A STUDY ON PELL EQUATION

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ABSTRACT:

Pell equation (alternatively called the Pell- Fermat equation) is a

type of a Diophantine equation of the form 2 -Dye 2 =1 for a natural number

D. If D is a perfect square, then pell equation can be rewritten as

y).(x+√y) = 1.similarly, the trivial solution (x,y) = (0,1)

is not very interesting. Therefore it is often assumed that D is not a square

and only nontrivial solution (non zero pairs of integers) are considered.

KEYWORDS:

Elementary number, Algebraic Number, Analytic Number.

INTRODUCTION:

Number theory, known as the queen of mathematics is the branch of mathematics that concerns about the positive integers 1, 2, 3, 4, 5 which are often called natural numbers and their appealing properties from antiquity, these natural numbers classified as odd numbers, even numbers, square numbers, prime numbers, Fibonacci numbers, triangular numbers, etc. Due to the dense of unsolved problems, number theory plays a significant role in mathematics. Theory depending upon the tools used to address the related problems.

DEFINITION:

NATURAL NUMBERS:

The numbers 1, 2, 3…..are called natural numbers. They are also counting numbers.

Since, they are used for counting objects. The collection of all natural numbers is denoted by N.

Thus N = (1, 2, 3……..)

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DEFINITION:

POSITIVE AND NEGATIVE INTEGER:

The natural number { -1,-2,-3,-4,-5……}is called negative integer.

The natural number { 1,2,3,4,5……}is called positive integer.

DEFINITION:

ORDER OF MODULO m:

Let m and a be any positive integer such (a, m) = 1 then the least positive exponent e there exist

$a^e \equiv 1\ (mod m)$is the order of modulo m.

DEFINITION:

PRIMITIVE ROOT MODULO m:

If an integer a has exponent ϕ (m) modulo m a positive integer then we say that **a** is a primitive root modulo m or a is a primitive root belonging to m.

EXAMPLE: 1.4

List the primitive root modulo 6

ϕ (6) = 2

Any integer (mod6) is congruent to one of

0, 1, 2, 4 or 5

Therefore the primitive root (mod6)

DEFINITION:

PELL EQUATION:

The Equation $x^2 - ny^2 = 1$with the integer n and unknowns xa, is usually called pell equation.

THEOREM:

THE PELL EQUATION $x^2 - 2y^2 = 1$

To show that this rule gives a new solution we first calculate

$x_3 and y_3$.Expanding the left hand side and collecting its Rational and

irrational parts, we find that,

$x_3 = x_1x_2 + 2y_1y_2,$ $y_3 =$

$x_1y_2 + y_1x_2$

It can then be checked by multiplication that

$(x_1x_2 + 2y_1y_2)^2 - 2(x_1y_2 + y_1x_2)^2 = (x_1^2 - 2y_1^2)(x_2^2 - 2y_2^2)$

$= 1 \times 1$

$= 1$

Hence $x_3^2 - 2y_3^2 = 1$ as required.

EXAMPLES:

Composing the solution (3,2) with itself, we get a new solution (x_3, y_3).

$$x_3 + y_3\sqrt{2} = (3 + 2\sqrt{2})^2$$

$$= 9 + 8 + 12\sqrt{2}$$

$$= 17 + 12\sqrt{2}$$

Equating rational and irrational parts $x_3 = 17$,

$y_3 = 12$which is indeed another solution. If we then compose (17,12) with (3,2) we get,

$$(17 + 12\sqrt{2})(3 + 2\sqrt{2}) = 51 + 48 + (36 + 34)\sqrt{2}$$

$$= 99 + 70\sqrt{2}.$$

Hence another solution is (99,70) and so on.

By this process we can obtain infinitely many integer solution.

But it is not clear how close we are to finding all integer solution. The situation becomes clearer when we observe that a group structure is present.

THE GENERAL PELL EQUATION AND $\mathbb{Z}[\sqrt{n}]$:

If n is a non square integer we define,

$$\mathbb{Z}[\sqrt{n}] = \{x + y\sqrt{n} : x, y \in \mathbb{Z}\}$$

as we used the numbers $x + y\sqrt{2}$ to study $x^2 - 2y^2 = 1$ we use the numbers $x + y\sqrt{n}$ to study $x^2 - ny^2 = 1$.

In fact, $x^2 - ny^2$ is what we call the norm of

$x + y\sqrt{n}$ in $\mathbb{Z}[\sqrt{n}]$, the product of $x + y\sqrt{n}$

by its conjugate $x - y\sqrt{n}$. Which is o f the form a-b$\sqrt{n}$ for some integers a and b, is therefore irrational and such that,

$$|a - b\sqrt{n}| < \frac{1}{B}$$

Also, b<B because b is the different of two positive integers less than B.

norm $(\boldsymbol{x} + \boldsymbol{y}\sqrt{\boldsymbol{n}}) = (\boldsymbol{x} - \boldsymbol{y}\sqrt{\boldsymbol{n}})(\boldsymbol{x} + \boldsymbol{y}\sqrt{\boldsymbol{n}}) = \boldsymbol{x}^2 - \boldsymbol{n}\boldsymbol{y}^2$

Thus finding solution of the pell equation is the same as finding elements of $\mathbb{Z}[\sqrt{n}]$ with norm 1.

The advantage of searching in $\mathbb{Z}[\sqrt{n}]$, rather than among pairs (x, y) of integers,is that we can use algebra on numbers in $\mathbb{Z}[\sqrt{n}]$.

DIRICHLET'S APPROXIMATION

THEOREM:

For any irrational $\sqrt{n}$ and integer B> 0 .there are integers a,b with $0<b<B$ and

$$|a - b\sqrt{n}| < \frac{1}{}$$

Proof:

For any integer B> 0 consider the B - 1 numbers

$\sqrt{n}, 2\sqrt{n} \ldots, (B-1)\sqrt{n}$.for each multiplier k choose the integer A_k. Such that,

$0 < A_k - k\sqrt{n} < 1$.

Since $\sqrt{n}$is irrational B-1 numbers $A_k - k\sqrt{n}$we are strictly between o and 1. They are all different for the same reason.

Thus we have B+1 different number $0, A-\sqrt{n}, A-2\sqrt{n} \ldots AB-1-(B-1)\sqrt{n}, 1$.in the interval from o to 1.

If we then divide this interval into B Sub intervals of length 1/B, it follows by the finite pigeonhole principle that atleast one subinterval contain two of the numbers.

CONCLUSION:

In this paper discussed about number theory using Pell equation. Also we have discussed Brahmagupta composition rule , the quaternion units , the Hurwitz integers , conjugates , a prime divisor property These are all the field of current research in number theory

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