DYNAMIC OF PARTICLES

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Abstract:

The studies on the dynamics of particles have recently experienced advances in experimental technique, numerical simulations and theoretical advances in experimental techniques, numerical simulations and theoretical understandings. This focus on collection aims to provide a snapshot of this fast –evolving field. We attempt to collect the cutting –edge achievements from many branches in physics and engineering, among which dynamics of particles in turbulence is the common interest.

Keywords: Force, keplers law, velocity , Newtons law

Introduction:

Firstly by the term ‘particle’ we mean that a finite mass occupying a point in the Euclidean space. In fact this is a purely mathematical concept. In physics by the term ‘particle’ we mean a mass occupying an infinitesimally small amount of volume approximately, compared to the distance between masses. For example the earth and the moon may be thought of as particles approximately, compared to their distance apart. Now we recall the Newton’s laws[1] of motion, which are the foundation of classical mechanics, as follows.

DEFINITION : 1.1

Force is basically an interaction between the objects because of which they make changes in their motion.

$$F = ma$$

F = Net force

m = mass

a = acceleration

DEFINITION: 1.2

The work done by a force is defined as the product of the force and the displacement of the particle in the direction of the force.

DEFINITION: 1.3

When a particle is subject to the action of a force which is always either towards or away from a fixed point, the particle is said to be under the action of a central force.

DEFINITION: 1.4

The path of a particle moving under a central force is called a central orbit.

DEFINITION : 1.5

A system of particles such that the distance between any two of them is always constant, is called a rigid body.

DEFINITION: 1.6

If momentum M of a moving particle at any time we mean the vector mv, where m is the mass and v is the velocity vector of the particle thus we write.

M = mv

DEFINITION: 1.7

Consider a particle of mass m, with position r and velocity V = r. .Its angular momentum (about r = 0) is defined as L = r x p = mr x v.

DEFINITION : 1.8

Energy is the capacity for doing work. It is of two types.

Potential energy.

Kinetic energy.

DEFINITION: 1.9

Potential energy of a particle is the energy that a body possesses in virtue of its position and it is measured by work done in moving the body from present position to some standard position.

DEFINITION: 1.10

The kinetic energy of a particle of mass m moving with velocity $\underline{V}$ is defined as $\frac{1}{2}m\underline{v}^2$.

First Law :

Every body continues in its state of rest, or of uniform motion in a straight line, except in so for as it be compelled by impressed force to change that state.

This law is known as the 'law of inertia'. This law gives us the definition of force.

Second Law :

The rate of change of momentum is proportional to the impressed force, and takes place in the direction in which the force acts.

From this law we get the very useful formula $\vec{P} = \frac{d}{dt}(m\vec{v})$, where m is the mass of the particle, $\vec{v}$ is the velocity vector, $\vec{P}$ is the applies force and m$\vec{v}$ is the momentum.

In general m is taken as constant and then the formula reduces to

$$\vec{P} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2} = m\vec{f}, where\ \vec{f} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This equation is known as the equation of motion. Here it is to be noted that m is not constant in the theory of relativity.

In Cartesian coordinate system, if $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ and $\vec{F} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ then the equations of motion are obtained as

$$m\frac{d^2x}{dt^2} = x, m\frac{d^2y}{dt^2} = y, m\frac{d^2z}{dt^2} = z.$$

Third law :

To every action there is an equal and opposite reaction.

Impulse : 2.1

The concept of impulse is used to describe the effect of a force on a body during a certain interval of time.

It is defined as the product of a force and the elementary time during which the force acts. Therefore its is a vector quantity.

Let $\vec{F}$ be a force acting on a body for a very small interval of time. We now assume that during the interval of time the acted force is constant.

Hence the elementary impulse during the time δt is $\vec{F}\delta t$. Therefore the total impulse (I) during a time interval [0, t] is

$$I = \int_0^t \vec{F} dt \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Using (1) in (2) we get

$$I = m\int_0^t \frac{d\vec{v}}{dt} dt$$

$$= m\int_0^t d\vec{v}$$

$$= [m\vec{v}]_0^t$$

$$= m\vec{v}(g) - m\vec{v}\,(o)$$

= change of momentum

This equation is sometimes known as the momentum equation. The force $\vec{F}$ used in the momentum equation is known as the impulsive force.

The phenomenon in which the velocity of a particle changes finitely in a very small interval of time is known as the impact. Also the impact of forces are known as the forces of impact or impulsive forces.

Firing a shot from a gun, hitting a cricket ball by a bat, striking, a nail by a hammer etc. are the examples of impact.

Note : 1

The impulse of an impulsive force is a finite quantity.The impulse of non – impulsive force in the small interval of time is very small, which can be ignored for all partical purposes.

Note : 2

The displacement of the particle during the small time interval under the action of impulsive forces is negligible.

Impulse of a force :

If a constant force F acts on particle during the interval (t_0 , t_1) then the vector I = ($t_1 - t_0$) F is called the impulse of the force F during the interval (t_0 , t_1). Direction of I is same as that of F.

Similarly if a variable force F(t) acts on a particle during the interval (t_0 , t_1) then the vector.

$I = \int_{t_0}^{t_1} Fdt$ is called the impulse of F(t) during the interval (t_0 , t_1). Direction of I is that of the time average of F over the interval (t_0 , t_1).

Impulse – momentum principle for a particle:2.1

Statement :

The change of linear momentum of a particle during a time interval is equal to the net impulse of the external forces during this interval.

Proof :

Let a force F act on a particle of mass m during the interval (t_0 , t_1) and let V be velocity of the particle at any time.

Let V_0 be the velocity of the particle of the beginning of the time interval (t_0 , t_1) and V_1 be the velocity at the end of this time interval.

Now, by the fundamental equation of dynamics, we have

$$m(dv/dt) = F$$

let I be the impulse vector of the force during the time interval (t_0 , t_1) then,

$$I = \int_{t_0}^{t_1} Fdt$$

$$= \int_{t_0}^{t_1} m\frac{dv}{dt}dt$$

$$= m\int_{v_0}^{v_1} dt$$

$$= m[V]_{v_2}^{v_1}$$

I = m V_1 – m V_0 = Change in the momentum vector.

which is known as the impulse – momentum principle.

Note :

The above principle is a re – statement of Newton's second law of motion in an integral form.

Example :

Prove that he change in the kinetic energy a particle during any time interval [0,T] is equal to the half of the dot product of the impulse and the sum of the initial and final velocities.

Proof :

Let $\vec{F}$ be the force acting on a particle of mass m. Also let under the action of the force the velocity of the particle changes from $\overrightarrow{v_0}$ to $\overrightarrow{v_T}$ during the time interval [0, T].

Now if $\vec{I}$ be the impulse of the force then we have

$$\vec{I} = m(\overrightarrow{v_T} - \overrightarrow{v_0})$$

Now the change in kinetic energy during [0, T] is equal to

$$\frac{1}{2}m[v(T)]^2 = \frac{1}{2}m[v(0)]^2 = \frac{1}{2}m[\overrightarrow{v_T}.\overrightarrow{v_T} - \overrightarrow{v_0}.\overrightarrow{v_0}]$$

Where V(T) = $[\vec{V}(T)]$

$$= \frac{1}{2}(\overrightarrow{v_T} - \overrightarrow{v_0}).(\overrightarrow{v_T} + \overrightarrow{v_0})$$

$$= \frac{1}{2}\vec{I}(\overrightarrow{v_T} + \overrightarrow{v_0})$$

This proved the result.

Work : 2.2

If a particle traverses along a curve (Space or planar) Γ with a velocity $\vec{v}$ under the action of a force $\vec{F}$, then we define the work done (w) by the force as

$$w = \int (\vec{F}.\vec{v})dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (1)$$

Where S is the arc length of Γ. Further we have

$$\vec{F}.\vec{v} = \vec{F}.\frac{d\vec{r}}{dt} = \left(\vec{F}.\frac{d\vec{r}}{ds}\right)ds = (\vec{F}.\hat{T})ds$$

In view of this relation we obtain from (3) that

$$w = \int_{\Gamma} \left(\vec{F}.\hat{T}\right)ds \text{ ……………………..(2)}$$

From the last relation it follows that if the force $\vec{F}$ acts along the normal to the path then no work will be done by the force.

Theorem : 2.2

The work done by a conservative force field in taking a particle from a point to another point is independent of the path.

Proof :

Let $\vec{F}$ be a conservative force field. Then there exists a scalar patential $\emptyset$ such that $\vec{F} = grad\emptyset$.

Also we assume that the force displaces a particle from a point A to another point B.

Now using the formula (2) for total work done we obtain

$$\int_A^B \vec{F}.d\vec{r} = \int_A^B grad\emptyset.\overrightarrow{dr} = \int_A^B d\emptyset = [\emptyset]_A^B = \emptyset(B) - \emptyset(A)$$

Which depends only on the initial and terminal points A and B hence is independent of the path joining A and B. This proves the theorem.

Internal and External Forces :

If a force originating from the inside of a body acts on that body, then it is known as internal forces.

If a force originating from the outside of a body acts on that body, then it is known as external force.

For example the mutual forces of action and reaction of the molecules of a body are the internal forces.

The gravitational force acting on a body is an example of an external force.

Theorem : 2.3

The change in the kinetic energy of a particle equals to the total work done by both the external and internal forces.

[This is known as the principle of work and energy.]

Proof :

Let us consider a particle of mass m acted by the external force $\vec{F}$ and internal force $\vec{f}$.

Also let W and w be the works done by the external and internal forces respectively.

Now if $\vec{v}$ be the velocity of the particle at any time t, then from the Newton's second law of motion we find that

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{f}$$

Taking dot product with $\vec{v}$ $we\ get$

$$m\frac{d\vec{v}}{dt}.\vec{v} = \vec{F}.\vec{v} + \vec{f}.\vec{v}$$

Which yields, on integration with respect to t over $0 \le t \le T$

$$\int_0^T m\frac{d\vec{v}}{dt}.\vec{v}dt = \int_0^T \vec{F}.\vec{v}dt + \int_0^T \vec{f}.\vec{v}dt$$

i.e) $m\int_0^T \vec{v}dt = W + w$

i.e) $\frac{1}{2}m\int_0^T d(\vec{v}.\vec{v}) = W + w$

i.e) $\frac{1}{2}m[\vec{v}.\vec{v}]_0^T = W + w$

i.e) $\frac{1}{2}m[v(T)]^2 - \frac{1}{2}m[v(0)]^2 = W + w, where\ v(t) = |\vec{v}(t)|$

This proves the theorem.

Note :1

The work done is said to be positive or negative according as the component of force and the displacement have the same or opposite direction.

Note : 2

The unit of work is S.I system is joule. This unit is called the absolute unit of work.

1 Joule = 1.0972 x 10^4 gm – cms. nearly = 0.73756 ft. -1bs nearly.

Central Force :

If a particle moves in a plane under a force which is always directed towards a fixed point in the plane then such a force is known as the central force, the fixed point is known as the centre of force and the path traced out by the particle is known as the central orbit.

Now we discuss the following problem relating to the concept of central force.

Motion Under Inverse Square Law :

Let us consider two particles of mass m and M of which one mass M is fixed at the origin.

Also let $\vec{r}$ be the position vector of the mass m at any time t. Now according to the Newton's law of gravitation, the force acting on the particle of mass m due to the particle of mass M is - $\frac{GmM}{r^2}$.

Where $= |\vec{r}|$, G is called the 'universal contact of gravitation' and is experimental value is 6.65 x 10^{-8} cm^3 / (gm. –sec^2), approximately.

The equation of motion for the particle of mass m is

$$m\frac{d\vec{v}}{dt} = \frac{-GmM}{r^2}\frac{\vec{r}}{r}, i.e., \frac{d\vec{v}}{dt} = -\frac{GM}{r^3}\vec{r}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where $\vec{v}$ is the velocity of the particle of mass m of any time t. Now in view of (1), we have

$$\frac{d}{dt}(\vec{r} \text{ x } \vec{v}) = \frac{d\vec{r}}{dt} \text{ x } \vec{v} + \vec{r} \text{ x } \frac{d\vec{v}}{dt} = \vec{v} \text{ x } \vec{v} - \frac{GM}{r^3}\vec{r} \text{ x } \vec{r} = \vec{0}$$

Which yields.

$\vec{r}$ x $\vec{v}$ = $\vec{r}$ x $\frac{d\vec{r}}{dt} = a$ constant vector = $\vec{h}$, say → (2)

Since we know that $\left|\vec{r}\ x\ \frac{d\vec{r}}{dt}\right|$ is twice the rate of change of sectorial area, so in view of (2) if follows that equal areas are traced out in equal intervals of time [Kepler's first law of planetary motion].

We have

$$\vec{h} \cdot \vec{r} = \left(\vec{r}\ x\ \frac{d\vec{r}}{dt}\right).\vec{r} = \left[\vec{r}, \vec{r}\ x\ \frac{d\vec{r}}{dt}\right] = 0$$

Which shows that the position vector of the particle is always perpendicular to the constant vector $\vec{h}$.

Therefore the motion is planar.

Now let $\hat{r}$ be a unit vector in the direction of $\vec{r}$. Then we have $\vec{r} = r\hat{r}$ and consequently we have $\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$ so that

$$\vec{h} = \vec{r}\ x\ \frac{d\vec{r}}{dt} = \vec{r}\ \text{x} \left(\frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}\right) = r\vec{r}\ \text{x}\ \frac{d\hat{r}}{dt}$$

Hence in view of (1) and the last relation we have

$\frac{d}{dt}\left(\vec{v}\ x\ \vec{h}\right) = \frac{d\vec{v}}{dt}\ x\ \vec{h}$ since $\vec{h}$ is a constant vector

$$= -\frac{GM}{r^2}\hat{r}\ x\ \left(r\vec{r}\ \ x\ \frac{d\hat{r}}{dt}\right)$$

$$= -GM\hat{r}\ \ x\ \left(\hat{r}\ \ x\ \frac{d\hat{r}}{dt}\right)$$

$$= GM\left[(\hat{r}.\hat{r})\frac{d\hat{r}}{dt} - \left(\hat{r}.\frac{d\hat{r}}{dt}\right)\hat{r}\right]$$

$$= GM\ \frac{d\hat{r}}{dt},$$

Which on integration gives

$$\vec{v}\ x\ \vec{h} = \mu\hat{r} + \ \vec{c}\ ,$$

Where $\mu = GM$ and $\vec{c}$ is the constant of integration. Since $\hat{r}$ and $\vec{v}\ x\ \vec{h}$ are parallel to the plane of the path of the particle, so $\vec{c}$ is also parallel to the plane of the orbit and the direction of it can be taken as the initial line.

Now let be the angle between $\vec{c}$ and $\vec{r}$. Then we have

$$\vec{r}\cdot\left(\vec{v}\,x\,\vec{h}\right)=\vec{r}\cdot(\mu\hat{r}+\ \vec{c})$$

Or, $\vec{h}\cdot(\vec{r}\,x\,\vec{v})=r\hat{r}.\mu\hat{r}+\ \vec{r}\vec{c}$

Or, $\vec{h}\,.\vec{h}=\ \mu r+cr\ cos\theta, where\ c=|\vec{c}|$

Or, $\frac{h^2}{r}=\mu+c\ cos\theta, where\ h=|\vec{h}|$

Or, $\frac{l}{r}=1+e\ cos\theta, where\ l=\ \frac{h^2}{\mu},\rho=\frac{c}{\mu}$

Represending a conic whose eccentricity is e, latus rectum is $\frac{h^2}{\mu}$, and one of whose foci is at the centre of force (Kepler's second law).

For e<1, the orbit will be an ellipse otherwise it will be a hyper bola or parabola according as e>1 or e = 1.

For the elliptic path (having $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as Cartesian equation), we find the length of the major axis as

$$2a=\frac{l}{1+e}+\frac{l}{1-e}=\frac{2l}{1-e^2}=\frac{2h^2}{\mu(1-e^2)}=\frac{2h^2}{GM(1-e^2)}$$

Or, $h^2=Gm\ (1-e^2)a.$

Let A be the area of the ellipse. Then we have

$A=\pi ab=\pi a^2\sqrt{1-e^2}$ and $2\frac{dA}{dt}=h.$

Which yields, on integration between $0\le t\le T$, T being the time period.

$$2\int_0^A\ dA=h\int_0^T\ dt,\quad ie.,hT=2A$$

From the last relation we get

$$T^2=\frac{4A^2}{h^2}=\frac{4\pi^2a^4(1-e^2)}{GM\ (1-e^2)a}=\frac{4\pi^2}{Gm}a^3,\qquad i.e.,T^2\alpha a^3$$

Which is Kepler's third law.

Summing up the above discussion we now deduce the following law[2] which were first discovered by German astronomer Johannes Kepler (1571 – 1630) connecting the motion of various planets about the sun.

Kepler’s First Law :

Each planet describes an ellipse with the sun at one of its foci.

Kepler’s Second Law :

The radius vector drawn from the sun to a planet sweeps out equal areas in equal interval of time.

Kepler’s Third Law :

The squares of the periodic times of the planents are proportional to the cubes of the mean distance from the sun.

Note :

Although to deduce kepler’s law on planetary motion we have used Newton’s inverse square law of gravitation but the fact that newton deduced the inverse square law of gravitational attraction between two bodies from Keplers observational facts.

Deduction of Differential Equation of the Central Orbit :

Let us consider a particle of mass m moving under the attraction of acentral force F(r) per unit mass, where $r = |\vec{r}|$.

Therefore the equation of motion is

$$m\frac{d^2\vec{r}}{dt^2} = -\,m\,F\frac{\vec{r}}{r}$$

i.e., $\frac{d^2\vec{r}}{dt^2} = -\,\frac{F}{r}\vec{r}$

Taking the dot product on both sides of the last relation with $2\frac{d\vec{r}}{dt}$ we get

$$2\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2} = -\,\frac{2F}{r}\vec{r}.\frac{d\vec{r}}{dt}$$

Or, $\frac{d}{dt}\left(\frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt}\right) = -\,\frac{2F}{r}r.\frac{dr}{dt} = -2F\frac{dr}{dt}$

Which on integration, yields

$$v^2 = K - 2\int F.dr \quad \text{………………….(1)}$$

Where $v = |\vec{v}| = \left|\frac{d\vec{r}}{dt}\right|$ and K is the constant of integration. Let p be the length of the perpendicular drawn from orgin to the tangent.

Since mv is the momentum, so the moment of momentum = Pmv and the angular momentum = mh.

Therefore we get

mvp = mh, i.e., vp = h ……………………..(2)

Eliminating v between (1) and (2) we obtain

$$\frac{h^2}{p^2} = k - 2\int \; F.dr$$

Which on differentiation with respect to r, leads to the following.

$$\frac{h^2}{p^3}\frac{dp}{dr} = F$$

This is the required differential equation of the central orbit.

Conclusion:

Though this work deal with only basic ideas and concept may be extended in the application of some field.

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