A STUDY ON PLANE AND SPHERE

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ABSTRACT

We will study some of the basic concepts about vectors various operations on vectors, their algebraic and geometric properties. These two types of properties, when considered to gather give full realizations to the concepts of vectors, and lead to their vital applicability in various areas.

KEYWORDS

Scalar, Vector, Collinear vector, Like vector, Unlike vector, coplanar

INTRODUCTION

Vector algebra is introduced in (1805-1865) by W.R. Hamilton.

However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction n (in which another player is positioned). Such quantities are called vectors. We frequently come across with both types of resistance etc... And vector quantities like displacement, velocity, acceleration, force, weight, and momentum electric field about vectors.

Vectors are main objects of study in multivariable calculus. They have the same role that numbers have in single-variable calculus. It is very important to gain a solid understanding of vectors before proceeding to multivariable calculus.

BASIC DEFINITIONS

Multiplications of a vector by a scalar:

Let m be any scalar and $\vec{a}$ be any vector, then the product $\boldsymbol{m\vec{a}}$ $(\boldsymbol{or})$ $\boldsymbol{\vec{a}m}$ of the vector $\vec{a}$ and the scalar m is a vector whose,

i. Magnitude is $|m|$ times that of $\vec{a}$.
ii. Support is the same or parallel to that of $\vec{a}$; and
iii. Sense is the same or opposite to that of $\vec{a}$ according as m is +ve or –ve.

DEFINITION:

A vector $\vec{r}$ is said to be a **linear combination** of the vectors $\vec{a}, \vec{b}, \vec{c}, \dots\dots$, if there exist scalars x, y, z....,

such that $\boldsymbol{\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}} + \cdots$

DEFINITION:

A system of vector $\overrightarrow{a_1}, \overrightarrow{a_2}, \dots\dots \overrightarrow{a_n}$ is said to be **linearly dependent** if there exist scalars $x_1 \dots\dots x_n$ (not all zero)

such that $x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots + x_n\overrightarrow{a_n} = 0$.

DEFINITION:

A system of vectors $a_1, a_2, \dots . a_n$ is said to be **linearly independent** if there exist scalars $x_1, x_2, \dots . x_n$ (all zero)

such that $x_1a_1 + x_2a_2 + \cdots + x_na_n = 0$.

DEFINITION:

We take in space any arbitrary point 0, to be called as the **origin of reference.**

Then the **position vector** of any point p, with respect to the origin 0 is the vector $\boldsymbol{\overrightarrow{OP}}$.

We usually denoted the **position vector** of the points A, B, C by $\vec{a}, \vec{b}, \vec{c}$ respectively.

CHARACTERISATION OF A VECTOR:

1. **Length**
2. **Support**
3. **Sense**

Length:

The length of the directed line segment $\overrightarrow{\boldsymbol{PQ}}$ is the distance between **P and Q.**

Support:

The line of an unlimited length of which a directed line segment is a part is called the **Support**.

Sense:

The **Sense** of the directed line segment is from its initial point to its terminal point.

DEFINITION:

DISPLACEMENT OF VECTOR:

If a point is displaced from position A to B then the displacement AB represents a vector $\overrightarrow{\boldsymbol{AB}}$ which is known as the **displacement vector.**

PLANE:

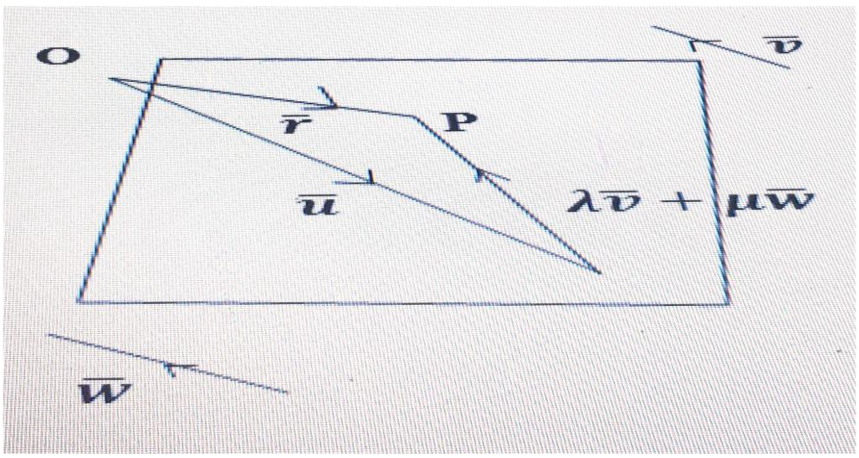
A plane is a surface which is purely flat such that if any two points are considered on the surface and joined by a straight line lies wholly on the surface. The surface is known as plane surface or flat surface. Now we study the plane under various conditions using the tools of vector algebra.

Plane Through a Point and Parallel to Two Vectors

We assume that the plane passes through the point A and is parallel to the given vectors $\vec{v}$ and $\vec{w}$. Let P be any point on the plane. Relative to a fixed point O chosen as origin, suppose that $\vec{u}$ and $\vec{r}$ are the position vector of A and P respectively. Then we have $\overrightarrow{AP} = \vec{r} - \vec{u}$. Since $\overrightarrow{AP}$ is parallel to $\vec{v}$ as well as $\vec{w}$, so it is parallel to a vector which is the linear combination of $\vec{v}$ and $\vec{w}$ and hence we must have $\overrightarrow{AP} = \lambda\vec{v} + \mu\vec{w}$, λ and μ are the scalar quantities. Thus from above we obtain

$$\vec{r} - \vec{u} = \lambda\vec{v} + \mu\vec{w}, \text{ i.e., } \vec{r} = \vec{u} + \lambda\vec{v} + \mu\vec{w}, \qquad \textbf{..... (3.1)}$$

Which is that the required vector equation of the plane.



Corollary 1:

If the plane passes through the origin, then the equation of the plane (3.1) takes the form

$$\vec{r} = \lambda\vec{v} + \mu\vec{w}.$$

Corollary 2:

If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}, \vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}, \vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ and $\vec{w} = w_1\vec{\imath} + w_2\vec{\jmath} + w_3\vec{k}$, then from (3.1) we obtain

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k} + \lambda\left(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}\right) + \mu(w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}),$$

Which yields

$$x = u_1 + \lambda v_1 + \mu w_1, y = u_2 + \lambda v_2 + \mu w_2, z = u_3 + \lambda v_3 + \mu w_3,$$

Which, on elimination of λ and μ, reduces to

$$\alpha x + \beta y + \gamma z = \delta,$$

Where α, β, γ and δ are scalars. Thus we obtain the Cartesian equation of the plane in three dimensional coordinate geometry.

Corollary 3: (Non-parametric form).

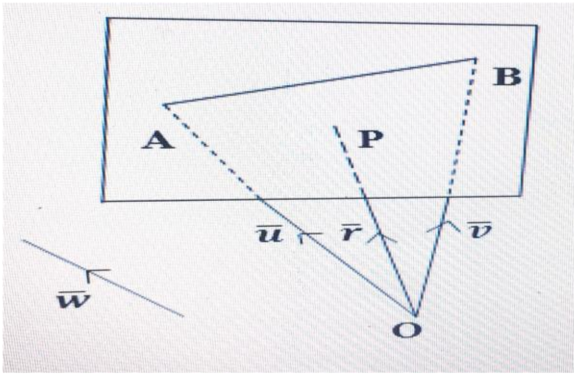
From the above discussion it follows that the vector $\vec{r} - \vec{u}$ is parallel to the vector $\vec{v}$ and $\vec{w}$ and hence is perpendicular to the vector $\vec{v} \times \vec{w}$. Hence as a result the vectors $\vec{r} - \vec{u}, \vec{v}$ and $\vec{w}$ are coplanar. Thus we obtain

$$[\vec{r} - \vec{u}, \vec{v}, \vec{w}] = 0, \text{ i.e., } [\vec{r}, \vec{v}, \vec{w}] = [\vec{u}, \vec{v}, \vec{w}], \quad \textbf{.... (3.2)}$$

Which is the equation of the plane in non-parametric form.

Plane Through Two Points and Parallel to a Vector

We consider A, B as the given points and $\vec{w}$ as the given vector. With respect to a fixed point O chosen as origin, we suppose that $\vec{u}$ and $\vec{v}$ are the position vectors of A and B respectively. Let P, with position vector $\vec{r}$, be any point on the plane.



Then we have $\overrightarrow{AP} = \vec{r} - \vec{u}$ and $\overrightarrow{AB} = \vec{v} - \vec{u}$. Since the required plane is parallel to $\vec{w}$ as well as $\vec{v} - \vec{u}$, so it is parallel to a vector which is the linear combination of $\vec{w}$ and $\vec{v} - \vec{u}$ and hence we must have $\overrightarrow{AP} = \lambda(\vec{v} - \vec{u}) + \mu\vec{w}, \lambda$ and μ being the scalar quantities. Thus we have

$$\vec{r} - \vec{u} = \lambda(\vec{v} - \vec{u}) + \mu\vec{w}, \text{ i.e., } \vec{r} = (1 - \lambda)\vec{u} + \lambda\vec{v} + \mu\vec{w}, \qquad \textbf{... (3.3)}$$

Which is that the required vector equation of the plane.

Corollary 4:

From the above discussion it is clear that the vector $\vec{r} - \vec{u}, \vec{v} - \vec{u}$ *and* $\vec{w}$ are coplanar. Hence we obtain

$$(\vec{r} - \vec{u}).(\vec{v} - \vec{u}) \times \vec{w} = 0,$$

Which on simplification yields

$$\vec{r}.(\vec{v} \times \vec{w} + \vec{w} \times \vec{u}) = [\vec{u}, \vec{v}, \vec{w}], \qquad \textbf{..... (3.4)}$$

Which is that the equation of the plane in non-parametric form.

SPHERE:

DEFINITION:

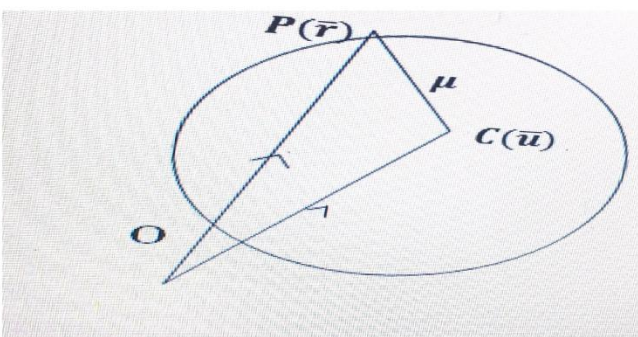
A sphere is that the locus of a variable point in space whose distance from a hard and fast point within the space is usually a continuing . The fixed point is known as the centre of the sphere and the constant length of any point on the sphere from the centre is known as its radius.

Now we will deduce the equations of sphere under different condition as follows:

Sphere with Given Radius and Centre

we consider a sphere whose centre is at C with position vector $\vec{u}$ relative to a fixed point O chosen as origin. Let P be any point with position vector $\vec{r}$ on the surface of the sphere whose radius is of ρ units. Then we have $\overrightarrow{CP} = \vec{r} - \vec{u}$. Now $\left|\overrightarrow{CP}\right|^2 = \rho^2$. Hence we have

$$|\vec{r} - \vec{u}|^2 = \rho^2 \Rightarrow |\vec{r}|^2 + |\vec{u}|^2 - 2\vec{r}.\vec{u} = \rho^2, \qquad \textbf{..... (4.1)}$$



Corollary 1:

If the centre of the sphere is at the origin then the equation of the sphere reduces to the following:

$$|\vec{r}|^2 = \rho^2, \text{ i.e., } |\vec{r}| = \rho \qquad \textbf{..... (4.2)}$$

Corollary 2:

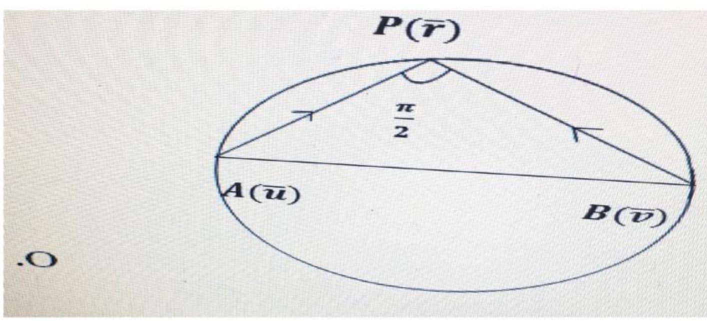
If the sphere passes through the origin O, then we have $|\overrightarrow{OC}| = |\vec{u}| = \rho$ and hence the equation of the sphere reduces to the following:

$$|\vec{r}|^2 - 2\vec{r}.\vec{u} = 0. \qquad \textbf{..... (4.3)}$$

Sphere with Given Extremities of a Diameter

Let P be any point with position vector $\vec{r}$ on the surface of the sphere. Further let A and B be the extremities of a diameter with position vector $\vec{u}$ and $\vec{v}$ respectively. Then we have $\overrightarrow{AP} = \vec{r} - \vec{u}$ and $\overrightarrow{BP} = \vec{r} - \vec{v}$. Now for the sphere, $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BP}$. Hence we have

$$(\vec{r} - \vec{u}).(\vec{r} - \vec{v}) = 0,$$

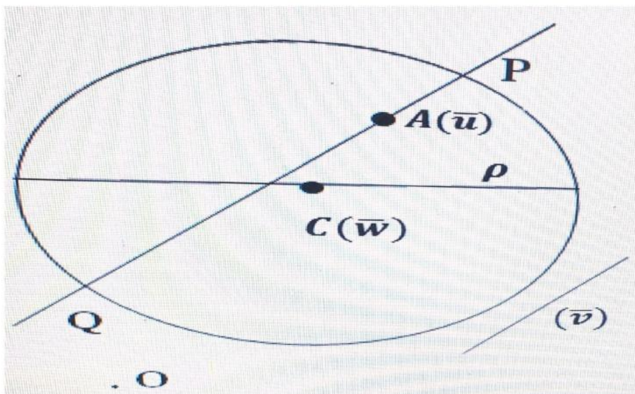


Intersection of a Line and a Sphere

We consider a line through a given point A and parallel to a given unit vector $\vec{v}$. Let $\vec{u}$ be the position vector of A. Further we consider a sphere whose centre is at C with position vector $\vec{w}$ and radius is of ρ units. Then the equation of the road and therefore the sphere are often respectively written as $\vec{r} = \vec{u} + \lambda\vec{v}$ and $|\vec{r}|^2 + |\vec{w}|^2 - 2\vec{r}.\vec{w} = \rho^2$, Where λ is a scalar. Eliminating $\vec{r}$ between the last two equations and using the fact that $|\vec{v}| = 1$, we obtain the required condition as follows:

$$\boldsymbol{\lambda^2 + 2\hat{v}.(\vec{u} - \vec{w})\lambda + |\vec{u}|^2 - 2\vec{w}.\vec{u} + |\vec{w}|^2 - \rho^2 = 0,} \quad \textbf{... (4.5)}$$

Which yields two values of λ corresponding to each of which we have the points of intersection as P and Q. Thus the position vectors of P and Q are $\vec{u} + \lambda_1\vec{v}$ and $\vec{u} + \lambda_2\vec{v}$ respectively, where $\lambda_1 = AP$ and $\lambda_2 = AQ$ are the roots of the quadratic



equation (4.5). Now the product of these roots is $|\vec{u}|^2 - 2\vec{w}.\vec{u} + |\vec{w}|^2 - \rho^2$, which is independent of $\vec{v}$ and hence it is same for all lines drawn through A to intersect the sphere. If P and Q coincide at a point T on the sphere, then the line will be a tangent and we find the relation

$$AT^2 = AP.AQ = |\vec{u}|^2 - 2\vec{w}.\vec{u} + |\vec{w}|^2 - \rho^2.$$

NOTE:

The tangent drawn from A generates a cone having its vertex at A and enveloping the sphere is known as the enveloping cone. Also the quantity $AT^2 = |\vec{u}|^2 - 2\vec{w}.\vec{u} + |\vec{w}|^2 - \rho^2$ is known as the power of the point A.

Tangent Plane to a Sphere

We consider that the point A (under consideration in the subsection lies on the sphere, then the points A, P, Q coincide, which implies that $\lambda_1 = 0 = \lambda_2$ and hence the sum and product of the roots of the quadratic (4.5) are zero. As a result we have

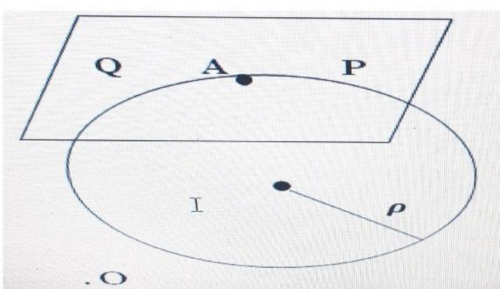
$$\vec{\boldsymbol{v}}.(\vec{\boldsymbol{u}} - \vec{\boldsymbol{w}}) = \mathbf{0} \text{ and } |\vec{\boldsymbol{u}}|^2 - 2\vec{\boldsymbol{w}}.\vec{\boldsymbol{u}} + |\vec{\boldsymbol{w}}|^2 - \boldsymbol{\rho}^2 = \mathbf{0}, \quad \textbf{.... (4.6)}$$

Which are the required conditions of tangency. Now from the first equation of **(4.6)** it follows that the line $\vec{r} = \vec{u} + \lambda\vec{v}$ is perpendicular to the vector $\vec{u} - \vec{w}$ and hence we get

$$(\vec{\boldsymbol{r}} - \vec{\boldsymbol{u}}).(\vec{\boldsymbol{u}} - \vec{\boldsymbol{w}}) = \mathbf{0}, \qquad \textbf{.... (4.7)}$$

Which shows that all tangents through the point A lies on it and it is the tangent plane to the sphere at A. Further using the second equation of **(4.6)** and **(4.7)**, we get the standard equation of the tangent plane at A to the sphere (4.1) as follows:

$$\vec{r}.(\vec{\boldsymbol{u}} - \vec{\boldsymbol{w}}) - \vec{\boldsymbol{u}}.\vec{\boldsymbol{w}} + |\vec{\boldsymbol{w}}|^2 - \boldsymbol{\rho}^2 = \mathbf{0} \qquad \textbf{.... (4.8)}$$



Corollary:3 (Condition of tangency)

If a plane $\vec{r}.\hat{n} = \lambda, \hat{n}$ being the unit normal touches the sphere given in (4.1), then the perpendicular distance of the plane from the centre of the sphere equals to the radius of the sphere. Thus we have the following:

$$(\boldsymbol{\lambda} - \vec{\boldsymbol{u}}.\hat{\boldsymbol{n}})^2 = \boldsymbol{\rho}^2.$$

CONCLUSION

The vector algebra play fundamental role in the straight line, plane and sphere. In this applications to various fields such as geometry, mechanics, physics, and engineering, scientific research.

The concept of vector algebra is explained lucidly with the geometric notations and physical motivations. Further the geometrical and physical applications through this chapters help one in the advanced reading of the subject which is becoming more and more abstract.

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