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Kenjiro Hori
Birkbeck, University of London

Stephen Wright
Birkbeck, University of London

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Kenjiro Hori and Stephen Wright[1]
Birkbeck, University of London

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[1]Both at Department of Economics, Mathematics and Statistics, Birkbeck, University of London, Malet Street, London, WC1E 7HX, UK. Email: k.hori@bbk.ac.uk and s.wright@bbk.ac.uk, respectively.

Abstract

We derive key properties of the actuarially fair contribution rate for defined benefit (DB) schemes, that equates scheme assets to liabilities for any given scheme member. The unpleasant actuarial arithmetic of both increased life expectancy and (especially) negative real yields has resulted in a massive rise in the fair contribution rate over recent decades. At present there appears to be little prospect of these rises being reversed. We analyse the implications for the viability of DB schemes, and consider the (potentially significant) impact of incorporating systematic risk into benefits.

JEL Classification: J32

Keywords: defined benefit, pension contribution rate

1 Introduction

Contribution rates to defined benefit (DB) pension schemes are usually fixed for prolonged periods, and invariant across scheme members. In this paper we derive some key properties of the actuarially fair contribution rate that, for any given scheme member, equates scheme liabilities to the present value of their contributions. The unpleasant actuarial arithmetic of the title of this paper is illustrated with reference to the defined benefit scheme for academics in UK universities (the Universities Superannuation Scheme, USS), over the period $1985 - 2020$: we show that, on current rules, fair contribution rates (by both employees and employers) would have risen roughly sixfold over this time period, to well over 40%. We analyse the implications of our results for the viability of DB schemes.

The exercise is conceptually relatively straightforward. DB schemes sell pensions that are simply deferred annuities (typically indexed) along with other future payments that are risk-free, contingent only on the scheme member's mortality risk. We therefore draw on the clear analogies with the literature on annuity pricing in deriving and assessing our results.[1]

The key (or indeed, arguably, only) difference in what DB schemes offer, compared to other annuity providers, is that, in contrast to the lump sum payments made for conventional annuities, DB scheme members pay for their deferred benefits by a sequence of payments as a fixed share c of earnings through the working life, with scheme liabilities accruing over time by a pre-announced formula. The *fair* contribution rate c_t^*, at the time, t, at which a scheme member enters the scheme, is then simply the ratio of the present value of scheme liabilities to the present value of earnings over the working life[2]). In our central case we work on the assumption that both accrued contributions and liabilities are zero beta, hence we can simply apply the law of large numbers, taking into account mortality risk, using risk-free yields to calculate present values, and hence c_t^*.

A key feature of this calculation is that it is entirely independent of the investment strategy of the fund's investments, or indeed of whether the scheme is funded at all. But we do consider, in light of the annuities literature, whether the present value calculations

[1] We draw in particular on Finkelstein & Poterba (2004); Koijen & Yogo (2015); Poterba and Solomon (2021); Cannon & Tonks (2016) and Verani and Yu (2021).

[2] While our central case assumes that the scheme member stays in the scheme into retirement, we also show that c_t^* is close to being unaffected by the risk of the scheme member leaving the scheme earlier.

made on this basis are consistent with viable hedging strategies by providers. We also note the implications for scheme fragility when actual contribution rates differ from fair rates.

While the exact calculation of c_t^* is somewhat convoluted, we derive an approximation that closely matches the exact calculation, and which provides strong intuition. Assume a pure pension liability p (payable from the date of retirement, $t+R$) is calculated on the basis of average salary through the working life, $\overline{w}$ by the formula $p = \frac{R}{Y}\overline{w}$, where $\frac{1}{Y}$ is the scheme's accrual rate. (Thus, if, eg, $Y = 80$ and the scheme member stays in the scheme for $R = 40$ years $p = \frac{\overline{w}}{2}$.) The approximation then takes the form

$$c_t^* \approx \frac{E_t}{Y} \times \mathcal{P}_t \times Q_t \qquad (1)$$

where E_t is life expectancy at retirement, in years. We note that the ratio $\frac{E_t}{Y}$ is closely related to the saving ratio in the original Modigliani (1966) life cycle model: it is the notional fair contribution rate if real yields and real wage growth were zero and the scheme member had a 100% probability of surviving into retirement.

In our approximation this notional "Modigliani" contribution rate $\frac{E_t}{Y}$ is scaled by two terms, $\mathcal{P}_t$ and Q_t. $\mathcal{P}_t$ is the value of a zero coupon bond with maturity that is close to (but somewhat higher than) half the scheme member's *current* life expectancy, $R + E_t$.[3] Q_t is close to (but somewhat higher than) the scheme member's probability of surviving into retirement.

The significant rise in our estimates of c_t^* is dominated by the impact of falls in risk-free yields on $\mathcal{P}_t$. In 1985, an appropriate measure of CPI-adjusted real yields was around 2.5% implying $\mathcal{P}_t \approx \frac{1}{3}$; whereas by 2020 the same measure of real yields was around -1.3%. implying $\mathcal{P}_t \approx \frac{3}{2}$. Thus the reduction in yields alone would have implied a roughly $4\frac{1}{2}$ fold increase in c_t^*, with the remainder of the sixfold increase driven by increases in E_t and (to a lesser extent) Q_t.

These calculations are a stark reminder of the unpleasant property that sustained negative yields imply that financial payoffs in the future are worth more, the further into the future they lie; and DB schemes make promises that are very long-dated: for a scheme member joining the USS with 40 years to retirement the duration of the notional bond in (1) is around 31 years.

The approximation also helps to illustrate the role of the age of the scheme member

[3] Or equivalently the average maturity of a hypothetical annuity paid at a constant rate over the scheme member's entire remaining life.

on joining. Only in the restricted "Modigliani" case would the fair contribution rate $\frac{E_t}{Y}$ be invariant to the age of the scheme member, since scheme assets and liabilities would accrue in exactly equal proportion through the working life. But both $\mathcal{P}_t$ and Q_t, and hence c_t^*, are dependent on the age of the scheme member on joining the scheme. Most crucially the duration of the notional bond is clearly decreasing in the age of the scheme member on joining: thus common contribution rates imply redistributions between scheme members of different ages. These are accentuated by additional benefits such as lump sums, spousal pensions and life insurance components, which also increase c_t^* nontrivially.[4]

Most DB schemes have similar characteristics, and thus face similar rises in c_t^*. While we show that, in the USS example, actual contribution rates and other scheme parameters have adjusted to c_t^* (albeit typically with a lag) the approximation in (1) has the strong implication that any scheme predicated on a constant contribution rate must be inherently fragile.[5] We argue that, to survive, schemes that offer risk-free payoffs must follow the example of annuity providers, and explicitly incorporate time variation in contribution rates, ideally along with some dependence on individual characteristics.

Our analysis can also straightforwardly be extended to consider the impact of pension payments that have some systematic risk. While we show that this could in principle imply significantly lower fair contribution rates, we note that the logistical implications of designing such a hybrid scheme are formidable.

The rest of the paper is organised as follows. In Section 2 we derive a general expression for the fair contribution rate, and investigate two key benchmark restricted cases that feed into our approximation. We then analyse the impact of shifts in the general level of interest rates and the (minimal) impact of wage growth during the working life. In Section 3 we derive our approximation. In Section 4 we use it analyse the key historical determinants of c_t^* in our applied example of the USS. Section 5 considers the robustness of our approach to a range of assumptions. Section 6 discusses the implications of our analysis for the viability of DB schemes, and illustrates the impact of incorporating systematic risk into the pension payment. Section 7 concludes the paper. Appendices provide proofs and details of data construction.

[4] We show that we can incorporate the impact of additional non-pension benefits into our approximation by adjusting actual life expectancy in retirement, E_t, for the effective impact, in years of pension, of these additional benefits.

[5] It thus sheds light on the recent debate on the viability of the USS scheme, in particular, eg, Marsh (2019), Wong (2021), Miles and Sefton (2021); Wolf (2021).

2 The Fair Contribution Rate

A scheme member joins a defined benefit (DB) scheme at at time t, at age A. If they survive, they will retire at age A_R (assumed to be a parameter of the scheme[6]).

2.1 The market environment

2.1.1 Wage income

Let $R = A_R - A$ be the scheme member's time to retirement on joining the scheme. The scheme member's real wage income $w_{t+\tau}$, from which contributions to the scheme will be drawn, is given by

$$w_{t+\tau} = w_t e^{g_t(\tau;A)\tau}, \tau \in [0, R) \tag{2}$$

The age- and time-specific growth path $g_t(\tau; A)$ is assumed to be deterministic and known at time t. We discuss below the implications of uncertainty in wage paths, but note here that in most contexts where DB schemes exist, job security is usually high and progression through salary scales is fairly predictable.

2.1.2 The yield curve

There is a real market yield curve described by the function $y_t(\tau)$ for $\tau \in [0, \infty]$, the continuously compounded risk-free real return to maturity on a zero-coupon bond issued at time t, and maturing with a risk-free real payoff at time τ (thus with price $\mathcal{P}_t(\tau) = e^{-y_t(\tau)\tau}$). The yield curve evolves stochastically through time, but at any point t is a known deterministic function in τ.

2.1.3 Mortality Risk

The scheme member will die at random date $t + \widetilde{A}_D - A$. Their probability of surviving from time t to time $t+ \tau$ is given by the time-varying survival function $S_t(\tau; A) \equiv \mathbb{P}_t\left(\widetilde{A}_D > A + \tau\right) = e^{-\Lambda_t(\tau;A)\tau}$, where

$$\Lambda_t(\tau; A) = \frac{1}{\tau}\int_0^\tau \lambda_t(u; A)\, du$$

[6]We show below in Section 5.1 that it is easy to account for members being allowed to retire, fully or partially, before this age.

given the time-varying, age-specific hazard rate function $\lambda_t(\tau; A) \equiv \lambda_t(A+\tau)$.

In most of what follows, for compactness of notation, we take the dependence of the hazard rate and wage growth profile on A as given, except where relevant to the analysis - as a result there is a clear symmetry between the functions $g_t(\tau)$, $\Lambda_t(\tau)$, and $y_t(\tau)$.

2.2 The Scheme

The scheme member's instantaneous contribution to the pension scheme at time $t+\tau \in [0, R)$ is assumed to be a proportion c_t of their wage $w_{t+\tau}$ where c_t is fixed at time t,but is assumed invariant over τ. The present value at t of the member's total payments into the scheme, $\mathcal{A}_t$ is then simply proportional to the present value $\mathcal{W}_t$ of the scheme member's lifetime earnings within the scheme

$$\mathcal{A}_t = c_t \mathcal{W}_t \tag{3}$$

In valuing both payments into the scheme, and the scheme's liabilities, we make the following assumption:

Assumption A1: *All contributions to, and contingent payoffs by the scheme have zero risk prices (are zero beta).*

Both payments into the scheme and liabilities relating to an individual scheme member are clearly stochastic, due to mortality risk. However if the payoffs are uncorrelated with market risk factors they will have zero risk prices so the no-arbitrage prices of both assets and liabilities for an individual scheme member can be calculated using expected payoffs, discounted using yields on risk-free bonds.

Assumption 1 simplifies, but is not essential to our analysis; we consider the impact of relaxing it in later sections.[7]

Given the assumed deterministic path for earnings, and using A1, we then have

$$\mathcal{W}_t = w_t \int_0^R e^{(g(\tau)-\Lambda(\tau)-y(\tau))\tau} d\tau. \tag{4}$$

In return for these contributions, the scheme provides contingent benefits, with implied contingent liabilities. In our central case we consider 3 forms of contingent liability.

[7] In Section 5.4 we consider the impact of relaxing A1 in this context. In 5.1 we also consider the (empirically nontrivial) probability that the scheme member may quit the scheme before retirement; but we show that this has a negligible effect on our results. In Section 6.2 we consider the implications benefits with systematic risk.

2.2.1 The pension liability

Conditional upon the member surviving into retirement, the scheme will pay a pension p_t, at a constant rate from $t+R$ until the scheme member's death. We assume in our central case that the pension is calculated on the basis of average salary,[8] as

$$p_t = \frac{R}{Y}\overline{w}_t \tag{5}$$

where Y is the notional number of years the member would need to remain in the scheme to accrue a pension equal to their average salary (although in practice in almost all schemes $Y >> R$), and

$$\overline{w}_t = \frac{w_t}{R}\int_0^R e^{g(\tau)\tau}d\tau \tag{6}$$

is the average salary over the working life.

Under Assumption A1 the implied pension liability at t is,

$$\mathcal{L}_t^p = p_t\int_R^\infty e^{-(y_t(\tau)+\Lambda_t(\tau))\tau}d\tau \equiv p_t C_t^p. \tag{7}$$

which is simply the present value at time t, of a deferred annuity at rate p_t, to be paid from period $t+R$, which can be expressed as C_t^p years' worth of annual pension, where C_t^p is a capitalisation factor.

2.2.2 The lump sum liability

The scheme will also make a lump-sum payout at $t+R$ of Z years' worth of the pension, Zp_t, again, conditional upon the scheme member surviving into retirement, with implied liability

$$\mathcal{L}_t^Z = p_t Z e^{-(y_t(R)+\Lambda_t(R))R} \equiv p_t C_t^Z \tag{8}$$

and capitalisation factor C_t^Z.

2.2.3 The spousal pension liability

If the scheme member has a spouse, who survives after the member's death, the spouse will receive a fixed fraction, s of the pension p_t that would have accrued had the scheme member survived until time $t+R$. We assume this will be payable from time $t+R$,

[8] We consider below (see Section 5.2) the impact of defining the pension in terms of final, rather than average salary.

or the date of the scheme member's death (whichever is latest) until the date of the spouse's own death.

In our central case we assume that the scheme member does indeed have a spouse,[9] with an identical age-related survival probability to the scheme member on joining the scheme.[10]

The associated liability is then

$$\mathcal{L}_t^s = sp_t \int_R^\infty \lambda(\tau) e^{-\Lambda_t(\tau)\tau} \left\{ \int_\tau^\infty e^{-(y_t(u)+\Lambda_t(u))u} du \right\} d\tau \equiv p_t C_t^s. \tag{9}$$

where the integrals over τ and u capture the probabilities of the scheme member's and spouse's deaths, respectively at a given time horizon. The present value of this liability can again be expressed as C_t^s year's worth of annual pension, where C_t^s is a third capitalisation factor.

2.2.4 Life insurance components

Defined benefit schemes commonly also offer some element of life insurance. However, to simplify our analytical framework we neglect the cost of any life insurance components in the scheme, in which we include both death-in-service payments and any liability to pay a spousal pension if the scheme member dies *before* retirement, since this is in effect an additional form of life insurance. We thus focus our analysis on the cost of pension provision, and other benefits provided after the retirement date. We do however illustrate the impact of these additional components in our applied example.

2.3 The Fair Contribution Rate

We define the *fair contribution rate* c_t^* as the contribution rate that equates $\mathcal{A}_t$ (the present value of the member's contributions) to $\mathcal{L}_t$, the present value of all contingent liabilities to a scheme member joining the scheme at time t. Thus, using (3),

$$c_t^* = \frac{\mathcal{L}_t}{\mathcal{W}_t}. \tag{10}$$

The fair contribution rate is therefore simply equal to the ratio of the present value of

[9] Clearly this assumption will not hold for all scheme members. In Section 6 we consider the distributional implications of this (and other) elements of the scheme.

[10] In the data survival probabilities are clearly not identical, so the calculations below can also be viewed as being calculated under a veil of ignorance on the sex of the scheme member.

the scheme's liabilities to the present value of their career earnings within the scheme.[11] As such, while, from the definition of $\mathcal{A}_t$ in (3), c_t^* is assumed constant, for any given scheme member, over their working life, its value will usually vary both over time and across characteristics of scheme members.

Note that this definition is quite general, and does *not* rely on Assumption 1 - a property we exploit below, in Section 6.2, when we consider the impact of possible amendments to DB schemes, in which either liabilities or assets may carry systematic risk. In the main body of the paper, however, we focus on the zero beta case.

Given

$$\mathcal{L}_t = \mathcal{L}_t^p + \mathcal{L}_t^Z + \mathcal{L}_t^s \tag{11}$$

and noting that, using (7) to (9), each of the conditional liabilities can be expressed as a multiple of the annual pension p_t, we then have:

Proposition 1 *Let C_t be an aggregate capitalisation factor (in years' worth of annual pension), such that $\mathcal{L}_t = p_t C_t$. Under A1, the fair contribution rate c_t^* can be expressed as*

$$c_t^* = \frac{C_t}{Y} U_t \tag{12}$$

where

$$C_t = C_t^p + C_t^Z + C_t^s,$$

$$\begin{aligned} C_t^p &= \int_R^\infty e^{-(y_t(\tau)+\Lambda_t(\tau))\tau} d\tau \\ C_t^Z &= Z e^{-(y_t(R)+\Lambda_t(R))R} \\ C_t^s &= s \int_R^\infty \lambda(\tau) e^{-\Lambda_t(\tau)\tau} \left\{ \int_\tau^\infty e^{-(y_t(u)+\Lambda_t(u))u} du \right\} d\tau. \end{aligned} \tag{13}$$

and

$$U_t = \frac{\int_0^R w_{t+\tau} d\tau}{\mathcal{W}_t} \tag{14}$$

where $\mathcal{W}_t$, as defined in (4) is the present value of career earnings.

[11] Note that we follow standard academic practice in ignoring the administrative costs of the scheme in calculating this fair rate. As such our estimates understate the true fair rate after costs. As a benchmark for comparison, for example, Koijen and Yogo (2015, Table 2, p451) show that, using risk-free valuation, average markups on life annuities in the United States since the early 1990s were around 8%, and were systematically positive except for brief period during the financial crisis - a dip which the authors attribute to regulatory distortions. We revert to the issue of a broader definition of fairness in Section 5.4.

Proof. *By substitution from (4) to (9).* ■

Proposition 1 factors the fair contribution rate into two ratios.

In the first ratio both C_t and Y are measured in years. Writing total liabilities as C_t years' worth of annual pension, C_t is the sum of three capitalisation factors (determined by market yields and mortality rates), for the three contingent liabilities of the fund (the member's pension, the lump sum Z and the spousal pension). The higher is $\frac{C_t}{Y}$ (where $1/Y$ is the accrual rate for the pension), the higher the fair contribution rate needs to be.

The second term, U_t is the ratio of the undiscounted sum of career earnings, if the scheme member survives to $t + R$, to the present value of career earnings, which, from (4) takes account of both discounting and mortality risk. Since both numerator and denominator scale in current wage income, w_t, it is evident that the fair contribution rate is invariant to w_t.

Other than this, Proposition 1 only limited intuition. We now proceed to analyse some key properties of c_t^* that both provide more intuition and allow us to derive our approximation.

2.4 The fair contribution rate with zero yields, and the "Modigliani Case"

We first consider two heavily restricted benchmark cases in which the elements in Proposition 1 simplify considerably. In addition to providing intuition, these provide key elements in our approximation. We then consider the impact of relaxing the restrictions.

Proposition 2 ***(The Zero Yield, Zero Growth, Pure Pension Case)*** *If real yields and wage growth are zero, $y_t(\tau) = g_t(\tau) = 0$ for all τ, then, for a pure pension liability $(s = Z = 0)$ the fair contribution rate c_t^* is given by*

$$c_{0,t}^* = \frac{E_t}{Y} \times Q_t, \tag{15}$$

where E_t is life expectancy at retirement, given by

$$E_t = \mathbb{E}_t \left[\widetilde{A}_D - A_R \middle| \widetilde{A}_D > A_R \right] \tag{16}$$

and

$$Q_t = S_t(R)\,U_{0,t} \in (S_t(R), 1) \tag{17}$$

$$U_{0,t} = \left(\frac{R}{S_t(R)\,R + (1 - S_t(R))\,\mathbb{E}_t\left[\widetilde{A}_D - A \,\middle|\, \widetilde{A}_D < A_R\right]} \right) > 1 \tag{18}$$

where $S_t(R) = e^{-\Lambda_t(R)R}$*, is the probability of survival until age* A_R*, at time* $t + R$ *(given* $R = A_R - A$*), and* $U_{0,t}$ *is the value of* U_t*, as defined in Proposition 1 under the same assumptions.*

Corollary 1 ***(The Modigliani Case)*** *If, additionally,* $S_t(R) = 1$ *(the scheme member will survive into retirement with certainty) then* c_t^* *is given by*

$$c_{M,t}^* = \frac{E_t}{Y}.$$

Proof. See Appendix A. ■

We focus first on the intuition provided by Corollary 1, in which the scheme member will survive the R years until retirement with certainty ($S_t(R) = 1$). By inspection of (17) and (18) this also implies $Q_t = 1$. We denote this the "Modigliani Case" since, as we show below, the insights are closely related to the simplified life cycle model as analysed in Modigliani (1966). In this simplest case the two ratios in Proposition 1 simplify considerably. The capitalisation factor C_t^p for the pension liability in Proposition 1 is then simply equal to the number of years the pension will be paid, in expectation, hence equals life expectancy at retirement, E_t, defined in (16).

The resulting fair contribution rate in this special case can thus be interpreted as the ratio of two time periods, both measured in years: the numerator, E_t, and the denominator Y, where the latter is the number of years the scheme member would notionally need to work to have a pension equal to their average salary. Note that while Y_t is transparently a scheme parameter, the numerator E_t is, from (16) also crucially determined both by mortality rates and by the scheme retirement age A_R.

Note that in this restricted special case the fair contribution rate is invariant to the scheme member's age, A, on joining the scheme, since $R = A_R - A$ drops out of the formula, and E_t is also invariant to A.

We refer to this as the "Modigliani Case" because there is a very close link between this formula and that for the saving rate in the simplest case of the life cycle model, as analysed in Modigliani (1966) in which both real interest rates and wage growth are also assumed equal to zero, and the date of death $\widetilde{A}_D$ is assumed known. If we interpret $R = A_R - A$ as the length of the working life, then in Modigliani's model, the saving rate ς that allows constant consumption before and after retirement is given setting $Y = R$, i.e. letting $\varsigma = \frac{E_t}{R} = c^*_{M,t}|_{Y=R}$, assuming all of the pension is consumed. Clearly to the extent that R is typically significantly less than Y, then $c^*_{M,t} \ll \varsigma$.[12]

Once we allow for a non-zero probability of death before retirement ($S_t(R) < 1$), then the main proposition implies that the size of the fair contribution rate is strictly less than in the Modigliani Case.[13] Note that that given that the probability of surviving to retirement $S_t(R) = S_t(A_R - A)$, this is the only way in which the age of the scheme member enters the formula in this restricted case: younger scheme members have a (moderately) greater risk of dying before receiving their pension, and hence a lower value of $c^*_{0,t}$. But apart from the impact of A on this term, the fair contribution rate in this restricted case is very close to being independent of age: there is only a modest cross-subsidy from young to old scheme members.

We have thus far considered only the case of a pure pension liability. But it is easy to show that Proposition 2 generalises to the the case where the scheme provides additional benefits:

Corollary 2 ***(The Zero Yield, Zero Growth Case with Additional Benefits).*** *If the scheme additionally offers a lump sum* ($Z > 0$) *and a post-retirement spousal pension* ($s > 0$) *then the fair contribution rate* $c^*_{0,t}$ *takes the same form as in Proposition 2, but replacing* E_t *with* $\widehat{E}_t$, *"Effective Life Expectancy in Retirement", defined by*

$$\widehat{E}_t = \mathbb{E}_t\left[\widetilde{A}_D - A_R \,\middle|\, \widetilde{A}_D > A_R\right] + Z + \frac{s}{2} S_t(R)\left(\mathbb{E}_t \widetilde{A}^S_D - \widetilde{A}_D \,\middle|\, \widetilde{A}^S_D > \widetilde{A}_D > A_R\right) \quad (19)$$

where $\widetilde{A}^S_D$ *is the random date of the death of the spouse.*

[12] The comparison with Modigliani's original analysis, which assumes constant consumption throughout life, casts an interesting sidelight on the design of DB schemes, where Y is normally set at a value considerably larger than R (e.g., on current USS rules, $Y = 75$), so that the maximum pension provided by the pension scheme is only a fraction of the scheme member's income while working. While this may in part reflect lower expenditures required to maintain a given living standard in retirement, it may also reflect the implicit assumption that scheme members would not optimally choose to hold all of their wealth in the risk-free form offered by DB schemes. We revert to this issue in Section 6 below.

[13] Relating back to the terms in Proposition 1, for this case the capitalisation factor $C_t = C^p_t = S_t(R)\,E_t$, but $U_t = \int_0^R w_{t+\tau} d\tau / \mathcal{W}_t = U_{0,t} > 1$. Hence in this case there are offsetting effects of mortality risk before retirement - unambiguously lowering C^p_t, but increasing the ratio of total lifetime earnings to wealth. But the proof of the proposition shows that the net effect is to reduce $c^*_{0,t}$ unambiguously compared to the Modigliani Case.

Relating this expression to the capitalisation factors in Proposition 1, given the restrictions, the capitalisation factor for the lump sum, C_t^Z, simply equals the number of years' extra pension provided as a lump sum, Z; and the capitalisation factor for the spouse's pension C_t^s equals the effective number of additional years the pension will be paid to the spouse, in expectation.[14] The total capitalisation factor C_t then equals $\widehat{E}_t$,which can be interpreted as "effective life expectancy at retirement". We show below that the quantitative impact of the additional benefits is nontrivial; however, given the conceptual equivalence in this benchmark case, in the remainder of our analytical results we focus primarily on the case of a pure pension liability.

2.5 An Illustrative Example: Key elements in Proposition 2 for the UK Universities' Superannuation Scheme (USS)

While the assumptions of zero real yields and zero wage growth in Proposition 2 are clearly restrictive, the resulting expression for the fair contribution rate turns out to provide a key element of our approximation, derived below. We therefore take a first look at our illlustrative example.

Table 1 illustrates using UK mortality rates and scheme parameters from the USS, the defined benefit scheme for academics in UK universities.[15] We illustrate for two cases, $R = 20$ (hence for a scheme member joining at age $A = 47$) and $R = 40$, implying $A = 27$.

[14] Noting that given our assumption of the veil of ignorance on the gender of the scheme member there is a 50:50 chance that the spouse will outlive them.

[15] For details of data and scheme parameters, see Appendix F.

Table 1.

Key Determinants of the Fair Contribution Rate in Proposition 2.[16]

(given $y(\tau) = g(\tau) = 0$)

Scheme retirement age	A_R	67
Life expectancy at retirement	$E_t = \mathbb{E}_t \left[\widetilde{A}_D - A_R \middle| \widetilde{A}_D > A_R \right]$	18.25 years
Probability of surviving to retirement	$S_t(R)$	$R = 20$ $R = 40$ 89.8% 88.0%
Inverse accrual rate	Y	75 years
Pure Pension Case $(s = Z = 0)$		
Fair contribution rate (Modigliani Case)	$c^*_{M,t} = \frac{E_t}{Y}$	24.3%
Fair contribution rate	$c^*_{0,t} = \frac{E_t}{Y} Q_t$	$R = 20$ $R = 40$ 22.7% 22.1%
Impact of Additional Benefits $(s > 0, Z > 0)$		
Lump sum as multiple of pension	Z	3
Surviving spouses's life expectancy	$\mathbb{E}_t \left[\widetilde{A}^S_D - \widetilde{A}_D \middle| \widetilde{A}^S_D > \widetilde{A}_D > A_R \right]$	6.1 years
Spousal pension percentage	s	50%
"Effective life expectancy at retirement"	$\widehat{E}_t = E_t + s \times \frac{S(R)}{2} \times 6.1$	$R = 20$ $R = 40$ 22.62 22.59
Fair contribution rate	$c^*_{0,t} = \frac{\widehat{E}_t}{Y} Q_t$	$R = 20$ $R = 40$ 28.15% 27.39%

The key features the table brings out are:

- The fair contribution rate $\frac{E_t}{Y}$ in the "Modigliani Case" (with a pure pension liability and $S_t(R) = 1$) is just under one quarter, and invariant to age.[17]
- The counterpart to this invariance reflects the fact that both the present value of earnings in the scheme, $\mathcal{W}_t$ and scheme liabilities $\mathcal{L}_t$ scale with $R = A_R - A$, hence the impact of age cancels out precisely.
- Once account is taken of mortality risk before retirement, the implied fair contribution rate $c^*_{0,t}$ for a pure pension liability is close-to-invariant to the scheme

[16] All estimates use mortality statistics from the UK Life Tables, based on data for 2017-2019, and all calculations are for the zero yield, zero wage growth case of Proposition 2 and corollaries.

[17] In terms of the original Modigliani framework, it corresponds to the hypothetical saving rate of an individual who wished to maintain a constant consumption level after retirement, but with a notional working life $R = Y = 75$ years. Since in practice $R << Y$ clearly, as noted above, in relation to Proposition 2, the implied pension level is always significantly lower than the average income during the working life.

member's age, but with a modestly smaller value of $c^*_{0,t}$ for those joining the scheme earlier in their career (hence with a higher value of R), reflecting a slightly lower probability of surviving into retirement.

- The additional benefits of a lump sum and the spousal pension increase "Effective Life Expectancy," $\widehat{E}_{t,}$ nontrivially relative to actual life expectancy in retirement, $E_{t,}$ and thus also increase $c^*_{0,t}$ by more than five percentage points compared to the case of a pure pension liability.[18]
- Clearly for any given accrual rate, Y, and scheme retirement age A_R, the implied values of of $c^*_{0,t}$ are increasing in life expectancy. Improvements in life expectancy have increased $c^*_{0,t}$ by around 7 percentage points over our sample period.[19]
- The latest implied values of $c^*_{0,t}$ are somewhat below the actual contribution rate for new members of 34.7% on the latest USS rules (made up of an employee contribution of 11% and the employers' contribution of 23.7%). However this comparison ignores both the impact of additional life insurance components in the USS, not allowed for in our analysis, and - crucially - the impact of interest rates.

Clearly the case analysed in Proposition 2 is very restrictive. We thus need to establish how c^*_t is affected by non-zero values of both yields $y_t(\tau)$ and wage growth $g_t(\tau)$.

2.6 The impact of interest rate changes on the fair contribution rate.

We first investigate this sensitivity of c^*_t to the general level of interest rates.

In line with common practitioner usage, define the (modified) durations of $\mathcal{W}_t$ (the present value of career earnings) and and $\mathcal{L}_t$, liabilities as

$$\delta^{\mathcal{W}}_t \equiv -\frac{\partial \ln \mathcal{W}_t}{\partial r_t}, \; \delta^{\mathcal{L}}_t \equiv -\frac{\partial \ln \mathcal{L}_t}{\partial r_t} \tag{20}$$

[18]Note that while actual life expectancy in retirement E_t is invariant to age, Table 1 shows that effective life expectancy in retirement $\widehat{E}_t$ decreases with $R = A_R - A$ (albeit to a very modest extent) given the dependence of the value of the spousal pension on $S_t(R)$: for a younger scheme member, it is somewhat less likely that their spouse will survive them.

[19]In Figure 1 below we plot our calculated value of $c^*_{0,t}$ over time.

given a parallel shift in the zero coupon yield curve, such that

$$\frac{\partial y_t(\tau)}{\partial r_t} = 1 \ \forall \tau \tag{21}$$

where $r_t = y_t(0)$ is the instantaneous spot rate at t. Then,

Proposition 3 *The durations at t of the scheme's wealth ($\mathcal{W}_t$) and liabilities ($\mathcal{L}_t$), as defined in (20), given (21), satisfy the following inequalities:*

$$\delta_t^{\mathcal{W}} < R < \delta_t^{\mathcal{L}}, \tag{22}$$

hence the semi-elasticity $\delta_t^{c^*} = -\frac{\partial \ln c_t^*}{\partial r_t}$, or "notional duration", of the fair contribution rate satisfies

$$\delta_t^{c^*} = -\frac{\partial \ln c_t^*}{\partial r_t} = \delta_t^{\mathcal{L}} - \delta_t^{\mathcal{W}} > 0. \tag{23}$$

Thus a parallel upward shift in yields unambiguously lowers c_t^*.

Proof. See Appendix B. ■

Since c_t^* is the ratio of liabilities to the present value of the scheme members' earnings, its "notional duration" $\delta_t^{c^*} = -\frac{\partial \ln c_t^*}{\partial r_t}$ is simply equal to the difference between the two durations. We show below that this notional duration is equal to the duration of the notional bond with price $\mathcal{P}_t$, referred to in the introduction, in the neighbourhood of the zero yield, zero growth case.

The intuition for the property that c_t^* is strictly decreasing in yields is very straightforward. Since all payouts from the scheme will occur after $t + R$, the liabilities of the scheme will have duration greater than R. On the other hand assets must have duration strictly less than R, and hence $\delta_t^{\mathcal{L}} \geq R > \delta_t^{\mathcal{W}}$. As a result the notional duration of c_t^* must be positive, hence an upward parallel shift in the yield curve must lower c_t^*.

Note that while Proposition 3 only establishes the sign of this notional duration, we show below that we can also quantify it, to a good approximation, in the neighbourhood of the zero yield, zero growth case.

2.7 The (minimal) impact of wage growth

While Proposition 3 makes it clear that interest rates play a crucial role in determining c_t^*, it is relatively straightforward to show that, in contrast, wage growth through the working life plays a very limited role.

Without much loss of generality we analyse the case where the salary growth rate is constant, $g(\tau) = g \ \forall\tau$, and focus on properties in the neighborhood of $g = 0$. It is then straightforward to show the following:

Proposition 4 *Letting*

$$\delta_g^{c^*} = \left.\frac{\partial \ln c_t^*}{\partial g}\right|_{g(\tau)=g=0 \ \forall\tau} \tag{24}$$

then

$$\delta_g^{c^*} = \frac{R}{2} - \left.\delta^{\mathcal{W}}\right|_{g=0} \tag{25}$$

which in general is of ambiguous sign. However,

$$y_t(\tau) + \Lambda_t(\tau) \gtreqless 0 \ \forall\tau \leq R \Rightarrow \delta_g^{c^*} \gtreqless 0 \tag{26}$$

Proof. See Appendix C. ■

The ambiguity of the impact of wage growth can be related fairly straightforwardly to the Modigliani Case analysed above. We show below, in relation to Proposition 5, that in this case $\delta^W = \frac{R}{2}$ and hence $\delta_g^{c^*} = 0$ precisely. Away from this restricted case, if both yields and hazard rates are positive, it is straightforward to show that $\delta^W < \frac{R}{2}$, hence $\delta_g^{c^*} > 0$. Additionally, since hazard rates are relatively low before retirement, the sign of $\delta_g^{c^*}$, is approximately equal to the sign of yields before retirement.

We show in Appendix F.5 that, due to the offsetting terms in Proposition 4, the magnitude of $\delta_g^{c^*}$ is always small, thus in our approximation we focus on the case where $g = 0$.[20]

3 Approximating the Fair Contribution Rate

We now exploit the results in Propositions 2, 3 and 4 to derive an approximation for the fair contribution rate derived in Proposition 1. To simplify the analysis, we focus on the case of a pure pension liability, and, exploiting the ambiguity in Proposition 4, we assume wage growth is zero.

[20] In section 5.1 we show that the case for ignoring the impact of wage growth is further accentuated if we allow for the possibility of scheme members leaving the scheme before retirement. In contrast, we also show, in Section 5.2, that if the scheme calculates the pension in terms of the final, rather than average salary, g has a strongly positive impact on c_t^*.

3.1 The Fair Contribution Rate for a pure pension liability with a flat yield curve

Key elements of the approximation can be derived from analysing the sensitivity of c_t^* to the general level of yields under the following assumption:

Assumption A2: *The yield curve is flat and wage growth is zero:* $y_t(\tau) = r_t$, $g_t(\tau) = 0, \forall \tau$.

We then have:

Proposition 5 *Under Assumption A2, the fair contribution rate c_t^* for a pure pension liability $(s = Z = 0)$ can be approximated by*

$$\ln c_t^*(r_t) = \ln c_{0,t}^* - \delta_{0,t}^{c^*} r_t + O\left(r_t^2\right) \tag{27}$$

where $c_{0,,t}^* = \frac{E_t}{Y} Q_t$, *is as defined in Proposition 2 and* $\delta_{0,t}^{c^*} \equiv -\frac{d \ln c^*}{dr_t}\Big|_{y_t(\tau)=g_t(\tau)=0}$ *is the notional duration of Proposition 3 for the zero yield, zero growth case of. Furthermore,*

$$\delta_{0,t}^{c^*} \approx \frac{R + E_t}{2} + \frac{1}{2}\left(\frac{\sigma_{D>R}}{\sqrt{3}} + \frac{R}{3}\left(\frac{1 - S_t(R)}{1 + S_t(R)}\right)\right) \tag{28}$$

where $\sigma_{D>R}^2$ is the variance of mortality risk, conditional upon surviving to retirement, for any survival probability function $S_t(\tau)$ sufficiently close to piecewise linearity over (t, R) and $(R, R + 2E_t)$.

Proof. See Appendix D. ■

The first approximation arises straightforwardly, since, under A2,, for a given hazard rate function $\lambda_t(\tau)$, and a given value of $R \equiv A_R - A$, c_t^* is simply a function $c_t^*(r_t)$ of the instantaneous interest rate, which admits a semilog-linear approximation in the neighbourhood of the zero yield, zero growth case.

The second approximation exploits the properties of c_t^* in the neighbourhood of the zero yield, zero growth case of Proposition 2 to derive an approximation for the notional duration $\delta_{0,t}^{c^*}$ itself, in the neighbourhood of this case.

For intuition on the approximation (28), consider first the case where the age of death A_D is known precisely (hence $\sigma_{D>R}^2 = 0$, and $S_t(R) = 1$), i.e., as in the Modigliani Case of Corollary 1. In this case only the first term in (28) applies, and the intuition is quite straightforward. Under the assumptions underlying Proposition 2, the present value of earnings would simply equal $w_t R$. The duration of wealth $\delta_{0,t}^{\mathcal{W}}$ would then simply be the average maturity of all wage payments, i.e. $\frac{R}{2}$. The duration of liabilities at $t+R$ in this case would simply be $\frac{E_t}{2}$, but a zero coupon bond that would provide the lump sum required to buy the annuity at time $t+R$ would have duration R at time t, hence $\delta_{0,t}^{\mathcal{L}} = R + \frac{E}{2}$, and hence in turn, from Proposition 3, $\delta_{0,t}^{c^*} = \delta_{0,t}^{\mathcal{L}} - \delta_0^{\mathcal{W}} = \frac{R+E_t}{2}$.

Given that $R \equiv A_R - A$, and in this case $E_t = A_D - A_R$, it folllows that $R + E_t = A_D - A$, so the notional duration of c_t^* itself would then be equal to the duration of a zero coupon bond with maturity equal to half the scheme member's life expectancy at time t, or, equivalently the average duration of a hypothetical annuity paid at a constant rate over the scheme member's *entire* remaining lifetime (i.e., not just after retirement).

The approximation in (28) shows that, absent any countervailing impact of higher moments, the notional duration $\delta_{0,t}^{c^*}$ is unambigously increasing in mortality risk, reflecting the impact of mortality risk on the duration of both the assets and liabilities of the scheme.

The first term in brackets, reflecting the impact on liabilities, has a nontrivial effect. For intuition here, consider the case where, after retirement, the date of death is uniformly distributed around E_t. In this special case the scheme liability would be the present value of a pension that will be paid for up to $2E_t$ years, with linearly declining probability, in contrast to the deterministic case where it will be paid with certainty for only E_t years. In the former case, $\delta_{0,t}^{L} = R + \frac{2}{3}E_t$, and the approximation in the proposition follows since under the same assumptions $\sigma_{D>R} = \frac{E_t}{\sqrt{3}}$.[21]

The second term, reflecting the impact of mortality risk on δ^W, is quantitatively much less significant, but reflects the property that mortality risk before retirement somewhat reduces the duration of the scheme's assets, and thus somewhat *in*creases the sensitivity of c_t^* to the general level of yields.

In Appendix F.6 we show that the empirical survival probability function is sufficiently close to piecewise linearity that (28) provides a close approximation.

Exploiting the approximation in (28), since $R + E_t = A_R + E_t - A$, it follow that $\delta_0^{c^*}$ is decreasing in A (but invariant to the scheme retirement age, A_R). Since $c_{0,t}^* = c_{M,t}^* Q_t$, and $c_{M,t}^* = \frac{E_t}{Y}$ is invariant to age, this means that age only enters the approximation via its impact on the notional duration $\delta_0^{c^*}$ and on $S_t(R)$, and hence Q_t in Proposition 2. For positive yields these terms reinforce, meaning that younger members have an unambiguously lower fair contibution rate; for negative yields they are offsetting.

[21] In the proof of the proposition the approximation in (28) is derived as an approximation around the uniform case.

3.2 Generalising the approximation.

While the assumption of a flat yield curve in Proposition 5 is clearly restrictive, the approach generalises fairly straightforwardly to a general yield curve structure, as follows:

$$c_t^* \approx \frac{E_t}{Y} Q_t \exp\left(-\widehat{\delta}_{0,t}^{c^*} y_t^{c^*}\right) \tag{29}$$

where $\widehat{\delta}_0^{c^*}$ is given by the approximation in (28), and $y_t^{c^*}$ is an appropriately weighted average level of yields, given by

$$y_t^{c^*} = y_t^{\mathcal{L}} + \frac{\delta_{0,t}^{\mathcal{W}}}{\widehat{\delta}_0^{c^*}} \left(y_t^{\mathcal{L}} - y_t^{\mathcal{W}}\right) \tag{30}$$

where $y_t^{\mathcal{L}}$ and $y_t^{\mathcal{W}}$ are average yields, weighted by life expectancy, for liabilities and wealth respectively, defined by

$$y_t^{\mathcal{L}} = \int_R^\infty \mu^{\mathcal{L}}(\tau)\, y_t(\tau)\, d\tau;\ \mu^{\mathcal{L}}(\tau) = \frac{\tau S_t(\tau)}{\int_R^\infty \tau S_t(\tau)}; \tag{31}$$

$$y_t^{\mathcal{W}} = \int_0^R \mu^{\mathcal{W}}(\tau)\, y_t(\tau)\, d\tau; \quad \mu^{\mathcal{W}}(\tau) = \frac{\tau S_t(\tau)}{\int_0^R \tau S_t(\tau)} \tag{32}$$

The approximation in (29) can be viewed in two complementary ways. It can be derived analogously to the approximation in Proposition 5 as a first-order semilog approximation with respect to points on the observed yield curve at time t. But it can also be viewed as a direct application of the approximation in Proposition 5 to a hypothetical flat yield curve with $r_t = y_t(\tau) = y_t^{c*}\ \forall\tau$. In this hypothetical case we can also write (29) as in the Introduction, as

$$c_t^* \approx \frac{E_t}{Y} \times Q_t \times \mathcal{P}_t \tag{33}$$

where $\mathcal{P}_t = \exp\left(-\widehat{\delta}_{0,t}^{c^*} y_t^{c^*}\right)$ is the price of a zero coupon bond with maturity $\widehat{\delta}_{0,t}^{c^*}$ in this hypothetical case.

A key feature of $y_t^{c^*}$ is that it is a weighted average of the two component yields but with a *negative* weight on $y_t^{\mathcal{W}}$. Thus if the yield curve slopes upwards ($y_t^{\mathcal{L}} > y_t^{\mathcal{W}}$) then the weighted yield is *higher* than the yield on liabilities (and hence typically above the asymptote of the yield curve).

4 The illustrative example revisited: the fair contribution rate for the UK Universities' Superannuation Scheme (USS)

4.1 The fair contribution rate for a pure pension liability

Figure 1 plots the fair contribution rate (the sum of employer's and employee's contributions) on a log scale, alongside the approximation in (29), for the case of a pure pension liability, for a scheme member joining the scheme 40 years before retirement, using the scheme parameters $Y = 75$, $A_R = 67$, as in the current USS scheme. The exact calculation uses the formula for c_t^* in Proposition 1, while the the approximation uses (29), in both cases setting $Z = s = 0$. It shows that the approximation captures the key features of the exact calculation.

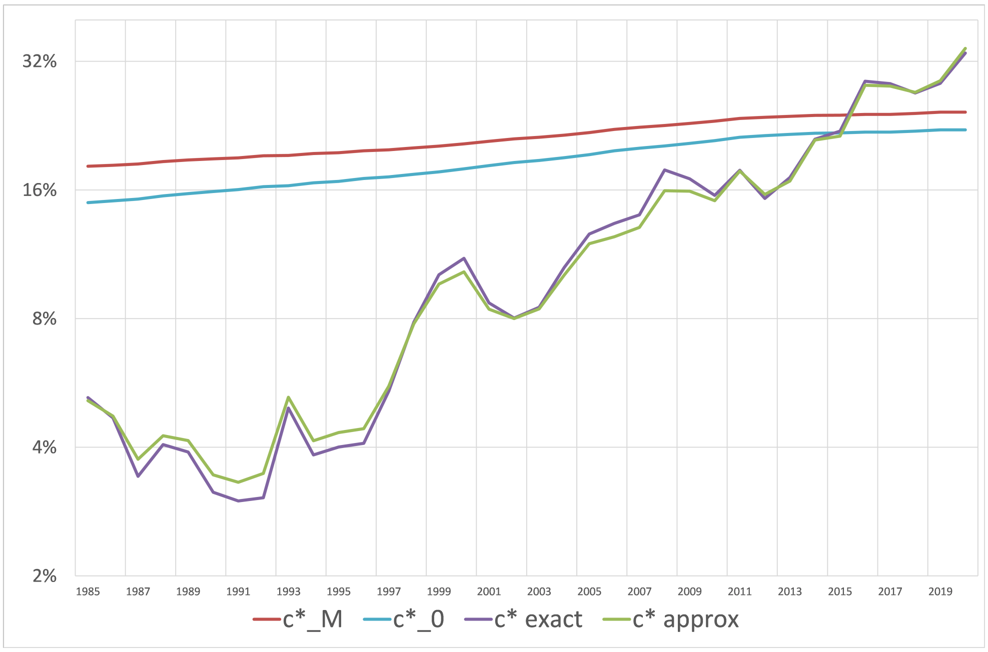

Figure 1: The fair contribution rate for a pure pension liability ($R = 40$, $Y = 75$, $A_R = 67$, $Z = s = 0$)

Figure 1 also shows the time path of the two special cases of the fair contribution rate from Proposition 2: the "Modigliani Case", $c_{M,t}^* = \frac{E_t}{Y}$, and the zero yield, zero growth case, $c_{0,t}^* = \frac{E_t}{Y} Q_t$. Both have risen by nontrivial amounts, driven by reductions in mortality risk; and the same determinants have also driven them closer together, given rises in $S_t(R)$, the probability of surviving into retirement.

But, significant as these rises are, they are dwarfed by the impact of falling real yields. Over the sample period shown, from 1985 to 2020, c_t^* on this basis increased more than sixfold. The dominant explanation, in terms of our approximation, lies in the price $\mathcal{P}_t$ of the notional bond that scales $c_{0,t}^*$ in (29). In 1985 the weighted average real yield $y_t^{c^*}$was around 2.5% (CPI-adjusted), implying $\mathcal{P}_t \approx \frac{1}{3}$ (implying that a fairly priced DB scheme would have been relatively "cheap"), whereas in the most recent data (end-2020) $y_t^{c^*}$was around -1.3% on the same basis, implying $\mathcal{P}_t \approx \frac{3}{2}$.Thus the reduction in yields alone would have required a roughly $4\frac{1}{2}$ fold increase in c_t^*, with the remainder of the required increase driven by increases in life expectancy.

Clearly the impact of yields is increasing in the duration $\delta_{0,t}^{c^*}$ of the notional bond in (33), The sensitivity to yields is accentuated in more recent data by increasing life expectancy, albeit only modestly,: for $R = 40$ (hence $A = 27$) the duration of the notional bond increase from 29 years in 1985 to 31 years in 2020.

4.2 The Impact of Age on Fair Contribution Rates

Our analysis shows that only under very special circumstances are fair contribution rates invariant to $R = A_R - A$, the number of years to retirement, and hence, for a given scheme retirement age, A_R, to age, A. Only in the Modigliani case, with $S_t(R) = 1$ does age drop out of the calculation.

In more general cases, with $S_t(R) < 1$, the first term, $Q_t \in (S_t(R), 1)$ in the approximation in (29) turns out to have only limited dependence on age in the data since, for a scheme member with, for example, 40 years to retirement, mortality rates in the first 20 years are very low. Therefore while $S_t(20) > S_t(40)$, the fair contribution rates in the zero yield case of Proposition 2, $c_{0,t}^* = \frac{E}{Y} Q_t$ plotted in Figure 2 show that the impact of R on the fair contribution rate is trivially small for yields close to zero. However, Figure 2 also shows that for non-zero yields the dependence on R (and hence inverse dependence on age) can be very significant. In the early part of the sample shown in the chart, strongly positive real yields meant that counterfactual fair contribution rates for young (high R) members joining the scheme would have been significantly lower than for older (low R) members, thus implying nontrivial redistributions between scheme members. This situation would only have been reversed in the most recent years, once real yields became negative.

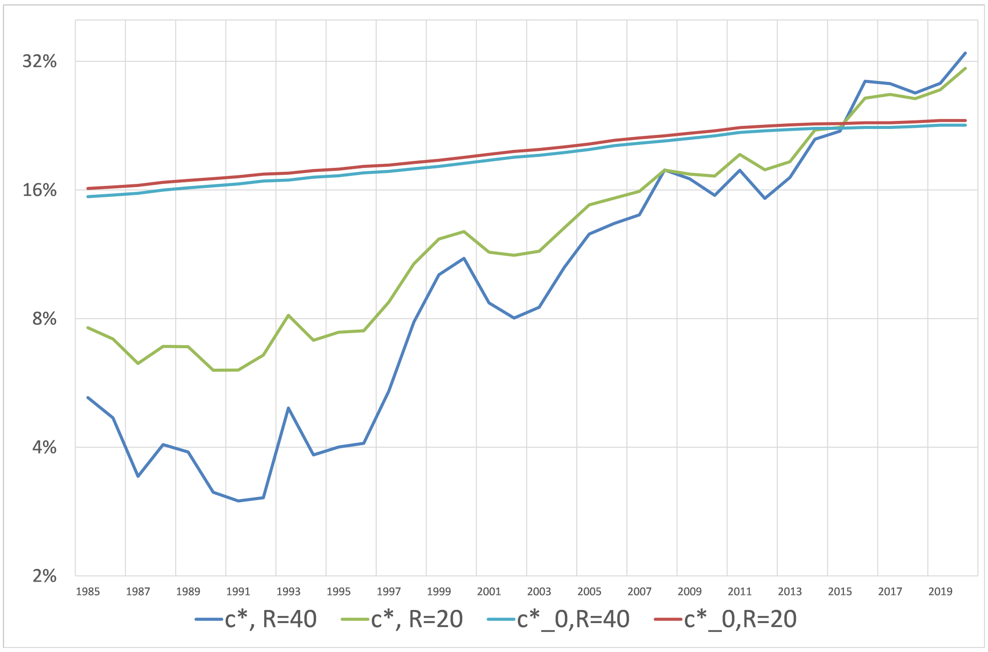

Figure 2: Counterfactual fair contribution rates under current USS rules for $R = 20$ vs $R = 40$

4.3 Impact of additional benefits on fair contribution rates

Figures 1 and 2 focussed on the simplest case of a pure pension liability. But DB schemes do not only provide pensions.

In our central case we also considered the impact of a promise on a lump sum of Z years' worth of pension, (in the USS case $Z = 3$) and the impact of a spousal pension at a rate of s times the accrued pension, payable if the scheme member predeceases their spouse after retirement (in the USS case $s = 0.5$). Corollary 2 showed that, in the zero yield case of Proposition 2 these increase the "effective life expectancy" of the scheme member, $\widehat{E}_t$, relative to the true life expectancy in retirement, E_t. Table 1 showed that, using the most recent UK life tables these additional benefits raise $\widehat{E}_t$ by around 4 years and hence, for given scheme parameter Y, increase both $c^*_{M,t}$ and $c^*_{0,t}$ by a factor of around $1\frac{1}{4}$. For non-zero yields, this scaling factor does not translate precisely into our approximation in (29) since the liabilities associated with these additional benefits do not have the same duration as the pension liability. To the extent that both payments increase the duration of the notional bond, in the most recent data they increase c^*_t somewhat more than proportionately to the increase in $c^*_{0,t}$.[22]

[22] The lump sum, for example, clearly has duration precisely equal to R. For relatively young scheme members this will be higher than the approximated value of $\delta^{c^*}_{0,t}$ for the pure pension liability in (28):

We also noted that many DB schemes also offer life insurance components. Clearly these must also increase c_t^*, although, since they are of shorter duration, they somewhat reduce the sensitivity to yields, and, in contrast to all other factors the impact on c_t^* has been somewhat reduced by improvements in life expectancy.[23]

Figure 3 shows how the cumulative contribution of each of these additional components has evolved over time. In the most recent data they increase c_t^* from around 33% for a pure pension liability, as in Figure 1, to close to 46%.

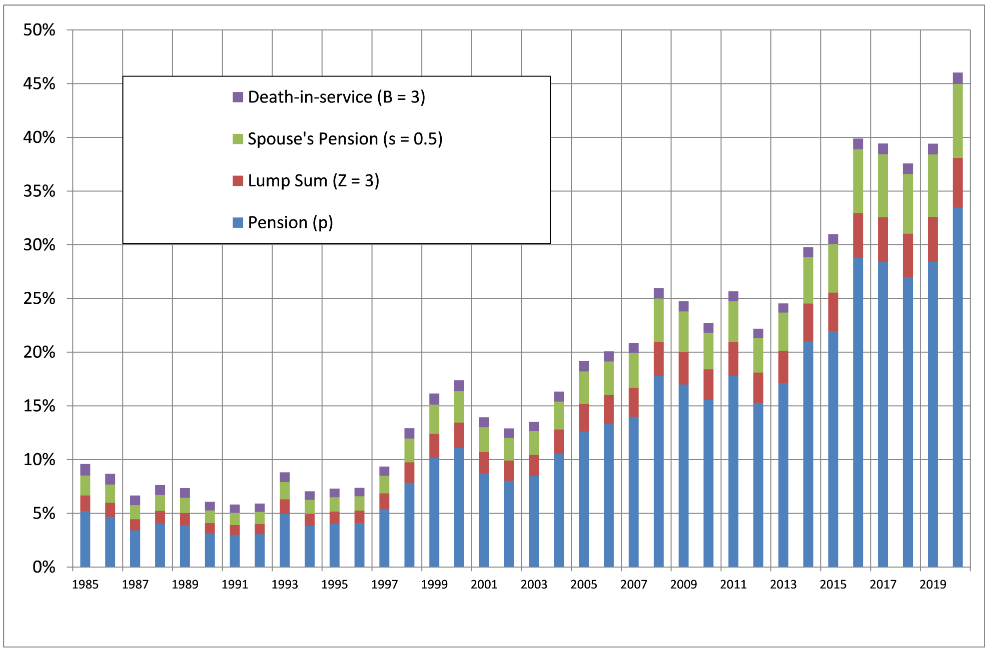

Figure 3: Decomposition of contribution rate for current USS rules, $R = 40$

4.4 Impact of Changes to USS Rules

The calculations underlying Figures 1, 2 and 3 were on a strictly counterfactual basis, on the basis of current scheme rules. In reality, USS scheme rules have changed over time, with at times significant impacts on implied fair contribution rates. Figure 4 summarises the impact of these changes, and also compares the resulting figures with the actual contribution rate paid by scheme members at any point in time.[24]

eg as noted above for $R = 40$, $\delta_{0,t}^{c^*} \approx 31$ years on most recent data. The duration of $\mathcal{L}_t^s$, the liability for the spousal pension, will also clearly be longer than that of the member's pension, since it it will only be paid after the member's death.

[23] Appendix F.3 describes the methodology by which we incorporate life insurance components into

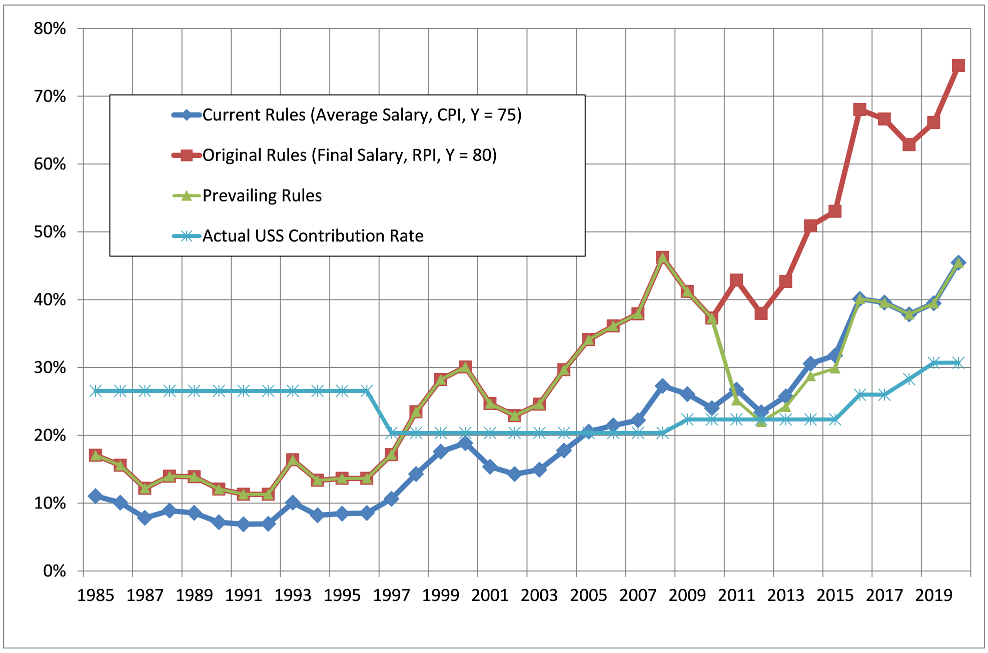

Figure 4: $c^*_{USS,t}$ under different scheme rules, $R = 40$, wage growth $g = 2\%$

The calculations in Figure 4 are carried out for a new scheme member joining the scheme, with $R = 40$ years to retirement. Note that, in contrast to our main analysis, the calculations here assume positive wage growth $g = 2\%$ at an annual rate through the working life.[25] While we have shown that wage growth has a minimal effect on fair contribution rates when pensions are based on career average earnings, wage growth has a very much stronger effect in final salary schemes,[26] which is relevant to earlier USS rules.

The first series (dark blue) shows the counterfactual value $c^*_{USS,t}$ under current scheme rules (CPI indexation, final salary, and $Y = 75$). It is almost identical to the same series in Figure 3[27]. The second series (red) shows the implied contribution rate on USS scheme rules until 2010, of final salary basis, RPI indexation and $Y = 80$. The third series (green) is the implied value of $c^*_{USS,t}$ under prevailing USS rules at any point in

our calculations.

[24] The details of how the scheme rules have changed are given in Appendix F.4.

[25] The assumption of $g = 2\%$ implies somewhat more than doubling of salaries in real terms over a 40 year working life - a quite conservative assumption given the combination of semi-automatic progression through salary scales, and, in most cases, promotions to higher salary scales, in the UK academic system.

[26] We show this formally below, in Section 5.2.

[27] In line with Proposition 4, in a career average scheme the impact of positive wage growth raises the counterfactual c^*_t somewhat in the earlier part of the sample, when yields were positive, and lowers it marginally in more recent years when yields were negative.

time.[28] As discussed in Section 5.2, given even a relatively modest growth rate of salaries the fair contribution rate with a final salary scheme is markedly higher. As a result, had the final salary component of the scheme been maintained (along with RPI indexation) the fair contribution rate would have been around 75% by the end of our sample.

Finally, Figure 3 also shows (light blue) the actual contribution rates for the USS scheme at any point in time. Strikingly, until the second half of the 1990s, this rate was well *above* the fair contribution rate on prevailing rules (shown in green), implying that scheme members and employers systematically overpaid for the benefits the scheme provided, and sometimes by a very wide margin.[29]

However, in all but one year since the late 1990s scheme members have paid less (often significantly less) than fair contribution rates. The contribution rate was increased to 34.7% in October 2021 (11.0% employee contribution, 23.7% employer contribution). It is worth noting that the scheme sponsor's recent proposed rise would increase the rate to 42.1% (13.6% and 28.5%, respectively), still somewhat below the estimated fair rate shown in the chart.

Without asserting any clear empirical regularity, it is fairly easy to see in Figure 4 a tendency for both contribution rates and scheme rules to be adjusted in response to the gap between actual and fair contribution rates in earlier years; but this is (perhaps unsurprisingly) distinctly more marked when actual rates were below fair rates, rather than above them. Thus, the cut in contribution rates in the second half of the 1990s can be viewed as a (modest) adjustment to the fact that scheme members had been systematically over-contributing in the previous decade and a half. Ironically, it occurred at a time when fair contribution rates were actually rising steeply, as real yields fell, such that the cut in contribution rate ended up accentuating the extent to which scheme members were *under*-contributing. In due course the gap was closed (if only for a very short while) by a combination of increased contribution rates and (more crucially) changes in scheme rules (most notably, as discussed above, the shift from a final to average salary scheme). But neither of these changes have been sufficient to offset the impact of continued rises in fair contribution rates as real yields have continued to fall.

[28] Between 2011 and 2015 the prevailing rules implied a slightly lower fair rate since during this period, $Y = 80$, compared to $Y = 75$ on current rules.

[29] It is striking that an implied markup of, occasionally, more than 100% during this period dwarfs the implied markups highlighted by the literature on "money's worth" for annuities (Koijen & Yogo, 2015; Cannon & Tonks, 2016, Poterba & Solomon, 2021; Verani and Yu, 2021).

5 Robustness Checks

In this section we we consider the robustness of our analysis to various modifications to our framework.

5.1 The impact of early leavers

The first modification, which allows for the possibility of early leavers from the scheme, strengthens the case for ignoring the impact of g in the approximation.

The calculation underpinning the general formula for c_t^* in Proposition 1 is predicated on the apparently restrictive assumption that the scheme member will stay in the scheme, with certainty, until retirement at $t+R$. However, a small amendment to our framework makes it straightforward to show that the impact of this assumption is minimal.

Consider the case where the scheme member has a constant instantaneous probability q of quitting the scheme at time $t+\tau$ for $\tau \in [0, R]$. Upon quitting, the scheme's pension liability is fixed at the level accrued at the time they quit, but so are accrued assets. The pension payout still starts at $t + R$. Again, for simplicity, assuming constant growth, $g(\tau) = g$, the expected pension is then given by,

$$\mathbb{E}_t [p_t] = \frac{1}{Y} \int_0^R e^{(g-q)\tau} d\tau \tag{34}$$

while the relevant measure of wealth, out of which contributions are accrued, similarly becomes

$$\mathcal{W}_t = w_t \int_0^R e^{(g-q-\Lambda(\tau)-y(\tau))\tau} d\tau. \tag{35}$$

But these expressions are identical to our central case, if g is replaced with $g - q$, and thus can be analysed using Proposition 4, which showed near-invariance to g. Since the impact of early leavers is to lower the effective growth rate of earnings of the typical scheme member, this reinforces the case for ignoring the impact of wage growth in our approximation.

5.2 A final salary scheme

Historically, DB pension schemes have also calculated pensions on the basis of final salary, rather than average salary, as in our main analysis. In such schemes, in marked contrast to our central case, the impact of wage growth cannot be ignored.

Again, for simplicity, we focus on the case of assuming a constant growth rate g. It is then straightforward to show

Corollary 3 *For a final salary scheme, and constant wage growth g, with pension given by*

$$p_t^{FS} = \frac{R}{Y} w_t e^{gR} \tag{36}$$

then

$$\delta_g^{c^{*FS}} = R - \delta^{\mathcal{W}}\big|_{g=0} > 0 \tag{37}$$

Proof. See Appendix E. ■

The impact of wage growth in this case is both unambiguous in sign (since Proposition 3 showed that $\delta^{\mathcal{W}} < R$) and quantitatively significant, since, as discussed previously, for a range of cases, $\delta^{\mathcal{W}} \approx \frac{R}{2} \Rightarrow \delta_g^{c^{*FS}} \approx \frac{R}{2}$. It is therefore clear that in the case of a final salary scheme the approximation would need to take this impact into account: ie, the implied value of c^* would be higher than in (29) by a factor of approximately $e^{\frac{Rg}{2}}$. Figure 4 showed that in the case of our USS example, had the scheme continued to pay benefits based on final salaries, fair contribution rates for $R = 40$, $g = 2\%$ would by 2020 have risen above 70%.

5.3 Robustness to Mortality Assumptions

Our approach to mortality risk can be criticised on a number of grounds.

There is a substantial literature on the pricing of annuities which, building on Rothschild & Stiglitz (1976) and Finkelstein and Poterba (2004) focusses on a potential adverse selection problem that those choosing to buy annuities may do so because they know themselves to have lower mortality risk. In the absence of a separating equilibrium this may lead to market prices of annuities being higher than if they were priced on the basis of mortality risk for the entire population. For example Poterba & Solmon (2021) show that money's worth calculations for US annuities are significantly impacted by whether mortality risk is estimated for the population as a whole, or for the self-selecting sample of annuitants, with implied prices for the latter being signficantly higher.

On the face of it, we might defend our use of population mortality risk from UK Life Tables by arguing that, since members of DB schemes such as USS are typically automatically enrolled, this may mitigate, or possibly eliminate, any adverse selection bias. However Poterba and Solomon (2021) also note that other characteristics of annuitants such as higher educational status or earnings power, may also help to explain their lower mortality risk. Since these characteristics are more likely also to apply to members of the USS scheme (and possibly members of DB schemes more generally) this source of potential bias cannot be so easily ignored, implying that our calculated fair contribution rates are likely to be underestimates.

To assess the quantitative significance of this, in Appendix F.1 we follow Cannon & Tonks (2016) and show the impact of using an alternative set of life tables based on mortality rates of members of DB schemes administered by life insurers. This results in an estimate of life expectancy at retirement, E_t around 10% longer than using ONS life tables, and thus, taken in isolation, suggests c_t^* may be underestimated by the same proportion.

An additional factor that we do not address is the role of the stochastic properties of mortality risk itself. While we assume the survival probability function $S_t(\tau)$ to be time-varying, implying that life expectancy in retirement E_t is also time-varying, at any time t, the $S_t(\tau)$ is treated as known for any time $t + \tau$. Poterba & Solomon (2021) note that insurance companies typically build in an allowance for falls in mortality rates based on historic trends; while Cannon & Tonks (2016) argue that uncertainty about future mortality may have a significant positive impact on the price of annuities sold by insurance companies, with a stronger effect, the more conservative issuers are required to be in their treatment of mortality risk.[30] They also show that the impact of uncertainty about mortality risk is increasing, the lower are market yields. By implication this points to a further argument that our estimates of recent rises in c_t^* are understated.

In summary it appears fairly clear that our treatment of mortality risk is, other things being equal, likely to lead to our understating fair contribution rates.

5.4 Robustness to discounting assumptions

5.4.1 Liabilities

In contrast to the fairly clear-cut implications of the annuities literature for mortality risk, there is no clear-cut consensus on the appropriate discount rate to be used in calculating implied liabilities. Some authors (eg, Cannon & Tonks, 2016; Koijen and Yogo, 2015) assume, as we do, that the appropriate discount rate for any scheme liability is the yield on a risk-free government bond of matching duration. But other more recent contributions to the literature (eg Poterba and Solomon, 2021; Verani and Yu (2021))

[30] They also argue that this impact is likely to be hard to distinguish from the adverse selection effect noted above.

argue that, since insurance companies - at least in the United States - typically back annuities with investments in relatively low risk corporate bonds, it is more appropriate to use corporate yield data.[31]

It is crucial to stress that both these authors take it as given that that fair valuations must reflect the risk-free nature of the promises made, and thus are in principle consistent with our approach; their arguments focus instead on the nature of the required arbitrage. Thus Verani and Yu (2021) argue that, since annuities are risk-free but illiquid, issuers of annuities need to compensate annuitants for the loss of the "convenience yield" from holding liquid risk-free assets, and thus must (and, at least in the United States, do) invest in higher yielding corporate bonds. But they also note that given the market friction that the duration of liabilities exceeds the duration of corporate bonds, issuers carry interest rate risk, which means they must hold higher reserves in order to guarantee payments, and thus in turn must in equilibrium charge prices that are higher than notional fair prices calculated using a corporate yield curve.[32]

But this in turn means that, in the context of more complex hedging strategies, the concept of "fairness" becomes less clear-cut, if some part of the apparent markup over notional fair present values reflects compensation for the necessary imperfections of the hedging process.

5.4.2 Assets/Present value of earnings

To the extent that there is covariation between wage growth and market risk factors, we would normally expect this to be positive. If so the implied value of $\mathcal{W}_t$, as defined in (4) represents an upper bound to the present value of the scheme member's earnings within the scheme, thus, in isolation, implying, from (10), some element of underestimation of the fair rate.

[31] Poterba and Solomon compare the impact of using BBB yields vs risk-free Treasury yields; Verani and Yu use the the HQM yield curve.

[32] Verani and Yu do however stress that this approach represents a valuation from the perspective of the shareholder of an insurance company issuing the annuity. But they also note that: "*the discount rate of an annuity shopper is likely very different from the discount rate of the owner of a life insurer. An annuity shopper seeking a safe longevity insurance contract may perceive an annuity contract to be relatively "safe" because of the existence, for example, of a state insurance guarantee fund.*" This provides an alternative justification for the use of risk-free yields, in calculating fair contribution rates from the perspective of scheme members.

5.5 Offsetting Biases?

We argued in Section 5.3 that our approach to mortality risk, taken in isolation, almost certainly leads to an underestimate of fair contribution rates; in contrast, there do appear to be valid arguments that using risk-free yields may in isolation lead to an offsetting *over*estimate. Thus it is an open question whether our relatively simple approach to estimating fair contribution rates results in estimates that are biased upwards or downwards. But one simple, if rather crude approach to answering this question exploits the results of Poterba and Solomon (2021). Their Tables 4 and 7 illustrate the impact of both biases. They shows that if calculating "money's worth" for annuities incorporating both sourcces of potential biase (population mortality and risk-free yields) then money's worth is always less than 100%. Hence arguably there are actually three components in annuity pricing that our analysis neglects: mortality risk, discounting and markups (100/money's worth); ie, actual annuities are not, in practice, fairly priced on our definition, but this lack of (actuarial) fairness may reflect compensation for a range of market imperfections. Hence, coupled with the likely over-estimate of the career value of earnings, it seems our simplistic approach may not be too far wrong, or if anything may understate the net impact of these three sources of bias.

6 Can Defined Benefit Pension Schemes be fixed?

6.1 DB schemes with risk-free liabilities

Our approximation shows that, for a given set of scheme parameters, fair contribution rates are driven by life expectancy in retirement, E_t and real yields. Both have shown significant drift over time. While the upward drift in life expectancy may level off, it is very unlikely to be reversed.[33] The apparent secular decline in real yields over recent decades is more obviously reversable in principle; but at the time of writing bond markets show no sign of expecting this.[34] Thus there appears to be little prospect of a return to the heyday of DB schemes, when, as Figure 4 showed, fair contribution rates were both low, and well below actual contribution rates for prolonged periods - effectively providing a significant financial cushion for scheme sponsors.

[33] On the assumption that the drop in measured life expectancy in 2020 and 2021 due to covid will prove to be temporary.

[34] A recent literature on long-term asset returns (eg Jorda et al, 2019; Anarkulovaa et al, 2021) also shows that in a range of developed ocuntries risk-free returns have typically been both close to zero, and with some evidence of a declining trend.

With this heyday clearly in the past, can DB schemes that make risk-free promises be fixed?

At present the only solutions on offer are either to close DB schemes entirely (as has largely been the case in the private sector) or, as illustrated in our example of the USS, by some combination of less generous scheme parameters and a rise in contribution rates towards the fair rates we calculate in this paper. But Figure 4 suggests that empirically such adjustments appear to come as a delayed response to the underlying determinants of fair rates, leaving schemes like the USS vulnerable to a nontrivial risk of defaulting on their obligations (Miles & Sefton, 2021).

Arguably the problems of DB schems are compounded by their typically complex, and hence distinctly non-transparent nature. We have shown that, while fair contribution rates for pure pension liabilities have risen significantly in recent years, the impact of additional benefits has also become ever more significant. While the nature of pension provision by DB schemes appears, currently at least, to occupy a unique market niche, this is far less obvious for some other components such as the lump sum and life insurance elements offered by the USS, which are readily available in alternative and reasonably well-functioning markets. Additionally, some components of the scheme, such as the spousal pension (and more generally payments to dependents) effectively imply cross-subsidies from scheme members who do not beneft from them. If scheme members were allowed to choose additional benefits, with a clear tradeoff between benefits and contribution rates, this might enable DB schemes - in their current form, at least - to focus their efforts on their core task of providing pensions, as well as being more transparently equitable.

But even such a consolidation to pure pension provision would still leave fair contribution rates at historically high levels, as Figure 1 showed. They are also, self-evidently, time-varying, and highly persistent, reflecting the same attributes in their determinants. But it should be stressed that our analysis is predicated on the assumption that a new scheme member of a given age has a contribution rate that would be fixed throughout their working life, with a known payoff in terms of pension in retirement. It seems reasonable to assume that both the predictable nature of contributions and the known formula for the risk-free pension add to the attractions of DB schemes. But our analysis also reveals the evolving market determinents of fair contribution rates - hence in our framework a scheme member of the same age, but joining at a later date, would pay a different contribution rate throughout their working life.

In practice, while Figure 4 showed that, in our example, USS contribution rates were constant for prolonged periods, scheme members have faced periodic adjustments in contribution rates; we would argue that these were effectively delayed responses to changes in fair rates. These adjustments have had two effects: first, a clear reduction in the degree of certainty scheme members can have about what they will get out of the scheme in future; and, second, an effective cross-subsidy between different cohorts of scheme members, with those with low fair contribution rates on joining the scheme cross-subsidising those with high fair rates.

This is a form of intertemporal risk-sharing between cohorts that does not obviously occur anywhere else: investors who bought bonds at recent high prices when yields were low did not get compensation from earlier investors who bought them when yields were high; homeowners taking on mortgages at current low rates are not obliged to compensate earlier borrowers who paid higher rates. Nor - at least as far as we are aware - is there any clear theoretical justification for such forms of risk-sharing.

Furthemore, even after stripping out additional benefits, Figure 2 showed that for a pure pension liability there is also a clear effect of the age of the scheme member on joining, driven primarily by the sign of real yields. So in addition to the cross-subsidies between cohorts of any given age on joining, there are also cross-subsidies between cohorts of different ages, at the same point in time.

In contrast, market annuity prices are largely free from such cross-subsidies - to the extent that any such cross-subsidies exist they are either driven by legislation or by unavoidable pooling effects in face of adverse selection. It is not clear why DB pension schemes should be different in this respect, since, as we stressed at the outset, their liabilities are simply deferred annuities.

Where DB schemes do offer something quite distinct is in the manner in which scheme members pay for these deferred annuities, by stable contribution rates out of earnings. But in this respect the cohort cross-subsidies actually detract from the benefits DB schemes offer, because they introduce time variation in contribution rates that would not occur in our framework, along with unavoidable fragilities in scheme viability due to the mis-match between scheme liabilities and assets (Miles and Sefton, 2021). DB schemes could of course in principle avoid such instability by increasing the frequency of changes in contribution rates (indeed Figure 4 showed that such changes have become distinctly more frequent in recent years); but in so doing so, in the limit, as changes in contribution rates became more frequent, scheme members would simply be investing in risk-free assets at prevailing market prices, thus eliminating the key distinctive feature of DB schemes.

The discussion above has focussed on the arguments for matching actual contribution rates to fair contribution rates, for given scheme parameters. But a further advantage of linking actual contribution rates more directly to fair contribution rates would be that in principle it would enable schemes to offer members choices over the parameters themselves. To the extent that these have changed over time, such changes appear to have been primarily responses to risks to scheme viability, rather than to members' preferences over the parameters themselves, which may well be both heterogeneous, and time-varying. Thus, in the face of very significant changes in prices of risk-free assets, it seems plausible that members (or at the very least, less risk-averse members) would choose to reduce the share of risk-free assets in their wealth. Supporting evidence for this can arguably be seen in recent declines in annuity purchases in the UK[35] and elsewhere But current DB schemes, by offering a fixed accrual rate ($1/Y$ in our analysis) effectively impose an identical risk-free share for all members, on at least that part of scheme members' wealth that is tied up in the scheme. Since, the fair contribution rate simply scales in inverse proportion to Y, in principle Y itself could be a parameter chosen by the scheme member,on joining the scheme.[36]

Overall, to the extent that DB schemes could evolve to more closely resemble annuity providers in the way they differentiate between measurable differences in the nature of the deferred annuities to be provided, by linking contribution rates more closely to member-specific fair rates, as in our calculations, this would arguably help to preserve the distinctive characteristic that DB schemes do offer, of stable contribution rates for individual scheme members, and clearly defined payoffs after retirement.

Such changes would not, however, change the unpleasant actuarial arithmetic that has driven fair contribution rates up so strikingly over recent decades, which is ultimately driven by the combination of unprecedentedly high prices of risk-free assets, and risk-free commitments. Thus it is also worth considering the implications for fair contribution rates of a generalisation of DB schemes, in which commitments are not entirely risk-free, an issue to which we now turn.

6.2 Hybrid ("Random Defined Benefit") Pension Schemes

The central analysis of this paper has taken it as given that the central concept of a defined benefit is taken literally: that the defined benefits it provides are risk-free,

[35] See https://www.fca.org.uk/data/retirement-income-market-data-2019-20.

[36] Note that this scaling arises from the general case of Proposition 1, so does not rely on our approximation.

other than through mortality risk. While this literal interpretation might be considered pedantic, even naive (and we have indeed noted a number of caveats in Section 5.4 above) we would argue that the commitments DB schemes such as the USS make are at least *presented* as risk-free; and we would further argue that in the recent debate around the USS in the UK, both the sponsors of the scheme and those protesting at proposed changes have both taken this feature as given.

However, if we are prepared to consider a broader class of pensions schemes, in which the pension may carry some element of systemic risk , then our analysis also points to some fairly straightforward implications.

To see the impact of systematic risk, consider again, our definition of the fair contribution rate (10), restated here:

$$c_t^* = \frac{\mathcal{L}_t}{\mathcal{W}_t}.$$

It is evident that anything that lowers (or raises) the ratio of $\mathcal{L}_t$ to $\mathcal{W}_t$ must in turn by definition imply a lower (or higher) value of c_t^*.

We have indeed already discussed, in Section 5, the possibility that to the extent that earnings (and hence contributions into the scheme) have some element of systemic risk, then, for a given risk-free pension (or other) commitment (hence a given value of $\mathcal{L}_t$) then we may be overstating $\mathcal{W}_t$, and hence we may be *under*stating c_t^*. Conversely, our discussion of the literature on marketed annuities can be framed in terms of its impact on $\mathcal{L}_t$: higher (or unpredictable) mortality risk raises $\mathcal{L}_t$, but some combination of illiquidity premia and financial market imperfections may raise the implied discount rate, lowering $\mathcal{L}_t$ and hence c_t^*.

But all of these possible modifications are all predicated on the pension obligation being entirely, or close to risk-free, and thus do not, in net terms, change the answer very much. In contrast, if we introduce the possibility that pension payments have systematic risk, the impact can in principle be much more significant, and always points in the same direction - towards lower fair contribution rates.

However, while the implications for fair contribution rates of pensions with systematic risk are easy to demonstrate in principle, they are very much harder to implement in practice.

To illustrate, consider the simple case of the flat yield curve, pure pension liability case of Proposition 5, but assume a simple CAPM world, in which all pension payoffs have a systematic component with CAPM beta β_p, and hence with appropriate discount rate $r_t + \beta_d RP_m$ where RP_m is the market risk premium, for simplicity also assumed constant. We can then straightforwardly generalise the definition of the fair contribution rate in this case as $c_t^*(\beta_d)$ (the analysis in the main paper therefore sets $\beta_d = 0$). Then, unpicking the approximation in Proposition 5, and using $\delta_{0,t}^{c^*} = \delta_{0,t}^{\mathcal{L}} - \delta_{0,t}^{\mathcal{W}}$, we have

$$\begin{aligned} \ln c_t^*(\beta_d) &\approx \ln \frac{E_t}{Y} Q_t - \delta_{0,t}^{\mathcal{L}} (r_t + \beta_d RP_m) + \delta_{0,t}^{\mathcal{W}} r_t \\ &\approx \ln \frac{E_t}{Y} Q_t - \delta_{0,t}^{c^*} r_t - \delta_{0,t}^{\mathcal{L}} \beta_d RP_m \\ &\approx \ln c_t^*(0) - \delta_{0,t}^{\mathcal{L}} \beta_d RP_m \end{aligned} \tag{38}$$

where $c_t^*(0)$ is the fair contribution rate with risk-free liabilities, as in the main paper. It should be fairly evident from this expression that, given the long duration of pension liabilities, even a modest element of systematic risk in the pension payment, as captured by β_d could in principle significantly reduce the implied fair contribution rate $c_t^*(\beta_d)$, relative to the estimates we have derived in the main body of the paper. Figure 5 illustrates.[37]

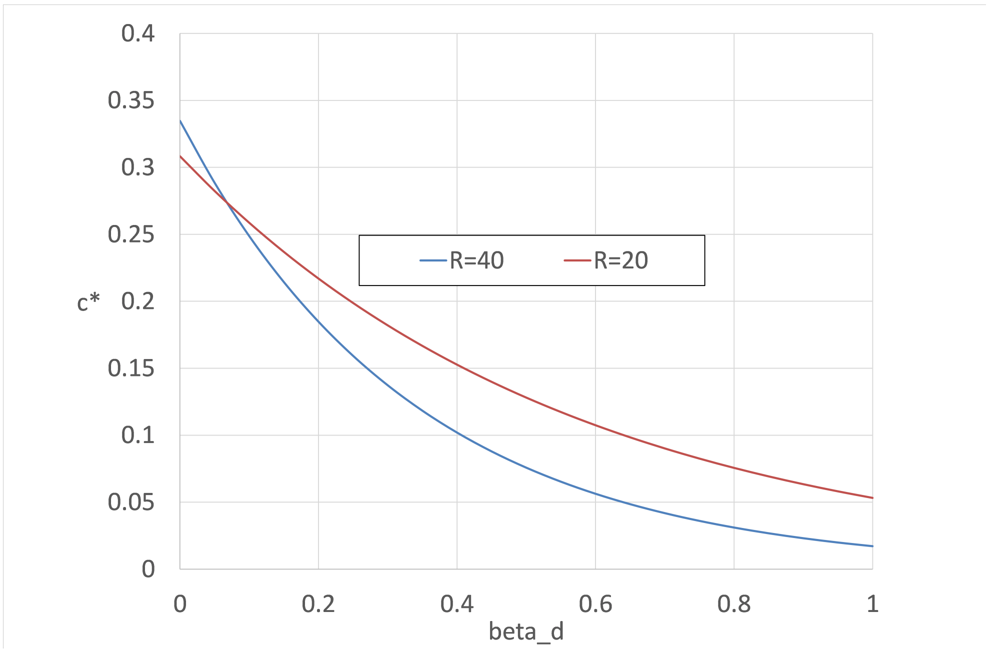

Figure 5: Approximated fair contribution rate $c_t^*(\beta_d)$ for a pension with systematic risk β_d

But simply demonstrating such a strong impact in principle does not tell us how to

[37] Assuming a notional flat yield curve as discussed in Section 3.2 in which $r_t = y_{2020}^{c^*} = -1.37\%$ ($R = 40$), $= -1.24\%$ ($R = 20$), where $y_t^{c^*}$ is as defined in (30) and assuming $E(R_m) = 4.5\%$, as in Miles and Sefton (2021).

design an implementable DB scheme in which the pension liability has systematic risk. The calculation above, for example, implicitly requires that the pension would have the same systematic risk at all points in retirement: it would thus be close to replicating the outcome from a defined contribution scheme, in which the pension payment throughout retirement is proportional to a diversified portfolio of stocks with CAPM beta=β_d, but in which the scheme undertook to insure the pensioner against mortality risk. Thus far such schemes are distinctly thin on the ground, unsurprisingly so since the informational and hedging requirements would be formidable.[38] As a result at present the only viable market equilibrium appears to be an essentially discrete choice between DB schemes with risk-free payoffs and defined contribution schemes.

Would simpler, and implementable, hybrid schemes be possible? Consider for example a broader definition of defined benefit schemes, in which the benefits would still be precisely defined in terms of some observable magnitude, but some element of the payoff would be risky (analogous to the payoffs on derivatives), hence these would be *random* defined benefit schemes (thus in principle addressing the concerns of, for example, Wolf (2021)).

Thus, consider a hybrid/random DB scheme with a pure pension liability, but where some share α of the pension, again payable at a constant rate from $t+R$ till death, is determined by observable market payoffs. Specifically consider the payoff on an investment in an observable stock market index M_t at time t, with random observable payoff $\widetilde{M}_{t+R}$, to be invested in an annuity at $t+R$, with random price $\widetilde{q}_{t+R}$.

Thus let the random pension payable from $t+R$ onwards be given by the $t+R$-dated random variable

$$\widetilde{p}_{t+R} = \frac{R\overline{w}_t}{Y}\left(1-\alpha+\alpha\widetilde{v}_{t+R}\right) \tag{39}$$

where $\widetilde{v}_{t+R} = \left(\widetilde{M}_{t+R}/\widetilde{q}_{t+R}\right)/\mathbb{E}_t\left(\widetilde{M}_{t+R}/\widetilde{q}_{t+R}\right)$, hence the *expected* pension, $\mathbb{E}_t\widetilde{p}_{t+R} = \frac{R\overline{w}_t}{Y}$ as in the main analysis, and for $\alpha = 0$ the fair contribution rate is unchanged from the main paper: again denote this value $c_t^*(0)$.

This component of the pension would thus be a random variable at time $t+R$, but constant thereafter, through retirement, but by pre-committing to buy an annuity at the annuity price $\widetilde{q}_{t+R}$ this component of the fund's liabilities would simply have duration R.

Both $\widetilde{M}_{t+R}$ and $\widetilde{q}_{t+R}$ can be expected to have positive systematic risk, indeed for

[38] Of course, even such a scheme existed, it is far from evident that there would be demand for a pension that had the same element of systematic risk throughout retirement - given the literature on optimal life cycle investment patterns (eg Bodie et al, 1992).

simplicity set $\beta_M = 1$. Then, assuming $\beta_q << 1$ the present value of v_{t+R} will be less than 1.

Thus for positive α we have, again, exploiting our approximation,

$$\begin{aligned} c_t^*(\alpha) &\approx (1-\alpha)\, c_t^*(0) + \alpha \frac{R\overline{w}_t}{Y} e^{-\left(1-\beta_q\right) RP_M \times R} \\ &\approx c_t^*(0)\left(1-\alpha+\alpha e^{-\left(1-\beta_q\right) RP_M \times R}\right) \end{aligned} \tag{40}$$

Thus we again find, as in the previous more general calculation, that the higher is α, and hence the more systematic risk the scheme member carries, the lower would be the implied contribution rate. The contrast with the more general case in (38) is that a) the random component in the pension would be known to the scheme member from $t+R$ onwards, and thus more likely to be in line with life cycle optimisation; b) the random component would in principle be hedgeable by the scheme; and hence c) regulation to ensure viability of the payment would be along similar lines to the regulation of any derivative security.

7 Concluding Remarks

The key conclusions of this paper are:

1. The unpleasant actuarial arithmetic of both increased life expectancy and (especially) negative real yields has resulted in a massive rise in implied fair contribution rates for defined benefit (DB) schemes. At present there appears to be little prospect of these rises being reversed.

2. DB schemes provide deferred annuities. But in contrast to prices of marketed annuities, actual contribution rates into DB schemes appear to adjust only with a lag to fair rates, increasing scheme fragility.

3. DB schemes with common, and slow-moving contribution rates imply significant redistributions between scheme members.

4. The significant rises in fair contribution rates reflect the risk-free nature of benefits provided. In principle a pre-defined explicit element of systematic risk in benefits could significantly lower fair contribution rates; but the logistical obstacles are formidable.

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Appendix

A Proof of Proposition 2

Proposition 1 shows that c_t^* is invariant to w_t. Hence let $w_{t+\tau} = 1 \ \forall \tau \in [0, R]$ and $y_t(\tau) = 0 \ \forall \tau \geq 0$ as assumed in Proposition 2.

In this proof we consider all elements of scheme liabilities and thus effectively derive the case in Corollary 2, which we show has the same form as in the main proposition, replacing actual life expectancy in retirement, E_t, with effective life expectancy in retirement, $\widehat{E}_t$; the main proposition then follows directly by setting $s = Z = 0$.

First, given the assumptions,

$$p_t = \frac{1}{Y} \int_0^R d\tau = \frac{R}{Y}.$$

Second, consider

$$\mathbb{E}_t \left[\widetilde{A}_D - A \middle| \widetilde{A}_D \leq A_R \right] = \frac{\int_0^R \tau \lambda_t(\tau) e^{-\Lambda_t(\tau)\tau} d\tau}{\int_0^R \lambda_t(\tau) e^{-\Lambda_t(\tau)\tau} d\tau}.$$

The denominator is the probability of dying before R, i.e. $1 - S_t(R)$. For the numerator, noting that $\frac{\partial}{\partial \tau}[\Lambda_t(\tau)\tau] = \frac{\partial}{\partial \tau}\left[\int_0^\tau \lambda_t(u)\, du\right] = \lambda_t(\tau)$, integration by parts yields

$$\int_0^R \tau \lambda_t(\tau) e^{-\Lambda_t(\tau)\tau} d\tau = -Re^{-\Lambda_t(R)R} + \int_0^R e^{-\Lambda_t(\tau)\tau} d\tau.$$

Given that $\int_0^R e^{-\Lambda_t(\tau)\tau} d\tau = \mathcal{W}_t$,

$$\mathbb{E}_t \left[\widetilde{A}_D - A \middle| \widetilde{A}_D \leq A_R \right] = \frac{-RS_t(R) + \mathcal{W}_t}{1 - S_t(R)}$$
$$\Leftrightarrow \mathcal{W}_t = (1 - S_t(R))\, \mathbb{E}_t \left[\widetilde{A}_D - A \middle| \widetilde{A}_D \leq A_R \right] + S_t(R)\, R.$$

Third, consider life expectancy in retirement, E_t, as in the main proposition,

$$E_t \equiv \mathbb{E}_t \left[\widetilde{A}_D - A_R \middle| \widetilde{A}_D > A_R \right] = \frac{\int_R^\infty (\tau - R)\, \lambda_t(\tau) e^{-\Lambda_t(\tau)\tau} d\tau}{\int_R^\infty \lambda_t(\tau) e^{-\Lambda_t(\tau)\tau} d\tau}.$$

The denominator is the probability of dying after R, i.e. $S_t(R)$. For the numerator, again by integration by parts,

$$\int_R^\infty (\tau - R)\,\lambda_t(\tau)\, e^{-\Lambda_t(\tau)\tau} d\tau = \int_R^\infty e^{-\Lambda_t(\tau)\tau} d\tau.$$

Then with $L_t^p = p_t \int_R^\infty e^{-\Lambda_t(\tau)\tau} d\tau$,

$$E_t = \frac{\frac{1}{p_t} L_t^p}{S_t(R)}$$
$$\Leftrightarrow L_t^p = p_t S_t(R)\, E_t.$$

Next, $L_t^Z = Z p_t e^{-\Lambda_t(R)R}$ means,

$$L_t^Z = Z p_t S_t(R).$$

Finally, consider

$$\mathbb{E}_t\left[\widetilde{A}_D^S - \widetilde{A}_D \,\middle|\, \widetilde{A}_D^S > \widetilde{A}_D > A_R\right] = \frac{\int_R^\infty \lambda_t(\tau)\, e^{-(\Lambda_t(\tau)+\Lambda_t'(\tau))\tau} \left\{ \frac{\int_\tau^\infty (u-\tau)\lambda_t'(u) e^{-\Lambda_t'(u)u} du}{\int_\tau^\infty \lambda_t'(u) e^{-\Lambda_t'(u)u} du} \right\} d\tau}{\int_R^\infty \lambda_t(\tau)\, e^{-(\Lambda_t(\tau)+\Lambda_t'(\tau))\tau} d\tau},$$

where $(\lambda_t(\tau), \lambda_t'(\tau))$ denote the instantaneous hazard rates for the scheme member and their spouse, respectively. The denominator is the probability of the spouse surviving the scheme member after retirement. Noting that $\int_\tau^\infty \lambda_t'(u)\, e^{-\Lambda_t'(u)u} du = S_t'(\tau) = e^{-\Lambda_t'(\tau)\tau}$ and as we know from above that $\int_\tau^\infty (u-\tau)\,\lambda_t'(u)\, e^{-\Lambda_t'(u)u} du = \int_\tau^\infty e^{-\Lambda_t'(u)u} du$, the numerator is,

$$\int_R^\infty \lambda_t(\tau)\, e^{-\Lambda_t(\tau)\tau} \left\{ \int_\tau^\infty e^{-\Lambda_t'(u)u} du \right\} d\tau.$$

The denominator is, by integration by parts,

$$\int_R^\infty \lambda_t(\tau)\, e^{-\Lambda_t(\tau)\tau} e^{-\Lambda_t'(\tau)\tau} d\tau = e^{-(\Lambda_t(R)+\Lambda_t'(R))R} - \int_R^\infty \lambda_t'(\tau)\, e^{-\Lambda_t'(\tau)\tau} e^{-\Lambda_t(\tau)\tau} d\tau.$$

Applying symmetry $\lambda_t(t+\tau) = \lambda_t'(t+\tau)$, this implies

$$\int_R^\infty \lambda_t(\tau)\, e^{-\Lambda_t(\tau)\tau} e^{-\Lambda_t'(\tau)\tau} d\tau = \frac{e^{-(\Lambda_t(R)+\Lambda_t'(R))R}}{2} = \frac{S_t(R)^2}{2}.$$

Therefore, where $L_t^s = sp_t \int_R^\infty \lambda_t(\tau) e^{-\Lambda_t(\tau)\tau} \left\{ \int_\tau^\infty e^{-\Lambda_t(u)u} du \right\} d\tau$,

$$\mathbb{E}_t \left[\widetilde{A}_D^S - \widetilde{A}_D \middle| \widetilde{A}_D^S > \widetilde{A}_D > A_R \right] = \frac{\frac{1}{sp_t} L_t^s}{\frac{S_t(R)^2}{2}}$$

$$\Leftrightarrow L_t^s = sp_t \frac{S_t(R)^2}{2} \mathbb{E}_t \left[\widetilde{A}_D^S - \widetilde{A}_D \middle| \widetilde{A}_D^S > \widetilde{A}_D > A_R \right].$$

Hence we have, for $w_{t+\tau} = 1 \ \forall \tau \in [0, R]$ and $y_t(\tau) = 0 \ \forall \tau \geq 0$,

$$\begin{aligned} c_{0,t}^* &= \frac{L_t^p + L_t^Z + L_t^s}{\mathcal{W}_t} \\ &= \frac{R}{Y} \frac{E_t + Z + \frac{sS_t(R)}{2} \mathbb{E}_t \left[\widetilde{A}_D^S - \widetilde{A}_D \middle| \widetilde{A}_D^S > \widetilde{A}_D > A_R \right]}{R + \left(\frac{1 - S_t(R)}{S_t(R)} \right) \mathbb{E}_t \left[\widetilde{A}_D - A \middle| \widetilde{A}_D \leq A_R \right]}. \end{aligned}$$

which can be written, as in Corollary 2, as

$$c_t^* = c_{0,t}^* = \frac{\widehat{E}_t}{Y} Q_t \tag{41}$$

where

$$\begin{aligned} Q_t &= S_t(R) \frac{R}{S_t(R) R + (1 - S_t(R)) \mathbb{E}_t \left[\widetilde{A}_D - A \middle| \widetilde{A}_D \leq A_R \right]} \\ &= S_t(R) U_{0,t} \end{aligned}$$

where E_t and $U_{0,t} = U_t$, as defined in Proposition 1, under the assumptions of Proposition 2.

Note that $U_{0,t} \geq 1$, hence

$$Q_t \begin{cases} \in (S_t(R), 1) & \text{for} \quad S_t(R) < 1 \\ = 1 & \text{for} \quad S_t(R) = 1. \end{cases}$$

If $s = Z = 0$ then $c_{0,t}^* = \frac{E_t}{Y} Q_t$, as in the main proposition; if additionally $\widetilde{A}_D > A_R$ with certainty, i.e. $S_t(R) = 1$, then $Q_t = 1$ and (41) simplifies to the Modigliani Case.■

B Proof of Proposition 3

First

$$\delta_r^{\mathcal{W}} = \frac{w_t}{\mathcal{W}_t}\left(\int_0^R \tau e^{(g_t(\tau)-y_t(\tau)-\Lambda_t(\tau))\tau} d\tau\right) \tag{42}$$
$$< \frac{w_t}{\mathcal{W}_t}\left(\int_0^R R e^{(g_t(\tau)-y_t(\tau)-\Lambda_t(\tau))\tau} d\tau\right) = \frac{R}{\mathcal{W}_t}\left(w_t\int_0^R e^{(g_t(\tau)-y_t(\tau)-\Lambda_t(\tau))\tau} d\tau\right) = R.$$

using the definition of $\mathcal{W}_t$ in (4).

Second, substituting from (7), (9) and (8),

$$\mathcal{L}_t = p_t\left(\int_R^\infty e^{-(y_t(\tau)+\Lambda_t(\tau))\tau} d\tau + Z e^{-(y_t(R)+\Lambda_t(R))R} + s\int_R^\infty e^{-(y_t(\tau)+\Lambda_t(\tau))\tau}\left\{\int_R^\tau \lambda_t(u)\, e^{-\Lambda_t(u)u} du\right\} d\tau\right). \tag{43}$$

Then, by similar reasoning to the derivation of the inequality for (42),

$$\delta_r^{\mathcal{L}} = \frac{p_t}{\mathcal{L}_t}\left(\int_R^\infty \tau e^{-(y_t(\tau)+\Lambda_t(\tau))\tau} d\tau + RZ e^{-(y_t(R)+\Lambda_t(R))R} + s\int_R^\infty \tau e^{-(y_t(\tau)+\Lambda_t(\tau))\tau}\left\{\int_R^\tau \lambda_t(u)\, e^{-\Lambda_t(u)u} du\right\} d\tau\right)$$
$$> \frac{R}{\mathcal{L}_t}\left(p_t\int_R^\infty e^{-(y_t(\tau)+\Lambda_t(\tau))\tau} d\tau + RZ e^{-(y_t(R)+\Lambda_t(R))R} + s\int_R^\infty R e^{-(y_t(\tau)+\Lambda_t(\tau))\tau}\left\{\int_R^\tau \lambda_t(u)\, e^{-\Lambda_t(u)u} du\right\} d\tau\right)$$
$$= R$$

given (43). Hence, as given in the proposition,

$$\delta_r^{\mathcal{W}} < R < \delta_r^{\mathcal{L}} \quad \blacksquare$$

C Proof of Proposition 4

Writing

$$
\begin{aligned}
c_t^* &= \frac{\mathcal{L}_t^p}{\mathcal{W}_t} = \frac{\frac{1}{Y}\int_0^R e^{g_t(\tau)\tau}d\tau \int_R^\infty e^{-(y_t(\tau)+\Lambda_t(\tau))\tau}d\tau}{\mathcal{W}_t} \\
&= \frac{\int_R^\infty e^{-(y_t(\tau)+\Lambda_t(\tau))\tau}d\tau}{Y} \times \frac{\int_0^R e^{g_t(\tau)\tau}d\tau}{\mathcal{W}_t} \qquad (44)
\end{aligned}
$$

the first term in (44) is invariant to g. Thus for $g(\tau) = g \forall \tau$, and in the neighbourhood of $g = 0$

$$
\frac{\partial \ln c_t^*}{\partial g} \approx \frac{\int_0^R \tau d\tau}{\int_0^R 1 d\tau} - \left.\delta^{\mathcal{W}}\right|_{g=0} = \frac{R}{2} - \left.\delta^{\mathcal{W}}\right|_{g=0} \qquad (45)
$$

which is of ambiguous sign in general. However, using (42),

$$
\begin{aligned}
y_t(\tau) + \Lambda_t(\tau) &> 0, \forall \tau < R \Rightarrow \left.\delta^{\mathcal{W}}\right|_{g=0} < \frac{R}{2} \\
&\Rightarrow \delta_g^{c^*} > 0 \ \blacksquare
\end{aligned}
$$

D Proof of Proposition 5

The approximation for c_t^* in (27) follows straightforwardly from Assumption A2: with a flat yield curve and zero wage growth, for a given hazard rate function c_t^* is simply a function of r_t, and hence can be approximated in the neighbourhood of $c_{0,t}^*$.

To derive the approximation for $\delta_0^{c^*}$ in (28) we proceed by analysing the special case of a piecewise linear distribution of the date of death, with two uniform densities, before and after retirement

$$
\begin{aligned}
f(\tau) &= \frac{1 - S_t(R)}{R}; \tau < R \qquad (46) \\
&= \frac{S_t(R)}{2E_t}; \tau \in (R, R + 2E_t) \qquad (47)
\end{aligned}
$$

which is specified to match the true values of both $S_t(R)$ and $E_t = \mathbb{E}_t\left(\widetilde{A}_D - A_R | \widetilde{A}_D - A_R\right)$.

This implies a piecewise linear approximation to the survival function $\widehat{S}_t(\tau)$ given by

$$\begin{aligned}\widehat{S}_t(\tau) &= 1-\frac{(1-S_t(R))}{R}\tau;\quad \tau<R &(48)\\ &= S_t(R)\left(1-\frac{1}{2E_t}(\tau-R)\right);\ \tau\in(R,R+2E_t) &(49)\end{aligned}$$

which in turn implies the duration of liabilities is given by

$$\widehat{\delta}_0^{\mathcal{L}}=\frac{\int_R^{R+2E_t}\tau\widehat{S}_t(\tau)\,d\tau}{\int_R^{R+2E_t}\widehat{S}(\tau)\,d\tau}=\frac{2}{3}E_t+R \tag{50}$$

and the duration of career earnings is given by

$$\widehat{\delta}_0^{\mathcal{W}}=\frac{\int_0^R\tau\widehat{S}_t(\tau)\,d\tau}{\int_0^R\widehat{S}(\tau)\,d\tau}=\frac{R}{2}\left(1-\frac{1}{3}\frac{1-S_t(R)}{1+S_t(R)}\right) \tag{51}$$

We also have, for the same piecewise uniform densities,

$$\widehat{\sigma}^2_{D>R}=var\left(\widetilde{A}_D-A_R|\widetilde{A}_D-A_R\right)=\frac{1}{3}E_t^2 \tag{52}$$

implying that we can write

$$\delta_0^{c^*}\equiv\widehat{\delta}_0^{\mathcal{L}}-\widehat{\delta}_0^{\mathcal{W}}\approx\frac{R+E_t}{2}+\frac{1}{2}\left(\frac{\sigma_{D>R}}{\sqrt{3}}+\frac{R}{3}\frac{1-S_t(R)}{1+S_t(R)}\right) \tag{53}$$

as in the proposition, for any survival function sufficiently close to piecewise linearity.[39] ■

E Proof of Corollary 3

In this case we can write

$$c_t^{*FS}=\frac{\int_R^\infty e^{-(y_t(\tau)+\Lambda_t(\tau))\tau}d\tau}{Y}\times\frac{R\,e^{g(R)R}}{\mathcal{W}_t} \tag{54}$$

implying, straightforwardly, given Proposition 3,

$$\frac{\partial\ln c_t^{*FS}}{\partial g}=R-\delta^{\mathcal{W}}\big|_{g=0}>0\ \ \blacksquare \tag{55}$$

[39] In Appendix F.6 we show that $S_t(\tau)$ is quite well approximated by $\widehat{S}_t(\tau)$, and that this results in a close match to $\delta_0^{c^*}$.

F Estimating Fair Contribution Rates for the UK Universities Superannuation Scheme (USS)

F.1 Hazard Rates

The Office of National Statistics releases the national life table (source: https://www.ons.gov.uk/) in September of each year. The implied hazard rates for 2018 − 20 are shown in Figure 6.

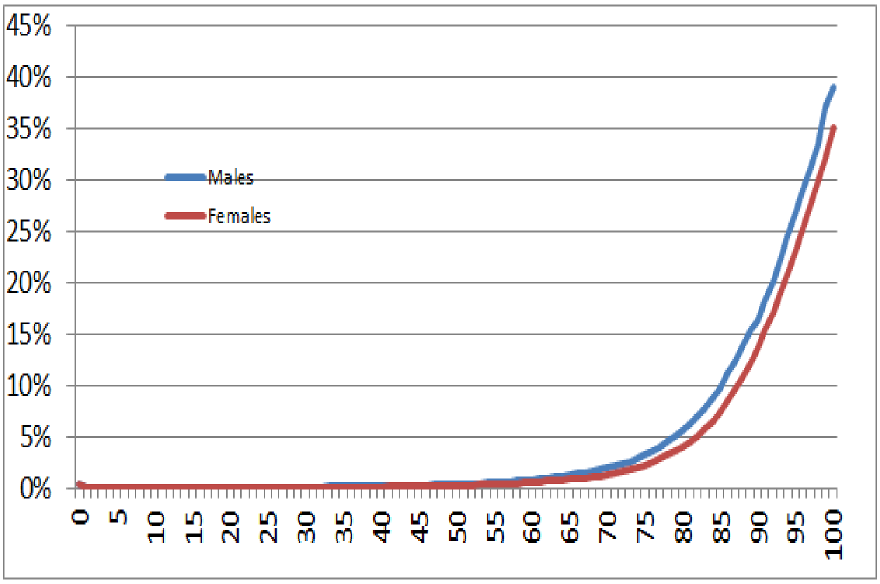

Figure 6: UK Hazard Rates, 2018 − 20 (source: ONS)

Using these data the average age of death for the UK population can be calculated as 79.1 years for male, 82.9 years for female and 80.9 years for the whole population. Using ONS's historical data the life expectancy at 67 can be shown to have increased by 4.4 years between 1985 and 2020, as shown in Figure 7. This also shows a drop of 0.2 years in 2018 − 2020 from 2017 − 2019 due to the COVID pandemic, the first time the life expectancy has fallen since at least 1985.

As a cross-check, we also examine the impact of using mortality statistics from a more relevant sample: following Cannon & Tonks (2016) we examine the impact of using the CMI PCMA00/PCFL00 tables based on mortality rates of members of DB schemes administered by life insurers in 1999-2002[40], rather than the closest equivalent

[40] Published by the Institute and Faculty of Actuaries (https://www.actuaries.org.uk/learn-and-develop/continuous-mortality-investigation/cmi-mortality-and-morbidity-tables/00-series-tables

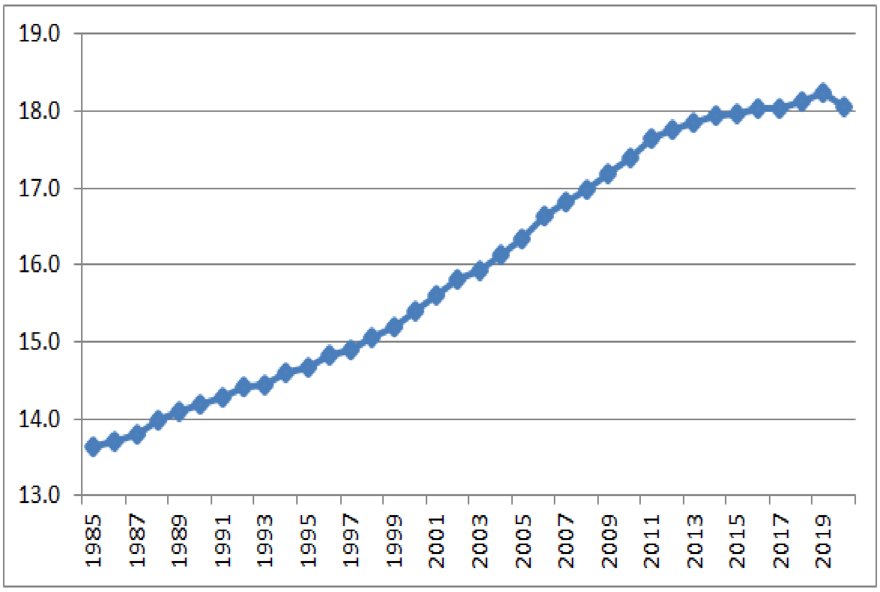

Figure 7: Life Expectancy at 67

figures (for 2000-2002) from the ONS Life Tables. The implied life expectancy for males in this period rises from 13.29 (ONS) to 15.50 years (CMI) and for females from 16.12 to 17.02 years, implying a gender-neutral average increase in life expectancy in retirement (E_t) from 14.71 to 16.3 years. Taken in isolation this would directly imply an equivalent proportionate increase in c^*_{0t}, in Proposition 2, for a pure pension liability. Assuming similar proportionate increases in E_t in more recent years, and applying our approximation locally would imply an increase in c^*_t in 2020, again for a pure pension liability form 33.47% to 37.0%.

F.2 UK Real Yields and Forward Rates

The Bank of England publishes real yields and instantaneous forward rates for RPI inflation (https://www.bankofengland.co.uk/statistics/yield-curves/) for years 2.5 to 40. In 2011 the USS changed the inflation index applied to adjust pension payouts from RPI and CPI; thus it is necessary to adjust this to real rates reflecting CPI inflation. To do this the RPI-CPI wedge is estimated using a 20 year rolling average. The wedge applied is between 52bp and 92bp, with 73bp for the most recent data (2020) (Figure 8)

This contrasts with the "long-run average difference between RPI and CPI inflation" of 80bp applied in Bank of England Quarterly Bulletin (2006), or the statement in their Staff Working Paper No.551 (2015) that, "[o]ver longer horizons, the expected RPI/CPI

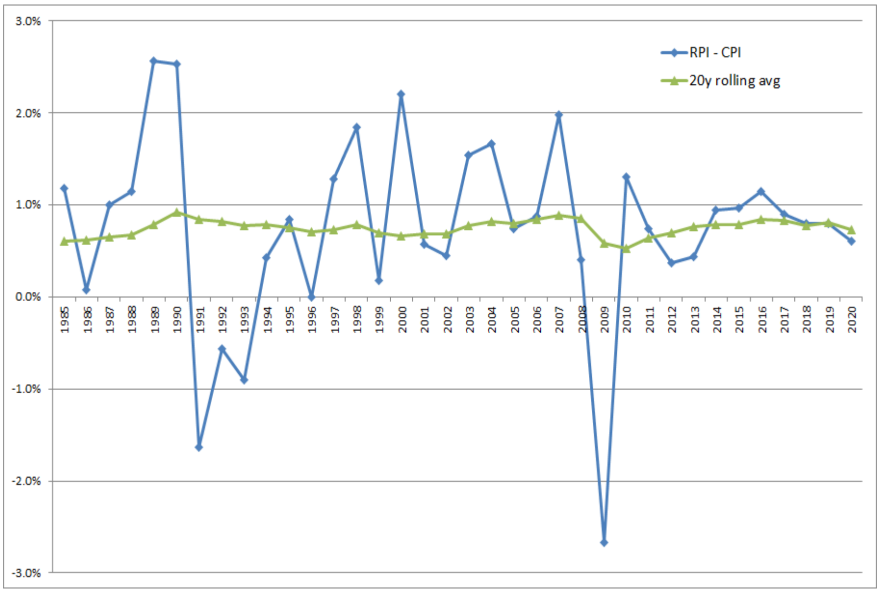

Figure 8: RPI - CPI Wedge and 20 year Rolling Average

wedge appears fairly stable at around 66 basis points". On the other hand, OBR (2015) quote their "new estimate of the long-run wedge between RPI and CPI inflation of 1.0 percentage points". Our estimate of 80bp for the most recent data lies within the appropriate range of these estimates.

F.3 Adjustments to fair contribution rates for life insurance components

In carrying out the calculations below for the specific example of the USS scheme, we need to include an additional term that adjusts the value of c_t^* derived in Proposition 1 to reflect the life insurance components provided by the scheme: these consist of two elements.

First, the USS scheme provides a lump-sum death-in-service payment to a named beneficiary of B times the annual wage at the time of death (In USS's case, $B = 3$). Assuming constant growth of wages, with $w_{t+\tau} = e^{g\tau}$, this liability is given by,

$$\mathcal{L}_t^{LI_B} = B \int_t^{t+R} \lambda(\tau)\, e^{\left(g - y(t,\tau) - \bar{\lambda}(t,\tau)\right)(\tau - t)} d\tau. \tag{56}$$

Second, we deliberately exclude from our theoretical analysis the effective life insurance component of the spousal pension - namely the value of the commitment to pay the spousal pension if the scheme member dies before retirement,given by

$$\mathcal{L}_t^{LI_s} = sp_t \int_0^R \lambda(\tau) e^{-\Lambda_t(\tau)\tau} \left\{ \int_\tau^R e^{-(y_t(u)+\Lambda_t(u))u} du \right\} d\tau \tag{57}$$

Then,

$$c_{USS,t}^* = c_t^* + \frac{\mathcal{L}_t^{LI_B} + \mathcal{L}_t^{LI_s}}{\mathcal{W}_t} \tag{58}$$

where c_t^* is as given in Proposition 1, and $\mathcal{W}_t$ is as given by (4).

F.4 Historical Scheme Parameters

The historical USS contribution rates for employees and employers, as shown in Figure 4 are sourced from Wikipedia. The period between 1974 and 1997 includes a 2% surcharge aimed at covering benefits for service prior to the scheme's inception in 1974. Between 2011 and 2016 the employee contribution rate was 7.5% for existing workers with final salary scheme and 6.35% for new workers with career average revalued earnings (CARE). In October 2021 the employee contribution rate was increased to 11.0% and the employer contribution rate to 23.7% (total 34.7%). The current proposal is to increase these to 13.6% and 28.5% (total 42.1%) in 2022, respectively.

F.5 The impact of different rates of wage growth

We noted in relation to Proposition 4 that, since hazard rates, and hence $\Lambda(\tau)$ are relatively low before retirement, to a good approximation c^* increases (decreases) with g when yields are positive (negative), but that the impact can be expected to be relatively small, given the offsetting terms in Proposition 4. Figure 9 shows that this is indeed the case for the exact calculation, for $R = 40$, implied by Proposition 1, on a counterfactual basis using current USS rules. It is worth noting that the range of constant growth rates assumed is quite wide: for example a 4% growth rate throughout a 40 year working life implies a nearly five-fold increase in salary.

F.6 A Piecewise Linear Approximation for $S_t(\tau)$

Proposition 5 derives an approximation for $\delta_0^{c^*}$ that is predicated on the actual survival function $S_t(\tau)$ being close to a piecewise linear approximation, $\widehat{S}_t(\tau)$. Figure ' shows

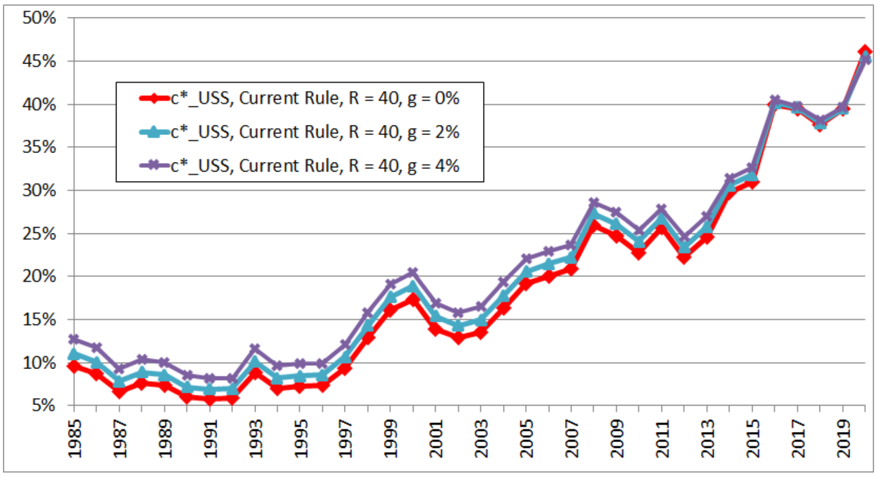

Figure 9: Simulated USS fair contribution rates under current rules for different wage growth rates, $R = 40$

that, using ONS life tables, the approximation (for $R = 40$) fits the actual survival probability from the UK Life Tables quite well, which helps to explain the closeness of the approximation for c_t^* itself.

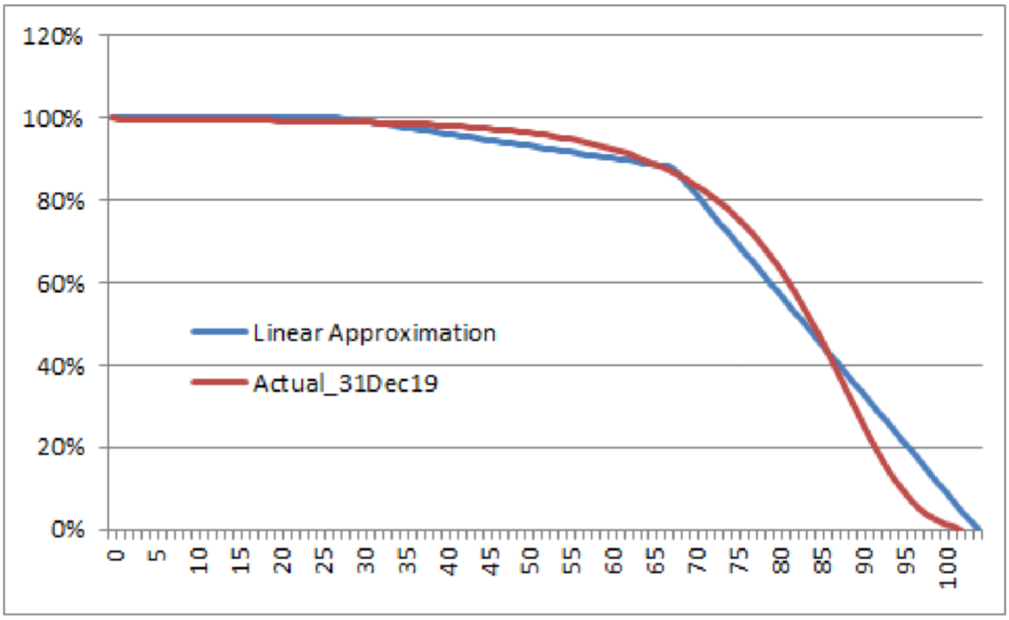

Figure 10: Survival Probability function $S_t(\tau)$ given $A = 27$, and piecewise approximation