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Rosenbrock's theorem for systems over von Neumann regular rings

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ABSTRACT

If (A, B) is a finite system over a commutative von Neumann regular ring R, the problem of searching for a matrix F such that the pencil $[sI - A - BF]$ has some prescribed Smith normal form is reduced to the case where R is a field, a problem which for controllable systems is described by a well-known theorem of Rosenbrock on pole assignment [12], and was then generalized to noncontrollable pairs [14]. It this paper, von Neumann regular rings are characterized as the class of commutative rings for which the solution of the above problem over the ring is equivalent to its solution in each residue field.

1. Introduction and notation

Let R be a commutative ring with 1. An m-input, n-dimensional linear system over R is a pair of matrices (A, B), where $A \in R^{n\times n}$ and $B \in R^{n\times m}$.

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Von Neumann regular rings appear in a natural way when trying to generalize to commutative rings some known results in systems theory over fields. The reader is referred to [4,13], and the references therein to see the applications of von Neumann regular rings to systems theory.

Given a system (A, B), it is customary to modify its structure by means of state feedback, i.e. A is replaced by $A + BF$, for some matrix $F \in R^{m\times n}$.

Let $R[s]$ be the polynomial ring over R in the indeterminate s. Given n polynomials $\phi_1(s), \ldots, \phi_n(s)$ in $R[s]$, such that $\phi_i(s)|\phi_{i+1}(s)$, for $i = 1, \ldots, n-1$, consider the $n \times n$ diagonal matrix $M = diag\{\phi_1(s), \ldots, \phi_n(s)\}$. This paper deals with the following invariant factor assignment problem:

$$(A, B, M): \qquad \text{Given } (A, B), \text{ find } F \text{ such that } M \sim sI - (A + BF)$$

where $\sim$ denotes equivalence over $R[s]$, i.e. we look for $n\times n$ invertible matrices $P(s), Q(s)$ in $R[s]$ such that $M = P(s)(sI - (A + BF))Q(s)$.

If $\mathfrak{m}$ is a maximal ideal of R, the reduction of (A, B, M) modulo $\mathfrak{m}$ (i.e. the image under the natural map $R \to R/\mathfrak{m}$) is denoted by $(A_\mathfrak{m}, B_\mathfrak{m}, M_\mathfrak{m})$ and is defined in the natural way: find a matrix $F_\mathfrak{m}$ with coefficients in the residue field $R/\mathfrak{m}$ such that $M_\mathfrak{m} \sim sI - (A_\mathfrak{m} + B_\mathfrak{m} F_\mathfrak{m})$ over $(R/\mathfrak{m})[s]$.

Since $R/\mathfrak{m}$ is a field, if $(A_\mathfrak{m}, B_\mathfrak{m})$ is controllable, Rosenbrock's theorem on pole assignment [12] states that a solution to $(A_\mathfrak{m}, B_\mathfrak{m}, M_\mathfrak{m})$ exists if and only if

$$\sum_{i=j}^{n} c_{i,\mathfrak{m}} \leq \sum_{i=j}^{n} d_{n+1-i,\mathfrak{m}}, \qquad j = 1, \ldots, n, \tag{1}$$

with equality for $j = 1$, where $c_{1,\mathfrak{m}} \geq \cdots \geq c_{n,\mathfrak{m}}$ are the controllability indices of $(A_\mathfrak{m}, B_\mathfrak{m})$ (see the definition in [14]) and $d_{i,\mathfrak{m}}$ is the degree of $\phi_{i,\mathfrak{m}}(s)$ (the reduction of $\phi_i(s)$ modulo $\mathfrak{m}$) for all i. For noncontrollable systems, the problem can be solved with the majorization and interlacing inequalities used by Zaballa in [14].

Of course, solvability of (A, B, M) implies solvability of $(A_\mathfrak{m}, B_\mathfrak{m}, M_\mathfrak{m})$ for all $\mathfrak{m}$. In this paper, we prove in Theorem 6 that commutative von Neumann regular rings are characterized as those rings satisfying the following local–global property: any problem (A, B, M) has a solution F over R if and only if $(A_\mathfrak{m}, B_\mathfrak{m}, M_\mathfrak{m})$ has a solution $F_\mathfrak{m}$ over $R/\mathfrak{m}$, for all maximal ideals $\mathfrak{m}$ in R. That is to say, the solution of the problem (A, B, M) over a ring R is a local–global property exactly when R is a von Neumann regular ring. The key tool to prove this is a local–global argument like those used by Pierce [11], Guralnick [9] and Costa [5].

2. Preliminaries

We shall see for which rings R and for which matrices M does it make sense to consider an invariant factor assignment problem (A, B, M), where (A, B) is an m-input, n-dimensional linear system over R.

2.1. Admissible rings

First, note that since we are asking if every matrix with coefficients in the ring $R[s]$ is equivalent to a diagonal matrix, this makes sense only if $R[s]$ is an elementary divisor ring, in the sense of Kaplansky [10]: every rectangular matrix is equivalent to a diagonal matrix D, with diagonal elements d_i satisfying $d_i|d_{i+1}$, for $i = 1, \dots, n-1$. By [1, Theorem 15], $R[s]$ is an elementary divisor ring if and only if R is von Neumann regular, i.e. for all a in R there exists x such that $a = a^2x$. This justifies why we choose to work over commutative von Neumann regular rings. See [7] and [8] for further properties and characterizations of regular rings.

2.2. Admissible structures

In the sequel, assume that the commutative ring R is von Neumann regular. When varying F, any matrix equivalent to $sI - (A + BF)$ will have determinant equal to $\chi(A + BF)$, up to multiplication by unit elements of R (the units of $R[s]$ are those of R, since R has no nonzero nilpotents). Therefore, the problem (A, B, M) over the ring R makes sense only if M has determinant of degree n and with leading coefficient invertible in R, in which case M will be called an admissible structure matrix for the system (A, B).

As the next example shows, an admissible structure does not need to have monic elements.

Example 1 *(Non-monic structures).* Let R be any von Neumann regular ring which is not a field. By [7, p. 7], the ideal of R generated by any nonunit and nonzero element is of the form eR, for some idempotent e different from 0 and 1. Then, $((1-e)s+e)(es+1-e) = s$ is a nontrivial factorization of s into two polynomials of degree 1. Therefore, $d_1 := (1-e)s + e$ is a divisor of $d_2 := s(es + 1 - e)$, neither of d_1, d_2 is monic (nor has leading coefficient invertible) but $d_1d_2 = s^2$, monic of degree 2. Consequently, the matrix $diag(d_1, d_2)$ is a Smith normal form over $R[s]$, and it is admissible, in the sense explained above. This justifies that, when working over von Neumann regular rings, it is too restrictive to consider only monic Smith normal forms.

3. Main results

Wee shall need the following result [5, Theorem 1.2]: a property definable by algebraic (polynomial) equations[1] holds for a commutative von Neumann regular ring if and only if it holds for every localization at a maximal ideal.

[1] The precise meaning of property definable by algebraic equations is given in [5, pp. 225–226] and includes, in particular, properties defined in terms of a finite set of algebraic equations in a finite set of unknowns, but the number of unknowns and equations involved may be (countably) infinite.

Recall that for a regular ring R and a maximal ideal $\mathfrak{m}$, the localization $R_\mathfrak{m}$ is a field [7, Theorem 3.1], and in fact it is isomorphic to the residue field $R/\mathfrak{m}$. Therefore, in [5, Theorem 1.2] we may replace localization by residue field.

We are ready to prove our first result.

Theorem 2. *Let R be a commutative von Neumann regular ring. Then, any problem (A, B, M) (with M an admissible structure) has a solution over R if and only if it has a solution modulo each maximal ideal.*

Proof. If a problem (A, B, M) admits a solution over R, there exist matrices $F, P(s)$, $P'(s), Q(s), Q'(s)$ such that

$$P(s)P'(s) = I, Q(s)Q'(s) = I, P(s)(sI - (A + BF))Q(s) = M \tag{2}$$

Reducing modulo some maximal ideal $\mathfrak{m}$ yields

$$P_\mathfrak{m}(s)P'_\mathfrak{m}(s) = I, Q_\mathfrak{m}(s)Q'_\mathfrak{m}(s) = I, P_\mathfrak{m}(s)(sI - (A_\mathfrak{m} + B_\mathfrak{m}F_\mathfrak{m}))Q_\mathfrak{m}(s) = M_\mathfrak{m},$$

so we have found a matrix $F_\mathfrak{m}$ and invertible matrices $P_\mathfrak{m}(s)$, $Q_\mathfrak{m}(s)$ which solve problem $(A_\mathfrak{m}, B_\mathfrak{m}, M_\mathfrak{m})$ over the field $R/\mathfrak{m}$.

Conversely, suppose that for each maximal ideal $\mathfrak{m}$, the problem reduced modulo $\mathfrak{m}$, $(A_\mathfrak{m}, B_\mathfrak{m}, M_\mathfrak{m})$, admits a solution over $R/\mathfrak{m}$.

Let us see that problem (A, B, M) is definable by algebraic equations, in the sense of [5]. The countable set of unknowns will be

$$X = \{f_{ij}\} \cup \{p_{ijk}\} \cup \{p'_{ijk}\} \cup \{q_{ijk}\} \cup \{q'_{ijk}\}$$

where (f_{ij}) are the coefficients of F, and p_{ijk} is the coefficient of s^k in the position (i, j) of $P(s)$, and similarly for $P'(s), Q(s), Q'(s)$. Analogously, the set of parameters defining the problem is $Y = \{a_{ij}\} \cup \{b_{ij}\} \cup \{\phi_{ik}\}$, where $A = (a_{ij}), B = (b_{ij})$ and ϕ_{ik} is the coefficient of s^k in $\phi_i(s)$.

Furthermore, for each positive integer N, denote by S_N the set of all equations resulting from equating coefficients of each power of s in all positions of the matricial equations (2), when $P(s), P'(s), Q(s), Q'(s)$ have degrees bounded by N, i.e. we consider only those variables in X with $k \leq N$ (see example below). It is clear that all equations in S_N are expressions in the polynomial ring $\mathbb{Z}[X \cup Y]$, i.e. sums of arbitrary products of certain variables from X and Y. Also, the problem (A, B, M) will have a solution if and only if all equations of S_N are satisfied for some N, but we do not know a priori an upper bound for N.

Now, the sets S_N have a very important property: if $N < N'$, then S_N is just the restriction of $S_{N'}$, when specializing to zero all variables appearing in $S_{N'}$ and not in S_N (concretely, all variables with an index k satisfying $N < k \leq N'$). This property gives a total order on the sets S_N, which in particular ensures the conditions required in the discussion preceeding Theorem 1.2 in [5]. Thus, the cited theorem can be applied, and a local solution is lifted to a global solution. □

As an immediate consequence, one obtains a version of Rosenbrock's theorem for von Neumann regular rings.

Corollary 3 *(Rosenbrock's theorem for von Neumann regular rings). Let (A, B) be a controllable system over a commutative von Neumann regular ring R. Then, a problem (A, B, M) has a solution over R if and only if the inequalities (1) hold over $R/\mathfrak{m}$, for each maximal ideal $\mathfrak{m}$ in R.*

The following example may help clarify the proof of Theorem 2.

Example 4. With the notations of Theorem 2, suppose that $P(s), P'(s)$ are of size 2×2, with unknown coefficients given by

$$P(s) = \begin{bmatrix} p_{110} + p_{111}s + p_{112}s^2 & p_{120} + p_{121}s + p_{122}s^2 \\ p_{210} + p_{211}s + p_{212}s^2 & p_{220} + p_{221}s + p_{222}s^2 \end{bmatrix}$$
$$P'(s) = \begin{bmatrix} p'_{110} + p'_{111}s + p'_{112}s^2 & p'_{120} + p'_{121}s + p'_{122}s^2 \\ p'_{210} + p'_{211}s + p'_{212}s^2 & p'_{220} + p'_{221}s + p'_{222}s^2 \end{bmatrix}$$

In the set S_2, for example, the equation corresponding to the coefficient of s^2 in position $(1, 1)$ of the matricial equation $P(s)P'(s) = I$ is

$$p_{110}p'_{112} + p_{111}p'_{111} + p_{112}p'_{110} + p_{120}p'_{212} + p_{121}p'_{211} + p_{122}p'_{210} = 0$$

Making zero the variables $p'_{212}, p_{122}, p'_{112}, p_{112}$ not appearing in S_1, we get $p_{111}p'_{111} + p_{121}p'_{211} = 0$, which is one of the equations of the set S_1, namely the one corresponding to s^2 in position $(1, 1)$.

Remark 5. As an example of how to take profit of properties definable by algebraic equations, we can prove that polynomials in one indeterminate over von Neumann regular rings satisfy the following property: if two elements generate the same ideal, then they are associates. This is precisely the condition needed to ensure that in an elementary divisor ring, the diagonal reduction (Smith normal form) obtained from a matrix is really canonical, up to multiplication by units. This topic is discussed e.g. in Chapter 15 of [3], and in [10, pp. 465–466], where examples are given of commutative rings not satisfying this property.

To prove this, the fact that two polynomials $f(s), g(s)$ generate the same ideal in $R[s]$ can be translated into the following polynomial condition: there exist $a(s), b(s)$ in $R[s]$ such that $f(s)a(s) = g(s)$ and $g(s)b(s) = f(s)$. Also, the fact that $f(s), g(s)$ are associates means that there exist u, v in R with $uv = 1$ and $f(s)u = g(s)$. Thus, the problem is clearly definable by algebraic equations, as required in the discussion preceding [5, Theorem 1.2]. Moreover, for each maximal ideal $\mathfrak{m}$ of R, the problem reduced modulo $\mathfrak{m}$ consists in finding a nonzero element $\bar{u}$ in the field $R/\mathfrak{m}$ such that $\bar{f}(s)\bar{u} = \bar{g}(s)$, subject to the condition that $\bar{f}(s), \bar{g}(s)$ generate the same ideal in the principal ideal domain $(R/\mathfrak{m})[s]$. Such an element $\bar{u}$ certainly exists in $R/\mathfrak{m}$, and by [5, Theorem 1.2], the existence of a residual solution $\bar{u}$ for all $\mathfrak{m}$ implies that there is a global solution u in R.

Next theorem gives a new characterization of commutative von Neumann regular rings.

Theorem 6. *For a commutative ring R, the following conditions are equivalent.*

(i) *R is von Neumann regular.*
(ii) *Any problem (A, B, M) admits a global solution over R if and only if it admits a residual solution over $R/\mathfrak{m}$, for each maximal ideal $\mathfrak{m}$.*

Proof. (i) ⇒ (ii). This has already been proved in Theorem 2.

(ii) ⇒ (i). Let R be a commutative ring satisfying (ii), and take an arbitrary element a of R. To prove that R is regular, we must find some x such that $a = a^2x$. Consider the following problem with sizes $n = 1, m = 1$: $(A = [0], B = [a^2], M = [s - a])$, where we look for some $F = [f]$ such that $sI - (A + BF) \sim M$. That is, we look for some unit element p in R satisfying $p(s - (0 + a^2 f))p^{-1} = s - a$, or equivalently, $a^2 f = a$.

Let $\mathfrak{m}$ be a maximal ideal of R. If $a \in \mathfrak{m}$, then also $a^2 \in \mathfrak{m}$, and the equation to be solved in $R/\mathfrak{m}$ is $\bar{0}\bar{f} = \bar{0}$ (here, $\bar{\ }$ denotes reduction modulo $\mathfrak{m}$), so any $\bar{f}$ is a solution. On the other hand, if a does not belong to $\mathfrak{m}$, then $\bar{a}$ is nonzero in the field $R/\mathfrak{m}$, and $\bar{a}^2\bar{f} = \bar{a}$ admits a solution $\bar{f} = \bar{a}^{-1}$. Thus, the problem has a solution in every residue field, hence by (ii) there must be a solution over R, i.e. $a^2 f = a$, as we wanted to prove. □

Example 7. Consider the finite ring $R = \mathbb{Z}/6\mathbb{Z}$ and the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 3 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3s+1 & 0 \\ 0 & 0 & 0 & \underbrace{4s^4 + 3s^3 + 3s^2 + 3s + 5}_{(3s+1)(4s^4+3s^2+5)} \end{bmatrix}.$$

Let us see if the problem (A, B, M) is solvable over R. It suffices to solve it in every residue field. R has two maximal ideals $2R, 3R$, with corresponding residue fields isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ respectively.

Over $R/2R$, we see that $(A_\mathfrak{m}, B_\mathfrak{m})$ is controllable and in Brunovsky canonical form, with controllability indices $(3,1,0,0)$. Besides, $M_\mathfrak{m} = diag(1,1,s+1,s^3+s^2+s+1)$, with degree sequence $(0,0,1,3)$. The inequalities (1) are satisfied, so the problem is solvable.

On the other hand, over $R/3R$, $(A_\mathfrak{m}, B_\mathfrak{m})$ is controllable and in Brunovsky canonical form, with controllability indices $(2,2,0,0)$, and $M_\mathfrak{m} = diag(1,1,1,s^4+2)$, with degree sequence $(0,0,0,4)$. Again, the problem is solvable.

An explicit global solution is obtained by combining the solutions in $R/2R \cong 3R$ and in $R/3R \cong 4R$, and then lifting via the decomposition $R = 3R + 4R$:

$$F = 3\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} + 4\begin{bmatrix}0 & 1 & 2 & 0\\ 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}3 & 1 & 5 & 0\\ 0 & 0 & 0 & 5\\ 0 & 0 & 0 & 0\end{bmatrix}$$

This matrix F satisfies $M \sim (sI - (A + BF))$.

Let us see an example of a von Neumann regular ring with infinitely many maximal ideals, where our results can be applied.

Example 8. Let $R = C(\mathbb{N}, \mathbb{R})$ be the ring of continuous real-valued functions over $\mathbb{N}$, the set of natural numbers, with the discrete topology. Note that the set of maximal ideals of R is infinite, it is in 1–1 correspondence with $\beta\mathbb{N}$, the Stone–Cěch compactification of $\mathbb{N}$, see [6]. That R is a von Neumann regular ring can be proved directly (showing that each finitely generated ideal is generated by an idempotent element), or applying the results in [6] (because $\mathbb{N}$ is a P-space).

Remark 9. Let R be a commutative von Neumann regular ring, and M, M' matrices over R, such that $xI - M$ and $xI - M'$ have invariant factors $\{(d_i(x))\}$ and $\{(d'_i(x))\}$, for $i = 1, \dots, n$. The following diagram summarizes the relationship among similarity, equivalence and invariant factors in this situation.

$$\begin{array}{ccccc} M \approx M' & \overset{1.}{\Leftrightarrow} & xI - M \sim xI - M' & \overset{2.}{\Leftrightarrow} & \{(d_i(x)) = (d'_i(x))\}_{i=1}^n \\ \Updownarrow 3. & & \Updownarrow 4. & & \Updownarrow 5. \\ \overline{M} \approx \overline{M'} & \overset{6.}{\Leftrightarrow} & xI - \overline{M} \sim xI - \overline{M'} & \overset{7.}{\Leftrightarrow} & \left\{(\overline{d_i}(x)) = (\overline{d'_i}(x))\right\}_{i=1}^n \end{array}$$

Here, $\sim$ denotes equivalence of matrices, $\approx$ denotes similarity, $(*)$ means the ideal generated by some element $*$, and I is the $n \times n$ identity matrix. In the last row, $^-$denotes reduction modulo some maximal ideal $\mathfrak{m}$, and statements hold for all $\mathfrak{m}$.

The equivalences 1. and 6. are consequence of [3, Corollary 16], 3. was proved by Guralnick [9], and so 4. follows from the commutativity of the diagram. On the other hand, 7. holds because $(R/\mathfrak{m})[x]$ is a principal ideal domain, and 2. because $R[x]$ is an elementary divisor domain, with the additional property discussed in Remark 5. Therefore, commutativity yields 5. and hence all statements are equivalent.

With the help of the above diagram, the reader can arrive at the main result of this paper via this alternative approach: given a problem (A, B, M), first prove that there exists a matrix A' with $sI - A' \sim M$, and then find F such that $A + BF \approx A'$. Applying the equivalence 3., the solution to this similarity problem over R is equivalent to its solution in each residue field.

Remark 10. The following is an example of a noncommutative von Neumann regular ring. Let $A = \mathbb{K}^{n \times n}$ be the ring of $n \times n$ matrices with coefficients in a field $\mathbb{K}$. Since A is noncommutative, for any $a \in A$ one must find x such that $a = axa$ (see [8]). Indeed, we put a in Hermite form: $a = PJQ$, with P, Q invertible and $J = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$. Note that $J^3 = J$, from which it follows that

$$a = PJQ = PJ^3Q = P(P^{-1}aQ^{-1})J(P^{-1}aQ^{-1})Q = a(Q^{-1}JP^{-1})a = axa$$

4. Conclusions

In this paper, we have proved that the invariant factor assignment problem which we called (A, B, M) has a solution over a commutative ring R if and only if R is von Neumann regular. Once more, like in [4,13], commutative von Neumann regular rings are characterized by a systems property, which shows that regular rings are a very appropriate class of commutative rings to which many results from systems theory over fields can be extended.

Although all our work deals with commutative rings, it is worth to note that systems over noncommutative rings have also been studied (see e.g. [2]), and there are plenty of noncommutative von Neumann regular rings, like the one of the previous remark.

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