L}+1}\right\} \subset$ $\mathcal{T}$ containing $m$ elements, we have

$$
S_{D-m}(\mathcal{T})=\left(-w_{m+1}\right)\left(-w_{m+2}\right) \ldots\left(-w_{D}\right)+\sum_{j=1}^{D-m} a_{m}(j)\left(-w_{1}\right)^{j} S_{D-m-j}(\mathcal{T})+\varepsilon_{D-m}(\kappa)
$$

where $\left|\varepsilon_{D-m}(\kappa)\right| \ll h^{D-m}$. For $m+1 \leq l \leq \tilde{L}+1$, replacing instances of $\left(-w_{l}\right)$ with $\left(-w_{1}\right)+$ $\left(\left(-w_{l}\right)-\left(-w_{1}\right)\right)$ and observing that $\left|w_{l}-w_{1}\right| \leq \epsilon h$, we find that

$$
\begin{gathered}
S_{D-m}(\mathcal{T}) \\
=\left(-w_{1}\right)^{\tilde{L}+1-m}\left(-w_{\tilde{L}+2}\right)\left(-w_{\tilde{L}+3}\right) \ldots\left(-w_{D}\right)+\sum_{j=1}^{D-m} a_{m}(j)\left(-w_{1}\right)^{j} S_{D-m-j}(\mathcal{T})+\varepsilon_{D-m}^{(1)},
\end{gathered}
$$

where $\left|\varepsilon_{D-m}^{(1)}\right| \lesssim \epsilon h^{D-m}$.
Dividing through by $\left(-w_{1}\right)^{\tilde{L}+1-m}$, we obtain

$$
=\left(-w_{\tilde{L}+2}\right) \ldots\left(-w_{D}\right)+\sum_{j=1}^{D-m} a_{m}(j)\left(-w_{1}\right)^{j+m-(\tilde{L}+1)} S_{D-m}(\mathcal{T}),
$$

where $\left|\varepsilon_{D-m}^{(2)}\right| \lesssim \epsilon h^{D-(\tilde{L}+1)}$.
Many of the terms appearing in the above sum are near vanishing. To pick out the significant terms, we observe $D-m-j \in \widetilde{\mathcal{D}}(\mathcal{T})$ requires that $j=D(\mathcal{T})-\tilde{D}_{i}(\mathcal{T})-m$ for $0 \leq i \leq \tilde{L}$. The relevant set of indices $j$ is precisely the positive elements of $\widetilde{\mathcal{K}}(\mathcal{T})-m$. Thus, we find

$$
\begin{align*}
& \left(-w_{1}\right)^{m-(\tilde{L}+1)} S_{D-m}(\mathcal{T}) \\
= & \left(-w_{\tilde{L}+2}\right) \ldots\left(-w_{D}\right)+\sum_{\substack{\tilde{\tilde{K}}(\mathcal{T})-m \\
j \geq 1}}^{D-m} a_{m}(j)\left(-w_{1}\right)^{j+m-(\tilde{L}+1)} S_{D-m-j}(\mathcal{T})  \tag{7.2.2}\\
+ & \varepsilon_{D-m}^{(3)},
\end{align*}
$$

where $\varepsilon_{D-m}^{(3)}$ is an error term with $\left|\varepsilon_{D-m}^{(3)}\right| \lesssim \epsilon h^{D-(\tilde{L}+1)}$. The normalised symmetric functions appearing in the sum are precisely $\left(-w_{1}\right)^{\tilde{k}_{i}-(\tilde{L}+1)} S_{\tilde{D}(\tilde{L}-i)}(\mathcal{T})$ where $\tilde{D}(\tilde{L}-i) \in \widetilde{\mathcal{D}}$ and $\tilde{D}(\tilde{L}-i)<$ $D-m$. Observe that, from Lemma 7.2.4, $a_{m}(0)=(-1)$ so that, if $D-m \in \widetilde{\mathcal{D}}$, then (7.2.2) can be rearranged to

$$
\begin{align*}
& \varepsilon_{D-m}^{(4)} \\
& \qquad=\left(-w_{\tilde{L}+2}\right) \ldots\left(-w_{D}\right)+\sum_{\substack{j \in \tilde{\mathcal{K}}(\mathcal{T})-m \\
j \geq 0}}^{D-m} a_{m}(j)\left(-w_{1}\right)^{j+m-(\tilde{L}+1)} S_{D-m-j}(\mathcal{T}), \tag{7.2.3}
\end{align*}
$$

where $\varepsilon_{D-m}^{(4)}$ is an error term with $\left|\varepsilon_{D-m}^{(4)}\right| \ll h^{D-(\tilde{L}+1)}$. In fact, (7.2.3) is valid for all $m$ since, if $D-m \notin \widetilde{\mathcal{K}}$, then $\left|\left(-w_{1}\right)^{m-(\tilde{L}+1)} S_{D-m}(\mathcal{T})\right| \ll h^{D-(\tilde{L}+1)}$.

We set

$$
\vec{v}=\left(\begin{array}{c}
\left(-w_{\tilde{L}+2}\right)\left(-w_{\tilde{\tilde{L}}+3}\right) \ldots\left(-w_{D}\right) \\
\left(-w_{1}\right)^{\tilde{k}_{1}-(\tilde{L}+1)} S_{D(\tilde{L}-1)} \\
\vdots \\
\left(-w_{1}\right)^{\tilde{k}_{\tilde{L}}-(\tilde{L}+1)} S_{\tilde{D}(\tilde{L}-\tilde{L})}
\end{array}\right), \quad \vec{\varepsilon}=\left(\begin{array}{c}
\varepsilon_{D-1}^{(4)} \\
\vdots \\
\varepsilon_{D-(\tilde{L}+1)}^{(4)}
\end{array}\right) .
$$

Bringing the equations (7.2.3) for $1 \leq m \leq \tilde{L}+1$ together, we have a matrix equation

$$
\begin{equation*}
\vec{\varepsilon}=M \vec{v}, \tag{7.2.4}
\end{equation*}
$$

with $M$ as specified below. Recall, from Lemma 7.2.4, that $a_{m}(j)=(-1)^{j-1}\binom{m-1+j}{m-1}$ so that $a_{m}(j-m)=(-1)^{j-m-1}\binom{j-1}{m-1}$.

Let us first give an example of $M$. When $\tilde{L}=2$, and $\tilde{k}_{1}>3$ we have

$$
M=\left(\begin{array}{ccc}
1 & \left(-1 \tilde{k}_{1}-2\right. & (-1)^{\tilde{k}_{2}-2} \\
1 & (-1)^{\tilde{k}_{1}-3}\left(\tilde{k}_{1}-1\right) & (-1)^{\tilde{k}_{2}-3}\left(\tilde{k}_{2}-1\right) \\
1 & (-1)^{\tilde{k}_{1}-4} \frac{1}{2}\left(\tilde{k}_{1}-2\right)\left(\tilde{k}_{1}-1\right) & (-1)^{\tilde{k}_{2}-4} \frac{1}{2}\left(\tilde{k}_{2}-2\right)\left(\tilde{k}_{2}-1\right)
\end{array}\right) .
$$

Let us give an example of $M$ when it contains some zero entries. This happens, for example, if $\tilde{L}=2$ and $\tilde{k}_{2}>3$ but $\tilde{k}_{1}=2$. In this case,

$$
M=\left(\begin{array}{ccc}
1 & (-1)^{\tilde{k}_{1}-2} & (-1)^{\tilde{k}_{2}-2} \\
1 & -1 & (-1)^{\tilde{k}_{2}-3}\left(\tilde{k}_{2}-1\right) \\
1 & 0 & (-1)^{\tilde{k}_{2}-4} \frac{1}{2}\left(\tilde{k}_{2}-2\right)\left(\tilde{k}_{2}-1\right)
\end{array}\right)
$$

In general, $M$ is an $(\tilde{L}+1) \times(\tilde{L}+1)$ matrix. According with (7.2.3), we must be sensitive to whether $\tilde{\mathcal{K}}-m$ contains negative elements distinct from $-m$ (as these do not appear in the sum). To account for these, it is useful to express

$$
\begin{equation*}
a_{m}\left(\tilde{k}_{b}-m\right)=\binom{\tilde{k}_{b}-1}{m-1}=\frac{1}{(m-1)!} \prod_{i=1}^{m-1}\left(\tilde{k}_{b}-i\right) \tag{7.2.5}
\end{equation*}
$$

For $b \geq 1$, the expression on the right hand side of (7.2.5) is equal to 0 when $\tilde{k}_{b}<m$. If $b \geq 1$ and $\tilde{k}_{b}-(\tilde{L}+1)<0$, then, for $m>\tilde{k}_{b}, \frac{1}{(m-1)!} \prod_{i=1}^{m-1}\left(\tilde{k}_{b}-i\right)=0$. Observe also that $1=\frac{(-1)^{-m-1}}{(m-1)!} \prod_{i=1}^{m-1}(0-i)$. Thus, for $1 \leq m \leq \tilde{L}+1$ and $1 \leq b \leq \tilde{L}+1$ the matrix entry $M(m, b)$ is given by

$$
\begin{equation*}
M(m, b)=\frac{(-1)^{\tilde{k}_{b-1}-m-1}}{(m-1)!} \prod_{i=1}^{m-1}\left(\tilde{k}_{b-1}-i\right) . \tag{7.2.6}
\end{equation*}
$$

We claim that $M$ is invertible. Let us suppose, for now, that the claim holds and see how the result follows. Using the fact that $M$ is invertible, we find from (7.2.4) that

$$
|\vec{\varepsilon}| \sim h^{D-(\tilde{L}+1)}
$$

Throughout, we have tracked error terms so that $|\vec{\varepsilon}| \ll h^{D-(\tilde{L}+1)}$. As such, we have a contradiction. Therefore, $B\left(w_{1}, \epsilon\left|w_{1}\right|\right)$ can contain at most $\tilde{L}$ roots.

Let us now prove the claim. First, lets multiply the $j$ th column of $M$ by $(-1)^{\tilde{k}_{j}}$ and call the resulting matrix $\widetilde{M}$. We work to show the column vectors of $\widetilde{M}$ are linearly independent via a Taylor expansion. Observe that the columns of $\widetilde{M}$ are evaluations of the vector polynomial

$$
\vec{p}(t)=\left(\begin{array}{c}
1 \\
-(t-1) \\
\frac{1}{2!}(t-1)(t-2) \\
\vdots \\
(-1)^{\tilde{L}} \frac{1}{\tilde{L}!}(t-1)(t-2) \ldots(t-\tilde{L})
\end{array}\right)
$$

at the points $t=0, \tilde{k}_{1}, \ldots \tilde{k}_{\tilde{L}}$. Considering the Maclaurin expansion, we can express $p(t)$ as a matrix transformation of the curve,

$$
\vec{q}(t)=\left(\begin{array}{c}
1 \\
t \\
\frac{1}{2!} t^{2} \\
\vdots \\
\frac{1}{\tilde{L}!} t^{\tilde{L}}
\end{array}\right) .
$$

Note that, since the $j$ th component of $\vec{p}$ is a degree $j-1$ polynomial, we can write

$$
\vec{p}(t)=T \vec{q}(t)
$$

where $T$ is an upper triangular matrix. It is easy to see that, along the diagonal, the entries are non-zero, so that $T$ is invertible. It is well known that $\tilde{L}$ distinct points on the moment curve

$$
\vec{r}(t)=\left(\begin{array}{c}
t \\
\frac{1}{2!} t^{2} \\
\vdots \\
\frac{1}{(\tilde{L})!} t^{\tilde{L}}
\end{array}\right)
$$

are in general position. We thus see that

$$
\begin{aligned}
& \operatorname{dim} \operatorname{span}\left\{\vec{q}(0), \vec{q}\left(\tilde{k}_{1}\right), \ldots, \vec{q}\left(\tilde{k}_{\tilde{L}}\right)\right\} \\
& =1+\operatorname{dim} \operatorname{span}\left\{\vec{r}\left(\tilde{k}_{1}\right), \ldots, \vec{r}\left(\tilde{k}_{\tilde{L}}\right)\right\} \\
& =\tilde{L}+1
\end{aligned}
$$

We also know, since $T$ is invertible, that

$$
\begin{gathered}
\operatorname{dim\operatorname {span}\{ T\vec {q}(0),T\vec {q}(\tilde {k}_{1}),\ldots ,T\vec {q}(\tilde {k}_{\tilde {L}})\} } \begin{aligned}
=\operatorname{dim} \operatorname{span}\left\{\vec{q}(0), \vec{q}\left(\tilde{k}_{1}\right), \ldots, \vec{q}\left(\tilde{k}_{\tilde{L}}\right)\right\} \\
=\tilde{L}+1
\end{aligned}
\end{gathered}
$$

Therefore, the column vectors of $\widetilde{M}$ are linearly independent, completing the proof that $\widetilde{M}$, and thus $M$, is invertible.

### 7.3 A series expansion lemma; proof of Lemma 7.2.4

In this section, we work to prove our main series expansion Lemma 7.2.4. The work is of a rather combinatorial flavour. Lemma 7.2.4 is a corollary of Lemmas 7.3.1 and 7.3.2.
Lemma 7.3.1. For $m$ highlighted roots, $\kappa$, including $w_{1}$ and $D-m$ excluded roots $e=\mathcal{T} \backslash \kappa$, the symmetric function $S_{D-m}(\mathcal{T})$ can be expressed as

$$
S_{D-m}(\mathcal{T})=S_{D-m}(e)+\sum_{j=1}^{D-m} a_{m}(j, m)\left(-w_{1}\right)^{j} S_{D-m-j}(\mathcal{T})+\varepsilon_{D-m}
$$

where $\left|\varepsilon_{D-m}\right| \lesssim \max _{w, w^{\prime} \in h}\left|w-w^{\prime}\right| h^{D-m}$, if $D \geq m \geq 2$, and $\varepsilon_{D-m}=0$, if $m=1$. The coefficients $a_{m}(j, m)$ are outlined in the following. We first set $b=\max \{1, m-(D-m)+1\}$. We have that $a_{m}(\cdot, \cdot)$ is the solution of the below recurrence relation:

$$
\begin{array}{r}
a_{m}(1, m)=\binom{m}{m-1}, a_{m}(1, m-1)=\binom{m}{m-2}, \ldots, a_{m}(1, b)=\binom{m}{b-1} \\
a_{m}(1, l)=0, \text { for } l \leq b-1 \text { or } l \geq m+1 \\
a_{m}(j+1, l)=a_{m}(j, l-1), \text { for } l \leq b-1 \text { or } l \geq m+1  \tag{7.3.1}\\
a_{m}(j+1, l)=-\binom{m}{l-1} a_{m}(j, m)+a_{m}(j, l-1), \text { for } b \leq l \leq m
\end{array}
$$

In the case that $m \leq D-m$, i.e. $b=1$, we include a figure representing a step of the recursion for those $1 \leq l \leq m$. Figure 7.1 represents the dynamics taking us one step from
$\left\{a_{m}(j, m), \ldots, a_{m}(j, 1)\right\}$ to $\left\{a_{m}(j+1, m), \ldots, a_{m}(j+1,1)\right\}$, where each arrow represents addition. We suppress the first argument of $a_{m}$.


Figure 7.1: A step of the recursion (7.3.1).

Lemma 7.3.2. Let $a_{m}(\cdot, \cdot)$ satisfy the recurrence relation (7.3.1). For $b \leq l \leq m$ and $1 \leq j \leq$ $D-m$,

$$
\begin{equation*}
a_{m}(j, l)=(-1)^{j-1}\binom{m-1+j}{l-1}\binom{m-1+j-l}{j-1} \tag{7.3.2}
\end{equation*}
$$

In particular,

$$
a_{m}(j, m)=(-1)^{j-1}\binom{m-1+j}{m-1}
$$

for $1 \leq j \leq D-m$.
Now we turn to the derivation of our desired series expansion in terms of the above specified recurrence relation.

Proof of Lemma 7.3.1. Take $m$ highlighted roots $\boldsymbol{\hbar}=\left\{w_{1}, w_{2}, \ldots, w_{m}\right\}$. We work from Lemma 7.2.3, which gives

$$
S_{D-m}(\mathcal{T})-S_{D-m}(e)=\sum_{l=1}^{\min \{D-m, m\}} S_{l}(\kappa) S_{D-m-l}(e) .
$$

We perform an iterative procedure to obtain the recurrence relation (7.3.1). It is obtained by expressing each of the highest order $S_{j}(e)$ in terms of lower order $S_{l}$ and each $S_{j-l}(\hbar)$ in terms of a distinguished root $w_{1}$. One step of this procedure corresponds to first replacing the instance of $S_{j}(e)$ with the largest index $j$ with $S_{j}(\mathcal{T})-\sum S_{i}(\kappa) S_{j-i}(e)$ and then replacing instances of $S_{i}(\hbar)$ with $\binom{m}{i}\left(-w_{1}\right)^{j-l}+\varepsilon_{i}(\hbar)$, where $\left|\varepsilon_{i}(\hbar)\right| \lesssim \max _{w, w^{\prime} \in h}\left|w-w^{\prime}\right| h^{i-1}$.

Let us first consider the simpler case of $m=1$. We see that

$$
S_{D-1}(\mathcal{T})-S_{D-1}(e)=\left(-w_{1}\right) S_{D-2}(e)=a_{1}^{\prime}(1,1)\left(-w_{1}\right) S_{D-2}(e)
$$

This gives our initialisation of the recurrence relation, with $a_{1}^{\prime}(1,1)=1$ with $a_{1}^{\prime}(1, l)=0$ for $l \neq 1$. To begin the recurrence, we substitute $S_{D-2}(e)=S_{D-2}(\mathcal{T})-\left(-w_{1}\right) S_{D-3}(e)$, which gives

$$
\begin{aligned}
& S_{D-1}(\mathcal{T})-S_{D-1}(e)=\left(-w_{1}\right) S_{D-2}(\mathcal{T})-\left(-w_{1}\right)^{2} S_{D-3}(e) \\
& \quad=a_{1}^{\prime}(2,2)\left(-w_{1}\right) S_{D-2}(\mathcal{T})+a_{1}^{\prime}(2,1)\left(-w_{1}\right)^{2} S_{D-3}(e),
\end{aligned}
$$

so that $a_{1}^{\prime}(2,2)=a_{1}^{\prime}(1,1)=1$ and $a_{1}^{\prime}(2,1)=-1=-a_{1}^{\prime}(1,1)+a_{1}^{\prime}(1,0)$. This continues until we have eliminated all appearances of the $S_{l}(\hbar)$ and $S_{l}(e)$ for $l>1$, the recurrence relation is seen to be $a_{1}^{\prime}(j+1,1)=-a_{1}^{\prime}(j, 1)=-a_{1}^{\prime}(j, 1)+a_{1}^{\prime}(j, 0)$ for $j \leq D-1$ and $a^{\prime}(j+1, l)=a^{\prime}(j, l-1)$ for $l \neq 1$.

In the case where $m \geq 2$, it is harder to work with exact expressions, so we will introduce appropriate error terms. We distinguish the root $w_{1} \in \mathcal{h}$. Note that, wherever $S_{l}(\mathcal{h})$ appears, we can write $S_{l}(\kappa)=\binom{m}{l}\left(-w_{1}\right)^{l}+\varepsilon_{l}(\kappa)$, where $\left|\varepsilon_{l}(\kappa)\right| \lesssim \max _{w, w^{\prime} \in h}\left|w-w^{\prime}\right| h^{l-1}$, if $m \geq 2$. This is seen by replacing any instance of $w_{j}$ for $2 \leq j \leq m$ with $w_{j}=w_{1}+\left(w_{j}-w_{1}\right)$ : the term resulting from the difference on the right hand side of this equation is cast into to the error term $\varepsilon_{l}(\hbar)$.

For our initialisation, we first modify all terms featuring highlighted roots by introducing an appropriate error and find that

$$
\begin{gather*}
S_{D-m}(\mathcal{T})-S_{D-m}(e)=\sum_{l=1}^{\min \{m, D-m\}} S_{l}(f) S_{D-m-l}(e) \\
=\sum_{l=1}^{\min \{m, D-m\}}\binom{m}{l}\left(-w_{1}\right)^{l} S_{D-m-l}(e)+\left(\sum_{l=1}^{\min \{m, D-m\}} \varepsilon_{l}(\kappa) S_{D-m-l}(e)\right) \\
=\sum_{l \geq 1} a_{m}^{\prime}(1, m+1-l)\left(-w_{1}\right)^{l} S_{D-m-l}(e)+\varepsilon_{D-m}^{(0)}, \tag{7.3.3}
\end{gather*}
$$

where $a_{m}^{\prime}(1, l)=\binom{m}{l}$ for $1 \leq l \leq \min \{m, D-m\}$ and $a_{m}^{\prime}(1, l)=0$ for $l>\min \{m, D-m\}$ or $l \leq 0$. Here $\varepsilon_{D-m}^{(0)}:=\sum_{l=1}^{\min \{m, D-m\}} \varepsilon_{l}(\kappa) S_{D-m-l}(e)$ is easily verified to satisfy the required error bound $\left|\varepsilon_{D-m}^{(0)}\right| \lesssim \max _{w, w^{\prime} \in \hbar}\left|w-w^{\prime}\right| h^{D-m-1}$. Note that, although $S_{j}(e)$ appearing in the sum (7.3.3) is not defined for $j>D-m$ or $j<0$, the corresponding terms should not be considered as part of the sum since the corresponding coefficients are 0 . We write the initialisation as an infinite series expansion in this way because it makes the expression of our recursion more convenient.

As the first step of the recursion, we will use the equation

$$
\begin{align*}
& S_{D-m-1}(e)=S_{D-m-1}(\mathcal{T})-\sum_{j=1}^{\min \{m, D-m-1\}} S_{j}(f) S_{D-m-1-j}(e) \\
= & S_{D-m-1}(\mathcal{T})-\sum_{j \in \mathbb{Z}} \mathbb{1}_{I_{1}}(j)\binom{m}{j}\left(-w_{1}\right)^{j} S_{D-m-1-j}(e)+\varepsilon_{D-m-1}^{(0,1)}(f), \tag{7.3.4}
\end{align*}
$$

where $I_{1}=[1, \min \{m, D-m-1\}]$ and $\varepsilon_{D-m-1}^{(0,1)}:=\sum_{j \in \mathbb{Z}} \mathbb{1}_{I_{1}}(j) \varepsilon_{j}(\kappa) S_{D-m-1-j}(e)$ satisfies $\left|\varepsilon_{D-m-1}^{(0,1)}(\kappa)\right| \lesssim \max _{\left\{w, w^{\prime} \in K\right\}}\left|w-w^{\prime}\right| h^{D-m-2}$.

To carry out the first step of our recursive procedure, we substitute (7.3.4) into (7.3.3). We end up with

$$
\begin{gathered}
S_{D-m}(\mathcal{T})-S_{D-m}(e) \\
=a_{m}^{\prime}(1, m)\left(-w_{1}\right) S_{D-m-1}(\mathcal{T})+\sum_{j \in \mathbb{Z}} \mathbb{1}_{I_{1}}(j)\binom{m}{j}\left(-a_{m}^{\prime}(1, m)\right)\left(-w_{1}\right)^{j+1} S_{D-m-1-j}(e) \\
+a_{m}^{\prime}(1, m)\left(-w_{1}\right) \varepsilon_{D-m-1}^{(0,1)}(f) \\
+\sum_{l \geq 2} a_{m}^{\prime}(1, m+1-l)\left(-w_{1}\right)^{l} S_{D-m-l}(e)+\varepsilon_{D-m}^{(0)}(\kappa) \\
=a_{m}^{\prime}(1, m)\left(-w_{1}\right) S_{D-m-1}(\mathcal{T})+\sum_{j \in \mathbb{Z}} \mathbb{1}_{I_{1}}(j)\binom{m}{j}\left(-a_{m}^{\prime}(1, m)\right)\left(-w_{1}\right)^{j+1} S_{D-m-1-j}(e) \\
+\sum_{j \geq 1} a_{m}^{\prime}(1, m-j)\left(-w_{1}\right)^{j+1} S_{D-m-1-j}(e)+\varepsilon_{D-m}^{(1)}(\kappa)
\end{gathered}
$$

$$
\begin{gathered}
=\sum_{l \leq 0} a_{m}^{\prime}(2, m+1-l)\left(-w_{1}\right)^{l+1} S_{D-m-1-l}(\mathcal{T}) \\
+\sum_{l \geq 1} a_{m}^{\prime}(2, m+1-l)\left(-w_{1}\right)^{l+1} S_{D-m-1-l}(e)+\varepsilon_{D-m}^{(1)}(\kappa)
\end{gathered}
$$

where $\left|\varepsilon_{D-m}^{(1)}(\kappa)\right| \ll h^{D-m}$ and

$$
\begin{equation*}
a_{m}^{\prime}(2, m+1-l)=-\mathbb{1}_{I_{1}}(l)\binom{m}{l} a_{m}^{\prime}(1, m)+a_{m}^{\prime}(1, m-l) \tag{7.3.5}
\end{equation*}
$$

The procedure continues. The recurrence relation we get is

$$
\begin{equation*}
a_{m}^{\prime}(j+1, m+1-l)=-\mathbb{1}_{I_{j}}(l)\binom{m}{l} a_{m}^{\prime}(j, m)+a_{m}^{\prime}(j, m-l) \tag{7.3.6}
\end{equation*}
$$

where

$$
\begin{equation*}
I_{j}:=[1, \min \{m, D-m-j\}], \tag{7.3.7}
\end{equation*}
$$

where, if $\min \{m, D-m-j\}<1$, (7.3.7) should be understood as a void interval, i.e. $I_{j}=\emptyset$. The indicator function $\mathbb{1}_{I_{j}}$ appearing in (7.3.6) simply ensures that we do not pick up any symmetric functions below $S_{0}(\mathcal{T})$ in our series expansion.

Looking at (7.3.6) and (7.3.7), we can see that, for $j \geq D-m, a_{m}^{\prime}(j+1, l)=a_{m}^{\prime}(j, l)$ for all $l$. This reflects the fact that we can only carry out the iteration procedure, where we replace instances of $S_{D-m-j}(e)$ with $S_{D-m-j}(\mathcal{T})$, an error, and terms involving $S_{D-m-j^{\prime}}(e)$ for $j^{\prime}>j$, $D-m-1$ times. The derived series expansion for $S_{D-m}(\mathcal{T})-S_{D-m}(e)$ is

$$
\begin{gathered}
S_{D-m}(\mathcal{T})-S_{D-m}(e) \\
=\sum_{l \leq 0} a_{m}^{\prime}(2, m+1-l)\left(-w_{1}\right)^{l+1} S_{D-m-1-l}(\mathcal{T})+\sum_{l \geq 1} a_{m}^{\prime}(2, m+1-l)\left(-w_{1}\right)^{l+1} S_{D-m-1-l}(e) \\
+\varepsilon_{D-m}^{(1)}(\mathfrak{f}) \\
=\sum_{l \leq 0} a_{m}^{\prime}(3, m+1-l)\left(-w_{1}\right)^{l+2} S_{D-m-2-l}(\mathcal{T})+\sum_{l \geq 1} a_{m}^{\prime}(3, m+1-l)\left(-w_{1}\right)^{l+2} S_{D-m-2-l}(e) \\
+\varepsilon_{D-m}^{(2)}(\mathfrak{f}) \\
=\ldots \\
=\sum_{l \leq 0} a_{m}^{\prime}\left(j^{\prime}+1, m+1-l\right)\left(-w_{1}\right)^{l+j^{\prime}} S_{D-m-j^{\prime}-l}(\mathcal{T})+\sum_{l \geq 1} a_{m}^{\prime}\left(j^{\prime}+1, m+1-l\right)\left(-w_{1}\right)^{l+j^{\prime}} S_{D-m-j^{\prime}-l}(e) \\
+\varepsilon_{D-m}^{\left(j^{\prime}\right)}(\kappa) .
\end{gathered}
$$

Taking $j^{\prime}=D-m-1$, we find that

$$
\begin{gathered}
S_{D-m}(\mathcal{T})-S_{D-m}(e)=\sum_{l \leq 0} a_{m}^{\prime}(D-m, m+1-l)\left(-w_{1}\right)^{l+D-m-1} S_{1-l}(\mathcal{T}) \\
+\sum_{l \geq 1} a_{m}^{\prime}(D-m, m+1-l)\left(-w_{1}\right)^{l+D-m-1} S_{1-l}(e)+\varepsilon_{D-m}^{(D-m-1)}(\kappa) \\
=\sum_{l \leq 0} a_{m}^{\prime}(D-m, m+1-l)\left(-w_{1}\right)^{l+D-m-1} S_{1-l}(\mathcal{T})+a_{m}^{\prime}(D-m, m)\left(-w_{1}\right)^{D-m} S_{0}(e)+\varepsilon_{D-m}^{(D-m-1)}(\kappa) \\
=\sum_{m-D+2 \leq l \leq 1} a_{m}^{\prime}(D-m+l-1, m)\left(-w_{1}\right)^{l+D-m-1} S_{1-l}(\mathcal{T})+\varepsilon_{D-m}^{(D-m-1)}(\kappa) .
\end{gathered}
$$

Now observe that, for $l \leq 0, a_{m}^{\prime}(j+1, m+1-l)=a_{m}^{\prime}(j, m-l)$, by definition (7.3.6). Therefore,
for $l \leq 0, a_{m}^{\prime}\left(j^{\prime}, m+1-l\right)=a_{m}^{\prime}\left(j^{\prime}+l-1, m\right)$. In particular, we have

$$
\begin{gathered}
S_{D-m}(\mathcal{T})-S_{D-m}(e) \\
=\sum_{1 \leq j \leq D-m} a_{m}^{\prime}(j, m)\left(-w_{1}\right)^{j} S_{D-m-j}(\mathcal{T})+\varepsilon_{D-m}^{(D-m-1)}(\kappa) .
\end{gathered}
$$

One can easily see that the recurrence relation (7.3.6) differs from the recurrence relation (7.3.1). However, the terms that appear in the series expansion are $a_{m}^{\prime}(j, m)$ for $1 \leq j \leq D-m$ and it can be verified that these coincide with the same $a_{m}(j, m)$.
Proof of Lemma 7.3.2. We want to show that, for $b \leq l \leq m$ and $1 \leq j \leq D-m$,

$$
\begin{equation*}
a_{m}(j, l)=(-1)^{j-1}\binom{m-1+j}{l-1}\binom{m-1+j-l}{j-1} \tag{7.3.8}
\end{equation*}
$$

where $a_{m}$ satisfies the recurrence relation (7.3.1). This is true for $j=1$ by definition. We now work by induction. We must show that the right hand side of (7.3.8) satisfies the recurrence relation (7.3.1). We apply the relevant forward expression to see that, for $b \leq l \leq m$ and $j \geq 1$,

$$
\begin{gathered}
-\binom{m}{l-1}(-1)^{j-1}\binom{m-1+j}{m-1}+(-1)^{j-1}\binom{m-1+j}{l-2}\binom{m+j-l}{j-1} \\
=(-1)^{j}\left(\frac{m!}{(l-1)!(m-l+1)!} \cdot \frac{(m-1+j)!}{j!(m-1)!}-\frac{(m+j-1)!}{(m+j-l+1)!(l-2)!} \cdot \frac{(m+j-l)!}{(j-1)!(m-l+1)!}\right) \\
=(-1)^{j}\left(\frac{(m-1+j)!}{(l-1)!(m+j-l+1)!}\right)\left(\frac{m(m+j-l+1)!}{(m-l+1)!j!}-\frac{(m+j-l)!(l-1)}{(j-1)!(m-l+1)!}\right) \\
=(-1)^{j}\left(\frac{(m-1+j)!}{(l-1)!(m+j-l+1)!}\right)\left(\frac{(m+j-l)!}{j!(m-l+1)!}\right)(m(m+j-l+1)-(l-1) j) \\
=(-1)^{j}\left(\frac{(m-1+j)!}{(l-1)!(m+j-l+1)!}\right)\left(\frac{(m+j-l)!}{j!(m-l+1)!}\right)((m+j)(m+1-l)) \\
=(-1)^{j}\binom{m+j}{l-1}\binom{m+j-l}{j},
\end{gathered}
$$

as required.

## Chapter 8

## Explicit root structure

In Section 7.1, we introduced certain reference heights. These reference heights provided an essential scaffold for our root structure analysis. Nevertheless, the tools developed so far tell us nothing explicit about even the size of the reference heights. The reference heights were given in terms of the size of certain roots and thus depended implicitly on the coefficients of our polynomial $\Psi$. In this section, we establish an explicit toolkit for estimating reference heights, from which we are then able to obtain refined structural statements.

We also consider the rough factorisation of monic $\Psi$ into polynomials corresponding with each tier. We provide an explicit factorised polynomial $\widetilde{\Psi}(t)=\prod_{r=1}^{s} \widetilde{\Psi}_{r}(t)$. Our analysis results in a further tool for root finding, in that a suitable small neighbourhood of the roots of the polynomial factor $\widetilde{\Psi}_{l}$ will contain the roots of $\Psi$ in the tier $\mathcal{T}_{l}$.

### 8.1 Root tier stratification

We have defined the tier structure in terms of reference heights about which we have no a priori information. The following lemma outlines a useful procedure for estimating the reference heights in an unknown regime.

In this section, we prove our explicit tool for estimating the heights of roots, Lemma 8.1.1. We also prove Theorem 6.0.3. This is a direct corollary of our later Proposition 8.1.2 considered with reference to Theorem 7.2.1.

Lemma 8.1.1. We fix a set of exponents $k_{1}<k_{2}<\ldots<k_{L}$ and consider polynomials

$$
\Psi(t)=x+y_{1} t^{k_{1}}+\ldots+y_{L} t^{k_{L}}
$$

with $y_{L} \neq 0$. The roots of $\Psi$ are structured into tiers, $\mathcal{T}_{1}, \mathcal{T}_{2}, \ldots \mathcal{T}_{s}$, according with Theorem 6.0.1. There exists an algorithm outputting a sequence of height estimates $\eta_{1} \geq \eta_{2} \geq \ldots \geq \eta_{a}$, where $1 \leq s \leq a \leq L$, which satisfy the following.

Associated with $\eta_{1}, \eta_{2}, \ldots \eta_{a}$ are indices $0=\alpha(0), \alpha(1), \alpha(2), \ldots, \alpha(a)$ such that

$$
\eta_{j}^{D(\alpha(j))-D(\alpha(j-1))}=\left|\frac{y_{L-\alpha(j)}}{y_{L-\alpha(j-1)}}\right| .
$$

For each $0 \leq b<a$, we proceed by setting

$$
\eta_{b+1}=\max _{\alpha(b)<j \leq L}\left|\frac{y_{L-j}}{y_{L-\alpha(b)}}\right|^{\frac{1}{D(j)-D(\alpha(b))}}=\left|\frac{y_{L-\alpha(b+1)}}{y_{L-\alpha(b)}}\right|^{\frac{1}{D(\alpha(b+1))-D(\alpha(b))}} .
$$

There exist $0=\beta(0), \beta(1), \beta(2), \ldots \beta(s)=a$ such that

$$
\eta_{j} \sim h\left(\mathcal{T}_{r}\right), \text { for } \beta(r-1)<j \leq \beta(r)
$$

Furthermore, $\alpha(\beta(r))=L(r)$, with the $l_{j}$ and $L(j)$ defined as in Definition 7.1.5.

We can reformulate the previous comparison as follows: for $\alpha(j)$ with $L(r-1)+1 \leq \alpha(j) \leq$ $L(r)$,

$$
\begin{equation*}
\eta_{j} \sim h\left(\mathcal{T}_{r}\right) \tag{8.1.1}
\end{equation*}
$$

Proof. Suppose we are in the regime indexed by $\left(l_{1}, l_{2}, \ldots, l_{s}\right)$, as in Definition 7.1.5. We can easily verify that

$$
\begin{equation*}
\left|S_{D(L(i))}(\mathcal{R})\right| \sim h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} \ldots h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)} . \tag{8.1.2}
\end{equation*}
$$

Indeed, $\left(-z_{1}\right)\left(-z_{2}\right) \ldots\left(-z_{D(L(i))}\right)$ is the term of largest magnitude appearing in the sum $S_{D(L(i))}(\mathcal{R})$ and it is comparable in magnitude to $h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} \ldots h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)}$. Since they must include a root from a smaller tier, all the remaining terms in the $\operatorname{sum} S_{D(L(i))}(\mathcal{R})$ are bounded in magnitude by

$$
h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} \ldots h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)-1} h\left(\mathcal{T}_{i+1}\right) \ll h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} \ldots h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)}
$$

Similarly, note that, for $L(t-1)+1 \leq j \leq L(t)$,

$$
\begin{equation*}
\left|S_{D(j)}(\mathcal{R})\right| \lesssim h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} h\left(\mathcal{T}_{2}\right)^{D\left(\mathcal{T}_{2}\right)} \ldots h\left(\mathcal{T}_{t-1}\right)^{D\left(\mathcal{T}_{t-1}\right)} h\left(\mathcal{T}_{t}\right)^{D(j)-D(L(t-1))} \tag{8.1.3}
\end{equation*}
$$

Set

$$
\eta_{1}=\max _{1 \leq j \leq L}\left|S_{D(j)}(\mathcal{R})\right|^{\frac{1}{D(j)}}=\left|S_{D(\alpha(1))}(\mathcal{R})\right|^{\frac{1}{D(\alpha(1))}}=\left|\frac{y_{L-\alpha(1)}}{y_{L}}\right|^{\frac{1}{D(\alpha(1))}} .
$$

Since all roots are bounded in magnitude by $h\left(\mathcal{T}_{1}\right)$ and, from (8.1.2), $S_{D(L(1))}(\mathcal{R}) \sim h\left(\mathcal{T}_{1}\right)^{D(L(1))}$, we then see that $\eta_{1} \sim h\left(\mathcal{T}_{1}\right)$.

Next, we set

$$
\begin{gathered}
\eta_{2}=\max _{\alpha(1)<j \leq L}\left|\frac{S_{D(j)}(\mathcal{R})}{S_{D(\alpha(1))}(\mathcal{R})}\right|^{\frac{1}{D(j)-D(\alpha(1))}} \\
=\left|\frac{S_{D(\alpha(2))}(\mathcal{R})}{S_{D(\alpha(1))}(\mathcal{R})}\right|^{\frac{1}{D(\alpha(2))-D(\alpha(1))}}=\left|\frac{y_{L-\alpha(2)}}{y_{L-\alpha(1)}}\right|^{\frac{1}{D(\alpha(2))-D(\alpha(1))}} .
\end{gathered}
$$

Having determined $\eta_{1}, \eta_{2}, \ldots, \eta_{i}$ and corresponding $\alpha(1), \alpha(2), \ldots, \alpha(i)<L$, we set

$$
\begin{gather*}
\eta_{i+1}=\max _{\alpha(i)<j \leq L}\left|\frac{S_{D(j)}(\mathcal{R})}{S_{D(\alpha(i))}(\mathcal{R})}\right|^{\frac{1}{D(j)-D(\alpha(i))}}  \tag{8.1.4}\\
=\left|\frac{S_{D(\alpha(i+1))}(\mathcal{R})}{S_{D(\alpha(i))}(\mathcal{R})}\right|^{\frac{1}{D(\alpha(i+1))-D(\alpha(i))}}=\left|\frac{y_{L-\alpha(i+1)}}{y_{L-\alpha(i)}}\right|^{\frac{1}{D(\alpha(i+1))-D(\alpha(i))}} .
\end{gather*}
$$

We wish to ensure that at least one height estimate corresponds to every reference height. This is ensured provided $h_{l(r)} \gg h_{l(r)+1}$ for each $1 \leq r \leq s$ with suitable constants in Definition 7.1.5. In the procedure defined above, we show that each $y_{L-L(t)}$ appears in one of the expressions for $\eta_{j}$, (8.1.4). Furthermore, for terms picked up in the procedure, the bound (8.1.3) can be upgraded to a comparison. We show this inductively.

First, take the largest $0 \leq \alpha\left(\beta_{1}-1\right)$ such that $\alpha\left(\beta_{1}-1\right)<L(1)$. Observe that

$$
\left|S_{D(L(1))}(\mathcal{R})\right|^{\frac{1}{D(L(1))}}=\left|\frac{y_{L-L(1)}}{y_{L}}\right|^{\frac{1}{D(L(1))}} \sim h\left(\mathcal{T}_{1}\right)
$$

by (8.1.2). Thus we can see, by definition of our height estimates, that, if $\alpha\left(\beta_{1}-1\right)>0$, then $\left|S_{D\left(\alpha\left(\beta_{1}-1\right)\right)}\right|^{\frac{1}{D\left(\alpha\left(\beta_{1}-1\right)\right)}} \gtrsim h\left(\mathcal{T}_{1}\right)$. Considering (8.1.3), we then see that either $\alpha\left(\beta_{1}-1\right)=0$ or

$$
\begin{equation*}
\left|S_{D\left(\alpha\left(\beta_{1}-1\right)\right)}\right|^{\frac{1}{D\left(\alpha\left(\beta_{1}-1\right)\right)}} \sim h\left(\mathcal{T}_{1}\right) . \tag{8.1.5}
\end{equation*}
$$

$\underset{\sim}{W}$ ne now wish to show that $\alpha\left(\beta_{1}\right)=L(1)$. Suppose for contradiction that $\alpha\left(\beta_{1}\right)>L(1)$. Set $\tilde{h}=h\left(\mathcal{T}_{2}\right)$ and $h=h\left(\mathcal{T}_{1}\right)$. We know that $\tilde{h} \ll h$. Using the equations (8.1.3) and (8.1.5) to
bound the size of the symmetric functions in terms of height estimates, we see that

$$
\begin{gathered}
\eta_{\beta_{1}}=\left\lvert\, \frac{\left.y_{L-\alpha\left(\beta_{1}\right)}^{y_{L-\alpha\left(\beta_{1}-1\right)}}\right|^{\frac{1}{D\left(\alpha\left(\beta_{1}\right)\right)-D\left(\alpha\left(\beta_{1}-1\right)\right)}}}{=\left|\frac{S_{D\left(\alpha\left(\beta_{1}\right)\right)}(\mathcal{R})}{S_{D\left(\alpha\left(\beta_{1}-1\right)\right)}(\mathcal{R})}\right|^{\frac{1}{D\left(\alpha\left(\beta_{1}\right)\right)-D\left(\alpha\left(\beta_{1}-1\right)\right)}}} \begin{array}{c}
\lesssim\left|\frac{\tilde{h}^{D\left(\alpha\left(\beta_{1}\right)\right)-D(L(1))} h^{D(L(1))}}{h^{D\left(\alpha\left(\beta_{1}-1\right)\right)}}\right|^{\frac{1}{D\left(\alpha\left(\beta_{1}\right)\right)-D\left(\alpha\left(\beta_{1}-1\right)\right)}} \\
\leq \left\lvert\, \frac{\tilde{h}^{D\left(\alpha\left(\beta_{1}\right)\right)-D(L(1))}}{\left.h^{D\left(\alpha\left(\beta_{1}\right)\right)-D(L(1))} h^{\left.D\left(\alpha\left(\beta_{1}\right)\right)\right)-D\left(\alpha\left(\beta_{1}-1\right)\right)}\right|^{\frac{1}{D\left(\alpha\left(\beta_{1}\right)\right)-D\left(\alpha\left(\beta_{1}-1\right)\right)}}}\right. \\
<h=\left|h^{D(L(1))-D\left(\alpha\left(\beta_{1}-1\right)\right)}\right|^{\frac{1}{D(L(1))-D\left(\alpha\left(\beta_{1}-1\right)\right)}} \\
\sim\left|\frac{S_{D(L(1))}(\mathcal{R})}{S_{D\left(\alpha\left(\beta_{1}-1\right)\right)}(\mathcal{R})}\right|^{\frac{1}{D(L(1))-D\left(\alpha\left(\beta_{1}-1\right)\right)}} \\
=\left|\frac{y_{L-L(1)}}{y_{L-\alpha\left(\beta_{1}-1\right)}}\right|^{D(L(1))-D\left(\alpha\left(\beta_{1}-1\right)\right)}
\end{array}\right.
\end{gathered}
$$

This contradicts the definition of $\eta_{\beta_{1}}$, so we must have that $\alpha\left(\beta_{1}\right)=L(1)$. Furthermore, for $0=\beta(0)<i \leq \beta(1)=\beta_{1}$,

$$
\eta_{i} \sim h\left(\mathcal{T}_{1}\right) .
$$

We proceed inductively. Fix some index $r$. Suppose that $\beta(r-1)$ is such that $\alpha(\beta(\tilde{r}))=L(\tilde{r})$ for $1 \leq \tilde{r} \leq r-1$ and, for $1 \leq \tilde{r}<r$ and $\beta(\tilde{r}-1)<i \leq \beta(\tilde{r})$,

$$
\begin{equation*}
\eta_{i} \sim h\left(\mathcal{T}_{\tilde{r}}\right) . \tag{8.1.6}
\end{equation*}
$$

We then choose $\beta_{r}$ maximally so that $\alpha(\beta(r-1))=L(r-1) \leq \alpha\left(\beta_{r}-1\right)<L(r)$. Similarly to our proof of (8.1.5), it is routine to verify that

$$
\left|\frac{y_{L(r)}}{y_{L(r-1)}}\right|^{\frac{1}{\overline{D(L(r))-D(\alpha(\beta(r-1)))}}}=\left|\frac{S_{D(L(r))}(\mathcal{R})}{S_{D(\alpha(L(r-1))}(\mathcal{R})}\right|^{\frac{1}{D(L(r))-D(\alpha(\beta(r-1)))}} \sim h\left(\mathcal{T}_{r}\right),
$$

we easily see that, for $\beta(r-1)<i \leq \beta(r)$,

$$
\begin{equation*}
\eta_{i} \sim h\left(\mathcal{T}_{r}\right) . \tag{8.1.7}
\end{equation*}
$$

As a consequence of (8.1.6) and (8.1.7), we find that, for $\beta(r-1)<i \leq \beta(r)$,

$$
\begin{gather*}
\left|S_{D(\alpha(i))}(\mathcal{R})\right|=\left|\prod_{l=1}^{i} \eta_{i}^{D(\alpha(l))-D(\alpha(l-1))}\right| \\
\sim h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} h\left(\mathcal{T}_{2}\right)^{D\left(\mathcal{T}_{2}\right)} \ldots h\left(\mathcal{T}_{r-1}\right)^{D\left(\mathcal{T}_{r-1}\right)} h\left(\mathcal{T}_{r}\right)^{D(\alpha(i))-D(L(r-1))} . \tag{8.1.8}
\end{gather*}
$$

As previously, we now wish to show that $\alpha\left(\beta_{r}\right)=L(r)$. Suppose, then, to find a contradiction, that $\alpha\left(\beta_{r}\right)>L(r)$. Similarly to (8.1.8), we find, setting $h=h\left(\mathcal{T}_{r}\right)$ and $\tilde{h}=h\left(\mathcal{T}_{r+1}\right)$, that

$$
\begin{gathered}
\eta_{\beta_{r}}=\left|\frac{\left.y_{L-\alpha\left(\beta_{r}\right)}^{y_{L-\alpha\left(\beta_{r}-1\right)}}\right|^{\frac{1}{D\left(\alpha\left(\beta_{r}\right)\right)-D\left(\alpha\left(\beta_{r}-1\right)\right)}}=\left|\frac{S_{D\left(\alpha\left(\beta_{r}\right)\right)}(\mathcal{R})}{S_{D\left(\alpha\left(\beta_{r}-1\right)\right)}(\mathcal{R})}\right|^{\frac{1}{D\left(\alpha\left(\beta_{r}\right)\right)-D\left(\alpha\left(\beta_{r}-1\right)\right)}}}{\lesssim} \begin{array}{|l}
h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} h\left(\mathcal{T}_{2}\right)^{D\left(\mathcal{T}_{2}\right)} \ldots h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{r}\right)} h\left(\mathcal{T}_{r+1}\right)^{D\left(\alpha\left(\beta_{r}\right)\right)-D(L(r))} \\
h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} h\left(\mathcal{T}_{2}\right)^{D\left(\mathcal{T}_{2}\right)} \ldots h\left(\mathcal{T}_{r-1}\right)^{D\left(\mathcal{T}_{r-1}\right)} h\left(\mathcal{T}_{r}\right)^{D\left(\alpha\left(\beta_{r}-1\right)\right)-D(L(r-1))}
\end{array}\right|^{\frac{1}{D\left(\alpha\left(\beta_{r}\right)\right)-D\left(\alpha\left(\beta_{r}-1\right)\right)}}
\end{gathered}
$$

$$
\begin{aligned}
& \sim\left|\frac{h^{D(L(r))-D(L(r-1))} \tilde{h}^{D\left(\alpha\left(\beta_{r}\right)\right)-D(L(r))}}{h^{D\left(\alpha\left(\beta_{r}-1\right)\right)-D(L(r-1))}}\right|^{\frac{1}{D\left(\alpha\left(\beta_{r}\right)\right)-D\left(\alpha\left(\beta_{r}-1\right)\right)}} \\
& =\left|h^{D\left(\alpha\left(\beta_{r}\right)\right)-D\left(\alpha\left(\beta_{r}-1\right)\right)} \frac{\tilde{h}^{D\left(\alpha\left(\beta_{r}\right)\right)-D(L(r))}}{h^{D\left(\alpha\left(\beta_{r}\right)\right)-D(L(r))}}\right|^{\frac{1}{D\left(\alpha\left(\beta_{r}\right)\right)-D\left(\alpha\left(\beta_{r}-1\right)\right)}} \\
& \quad \ll h \sim\left|\frac{y_{L(r)}}{y_{L(r-1)}}\right|^{\frac{1}{D(L(r))-D(\alpha(\beta(r-1)))}},
\end{aligned}
$$

which contradicts the definition of $\eta_{\beta_{r}}$.

We can use the procedure from Lemma 8.1.1 to obtain the refined structural result, Theorem 6.0.3. Here, with additional restrictions on the coefficients, we have stronger control on the root structure as a consequence of the explicit estimates for the reference heights. The following proposition feeds directly into our result on the structure of roots within a given tier, Theorem 7.2.1, to give the refined structural result, Theorem 6.0.3, as a corollary.

Proposition 8.1.2. Fix the set of exponents $k_{1}<k_{2}<\ldots<k_{L}$. We consider polynomials

$$
\Psi(t)=x+y_{1} t^{k_{1}}+\ldots+y_{L} t^{k_{L}}
$$

such that

$$
\max _{1 \leq j \leq L}\left|y_{j}\right|^{\frac{1}{k_{j}}} \leq 1
$$

Take some $\gamma \in(0,1]$ and suppose, additionally, that

$$
\left|y_{m}\right|^{\frac{1}{k_{m}}} \geq \gamma
$$

and, for $n>m$,

$$
\left|y_{n}\right|^{\frac{1}{k_{n}}} \leq \delta,
$$

for some $m$ and some suitably small $\delta=\delta(\gamma)>0$. Then the roots of $\Psi$ can be classified into large and small tiers $\mathcal{T}_{1}, \ldots \mathcal{T}_{s(1)}$ and $\mathcal{T}_{s(1)+1}, \ldots \mathcal{T}_{s(1)+s(2)}$ which satisfy the following.

First, we have that $0 \leq s(1) \leq s(1)+s(2)=s \leq L$ and, additionally $s(1) \leq L-m+1$. If $s(2) \geq 1$, then $s(1) \leq L-m$.

We refer to those tiers $\mathcal{T}_{r}$ with $1 \leq r \leq s(1)$ as the large tiers. For any root $w$ in a large tier, we have that $|w| \gtrsim_{\gamma} 1$. In the case that $s(2) \geq 1$, we refer to those tiers $\mathcal{T}_{r}$ with $s(1)+1 \leq r \leq s(1)+s(2)$ as the small tiers. For any root $w$ in a small tier, we have that $|w| \lesssim_{\gamma} 1$.

The tiers are well separated: for any choice of $w_{j} \in \mathcal{T}_{j}$, we have that

$$
\left|w_{1}\right| \ll\left|w_{2}\right| \ll \ldots \ll\left|w_{s}\right| .
$$

Finally, we have the following. In the case that $s(2) \geq 1$, we have that $L(s(1))=L-m$. If $s(2)=0$, then, $l_{s(1)} \geq m$ and, for $L-m<L(s-1)+j<L$,

$$
\left|S_{D(L(s-1)+j)}(\mathcal{R})\right| \ll h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} h\left(\mathcal{T}_{2}\right)^{D\left(\mathcal{T}_{2}\right)} \ldots h\left(\mathcal{T}_{s}\right)^{D(L(s-1)+j)-D(L(s-1))}
$$

Before proceeding with the proof of Proposition 8.1.2, let us show how Theorem 6.0.3 is obtained as a corollary.

Proof of Theorem 6.0.3. Proposition 8.1.2 already gives the large and small tiers, and the required control on their size. It remains to determine the way in which roots may cluster. To do so, we work with reference to the final paragraph of the proposition's statement.

Let us first consider the case that $s(2) \geq 1$. Here we can see, since $L(s(1))=L-m$, that for $1 \leq r \leq s(1), l_{r} \leq L(s(1)) \leq L-m$. Therefore $\left|\mathcal{D}\left(\mathcal{T}_{r}\right)\right|-1=l_{r} \leq L-m$ and, by Theorem 7.2.1, $B(w, \epsilon|w|)$ contains at most $L-m$ roots for any root $w$ in a large tier $\mathcal{T}_{r}$. Likewise, for
$s(1)+1 \leq r \leq s(1)+s(2), l_{r} \leq L-L(s(1))=m$ so that, for any root $w$ in a small tier $\mathcal{T}_{r}$, $B(w, \epsilon|w|)$ contains at most $m$ roots.

In the case that $s(2)=0$, we work as follows. We consider a tier $\mathcal{T}_{r}$. If $l_{r} \leq L-m+1$, then, as above, one has that for any root $w$ the tier $\mathcal{T}_{r}, B(w, \epsilon|w|)$ contains at most $L-m+1$ roots. Otherwise, $L(r) \geq l_{r}>L-m+1$ so that $L(s)-L(r)=L-L(r)<m-1$. Since $l_{s} \geq m$ we must then have that $r=s$ because otherwise we would have $m \leq l_{s} \leq L(s)-L(r)<m-1$. According with Lemma 7.1 .8 , for $1 \leq j \leq l_{s}$, we then approximate $S_{D_{j}\left(\mathcal{T}_{s}\right)}\left(\mathcal{T}_{s}\right)$ by

$$
\frac{S_{D(L(s-1)+j)}(\mathcal{R})}{S_{D(L(s-1))}\left(\mathcal{T}_{1} \cup \ldots \cup \mathcal{T}_{s-1}\right)} .
$$

In particular, we have for $1 \leq j \leq l_{s}$ that

$$
\left|S_{D_{j}\left(\mathcal{T}_{s}\right)}\left(\mathcal{T}_{s}\right)-\frac{S_{D(L(s-1)+j)}(\mathcal{R})}{S_{D(L(s-1))}\left(\mathcal{T}_{1} \cup \ldots \cup \mathcal{T}_{s-1}\right)}\right| \ll h\left(\mathcal{T}_{s}\right)^{D_{j}\left(\mathcal{T}_{s}\right)}
$$

Combining this with Proposition 8.1.2, for $L-m<L(s-1)+j<L$, we see that

$$
\left|S_{D_{j}\left(\mathcal{T}_{s}\right)}\left(\mathcal{T}_{s}\right)\right| \ll h\left(\mathcal{T}_{s}\right)^{j}
$$

From this, it is a matter of counting to see that at most $L-m+1$ non-trivial symmetric functions $S_{j}\left(\mathcal{T}_{s}\right)$ are substantial: we can choose $\widetilde{\mathcal{D}}\left(\mathcal{T}_{s}\right) \subset \mathcal{D}\left(\mathcal{T}_{s}\right)$ in Theorem 7.2.1, with $\left.\mid \widetilde{\mathcal{D}}\left(\mathcal{T}_{s}\right)\right) \mid-1 \leq$ $L-m+1$, so that, for $j \notin \widetilde{\mathcal{D}}\left(\mathcal{T}_{s}\right),\left|S_{j}\left(\mathcal{T}_{s}\right)\right| \ll h\left(\mathcal{T}_{s}\right)^{j}$. Therefore, by Theorem 7.2.1, for any root $w, B(w, \epsilon|w|)$ contains at most $L-m+1$ roots.

Proof of Proposition 8.1.2. We consider what the supposed conditions tell us under the height estimation procedure, Lemma 8.1.1. We obtain a sequence of reference heights $\eta_{1}, \eta_{2}, \ldots$ and corresponding indices $\alpha(1), \alpha(2), \ldots$ satisfying the conditions of that lemma.

In the case that $m=L$, there is nothing to prove. The height estimates are all $\lesssim_{\gamma} 1$. We set $s(1)=0$ so that $L(s(1))=0$ and each tier is a small tier. In what follows, we consider the case $m<L$.

Firstly, observe that $\left|\frac{y_{m}}{y_{L}}\right|^{\frac{1}{D(L-m)}} \geq\left|\frac{\gamma^{k_{m}}}{\delta^{k} L}\right|^{\frac{1}{D(L-m)}} \gg 1$, provided $\delta$ is sufficiently small. As such, we are guaranteed to pick up a number of large height estimates $\eta_{j} \gtrsim_{\gamma} 1$.

Let $0 \leq i_{0}$ be chosen maximally so that $\alpha\left(i_{0}\right) \leq L-m$. We either have that $\alpha\left(i_{0}\right)=L-m$ or $\alpha\left(i_{0}\right)<L-m$ and we first split our analysis by these cases. Since, for $n>m$,

$$
\left|\frac{y_{m}}{y_{n}}\right|^{\frac{1}{D(L-m)-D(L-n)}} \gtrsim \gamma 1
$$

it is easy to see that

$$
\eta_{\max \left\{i_{0}, 1\right\}} \gtrsim \gamma 1
$$

In the first case, where $L-\alpha\left(i_{0}\right)=m$, note that $i_{0} \geq 1$. We then observe that, provided we choose $\delta$ is chosen sufficiently small depending on $\gamma$, for $1 \leq j<m$,

$$
\begin{align*}
& \left|\frac{y_{j}}{y_{m}}\right|^{\frac{1}{D(L-j)-D(L-m)}} \leq\left|\frac{1}{\gamma^{k_{m}}}\right|^{\frac{1}{D(L-j)-D(L-m)}} \\
& \ll\left|\frac{\gamma^{k_{m}}}{\delta^{k_{L-\alpha\left(i_{0}-1\right)}}}\right|^{\frac{1}{D(L-m)-D\left(\alpha\left(i_{0}-1\right)\right)}} \leq\left|\frac{y_{m}}{y_{L-\alpha\left(i_{0}-1\right)}}\right|^{\frac{1}{\overline{D(L-m)-D\left(\alpha\left(i_{0}-1\right)\right)}}=\eta_{i_{0}}, ~, ~, ~, ~ . ~} \tag{8.1.9}
\end{align*}
$$

by definition of the height estimates. Thus we see that either $\eta_{i_{0}+1} \ll \eta_{i_{0}}$ or, if $\eta_{i_{0}+1} \gtrsim \eta_{i_{0}}$, we must have that $\eta_{i_{0}+1}=\left|\frac{x}{y_{m}}\right|^{\frac{1}{D(L)-D(L-m)}}$.

We now consider what happens in the height estimation procedure in the case that $\alpha\left(i_{0}\right)<$
$L-m$. We observe that, for $1 \leq j<m$,

$$
\begin{align*}
& \left|\frac{y_{j}}{y_{L-\alpha\left(i_{0}\right)}}\right|^{\frac{1}{D(L-j)-D\left(\alpha\left(i_{0}\right)\right)}} \leq\left|\frac{1}{y_{L-\alpha\left(i_{0}\right)}}\right|^{\frac{1}{D(L-j)-D\left(\alpha\left(i_{0}\right)\right)}} \\
& \quad \ll\left|\frac{y_{m}}{y_{L-\alpha\left(i_{0}\right)}}\right|^{\frac{1}{D(L-m)-D\left(\alpha\left(i_{0}\right)\right)}} \tag{8.1.10}
\end{align*}
$$

where one can verify that the last inequality holds because it is satisfied in the extreme:

$$
\left|\frac{1}{\delta^{k_{L-\alpha\left(i_{0}\right)}}}\right|^{\frac{1}{D(L-j)-D\left(\alpha\left(i_{0}\right)\right)}} \ll\left|\frac{\gamma^{k_{j}}}{\delta^{k_{L-\alpha\left(i_{0}\right)}}}\right|^{\frac{1}{D(L-m)-D\left(\alpha\left(i_{0}\right)\right)}} .
$$

Thus, in the case that $\alpha\left(i_{0}\right)<L-m$, we necessarily have that $\eta_{i_{0}+1}=\left|\frac{x}{y_{L-\alpha\left(i_{0}\right)}}\right|^{\frac{1}{D(L)-D\left(\alpha\left(i_{0}\right)\right)}}$ and $\alpha\left(i_{0}+1\right)=L$, since we have specified that $y_{m}$ contributes to no height estimate by the condition $\alpha\left(i_{0}\right)<L-m$.

We continue our analysis by splitting according to the control between height estimates $\eta\left(i_{0}\right)$ and $\eta\left(i_{0}+1\right)$. Firstly, we analyse the situation where $\eta\left(i_{0}+1\right) \ll \eta\left(i_{0}\right)$, which can only occur if $L-m=\alpha\left(i_{0}\right)$. Secondly, we analyse the situation where either $\eta\left(i_{0}+1\right) \gtrsim \eta\left(i_{0}\right)$ or where $i_{0}=0$, which can occur with $L-m=\alpha\left(i_{0}\right)$ or with $L-m>\alpha\left(i_{0}\right)$.

The first scenario is where $\eta\left(i_{0}+1\right) \ll \eta\left(i_{0}\right)$. Here, we have that $\alpha\left(i_{0}\right)=L-m$. This is the $s(2) \geq 1$ case. According with Lemma 8.1.1, we choose $s(1)$ so that the large tiers $\mathcal{T}_{r}$, for $1 \leq r \leq s(1)$ are those corresponding with the height estimates $\eta_{1}, \eta_{2}, \ldots, \eta_{i_{0}}$. We must have, by (8.1.1) from Lemma 8.1.1, that $L(s(1))=\alpha\left(i_{0}\right)=L-m$. We can also see that

$$
\eta_{i_{0}}=\left|\frac{y_{m}}{y_{L-\alpha\left(i_{0}-1\right)}}\right|^{\frac{1}{\overline{D(L-m)-D\left(\alpha\left(i_{0}-1\right)\right)}}} \gtrsim_{\gamma} 1,
$$

and, by Lemma 8.1.1, for roots $w$ in large tiers,

$$
|w| \gtrsim_{\gamma} 1
$$

There are also small tiers of roots: following Lemma 8.1.1, these are the tiers corresponding with the height estimates $\eta_{i_{0}+1}, \eta_{i_{0}+2}, \ldots \eta_{a}$. It is easy to see that $1 \gtrsim_{\gamma} \eta_{i_{0}+1}$, since

$$
\eta_{i_{0}+1}=\left|\frac{y_{L-\alpha\left(i_{0}+1\right)}}{y_{m}}\right|^{\frac{1}{D\left(\alpha\left(i_{0}+1\right)\right)-D(L-m)}} \leq\left|\frac{1}{\gamma}\right|^{\frac{1}{D\left(\alpha\left(i_{0}+1\right)\right)-D(L-m)}},
$$

and, by Lemma 8.1.1, for roots $w$ in small tiers,

$$
|w| \lesssim_{\gamma} 1
$$

In the second scenario, $\eta\left(i_{0}\right) \sim \eta\left(i_{0}+1\right)$ or there is only one height estimate and $\alpha(1)=$ $L$. In either case, $\alpha\left(i_{0}+1\right)=L$, as we previously showed how, in this case, we must have $\eta_{i_{0}+1}=\left|\frac{x}{y_{m}}\right|^{\frac{1}{D(L)-D(L-m)}}$. This is the $s(2)=0$ case, where all tiers are large. By Lemma 8.1.1, $l_{s} \geq \alpha\left(i_{0}+1\right)-\alpha\left(i_{0}\right) \geq L-(L-m)=m$. In this case, to conclude the proof, we must establish control the size of the symmetric functions $S_{D(L-j)}(\mathcal{R})$ for $1 \leq j<m$. For these $j$, note that $L-j>L-m \geq L-l_{s}=L(s-1)$. We work to show that that, for $1 \leq j<m$,

$$
\left|S_{D(L-j)}(\mathcal{R})\right| \ll h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} h\left(\mathcal{T}_{2}\right)^{D\left(\mathcal{T}_{2}\right)} \ldots h\left(\mathcal{T}_{s}\right)^{D(L-j)-D(L(s-1))}
$$



$$
\begin{equation*}
\eta_{i_{0}+1} \sim h\left(\mathcal{T}_{s}\right) \tag{8.1.11}
\end{equation*}
$$

by Lemma 8.1.1. We claim that that, for $1 \leq j<m$,

$$
\begin{equation*}
\left|\frac{y_{j}}{y_{L-\alpha\left(i_{0}\right)}}\right|^{\frac{1}{D(L-j)-D\left(L-\alpha\left(i_{0}\right)\right)}} \ll h\left(\mathcal{T}_{s}\right) \tag{8.1.12}
\end{equation*}
$$

Assuming for now that (8.1.12) holds, also using the inequality (8.1.1) from the statement of Lemma 8.1.1, we see that, for $1 \leq j<m$,

$$
\begin{gathered}
\left|S_{D(L-j)}(\mathcal{R})\right|=\left|\frac{y_{j}}{y_{L}}\right| \\
=\left|\frac{y_{\alpha(1)}}{y_{L}}\right| \ldots \left\lvert\, \frac{\left.y_{L-\alpha\left(i_{0}\right)}^{y_{L-\alpha\left(i_{0}-1\right)}}| | \frac{y_{j}}{y_{L-\alpha\left(i_{0}\right)}} \right\rvert\,}{\ll h\left(\mathcal{T}_{1}\right)^{D\left(\mathcal{T}_{1}\right)} \ldots h\left(\mathcal{T}_{s-1}\right)^{D\left(\mathcal{T}_{s-1}\right)} h\left(\mathcal{T}_{s}\right)^{D(L-j)-D(L(s-1))},}\right.
\end{gathered}
$$

which, after reindexing, is the desired error bound. To see this, we consider $1 \leq j^{\prime}<l_{s}$ such that $L-\left(L(s-1)+j^{\prime}\right)=l_{s}-j^{\prime}<m$ : we set $j=L-\left(L(s-1)+j^{\prime}\right)$, which ranges between 1 and $m-1$, as in the proposition's statement.

To conclude, we prove our claimed inequality (8.1.12). In the case that $L-\alpha\left(i_{0}\right)=m$, this is a direct consequence of (8.1.9) upon observing from Lemma 8.1.1 that $\eta_{i_{0}} \sim h\left(\mathcal{T}_{s}\right)$. In the case that $\alpha\left(i_{0}\right)<L-m$, we use (8.1.10) and Lemma 8.1.1: if $i_{0}=0$, then there is one height estimate $\eta_{1}=\left|\frac{x}{y_{L}}\right|^{\frac{1}{k_{L}}} \geq\left|\frac{y_{m}}{y_{L}}\right|^{\frac{1}{D(L-m)}} \gg\left|\frac{y_{j}}{y_{L-\alpha\left(i_{0}\right)}}\right|^{\frac{1}{D(L-j)-D\left(L-\alpha\left(i_{0}\right)\right)}}$, if $i_{0} \geq 1$, then $\eta_{i_{0}+1} \geq\left|\frac{y_{m}}{y_{L-\alpha\left(i_{0}\right)}}\right|^{\frac{1}{\overline{D(L-m)-D\left(\alpha\left(i_{0}\right)\right)}}} \gg\left|\frac{y_{j}}{y_{L-\alpha\left(i_{0}\right)}}\right|^{\frac{1}{D(L-j)-D\left(L-\alpha\left(i_{0}\right)\right)}}$. Referring to (8.1.11), the inequality follows.

Remark 8.1.3. As for Proposition 8.1.2, Lemma 8.1.1 and the procedure it outlines can be used to obtain other refinements of the main structural result, Theorem 6.0.1. For example, if we had that

$$
\left|\frac{y_{L-1}}{y_{L}}\right|^{\frac{1}{d(1)}} \gg\left|\frac{y_{L-2}}{y_{L-1}}\right|^{\frac{1}{d(2)}} \gg \ldots \gg\left|\frac{x}{y_{1}}\right|^{\frac{1}{d(L)}}>0
$$

then we would have $L$ tiers of roots which are separated and, for sufficiently small $\epsilon$ and some root $z \in \mathcal{R}, B(z, \epsilon|z|)$ contains only the root $z$.

### 8.2 Rough factorisation

For notational reasons, we consider monic polynomials in this section:

$$
\Psi(t)=\sum_{j=0}^{L} y_{j} t^{k_{j}},
$$

where $k_{0}=0$ and $y_{L}=1$. In this section, we provide a rough factorisation of monic polynomials with a well separated tier structure. To this end, let us suppose that throughout this section we are in the regime indexed by $\left(l_{1}, l_{2}, \ldots, l_{s}\right)$, as in Definition 7.1.5. Here there are $s$ tiers, $\mathcal{T}_{1}$, $\ldots, \mathcal{T}_{s}$, containing $D\left(\mathcal{T}_{1}\right)=\left|\mathcal{T}_{1}\right|, \ldots, D\left(\mathcal{T}_{s}\right)=\left|\mathcal{T}_{s}\right|$ roots, respectively. The tier regime may be roughly characterised by

$$
\begin{equation*}
h\left(\mathcal{T}_{1}\right) \gg h\left(\mathcal{T}_{2}\right) \gg \ldots \gg h\left(\mathcal{T}_{s}\right) \tag{8.2.1}
\end{equation*}
$$

for some suitable choice of constants.
Recall the definition of the $k_{j}\left(\mathcal{T}_{r}\right)$ exponents for a given tier, Definition 7.2.6. More explicitly, for $0 \leq j \leq l_{r}$, we can write $k_{j}\left(\mathcal{T}_{r}\right)=k_{L-L(r)+j}-k_{L-L(r)}$. Recall also the Definition 7.1.7 of the distinguished indices $D_{j}\left(\mathcal{T}_{r}\right)$, which we can write more explicitly as $D_{j}\left(\mathcal{T}_{r}\right)=D(L(r-1)+$ $j)-D(L(r-1))=k_{L-L(r-1)}-k_{L-L(r-1)-j}$.

Definition 8.2.1. If we are in the regime given in Definition 7.1 .5 indexed by $\left(l_{1}, l_{2}, \ldots, l_{s}\right)$, then, for $1 \leq j \leq s$, we define the monic polynomial

$$
\begin{equation*}
\widetilde{\Psi}_{j}(t):=\frac{1}{y_{L-L(j-1)}} \sum_{i=0}^{l_{j}} y_{L-L(j)+i} t^{k_{i}\left(\mathcal{T}_{j}\right)} \tag{8.2.2}
\end{equation*}
$$

We define the rough factorisation of $\Psi(t)$ by

$$
\begin{equation*}
\widetilde{\Psi}(t):=\prod_{j=1}^{s} \widetilde{\Psi}_{j}(t) \tag{8.2.3}
\end{equation*}
$$

We denote the roots of $\widetilde{\Psi}_{j}$ by $\widetilde{\mathcal{T}}_{j}$ and the roots of $\widetilde{\Psi}$ by $\widetilde{\mathcal{R}}$.
Away from the zeros of the polynomial, the rough factorisation, (8.2.3), we seek should be quantifiably close to the original $\Psi$. Furthermore, the root structure of $\Psi$ and the root structure of its rough factorisation should be closely related. In particular, we have Theorem 8.2.2.
Theorem 8.2.2. There exists a polynomial $\widetilde{\Psi}=\prod_{l=1}^{s} \widetilde{\Psi}_{l}(t)$, with $\widetilde{\Psi}_{l}(t)$ given by (8.2.2), which roughly factorises $\Psi$ in the following sense.

The polynomial $\widetilde{\Psi}_{l}$ has roots, $\widetilde{\mathcal{T}}_{l}$, which are all of comparable magnitude. Furthermore, for roots $w_{j} \in \widetilde{\mathcal{T}}_{j}$, we have that

$$
\left|w_{1}\right| \gg\left|w_{2}\right| \gg \ldots \gg\left|w_{s}\right|
$$

There exists a covering, $N(\mathcal{R})$, of the roots, $\mathcal{R} \subset \mathbb{C}$, of $\Psi$ which satisfies the following. Each connected component of $N(\mathcal{R})$, which we call a cell, is given by a ball. Each cell containing non-zero roots contains at most $L$ roots. For a cell $B$ containing exactly $m$ roots of $\Psi, B$ contains exactly $m$ roots of $\widetilde{\Psi}$.

For $t \notin N(\mathcal{R})$,

$$
|\Psi(t)-\widetilde{\Psi}(t)| \ll|\Psi(t)|
$$

The analysis in this section requires strong separation of the height estimates in the specification of the tier regime, Definition 7.1.5, as we will see. It should be noted, however, that the results of Chapter 7 do not require such strong separation, although this is not something we specify in this thesis. All of the results in this section should be understood as valid with respect to a tier regime specified with sufficiently strong separation in (8.2.1).

To begin with, let us bound the difference of $\Psi(t)$ and $\widetilde{\Psi}(t)$.
Lemma 8.2.3. With

$$
\begin{equation*}
E(t):=\widetilde{\Psi}(t)-\Psi(t) \tag{8.2.4}
\end{equation*}
$$

we have that, for each $1 \leq r \leq s$ and $h\left(\mathcal{T}_{r+1}\right) \ll|t| \lesssim h\left(\mathcal{T}_{r}\right)$, that

$$
\begin{equation*}
|E(t)| \leq \epsilon_{f}\left(\prod_{i=1}^{r} h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)}\right)|t|^{\sum_{j=r+1}^{s} D\left(\mathcal{T}_{j}\right)} \tag{8.2.5}
\end{equation*}
$$

including for $r=s$ and $r=1$, subject to the understanding that $h\left(\mathcal{T}_{0}\right)=\infty$ and $h\left(\mathcal{T}_{s+1}\right)=0$. Here the constant $\epsilon_{f}$ can be taken arbitrarily small, provided we make a suitably strong choice of separation constants in the specification of the tier regime, (8.2.1).
Proof. For $1 \leq i \leq l_{j}$, we wish to estimate

$$
\left|\frac{y_{L-L(j)+i}}{y_{L-L(j-1)}}\right|
$$

As a consequence of the height estimation lemma, Lemma 8.1.1, we have that

$$
\begin{equation*}
\left|\frac{y_{L-L(j)+i}}{y_{L-L(j-1)}}\right| \lesssim h\left(\mathcal{T}_{j}\right)^{D_{l_{j}-i}\left(\mathcal{T}_{j}\right)} \tag{8.2.6}
\end{equation*}
$$

We use this to bound the error term.

We can write

$$
\widetilde{\Psi}(t)=\prod_{j=1}^{s} \frac{1}{y_{L-L(j-1)}}\left(\sum_{i_{j}=0}^{l_{j}} y_{L-L(j)+i_{j}} t^{k_{i_{j}}\left(\mathcal{T}_{s}\right)}\right)
$$

To avoid a proliferation of nested sub and super-scripts, we will sometimes write $i(j)$ and $l(j)$ in place of $i_{j}$ and $l_{j}$, respectively. If we expand the above product expression for $\widetilde{\Psi}$, we obtain a sum that we will refer to throughout this proof. Each term in the resulting sum can be indexed by $\left(i_{1}, i_{2}, \ldots, i_{s}\right)$, where the index $i_{j}$ ranges over $\left\{0,1, \ldots, l_{j}\right\}$ for each $1 \leq j \leq s$.

We first consider those terms which sum to $\Psi$. For $1 \leq a \leq s$ one can see that the term indexed by $\left(0,0, \ldots, i_{a}, l_{a+1}, l_{a+2}, \ldots, l_{s}\right)$, with $0 \leq i_{a}<l_{a}$ is exactly

$$
\begin{gathered}
\left(\prod_{j=1}^{a-1} \frac{y_{L-L(j)}}{y_{L-L(j-1)}}\right)\left(\frac{1}{y_{L-L(a-1)}} y_{L-L(a)+i(a)} t^{k_{i(a)}\left(\mathcal{T}_{a}\right)}\right)\left(\prod_{i=a+1}^{s} t^{k_{l(i)}\left(\mathcal{T}_{i}\right)}\right) \\
=y_{L-L(a)+i(a)} t^{k_{L-L(a)+i(a)}}
\end{gathered}
$$

As an example, corresponding to $a=s$, we have that the term indexed by $\left(0,0, \ldots, 0, i_{s}\right)$ is

$$
\begin{gathered}
\left(\prod_{j=1}^{s-1} \frac{y_{L-L(j)}}{y_{L-L(j-1)}}\right)\left(\frac{1}{y_{L-L(s-1)}} y_{L-L(s)+i(s)} t^{k_{i(s)}\left(\mathcal{T}_{s}\right)}\right) \\
=y_{i(s) t^{k_{i(s)}}} .
\end{gathered}
$$

We also have that the term indexed by $\left(l_{1}, l_{2}, \ldots, l_{s}\right)$ is

$$
\left(\prod_{i=1}^{s} t^{k_{l(i)}\left(\mathcal{T}_{i}\right)}\right)=t^{k_{L}}
$$

In this way, we have uniquely indexed all of the terms appearing in the expansion of $\Psi(t)$. The remaining terms are exactly those which sum to $E(t)=\widetilde{\Psi}(t)-\Psi(t)$. Before proceeding to bound $E$, let us consider the size of the terms we have just indexed. This will inform us as to the bounds we should shoot for on the error term. For the term indexed by $\left(0,0, \ldots, i_{a}, l_{a+1}, l_{a+2}, \ldots, l_{s}\right)$, we see using (8.2.6) that it is bounded in magnitude

$$
\lesssim\left(\prod_{j=1}^{a-1} h\left(\mathcal{T}_{j}\right)^{D_{l_{j}}\left(\mathcal{T}_{j}\right)}\right)\left(h\left(\mathcal{T}_{a}\right)^{D_{l(a)-i(a)}\left(\mathcal{T}_{a}\right)}|t|^{k_{i(a)}\left(\mathcal{T}_{a}\right)}\right)\left(\prod_{j=a+1}^{s}|t|^{k_{l(j)}\left(\mathcal{T}_{j}\right)}\right)
$$

Now, if we are considering $h\left(\mathcal{T}_{r+1}\right) \ll|t| \lesssim h\left(\mathcal{T}_{r}\right)$, for $1 \leq a \leq L$, these terms can be uniformly bounded

$$
\lesssim\left(\prod_{j=1}^{r} h\left(\mathcal{T}_{j}\right)^{D_{l_{j}}\left(\mathcal{T}_{j}\right)}\right)\left(\prod_{j=r+1}^{s}|t|^{k_{l(j)}\left(\mathcal{T}_{j}\right)}\right)=\left(\prod_{j=1}^{r} h\left(\mathcal{T}_{j}\right)^{D\left(\mathcal{T}_{j}\right)}\right)|t|^{\sum_{j=r+1}^{s} D\left(\mathcal{T}_{j}\right)}
$$

In what follows, we consider those terms that we did not specify as summing to $\Psi$ above. We refer to these as the remainder terms. Let us fix $1 \leq r \leq s$ and consider

$$
\begin{equation*}
h\left(\mathcal{T}_{r+1}\right) \ll|t| \lesssim h\left(\mathcal{T}_{r}\right) . \tag{8.2.7}
\end{equation*}
$$

As above, for the term indexed by $\left(i_{1}, i_{2}, \ldots, i_{s}\right)$, we can estimate the size of each of the $s$ factors using (8.2.6) and (8.2.7). For $1 \leq j \leq r$,

$$
\begin{equation*}
\left|\frac{1}{y_{L-L(j-1)}} y_{L-L(j)+i(j)} t^{k_{i(j)}\left(\mathcal{T}_{j}\right)}\right| \lesssim h\left(\mathcal{T}_{j}\right)^{D\left(\mathcal{T}_{j}\right)} \tag{8.2.8}
\end{equation*}
$$

For $r<j \leq s$,

$$
\begin{equation*}
\left|\frac{1}{y_{L-L(j-1)}} y_{L-L(j)+i(j)} t^{k_{i(j)}\left(\mathcal{T}_{j}\right)}\right| \lesssim h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{j}\right)} \tag{8.2.9}
\end{equation*}
$$

To bound $E$ as an error term, we require stronger control on the remainder terms. We observe that those terms summing to $E$ are indexed by $\left(i_{1}, i_{2}, \ldots, i_{s}\right)$, such that if $a$ is the smallest index such that $i_{a} \neq 0$ (or $a=0$ if no such index $i_{j}$ exists), then there exists a minimal $a^{\prime}>a$ for which $i_{a^{\prime}}<l_{a^{\prime}}$. The corresponding term is

$$
\left(\prod_{j=1}^{a-1} \frac{y_{L-L(j)}}{y_{L-L(j-1)}}\right)\left(\frac{1}{y_{L-L(a-1)}} y_{L-L(a)+i(a)} t^{k_{i(a)}\left(\mathcal{T}_{a}\right)}\right)\left(\prod_{j=a+1}^{s} \frac{1}{y_{L-L(j-1)}} y_{L-L(j)+i(j)} t^{k_{i(j)}\left(\mathcal{T}_{j}\right)}\right)
$$

For each of these remainder terms, one of the factors appearing in the above expression will allow us to establish the error bound. Let us now fix some remainder term and its corresponding index $\left(i_{1}, \ldots, i_{s}\right)$. We split our analysis according to whether $r \leq a<s$ or $1 \leq a<r$.

If $r \leq a<s$, then, using (8.2.6) and the fact that $|t| \gg h\left(\mathcal{T}_{r+1}\right) \geq h\left(\mathcal{T}_{a^{\prime}}\right)$, we can strongly bound the factor indexed by $i_{a^{\prime}}$

$$
\begin{gathered}
\left|\frac{1}{y_{L-L\left(a^{\prime}-1\right)}} y_{L-L\left(a^{\prime}\right)+i\left(a^{\prime}\right)} t^{k_{i\left(a^{\prime}\right)}\left(\mathcal{T}_{a^{\prime}}\right)}\right| \lesssim h\left(\mathcal{T}_{a^{\prime}}\right)^{D_{l\left(a^{\prime}\right)-i\left(a^{\prime}\right)}\left(\mathcal{T}_{a^{\prime}}\right)} t^{k_{i\left(a^{\prime}\right)}\left(\mathcal{T}_{a^{\prime}}\right)} \\
\ll|t|^{k_{l\left(a^{\prime}\right)}}=|t|^{D\left(\mathcal{T}_{a^{\prime}}\right)},
\end{gathered}
$$

since $D_{l\left(a^{\prime}\right)-i\left(a^{\prime}\right)}>0$. Putting this together with (8.2.8) and (8.2.9), we can bound the magnitude of the remainder term indexed by $\left(i_{1}, \ldots, i_{s}\right)$

$$
\begin{gathered}
\ll\left(\prod_{j=1}^{r} h\left(\mathcal{T}_{j}\right)^{D_{l_{j}}\left(\mathcal{T}_{j}\right)}\right)|t|^{D\left(\mathcal{T}_{a^{\prime}}\right)}\left|\prod_{j=r+1, j \neq a^{\prime}}^{s} \frac{1}{y_{L-L(j-1)}} y_{L-L(j)+i(j)} t^{k_{i(j)}\left(\mathcal{T}_{j}\right)}\right| \\
\left.\lesssim\left(\prod_{j=1}^{r} h\left(\mathcal{T}_{j}\right)^{D_{l_{j}}\left(\mathcal{T}_{j}\right)}\right)\left|\prod_{j=r+1}^{s}\right| t\right|^{D\left(\mathcal{T}_{j}\right)} \mid
\end{gathered}
$$

If $1 \leq a<r$, then we can strongly bound the factor indexed by $i_{a}$

$$
\left|\frac{1}{y_{L-L(a-1)}} y_{L-L(a)+i(a)} t^{k_{i(a)}\left(\mathcal{T}_{a}\right)}\right| \ll h\left(\mathcal{T}_{a}\right)^{k_{l(a)}}=h\left(\mathcal{T}_{a}\right)^{D\left(\mathcal{T}_{a}\right)}
$$

since $|t| \lesssim h\left(\mathcal{T}_{r}\right) \ll h\left(\mathcal{T}_{a}\right)$ and $k_{i(a)}\left(\mathcal{T}_{a}\right)>0$. Therefore, also using (8.2.8) and (8.2.9), we can bound the remainder term indexed by $\left(i_{1}, \ldots, i_{s}\right)$

$$
\ll\left(\prod_{j=1, j \neq a}^{r} h\left(\mathcal{T}_{j}\right)^{D_{l_{j}}\left(\mathcal{T}_{j}\right)}\right)\left(h\left(\mathcal{T}_{a}\right)^{D\left(\mathcal{T}_{a}\right)}\right)\left(\prod_{j=r+1}^{s}|t|^{D_{l_{j}}\left(\mathcal{T}_{j}\right)}\right) .
$$

Summing the bounds on each of the remainder terms gives the desired estimate.
For a given tier, $\mathcal{T}_{r}$, there are two important scale parameters appearing in the previous analysis. Provided the separation in (8.2.1) is strong enough, we can set our fine parameter $\epsilon_{f}$ in Lemma 8.2.3 as small as we like. The other important parameter appearing in our analysis is the coarse scale parameter $\epsilon_{c}$, which we now define. We choose $\epsilon_{c}>0$ so that, according with Theorem 7.2.2, at most $L\left(\mathcal{T}_{r}\right)$ roots from $\mathcal{T}_{r}$ can appear in the ball $B\left(w, 3 \epsilon_{c} h\left(\mathcal{T}_{r}\right)\right)$ for any choice of $w \in \mathcal{T}_{r}$.

Before giving the proof of our main result, we require a covering lemma. This covering lemma essentially provides a partition of the roots into clusters of size up to $L$, with strong separation between distinct clusters. In place of clusters, which are finite collection of roots, we use cells, which are suitable open balls containing these roots. The proof gives a recursive construction of these cells. Associated with this construction are a well separated sequence of
parameters, $\epsilon_{f} \ll \epsilon_{1} \ll \epsilon_{2} \ll \ldots \ll \epsilon_{L} \ll \epsilon_{L+1} \ll \epsilon_{c}$, which we now define.
Definition 8.2.4. We set $\epsilon_{1}=\epsilon_{f}^{\frac{1}{L}}$ and, for $1 \leq j \leq L$, we set $\epsilon_{j+1}=\epsilon_{j}^{\frac{1}{L}}$.
Remark 8.2.5. We can achieve strong separation of the parameters $\epsilon_{j}$ provided we start from a suitable fine error parameter, $\epsilon_{f}$. Indeed, to have that $\epsilon_{j} \ll \epsilon_{j+1}$ and $\epsilon_{L} \ll \epsilon_{c}$, the two things we require are that

$$
\epsilon_{j} / \epsilon_{j+1}=\epsilon_{j}^{1-\frac{1}{L}}=\epsilon_{1}^{\frac{L-1}{L j}} \ll 1
$$

and

$$
\epsilon_{L+1}=\epsilon_{1}^{\frac{1}{L^{L}}} \ll \epsilon_{c} .
$$

This is possible provided we can take $\epsilon_{f}$ sufficiently small, which is something we can achieve if we specify the tier regime with strong separation of the reference heights.

Definition 8.2.6. For points $w_{1}, \ldots, w_{a} \in \mathbb{C}$, we denote by $A\left(w_{1}, \ldots, w_{a}\right)$ their arithmetic mean:

$$
A\left(w_{1}, \ldots, w_{a}\right):=\frac{1}{a} \sum_{i=1}^{a} w_{i} .
$$

We can now state our root cell covering lemma.
Lemma 8.2.7. There exists a covering, $N(\mathcal{R})$, of the roots, $\mathcal{R}$, of $\Psi$ which satisfies the following.

Each connected component of $N(\mathcal{R})$, which we call a cell, B, contains only roots from one tier. Furthermore, if a cell $B$ contains only the roots $w_{1}, \ldots, w_{b}$ in the tier $\mathcal{T}_{r}$, then $B=B\left(A\left(w_{1}, \ldots, w_{b}\right), \epsilon_{b} h\left(\mathcal{T}_{r}\right)\right)$. For cells containing non-zero roots, each cell can contain at most $L$ roots.

For distinct cells $B=B\left(A\left(w_{1}, \ldots, w_{b}\right), \epsilon_{b} h\left(\mathcal{T}_{r}\right)\right)$ and $B^{\prime}=B\left(A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right), \epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)\right)$ with $b \geq b^{\prime}$,

$$
d\left(B, B^{\prime}\right) \geq \frac{1}{4} \epsilon_{b+1} h\left(\mathcal{T}_{r}\right)
$$

In particular, for any root $w^{\prime}$ outwith the cell $B=B\left(A\left(w_{1}, \ldots, w_{b}\right), \epsilon_{b} h\left(\mathcal{T}_{r}\right)\right)$ and any root $w \in B$,

$$
\left|w-w^{\prime}\right| \gtrsim \epsilon_{b+1} h\left(\mathcal{T}_{r}\right) .
$$

Figure 8.1 is a sketch of 4 nearby root cells from a root cell covering. Roots are marked with a cross. Note that the larger cells are at a larger distance from adjacent cells.


Figure 8.1: Four cells from a root cell covering.

Proof. Let us carry out the construction in each tier separately. Let $\mathcal{T}=\mathcal{T}_{r}$ for some $r$ and let $h=h\left(\mathcal{T}_{r}\right)$ denote the corresponding reference height.

The construction is first outlined with reference to a particular choice of $w_{1} \in \mathcal{T}$. For each $w_{1} \in \mathcal{T}$, we construct an appropriate ball $N\left(w_{1}\right)$ containing $w_{1}$ and a number of other roots in
$\mathcal{T}$. The root cell covering $N(\mathcal{R})$ in the lemma statement is then given as the union of the balls $N\left(w_{1}\right)$.

For now, let us fix some $w_{1}$. If for all remaining $w^{\prime} \in \mathcal{T},\left|w_{1}-w^{\prime}\right| \geq \epsilon_{2} h$, then we set $N\left(w_{1}\right)=B\left(w_{1}, \epsilon_{1} h\right)$.

Continuing the construction, it remains to consider the case where there exists $w_{2}$ with $\left|w_{1}-w_{2}\right|<\epsilon_{2} h$. Fix some choice of such $w_{2}$. We then divide our analysis with reference to $A\left(w_{1}, w_{2}\right)=\frac{w_{1}+w_{2}}{2}$. If, for all remaining $w^{\prime} \in \mathcal{R}$, we have that $\left|w^{\prime}-A\left(w_{1}, w_{2}\right)\right| \geq \epsilon_{3} h$, then we set $N\left(w_{1}\right)=B\left(A\left(w_{1}, w_{2}\right), \epsilon_{2} h\right)$. Otherwise, distinct from $w_{1}$ and $w_{2}$, there exists $w_{3}$ such that $\left|w_{3}-A\left(w_{1}, w_{2}\right)\right| \leq \epsilon_{3} h$ and we continue as previously, fixing some choice of $w_{3}$ and then working with reference to $A\left(w_{1}, w_{2}, w_{3}\right)$ and the scale $\epsilon_{4} h$.

The procedure continues; we consider the distance of remaining roots to the average of the roots already picked up by our procedure. Since there are finitely many roots, we know that the construction will terminate and we will refer to the resulting balls as terminal. For each choice of $w_{1}$, we construct a terminal ball $N\left(w_{1}\right)=B\left(A\left(w_{1}, w_{2}, \ldots, w_{b}\right), \epsilon_{b} h\right)$. By definition, the terminal ball $N\left(w_{1}\right)$ is constructed so that, for $w^{\prime} \in \mathcal{R}$ with $w^{\prime} \notin N\left(w_{1}\right)$, $\left|w^{\prime}-A\left(w_{1}, w_{2}, \ldots, w_{b}\right)\right| \geq \epsilon_{b+1} h$.

For each $w_{1} \in \mathcal{T}$, we can construct a ball $N\left(w_{1}\right)$ following the above procedure. The root cell covering $N(\mathcal{R})$ is simply given as the union of the sets $N\left(w_{1}\right)$. It remains to show that each connected component of $N(\mathcal{R})$ is given by a ball containing at most $L(\mathcal{T})$ roots, in particular that it is given by $N\left(w_{1}\right)$ for some $w_{1}$.

Let us first determine how close roots $w_{j}$ in a cell are to its centre. From the cell $B\left(A\left(w_{1}, \ldots, w_{b}\right), \epsilon_{b} h\left(\mathcal{T}_{r}\right)\right)$, we take the root $w_{a}$. We see that

$$
\begin{gather*}
\left.\left|w_{a}-A\left(w_{1}, \ldots, w_{b}\right)\right| \leq \mid w_{a}-A\left(w_{1}, \ldots, w_{a}\right)\right)\left|+\sum_{j=a+1}^{b}\right| A\left(w_{1}, \ldots, w_{j-1}\right)-A\left(w_{1}, \ldots, w_{j}\right) \mid \\
=\left|w_{a}-\frac{1}{a}\left((a-1) A\left(w_{1}, \ldots, w_{a-1}\right)+w_{a}\right)\right| \\
+\sum_{j=a+1}^{b} \frac{1}{j}\left|j A\left(w_{1}, \ldots, w_{j-1}\right)-\left((j-1) A\left(w_{1}, \ldots, w_{j-1}\right)+w_{j}\right)\right| \\
=\frac{a-1}{a}\left|w_{a}-A\left(w_{1}, \ldots, w_{a-1}\right)\right|+\sum_{j=a+1}^{b} \frac{1}{j}\left|A\left(w_{1}, \ldots, w_{j-1}\right)-w_{j}\right| \\
\leq \frac{a-1}{a} \epsilon_{a} h\left(\mathcal{T}_{r}\right)+\sum_{j=a+1}^{b} \frac{1}{j} \epsilon_{j} h\left(\mathcal{T}_{r}\right) \\
\leq \frac{b-1}{b} \epsilon_{b} h\left(\mathcal{T}_{r}\right) . \tag{8.2.10}
\end{gather*}
$$

We can verify that the construction of the terminal balls takes at most $L(\mathcal{T})$ steps. Indeed, suppose that this were not the case and consider the step from $L(\mathcal{T})$ to $L(\mathcal{T})+1$. We then know we can find $L(\mathcal{T})+1$ roots $w_{1} \ldots w_{L(\mathcal{T})+1}$ contained in $B\left(A\left(w_{1}, w_{2}, \ldots, w_{L(\mathcal{T})+1}\right), \epsilon_{L(\mathcal{T})+1} h\right)$. This contradicts the regime specific structure theorem, Theorem 7.2 .2 , since

$$
B\left(A\left(w_{1}, w_{2}, \ldots, w_{L(\mathcal{T})+1}\right), \epsilon_{L(\mathcal{T})+1} h\right) \subset B\left(w_{1}, 3 \epsilon_{L(\mathcal{T})+1} h\right) \subset B\left(w_{1}, \epsilon_{c} h\right)
$$

and $B\left(w_{1}, \epsilon_{c} h\right)$ contains at most $L(\mathcal{T})$ roots.
We are now able to show that each cell of $N(\mathcal{R})$ is a ball. To this end, let us take two terminal balls $B=B\left(A\left(w_{1}, \ldots, w_{b}\right), \epsilon_{b} h\left(\mathcal{T}_{r}\right)\right)$ and $B^{\prime}=B\left(A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right), \epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)\right)$ obtained by the construction outlined above. We suppose that these balls are such that

$$
\begin{equation*}
B\left(A\left(w_{1}, \ldots, w_{b}\right), \epsilon_{b} h\left(\mathcal{T}_{r}\right)\right) \cap B\left(A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right), \epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)\right) \neq \emptyset \tag{8.2.11}
\end{equation*}
$$

Without loss of generality, suppose that $b^{\prime} \leq b$. We find that, for any $w_{a^{\prime}} \in\left\{w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right\}$,

$$
\left|w_{a}^{\prime}-A\left(w_{1}, \ldots, w_{b}\right)\right|
$$

$$
\begin{gather*}
\leq\left|w_{a}^{\prime}-A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right)\right|+\left|A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right)-A\left(w_{1}, \ldots, w_{b}\right)\right| \\
\leq \frac{b^{\prime}-1}{b^{\prime}} \epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)+\epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)+\epsilon_{b} h\left(\mathcal{T}_{r}\right) \\
<\epsilon_{b+1} h\left(\mathcal{T}_{r}\right) . \tag{8.2.12}
\end{gather*}
$$

In particular, we must have that $w_{a}^{\prime} \in\left\{w_{1}, \ldots, w_{b}\right\}$, because otherwise the construction we outlined above must continue to account for $w_{a^{\prime}}$. Therefore $\left\{w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right\} \subset\left\{w_{1}, \ldots, w_{b}\right\}$. It is then easy to verify that $B^{\prime} \subset B$. This is obvious if $b=b^{\prime}$. To show this when $b^{\prime}<b$, let us take any complex number $w \in B^{\prime}$. Relating $w$ to $B^{\prime}$ and $B^{\prime}$ to $w_{1}^{\prime}$, which we know is an element of $B^{\prime}$ and of $B$, we find that

$$
\begin{gathered}
d\left(w, A\left(w_{1}, \ldots, w_{b}\right)\right) \\
\leq d\left(w, A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right)\right)+d\left(A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right), w_{1}^{\prime}\right)+d\left(w_{1}^{\prime}, A\left(w_{1}, \ldots, w_{b}\right)\right) \\
\leq \epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)+\frac{b^{\prime}-1}{b^{\prime}} \epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)+\frac{b-1}{b} \epsilon_{b} h\left(\mathcal{T}_{r}\right) \\
<\epsilon_{b} h\left(\mathcal{T}_{r}\right),
\end{gathered}
$$

so that $w \in B$. Therefore, $B^{\prime} \subset B$.
In fact, the above argument showing that cells are given by terminal balls can be strengthened. We can show that distinct cells are strongly separated. In particular, to conclude, we show that, for cells $B=B\left(A\left(w_{1}, \ldots, w_{b}\right), \epsilon_{b} h\left(\mathcal{T}_{r}\right)\right)$ and $B^{\prime}=B\left(A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right), \epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)\right)$ with $b^{\prime} \leq b$,

$$
d\left(B, B^{\prime}\right) \gtrsim \epsilon_{b+1} h\left(\mathcal{T}_{r}\right)
$$

Let us suppose, for a contradiction, that there exists a complex number

$$
\begin{equation*}
w \in B\left(A\left(w_{1}, \ldots, w_{b}\right), \frac{1}{3} \epsilon_{b+1} h\left(\mathcal{T}_{r}\right)\right) \cap B\left(A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right), \frac{1}{3} \epsilon_{b^{\prime}+1} h\left(\mathcal{T}_{r}\right)\right) \tag{8.2.13}
\end{equation*}
$$

We find that

$$
\begin{gathered}
d\left(w_{1}^{\prime}, A\left(w_{1}, \ldots, w_{b}\right)\right) \\
\leq d\left(w_{1}^{\prime}, A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right)\right)+d\left(A\left(w_{1}^{\prime}, \ldots, w_{b^{\prime}}^{\prime}\right), w\right)+d\left(w, A\left(w_{1}, \ldots, w_{b}\right)\right) \\
\leq \frac{b^{\prime}-1}{b^{\prime}} \epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)+\epsilon_{b^{\prime}} h\left(\mathcal{T}_{r}\right)+\epsilon_{b} h\left(\mathcal{T}_{r}\right) \\
<\epsilon_{b+1} h\left(\mathcal{T}_{r}\right)
\end{gathered}
$$

This inequality contradicts our assumption that $B=B\left(A\left(w_{1}, \ldots, w_{b}\right), \epsilon_{b} h\left(\mathcal{T}_{r}\right)\right)$ was a terminal ball, since it implies that the construction should continue to account for the root $w_{1}^{\prime} \notin\left\{w_{1}, \ldots, w_{b}\right\}$. Therefore (8.2.13) can not hold and, in particular,

$$
d\left(B, B^{\prime}\right) \geq \frac{1}{4} \epsilon_{b+1} h\left(\mathcal{T}_{r}\right)
$$

Lemma 8.2.8. The roots, $\widetilde{\mathcal{T}}_{j}$, of the polynomial $\widetilde{\Psi}_{j}(t)$ are all comparable in magnitude to $h\left(\mathcal{T}_{j}\right)$.
Proof. This is a simple consequence of Lemma 8.1.1, which can be applied with reference to $\Psi$ or to $\widetilde{\Psi}_{j}$. In either case, we see exactly the same expressions appearing as height estimates and the lemma tells us that these are comparable to $h\left(\mathcal{T}_{j}\right)$.

We are now ready to prove Theorem 8.2.2.
Proof. In the case that $s=1$, the factorisation is our initial polynomial and there is nothing to prove. Henceforth, we suppose that $s>1$.

We make use of Lemma 8.2.8 and consider the factorisations of $\Psi(t)$ and $\widetilde{\Psi}(t)$ in terms of
their roots:

$$
\begin{equation*}
\Psi(t)=\prod_{j=1}^{s} \prod_{w \in \mathcal{T}_{j}}(t-w) \quad \text { and } \quad \widetilde{\Psi}(t)=\prod_{j=1}^{s} \widetilde{\Psi}_{j}(t)=\prod_{j=1}^{s} \prod_{\tilde{w} \in \widetilde{\mathcal{T}}_{j}}(t-\tilde{w}) . \tag{8.2.14}
\end{equation*}
$$

Throughout, we appeal to the regime specific structure theorem, Theorem 7.2.2, and the root cell covering lemma, Lemma 8.2.7. Let us consider a specific $t \notin N(\mathcal{R})$ with $|t| \sim h\left(\mathcal{T}_{r}\right)$. By the definition of our covering in Lemma 8.2.7, the closest $t$ can be to a root $w \in \mathcal{T}_{r}$ is $\epsilon_{1} h\left(\mathcal{T}_{r}\right)$ and, furthermore, we can check that there are at most $L\left(\mathcal{T}_{r}\right)$ roots $w$ with $\epsilon_{1} h\left(\mathcal{T}_{r}\right)<|t-w|<\epsilon_{c} h\left(\mathcal{T}_{r}\right)$. Indeed, if $w \in \mathcal{T}_{r} \cap B\left(t, \epsilon_{c} h\left(\mathcal{T}_{r}\right)\right)$, then $B\left(t, \epsilon_{c} h\left(\mathcal{T}_{r}\right)\right) \subset B\left(w, 3 \epsilon_{c} h\left(\mathcal{T}_{r}\right)\right)$, which can contain at most $L\left(\mathcal{T}_{r}\right)$ roots, by Theorem 7.2.2 and our definition of $\epsilon_{c}$. The remaining roots in $\mathcal{T}_{r}$ are roots $w^{\prime} \notin B\left(t, \epsilon_{c} h\left(\mathcal{T}_{r}\right)\right)$ and we know there are at least $D\left(\mathcal{T}_{r}\right)-L\left(\mathcal{T}_{r}\right)$ of these. We thus find, using the factorisation (8.2.14), that, for $t \notin N(\mathcal{R})$ with $|t| \sim h\left(\mathcal{T}_{r}\right)$,

$$
\begin{gathered}
|\Psi(t)| \gtrsim\left(\epsilon_{c}^{D\left(\mathcal{T}_{r}\right)-L\left(\mathcal{T}_{r}\right)} \epsilon_{1}^{L\left(\mathcal{T}_{r}\right)} h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{r}\right)}\right) \prod_{i=r+1}^{s} h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{i}\right)} \prod_{i=1}^{r-1} h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)} \\
\gg \epsilon_{f} h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{r}\right)+D\left(\mathcal{T}_{r+1}\right)+\ldots+D\left(\mathcal{T}_{s}\right)} \prod_{i=1}^{r-1} h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)}
\end{gathered}
$$

provided $\epsilon_{f}$ has been taken small enough, since $\epsilon_{1}^{L\left(\mathcal{T}_{r}\right)} \epsilon_{c}^{D\left(\mathcal{T}_{r}\right)-L\left(\mathcal{T}_{r}\right)}=\epsilon_{f}^{\frac{L\left(\mathcal{T}_{r}\right)}{L}} \epsilon_{c}^{D\left(\mathcal{T}_{r}\right)-L\left(\mathcal{T}_{r}\right)} \gg \epsilon_{f}$. We now have an explicit lower estimate on the size of $\Psi(t)$ for $t \notin N(\mathcal{R})$. It is now possible to make sense of Lemma 8.2.3 as an error expression. Indeed, for $|t| \sim h\left(\mathcal{T}_{r}\right)$, we see that

$$
\begin{equation*}
|E(t)| \leq \epsilon_{f} h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{r}\right)+D\left(\mathcal{T}_{r+1}\right)+\ldots+D\left(\mathcal{T}_{s}\right)} \prod_{i=1}^{r-1} h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)} \tag{8.2.15}
\end{equation*}
$$

so that for $|t| \sim h\left(\mathcal{T}_{r}\right)$ with $t \notin N(\mathcal{R})$,

$$
|E(t)| \ll|\Psi(t)| .
$$

In particular, for $t \notin N(\mathcal{R})$,

$$
\widetilde{\Psi}(t)=\Psi(t)+E(t) \neq 0
$$

so that, for the roots of $\widetilde{\Psi}$,

$$
\widetilde{\mathcal{R}} \subset N(\mathcal{R})
$$

It remains for us to show that each cell of $N(\mathcal{R})$ contains the same number of roots of $\Psi$ and $\widetilde{\Psi}$. If there was no error term and all roots were isolated, this would be easy as both functions would be equal to zero on $\mathcal{R} \subset N(\mathcal{R})$. In fact, the argument requires more precision. In order to account for the error term and cells containing multiple roots, we must consider the size of the functions $\Psi$ and $\widetilde{\Psi}$ at a suitable distance from the roots $\mathcal{R}$. In particular, we estimate the size of the functions close to the boundary of $N(\mathcal{R})$. We know that $\mathcal{R} \subset N(\mathcal{R})$ and also that $\widetilde{\mathcal{R}} \subset N(\mathcal{R})$, which will allow us to estimate the size of the functions using their factorisations, (8.2.14).

For the remainder of the proof, we fix a cell $B=B(u, R)$, containing $m$ roots, with centre $u$ and radius $R=\epsilon_{m} h\left(\mathcal{T}_{r}\right)$. We consider the size of the functions $\Psi$ and $\widetilde{\Psi}$ at the boundary of $B^{*}$, where $B^{*}=B(u, 2 R)$ is the double of $B$. For $t \in \partial B^{*}$, using the factorisation (8.2.14) and the fact that $d(t, B)=\epsilon_{m} h\left(\mathcal{T}_{r}\right)$ with $B$ containing $m$ roots,

$$
\begin{equation*}
|\Psi(t)| \lesssim \epsilon_{m}^{m} h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{s}\right)+\ldots+D\left(\mathcal{T}_{r}\right)} \prod_{i=1}^{r-1} h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)} \tag{8.2.16}
\end{equation*}
$$

By the covering Lemma 8.2.7, we can verify that points on $\partial B^{*}$ are well separated from cells other than $B$. Indeed, $d\left(\partial B^{*}, B\right)=R=\epsilon_{m} h\left(\mathcal{T}_{r}\right)$ and, if we take a distinct cell $B^{\prime}$, then Lemma 8.2.7 tells us that $d\left(B, B^{\prime}\right) \geq \frac{1}{4} \epsilon_{m+1} h\left(\mathcal{T}_{r}\right)$ so that $d\left(\partial B^{*}, B^{\prime}\right) \gtrsim \epsilon_{m+1} h\left(\mathcal{T}_{r}\right)$. In particular, since
$\widetilde{\mathcal{R}} \subset N(\mathcal{R})$, for $t \in \partial B^{*}$ and any root $\tilde{w}^{\prime} \in \widetilde{\mathcal{T}}_{r}$ with $\tilde{w}^{\prime} \notin B$,

$$
\begin{equation*}
\left|w^{\prime}-t\right| \gtrsim \epsilon_{m+1} h\left(\mathcal{T}_{r}\right) . \tag{8.2.17}
\end{equation*}
$$

We also note that, by Theorem 7.2 .2 , for $t \in \partial B^{*}, B\left(t, \epsilon_{c} h\left(\mathcal{T}_{r}\right)\right)$ can contain at most $L\left(\mathcal{T}_{r}\right)$ roots of $\widetilde{\Psi}$. Let us now suppose that the given cell, $B$, contains only $\tilde{m}<m$ roots of $\widetilde{\Psi}$. For $t \in \partial B^{*} \subset N(\mathcal{R})^{c}$, there can be at most $L\left(\mathcal{T}_{r}\right)-\tilde{m}$ roots $\tilde{w}^{\prime} \notin B$ for which $\tilde{w}^{\prime} \in B\left(t, \epsilon_{c} h\left(\mathcal{T}_{r}\right)\right)$. We know that, for $t \in \partial B^{*},|t| \sim h\left(\mathcal{T}_{r}\right)$. Therefore, for $t \in \partial B^{*}$, using the factorisation (8.2.14) and also the distance estimate (8.2.17),

$$
\begin{gather*}
|\widetilde{\Psi}(t)| \gtrsim\left(\epsilon_{c}^{D\left(\mathcal{T}_{r}\right)-L\left(\mathcal{T}_{r}\right)} \epsilon_{m}^{\tilde{m}} \epsilon_{m+1}^{L\left(\mathcal{T}_{r}\right)-\tilde{m}} h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{r}\right)}\right) h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{s}\right)+\ldots+D\left(\mathcal{T}_{r+1}\right)} \prod_{i=1}^{r-1} h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)} \\
\gtrsim\left(\epsilon_{c}^{D\left(\mathcal{T}_{r}\right)-L\left(\mathcal{T}_{r}\right)} \epsilon_{m}^{m-1} \epsilon_{m+1}^{L\left(\mathcal{T}_{r}\right)-(m-1)} h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{r}\right)}\right) h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{s}\right)+\ldots+D\left(\mathcal{T}_{r+1}\right)} \prod_{i=1}^{r-1} h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)} \\
\gg \epsilon_{m}^{m} h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{s}\right)+\ldots+D\left(\mathcal{T}_{r}\right)} \prod_{i=1}^{r-1} h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)} \tag{8.2.18}
\end{gather*}
$$

because $\epsilon_{c}^{D\left(\mathcal{T}_{r}\right)-L\left(\mathcal{T}_{r}\right)} \epsilon_{m+1}^{L\left(\mathcal{T}_{r}\right)-(m-1)} \gg \epsilon_{m}$. This contradicts (8.2.15): considering (8.2.16) and (8.2.18) together, we have that

$$
|E(t)| \gtrsim|\widetilde{\Psi}(t)|-|\Psi(t)| \gg \epsilon_{m}^{m} h\left(\mathcal{T}_{r}\right)^{D\left(\mathcal{T}_{s}\right)+\ldots+D\left(\mathcal{T}_{r}\right)} \prod_{i=1}^{r-1} h\left(\mathcal{T}_{i}\right)^{D\left(\mathcal{T}_{i}\right)}
$$

Therefore, $B$ must contain $m$ roots of $\widetilde{\Psi}$.

## Chapter 9

## Oscillatory integral estimates

In this section, we prove the oscillatory integral estimates given by Theorems 6.0.5 and 6.0.7. Let us first present an example which shows why the condition $k_{L-s} \geq s+1$ is required in Theorem 6.0.7.

Proposition 9.0.1. For $k \geq m$, define $\Phi(t)$ by

$$
\begin{equation*}
\Phi(0)=0 \text { and } \Phi^{\prime}(t)=y_{L}\left(t^{k}-1\right)^{L-m+1} \tag{9.0.1}
\end{equation*}
$$

We set $y_{1}=\ldots y_{m-1}=0$, the remaining $(x, y)$ parameters are defined implictly by $\Phi^{\prime}(t)=$ $x+\sum_{j=m}^{L} y_{j} t^{(j-m+1) k}$. For this polynomial, we have that

$$
\left|\int e^{i \Phi(t)} d t\right| \gtrsim\left|y_{m}\right|^{-\frac{1}{L-m+2}},
$$

for $(x, y)$ in a region $R$ containing arbitrarily large $y_{m}$. In particular, for the estimate (6.0.3) to hold in the region $R$, we require that $k=k_{m} \geq L-m+1$.

Remark 9.0.2. Proposition 9.0.1 is a direct consequence of the following (equivalent) proposition, which amounts to a change in notation, and our testing the inequality

$$
\left|y_{m}\right|^{-\frac{1}{L-m+2}} \lesssim\left|y_{m}\right|^{-\frac{1}{k_{m}+1}},
$$

as $\left|y_{m}\right| \rightarrow \infty$. The inequality leads to the necessary condition $k_{m} \geq L-m+1$.
Proposition. Here, we set $\tilde{L}=L-m+1$. For $k \geq m$, define $\Phi(t)$ by

$$
\begin{equation*}
\Phi(0)=0 \text { and } \Phi^{\prime}(t)=y_{\tilde{L}}\left(t^{k}-1\right)^{l} \tag{9.0.2}
\end{equation*}
$$

The $(x, y)$ parameters are defined implictly by $\Phi^{\prime}(t)=x+\sum_{j=1}^{\tilde{L}} y_{j} t^{j k}$. For this polynomial, we have that

$$
\left|\int e^{i \Phi(t)} d t\right| \gtrsim\left|y_{1}\right|^{-\frac{1}{l+1}}
$$

for $(x, y)$ in a region $R$ containing arbitrarily large $y_{1}$.
Proof. Note that $\Phi^{\prime}$ has $l$ roots at each of the $k$ th roots of unity. The parameters $x$ and $y$ are defined implicitly by the equation (9.0.2). For example, $x=y_{\tilde{L}}(-1)^{l}$. Henceforth, we vary $\left|y_{1}\right| \gtrsim 1$ (corresponding to varying $\left.y_{\tilde{L}}\right)$. We also see that

$$
\begin{equation*}
\Phi^{(l+1)}(1)=y_{\tilde{L}} \prod_{w \in \mathcal{R}, w \neq 1}(w-1)=e^{\pi i(k-1)} y_{\tilde{L}}, \tag{9.0.3}
\end{equation*}
$$

where $\mathcal{R}$ is the set of roots of $\Phi^{\prime}$.
Stationary phase analysis will be used to analyse the contributions to the oscillatory integrals from the integration about the real critical points of the phase. Note that 1 is a real critical
point with multiplicity $l$ (this may also be true for -1 if $k$ is even). We are interested in taking large $y_{1}$. Heuristically, we expect to have an asymptotic expression for $I(x, y)$ as the sum of one or two oscillating terms, depending on whether -1 is also a critical point of the phase. In particular, we hope to find that, in some appropriate sense,

$$
I(x, y) \sim c_{-1} e^{i \Phi(-1)} \frac{1}{\left|\Phi^{(l+1)}(-1)\right|^{\frac{1}{l+1}}}+c_{1} e^{i \Phi(1)} \frac{1}{\left\lvert\, \Phi^{(l+1)}(1)^{\frac{1}{l+1}}\right.},
$$

with $c_{1}$ a non-zero real constant.
Let us now carry out this procedure explicitly. It is natural to expand $\Phi$ about the critical points and then find suitable local coordinates adapted to the degeneracy of the critical point. We integrate by parts to obtain

$$
\begin{gathered}
\Phi(1+t)-\Phi(1)=t \int_{0}^{1} \Phi^{\prime}(1+s t) d s \\
=t \int_{0}^{1} \frac{(-1)^{l}}{l!} \frac{d^{l}}{d s^{l}}\left((1-s)^{l}\right) \Phi^{\prime}(1+s t) d s \\
=\frac{t^{l+1}}{l!} \int_{0}^{1}(1-s)^{l} \Phi^{(l+1)}(1+s t) d s \\
=\frac{t^{l+1}}{(l+1)!} \Phi^{(l+1)}(1) G(t) \\
=\lambda t^{l+1} G(t)
\end{gathered}
$$

where $G(t)=(l+1) \int_{0}^{1}(1-s)^{l} \frac{\Phi^{(l+1)}(1+s t)}{\Phi^{(l+1)}(1)} d s$ and $\lambda=\frac{1}{(l+1)!} \Phi^{(l+1)}(1)=\frac{c}{(l+1)!} y_{\tilde{L}}$, by (9.0.3). Note that $G(0)=1$. We restrict our attention to $y_{\tilde{L}}>0$ or $y_{\tilde{L}}<0$, depending on $l$ and $k$, to ensure that $\lambda>0$.

Let us set $u=t G(t)^{\frac{1}{l+1}}$. We show that this defines a sensible change of variables for $|t| \leq \delta_{1}$, where $\delta_{1}$ is some suitable small parameter. We see that

$$
\begin{equation*}
\frac{d u}{d t}=G(t)^{\frac{1}{l+1}}+\frac{t}{l+1} G^{\prime}(t) G(t)^{-\frac{l-1}{l}} \tag{9.0.4}
\end{equation*}
$$

For $|t| \leq \delta_{1}$, we easily see that $\frac{d u}{d t} \sim 1$, so the change of variables is well defined. Indeed, $G(0)=1, G(t) \sim 1$, and $\left|G^{\prime}(t)\right| \lesssim 1$, for sufficiently small $|t|<\delta_{1}$. Thus, for $|t|<\delta_{1}$, we have that $\frac{d u}{d t} \sim 1$. Corresponding to each $u \in\left[-\delta_{1} G\left(-\delta_{1}\right)^{\frac{1}{l+1}}, \delta_{1} G\left(\delta_{1}\right)^{\frac{1}{l+1}}\right]$, there is a unique $t=t(u)$. Indeed, the whole situation regarding $G$ is wholly analogous to the proof of Proposition 4.1.1 and we omit these details. We define the Jacobian factor

$$
h(u)=\frac{d t}{d u}=\frac{1}{G(t(u)))^{\frac{1}{l+1}}+\frac{t}{l+1} G^{\prime}(t(u)) G(t(u))^{-\frac{l-1}{l}}} .
$$

We introduce a bump function $\phi \in C_{c}(\mathbb{R})$ with supp $\phi \subset\left[-\delta_{1}, \delta_{1}\right]$ and $\phi(t)=1$ for $|t| \leq \delta_{1} / 2$ to localise the oscillatory integral about the critical points. In the case that $k$ is even, there are two critical points, at 1 and -1 , and

$$
\begin{gather*}
I(x, y)=\int e^{i \Phi(t)} d t \\
=\int e^{i \Phi(t)}(1-\phi(1+t)-\phi(-1+t)) d t+\int e^{i \Phi(t)} \phi(1+t) d t+\int e^{i \Phi(t)} \phi(-1+t) d t \\
=: I_{\mathrm{err}}(x, y)+I_{-1}(x, y)+I_{1}(x, y) \tag{9.0.5}
\end{gather*}
$$

Recall from (9.0.3) that $\Phi^{(l+1)}(1)=c y_{\tilde{L}}$. Set $\lambda=\frac{c}{(l+1)!} y_{\tilde{L}}=\frac{1}{(l+1)!} \Phi^{(l+1)}(1)$. We get

$$
I_{1}(x, y)=e^{i \Phi(1)} \int e^{i(\Phi(1+t)-\Phi(1))} \phi(t) d t
$$

$$
\begin{gathered}
=e^{i \Phi(1)} \int e^{i \frac{u^{l+1}}{(l+1)!} \Phi^{(l+1)}(1)} \phi(t(u)) h(u) d u \\
=e^{i \Phi(1)} \int e^{i \lambda u^{l+1}} \phi(t(u)) h(u) d u \\
=e^{i \Phi(1)} \int e^{i \lambda u^{l+1}} F(u) d u \\
=e^{i \Phi(1)} \int e^{i \lambda u^{l+1}} F(0) d u+e^{i \Phi(1)} \int e^{i \lambda u^{l+1}}(F(u)-F(0)) d u
\end{gathered}
$$

where $F(u)=\phi(t(u)) h(u)$. The expression is the sum of a principal term and an error term, $p_{1}\left(y_{\tilde{L}}\right)+e_{1}\left(y_{\tilde{L}}\right)$, where

$$
p_{1}\left(y_{\tilde{L}}\right)=e^{i \Phi(1)} \int e^{i \lambda u^{l+1}} F(0) d u
$$

We control the error carefully by a dyadic decomposition and bounds obtained from the integral's non-stationary phase. The argument works provided we take $y_{\tilde{L}}$ large enough.

We take a bump function $\psi_{0} \in C_{c}^{\infty}(\mathbb{R})$ with $\psi_{0}(u)=1$ for $|u| \leq E$ and $\psi_{0}(u)=0$ for $|u|>2 E$, for some large constant $E$. Additionally, we set $\psi(u)=\psi_{0}(u)-\psi_{0}(2 u)$ and $\psi_{j}(u)=$ $\psi\left(2^{-j} u\right)$ so that $1=\sum_{j \geq 0} \psi_{j}(u)$. Then we see that

$$
\begin{gathered}
e_{1}\left(y_{\tilde{L}}\right)=\int e^{i \lambda u^{l+1}}(F(u)-F(0)) d u \\
=\lambda^{-\frac{1}{l+1}} \int e^{i v^{l+1}}\left(F\left(\lambda^{-\frac{1}{l+1}} v\right)-F(0)\right) d v \\
=\lambda^{-\frac{1}{l+1}} \sum_{j \geq 0} \int e^{i v^{l+1}}\left(F\left(\lambda^{-\frac{1}{l+1}} v\right)-F(0)\right) \psi_{j}(v) d v \\
=\sum_{j \geq 0} I_{j}(x, y) .
\end{gathered}
$$

For $j=0$ we see that, provided $y_{\tilde{L}}$ (and correspondingly $\lambda$ ) is taken large enough, by continuity of $F$,

$$
\begin{gather*}
\left|I_{0}(x, y)\right|=\lambda^{-\frac{1}{l+1}}\left|\int e^{i v^{l+1}}\left(F\left(\lambda^{-\frac{1}{l+1}} v\right)-F(0)\right) \psi_{0}(v) d v\right| \\
=\lambda^{-\frac{1}{l+1}} \int\left|\left(F\left(\lambda^{-\frac{1}{l+1}} v\right)-F(0)\right) \psi_{0}(v)\right| d v \\
\ll \lambda^{-\frac{1}{l+1}} \tag{9.0.6}
\end{gather*}
$$

Next, we see

$$
\begin{aligned}
& I_{j}(x, y)=\lambda^{-\frac{1}{l+1}} \int e^{i v^{l+1}}\left(F\left(\lambda^{-\frac{1}{l+1}} v\right)-F(0)\right) \psi_{j}(v) d v \\
= & \lambda^{-\frac{1}{l+1}} 2^{j} \int e^{i 2^{j(l+1)} u^{l+1}}\left(F\left(\lambda^{-\frac{1}{l+1}} 2^{j} u\right)-F(0)\right) \psi(u) d u
\end{aligned}
$$

Utilising the non-stationary phase, we can integrate by parts. However, note that, in order to have $\lambda^{-\frac{1}{l+1}} 2^{j} u \in \operatorname{supp} F \subset B\left(0, \delta_{1}\right)$ for $u \in \operatorname{supp} \psi \subset A_{\frac{1}{4} E, 2 E}(0)$, we require $\lambda^{-\frac{1}{l+1}} 2^{j-2} E \leq$ $\delta_{1}$. For these small $j$, we must be more careful in our integration by parts, because of the contribution of the function $F\left(\lambda^{-\frac{1}{l+1}} 2^{j}.\right)$. In fact, let us more precisely integrate by parts twice. We see that

$$
\left|I_{j}(x, y)\right|=2^{j}\left|\int e^{i 2^{j(l+1)} u^{l+1}}\left(F\left(\lambda^{-\frac{1}{l+1}} 2^{j} u\right)-F(0)\right) \psi(u) d u\right|
$$

$$
\begin{gather*}
=2^{j}\left|\int \frac{1}{(l+1) 2^{j(l+1)} u^{l}} \frac{d}{d u}\left(e^{i 2^{j(l+1)} u^{l+1}}\right)\left(F\left(\lambda^{-\frac{1}{l+1}} 2^{j} u\right)-F(0)\right) \psi(u) d u\right| \\
=2^{j}\left|\int e^{i 2^{j(l+1)} u^{l+1}} \frac{d}{d u}\left(\left(F\left(\lambda^{-\frac{1}{l+1}} 2^{j} u\right)-F(0)\right) \frac{1}{(l+1) 2^{j(l+1)} u^{l}} \psi(u)\right) d u\right| \\
=2^{j}\left|\int \frac{1}{(l+1) 2^{j(l+1)} u^{l}} \frac{d}{d u}\left(e^{i 2^{j(l+1)} u^{l+1}}\right) \frac{d}{d u}\left(\left(F\left(\lambda^{-\frac{1}{l+1}} 2^{j} u\right)-F(0)\right) \frac{1}{(l+1) 2^{j(l+1)} u^{l}} \psi(u)\right) d u\right| \\
=2^{j} \frac{1}{\left((l+1) 2^{j(l+1)}\right)^{2}}\left|\int e^{i 2^{j(l+1)} u^{l+1}} \frac{d}{d u}\left(\frac{1}{u^{l}} \frac{d}{d u}\left(\left(F\left(\lambda^{-\frac{1}{l+1}} 2^{j} u\right)-F(0)\right) \frac{1}{u^{l}} \psi(u)\right)\right) d u\right| \\
\leq 2^{j} \frac{1}{\left((l+1) 2^{j(l+1)}\right)^{2}} \int\left|\frac{d}{d u}\left(\frac{1}{u^{l}} \frac{d}{d u}\left(\left(F\left(\lambda^{-\frac{1}{l+1}} 2^{j} u\right)-F(0)\right) \frac{1}{u^{l}} \psi(u)\right)\right)\right| d u \\
\quad \lesssim 2^{j} \frac{1}{2^{2 j(l+1)}} \int_{\operatorname{supp} \psi}\left\|F\left(\lambda^{-\frac{1}{l+1}} 2^{j} \cdot\right)-F(0)\right\|_{L^{\infty}(\operatorname{supp} \psi)} d u  \tag{9.0.7}\\
+2^{j} \frac{1}{2^{2 j(l+1)}} \int_{\operatorname{supp} \psi}\left(\lambda^{-\frac{1}{l+1}} 2^{j}\left\|F^{\prime}\left(\lambda^{-\frac{1}{l+1}} 2^{j} \cdot\right)\right\|_{L^{\infty}(\operatorname{supp} \psi)}+\lambda^{\left.-\frac{2}{l+1} 2^{2 j}\left\|F^{\prime \prime}\left(\lambda^{-\frac{1}{l+1}} 2^{j} \cdot\right)\right\|_{L^{\infty}(\operatorname{supp} \psi)}\right) d u .} .\right. \tag{9.0.8}
\end{gather*}
$$

We need to bound the sum of these terms over $j \geq 1$.
We readily obtain an error bound on the terms in (9.0.8). Here we use the bounds $\left\|F^{\prime}\right\|_{L^{\infty}} \lesssim$ 1 and $\left\|F^{\prime \prime}\right\|_{L^{\infty}} \lesssim 1$, which can be established as previously for (4.1.25). Then, by ensuring $y_{\tilde{L}}$ is large enough, we see that, the sum over $j \geq 1$ of (9.0.8) is

$$
\begin{gather*}
\sum_{j \geq 1} 2^{j} \frac{1}{2^{2 j(l+1)}}\left(\lambda^{-\frac{1}{l+1}} 2^{j}+\lambda^{-\frac{2}{l+1}} 2^{2 j}\right) \\
\lesssim \sum_{j \geq 1} 2^{3 j} \frac{1}{2^{4 j}} \lambda^{-\frac{1}{l+1}} \ll 1 \tag{9.0.9}
\end{gather*}
$$

To bound the sum over $j \geq 1$ of the remaining term, (9.0.7), we split the summation. We first choose $j_{0}$ large enough so that, for $j \geq j_{0}, \frac{2^{j}}{2^{2 j(l+1)}} \ll 1$. In particular, using the bound $\|F\|_{L^{\infty}(\operatorname{supp} \psi)} \lesssim 1$, we then bound the corresponding piece of the sum of (9.0.7) by

$$
\begin{equation*}
\lesssim \sum_{j \geq j_{0}} 2^{j} \frac{2^{j}}{2^{2 j(l+1)}} \ll 1 \tag{9.0.10}
\end{equation*}
$$

Now, since $j_{0}$ is fixed, provided $y_{\tilde{L}}$ is chosen large enough depending on $j_{0}, \| F\left(\lambda^{-\frac{1}{l+1}} 2^{j}.\right)-$ $F(0) \|_{L^{\infty}(\operatorname{supp} \psi)} \ll 1$ for $1 \leq j \leq j_{0}$, by continuity, so that summing the remaining expression from (9.0.7), we have the bound

$$
\begin{equation*}
\lesssim \sum_{1 \leq j \leq j_{0}} 2^{j} \frac{1}{2^{j(l+1)}}\left\|F\left(\lambda^{-\frac{1}{l+1}} 2^{j} \cdot\right)-F(0)\right\|_{L^{\infty}(\operatorname{supp} \psi)} \ll 1 . \tag{9.0.11}
\end{equation*}
$$

Bringing together the bounds (9.0.6), (9.0.9), (9.0.10), and (9.0.11), we have that

$$
\left|e_{1}\left(y_{\tilde{L}}\right)\right| \ll \lambda^{-\frac{1}{l+1}} .
$$

In the case that $k$ is even, there is another principal term, $e_{-1}$, for which our above analysis bounding $e_{1}\left(y_{\tilde{L}}\right)$ can be repeated. In particular, for $j \in\{-1,1\}$, we can find that

$$
\begin{equation*}
\left|e_{j}\left(y_{\tilde{L}}\right)\right| \ll \lambda^{-\frac{1}{l+1}} \tag{9.0.12}
\end{equation*}
$$

We must still consider the error term $I_{\text {err }}$, (9.0.5). For $t \notin B\left(1, \frac{\delta_{1}}{2}\right) \cup B\left(-1, \frac{\delta_{1}}{2}\right)$, it is easy to see that $\left|\Phi^{\prime}(t)\right| \gtrsim\left|y_{\tilde{L}}\right| \delta_{1}$. To analyse $I_{\text {err }}$, which is conditionally convergent by truncation with a smooth cutoff, we dyadically decompose the integration. We use the same partition of unity
$\sum_{j \geq 0} \psi_{j}(t)=1$ as above. We have that

$$
I_{\mathrm{err}}(x, y)=\lim _{a \rightarrow \infty} \sum_{j=0}^{a} I_{\mathrm{err}, j}(x, y)
$$

where

$$
\left.I_{\mathrm{err}, j}(x, y)=\int e^{i \Phi(t)}(1-\phi(1+t)-\phi(-1+t))\right) \psi_{j}(t) d t
$$

We use that principle of non-stationary phase to bound each of the summands. Making the change of variables $s=2^{-j} t$ and setting $y_{\tilde{L}} \Phi_{j}(s)=2^{-j k l} \Phi\left(2^{j} s\right)$, we see that, for $j>1$,

$$
I_{\mathrm{err}, j}(x, y)=2^{j} \int e^{i 2^{j k l} \Phi_{j}(s)}\left(1-\phi\left(1+2^{j} s\right)-\phi\left(-1+2^{j} s\right)\right) \psi(s) d s
$$

For $j \geq 1$, and $|s| \sim 1$, we can see that $\left|\Phi_{j}^{\prime}(s)\right| \gtrsim 1$ and also $\left\|\Phi_{j}\right\|_{C^{2}(\operatorname{supp} \psi)} \lesssim 1$. Note that, for $s \in \operatorname{supp} \psi$, since $|s| \sim E$ for some large $E, s \notin \operatorname{supp} \phi\left(1+2^{j} \cdot\right) \cup \phi\left(-1+2^{j} \cdot\right)$. Therefore, for $j \geq 1$, by the non-stationary phase lemma, Lemma 0.0.1,

$$
\begin{equation*}
\left|I_{\mathrm{err}, j}(x, y)\right| \lesssim 2^{j} \frac{1}{2^{j k l}} \frac{1}{\left|y_{\tilde{L}}\right|} . \tag{9.0.13}
\end{equation*}
$$

It remains to consider the term $I_{\text {err }, 0}(x, y)$. We previously discussed how, for $j=0,\left|\Phi_{0}^{\prime}(s)\right| \gtrsim \delta_{1}$ over the region of integration. Thus we can apply the non-stationary phase lemma to see that

$$
\begin{equation*}
\left|I_{\text {err }, 0}(x, y)\right| \lesssim \delta_{1} \frac{1}{\left|y_{\tilde{L}}\right|} \tag{9.0.14}
\end{equation*}
$$

Summing the bounds (9.0.13) and (9.0.14), we obtain that

$$
\left|I_{\mathrm{err}}(x, y)\right| \lesssim \delta_{1} \frac{1}{\left|y_{\tilde{L}}\right|}
$$

In particular, provided we take $y_{\tilde{L}}$ large enough (depending on $\delta_{1}$ ),

$$
\begin{equation*}
\left|I_{\mathrm{err}}(x, y)\right| \ll \lambda^{-\frac{1}{l+1}} \tag{9.0.15}
\end{equation*}
$$

When we have taken all the error terms into account we have that

$$
I(x, y)=I_{\mathrm{err}}(x, y)+p_{1}\left(y_{\tilde{L}}\right)+e_{1}\left(y_{\tilde{L}}\right)+p_{-1}\left(y_{\tilde{L}}\right)+e_{-1}\left(y_{\tilde{L}}\right),
$$

with

$$
p_{1}\left(y_{\tilde{L}}\right)=e^{i \Phi(1)} \int e^{i \lambda u^{l+1}} F(0) d u
$$

and, accounting for the fact there is no critical point at -1 if $k$ is odd,

$$
p_{-1}\left(y_{\tilde{L}}\right)=\mathbb{1}_{2 \mathbb{Z}}(k) e^{i \Phi(-1)} \int e^{i \lambda u^{l+1}} F(0) d u .
$$

From the bounds (9.0.12) and (9.0.15) we have that

$$
\left|I_{\mathrm{err}}(x, y)+e_{1}\left(y_{\tilde{L}}\right)+e_{-1}\left(y_{\tilde{L}}\right)\right| \ll \lambda^{-\frac{1}{l+1}}
$$

In the case that $k$ is odd, we see that $\left|p_{1}\left(y_{\tilde{L}}\right)\right| \sim \lambda^{-\frac{1}{l+1}}$ so that

$$
|I(x, y)| \sim \lambda^{-\frac{1}{l+1}}
$$

as desired. In the case that $k$ is even, we must consider the interaction of $p_{1}\left(y_{\tilde{L}}\right)$ and $p_{-1}\left(y_{\tilde{L}}\right)$.

We then see that $\Phi$ is odd from

$$
\Phi(0)=0 \text { and } \Phi^{\prime}(t)=y_{\tilde{L}}\left(t^{k}-1\right)^{L-m+1}
$$

Thus we have that $\Phi(-1)=-\Phi(1)$ and, by considering the real component of $e^{i \Phi(1)}$, we see that $\left|e^{i \Phi(1)}+e^{i \Phi(-1)}\right| \sim 1$ so that

$$
|I(x, y)| \gtrsim\left|p_{1}\left(y_{\tilde{L}}\right)+p_{-1}\left(y_{\tilde{L}}\right)\right|-\left|I_{\mathrm{err}}(x, y)+e_{1}\left(y_{\tilde{L}}\right)+e_{-1}\left(y_{\tilde{L}}\right)\right| \sim \lambda^{-\frac{1}{l+1}} .
$$

In either case,

$$
\begin{equation*}
|I(x, y)| \gtrsim \lambda^{-\frac{1}{l+1}} . \tag{9.0.16}
\end{equation*}
$$

Let us now turn to the proof of the oscillatory integral bounds of Hickman and Wright.
Proof of Theorems 6.0.5 and 6.0.7. We first give the proof in the case that $k_{m} \geq L$ (Theorem 6.0.5) and later give the technical case splitting required to obtain the result when $k_{m} \geq L-m+1$ (Theorem 6.0.7).

Without loss of generality, we can prove the result for $\Phi$ as in (6.0.2) chosen such that $\max _{1 \leq j \leq L}\left|y_{j}\right|=1$. To see this, suppose we have a phase $\widetilde{\Phi}=\widetilde{\Phi}_{\tilde{x}, \tilde{y}}$ of the same form as (6.0.2), where $\tilde{y} \in \mathbb{R}^{L}$ is unrestricted. We wish to show $|J(\tilde{x}, \tilde{y})| \lesssim \min _{j}\left|\tilde{y}_{j}\right|^{-\frac{1}{k_{j}+1}}$, where $J(\tilde{x}, \tilde{y})=\int e^{i \tilde{\Phi}(s)} d s$, with $\tilde{\Phi}(s)=\tilde{x} s+\frac{\tilde{y}_{1}}{k_{1}+1} s^{k_{1}+1}+\ldots+\frac{\tilde{y}_{L}}{k_{L}+1} s^{k_{L}+1}$. By making a change of variables in the $s$ coordinate, it suffices to prove $|J(x, y)| \lesssim 1$, where $J(x, y)=\int e^{i \Phi(s)} d s$, with $\Phi(s)=x s+\frac{y_{1}}{k_{1}+1} s^{k_{1}+1}+\ldots+\frac{y_{L}}{k_{L}+1} s^{k_{L}+1}$ such that $\max _{1 \leq j \leq L}\left|y_{j}\right|=1$. Indeed, we set $\sigma=\max _{j}\left|\tilde{y}_{j}\right|^{\frac{1}{k_{j}+1}}$ and make the change of variables $\sigma s=t$ in the integral expression for $J$. Writing $\widetilde{\Phi}(s)$ in terms of $t$ we see that

$$
\begin{gathered}
\widetilde{\Phi}(t)=\tilde{x} t+\frac{\tilde{y}_{1}}{k_{1}+1} t^{k_{1}+1}+\ldots+\frac{\tilde{y}_{L}}{k_{L}+1} t^{k_{L}+1} \\
=x s+\frac{y_{1}}{k_{1}+1} s^{k_{1}+1}+\ldots+\frac{y_{L}}{k_{L}+1} s^{k_{L}+1}
\end{gathered}
$$

where $x=\sigma^{-1} \tilde{x}$ and $y_{j}=\sigma^{-\left(k_{j}+1\right)} \tilde{y}_{j}$. By definition of $\sigma, \max _{1 \leq j \leq L}\left|y_{j}\right|=1$.
We now use the root structure Theorem 6.0.1 to prove the desired inequality

$$
|J(x, y)|=\left|\int e^{i \Phi(t)} d t\right| \lesssim 1
$$

Let us consider the case where $\max _{1 \leq j \leq L}\left|y_{j}\right|^{\frac{1}{k_{j}}}=\left|y_{m}\right|^{\frac{1}{k_{m}}}=1$. Here, the appropriate cluster estimate to consider is the $k_{m}$-cluster estimates, namely the bound we desire is obtained with a suitable choice of $\mathcal{C}$ such that $|\mathcal{C}|=k_{m}$. We work to show that, for any $z_{j}$, there exists a $k_{m}$-cluster $\mathcal{C}$ such that

$$
\begin{equation*}
\prod_{l \notin \mathcal{C}}\left|z_{j}-z_{l}\right| \gtrsim\left|z_{1} z_{2} \ldots z_{k_{L}-k_{m}}\right| \tag{9.0.17}
\end{equation*}
$$

Once this has been established we note that

$$
\left|z_{1} z_{2} \ldots z_{k_{L}-k_{m}}\right| \gtrsim\left|S_{k_{L}-k_{m}}\right|=\left|\frac{y_{m}}{y_{L}}\right|=\left|\frac{1}{y_{L}}\right|
$$

so that we can apply the Phong and Stein estimate, Theorem 6.0.6, to establish that

$$
|J(x, y)| \lesssim\left(\frac{1}{\left|y_{L} z_{1} z_{2} \ldots z_{k_{L}-k_{m}}\right|}\right)^{\frac{1}{k_{m}+1}} \lesssim 1
$$

By Theorem 6.0.1, given a root of $\Phi^{\prime}, z$, there are at most $L$ other roots in $B(z, \epsilon|z|)$, where
$\epsilon$ is some suitable small constant. For a root $z_{j} \notin B(z, \epsilon|z|)$,

$$
\begin{equation*}
\left|z_{j}-z\right| \geq \max \left\{\epsilon|z|,\left|z_{j}\right|-|z|\right\} . \tag{9.0.18}
\end{equation*}
$$

We now take an arbitrary root $z$ and construct an appropriate $k_{m}$-cluster containing $z$. The cluster we will construct will depend on what tier $z$ is in. If there are not enough roots smaller than $z$, then we will just put the smallest $k_{m}$ roots in the cluster. As we will see, this will necessarily include all those roots in $B(z, \epsilon|z|)$. If there are many roots smaller than $z$, we will choose our cluster to contain all roots in $B(z, \epsilon|z|)$, with the remaining elements taken to be any small roots.

Recall that $D\left(\mathcal{T}_{i}\right)=\left|\mathcal{T}_{i}\right|$. Let $r$ be chosen such that $z \in \mathcal{T}_{r}$. In the case that $D\left(\mathcal{T}_{r}\right)+$ $D\left(\mathcal{T}_{r+1}\right)+\ldots+D\left(\mathcal{T}_{s}\right) \leq k_{m}$ then we choose our cluster $\mathcal{C}=\left\{z_{k_{L}}, z_{k_{L}-1}, \ldots, z_{k_{L}-k_{m}+1}\right\} \supset$ $\mathcal{T}_{r} \cup \mathcal{T}_{r+1} \cup \ldots \cup \mathcal{T}_{s}$ so that

$$
\prod_{j \notin \mathcal{C}}\left|z-z_{j}\right| \sim\left|z_{1} z_{2} \ldots z_{k_{L}-k_{m}}\right|
$$

It remains to consider the case that $D\left(\mathcal{T}_{r}\right)+D\left(\mathcal{T}_{r+1}\right)+\ldots+D\left(\mathcal{T}_{s}\right)>k_{m}$. After taking roots $\mathcal{C}_{r}=B(z, \epsilon|z|) \cap \mathcal{T}_{r}$, any choice of $k_{m}-\left|\mathcal{C}_{r}\right|$ roots from $\mathcal{T}_{r} \cup \ldots \cup \mathcal{T}_{s}$ suffices to complete our cluster $\mathcal{C}$. Indeed, we find, by (9.0.18), that

$$
\begin{gathered}
\prod_{z_{j} \notin \mathcal{C}}\left|z-z_{j}\right| \\
=\left(\prod_{z_{j} \in \mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \ldots \cup \mathcal{T}_{r-1}}\left|z-z_{j}\right|\right)\left(\prod_{j \notin \mathcal{C}, z_{j} \in \mathcal{T}_{r} \cup \mathcal{T}_{r+1} \cup \ldots \cup \mathcal{T}_{s}}\left|z-z_{j}\right|\right) \\
\gtrsim \epsilon\left(\left|z_{1} z_{2} \ldots z_{D\left(\mathcal{T}_{1}\right)+\ldots+D\left(\mathcal{T}_{r-1}\right)}\right|\right)\left(|z|^{D\left(\mathcal{T}_{r}\right)+\ldots+D\left(\mathcal{T}_{s}\right)-k_{m}}\right) \\
\gtrsim \mid z_{1} z_{2} \ldots z_{k_{L}-k_{m} \mid}
\end{gathered}
$$

For these calculations to be valid we require that there are enough spaces in $\mathcal{C}$ to contain all of those roots in $B(z, \epsilon|z|)$, we require $|\mathcal{C}|=k_{m} \geq L$, which is true by assumption.

Now, we consider the case where we can weaken the condition on $k_{m}$ to $k_{m} \geq L-m+1$. It is here we apply Theorem 6.0.3. Recall $\left|y_{m}\right|=1$. Set $\delta_{0}=1$. Since $\left|y_{m}\right| \geq \delta_{0}$, Theorem 6.0.3 applies relative to $\left|y_{m}\right|$ and $\delta_{0}$ provided that,

$$
\delta_{1} \geq\left|y_{n}\right|^{\frac{1}{k_{n}}}, \quad \text { for } \quad n>m
$$

with a suitable constant $\delta_{1}=\delta\left(\delta_{0}\right)>0$. However, in general we only have that $1 \geq\left|y_{n}\right|^{\frac{1}{k_{n}}}$ for $n>m$. Nevertheless, we will still be able to apply Theorem 6.0 . 3 by a suitable inductive procedure. Let $m(0)=m, \delta_{0}=1$, and $\delta_{1}=\delta\left(\delta_{0}\right)$ be as above. Suppose, for induction, that $m(0)<m(1)<\ldots<m(j)$ and $\delta_{0}, \delta_{1}, \ldots, \delta_{j}$ have already been defined by the inductive procedure and that $\left|y_{m(j)}\right|^{\frac{1}{k_{m(j)}}}>\delta_{j}=\delta\left(\delta_{j-1}\right)$. There are two cases. In the first case, the coefficients are such that

$$
\left|y_{n}\right|^{\frac{1}{k_{n}}} \leq \delta_{j+1}, \quad \text { for } \quad n>m(j)
$$

where $\delta_{j+1}=\delta\left(\delta_{j}\right)$ is such that Theorem 6.0.3 applies relative to the coefficient $\left|y_{m(j)}\right|>\delta_{j}^{k_{m(j)}}$. If we are in this case, we terminate the inductive procedure. Otherwise, in the second case, there exists some $m(j+1)>m(j)$ such that $\left|y_{m(j+1)}\right|^{\frac{1}{k_{m(j+1)}}}>\delta_{j+1}$, and we proceed with the induction. The process must terminate, since there are finitely many coefficients. We can thus apply Theorem 6.0.3 relative to some $\left|y_{m^{\prime}}\right|^{\frac{1}{k_{m^{\prime}}}}>\gamma^{\prime}$, for some $m^{\prime} \geq m$, with

$$
\left|y_{n}\right|^{\frac{1}{k_{n}}} \leq \delta\left(\gamma^{\prime}\right), \quad \text { for } \quad n>m^{\prime}
$$

By Theorem 6.0.3 we know that at most $L-m^{\prime}+1 \leq L-m+1$ roots can be contained in $B(w, \epsilon|w|)$ for roots $w$ in the large tiers $\mathcal{T}_{r}$. It is also a consequence of that theorem that there are at most $k_{m^{\prime}}$ roots in the small tiers, with such roots $|w| \lesssim \delta_{m^{\prime}}$. By our assumption, we
also know that $k_{m} \geq L-m+1$ so that $k_{m^{\prime}} \geq L-m^{\prime}+1$. The above argument thus carries through, constructing appropriate $k_{m^{\prime}}$-clusters $\mathcal{C}$ such that

$$
\prod_{l \notin \mathcal{C}}\left|z_{j}-z_{l}\right| \gtrsim\left|z_{1} z_{2} \ldots z_{k_{L}-k_{m^{\prime}}}\right|
$$

Once this has been established we note that

$$
\left|z_{1} z_{2} \ldots z_{k_{L}-k_{m^{\prime}}}\right| \gtrsim\left|S_{k_{L}-k_{m^{\prime}}}\right|=\left|\frac{y_{m^{\prime}}}{y_{L}}\right| \gtrsim \delta_{1}, \ldots, \delta_{m^{\prime}-1}\left|\frac{1}{y_{L}}\right| .
$$

The result follows by applying the Phong-Stein estimate, Theorem 6.0.6.
If we further restrict the region we consider, it is possible to strengthen the oscillatory integral bound (6.0.3).

Proposition 9.0.3. Set

$$
I(x, y)=\int e^{i \Phi(t)} d t
$$

Then, it is possible to bound

$$
|I(x, y)| \lesssim \epsilon \min \left|y_{j}\right|^{-\frac{1}{k_{j}+1}},
$$

provided we take $|x| \gg \max \left|y_{j}\right|^{\frac{1}{k_{j}+1}}$ with a sufficiently large constant depending on $\epsilon$.
Proof. We rescale as we have done previously, setting $s=\sigma t$ where $\sigma=\max _{j}\left|y_{j}\right|^{\frac{1}{k_{j}+1}}$ and $\tilde{y_{j}}=\sigma^{-\left(k_{j}+1\right)} y_{j}$. Note that, after rescaling, $|\tilde{x}| \gg 1$ by our assumption. It suffices for us to prove that

$$
|J(\tilde{x}, \tilde{y})|=\left|\int e^{i \tilde{\Phi}(s)} d s\right| \lesssim \epsilon,
$$

provided $|\tilde{x}| \gg 1$ with a large enough constant.
By Lemma 8.1.1, we know that all roots of $\widetilde{\Phi}^{\prime}$ are comparable to

$$
\eta=\left|\frac{\tilde{x}}{\tilde{y}_{L}}\right|^{\frac{1}{k_{L}}}
$$

Given a root $z$, we can construct a singleton cluster such that

$$
\prod_{j \notin \mathcal{C}}\left|z-z_{j}\right| \sim \eta^{k_{L}-1} \gg \eta^{k_{L}-k_{m}} \gtrsim\left|S_{k_{L}-k_{m}}\right|=\left|\frac{1}{\tilde{y}_{L}}\right| .
$$

Applying the Phong-Stein bound then gives the required result.

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[^0]:    ${ }^{1}$ This can be verified by the application of Lemma 2.1.5

[^1]:    ${ }^{2}$ This is a technical assumption we require to ensure that the graphing functions and their derivatives are suitably linearly independent.

[^2]:    ${ }^{1}$ Consider, for example, the condition (3.1.4), coming from our later Knapp example, as $D$ increases for $L=1$

[^3]:    ${ }^{2}$ Note that Theorem 5.0 .5 uses slightly different notation: it is expressed in terms of $(x, y) \in \mathbb{R} \times \mathbb{R}^{L}$, rather than $(x \cdot \omega, y) \in \mathbb{R} \times \mathbb{R}^{\tilde{L}}$.

[^4]:    ${ }^{1}$ In fact, a certain hypothesis, which amounts to a monotonicity assumption on (finitely many) subintervals of $\operatorname{supp} \phi_{0} \subset[-\delta, \delta]$, must be satisfied. However, with a simple case splitting procedure, the lemma is applicable here. The interested reader is referred to the proofs of Lemmas 4.2.3 and 4.2.4.

