Coherent States for infinite homogeneous waveguide arrays

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ABSTRACT

Perelomov coherent states for equally spaced, infinite homogeneous waveguide arrays with Euclidean E(2) symmetry are defined, and new resolutions of the identity are constructed in Cartesian and polar coordinates. The key point to construct these resolutions of the identity is the fact that coherent states satisfy Helmholtz equation (in coherent states labels) an thus a non-local scalar product with a convolution kernel can be introduced which is invariant under the Euclidean group. It is also shown that these coherent states for the Eucliean E(2) group have a simple and natural physical realization in these waveguide arrays.

Keywords: Coherent states, Euclidean group, waveguide arrays, Helmholtz equation

I. INTRODUCTION

Waveguides (see, for instance, [1]) are optical devices made of optical fibres, i.e. an infinite cylinder of dielectric material (the core, with a high index of refraction) surrounded

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by another material (the cladding, with a lower index of refraction) built up in such a way that they guide light in the core (by total reflection on the cladding) in the longitudinal direction, whereas the transverse size of the core is of the order of magnitude of the wavelength of the light. A parallel array of waveguides is a set of parallel waveguides in such a way that light couples between nearest waveguides by evanescent fields [1, 2]. We shall restrict ourselves to the case of parallel waveguides containing a single guided mode which are easier to describe.

The simplest parallel waveguide array is that with equally spaced waveguides and of homogeneous properties in the longitudinal direction. An the simplest among them is the case of an infinite array, since it possesses translational symmetry in longitudinal direction and (discrete) translational symmetry in the transverse direction. We shall see later that these infinite arrays possess the symmetry of the Euclidean E(2) group. The cases of semi-infinite and finite waveguide arrays will be considered in [3].

We shall see that the differential equations describing the amplitude of the electric field in each waveguide for the propagation along the longitudinal direction can be expressed in terms of the generators of the Euclidean E(2) group, known as (true) amplitude and phase operators [4, 5], which are naturally unitary and act on an extended Fock space (obtained by adding to the ordinary Fock number states the negative number states). Note that the same group E(2) is the symmetry of the quantum mechanics of a particle on the circle [6]. We shall see that in fact both system are intimately related.

Since the Eucliedan groups do not possess square integrable representations [7, 8], coherent states for the group $E(2)$ (and the higher dimensional extensions $E(n)$) of the Perelomov type [9] do not admit a resolution of the identity. Diverse techniques have been devised to circumvent this problem, like in [7], where the labels of the coherent states are restricted to the cylinder and further admissibility conditions are required on the fiducial or vacuum state (see also [10] for applications on signal processing on the circle). See [11, 12] for a recent account of these states. In [8], a reducible representation of E(2) is used summing up all irreducible representations with the radius of the circle belonging to an interval (considering in fact a label space defined by an annuls times a line). There are other approaches for the definition of coherent states on the circle, which are not related to the E(2) group, like [13] or [14]. See the review [15] for details on these and other families of coherent states on the circle.

In this paper we shall address the issue of constructing a new resolution of the identity for coherent states of the E(2) group (in the flavor of functions representing distributions of light amplitudes on an infinite waveguide array) modifying the usual integration on the phase space (the coadjoint orbit of the group) with an invariant measure under E(2) by a double integral with a convolution kernel on a non-homogeneous subspace, the one used to define a Cauchy initial value problem for the Helmholtz equation [16]. Helmholtz equation appears here as the eigenvalue equation of the Casimir of E(2) [17]. This construction can also be performed in the usual representation of E(2) in terms of functions on the circle, and can be easily generalized to higher dimensional cases (see [18] for details).

The content of the paper is as follows. In Section II we review the study of an infinite set of equally spaced homogeneous waveguide arrays, obtaining the differential equations describing the amplitudes of light along the waveguides and computing the propagator. In Section III we recall that the symmetry of this system is the Euclidean group and we construct coherent states of the Perelomov type. In Section IV resolutions of the Identity for these coherent states are built in both Cartesian and polar coordinates using the fact that coherent states satisfy Helmholtz equation. In Section V we present some conclusions of the present work.

II. EQUALLY SPACED, INFINITE HOMOGENEOUS WAVEGUIDE ARRAYS

Waveguide arrays are a good testbed to simulate both classical and quantum phenomena [19–23], and in this case we shall use them to realize coherent states of the Euclidean E(2) group. We shall focus in the case of an infinite number of equally spaced and homogeneous parallel waveguide arrays.

The equations describing light propagation along the z-direction are given by (see [1]):

$$i\frac{dA_n}{dz} = A_{n+1} + A_{n-1}\,, \qquad n \in \mathbb{Z}\,, \tag{1}$$

where $A_n(z)$ is the electric modal field in the n-th waveguide at position z. The distance z is measured in units of the coupling constant between waveguides (supposed constant) and the propagation constant has been removed by a suitable transformation (see [24]).

To describe algebraically this situation, let us use Dirac notation (see [23]) and introduce an abstract Hilbert space $\mathcal{H}$, expanded by the extended Fock basis $\bar{\mathcal{F}} = \{|n\rangle,\, n \in \mathbb{Z}\}$, where the Fock state $|n\rangle$ represents the n-th waveguide, and that is isomorphic to $\ell^2(\mathbb{Z})$.

Therefore the infinite length vector $\mathcal{A} = (\ldots, A_0, \ldots)$ can be represented (in Dirac's notation) as:

$$|\mathcal{A}\rangle = \sum_{n\in\mathbb{Z}} A_n |n\rangle\,, \tag{2}$$

Elements in $\mathcal{H}$ represent the amplitude of light distributions along the whole waveguide with finite total energy for each value of z.

$$||\mathcal{A}(z)||^2 = |||\mathcal{A}(z)\rangle||^2 = \sum_{n\in\mathbb{Z}} |A_n(z)|^2 < \infty\,. \tag{3}$$

The Eqs. (1) can be written as a Schrödinger equation, with $-z$ playing the role of time:

$$-i\frac{d}{dz}|\mathcal{A}\rangle = \hat{H}|\mathcal{A}\rangle\,, \tag{4}$$

with the Hamiltonian given by $\hat{H} = \hat{V}^\dagger + \hat{V}$, where $\hat{V}^\dagger$ and $\hat{V}$ are step operators:

$$\hat{V}|n\rangle = |n-1\rangle\,, \qquad \hat{V}^\dagger|n\rangle = |n+1\rangle\,, \qquad n \in \mathbb{Z}\,. \tag{5}$$

The propagator $\hat{U}(z)$, verifying $|\mathcal{A}(z)\rangle = \hat{U}(z)|\mathcal{A}(0)\rangle$, turns out to be:

$$\hat{U}(z) = e^{iz\hat{H}} = \sum_{n,m\in\mathbb{Z}} i^{n-m} J_{n-m}(2z)|n\rangle\langle m|\,. \tag{6}$$

where the commutation relations of the operators (5), given in eq. (10), have been used. It is interesting to note that if we define the *twisted* Hamiltonian[1]:

$$\hat{H}^\theta = e^{i(\theta-\frac{\pi}{2})\hat{n}}\hat{H}e^{-i(\theta-\frac{\pi}{2})\hat{n}} = -i(e^{i\theta}\hat{V}^\dagger - e^{-i\theta}\hat{V})\,, \tag{7}$$

where $\hat{n}|n\rangle = n|n\rangle$, $n \in \mathbb{Z}$, is the number operator, then the corresponding *twisted* propagator is

$$\hat{U}^\theta(z) = e^{iz\hat{H}^\theta} = e^{z(e^{i\theta}\hat{V}^\dagger - e^{-i\theta}\hat{V})} = \sum_{n,m\in\mathbb{Z}} e^{i(n-m)\theta} J_{n-m}(2z)|n\rangle\langle m|\,. \tag{8}$$

[1] This kind of Hamiltonian describes waveguides that are twisted around each other [25], although this is only physically realizable in the case of a finite number of waveguides. For convenience, the twisting has been shifted by $\frac{\pi}{2}$, so the non twisted case is recovered for $\theta = \frac{\pi}{2}$.

In particular, the original Hamiltonian $\hat{H}$ and propagator $\hat{U}$ are recovered for $\theta = \frac{\pi}{2}$. Also note that we can restrict the twisting θ to $[0, \pi)$, since the case $\theta \in [\pi, 2\pi)$ is obtained by backward propagation with $\theta - \pi$.

In Sec. III we discuss E(2) coherent states, which are generated by a displacement operator that coincides with $\hat{U}^\theta$.

III. E(2) COHERENT STATES

In the setting of previous Sec. II, we have that the step operators $\hat{V}^\dagger$ and $\hat{V}$ satisfy the following unitarity property:

$$\hat{V}\hat{V}^\dagger = \hat{V}^\dagger\hat{V} = \hat{I}_{\mathcal{H}} \tag{9}$$

Therefore these operators, together with the number operator $\hat{n}$, constitute a unitary and irreducible realization of the Euclidean algebra $E(2)$, with commutators:

$$\begin{aligned} \left[\hat{n}, \hat{V}\right] &= -\hat{V} \\ \left[\hat{n}, \hat{V}^\dagger\right] &= \hat{V}^\dagger \\ \left[\hat{V}, \hat{V}^\dagger\right] &= 0\,. \end{aligned} \tag{10}$$

It is clear that the unitarity property (9) is the eigenvalue equation (with eigenvalue 1) for the quadratic Casimir of the Euclidean algebra $E(2)$, $\hat{C}_2 = \hat{V}\hat{V}^\dagger = \hat{V}^\dagger\hat{V}$, i.e. $\hat{C}_2 = \hat{I}_{\mathcal{H}}$.

Perelomov-type coherent states can be introduced in the usual way, as the action of the Displacement operator:

$$\hat{D}(\alpha) = e^{\alpha\hat{V}^\dagger - \alpha^*\hat{V}}\,, \qquad \alpha \in \mathbb{C}\,, \tag{11}$$

on the vacuum, which in this case[2] is the state $|0\rangle$. If $\alpha = re^{i\theta}$, then we have that $\hat{D}(\alpha) = \hat{U}_\theta(r)$.

The Displacement operator satisfies:

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha + \beta)\,, \qquad \hat{D}(\alpha)^\dagger = D(-\alpha)\,. \tag{12}$$

[2] The vacuum in the Hilbert space $\bar{\mathcal{H}}$ is the most symmetrical state, which is the state $|0\rangle$ since $\hat{n}|0\rangle = 0$.

We define $E(2)$ coherent states as:

$$
\begin{aligned}
|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{\alpha\hat{V}^\dagger - \alpha^*\hat{V}}|0\rangle &= e^{r\left(e^{i\theta}\hat{V}^\dagger - \left(e^{i\theta}\hat{V}^\dagger\right)^{-1}\right)}|0\rangle \\
&= \sum_{n=-\infty}^{\infty} J_n(2r)e^{in\theta}\hat{V}^{\dagger n}|0\rangle \\
&= \sum_{n=-\infty}^{\infty} \alpha^n 2^n k_n(2|\alpha|)|n\rangle \equiv \sum_{n=-\infty}^{\infty} c_n(r,\theta)|n\rangle \,,
\end{aligned}
\tag{13}
$$

where we have used the generating function of Bessel functions [26]. Here $k_n(x) = \frac{J_n(x)}{x^n}$ are Bochner-Riesz integral kernels [27]. Thus $E(2)$ coherent states are coherent states of the AN type [28] $|\alpha\rangle = \sum_{n=0}^{\infty} \alpha^n h_n(|\alpha|^2)|n\rangle$ with $h_n(x) = 2^n k_n(2\sqrt{x})$.

Using the asymptotic behaviour of Bessel functions for large n:

$$
J_n(x) \approx \frac{1}{\sqrt{2\pi n}} \left(\frac{ex}{2n}\right)^n \,, \tag{14}
$$

we can check that the radius of convergence of the series in (14) is indeed infinite, therefore $\alpha \in \mathbb{C}$ and $E(2)$ coherent states are defined on the entire complex plane.

The overlap of two coherent states is:

$$
\langle\alpha|\alpha'\rangle = \langle re^{i\theta}|r'e^{i\theta'}\rangle = \sum_{n=-\infty}^{\infty} J_n(2r)J_n(2r')e^{in(\theta'-\theta)} = J_0(2R) \,, \tag{15}
$$

with $R = \sqrt{r^2 + r'^2 - 2rr'\cos(\theta'-\theta)}$, and in the last step we have made use of the *summation theorem* for Bessel functions (see [26], eq. 8.530).

Particular cases for the overlap are:

$$
\langle re^{i\theta}|r'e^{i\theta}\rangle = J_0(2|r'-r|) \tag{16}
$$

$$
\langle re^{i\theta}|r'e^{i(\theta+\frac{\pi}{2})}\rangle = J_0(2\sqrt{r^2+r'^2}) \tag{17}
$$

$$
\langle re^{i\theta}|r'e^{i(\theta+\pi)}\rangle = J_0(2(r+r')) \tag{18}
$$

$$
\langle re^{i\theta}|r'e^{i(\theta+\frac{3\pi}{2})}\rangle = J_0(2\sqrt{r^2+r'^2}) \tag{19}
$$

$$
\langle re^{i\theta}|re^{i\theta'}\rangle = J_0(4r\sin\left(\frac{\theta'-\theta}{2}\right)) \tag{20}
$$

$$
\langle re^{i\theta}|re^{i\theta}\rangle = J_0(0) = 1 \,. \tag{21}
$$

therefore the states are normalized.

IV. RESOLUTION OF THE IDENTITY

Once coherent states have been defined, we need to provide a resolution of the identity operator. We shall recover the well known result that the usual construction fails, the reason being that the group E(2) do not possesses square integrable representations (see for instance [8]). Let us see it with detail.

A. Naive resolution of the Identity

Coherent states of E(2) constitute an overcomplete family for the Hilbert space $\mathcal{H}$. In the context of the example of Sec. II, any finite energy light distribution in the infinite array, for each value of z, can be expanded in terms of the family of coherent states $|\alpha\rangle, \alpha \in \mathbb{C}$, apart from the fact that the coherent states $|ze^{i\theta}\rangle$ represent themselves the propagation in the longitudinal direction Z of light impinged at $z = 0$ at the waveguide $n = 0$ under the Hamiltonian $\hat{H}_\theta$ given in Eq. (7).

With the standard construction of a resolution of the identity for Perelomov-type coherent states, and taking into account that the invariant measure under $E(2)$ on the complex plane is $d\alpha d\alpha^* = rdrd\theta$, we can try with:

$$\hat{A} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} rdr|re^{i\theta}\rangle\langle re^{i\theta}| = \int_0^{\infty} rdr \sum_{n=-\infty}^{\infty} J_n(2r)^2|n\rangle\langle n| \,. \tag{22}$$

But each of the integrals $\int_0^\infty rdrJ_n(2r)^2$ is divergent. We can try with the measure $drd\theta$, but this leads again to divergent integrals. With measures $d\theta\frac{dr}{r^\lambda}$, with $\lambda > 0$, the integrals converge (see [26], formula 6.574-2), but the result depends on n, i.e.:

$$\hat{A}_\lambda = \int_0^{\infty} \frac{dr}{r^\lambda} \sum_{n=-\infty}^{\infty} J_n(2r)^2|n\rangle\langle n| = \frac{\Gamma(\lambda)}{2\Gamma(\frac{\lambda+1}{2})^2} \sum_{n=-\infty}^{\infty} \frac{\Gamma(|n| + \frac{1-\lambda}{2})}{\Gamma(|n| + \frac{1+\lambda}{2})}|n\rangle\langle n| \,. \tag{23}$$

Note that

$$\lim_{\lambda\to 0} \frac{2\Gamma(\frac{\lambda+1}{2})^2}{\Gamma(\lambda)}\hat{A}_\lambda = \hat{I}_{\mathcal{H}} \,. \tag{24}$$

However, this limit procedure can cause problems in explicit computations.

Therefore, it is difficult to find an integration measure of this form. Following [17, 18], we can introduce a scalar product for the Coherent State representation and a resolution of the identity based on the fact that $E(2)$ coherent states are solutions of Helmholtz equation.

B. Differential realization and Helmholtz Equation

A differential realization for the $E(2)$ generators acting on the label space of the coherent states can be obtained in the usual way (see, for instance [29]), resulting in:

$$\begin{aligned}\hat{n}_d &= -i\frac{\partial}{\partial\theta} \\ \hat{V}_d &= -\frac{1}{2}e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right) \\ \hat{V}_d^\dagger &= \frac{1}{2}e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right) .\end{aligned} \tag{25}$$

Note that the action of these operators on the coefficients of the Coherent States are:

$$\begin{aligned}\hat{n}_d\, c_n &= n c_n \\ \hat{V}_d\, c_n &= c_{n+1} \\ \hat{V}_d^\dagger\, c_n &= c_{n-1} ,\end{aligned} \tag{26}$$

thus the role of lowering and raising operators are interchanged when acting on the coefficients instead of the ket vectors (as it is usual).

We can also obtain the differential realization of the operators in Sec. II:

$$\begin{aligned}\hat{H}_d &= \hat{V}_d^\dagger + \hat{V}_d = -i\frac{\partial}{\partial y} \\ \hat{H}_{\theta d} &= -i(e^{i\theta}\hat{V}_d^\dagger - e^{-i\theta}\hat{V}_d) = \frac{\partial}{\partial r} ,\end{aligned} \tag{27}$$

with $\alpha = re^{i\theta} = x + iy$.

Therefore, we recover that $\hat{H}_d$ is the generator of propagation for parallel waveguides ($\theta = \frac{\pi}{2}$) whereas $\hat{H}_{\theta d}$ generates the propagation for a twisted waveguide with angle θ.

From the differential realization we derive that the eigenvalue equation for the quadratic Casimir of the Euclidean algebra, $\hat{C}_2 = \hat{V}_d\hat{V}_d^\dagger = \hat{V}_d^\dagger\hat{V}_d = \hat{I}$, leads to Helmholtz equation in the plane in polar coordinates:

$$\hat{C}_2|\alpha\rangle = |\alpha\rangle \qquad \Rightarrow \qquad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + k^2\right)|\alpha\rangle = 0\,, \tag{28}$$

where the wave number is $k = 2$. Note also that each of the coefficients $c_n(r,\theta)$ satisfy Helmholtz equation, constituting a basis for the Hilbert space $\mathcal{H}_{\rm osc}$ of bounded oscillatory solutions of Helmholtz equation [16, 17], which is isomorphic to $\mathcal{H}$. We remark that $c_n(r,\theta)$ include only oscillatory solutions of the Helmholtz equation, which is key for the isomorphism between $\mathcal{H}$ and $\mathcal{H}_{\rm osc}$.

It should be stressed that only regular solutions at the origin appear in the coherent states, thus Bessel functions of the second kind Y_n are discarded.

Seing the complex number $\alpha = re^{i\theta} = x + i\,y$ as a vector $\vec{\alpha} = (x,y) \in \mathbb{R}^2$, Eq. (28) is written as

$$\left[\Delta_{\vec{\alpha}} + k^2\right]|\alpha\rangle = 0\,. \tag{29}$$

C. Resolution of the identity: Cartesian coordinates

The fact that coherent states verify Helmholtz equation (28) or (29) implies that any coherent state can be obtained from coherent states in a line in the space of coherent states labels $\alpha \in \mathbb{C}$ (as a Cauchy initial value problem, see [16] and Appendix B of [17]) or a circle (as a boundary value problem). This redundancy explains why integration on the whole E(2) group (see eq. (22)) causes divergences. We can try with different integration measures, but the problem is not the measure, but the redundancy.

In fact, any oscillatory solution of Helmholtz equation (29) can be expressed in terms of its values and those of its derivative with respect to y at $y = 0$:

$$\begin{aligned}\psi(x,y) &= \int_{\mathbb{R}} dx' \left[\Delta(x-x',y-y')\frac{\partial\psi(x',y')}{\partial y'} - \frac{\partial\Delta(x-x',y-y')}{\partial y'}\psi(x',y')\right]_{y'=0} \\ &= \int_{\mathbb{R}} dx' \left(\Delta(x-x',y)\,\overset{\bullet}{\psi}(x') + \frac{\partial\Delta(x-x',y)}{\partial y}\,\overset{\circ}{\psi}(x')\right)\,,\end{aligned} \tag{30}$$

where

$$\Delta(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-k}^{k} d\epsilon e^{i\epsilon x}\frac{\sin(\sqrt{k^2-\epsilon^2}y)}{\sqrt{k^2-\epsilon^2}}$$

$$
\begin{aligned}
&= \sqrt{\frac{2}{\pi}} R \frac{\sin(y\sqrt{k^2 + \frac{\partial^2}{\partial x^2}})}{\sqrt{k^2 + \frac{\partial^2}{\partial x^2}}} j_0(Rx) \\
&= \sqrt{\frac{2}{\pi}} Ry \sum_{n=0}^{\infty} (-1)^n \frac{n!}{(2n+1)!} \left(\frac{2Ry^2}{x}\right)^n j_n(Rx)
\end{aligned} \tag{31}
$$

is the Helmholtz propagator, and $\overset{\circ}{\psi}(x)$ and $\overset{\bullet}{\psi}(x)$ are the values of $\psi(x,y)$ and its derivative at $y=0$, respectively. In the last formula j_n is the spherical Bessel function, which for $n=0$ coincides with the *sinc* function $\frac{\sin x}{x}$. Using (14), it is easy to check that the series is absolutely convergent for all values of x and y, therefore the Helmholtz propagator is well defined.

From this expression, it is clear that any integration involving $\psi(x,y)$ with the invariant measure under $E(2)$, $dxdy$, will be divergent.

To avoid this redundancy, we follow the ideas in [16] (see also [17, 18]), where a scalar product is introduced involving only the values of the functions and their derivatives with respect to y at $y=0$, and with a double integral with a convolution kernel. For this purpose, let us introduce two subsets of states:

$$
\begin{aligned}
|\overset{\circ}{x}\rangle &= |x + i\,y\rangle|_{y=0} = \sum_{n=-\infty}^{\infty} J_n(2x)|n\rangle \\
|\overset{\bullet}{x}\rangle &= -\frac{i}{\sqrt{2}}\frac{\partial}{\partial y}|x + i\,y\rangle|_{y=0} = \frac{1}{\sqrt{2}}\hat{H}_d|x + i\,y\rangle|_{y=0} = \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} n\frac{J_n(2x)}{x}|n\rangle\,.
\end{aligned} \tag{32}
$$

The overlaps among these states are:

$$
\begin{aligned}
\langle\overset{\circ}{x}\,|\,\overset{\circ}{x}\,'\rangle &= J_0(2|x'-x|) = k_0(2|x'-x|) \\
\langle\overset{\bullet}{x}\,|\,\overset{\bullet}{x}\,'\rangle &= \frac{1}{2xx'}\sum_{n=-\infty}^{\infty} n^2 J_n(2x)J_n(2x') = \frac{J_1(2|x'-x|)}{|x'-x|} = 2k_1(2|x'-x|) \\
\langle\overset{\circ}{x}\,|\,\overset{\bullet}{x}\,'\rangle &= 0\,,
\end{aligned} \tag{33}
$$

from which we deduce that $|\overset{\circ}{x}\rangle$ and $|\overset{\bullet}{x}\rangle$ are normalized and orthogonal with respect to each other. Moreover: according to (30), any coherent state can be expressed in terms of them, constituting a generating system.

Then we have that (see [17, 18]) a resolution of the identity is obtained as follows:

$$\hat{A} = \frac{\pi^{3/2}}{2\sqrt{2}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \left[4k_1(2|x'-x|)|\overset{\circ}{x}\rangle\langle\overset{\circ}{x}{}'| + k_0(2|x'-x|)|\overset{\bullet}{x}\rangle\langle\overset{\bullet}{x}{}'| \right] = \hat{I}_{\mathcal{H}} \,. \tag{34}$$

The proof of the validity of this statement is easily accomplished by considering the matrix elements in the $|\overset{\circ}{x}\rangle$ and $|\overset{\bullet}{x}\rangle$ basis:

$$\begin{aligned} \langle\overset{\circ}{x}|\hat{A}|\overset{\circ}{x}{}'\rangle &= k_0(2|x-x'|) \\ \langle\overset{\bullet}{x}|\hat{A}_{|}\overset{\bullet}{x}{}'\rangle &= 2k_1(2|x-x'|) \\ \langle\overset{\circ}{x}|\hat{A}|\overset{\bullet}{x}{}'\rangle &= 0\,, \end{aligned} \tag{35}$$

which implies that $\hat{A}_H = \hat{I}$ on $\mathcal{H}_{\text{osc}}$, and where we have used that

$$\int_{\mathbb{R}} dx'' k_\alpha(|x-x''|)k_\beta(|x''-x'|) = \frac{N_{\alpha-1/2}N_{\beta-1/2}}{N_{\alpha+\beta-1}} k_{\alpha+\beta-1/2}(|x-x'|) \tag{36}$$

with $N_\alpha = \frac{1}{2^\alpha\Gamma(\alpha+1)}$ (see Appendix D of Ref. [17]).

Due to the Euclidean symmetry E(2) of this scalar product [16, 17], the construction of coherent states should be valid using as initial Cauchy set any line obtained by translation and/or rotation of the line $y=0$ used in Eqs. (32). Thus, we can also use the families of coherent states:

$$\begin{aligned} |\pm \overset{\circ}{r}\rangle &= |re^{i(\theta+\frac{1\mp1}{2}\pi)}\rangle|_{\theta=\theta_0} = \sum_{n=-\infty}^{\infty} e^{in\theta_0} J_n(\pm 2r)|n\rangle \\ |\pm \overset{\bullet}{r}\rangle &= -\frac{i}{\sqrt{2}r}\frac{\partial}{\partial\theta}|re^{i(\theta+\frac{1\mp1}{2}\pi)}\rangle|_{\theta=\theta_0} = \frac{1}{\sqrt{2}}(e^{i\theta}\hat{V}_d^\dagger + e^{-i\theta}\hat{V}_d)|re^{i(\theta+\frac{1\mp1}{2}\pi)}\rangle|_{\theta=\theta_0} \\ &= \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} ne^{in\theta_0}\frac{J_n(\pm 2r)}{r}|n\rangle \end{aligned} \tag{37}$$

for $\theta_0 \in [0,\pi)$. Then equations (33) and (34) are valid substituting the coherent states in (32) by the ones in (37).

In particular, for $\theta_0 = \frac{\pi}{2}$, we recover that the coherent states $|\pm \overset{\circ}{r}\rangle$ are the states obtained by the (forward/backward) propagation of light impinged at $r=0$ at the waveguide $n=0$ under the Hamiltonian $\hat{H}$. This is a remarkable fact: these coherent states have a natural physical realization in a waveguide array (see [30] and also [31]). In addition, in order to obtain a resolution of the Identity in $\mathcal{H}$, we also need the states $|\pm \overset{\bullet}{r}\rangle$, which are obtained by acting with the operator $\frac{i}{\sqrt{2}}(\hat{V}_d^\dagger - \hat{V}_d) = \frac{i}{\sqrt{2}}\frac{\partial}{\partial x}$ on the coherent states and restricting to $x=0$.

D. Resolution of the identity: polar coordinates

In the previous section a resolution of the identity has been constructed in terms of coherent states at $y = 0$, and the derivatives of the coherent states with respecto to y at $y = 0$, denoted as $|\overset{\circ}{x}{}'\rangle$ and $|\overset{\bullet}{x}{}'\rangle$, respectively.

Using the fact that Helmholtz equation is separable also in polar coordinates, and that any regular solution (i.e. without Y_n terms) can be obtained from its values at a circle centered at the origin, we can try to obtain a resolution of the identity involving only coherent states on a circle.

In fact, if $\psi(r, \theta)$ is a solution regular at the origin, then

$$\psi(r, \theta) = \int_{-\pi}^{\pi} d\theta' \Delta_{r_0}(\theta - \theta', r) \overset{\circ}{\psi}(\theta') \,, \tag{38}$$

where $\overset{\circ}{\psi}(\theta) = \psi(r_0, \theta)$ and $r_0 > 0$ is fixed. The Helmholtz propagator in polar coordinates is given by:

$$\Delta_{r_0}(\theta, r) = \sum_{n\in\mathbb{Z}} \frac{J_n(kr)}{J_n(kr_0)} e^{in\theta} \,. \tag{39}$$

Choosing $0 < r_0 < \frac{z_{0,1}}{2}$, where $z_{n,j}$ indicates the j-th zero of $J_n(x)$, it is guaranteed that all Fourier coefficients of Δ_{r_0} are finite.

Using the asymptotic expression of Bessel functions (14), we have that $\frac{J_n(kr)}{J_n(kr_0)} \approx \left(\frac{r}{r_0}\right)^n$ for $n >> 1$. Then the Fourier series (39) only converges absolutely when $r < r_0$. For $r = r_0$ the propagator equals a Dirac comb and Eq. (38) is trivial. For $r > r_0$ the propagator should be understood in the sense of distributions.

The situation is similar to that of standard coherent states, when any function in phase space is expressed in terms of coherent states on a circle (see [32] and references therein). There, the problem is solved restricting the Hilbert space to *physical states*, those with finite values of any moment, since for these states the coefficients in the Fock basis decay fast enough to compensate the divergences appearing in the integration over the circle.

In the present case there is no need to restrict the Hilbert space $\mathcal{H}_{\rm osc}$, since regular solutions of the Helmholtz equation have Fourier coefficients decaying fast enough to compensate the divergence of the propagator, and thus (38) is well defined for all values of r.

Following again [17], let us introduce the following set of states:

$$| \overset{\circ}{\theta} \rangle = |r_0 e^{i\theta}\rangle \,. \tag{40}$$

The overlap of two such states is:

$$\langle \overset{\circ}{\theta} \mid \overset{\circ}{\theta}{}' \rangle = \sum_{n=-\infty}^{\infty} J_n(2r_0)^2 e^{in(\theta'-\theta)} = J_0(4r_0 \sin\left(\frac{\theta'-\theta}{2}\right)) \,. \tag{41}$$

Then a resolution of the identity in polar coordinates is given as:

$$\hat{A}_p = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\theta' K(\theta-\theta') | \overset{\circ}{\theta} \rangle \langle \overset{\circ}{\theta}{}' | = \hat{I}_{\mathcal{H}} \,, \tag{42}$$

where $K(\theta)$ is the inverse function under convolution of the ovelap (41), given by its Fourier series:

$$K(\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{J_n(2r_0)^2} e^{in\theta} \,. \tag{43}$$

Note that with the choice of r_0 as before, all Fourier coefficients of K are finite. However, they diverge very fast as n increases. As with the propagator (39), taking into account that $K(\theta)$ will be always integrated multiplied by two regular solutions of Helmholtz equation, the eq. (42) is well-defined.

The fact that $K(\theta)$ is the inverse under convolution of the overlap (41) guarantees that:

$$\langle \overset{\circ}{\theta} | \hat{A}_p | \overset{\circ}{\theta}{}' \rangle = J_0(4r_0 \sin\left(\frac{\theta'-\theta}{2}\right)) \tag{44}$$

proving that $\hat{A}_p$ is in fact the identity since the coherent states in the circle constitute a generating system.

V. CONCLUSION

In this paper we have reviewed the construction of Perelemov coherent states for an infinite array of homogeneous and equally spaced waveguide arrays. Although the symmetry group is the Euclidean E(2) group, the coherent states are labeled only for a complex number z, or the pair (x, y) with $z = x + iy$, that represents the momentum space of the E(2) group (see [17]). The angle variable $\varphi \in [0, 2\pi)$ of the Euclidean group, representing configuration space, is absent, the reason being that we have chosen as *fiducial state* the most *symmetrical* one, i.e. the state $|0\rangle$ satisfying $\hat{n}|0\rangle = 0$, and $\hat{n}$ is precisely the generator of the compact subgroup of rotations[3] in $E(2)$.

In Sec. IV C we have provided a resolution of the identity in Cartesian coordinates involving just coherent states on half the coherent state label space (at the price of including also the derivatives), i.e. in terms of the initial data, Eq. (32), for oscillatory solutions of the Helmholtz equation (28), or (29). This construction is the same as the one provided in [17] (eq. (53)) for the momentum representation of a particle on the sphere (there on S^3, but this applies to any Euclidean group $E(n)$). In fact, the states constructed there satisfy the same overlap (cf. eq. (52) of [17]) as in (33), except for a shift in the index of the kernels due to the dimensionality of the sphere.

Due to this analogy, it would be interesting to construct the generalized Fourier transform (see [17]) connecting functions on the label space with a representation in the circle (the variable of rotations in $E(2)$), and obtain a realization in terms of functions on the circle (and the corresponding coherent states) for solutions on the infinite waveguide array, providing an interpretation of it. This is work in progress [18].

With respect to the other resolution of the identity in terms of coherent states on the circle (again, this is a circle in the label space, not the circle associated with rotations in E(2)), it would be interesting to further study its properties, and try to generalize to other curves in the label space.

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[3] The reader should not confuse this fact with the expression of $\hat{n}$ in the differential realization $\hat{n}_d$ acting on functions on the label space variable $\alpha = re^{i\theta} \in \mathbb{C}$. Due to the semidirect action of rotations on the Euclidean plane, $\hat{n}$ acts both on the compact variable φ of E(2) and as rotations on the plane, (25), and this last one is the only surviving the condition $\hat{n}|0\rangle = 0$.

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