STRATIFICATION AND DOMINATION IN GRAPHS

by

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Abstract

In a recent manuscript (Stratification and domination in graphs. Discrete Math. 272 (2003), 171–185) a new mathematical framework for studying domination is presented. It is shown that the domination number and many domination related parameters can be interpreted as restricted 2-stratifications or 2-colorings. This framework places the domination number in a new perspective and suggests many other parameters of a graph which are related in some way to the domination number. In this thesis, we continue this study of domination and stratification in graphs.

Let F be a 2-stratified graph with one fixed blue vertex v specified. We say that F is rooted at the blue vertex v. An F-coloring of a graph G is a red-blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of F (not necessarily induced in G) rooted at v. The F-domination number $\gamma_F(G)$ of G is the minimum number of red vertices of G in an F-coloring of G.

Chapter 1 is an introduction to the chapters that follow. In Chapter 2, we investigate the X-domination number of prisms when X is a 2-stratified 4-cycle rooted at a blue vertex where a prism is the cartesian product $C_n \times K_2$, $n \geq 3$, of a cycle C_n and a K_2.

In Chapter 3 we investigate the F-domination number when (i) F is a 2-stratified path P_3 on three vertices rooted at a blue vertex which is an end-vertex of the P_3 and is adjacent to a blue vertex and with the remaining vertex colored red. In particular, we show that for a tree of diameter at least three this parameter is at most two-thirds its order and we characterize the trees attaining this bound. (ii)

We also investigate the F-domination number when F is a 2-stratified K_3 rooted at a blue vertex and with exactly one red vertex. We show that if G is a connected graph of order n in which every edge is in a triangle, then for n sufficiently large this parameter is at most $(n - \sqrt{n})/2$ and this bound is sharp.

In Chapter 4, we further investigate the F-domination number when F is a 2-stratified path P_3 on three vertices rooted at a blue vertex which is an end-vertex of the P_3 and is adjacent to a blue vertex with the remaining vertex colored red. We show that for a connected graph of order n with minimum degree at least two this parameter is bounded above by $(n-1)/2$ with the exception of five graphs (one each of orders four, five and six and two of order eight). For $n \geq 9$, we characterize those graphs that achieve the upper bound of $(n-1)/2$.

In Chapter 5, we define an $\mathcal{F}$-coloring of a graph to be a red-blue coloring of the vertices such that every blue vertex is adjacent to a blue vertex and to a red vertex, with the red vertex itself adjacent to some other red vertex. The $\mathcal{F}$-domination number $\gamma_{\mathcal{F}}(G)$ of a graph G is the minimum number of red vertices of G in an $\mathcal{F}$-coloring of G. Let G be a connected graph of order $n \geq 4$ with minimum degree at least 2. We prove that (i) if G has maximum degree Δ where $\Delta \leq n-2$, then $\gamma_{\mathcal{F}}(G) \leq n - \Delta + 1$, and (ii) if $G \neq C_7$, then $\gamma_{\mathcal{F}}(G) \leq 2n/3$.

In Chapter 6, we study total restrained domination in graphs. A set S of vertices in a graph $G = (V, E)$ is a total restrained dominating set of G if every vertex is adjacent to a vertex in S and every vertex of $V \setminus S$ is adjacent to a vertex in $V \setminus S$. The minimum cardinality of a total restrained dominating set of G is the total restrained domination number of G, denoted by $\gamma_{\mathrm{tr}}(G)$. Let G be a connected graph with minimum degree at least 2. We prove that (i) if G has order $n \geq 4$ with maximum degree Δ where $\Delta \leq n-2$, then $\gamma_{\mathrm{tr}}(G) \leq n - \frac{\Delta}{2} - 1$, and (ii) if G is a bipartite graph of order $n \geq 5$ with maximum degree Δ where $3 \leq \Delta \leq n-2$, then $\gamma_{\mathrm{tr}}(G) \leq n - \frac{2}{3}\Delta - \frac{2}{9}\sqrt{3\Delta - 8} - \frac{7}{9}$. Both bounds are shown to be sharp.

Dedicated

To my wife, Vanessa.

Preface

The work described in this thesis was carried out under the supervision and direction of Professor Michael A. Henning, School of Mathematical Sciences, University of KwaZulu-Natal, Pietermaritzburg campus from January 2002 to April 2006.

The thesis represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

Jacob Elgin Maritz

Professor Michael A. Henning (Supervisor)

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I wish to thank ...

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Chapter 1

INTRODUCTION

In the first section of this chapter we present the notation and give some basic definitions that will be used throughout this thesis. In Section 1.2, we give some background to the concepts of domination and stratification of a graph. We then give a formal definition of the concepts domination and stratification of a graph and also state some of the many results that have already been established in this research field. Finally, in Section 1.4, we give an overview of the remainder of this thesis.

1.1 Basic Definitions

A *graph* G consists of a finite nonempty set of *vertices* (the singular is *vertex*) and a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$ (or V if no confusion is likely), while the *edge set* of G is denoted by $E(G)$ (or E). The number of vertices in $V(G)$ is denoted by $n(G)$ which is also known as the *order* of the graph G, while the number of edges in $E(G)$ is denoted by $m(G)$. A graph G is *trivial* if $n(G) = 1$ and *non-trivial* if

$n(G) \geq 2$. For a graph G, if $n(G) = n$ and $m(G) = m$, then G is called a (n, m)-graph. Unless otherwise specified, the symbols n and m (or $n(G)$ and $m(G)$) will be reserved exclusively for the order and number of edges, respectively, of a graph G. We write $G = (V, E)$ to mean that the graph G has vertex set V and edge set E.

The edge $e = uv$ is said to *join* the vertices u and v. If $e = uv$ is an edge of G, then u and v are *adjacent vertices*, while u and e are *incident* as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*.

A graph G is called *complete* if every two vertices of G are adjacent. We denote a complete graph of order n by K_n. The *degree* of a vertex v in G is the number of edges incident with v and is denoted $\deg_G v$ (or $\deg v$ if there is no confusion). The *minimum degree* (resp., *maximum degree*) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). A vertex of degree k we call a *degree-k vertex*. If there is a vertex $v \in V(G)$ such that $\deg v = 0$, then v is called an *isolated vertex*, if $\deg v = 1$, then v is called an *end-vertex* and if $\deg v \geq 2$, then v is called an *internal vertex* of G. A vertex is called *odd* or *even* depending on whether its degree is odd or even.

A *subgraph* H of a graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *proper subgraph* of G is a subgraph of G that is different from G. A subgraph H is called a *spanning subgraph* of G if $V(H) = V(G)$. For a set $S \subseteq V(G)$, the *subgraph induced by* S is denoted by $G[S]$ and the subgraph obtained from G by deleting the vertices in S (and all edges incident with vertices in S) is denoted by $G - S$. For a vertex v (resp. an edge e) of G we denote by $G - v$ (resp. $G - e$) the graph obtained from G by deleting the vertex v (resp. the edge e).

Let u and v be (not necessarily distinct) vertices of a graph G. A *u-v walk* of G is a finite, alternating sequence $u = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = v$ of vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, n$. The number n (the number of occurrences of edges) is called the *length* of the walk. A *trivial walk* contains no edges. Often only the vertices of a walk are indicated since the edges present are then evident. A u-v walk is *closed* or *open* depending on whether $u = v$ or $u \neq v$. A *u-v trail* is a u-v walk in which no edge is repeated, while a *u-v path* is a u-v walk in which no vertex is repeated. A nontrivial closed trail of a graph G is referred to as a *circuit* of G, and a circuit $v_1, v_2, \ldots, v_n, v_1$ $(n \geq 3)$ whose n vertices are distinct is called a *cycle*. A graph of order n that is a path (or a cycle) is denoted by P_n (or C_n), respectively. Therefore, $P_n : v_1, v_2, \ldots, v_n$ indicates a path of length $n-1$ on the vertices $v_1, v_2, \ldots, v_n$, while C_n indicates a cycle of length n on the same vertices.

The *distance* between u and v, denoted by $d_G(u, v)$ (or $d(u, v)$ if there is no confusion) is the length of a shortest u-v path in G if such a path exist. A set S of vertices in a graph G is called a *packing* in G if the vertices in S are pairwise at distance at least 3 apart in G, i.e., if $u, v \in S$, then $d(u, v) \geq 3$.

Let u and v be distinct vertices of G. We say that u is *connected* to v if there exist a u-v path in G. The relation 'is connected to' is an equivalence relation on the vertex set of every graph G. The graph G is itself *connected* if u is connected to v for every pair u, v of vertices of G. A graph that is not connected is called *disconnected*. The trivial graph, then, is connected. A subgraph H of a graph G is a *component* of G if H is a maximal connected subgraph of G. An edge e of G is called a *bridge* if $G - e$ is disconnected while v is called a *cut-vertex* if $G - v$ is disconnected.

For a graph $G = (V, E)$, let $v \in V$ and let $S \subseteq V$. The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. The *open neighborhood* of S is defined by $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S by $N[S] = N(S) \cup S$. If $v \in S$, then a vertex $w \in V$ is a *private neighbor of* v (with respect to S) if $N[w] \cap S = \{v\}$. The *private neighbor set of v with respect to S*, denoted $\mathrm{pn}(v, S)$, is the set of all private neighbors of v. The *external private* *neighbor set of v with respect to S* is the set $\mathrm{epn}(v, S) = \mathrm{pn}(v, S) \cap (V \setminus S)$.

A graph G is *r-partite*, $r \geq 1$, if it is possible to partition V into r subsets $V_1, V_2, \ldots, V_r$ (called *partite sets*) such that every element of E joins a vertex of V_i to a vertex of $V_j, i \neq j$. If G is a 1-partite graph of order n, then $G \cong \overline{K}_n$. For $r = 2$, such graphs are called *bipartite graphs*, and where the specification of r is of no significance, an r-partite graph is also referred to as a *multipartite graph*. A *complete r-partite graph* G is an r-partite graph with partite sets $V_1, V_2, \ldots, V_r$ having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. If $|V_i| = n_i$, then this graph is denoted by $K(n_1, n_2, \ldots, n_r)$. (The order of the numbers $n_1, n_2, \ldots, n_r$ is not important.) A complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$, is denoted by $K(m, n)$ or $K_{m,n}$.

A *tree* is a connected graph which has no cycles. A *leaf* of a tree T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. A support vertex adjacent to two or more leaves is called a *strong support vertex*. A *star* is the tree $K_{1,n-1}$ of order $n \geq 2$. A *subdivided star* is a star where each edge is subdivided exactly once. A tree is a *doublestar* if it contains exactly two vertices that are not leaves; if one of these vertices is adjacent to r leaves and the other to s leaves, then we denote the double star by $S_{r,s}$. We call a path of maximum length in a tree a *diametrical path* in the tree.

A *prism* is the cartesian product $G = C_n \times K_2$, $n \geq 3$, of a cycle C_n and a K_2. Our prism G consists of two n-cycles $v_1, v_2, \ldots, v_n, v_1$ and $u_1, u_2, \ldots, u_n, u_1$ with $u_i v_i$ an edge for all $i = 1, 2, \ldots, n$.

We define a vertex as *small* if it has degree 2, and *large* if it has degree more than 2. We define a *ray* as a path (not necessarily induced) of length 3 the two internal vertices of which are small vertices.

Let G be a graph with minimum degree at least two, and let $\mathcal{L}$ be the set of all large vertices of G. Suppose $|\mathcal{L}| \geq 1$ and let C be any component of $G - \mathcal{L}$; it is a path. If C has only one vertex, or has at least two vertices but the two ends of C are adjacent in G to different large vertices, then we say that C is a 2-*path*. Otherwise we say that C is a 2-*handle*.

Other definitions will be given where they are needed. For notation and graph theory terminology that have not been defined here we in general follow [30].

1.2 Background

The earliest ideas of dominating sets date back to the origins of the game of chess in India over 400 years ago, in which one wishes to cover or dominate various opposing pieces or various squares of the chessboard. In 1862 de Jaenisch [16] posed the problem of finding the minimum number of queens that can be placed on a chessboard so that each square of the chessboard is attacked or dominated by at least one of the queens. A graph may be formed from an $n \times n$ chessboard by taking the squares as the vertices and two vertices are adjacent if a chess piece situated on one square covers the other.

The classical problems of covering chessboards with the minimum number of chess pieces rekindled interest in dominating concepts. Ultimately the theory of domination was formalized by Berge [2] in 1958 and Ore [46] in 1962. Ore coined the term 'domination number', but Berge was the first to define it as a parameter (coefficient of external stability).

The notion of domination is a standard one in coding theory. If one defines a graph whose vertices are the n-dimensional vectors with coordinates from $(1, \ldots, p)$ and two vertices are adjacent if they differ in one coordinate, then sets of vectors which are (n, p)-covering sets, single error correcting codes, or perfect covering sets are all dominating sets of a graph with certain additional properties. See for example

Kalfleisch, Stanton, and Horton [43].

As a further example, to illustrate the idea of dominating sets, consider a graph G representing a city road system where the vertices correspond to street intersections (see Figure 1.1). Two vertices are adjacent if and only if they correspond to adjacent intersections. We wish to place law officers at various intersections, so that at every intersection, there is a law officer located no more than one block away. This is equivalent to locating a dominating set in the graph G. One possible dominating set is shown in Figure 1.1, where the vertices in the dominating set of G are darkened. Actually, only four law officers are required to dominate G.

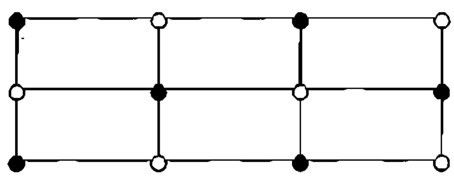


Figure 1.1: A graph G representing a road system with a dominating set.

In this thesis we continue the study of stratification and domination in graphs started by Chartrand, Haynes, Henning and Zhang [8]. A graph G whose vertex set has been partitioned is called a *stratified graph*. If the partition is $V(G) = \{V_1, V_2, \ldots, V_k\}$, then G is a k-stratified graph. The sets $V_1, V_2, \ldots, V_k$ are called the *strata* or sometimes the *color classes* of G. If $k = 2$, we ordinarily color the vertices of V_1 red and the vertices of V_2 blue. In what follows, we will restrict our attention to 2-stratified graphs.

In [47], Rashidi studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [5, 6, 11].

In [8] a new mathematical framework for studying domination is presented. It is shown that the domination number and many domination related parameters can be interpreted as restricted 2-stratifications or 2-colorings, with the red vertices forming the dominating set. This framework places the domination number in a new perspective and suggests many other parameters of a graph which are related in some way to the domination number. The book by Chartrand and Zang [13] includes a section on domination and stratification.

1.3 Known Results

In this section, we state some of the many known results in the theory on domination and 2-stratification of a graph. We begin with a formal definition of a dominating set and the domination number of a graph.

A set $S \subseteq V(G)$ of a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a *γ-set* of G. A dominating set S in a graph is a minimal dominating set if and only if for each $v \in S$, we have $\mathrm{pn}(v, S) \neq \emptyset$.

Early work on the topic of domination focussed on properties of minimal dominating sets. We give two classical results of Ore [46].

Theorem 1.1 (Ore [46]) *Let D be a dominating set of a graph G. Then D is a minimal dominating set of G if and only if each $v \in D$ has at least one of the following two properties.*
P_1: There exists a vertex $w \in V(G) \setminus D$ such that $N(w) \cap D = \{v\}$;
P_2: The vertex v is adjacent to no other vertex of D.

Theorem 1.2 (Ore [46]) *If G is a graph with no isolated vertex and D is a minimal dominating set of G, then $V(G) \setminus D$ is a dominating set of G.*

Bollobás and Cockayne [3] established the following property of minimal (or minimum) dominating sets in graphs.

Theorem 1.3 (Bollobás and Cockayne [3]) *If G is a graph with no isolated vertex, then there exists a minimum dominating set D of vertices of G in which every vertex has property P_1.*

We remark that the result of Theorem 1.3 can be formulated in terms of a set and the external private neighborhood of its members.

Theorem 1.3. *If G is a graph with no isolated vertex, then there exists a $\gamma(G)$-set S such that $|\mathrm{epn}(v, S)| \geq 1$ for every $v \in S$.*

As an immediate consequence of Theorems 1.2 and 1.3, we have the following upper bound on the domination number of a graph due to Ore [46].

Theorem 1.4 (Ore [46]) *If G is a graph of order n with no isolated vertex, then $\gamma(G) \leq n/2$.*

Let $G = (V, E)$ be a graph. A *total dominating set* (abbreviated, TDS) in G is a subset $S \subseteq V$ such that every vertex of G is adjacent to a vertex of S. Every graph G without isolated vertices has a total dominating set since $S = V(G)$ is such a set. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [14] and further studied, for example, in [1, 15, 21, 22, 34, 35, 49].

The following result is due to Cockayne, Dawes, and Hedetniemi [14].

Theorem 1.5 (Cockayne, Dawes, and Hedetniemi [14]) *If G is a graph of order $n \geq 3$ with no isolated vertices, then $\gamma_t(G) \leq 2n/3$.*

Let $G = (V, E)$ be a graph. A *restrained dominating set* (abbreviated, RDS) in G is a subset $S \subseteq V$ such that every vertex not in S is adjacent to a vertex in S and to a vertex in $V \setminus S$. The *restrained domination number* $\gamma_r(G)$ of G is the minimum cardinality of a RDS. Restrained domination was introduced by Telle and Proskurowski [48], albeit indirectly, as vertex partitioning problem and further studied, for example, in [18, 19, 20, 29, 36].

Let $G = (V, E)$ be a graph. If a set S of vertices in G is simultaneously a TDS and a RDS, then S is called a *total restrained dominating set* (abbreviated, TRDS). Thus if S is a TRDS of G, then every vertex of G is adjacent to a vertex in S and every vertex of $V \setminus S$ is adjacent to a vertex in $V \setminus S$. The minimum cardinality of a TRDS of G is the *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$. The concept of total restrained domination in graphs was also introduced in [48], albeit indirectly, as a vertex partitioning problem and has been studied, for example, in [28, 17, 50].

For $k \geq 1$, a *k-dominating set* in G is a subset $S \subseteq V$ such that every vertex not in S is adjacent to at least k vertices in S. The *k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a k-dominating set of G. In particular, the parameter $\gamma_2(G)$ is the minimum cardinality of a *2-dominating set.*

There are many other domination related parameters that are beyond the scope of this thesis. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The book by Chartrand and Lesniak [12] includes a chapter on domination. For a more thorough study of domination in graphs, see Haynes, Hedetniemi, and Slater [30, 31].

Next, we define the concepts associated with a 2-stratification or 2-coloring of a graph. Let F be a 2-stratified graph rooted at some blue vertex v and containing at least one red vertex. We define an *F-coloring* of a graph G to be a red-blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of F rooted at v. The *F-domination number* $\gamma_F(G)$ of G is the minimum number of red vertices of G in an F-coloring of G. We call an F-coloring of G that colors $\gamma_F(G)$ vertices red a *γ_F-coloring* of G. The set of red vertices in a γ_F-coloring is called a *γ_F-set*. If G has order n and G has no copy of F, then certainly $\gamma_F(G) = n$.

Let F be a K_2 rooted at a blue vertex v that is adjacent to a red vertex. An F-coloring of G is then a red-blue coloring of the vertices of G with the property that every blue vertex is adjacent to a red vertex. Notice that the red vertices of G correspond to a dominating set of G. Hence, $\gamma(G) \leq \gamma_F(G)$. On the other hand, given a γ-set of G we color the vertices in this set red and all remaining vertices blue. This red-blue coloring of the vertices of G has the property that every blue vertex is adjacent to a red vertex and is therefore an F-coloring of G (where F is a 2-stratified K_2). Thus, $\gamma_F(G) \leq \gamma(G)$. Consequently, if F is a 2-stratified K_2, then $\gamma_F(G) = \gamma(G)$.

Thus domination can be interpreted as a restricted 2-stratification or 2-coloring, with the red vertices forming the dominating set. Clearly, this F-coloring is the only well-defined one for connected graphs F with order 2.

Let F be a 2-stratified P_3 rooted at a blue vertex v. The five possible choices for the graph F are shown in Figure 1.2. (The red vertices in Figure 1.2 are darkened.)

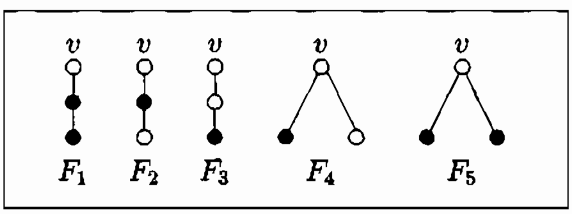


Figure 1.2: The five 2-stratified graphs P_3.

An example of a γ_F-coloring of $G = P_4 \circ K_1$ (the darkened vertices are the red vertices) is illustrated in Figure 1.3 where $F \in \{F_1, F_2, \ldots, F_5\}$.

$\gamma_{F_1}(G) = 4$ $\gamma_{F_2}(G) = 4$ $\gamma_{F_3}(G) = \gamma_{F_4}(G) = 4$ $\gamma_{F_5}(G) = 6$

Figure 1.3: A γ_F-coloring of a graph G.

The following result is established in [8].

Theorem 1.6 ([8]) *If G is a connected graph of order at least 3, then for $i \in \{1, 2, 4, 5\}$, the parameter $\gamma_{F_i}(G)$ is given by the following table:*

i	1	2	4	5
$\gamma_{F_i}(G) =$	$\gamma_t(G)$	$\gamma(G)$	$\gamma_r(G)$	$\gamma_2(G)$

.

Table 1.4: The parameter $\gamma_{F_i}(G)$.

Since the parameter $\gamma_{F_1}(G)$ is defined for all graphs G, while the parameter $\gamma_t(G)$ is defined only for graphs without isolated vertices, Theorem 1.6 suggests that the definition of $\gamma_{F_1}(G)$ may be preferable to that of $\gamma_t(G)$.

The parameter $\gamma_{F_3}(G)$ appears to be new and is further investigated in Chapter 3.

1.4 Overview

In Chapter 2, our aim is to determine the X-domination number of *a prism* when X is a 2-stratified cycle C_4.

In Chapter 3, we investigate the F-domination number when F is a 2-stratified path P_3 on three vertices rooted at a blue vertex which is an end-vertex of the P_3 and is adjacent to a blue vertex and with the remaining vertex colored red. (See Figure 1.2.) We also investigate the F-domination number when F is a 2-stratified K_3 rooted at a blue vertex and with exactly one red vertex.

In Chapter 4, we continue the study of the F_3-domination number of a graph. We have two immediate aims: Firstly to establish an upper bound on the F_3-domination number of a connected graph with minimum degree at least two in terms of the order of the graph and to characterize those graphs achieving equality in this bound. Secondly, to characterize connected graphs of sufficiently large order with maximum possible F_3-domination number.

In Chapters 5 and 6, we focus on two variations on the domination theme that are well studied in graph theory called total domination and restrained domination.

Chapter 2

STRATIFICATION AND DOMINATION IN PRISMS

2.1 Introduction

Recall, a *prism* is the cartesian product $G = C_n \times K_2$, $n \geq 3$, of a cycle C_n and a K_2. Our prism G consists of two n-cycles $v_1, v_2, \ldots, v_n, v_1$ and $u_1, u_2, \ldots, u_n, u_1$ with $u_i v_i$ an edge for all $i = 1, 2, \ldots, n$. In this chapter our aim is to determine the X-domination number of a prism when X is a 2-stratified cycle C_4. Recall a vertex $w \in V$ is a *private neighbor of v* (with respect to S) if $N[w] \cap S = \{v\}$; and the *private neighbor set of v with respect to S*, denoted $\mathrm{pn}(v, S)$, is the set of all private neighbors of v. Results on domination in prisms can be found, for example, in [4, 7].

2.2 A 2-stratified C_4

Let X be a 2-stratified C_4 rooted at a blue vertex v. The five possible choices for the graph X are shown in Figure 2.1. (The red vertices in Figure 2.1 are darkened.)

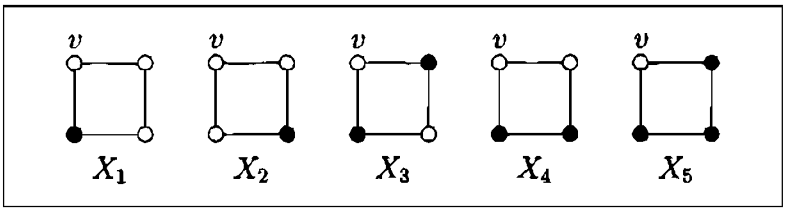


Figure 2.1:

2.3 Stratification in Prisms

The total domination number of grid graphs (i.e. a graph that is the cartesian product of two paths) is given in [26]. We state two important results from [26].

Proposition 2.1 (Gravier [26]) *For any $n \geq 4$,*

$$\gamma_t(P_4 \times P_n) = \begin{cases} \left\lfloor \dfrac{6n+8}{5} \right\rfloor & \text{if } n \equiv 1, 2, 4 \pmod 5 \\ \\ \left\lfloor \dfrac{6n+8}{5} \right\rfloor + 1 & \text{if } n \equiv 10, 3 \pmod 5). \end{cases}$$

Theorem 2.2 (Gravier [26]) *If k and n are two integers greater than 16, then*

$$\frac{3kn + 2(k+n)}{12} \leq \gamma_t(P_k \times P_n) \leq \left\lfloor \frac{(k+2)(n+2)}{4} \right\rfloor - 4.$$

In this chapter, we focus on prisms and we investigate the possible 2-stratifications of prisms. In all but one of the five possible choices for a 2-stratified C_4 (see Figure 2.1), the red vertices form a dominating set in the graph. Hence we have the following observation.

Observation 2.3 *For $i \in \{1, 3, 4, 5\}$ and for any graph G, $\gamma(G) \leq \gamma_{X_i}(G)$.*

Theorem 2.4 *For $n \geq 3$, let $G = C_n \times K_2$. Then for $i \in \{1,2,3,4,5\}$, the parameter $\gamma_{X_i}(G)$ is given by the following table:*

i	$\gamma_{X_i}(G)$	$\gamma_{X_i}(G)$
1	$\lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$	$\gamma(G)$
2	$\begin{cases} 2 & \text{if } n = 4 \\ 2n & \text{otherwise} \end{cases}$	
3	n	$\gamma_2(G)$
4	$2\left\lceil \frac{n}{3} \right\rceil$	$\begin{cases} \gamma_t(G) + 1 & \text{if } n \equiv 1 \pmod 6 \\ \gamma_t(G) & \text{otherwise.} \end{cases}$
5	$\left\lceil \frac{4n}{3} \right\rceil$	$\begin{cases} \gamma_{\times 2}^t(G) - 1 & \text{if } n \equiv 2 \pmod 6 \\ \gamma_{\times 2}^t(G) & \text{otherwise.} \end{cases}$

Table 2.2: The parameter $\gamma_{X_i}(G)$.

where $\gamma_2(G)$ denotes the 2-domination number, $\gamma_t(G)$ denotes the total domination number, and $\gamma_{\times 2}^t(G)$ denotes the double total domination number (which we define in Subsection 2.3.5).

Given a graph $G = (V, E)$ and a subset $S \subseteq V$, we call the coloring of G that colors the vertices of S red and the vertices of $V \setminus S$ blue the *red-blue coloring* *associated with S.*

Throughout Section 2.3, we let $G = C_n \times K_2$. The proof of Theorem 2.4 follows from Propositions 2.6, 2.9, 2.14, 2.15, 2.17, 2.18, 2.21 and 2.22. In some of the proofs that follow we will need the following lemma.

Lemma 2.5 *Let S be a set of vertices in G. If G can be partitioned into n/s subgraphs $H = P_s \times K_2$, each containing k vertices that belongs to S, then $|S| = nk/s$.*

2.3.1 X_1-stratification and the domination number

Proposition 2.6 *For $n \geq 3$, $\gamma_{X_1}(G) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

Proof. The desired result follows from Claims 2.7 and 2.8.

Claim 2.7 $\gamma_{X_1}(G) \geq \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Proof. In any X_1-coloring of a graph, every vertex colored blue is rooted at a copy of X_1. Hence as an immediate consequence of the definition of an X_1-coloring, any X_1-coloring of G colors at least one vertex from every 4-cycle red.

Suppose n is odd. Consider any given X_1-coloring of G. Renaming vertices if necessary, we may assume v_1 is colored red. Since $G - \{u_1, v_1\}$ contains $(n-1)/2$ disjoint 4-cycles, each of which contains at least one red vertex, our given X_1-coloring contains at least $(n+1)/2$ red vertices. Thus, $\gamma_{X_1}(G) \geq (n+1)/2$.

Suppose n is even. Then, G has $n/2$ disjoint 4-cycles, and therefore has at least $n/2$ red vertices. Thus, $\gamma_{X_1}(G) \geq n/2$. Further, suppose $n \equiv 2 \pmod 4$ and that exactly $n/2$ vertices are colored red. Then, every 4-cycle in G contains exactly one red vertex. In particular, v_1 is the only red vertex in the 4-cycle v_1, u_1, u_2, v_2, v_1. Since u_2 is rooted in a copy of X_1, the vertex u_3 is colored red, and so u_3 is the only red vertex in the 4-cycle u_3, v_3, v_4, u_4, u_3. Since v_4 is rooted in a copy of X_1, the vertex v_5 is colored red, and so v_5 is the only red vertex in the 4-cycle v_5, u_5, u_6, v_6, v_5. Proceeding in this manner, v_{n-1} is the only red vertex in the 4-cycle $v_{n-1}, u_{n-1}, u_n, v_n, v_{n-1}$. But then u_n is not rooted at a copy of X_1, a contradiction. Hence, if $n \equiv 2 \pmod 4$, then at least $n/2 + 1$ vertices are colored red. □

Claim 2.8 $\gamma_{X_1}(G) \leq \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Proof. If $n = 3$, then $\{v_1, u_3\}$ is an X_1-coloring of G, and the desired upper bound follows. Hence we may assume $n \geq 4$. Suppose first that $n \not\equiv 2 \pmod 4$. Let

$$S = \bigcup_{i=0}^{\lfloor n/4 \rfloor - 1} \{v_{4i+1}, u_{4i+3}\}.$$

If $n \equiv 0 \pmod 4$, let $D = S$. If $n \equiv 1 \pmod 4$, let $D = S \cup \{v_n\}$. If $n \equiv 3 \pmod 4$, let $D = S \cup \{u_n, v_{n-2}\}$. In all cases, coloring the vertices in D red and coloring all remaining vertices blue, produces an X_1-coloring of G, and so $\gamma_{X_1}(G) \leq |D| = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Suppose, secondly, that $n \equiv 2 \pmod 4$. If $n = 6$, let $S = \emptyset$, while if $n \geq 10$, let

$$S = \bigcup_{i=0}^{\lfloor n/4 \rfloor - 2} \{v_{4i+1}, u_{4i+3}\}.$$

Let $R = \{v_{n-5}, v_{n-4}, u_{n-2}, u_{n-1}\}$. Coloring the vertices in $R \cup S$ red and coloring all remaining vertices blue, produces an X_1-coloring of G, and so $\gamma_{X_1}(G) \leq |R| + |S| = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. □

Proposition 2.9 *For* $n \geq 3$, $\gamma(G) = \gamma_{X_1}(G)$.

Proof. By Observation 2.3, $\gamma(G) \leq \gamma_{X_1}(G)$. Hence it suffices for us to show that $\gamma(G) \geq \gamma_{X_1}(G)$. Among all $\gamma(G)$-sets, let S be chosen so that

(1) $G[S]$ has minimum size.

(2) Subject to (1), the red-blue coloring associated with S contains the maximum number of blue vertices that are rooted at a copy of X_1.

We proceed further by proving three claims.

Claim 2.10 $|N(v) \cap S| \leq 1$ *for all* $v \in S$.

Proof. Suppose there exists a vertex $v_i \in S$ such that $|N(v_i) \cap S| \geq 2$. If $u_i \in S$, then by symmetry we may assume that $v_{i+1} \in S$. But then $(S - \{u_i, v_i\}) \cup \{u_{i-1}\}$ is a dominating set of G of cardinality less than $\gamma(G)$, which is impossible. Hence, $u_i \notin S$; that is, $\{v_{i-1}, v_{i+1}\} \subset S$. Then, $u_i \in \text{pn}(v_i, S)$, and so $u_{i-1} \notin S$ and $u_{i+1} \notin S$. Hence, $(S - \{v_i\}) \cup \{u_i\}$ is a $\gamma(G)$-set that induces a subgraph of G with fewer edges than $G[S]$, contradicting our choice of S. □

Claim 2.11 $|\{u_i, v_i\} \cap S| \leq 1$ *for* $i = 1, 2, \ldots, n$.

Proof. Suppose that $\{u_i, v_i\} \subseteq S$ for some i, $1 \leq i \leq n$. By Claim 2.10, $S \cap \{u_{i-1}, v_{i-1}, u_{i+1}, v_{i+1}\} = \emptyset$. By the minimality of S, $\text{pn}(v_i, S) \subseteq \{v_{i-1}, v_{i+1}\}$ and $\text{pn}(u_i, S) \subseteq \{u_{i-1}, u_{i+1}\}$. Suppose that $v_{i-1} \in \text{pn}(v_i, S)$ and $u_{i+1} \in \text{pn}(u_i, S)$. Then, $S \cap \{u_{i+2}, v_{i-2}\} = \emptyset$. Hence, $(S - \{u_i, v_i\}) \cup \{u_{i+1}, v_{i-1}\}$ is a $\gamma(G)$-set that induces a subgraph of G with fewer edges than $G[S]$, contradicting our choice of S. Similarly we have a contradiction if $v_{i+1} \in \text{pn}(v_i, S)$ and $u_{i-1} \in \text{pn}(u_i, S)$. Hence, by symmetry, we may assume $\text{pn}(v_i, S) = \{v_{i+1}\}$ and $\text{pn}(u_i, S) = \{u_{i+1}\}$. Hence,

$\{u_{i-2}, v_{i-2}\} \subset S$ while $S \cap \{u_{i+2}, v_{i+2}\} = \emptyset$. But then $(S - \{v_i\}) \cup \{v_{i+1}\}$ is a $\gamma(G)$-set that induces a subgraph of G with fewer edges than $G[S]$, contradicting our choice of S. □

Claim 2.12 *The red-blue coloring associated with S is an X_1-coloring of G.*

Proof. Suppose not. Then, renaming vertices if necessary, we may assume that v_1 is a blue vertex that is not rooted at a copy of X_1 in the red-blue coloring associated with S. Since S is a dominating set, at least one neighbor of v_1 is in S. If $v_2 \in S$, then by Claim 2.11, $u_2 \notin S$. Since v_1 is not rooted at a copy of X_1 in the red-blue coloring associated with S, we must have $u_1 \in S$. Similarly, if $v_n \in S$, then $u_1 \in S$. Hence, $u_1 \in S$.

If $S \cap \{v_2, v_n\} = \emptyset$, then $\{u_2, u_n\} \subset S$, and so $|N(u_1) \cap S| = 2$, contradicting Claim 2.10. Hence at least one of v_2 and v_n is in S. By symmetry, we may assume $v_2 \in S$.

By Claim 2.11, $u_2 \notin S$. If $u_n \in S$, then $S - \{u_1\}$ is a dominating set of cardinality less than $\gamma(G)$, which is impossible. Hence, $u_n \notin S$, and so $v_n \in S$ (since v_1 is not rooted at a copy of X_1). If $v_3 \in S$, then $S - \{v_2\}$ is a dominating set, which is impossible. If $u_3 \in S$, then $(S - \{u_1, v_2\}) \cup \{u_2\}$ is a dominating set of cardinality less than $\gamma(G)$, which is impossible. Hence, $S \cap \{u_3, v_3\} = \emptyset$. In order to dominate u_3, we have $u_4 \in S$. Thus by Claim 2.11, $v_4 \notin S$.

By Claim 2.11, $|S \cap \{u_5, v_5\}| \leq 1$. If $u_5 \notin S$ and $v_5 \in S$, then $(S - \{u_1, u_4, v_2\}) \cup \{u_2, v_4\}$ is a dominating set of cardinality less than $\gamma(G)$, which is impossible. If $u_5 \in S$ and $v_5 \notin S$, then $(S - \{u_1, u_4, v_2\}) \cup \{u_2, v_3\}$ is a dominating set of cardinality less than $\gamma(G)$, which is impossible. Hence, $S \cap \{u_5, v_5\} = \emptyset$. Let $S' = (S - \{v_2\}) \cup \{v_3\}$. Then, S' is a $\gamma(G)$-set such that $G[S']$ has the same size as $G[S]$ and the red-blue coloring associated with S' contains one more blue vertex that is rooted at a copy of X_1 than does the red-blue coloring associated with S.

This contradicts our choice of the set S. □

By Claim 2.12, the red-blue coloring associated with S is an X_1-coloring of G. Hence, $\gamma_{X_1}(G) \leq \gamma(G)$, thus completing the proof of Proposition 2.9. □

As a consequence of the proof of Proposition 2.9, we have the following result.

Corollary 2.13 *For $n \geq 3$, there exists a $\gamma(G)$-set whose associated red-blue coloring is a minimum X_1-coloring in G.*

2.3.2 X_2-stratification

Proposition 2.14 *For $n \geq 3$, $\gamma_{X_2}(G) = 2n$, unless $n = 4$ in which case $\gamma_{X_2}(G) = 2$.*

Proof. In any X_2-coloring of a graph, every vertex colored blue is rooted at a copy of X_2. Consider an X_2-coloring of G. Suppose there is a vertex v of G colored blue. Renaming vertices if necessary, we may assume $v = v_1$ and that u_1 and v_2 are colored blue and u_2 is colored red. If $n \neq 4$, then v_2 is not rooted at a copy of X_2, a contradiction. Hence, $n = 4$. But then v_4 is the only other vertex colored red. Hence either every vertex is colored red or $n = 4$ and exactly two vertices (at distance 3 apart) are colored red. □

2.3.3 X_3-stratification and the 2-domination number

Proposition 2.15 *For $n \geq 3$, $\gamma_{X_3}(G) = \gamma_2(G) = n$.*

Proof. Clearly, $\gamma_2(G) \leq \gamma_{X_3}(G)$ for all graphs. For a 2-dominating set, every $P_2 \times K_2$ has at least two red vertices, and so by Lemma 2.5, $\gamma_2(G) \geq n$. For a

γ_{X_3}-set take one red vertex from every rung, alternating sides except possibly for the end, and so $\gamma_{X_3} \leq n$. □

As a consequence of the proof of Proposition 2.15, we have the following result.

Corollary 2.16 *For $n \geq 3$, there exists a $\gamma_2(G)$-set whose associated red-blue coloring is a minimum X_3-coloring in G.*

2.3.4 X_4-stratification and the total domination number

Proposition 2.17 *For $n \geq 3$, $\gamma_{X_4}(G) = 2\left\lceil \frac{n}{3} \right\rceil$.*

Proof. In any X_4-coloring of a graph, every vertex colored blue is rooted at a copy of X_4. Hence as an immediate consequence of the definition of an X_4-coloring, any X_4-coloring of G colors at least two vertices from every subgraph $H = P_3 \times K_2$ of G red. Consider any given X_4-coloring of G.

Suppose $n \equiv 0 \pmod 3$. Then, G contains $n/3$ disjoint copies of H, and so by Lemma 2.5, our given X_4-coloring colors at least $2n/3 = 2\lceil n/3 \rceil$ vertices red.

Suppose $n \equiv 2 \pmod 3$. If every vertex of G is colored red, then the required lower bound follows. Hence, renaming vertices if necessary, we may assume that our given X_4-coloring of G colors v_1 blue. Since every blue vertex is rooted at a copy of X_4, the vertex v_1 belongs to a 4-cycle, say v_1, v_2, u_2, u_1, v_1, containing two red vertices. Thus, $G - \{v_1, v_2, u_1, u_2\}$ can be partitioned into $(n-2)/3$ disjoint copies of H, and so by Lemma 2.5, our given X_4-coloring of G colors at least $2 + 2(n-2)/3 = 2(n+1)/3 = 2\lceil n/3 \rceil$ vertices red.

Finally, suppose $n \equiv 1 \pmod 3$. Suppose at most one of u_i and v_i is colored red for every $i = 1, 2, \ldots, n$. With this assumption, if both u_i and v_i are colored blue for some i, $1 \leq i \leq n$, then u_i or v_i is not rooted at a copy of X_4, a contradiction. Thus, exactly one of u_i and v_i is colored red for every i, and so exactly n vertices are colored red. On the other hand, suppose both u_i and v_i are colored red for some $i = 1, 2, \ldots, n$. Now, $G - \{u_i, v_i\}$ can be partitioned into $(n-1)/3$ disjoint copies of H, each of which contains at least two red vertices, and so by Lemma 2.5, our given X_4-coloring of G colors at least $2 + 2(n-1)/3 = 2(n+2)/3 = 2\lceil n/3 \rceil$ vertices red.

In all three cases, our given X_4-coloring of G colors at least $2\lceil n/3 \rceil$ vertices red. Thus, $\gamma_{X_4}(G) \geq 2\lceil n/3 \rceil$. We show next that $\gamma_{X_4}(G) \leq 2\lceil n/3 \rceil$. Let

$$D = \bigcup_{i=0}^{\lceil n/3 \rceil - 1} \{v_{3i+1}, u_{3i+1}\}.$$

Then coloring the vertices in D red and coloring all remaining vertices blue produces an X_4-coloring of G, and so $\gamma_{X_4}(G) \leq |D| = 2\lceil n/3 \rceil$. □

Recall, a set $S \subseteq V$ in a graph $G = (V, E)$ is a *total dominating set* (TDS) if every vertex is adjacent to at least one vertex of S.

Proposition 2.18 *For $n \geq 3$,*

$$\gamma_{X_4}(G) = \begin{cases} \gamma_t(G) + 1 & \text{if } n \equiv 1 \pmod 6 \\ \\ \gamma_t(G) & \text{otherwise.} \end{cases}$$

Proof. Any TDS of G contains at least two vertices from every subgraph $H = P_3 \times K_2$ of G (since the two vertices of degree 3 in H have disjoint open neighborhoods, each of which contains at least one vertex from any TDS). Let S be a $\gamma_t(G)$-set.

Suppose, first, that $n \equiv 1 \pmod 6$. Renaming vertices if necessary, we may assume $v_1 \notin S$. To dominate v_1, the set S contains at least one neighbor of v_1. If $u_1 \in S$, then $G - \{u_1, v_1\}$ can be partitioned into $(n-1)/3$ disjoint copies of H, each of which contains at least two vertices of S, and so $|S| \geq 1 + 2(n-1)/3 = (2n+1)/3$. If $v_2 \in S$, then $G - \{u_2, v_2\}$ can be partitioned into $(n-1)/3$ disjoint copies of H, and so once again $|S| \geq (2n+1)/3$. Similarly, if $v_n \in S$, then $|S| \geq (2n+1)/3$. Hence, $\gamma_t(G) \geq (2n+1)/3 = 2\lceil n/3 \rceil - 1$. On the other hand, the set

$$\left(\bigcup_{i=0}^{(n-7)/6} \{u_{6i+2}, u_{6i+3}, v_{6i+5}, v_{6i+6}\} \right) \cup \{u_1\}$$

is a TDS of G of cardinality $(2n+1)/3$, and so $\gamma_t(G) \leq (2n+1)/3 = 2\lceil n/3 \rceil - 1$. Consequently, $\gamma_t(G) = 2\lceil n/3 \rceil - 1$, and so, by Proposition 2.17, $\gamma_t(G) = \gamma_{X_4}(G) - 1$.

Suppose, then, that $n \not\equiv 1 \pmod 6$. The red vertices in any X_4-coloring of G form a TDS of G, and so $\gamma_t(G) \leq \gamma_{X_4}(G)$. Hence it suffices for us to show that $|S| = \gamma_t(G) \geq \gamma_{X_4}(G)$.

Suppose $n \equiv 0 \pmod 3$. Then, G contains $n/3$ disjoint copies of H, each of which contains at least two vertices of S, and so $|S| \geq 2n/3 = 2\lceil n/3 \rceil$. Hence by Proposition 2.17, $\gamma_t(G) \geq \gamma_{X_4}(G)$.

Suppose $n \equiv 2 \pmod 3$. Renaming vertices if necessary, we may assume $v_1 \notin S$. If $u_1 \in S$, then to totally dominate u_1 we may assume by symmetry that $u_2 \in S$, and so the 4-cycle $C'\colon v_1, v_2, u_2, u_1, v_1$ contains at least two vertices of S. On the other hand, if $u_1 \notin S$, then we may assume by symmetry that $v_2 \in S$ (to dominate v_1). To totally dominate v_2, at least one of u_2 or v_3 is in S, and so the 4-cycle $C'\colon v_2, v_3, u_3, u_2, v_2$ contains at least two vertices of S. In both cases the cycle C' contains at least two vertices of S and $G - V(C')$ can be partitioned into $(n-2)/3$ disjoint copies of H, each of which contains at least two vertices of S, and so $|S| \geq 2 + 2(n-2)/3 = 2(n+1)/3 = 2\lceil n/3 \rceil$. Hence by Proposition 2.17, $\gamma_t(G) \geq \gamma_{X_4}(G)$.

We show next that if $n \equiv 4 \pmod 6$, then $\gamma_t(G) \geq 2\lceil n/3 \rceil$ (and so, by Proposition 2.17, $\gamma_t(G) \geq \gamma_{X_4}(G)$). We proceed by induction on $n \geq 4$. If $n = 4$, then $\gamma_t(G) = 4 = 2\lceil n/3 \rceil$. This establishes the base case. Assume, then, that $n \geq 10$ and that for all integers $n' \equiv 4 \pmod 6$ with $4 \leq n' < n$ that $\gamma_t(C_{n'} \times K_2) \geq 2\lceil n'/3 \rceil$. Among all $\gamma_t(G)$-sets, let S be chosen to contain as many pairs $\{u_i, v_i\}$ as possible. We show that S contains at least one such pair. Assume, to the contrary, that $|S \cap \{u_i, v_i\}| \leq 1$ for all $i = 1, 2, \ldots, n$. Let $\mathcal{C}$ be the red-blue coloring associated with S. If every blue vertex in $\mathcal{C}$ is rooted at a copy of X_4, then $\gamma_t(G) \geq \gamma_{X_4}(G)$, as desired. Hence we may assume, renaming vertices if necessary, that v_1 is a blue vertex that is not rooted at a copy of X_4 in $\mathcal{C}$. If $u_1 \in S$, then to totally dominate u_1, we may assume $u_2 \in S$. By assumption, $|S \cap \{u_2, v_2\}| \leq 1$, and so $v_2 \notin S$. But then v_1 is rooted at a copy of X_4, a contradiction. Hence, $u_1 \notin S$.

By symmetry, we may assume $v_2 \in S$ (to dominate v_1), implying that $v_3 \in S$ and $S \cap \{u_2, u_3\} = \emptyset$. To dominate u_1, it follows from our choice of the set S that $S \cap \{u_{n-1}, u_n, v_{n-1}, v_n\} = \{u_{n-1}, u_n\}$. If $u_4 \in S$ or if $v_5 \in S$, then $(S - \{v_3\}) \cup \{u_2\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_2, v_2\}$, contrary to our choice of S. Hence, $S \cap \{u_4, v_5\} = \emptyset$.

Claim 2.19 $v_4 \notin S$.

Proof. Suppose $v_4 \in S$. If $u_5 \in S$, then $(S - \{v_4\}) \cup \{v_5\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_5, v_5\}$, contrary to our choice of S. Hence, $u_5 \notin S$, and so $u_6 \in S$ (to dominate u_5). Further, $u_7 \in S$ to totally dominate u_6. By our choice of S, $S \cap \{v_6, v_7\} = \emptyset$. If $u_8 \in S$, then $(S - \{v_4, u_6\}) \cup \{u_5, v_5\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_5, v_5\}$, contrary to our choice of S. Hence, $u_8 \notin S$. If $v_8 \in S$, then $(S - \{u_7\}) \cup \{v_6\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_6, v_6\}$, contrary to our choice of S. Hence, $v_8 \notin S$, implying that $S \cap \{u_9, u_{10}, v_9, v_{10}\} = \{v_9, v_{10}\}$. If $u_{11} \in S$, then $(S - \{v_{10}\}) \cup \{u_9\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_9, v_9\}$, a contradiction. Hence, $u_{11} \notin S$. If $v_{11} \in S$, then $(S - \{v_4, u_7, v_9\}) \cup \{u_5, u_8, v_8\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_8, v_8\}$, a contradiction. Hence, $v_{11} \notin S$, implying that $S \cap \{u_{12}, v_{12}, u_{13}, v_{13}\} = \{u_{12}, u_{13}\}$. Continuing in this way, we have that for each i where $1 \le i \le (n-4)/6$,

$$S \cap \left(\bigcup_{j=-1}^{4} \{u_{6i+j}, v_{6i+j}\} \right) = \{u_{6i}, u_{6i+1}, v_{6i+3}, v_{6i+4}\}.$$

This implies that $S \cap \{u_{n-1}, v_{n-1}, u_n, v_n\} = \{v_{n-1}, v_n\}$. But then the vertex u_1 is not dominated by S, a contradiction. □

By Claim 2.19, $v_4 \notin S$, implying that $S \cap \{u_4, v_4, u_5, v_5, u_6, v_6\} = \{u_5, u_6\}$. If $v_7 \in S$ or if $u_8 \in S$, then $(S - \{u_6\}) \cup \{v_5\}$ is a $\gamma_t(G)$-set that contains the pair $\{u_5, v_5\}$, contrary to our choice of S. Hence, $S \cap \{v_7, u_8\} = \emptyset$. Thus if $u_7 \notin S$, then $u_8 \in S$ to dominate v_7. Let $V' = \{u_1, v_1, u_2, v_2, \ldots, u_6, v_6\}$. Then, $S' = S \cap V' = \{v_2, v_3, u_5, u_6\}$, and $\{u_{n-1}, u_n\} \subset S$. Let G' be the prism $C_{n-6} \times K_2$ obtained from $G - V'$ by adding the edges v_7v_n and u_7u_n. Since S is a TDS of G, the set $S - S'$ is a TDS of G'. Thus, by the induction hypothesis, $|S| - 4 = |S - S'| \ge \gamma_t(G') \ge 2\lceil (n-6)/3 \rceil$, and so $|S| \ge 2\lceil n/3 \rceil$, as desired. Hence by Proposition 2.17, if $n \equiv 4 \pmod 6$, then $\gamma_t(G) \ge \gamma_{X_4}(G)$. □

Since the red vertices in any X_4-coloring of G form a TDS of G, as an immediate consequence of Proposition 2.18 we have the following result.

Corollary 2.20 *For $n \ge 3$ with $n \not\equiv 1 \pmod 6$, there exists a $\gamma_t(G)$-set whose associated red-blue coloring is a minimum X_4-coloring in G.*

2.3.5 X_5-stratification and the double total domination number

Proposition 2.21 *For $n \ge 3$, $\gamma_{X_5}(C_n \times K_2) = \left\lceil \dfrac{4n}{3} \right\rceil$.*

Proof. In any X_5-coloring of a graph, every vertex colored blue is rooted at a copy of X_5. Hence as an immediate consequence of the definition of an X_5-coloring, any X_5-coloring of G colors at least four vertices from every subgraph $H = P_3 \times K_2$ of G red. Furthermore, if it colors a vertex v blue, then v lies on a 4-cycle with three red vertices.

Consider any given X_5-coloring of G. If every vertex of G is colored red, then the required lower bound follows. Hence, renaming vertices if necessary, we may assume that our given X_5-coloring of G colors v_1 blue. Thus, v_1 lies on a 4-cycle in which the other three vertices are colored red. Renaming vertices if necessary, we may therefore assume that the vertices u_1, u_2 and v_2 are all colored red.

If $n \equiv 0 \pmod 3$, then G contains $n/3$ disjoint copies of H, each of which contains at least four red vertices, and so our given X_5-coloring contains at least $4n/3 = \lceil 4n/3 \rceil$ red vertices. If $n \equiv 1 \pmod 3$, then $G - \{u_2, v_2\}$ can be partitioned into $(n-1)/3$ disjoint copies of H, each of which contains at least four red vertices, and so our given X_5-coloring of G colors at least $2+4(n-1)/3 = (4n+2)/3 = \lceil 4n/3 \rceil$ vertices red. Finally, if $n \equiv 2 \pmod 3$, then $G - \{u_1, u_2, v_1, v_2\}$ can be partitioned into $(n-2)/3$ disjoint copies of H, each of which contains at least four red vertices, and so our given X_5-coloring of G colors at least $3+4(n-2)/3 = (4n+1)/3 = \lceil 4n/3 \rceil$ vertices red.

In all three cases, our given X_5-coloring of G colors at least $\lceil 4n/3 \rceil$ vertices red. Thus, $\gamma_{X_5}(G) \geq \lceil 4n/3 \rceil$. We show next that $\gamma_{X_5}(G) \leq \lceil 4n/3 \rceil$. Let

$$S = \bigcup_{i=0}^{\lfloor n/3 \rfloor - 1} \{v_{3i+2}, v_{3i+3}\}.$$

If $n \not\equiv 2 \pmod 3$, let $D = V(G) - S$. If $n \equiv 2 \pmod 3$, let $D = V(G) - (S \cup \{v_n\})$. Then coloring the vertices of D red and coloring all remaining vertices of G blue produces an X_5-coloring of G. Thus, $\gamma_{X_5}(G) \leq |D| = \lceil 4n/3 \rceil$. □

Next we consider a generalization of total domination in graphs which we call double total domination (defined in a similar way as that of double domination introduced by Harary and Haynes [27]). Let $G = (V, E)$ be a graph and let $S \subseteq V$. We say that a vertex $v \in V$ is *double totally dominated* by S if $|N(v) \cap S| \geq 2$. If every vertex of V is double totally dominated by S, then we call S a *double total dominating set* (DTDS) of G. The *double total domination number* $\gamma^t_{\times 2}(G)$ is the minimum cardinality of a DTDS of G. A DTDS of cardinality $\gamma^t_{\times 2}(G)$ we call a $\gamma^t_{\times 2}(G)$-set. We shall prove:

Proposition 2.22 *For* $n \geq 3$,

$$\gamma_{X_5}(G) = \begin{cases} \gamma^t_{\times 2}(G) - 1 & \text{if } n \equiv 2 \pmod 6 \\ \\ \gamma^t_{\times 2}(G) & \text{otherwise.} \end{cases}$$

Proof. Let S be any $\gamma^t_{\times 2}(G)$-set of G. Since every vertex is adjacent to at least two vertices in S, the set S contains at least four vertices from every subgraph $H = P_3 \times K_2$ of G (since the two vertices of degree 3 in H have disjoint open neighborhoods, each of which contains at least two vertices from any DTDS).

We show first that if $n \equiv 2 \pmod 6$, then $\gamma^t_{\times 2}(G) \geq \lceil 4n/3 \rceil + 1$. We proceed by induction on $n \geq 8$. If S contains four vertices that belong to a common 4-cycle in G, then removing these vertices from G we can partition the resulting graph into $(n-2)/3$ disjoint copies of H, and so $|S| \geq 4 + 4(n-2)/3 = \lceil 4n/3 \rceil + 1$. Hence we may assume that S contains at most three vertices from every 4-cycle in G, for otherwise the desired lower bound follows. Suppose that for every vertex $v \notin S$, we have $N(v) \subset S$. Then S contains exactly three vertices from every 4-cycle in G. Since we can partition G into $n/2$ disjoint 4-cycles, and since $n \geq 8$, $|S| \geq 3n/2 \geq 4(n+1)/3 = \lceil 4n/3 \rceil + 1$, as desired. Hence, renaming vertices if necessary, we may assume that $S \cap \{v_3, v_4\} = \emptyset$ (and still S contains at most three vertices from every 4-cycle in G), for otherwise the desired lower bound follows. Let $V' = \{u_1, v_1, u_2, v_2, \ldots, u_6, v_6\}$. Then, $S' = S \cap V' = V' - \{u_1, v_3, v_4, u_6\}$, and $\{v_n, v_7\} \subset S$. If $n = 8$, then to double totally dominate each of u_7 and u_8 we must have $\{u_7, u_8\} \subset S$, whence $|S| = 12 = \lceil 4n/3 \rceil + 1$, as desired. This establishes the base case of the induction. Assume, then, that $n \geq 14$ and that for all integers $n' \equiv 2 \pmod 6$ with $8 \leq n' < n$ that $\gamma^t_{\times 2}(C_{n'} \times K_2) \geq \lceil 4n'/3 \rceil + 1$. Let G' be the prism $C_{n-6} \times K_2$ obtained from $G - V'$ by adding the edges v_7v_n and u_7u_n. Since S is a DTDS of G, the set $S - S'$ is a DTDS of G'. Thus, by the induction hypothesis, $|S| - 8 = |S| - |S'| \geq \gamma^t_{\times 2}(G') \geq \lceil 4(n-6)/3 \rceil + 1$, and so $|S| \geq \lceil 4n/3 \rceil + 1$, as desired.

We show next that if $n \not\equiv 2 \pmod 6$, then $\gamma^t_{\times 2}(G) \geq \lceil 4n/3 \rceil$. If $S = V$, then $S - \{v_1\}$ is also a DTDS of G, contradicting the minimality of S. Hence, renaming vertices if necessary, we may assume that $v_1 \notin S$. To double totally dominate the vertex v_1, we have $|S \cap \{u_1, v_2, v_n\}| \geq 2$. Hence at least one of v_2 and v_n is in S. By symmetry, we may assume $v_2 \in S$. To double totally dominate v_2, we have $\{u_2, v_3\} \subset S$. If $n \equiv 0 \pmod 3$, then G contains $n/3$ disjoint copies of H, each of which contains at least four vertices of S, and so $|S| \geq 4n/3 = \lceil 4n/3 \rceil$. If $n \equiv 1 \pmod 3$, then $G - \{u_2, v_2\}$ can be partitioned into $(n-1)/3$ disjoint copies of H, and so $|S| \geq 2 + 4(n-1)/3 = (4n+2)/3 = \lceil 4n/3 \rceil$. If $n \equiv 2 \pmod 3$, then $G - \{u_2, u_3, v_2, v_3\}$ can be partitioned into $(n-2)/3$ disjoint copies of H, and so $|S| \geq 3 + 4(n-2)/3 = (4n+1)/3 = \lceil 4n/3 \rceil$. In all three cases, $|S| \geq \lceil 4n/3 \rceil$, i.e., $\gamma^t_{\times 2}(G) \geq \lceil 4n/3 \rceil$. Thus we have shown that

$$\gamma^t_{\times 2}(G) \geq \begin{cases} \lceil 4n/3 \rceil + 1 & \text{if } n \equiv 2 \pmod 6 \\ \\ \lceil 4n/3 \rceil & \text{otherwise.} \end{cases}$$

Next we establish upper bounds on $\gamma^t_{\times 2}(G)$. Suppose first that $n \not\equiv 3 \pmod 6$. For $n = 4$, let $D = V - \{v_1, v_2\}$. For $n = 5$, let $D = V - \{v_1, v_2, u_4\}$. For $n \geq 6$, let

$$U = \bigcup_{i=0}^{\lfloor n/6 \rfloor - 1} \{v_{6i+1}, v_{6i+2}, u_{6i+4}, u_{6i+5}\}.$$

If $n \equiv 0$ or 1 or $2 \pmod 6$ (and still $n \geq 6$), let $D = V - U$. If $n \equiv 4 \pmod 6$, let $D = V - U - \{v_{n-3}, v_{n-2}\}$. If $n \equiv 5 \pmod 6$, let $D = V - U - \{v_{n-4}, v_{n-3}, u_{n-1}\}$. In all cases, D is a DTDS of G such that the red-blue coloring associated with D is an X_5-coloring in G. Furthermore, if $n \equiv 2 \pmod 6$, then $|D| = \lceil 4n/3 \rceil + 1$; otherwise, $|D| = \lceil 4n/3 \rceil$.

For $n \equiv 3 \pmod 6$, let

$$W = \bigcup_{i=0}^{n/3-1} \{u_{3i+2}, v_{3i+2}\},$$

and let $D = V - W$. Then, D is a DTDS of G with $|D| = \lceil 4n/3 \rceil$. Thus we have shown that

$$\gamma^t_{\times 2}(G) \leq |D| = \begin{cases} \lceil 4n/3 \rceil + 1 & \text{if } n \equiv 2 \pmod 6 \\ \\ \lceil 4n/3 \rceil & \text{otherwise.} \end{cases}$$

Consequently,

$$\gamma^t_{\times 2}(G) = \begin{cases} \lceil 4n/3 \rceil + 1 & \text{if } n \equiv 2 \pmod 6 \\ \\ \lceil 4n/3 \rceil & \text{otherwise.} \end{cases}$$

The desired result now follows from Proposition 2.21. □

As a consequence of the proof of Proposition 2.22, we have the following result.

Corollary 2.23 *For $n \geq 3$ with $n \not\equiv 2$ or $3 \pmod 6$, there exists a $\gamma^t_{\times 2}(G)$-set whose associated red-blue coloring is a minimum X_5-coloring in G.*

Chapter 3

STRATIFICATION AND DOMINATION IN GRAPHS

3.1 Introduction

Let F be a 2-stratified P_3 rooted at a blue vertex v. The five possible choices for the graph F are shown in Figure 1.2 and the F-domination number, in each case, with the exception of one, is characterized by Theorem 1.6. In this chapter our aim is twofold. Firstly, we investigate the F-domination number for a tree G where $F = F_3$. In particular, we show that for a tree of diameter at least three this parameter is at most two-thirds its order and we characterize the trees attaining this bound. Secondly, we investigate the F-domination number when F is a 2-stratified K_3 that is rooted at a blue vertex with exactly one red vertex. We show that for n sufficiently large, if G is a connected graph of order n in which every edge is in a triangle, then this parameter is at most $(n - \sqrt{n})/2$ and this bound is sharp.

3.2 The parameter $\gamma_{F_3}(G)$

The parameter $\gamma_{F_3}(G)$ (see Figure 1.2) appears to be new. As pointed out in [8], F_3-domination is not the same as the distance domination parameter called k-*step domination* introduced in [45]. The difference in 2-step domination and F_3-domination is that in F_3-domination every blue vertex must have a blue-blue-red path (of length two) to some red vertex. Thus, every F_3-dominating set is a 2-step dominating set, but not every 2-step dominating set is a F_3-dominating set. If T is a star $K_{1,n-1}$ of order $n \geq 3$, then $\gamma_{F_3}(T) = n$ since the central vertex of T must be colored red in any F_3-coloring of T. However the 2-step domination number of T equals 2 (the set consisting of the central vertex and any leaf of T is a 2-step dominating set of T). A survey of results on distance domination in graphs can be found in §7.4 of [30]. For a more comprehensive survey, the reader is referred to [32].

Our aim in this section is to investigate the F_3-domination number of a tree. In particular, we establish an upper bound on the F_3-domination number of a tree in terms of its order and we characterize the trees attaining this bound.

3.2.1 Paths

First we establish the F_3-domination number of a path P_n on n vertices.

Proposition 3.1 *For* $n \geq 1$, $\gamma_{F_3}(P_n) = \left\lfloor \frac{n+7}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor - \left\lceil \frac{n}{3} \right\rceil$.

Proof. We proceed by induction on the order n of a path P_n. The result is straightforward to verify for $n \leq 3$. Assume then that $n \geq 4$. Consider a path $P\colon v_1, v_2, \ldots, v_n$.

We show first that there exists a γ_{F_3}-coloring of P that colors v_1 and v_4 red and v_2 and v_3 blue. Consider a γ_{F_3}-coloring of P. If v_1 is colored blue, then there is a copy of F_3 rooted at v_1, and so v_2 is colored blue and v_3 red. But then there is no copy of F_3 rooted at v_2, a contradiction. Hence, v_1 is colored red. If v_2 is colored blue, then since there is a copy of F_3 rooted at v_2, v_3 is blue and v_4 red as desired. Suppose then that v_2 is colored red. If now v_3 is blue, then v_4 must be blue and v_5 red. But then interchanging the colors of v_2 and v_4 produces a new γ_{F_3}-coloring of P that colors v_1 and v_4 red and v_2 and v_3 blue, as desired. On the other hand, if v_3 is red, then v_4 must be blue (otherwise if v_4 is red we can recolor v_2 and v_3 blue to produce an F_3-coloring that colors $\gamma_{F_3} - 2$ vertices red, a contradiction) and therefore v_5 is blue and v_6 is red. But then recoloring v_2 and v_3 blue and recoloring v_4 and v_5 red produces a new γ_{F_3}-coloring of P that colors v_1 and v_4 red and v_2 and v_3 blue, as desired.

Let C be a γ_{F_3}-coloring of P that colors v_1 and v_4 red and v_2 and v_3 blue. Let $P' = P - \{v_1, v_2, v_3\}$. Then the restriction of C to the path P' is an F_3-coloring of P' that colors $\gamma_{F_3}(P) - 1$ vertices red. Hence, $\gamma_{F_3}(P') \leq \gamma_{F_3}(P) - 1$. On the other hand, any γ_{F_3}-coloring of P' colors its end-vertices v_4 and v_n red and can therefore be extended to an F_3-coloring of P by coloring v_1 red and v_2 and v_3 blue. Thus, $\gamma_{F_3}(P) \leq \gamma_{F_3}(P') + 1$. Consequently, $\gamma_{F_3}(P) = \gamma_{F_3}(P') + 1$. Since $P \cong P_n$ and $P' \cong P_{n-3}$, the result now follows by applying the inductive hypothesis to the path P'. □

3.2.2 The Family $\mathcal{T}$

Let $H_1 = P_6$ and for $k \geq 2$, let H_k be the tree obtained from the disjoint union of a star $K_{1,k+1}$ and a subdivided star $K^*_{1,k}$ by joining a leaf of the star to the central vertex of the subdivided star. The tree H_3 is illustrated in Figure 3.1.

Let $\mathcal{T} = \{H_k \mid k \geq 1\}$. The following lemma establishes some properties of trees in the family $\mathcal{T}$.

H_3 :

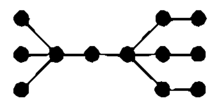


Figure 3.1:

Lemma 3.2 *If $T \in \mathcal{T}$ has order n, then* $\text{diam}(T) = 5$ *and* $\gamma_{F_3}(T) = 2n/3$. *Further, every vertex of T belongs to some γ_{F_3}-set of T.*

Proof. If $T = H_1$, then $T = P_6$ and $\gamma_{F_3}(T) = 2n/3$ and clearly every vertex of T belongs to some γ_{F_3}-set of T. Suppose then that $T = H_k$ for some $k \geq 2$. Then, $n = 3(k+1)$. Let u and w denote the central vertices of the star and subdivided star, respectively. If some F_3-coloring of T colors w red, then all vertices of the subdivided star must be colored red and at least one leaf of T in the star is colored red. On the other hand, if some F_3-coloring of T colors w blue, then u and all the leaves of T are colored red. Further, at least one neighbor of w must be colored red. Hence irrespective of whether w is colored red or blue, any F_3-coloring of T colors at least $2(k+1)$ vertices red. However coloring all vertices of the subdivided star red and exactly one leaf of T in the star red and all other vertices blue, produces an F_3-coloring of T that colors $2(k+1)$ vertices red. Hence, $\gamma_{F_3}(T) = 2n/3$. Further, coloring all vertices of the star red and exactly one leaf of the subdivided star red and all other vertices blue, produces another F_3-coloring of T that colors $2(k+1)$ vertices red. Hence every vertex of T belongs to some γ_{F_3}-set of T. □

3.2.3 Trees with maximum γ_{F_3}

As observed earlier, if T is a star $K_{1,n-1}$ of order $n \geq 3$, then $\gamma_{F_3}(T) = n$. Hence in what follows we consider trees of diameter at least 3. Our main result establishes an upper bound on the F_3-domination number of a tree of diameter at least 3 in terms of its order.

Theorem 3.3 *If T is a tree of order n with* $\mathrm{diam}(T) \geq 3$, *then* $\gamma_{F_3}(T) \leq 2n/3$ *with equality if and only if* $T \in \mathcal{T}$.

Proof. We proceed by induction on the order $n \geq 4$ of a tree T with diameter at least 3. If $n = 4$, then $T = P_4$ and $\gamma_{F_3}(T) = 2 < 2n/3$. This establishes the base case. Assume then that $n \geq 5$ and that all trees T' of order $n' < n$ with $\mathrm{diam}(T') \geq 3$ satisfy $\gamma_{F_3}(T') \leq 2n'/3$ with equality if and only if $T' \in \mathcal{T}$. Let T be a tree of order n with $\mathrm{diam}(T) \geq 3$. We proceed further with four claims.

Claim 3.4 *If* $\mathrm{diam}(T) = 3$, *then* $\gamma_{F_3}(T) = 2 \leq n/2$.

Proof. The tree T is a double star and coloring any two leaves at distance 3 apart red and coloring all other vertices blue produces an F_3-coloring of T. Thus, $\gamma_{F_3}(T) \leq 2 \leq n/2$. Since $\gamma_{F_3}(G) \geq 2$ for any nontrivial tree G, the desired result follow. □

Claim 3.5 *If* $\mathrm{diam}(T) = 4$, *then* $\gamma_{F_3}(T) \leq (n+1)/2$.

Proof. The tree T can be obtained from $k \geq 2$ disjoint nontrivial stars by adding a new vertex w and joining w to a vertex of maximum degree in each star and adding $t \geq 0$ new vertices and joining them to w. For $i = 1, 2, \ldots, k$, let $T_i = K_{1,n_i-1}$, $n_i \geq 2$, be the k disjoint stars and let v_i be the vertex of T_i joined to w (if $n_i \geq 3$, then v_i is the central vertex of T_i). Thus, $n = t + 1 + \sum_{i=1}^{k} n_i$.

Suppose that $t \geq 1$. Then, $n \geq 2(k+1)$. We now select any packing S of T (i.e., if $u, v \in S$, then $d(u,v) \geq 3$) that consist of $k+1$ leaves (and so S consists of one leaf from each of the k stars and one leaf adjacent to w). Coloring each vertex of S red and coloring all other vertices blue, produces an F_3-coloring of T. Hence, $\gamma_{F_3}(T) \leq k+1 \leq n/2$.

Suppose that $t = 0$. We may assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. Coloring all n_1 vertices in T_1 red and one leaf of T_i red for $i = 2, \ldots, k$, and coloring all remaining vertices blue, produces an F_3-coloring of T. Hence, $\gamma_{F_3}(T) \leq n_1 + k - 1$. However, $n+1 = 2+\sum_{i=1}^{k} n_i \geq 2+2n_1+(k-2)2 = 2(n_1+k-1)$, and so $\gamma_{F_3}(T) \leq (n+1)/2$. □

Claim 3.6 *If* $\text{diam}(T) = 5$, *then* $\gamma_{F_3}(T) \leq 2n/3$ *with equality if and only if* $T \in \mathcal{T}$.

Proof. If $n = 6$, then $T = P_6$ and $\gamma_{F_3}(T) = 2n/3$ and $T \in \mathcal{T}$. Assume then that $n \geq 7$. Let $P\colon u, v, w, x, y, z$ be a diametrical path in T. Let T_w and T_x denote the components of $T - wx$ containing w and x, respectively, and let $n_1 = |V(T_w)|$ and $n_2 = |V(T_x)|$. We consider three possibilities.

Suppose $\deg w \geq 3$ and $\deg x \geq 3$. Then each of T_w and T_x has diameter 3 or 4. Hence by Claims 3.4 and 3.5, $\gamma_{F_3}(T_w) \leq (n_1 + 1)/2$ and $\gamma_{F_3}(T_x) \leq (n_2 + 1)/2$. Combining an F_3-coloring of T_w and an F_3-coloring of T_x produces an F_3-coloring of T, and so $\gamma_{F_3}(T) \leq (n + 2)/2 < 2n/3$.

Suppose $\deg w = \deg x = 2$. Then each of T_w and T_x is a star with central vertices v and y, respectively. We may assume that $n_1 \leq n_2$, and so $n \geq 2n_1$. Coloring all n_1 vertices in T_w red and one leaf of T in T_x red and coloring all other vertices blue, produces an F_3-coloring of T. Hence, $\gamma_{F_3}(T) \leq n_1 + 1 \leq (n + 2)/2 < 2n/3$.

By symmetry, we may therefore assume that $\deg w = 2$ and $\deg x \geq 3$. Coloring all n_2 vertices in T_x red and one leaf of T in T_w red and coloring all other vertices blue, produces an F_3-coloring of T, and so $\gamma_{F_3}(T) \leq n_2 + 1$. Thus if $n_2 \leq (2n-4)/3$, then $\gamma_{F_3}(T) < 2n/3$. Hence we may assume that $n_2 \geq 2n/3 - 1 = 2(n_1 + n_2)/3 - 1$ or, equivalently, $n_2 \geq 2n_1 - 3$. Thus, $n = n_1 + n_2 \geq 3(n_1 - 1)$, i.e., $n_1 \leq n/3 + 1$.

Suppose $\text{diam}(T_x) = 3$. Then, T_x is a double star with central vertices x and y. Coloring all n_1 vertices in T_w red and coloring one leaf in T_x at distance 2 from

x red, and coloring all other vertices blue, produces an F_3-coloring of T, and so $\gamma_{F_3}(T) \leq n_1 + 1 \leq n/3 + 2 < 2n/3$.

Suppose finally that $\mathrm{diam}(T_x) = 4$. Then, $T_x - x$ consists of $k \geq 2$ disjoint stars each of order at least 2 and possibly isolated vertices. Thus, $n_2 \geq 2k + 1$. Coloring all n_1 vertices in T_w red and coloring one leaf of T from each of the k disjoint stars in $T_x - x$ red, and coloring all other vertices blue, produces an F_3-coloring of T, and so, since $n_1 \leq n/3 + 1$, $\gamma_{F_3}(T) \leq n_1 + k \leq n_1 + (n_2 - 1)/2 = (n + n_1 - 1)/2 \leq (n + n/3)/2 = 2n/3$. Furthermore, if $\gamma_{F_3}(T) = 2n/3$, then we must have equality throughout this inequality chain. In particular, $n_2 = 2k + 1$ and $n_1 = n/3 + 1$, and so $n = 3(k+1)$ and $n_1 = k + 2$. Thus, $T_w = K_{1,k+1}$ and $T_x = K^*_{1,k}$, i.e., $T = H_k \in \mathcal{T}$ as desired. □

Claim 3.7 *If* $\mathrm{diam}(T) = 6$, *then* $\gamma_{F_3}(T) < 2n/3$.

Proof. Among all vertices that belong to a diametrical path and are at distance 2 from a leaf of this path, let y be chosen to have maximum possible degree. Let $P\colon u, v, w, x, y, z, z'$ be such a diametrical path. Then, x is the central vertex of T.

Suppose $\deg y \geq 3$. Let T_1 and T_2 denote the two components of $T - xy$, where $y \in V(T_1)$. Then, $\mathrm{diam}(T_1) \in \{3, 4\}$ and $\mathrm{diam}(T_2) \geq 3$. For $i = 1, 2$, let $|V(T_i)| = n_i$. By the inductive hypothesis, $\gamma_{F_3}(T_1) < 2n_1/3$ and $\gamma_{F_3}(T_2) \leq 2n_2/3$, whence $\gamma_{F_3}(T) \leq \gamma_{F_3}(T_1) + \gamma_{F_3}(T_2) < 2n/3$. Thus we may assume that $\deg y = 2$. Hence by our choice of y, every vertex adjacent to x on a path of length 3 emanating from x has degree 2. In particular, $\deg w = 2$.

Suppose there is no leaf at distance 2 from x. Then, $T_x = T - x$ consists of possibly isolated vertices and $k \geq 2$ disjoint stars each of order at least 3 that are joined to x by one of their leaves. Let $r = |N[x]|$. If $r \leq n/2$, then coloring x and every leaf adjacent to x red and coloring one leaf of T from each of the k disjoint stars in $T_x - x$ red, and coloring all other vertices blue, produces an F_3-coloring of

T that colors exactly r vertices red, and so $\gamma_{F_3}(T) \le r \le n/2$. On the other hand if $r \ge (n+1)/2$, then coloring all vertices of $V - N[x]$ red and coloring one non-leaf neighbor of x red produces an F_3-coloring of T that colors exactly $n - r + 1$ vertices red, and so $\gamma_{F_3}(T) \le n - r + 1 \le (n+1)/2$. Hence we may assume that that there is at least one leaf at distance 2 from x, i.e., at least one neighbor of x is a support vertex. By our choice of y, every neighbor other than x of such a support vertex is a leaf.

Suppose x has a neighbor x' of degree at least 3. Then x' is a support vertex every neighbor of which different from x is a leaf. Let v' be a leaf adjacent to x'.

Suppose first that there is a strong support vertex at distance 2 from x. We may assume z is such a vertex. Consider the tree $T' = T - \{v', z'\}$ of order $n' = n - 2$. By the inductive hypothesis, $\gamma_{F_3}(T') \le (2n' - 1)/3 = (2n - 5)/3$. Consider a γ_{F_3}-coloring C' of T'. If x is colored red, then necessarily y and z are colored blue and exactly one leaf adjacent to z in T' is colored red. Hence C' can be extended to an F_3-coloring of T by coloring z' blue and v' red. On the other hand, suppose x is colored blue. Then every support vertex at distance 2 from x is colored red and therefore every leaf at distance 3 from x is colored red. Hence we may assume that x' is colored blue (since if x' is colored red, we can recolor y, for example, red and recolor x' blue). Further exactly one leaf adjacent to x' in T' is colored red. Hence C' can be extended to an F_3-coloring of T by coloring z' red and v' blue. In both cases, $\gamma_{F_3}(T) \le \gamma_{F_3}(T') + 1 \le 2(n-1)/3$.

Suppose secondly that every support vertex at distance 2 from x has degree 2. In particular, $\deg z = 2$. Let $T' = T - \{v', y, z, z'\}$ and let T' have order n'. Since $\operatorname{diam}(T') \in \{5, 6\}$, $\gamma_{F_3}(T') \le 2n'/3 = 2(n-4)/3$. Consider a γ_{F_3}-coloring C' of T'. If x is colored red, then we can extended C' to an F_3-coloring of T by coloring v' and z' red and coloring y and z blue. On the other hand, if x is colored blue, then we can choose C' so that x' is colored blue and exactly one leaf adjacent to x' in T'

is colored red. Hence we can extended C' to an F_3-coloring of T by coloring v' and y blue and coloring z and z' red. In both cases, $\gamma_{F_3}(T) \le \gamma_{F_3}(T') + 2 \le 2(n-1)/3$. Thus we may assume that every neighbor of x has degree at most 2, for otherwise $\gamma_{F_3}(T) < 2n/3$. In particular, every support vertex adjacent to x has degree 2. Let x' be a support vertex adjacent to x and let v' be the leaf adjacent to x'.

Suppose a support vertex at distance 2 from x has degree 2. We may assume that $\deg z = 2$. Let $T' = T - \{v', x', y, z, z'\}$ and let the tree T' have order n'. By the inductive hypothesis, $\gamma_{F_3}(T') \le 2n'/3$. Consider a γ_{F_3}-coloring C' of T'. If x is colored red, then we can extended C' to an F_3-coloring of T by coloring v', x' and z' red and coloring y and z blue. On the other hand, if x is colored blue, then since every neighbor of x has degree at most 2, at least one neighbor of x in T' is colored red. Hence we can extended C' to an F_3-coloring of T by coloring v', z and z' red and coloring x' and y blue. In both cases, $\gamma_{F_3}(T) \le \gamma_{F_3}(T') + 3 \le 2n'/3 + 3 < 2n/3$. Hence we may assume that every support vertex at distance 2 from x is a strong support vertex. Further, renaming if necessary, we may assume that z is such a strong support vertex of smallest degree.

Let $T' = T - \{v', x', z'\}$ and let T' have order $n' = n - 3$. Then, $\text{diam}(T') = 6$. By the inductive hypothesis, $\gamma_{F_3}(T') < 2n'/3$. Consider a γ_{F_3}-coloring C' of T'. If x is colored red, then necessarily y and z are colored blue and exactly one leaf adjacent to z in T' is colored red. Hence we can extended C' to an F_3-coloring of T by coloring v' and x' red and coloring z' blue. On the other hand, if x is colored blue, then since every neighbor of x has degree at most 2, at least one neighbor of x in T' is colored red. Hence we can extend C' to an F_3-coloring of T by coloring v' and z' red and coloring x' blue. In both cases, $\gamma_{F_3}(T) \le \gamma_{F_3}(T') + 2 \le (2n' - 1)/3 + 2 < 2n/3$. This completes the proof of Claim 3.7. □

We now return to the proof of Theorem 3.3. By Claims 3.4 to 3.7, we may assume that $\mathrm{diam}(T) \geq 7$. We show that $\gamma_{F_3}(T) < 2n/3$. Let $P: v_1, v_2, \ldots, v_{\mathrm{diam}(T)+1}$ be a diametrical path in T. Let T_1 and T_2 denote the two components of $T - v_4v_5$ of orders n_1 and n_2, respectively, where $v_4 \in V(T_1)$. For $i = 1, 2$, $\mathrm{diam}(T_i) \geq 3$. Applying the inductive hypothesis to T_i, $\gamma_{F_3}(T_i) \leq 2n_i/3$ with equality if and only if $T_i \in \mathcal{T}$. Hence, $\gamma_{F_3}(T) \leq \gamma_{F_3}(T_1) + \gamma_{F_3}(T_2) \leq 2n_1/3 + 2n_2/3 = 2n/3$. Further if $\gamma_{F_3}(T) = 2n/3$, then we must have equality throughout this inequality chain. In particular for $i = 1, 2$, $\gamma_{F_3}(T_i) = 2n_i/3$, and so $T_i \in \mathcal{T}$. By Lemma 3.2, there is a γ_{F_3}-coloring $\mathcal{C}$ of T_2 that colors v_5 red. Since $T_1 \in \mathcal{T}$, $T_1 = H_k$ for some $k \geq 1$ and by Lemma 3.2, $\gamma_{F_3}(T_1) = 2(k+1)/3$. Let u, v, w, x, y, z be a path in T_1 where v and x denote the central vertices of the star and subdivided star, respectively, in T_1. Since P is a longest path in T, either $v_4 = w$ or $v_4 = x$.

If $v_4 = w$, then we can extend $\mathcal{C}$ to an γ_{F_3}-coloring of T by coloring all leaves in the subdivided star of T_1 red, coloring one vertex in the subdivided star adjacent to a leaf red, and coloring one leaf of T_1 in the star red and coloring all other vertices blue. Hence, $\gamma_{F_3}(T) \leq k + 2 + \gamma_{F_3}(T_2) < \gamma_{F_3}(T_1) + \gamma_{F_3}(T_2) = 2n/3$. On the other hand, if $v_4 = x$, then we can extend $\mathcal{C}$ to an γ_{F_3}-coloring of T by coloring all leaves in T_1 red and coloring v red and coloring all other vertices blue. Hence, $\gamma_{F_3}(T) \leq 2k + 1 + \gamma_{F_3}(T_2) < n/3$. □

We close this subsection with the following consequence of Theorem 3.3.

Corollary 3.8 *If T is a tree of order n with* $\mathrm{diam}(T) = \ell \geq 6$, *then* $\gamma_{F_3}(T) < 2n/3$, *and this bound is best possible for each fixed ℓ.*

Proof. By Theorem 3.3, $\gamma_{F_3}(T) < 2n/3$. That this bound is asymptotically best possible may be seen as follows: Let $\ell \geq 3$ be a *fixed* integer and let k be a very large integer. Let T be the tree obtained from H_k by attaching a path of length ℓ to the central vertex, w say, of the subdivided star in H_k. Let P be the resulting path emanating from w (on $\ell+1$ vertices). Then, $\operatorname{diam}(T) = \ell+3 \geq 6$ and $n = |V(T)| = 3k+\ell+3$.

If some F_3-coloring of T colors w red, then all vertices of the subdivided star must be colored red, at least one leaf of T in the star is colored red and at least $(|V(P)|+2)/3 = (\ell+3)/3$ vertices of P (including w) must be colored red. On the other hand, if some F_3-coloring of T colors w blue, then all the leaves of T in both the subdivided star and the star are colored red, the center of the star is colored red, and at least $(\ell+2)/3$ additional vertices including at least one neighbor of w (possibly on P) are colored red. It follows that any F_3-coloring of T colors at least $2k+(\ell+5)/3$ vertices red. Hence as $k \longrightarrow \infty$,

$$\frac{\gamma_{F_3}(T)}{n} = \frac{6k+\ell+5}{9k+3\ell+9} = \frac{6+\frac{\ell}{k}+\frac{5}{k}}{9+\frac{3\ell}{k}+\frac{9}{k}} \longrightarrow \frac{2}{3}.$$

Therefore the bound $\gamma_{F_3}(T) < 2n/3$ is asymptotically best possible. □

3.3 A 2-stratified K_3

The two 2-stratified graphs K_3 rooted at a blue vertex v are shown in Figure 3.2, where the red vertices are indicated by darkened vertices.

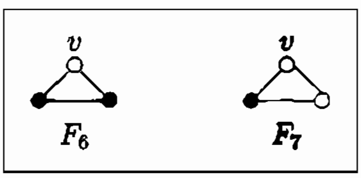


Figure 3.2:

Obviously, in any F_6-coloring or F_7-coloring of G, every vertex not on a triangle of G must be colored red. The following two results were established by Chartrand et al. [8].

Theorem 3.9 ([8]) *If G is a graph of order n in which every vertex is in a triangle, then $\gamma_{F_8}(G) \le 2n/3$, and this bound is sharp.*

Theorem 3.10 ([8]) *If G is a graph of order n in which every vertex is in a triangle, then $\gamma_{F_7}(G) < n/2$, and this bound is asymptotically best possible.*

Our aim in this section is twofold: First to present an alternative proof (using counting arguments) of Theorem 3.10 to that presented in [8] (which is by induction on the order of a graph in which every vertex is in a triangle), and secondly to show that this new proof can be used to obtain a sharp upper bound on the F_7-domination number of a graph with small domination number relative to its order. We will need the following result in [41].

Theorem 3.11 ([41]) *If G is a graph of order n in which every vertex is in a triangle, then $\gamma(G) \le n/3$, and this bound is sharp.*

Theorem 3.12 *If G is a graph of order $n \ge 3$ in which every vertex is in a triangle, then $\gamma_{F_7}(G) < n/2$. Further, if $2\lfloor(\gamma(G)+1)/2\rfloor \le (\sqrt{8n+1}-1)/4$, then*

$$\gamma_{F_7}(G) \le \frac{n}{2} - \frac{1}{8}\left(\sqrt{8n+1}-1\right),$$

and this bound is sharp.

Proof. Let $g(n) = (\sqrt{8n+1}-1)/4$ and let $f(n) = \frac{1}{2}(n - g(n))$. Then, $f(n) = (g(n))^2$. For $n \ge 3$, $f(n)$ is an increasing function in n and $f(n) \ge n/3$. If $\gamma(G) = 1$, then $\gamma_{F_7}(G) = 1 = f(3) \le f(n)$. Hence in what follows we assume that $\gamma(G) \ge 2$. Removing edges of G, if any, that do not belong to a triangle produces a graph with the same F_7-domination number as that of G. Hence in what follows we assume that every edge of G is in a triangle.

By Theorem 1.3, there exists a $\gamma(G)$-set S such that $|\mathrm{epn}(v, S)| \geq 1$ for every $v \in S$. Let $|S| = s$. We define S_1 and S_2 to be a *balanced partition of* S if S_1 and S_2 is a partition of S into two subsets such that $|S_1| = \lceil s/2 \rceil$ and $|S_2| = \lfloor s/2 \rfloor$.

Consider a balanced partition S_1 and S_2 of S. Let $X = \{v \in V - S \mid v$ belongs to no triangle that contains two vertices of $S\}$. For each vertex $v \in X$, we select one triangle T_v that contains v and a vertex of S. Let $X_1 = \{v \in X\colon |V(T_v) \cap S_1| = 1\}$ and let $X_2 = X - X_1$. Note that every external private neighbor (with respect to S) of a vertex of S_1 (resp., S_2) is in X_1 (resp., X_2). Let

$$Y = \bigcup_{v \in X} V(T_v).$$

Then, $X \cup S \subset Y$. Let $C = V - Y$. Then each vertex of C belongs to a triangle that contains two vertices of S. If $C = \emptyset$, then S is an F_7-coloring of G, and so, by Theorem 3.11, $\gamma_{F_7}(G) \leq |S| \leq n/3 \leq f(n)$. Hence we may assume that $C \neq \emptyset$. For each $v \in C$, we select one triangle T_v that contains it and two vertices of S and we associate these two vertices of S with v. Let

$$E_F = \bigcup_{v \in C \cup X} E(T_v),$$

and let F be the subgraph of G induced by the subset E_F of edges of G. By construction, F is a spanning subgraph of G every vertex of which belongs to a triangle (in F) with some vertex of S. Since $\gamma_{F_7}(G) \leq \gamma_{F_7}(F)$, it suffices to prove that $\gamma_{F_7}(F) < n/2$ and that if $2\lfloor (s+1)/2 \rfloor \leq g(n)$, then $\gamma_{F_7}(F) \leq f(n)$. Let $D = \{w \in Y \mid w$ is adjacent in F to a vertex of X_1 and to a vertex of $X_2\}$ (possibly, $D = \emptyset$), and let

$$A = \left(\bigcup_{v \in X_1} V(T_v) \right) - (D \cup S) \text{ and } B = \left(\bigcup_{v \in X_2} V(T_v) \right) - (D \cup S).$$

Every vertex of A is an external private neighbor of some vertex of S_1 and every vertex of B is an external private neighbor of some vertex of S_2. Thus A and B are disjoint.

Let $|A| = a$, $|B| = b$, $|C| = c$, and $|D| = d$. Note that $V - S = A \cup B \cup C \cup D$, and so $n = s + a + b + c + d$.

We say that $v \in C$ is **good** relative to S_1 and S_2 if v has one of its associated vertices in S_1 and the other in S_2, and **bad** otherwise. Hence if S_1 and S_2 is a random balanced partition of S, then the probability that v is good is

$$\left(\left\lceil\frac{s}{2}\right\rceil \cdot \left\lfloor\frac{s}{2}\right\rfloor\right) / \binom{s}{2}.$$

Let $k = \lfloor (s+1)/2 \rfloor$. Thus, $s = 2k$ or $s = 2k - 1$ (depending on the parity of s). Then the probability that v is good is $k/(2k-1)$. Hence the expected number of good vertices in C relative to S_1 and S_2 is $kc/(2k-1)$. Now among all balanced partitions of S, we choose a balanced partition S_1 and S_2 with the maximum number of good vertices.

Let C_g denote the set of all good vertices of C (relative to our partition S_1 and S_2) and let C_b denote the set of all bad vertices of C. Let $|C_g| = c_g$ and $|C_b| = c_b$. Then, $c_b + c_g = c$. Furthermore, by our choice of the partition S_1 and S_2,

$$c_g \geq \left(\frac{k}{2k-1}\right) c \quad \text{and} \quad c_b \leq \left(\frac{k-1}{2k-1}\right) c.$$

We proceed further by proving the following claim.

Claim 3.13 $\gamma_{F_7}(F) \leq \frac{1}{2}\left(n - \frac{c}{2k-1}\right)$.

Proof. By construction, every edge of F that joins a vertex of C and a vertex of S belongs to a triangle of F that contains two vertices of S, while every edge of F that joins two vertices of S belongs to a triangle that contains a vertex of C. In particular, each vertex of S that is isolated in $G[S]$ is adjacent to no vertex of C. For $i = 1, 2$, let S_i' be the set of vertices of S_i that are isolated in $G[S]$. Let A' be the set of vertices of A that are adjacent in F to a vertex in S_1', and let B' be the set of vertices of B that are adjacent in F to a vertex in S_2'. Since $|\mathrm{epn}(v,S)| \geq 1$ for every $v \in S$, $|A'| \geq |S_1'|$ and $|B'| \geq |S_2'|$. Now the set $S_1 \cup S_2' \cup C_b \cup (B - B')$ is an F_7-coloring of F, and so $\gamma_{F_7}(F) \leq k + c_b + b + |S_2'| - |B'| \leq k + c_b + b$. Further, the set $S_2 \cup S_1' \cup C_b \cup (A - A')$ is an F_7-coloring of F, and so $\gamma_{F_7}(F) \leq \lfloor s/2 \rfloor + c_b + a + |S_1'| - |A'| \leq \lfloor s/2 \rfloor + c_b + a$.

Suppose s is even. Then, $\gamma_{F_7}(F) \leq \min(k + c_b + b, k + c_b + a)$. By symmetry, we may assume that $a \geq b$. Thus,

$$\begin{aligned}
\gamma_{F_7}(F) &\leq k + c_b + b \\
&\leq k + \left(\frac{k-1}{2k-1}\right) c + b \\
&\leq \frac{1}{2}\left(2k + 2b + c - \frac{c}{2k-1}\right) \\
&\leq \frac{1}{2}\left(s + a + b + c + d - \frac{c}{2k-1} - d\right) \\
&= \frac{1}{2}\left(n - \frac{c}{2k-1} - d\right).
\end{aligned}$$

Suppose s is odd. Then, $\gamma_{F_7}(F) \leq \min(k + c_b + b, k - 1 + c_b + a)$. If $b \leq a - 1$, then,

$$\begin{aligned}
\gamma_{F_7}(F) &\leq k + c_b + b \\
&\leq \frac{1}{2}\left(2k + 2b + c - \frac{c}{2k-1}\right) \\
&\leq \frac{1}{2}\left(2k + (a-1) + b + c + d - \frac{c}{2k-1} - d\right) \\
&= \frac{1}{2}\left(n - \frac{c}{2k-1} - d\right).
\end{aligned}$$

On the other hand, if $a \leq b$ (and still s is odd), then since $\gamma_{F_7}(F) \leq k - 1 + c_b + a$, it can readily be established that

$$\gamma_{F_7}(F) < \frac{1}{2}\left(n - \frac{c}{2k-1} - d\right).$$

Since $d \geq 0$, the desired upper bound follows. □

We now return to the proof of Theorem 3.12. Since $c \geq 1$, it follows from Claim 3.13 that $\gamma_{F_7}(F) < n/2$.

Suppose further that $2\lfloor (s+1)/2 \rfloor \leq g(n)$, i.e., suppose that $2k \leq g(n)$. If $c/(2k-1) \geq g(n)$, then it follows from Claim 3.13 that $\gamma_{F_7}(F) \leq f(n)$, as desired. Thus we may assume that $c/(2k-1) < g(n)$, i.e., $c < (2k-1)g(n)$. The set $S \cup C$ is an F_7-coloring of F, and so

$$\gamma_{F_7}(F) \leq s + c < 2k + (2k-1)g(n) \leq (g(n))^2 = f(n),$$

as desired. □

We close this chapter with the following.

Conjecture 3.14 *If G is a graph of order n in which every vertex is in a triangle, then*

$$\gamma_{F_7}(G) \leq \frac{n}{2} - \frac{1}{8}\left(\sqrt{8n+1} - 1\right).$$

As shown in Theorem 3.12, Conjecture 3.14 is true for graphs with small domination number relative to their order. If Conjecture 3.14 is true, then the upper bound is sharp as may be seen as follows. For $t \geq 2$ even, let G be the graph of order $n = t + \binom{t}{2}$ obtained from a complete graph K_t on t vertices by adding a new vertex adjacent to each pair of vertices in the complete graph K_t. Then G has $t+1$ different minimal F_7-colorings (where an F_7-coloring is minimal if no proper subset of the red vertices produces an F_7-coloring) up to isomorphism, and depending on how many vertices of K_t are colored red. For $0 \leq x \leq t$, let

$$h(x) = x + \binom{x}{2} + \binom{t-x}{2}.$$

Then a minimal F_7-coloring that colors exactly x vertices of K_t red colors exactly $h(x)$ vertices of G red. A straightforward calculus argument shows that if x is a real number, then $h(x)$ is minimized when $x = (t-1)/2$. Hence, since x is an integer and t is even, and since $h(x)$ is a quadratic in x, $h(x)$ is minimized when x is the nearest integer to $(t-1)/2$, i.e., when $x = (t-2)/2$ or $x = t/2$. Thus since $h((t-2)/2) = h(t/2) = t^2/4$,

$$\gamma_{F_7}(G) = \frac{t^2}{4} = \frac{n}{2} - \frac{1}{8}\left(\sqrt{8n+1} - 1\right).$$

Chapter 4

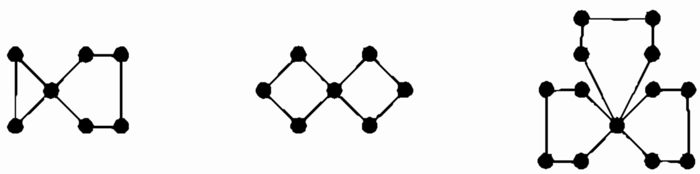
STRATIFIED GRAPHS WITH MINIMUM DEGREE TWO

4.1 Introduction

In this chapter we continue the study of the F_3-domination number of a graph by considering connected graphs with minimum degree at least 2. We show that $\gamma_{F_3}(G) \leq (n-1)/2$ for such graphs, where n is their order. Indeed, we show that $\gamma_{F_3}(G) \leq (n-1)/2$ except for five exceptional graphs. The proof rests on what we call the F_3-minimal graphs. We also characterize connected graphs of sufficiently large order with maximum possible F_3-domination number.

We will refer to a graph G as an **F_3-minimal graph** if G is edge-minimal with respect to satisfying the following three conditions: (i) $\delta(G) \geq 2$, (ii) G is connected, and (iii) $\gamma_{F_3}(G) \geq (n-1)/2$, where n is the order of G. To achieve our aims, we characterize F_3-minimal graphs. To do this, we define four families of graphs.

A *daisy* with $k \geq 2$ *petals* is a connected graph that can be constructed from $k \geq 2$ disjoint cycles by identifying a set of k vertices, one from each cycle, into one vertex. In particular, if the k cycles have lengths $n_1, n_2, \ldots, n_k$, we denote the daisy by $D(n_1, n_2, \ldots, n_k)$. Further, if $n_1 = n_2 = \cdots = n_k$, then we write $D(n_1, n_2, \ldots, n_k)$ simply as $D_k(n_1)$. The daisies $D(3,5)$, $D(4,4)$ and $D_3(5) = D(5,5,5)$ are shown in Figure 4.1.



$D(3,5)$ $D(4,4)$ $D_3(5)$

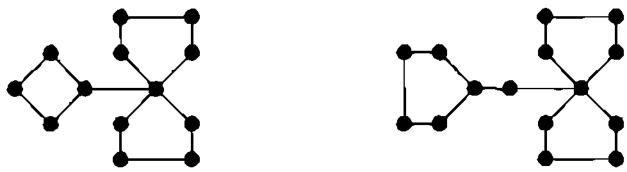
Figure 4.1: The daisies $D(3,5)$, $D(4,4)$ and $D_3(5)$.

For integers $n_1 \geq n_2 \geq 3$ and $k \geq 0$, we define a *dumbbell* $D_b(n_1, n_2, k)$ to be the graph of order $n = n_1 + n_2 + k$ obtained from the cycles C_{n_1} and C_{n_2} by joining a vertex of C_{n_1} to a vertex of C_{n_2} and subdividing the resulting edge k times. The dumb-bells $D_b(5,4,0)$ and $D_b(5,5,1)$ are shown in Figure 4.2.

$D_b(5,4,0)$ $D_b(5,5,1)$

Figure 4.2: The dumb-bells $D_b(5,4,0)$ and $D_b(5,5,1)$.

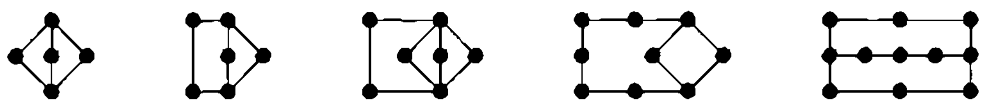
Let $A_1(4) = D_b(5,4,0)$ and $A_1(5) = D_b(5,5,1)$ be the two dumb-bells shown in Figure 4.2. For $k \geq 2$, let $A_k(4)$ be the graph obtained from a daisy $D_k(5)$ by adding a 4-cycle and joining the central vertex of the daisy to a vertex of the added 4-cycle. The graph $A_2(4)$ is shown in Figure 4.3(a). For $k \geq 2$, let $A_k(5)$ be the graph obtained from a daisy $D_k(5)$ by adding a 5-cycle and then adding a new vertex and joining it to the central vertex of the daisy and to a vertex of the added 5-cycle. The graph $A_2(5)$ is shown in Figure 4.3(b). Let $\mathcal{A} = \{A_k(4) \mid k \geq 1\} \cup \{A_k(5) \mid k \geq 1\}$.



(a) $A_2(4)$ (b) $A_2(5)$

Figure 4.3: The graphs $A_2(4)$ and $A_2(5)$ in the family $\mathcal{A}$.

Let $\mathcal{B} = \{B_1, B_2, B_3, B_4, B_5\}$ where B_1, B_2, B_3, B_4 and B_5 are the five graphs shown in Figure 4.4. We call each graph in the family $\mathcal{B}$ a *bad graph*.



B_1 B_2 B_3 B_4 B_5

Figure 4.4: The five bad graphs B_1, B_2, B_3, B_4 and B_5.

Next we define a subfamily $\mathcal{C}$ of cycles and a subfamily $\mathcal{D}$ of daisies by

$$\mathcal{C} = \{C_3, C_4, C_5, C_7, C_8, C_{11}\}$$

and

$$\mathcal{D} = \{D_k(5) \mid k \geq 2\} \cup \{D(3,5), D(4,4)\}.$$

4.2 Main Results

The following result, a proof of which is given in Section 4.4, characterizes F_3-minimal graphs.

Theorem 4.1 *A graph G is an F_3-minimal graph if and only if $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$.*

Let H_1 (respectively, H_2) be the graph obtained from C_8 (resp., C_{11}) by adding an edge joining two vertices at distance four apart on the cycle. The graphs H_1 and H_2 are shown in Figure 4.5.

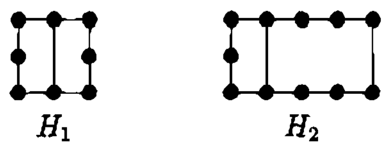


Figure 4.5: The graphs H_1 and H_2.

As a consequence of Theorem 4.1 we have our first main result, a proof of which is given in Section 4.5.

Theorem 4.2 *If G is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma_{F_3}(G) \leq (n-1)/2$ unless $G \in \{B_2, C_4, C_8, H_1\}$, in which case $\gamma_{F_3}(G) = n/2$, or $G = C_5$, in which case $\gamma_{F_3}(G) = (n+1)/2$.*

Our second main result provides a characterization of connected graphs with minimum degree at least two and order at least nine that have maximum possible F_3-domination number. A proof of Theorem 4.3 is given in Section 4.6.

Theorem 4.3 *If G is a connected graph of order $n \geq 9$ with $\delta(G) \geq 2$, then $\gamma_{F_3}(G) \leq (n-1)/2$ with equality if and only if $G \in \mathcal{A} \cup (\mathcal{D} - \{D(3,5), D(4,4)\})$ or $G \in \{B_4, B_5, C_{11}, H_2\}$.*

4.3 Preliminary Results

Our aim in this section is to establish some preliminary results that we will need later when proving our main results. We begin with the following observation, a proof of which is presented in Subsection 4.3.1.

Observation 4.4 *Let G be a connected graph with $\delta(G) \geq 2$ and let F be obtained from G by subdividing any edge four times. Then, $\gamma_{F_3}(F) \leq \gamma_{F_3}(G) + 2$.*

Next we establish the value of $\gamma_{F_3}(C_n)$ for a cycle C_n. A proof of Proposition 4.5 is presented in Subsection 4.3.2.

Proposition 4.5 *For $n \geq 3$, $\gamma_{F_3}(C_n) = \lceil (n-1)/3 \rceil + \lceil n/3 \rceil - \lfloor n/3 \rfloor$.*

Equivalently, Proposition 4.5 states that $\gamma_{F_3}(C_n) = \lceil n/3 \rceil + 1$ if $n \equiv 2 \pmod 3$ and $\gamma_{F_3}(C_n) = \lceil n/3 \rceil$ otherwise. For an example of a γ_{F_3}-coloring of an n-cycle $C_n\colon v_1, v_2, \ldots, v_n, v_1$, let $R = \{v_i \mid i \equiv 1 \pmod 3\}$, and so $|R| = \lceil n/3 \rceil$. If $n \equiv 2 \pmod 3$, then coloring the vertices of $R \cup \{v_n\}$ red and coloring all other vertices blue produces an F_3-coloring of C_n. If $n \not\equiv 2 \pmod 3$, then coloring the vertices of R red and coloring all other vertices blue produces an F_3-coloring of C_n. As an immediate consequence of Proposition 4.5 we can characterize the F_3-minimal graphs that are cycles.

Corollary 4.6 *A cycle G is an F_3-minimal graph if and only if $G \in \mathcal{C}$.*

Next we characterize the F_3-minimal graphs that are daisies. A proof of Proposition 4.7 is presented in Subsection 4.3.3.

Proposition 4.7 *If G is a daisy of order n, then $\gamma_{F_3}(G) \leq (n-1)/2$. Furthermore, $\gamma_{F_3}(G) = (n-1)/2$ if and only if $G = D(3,5)$, $G = D(4,4)$ or $G = D_k(5)$ for some $k \geq 2$.*

As an immediate consequence of Proposition 4.7 we can characterize the F_3-minimal graphs that are daisies.

Corollary 4.8 *A daisy G is an F_3-minimal graph if and only if $G \in \mathcal{D}$.*

Next we characterize the F_3-minimal graphs that are dumb-bells. We begin with the following result, a proof of which is presented in Subsection 4.3.4.

Proposition 4.9 *If G is a dumbbell of order n, then $\gamma_{F_3}(G) \leq (n-1)/2$ with equality if and only if $G \in \{A_1(4), A_1(5)\}$.*

Corollary 4.10 *A dumbbell G is an F_3-minimal graph if and only if $G \in \{A_1(4), A_1(5)\}$.*

The following two observations about graphs in the families $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ will prove to be useful.

Observation 4.11 *Let $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ have order n. Then, G is a connected graph with $\delta(G) = 2$, and*

$$\gamma_{F_3}(G) = \begin{cases} \dfrac{n+1}{2} & \text{if } G = C_5, \\[2ex] \dfrac{n}{2} & \text{if } G \in \{B_2, C_4, C_8\}, \\[2ex] \dfrac{n-1}{2} & \text{otherwise.} \end{cases}$$

Corollary 4.12 *Each graph in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ is an F_3-minimal graph.*

Observation 4.13 *Let $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$. Then for any vertex v of G, there is a minimum F_3-coloring in which v is colored blue and in which every blue vertex is adjacent to a red vertex. Further for any vertex v of G, except for the central vertex of a daisy $D_k(5)$, there is a minimum F_3-coloring of G in which v is colored red.*

We close our preliminary results with a characterization of F_3-minimal graphs of small order. A proof of Lemma 4.14 is presented in Subsection 4.3.5.

Lemma 4.14 *If G is an F_3-minimal graph of order n, $3 \leq n \leq 6$, then $G \in \{B_1, B_2, C_3, C_4, C_5\}$.*

4.3.1 Proof of Observation 4.4

Let uv be the edge of G that is subdivided four times to obtain the graph F, and let u, u_1, u_2, u_3, u_4, v be the resulting path in F. Any minimum F_3-coloring of G can be extended to an F_3-coloring of F as follows. If both u and v are colored red, then color u_1 and u_4 red and u_2 and u_3 blue. If exactly one of u and v, say u, is colored red, then color u_3 and u_4 red and u_1 and u_2 blue. Suppose both u and v are colored blue. If each of u and v has a neighbor colored red, then color u_2 and u_3 red and u_1 and u_4 blue. If exactly one of u and v, say u, has a neighbor colored red, then color u_2 red, color u_1, u_3 and u_4 blue, and recolor v red (note that in any F_3-coloring of a graph H that colors a vertex w and all its neighbors blue, we can recolor w red and leave all other vertices unchanged to produce a new F_3-coloring of H). If neither u nor v has a neighbor colored red, then color u_1 and u_4 red and u_2 and u_3 blue. In all cases, we produce an F_3-coloring of F that colors exactly two more vertices red than does the original F_3-coloring of G. It follows that $\gamma_{F_3}(F) \leq \gamma_{F_3}(G) + 2$.

4.3.2 Proof of Proposition 4.5

We proceed by induction on the order n of a cycle C_n. The result is straightforward to verify for $n \in \{3, 4, 5\}$. Assume then that $n \geq 6$ and that the result of the proposition is true for all cycles on fewer than n vertices. Consider a cycle $C: v_1, v_2, \ldots, v_n, v_1$. Let $\mathcal{C}$ be a γ_{F_3}-coloring of C. Since every blue vertex is rooted at a copy of F_3, every blue vertex on the cycle is adjacent to a red vertex and a blue vertex. Renaming if necessary, we may assume that $\mathcal{C}$ colors v_2 and v_3 blue, and therefore colors v_1 and v_4 red. Let C' be the cycle obtained from C by deleting the vertices v_1, v_2 and v_3 and adding the edge v_4v_n, i.e., $C' = (C - \{v_1, v_2, v_3\}) \cup \{v_4v_n\}$.

Then C' is a cycle of order $n-3 \geq 3$ and the restriction of $\mathcal{C}$ to C' is an F_3-coloring of C' that colors $\gamma_{F_3}(C)-1$ vertices red. Hence, $\gamma_{F_3}(C') \leq \gamma_{F_3}(C)-1$. On the other hand, any γ_{F_3}-coloring $\mathcal{C}'$ of C' can be extended to a γ_{F_3}-coloring of C by coloring exactly one of v_1, v_2 and v_3 red: If $\mathcal{C}'$ colors v_n and v_4 blue, then color v_2 red and v_1 and v_3 blue. If $\mathcal{C}'$ colors v_n red and v_4 blue or if $\mathcal{C}'$ colors v_n and v_4 red, then color v_1 and v_2 blue and v_3 red. If $\mathcal{C}'$ colors v_n blue and v_4 red, then color v_1 red and v_2 and v_3 blue. Thus, $\gamma_{F_3}(C) \leq \gamma_{F_3}(C')+1$. Consequently, $\gamma_{F_3}(C) = \gamma_{F_3}(C')+1$. Since $C \cong C_n$ and $C' \cong C_{n-3}$, the result now follows by applying the inductive hypothesis to the cycle C'.

4.3.3 Proof of Proposition 4.7

We proceed by induction on the order $n \geq 5$ of a daisy G to show that $\gamma_{F_3}(G) \leq (n-1)/2$. If $n = 5$, then $G = D(3,3)$ and $\gamma_{F_3}(G) = 1 < (n-1)/2$, while if $n = 6$, then $G = D(3,4)$ and $\gamma_{F_3}(G) = 2 < (n-1)/2$. This establishes the base cases. Assume, then, that $n \geq 7$ and that if G' is a daisy of order $n' < n$, then $\gamma_{F_3}(G') \leq (n'-1)/2$. Let G be a daisy of order n with $k \geq 2$ petals.

Suppose first that $k = 2$. Let $G = D(n_1+1, n_2+1)$, and so $n = n_1+n_2+1$. Let v denote the vertex of degree 4 in G and let F_1 and F_2 denote the two cycles passing through v, where $F_i \cong C_{n_i+1}$ for $i = 1, 2$. Let F_1 be the cycle $v, v_1, v_2, \ldots, v_{n_1}, v$ and let F_2 be the cycle $v, u_1, u_2, \ldots, u_{n_2}, v$. We consider four possibilities.

Case 1. $n_i \equiv 2 \pmod 3$ for $i = 1$ or $i = 2$.

We may assume $n_1 \equiv 2 \pmod 3$. Let $R_1 = \{v_i \mid i \equiv 0 \pmod 3\}$, and so $|R_1| = (n_1 - 2)/3$. Let C_2 be a γ_{F_3}-coloring of F_2 that colors v red. By Proposition 4.5, if $n_2 \not\equiv 1 \pmod 3$, then C_2 colors at most $(n_2 + 3)/3$ vertices red, while if $n_2 \equiv 1 \pmod 3$, then C_2 colors $(n_2 + 5)/3$ vertices red. We can extend C_2 to an F_3-coloring C of G by coloring the vertices in R_1 red and all remaining uncolored vertices of F_1 blue. If $n_2 \not\equiv 1 \pmod 3$, then C colors at most $(n_1 - 2)/3 + (n_2 + 3)/3 = n/3 < (n-1)/2$ vertices red. If $n_2 \equiv 1 \pmod 3$, then C colors $(n_1 - 2)/3 + (n_2 + 5)/3 = (n+2)/3 \leq (n-1)/2$ vertices red with strict inequality if $n > 7$. Hence, $\gamma_{F_3}(G) < (n-1)/2$ unless $G = D(3,5)$, in which case $\gamma_{F_3}(G) = 3 = (n-1)/2$.

Case 2. $n_i \equiv 0 \pmod 3$ for $i = 1, 2$.

Let $R_1 = \{v_i \mid i \equiv 1 \pmod 3\}$, and so $|R_1| = n_1/3$. Let C_2 be a γ_{F_3}-coloring of F_2 that colors v red. By Proposition 4.5, C_2 colors $(n_2 + 3)/3$ vertices red. We can extend C_2 to an F_3-coloring C of G by coloring the vertices in R_1 red and all remaining uncolored vertices of F_1 blue. Then, C colors $(n_1 + n_2 + 3)/3 = (n+2)/3 \leq (n-1)/2$ vertices red with strict inequality if $n > 7$. Hence, $\gamma_{F_3}(G) < (n-1)/2$ unless $G = D(4,4)$, in which case $\gamma_{F_3}(G) = 3 = (n-1)/2$.

Case 3. $n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 1 \pmod 3$.

Then, $n \geq 8$. Let $R_1 = \{v_i \mid i \equiv 2 \pmod 3\}$, and so $|R_1| = n_1/3$. Let $R_2 = \{u_i \mid i \equiv 1 \pmod 3\}$, and so $|R_2| = (n_2 + 2)/3$. Then coloring the vertices in $R_1 \cup R_2$ red and all remaining uncolored vertices blue produces an F_3-coloring of G

that colors $(n_1 + n_2 + 2)/3 = (n + 1)/3 < (n - 1)/2$ vertices red.

Case 4. $n_i \equiv 1 \pmod 3$ for $i = 1, 2$.

Then, $n \geq 9$. Let $R_1 = \{v_i \mid i \equiv 1 \pmod 3\}$, and so $|R_1| = (n_1 + 2)/3$. Let $R_2 = \{u_i \mid i \equiv 2 \pmod 3\} \cup \{u_{n_2-1}\}$, and so $|R_2| = (n_2 + 2)/3$. Then coloring the vertices in $R_1 \cup R_2$ red and all remaining uncolored vertices blue produces an F_3-coloring of G that colors $(n_1 + n_2 + 4)/3 = (n + 3)/3 \leq (n - 1)/2$ vertices red with strict inequality if $n > 9$. Hence, $\gamma_{F_3}(G) < (n - 1)/2$ unless $G = D(5, 5)$, in which case $\gamma_{F_3}(G) = 4 = (n - 1)/2$.

Hence we may assume $k \geq 3$. Let v denote the vertex of degree $2k$ in G, and let $F_1, F_2, \cdots, F_k$ denote the k cycles passing through v, where $F_i \cong C_{n_i+1}$ for $i = 1, 2, \ldots, k$. Thus, $n = 1 + \sum_{i=1}^{k} n_i$. Let F_1 be the cycle $v, v_1, v_2, \ldots, v_{n_1}, v$.

Let $G' = D(n_2, \ldots, n_k)$. Then, G' is a daisy of order $n' = n - n_1$. Applying the inductive hypothesis to G', $\gamma_{F_3}(G') \leq (n' - 1)/2 = (n - n_1 - 1)/2$. Let C' be a γ_{F_3}-coloring of G'. Note that if C' colors v blue, then v must be adjacent to at least one vertex colored red under C'. We extend C' to an F_3-coloring of G as follows. If $n_1 \equiv 2 \pmod 3$, let $R = \{v_i \mid i \equiv 0 \pmod 3\}$, and so $|R| = (n_1 - 2)/3$. If $n_1 \equiv 0 \pmod 3$ and v is colored blue in C', let $R = \{v_i \mid i \equiv 2 \pmod 3\}$, and so $|R| = n_1/3$. In all other cases, let $R = \{v_i \mid i \equiv 1 \pmod 3\}$, and so $|R| = \lceil n_1/3 \rceil$. Then, C' can be extended to an F_3-coloring C of G by coloring the vertices in R red and all remaining uncolored vertices of F_1 blue. If $n_1 \equiv 2 \pmod 3$, then C colors at most $|R| + (n' - 1)/2 = (n_1 - 2)/3 + (n - n_1 - 1)/2 < (n - 1)/2$ vertices red. If $n_1 \equiv 0 \pmod 3$, then C colors at most $n_1/3 + (n - n_1 - 1)/2 < (n - 1)/2$ vertices red. If $n_1 \equiv 1 \pmod 3$, then C colors at most $(n_1 + 2)/3 + (n - n_1 - 1)/2 = (3n - n_1 + 1)/6 \leq (n - 1)/2$ vertices red with strict inequality if $n_1 > 4$. Hence in all cases, C colors strictly less than $(n - 1)/2$ vertices red unless $n_1 = 4$ (and so,

$F_1 = C_5$) and $\gamma_{F_3}(G') = (n'-1)/2$. An identical argument shows that if $n_i \neq 4$ for some i, $1 \leq i \leq k$, then there is an F_3-coloring of G that colors strictly less than $(n-1)/2$ vertices red. Thus we have shown that $\gamma_{F_3}(G) < (n-1)/2$ unless $G = D_k(5)$, in which case $\gamma_{F_3}(G) \leq (n-1)/2$.

We show next that $\gamma_{F_3}(G) = (n-1)/2$ if and only if $G = D(3,5)$, $G = D(4,4)$ or $G = D_k(5)$ for some $k \geq 2$. The result is proven if G is a daisy with two petals. Hence we may assume G has at least three petals. If $\gamma_{F_3}(G) = (n-1)/2$, then we have shown that $G = D_k(5)$. Conversely, suppose $G = D_k(5)$. Then G has order $n = 4k+1$ and any F_3-coloring of G colors at least two vertices from each F_i, $1 \leq i \leq k$, red, and so $\gamma_{F_3}(G) \geq 2k = (n-1)/2$. On the other hand, if v denotes the vertex of degree $2k$ in G and if v, v_1, v_2, v_3, v_4, v denotes a 5-cycle in G, then coloring all vertices in $(N[v] - \{v_1, v_4\}) \cup \{v_2, v_3\}$ blue and all remaining vertices red, produces an F_3-coloring of G that colors exactly $2k = (n-1)/2$ vertices red, and so $\gamma_{F_3}(G) \leq (n-1)/2$. Consequently, $\gamma_{F_3}(G) = (n-1)/2$.

4.3.4 Proof of Proposition 4.9

We proceed by induction on the order $n \geq 6$. If $n = 6$, then $G = D_b(3,3,0)$ and $\gamma_{F_3}(G) = 2 = (n-2)/2$. Let $n \geq 7$, and assume that the result is true for all dumb-bells of order n', where $n' < n$. Let $G = D_b(n_1, n_2, k)$ be a dumbbell of order $n = n_1 + n_2 + k$. Suppose G contains a path on six vertices each internal vertex of which has degree 2 in G and whose end-vertices, say u and v, are not adjacent. Let G' be the graph obtained from G by removing the four internal vertices of this path and adding the edge uv. Then, G' is a dumbbell of order $n' = n - 4$. By Observation 4.4, $\gamma_{F_3}(G) \leq \gamma_{F_3}(G') + 2$. Applying the inductive hypothesis to G', $\gamma_{F_3}(G') \leq (n'-1)/2$. If $\gamma_{F_3}(G') < (n'-1)/2$, then $\gamma_{F_3}(G) < (n-1)/2$. On the other hand, if $\gamma_{F_3}(G') = (n'-1)/2$, then by the inductive hypothesis, $G' \in \{A_1(4), A_1(5)\}$. Now G is obtained from G' by subdividing the edge uv of G' four times. Irrespective of whether the edge uv is a cycle edge or a bridge of G', it is straightforward to check that $\gamma_{F_3}(G) \le (n-3)/2$. Hence we may assume that G contains no path on six vertices each internal vertex of which has degree 2 in G and whose end-vertices are not adjacent, for otherwise $\gamma_{F_3}(G) < (n-1)/2$. With this assumption, $3 \le n_1, n_2 \le 6$ and $0 \le k \le 3$. It is now a simple exercise to check that $\gamma_{F_3}(G) \le (n-1)/2$ with equality if and only if $(n_1, n_2, k) \in \{(5,4,0), (5,5,1)\}$.

4.3.5 Proof of Lemma 4.14

Let $G = (V, E)$. Let u be a vertex of maximum degree in G. If $n \in \{3,4\}$, then $G = C_n$. Suppose $n = 5$. If $\Delta(G) = 4$, then coloring u red and coloring all other vertices blue produces an F_3-coloring of G, and so $\gamma_{F_3}(G) = 1 < (n-1)/2$, a contradiction. If $\Delta(G) = 3$, then it follows from Observation 4.11 that $G = B_1$. If $\Delta(G) = 2$, then $G = C_5$.

Suppose $n = 6$. If $\Delta(G) = 2$, then $\gamma_{F_3}(G) \le 2 < (n-1)/2$, a contradiction. If $\Delta(G) = 4$, let $V - N[u] = \{v\}$. Then, u and v have at least two common neighbors. Coloring v and any neighbor of u red and coloring all other vertices blue produces an F_3-coloring of G, and so $\gamma_{F_3}(G) \le 2 < (n-1)/2$, a contradiction. If $\Delta(G) = 5$, then coloring u red and coloring all other vertices blue produces an F_3-coloring of G, and so $\gamma_{F_3}(G) = 1 < (n-1)/2$, a contradiction. Hence, $\Delta(G) = 3$. Let $V - N[u] = \{v, w\}$. If v and w have a common neighbor x, let $y \in N(u) - \{x\}$. Coloring v and y red and coloring all other vertices blue produces an F_3-coloring of G, and so $\gamma_{F_3}(G) \le 2 < (n-1)/2$, a contradiction. Hence, v and w have no common neighbor, whence $G = B_2$.

4.4 Proof of Theorem 4.1

The sufficiency follows from Corollary 4.12. To prove the necessary, we proceed by induction on the order $n \geq 3$ of an F_3-minimal graph. By Lemma 4.14, the result is true for $n \leq 6$. Let $n \geq 7$, and assume that the result is true for all F_3-minimal graphs of order less than n. Let $G = (V, E)$ be an F_3-minimal graph of order n. If e is an edge of G, then $\gamma_{F_3}(G - e) \geq \gamma_{F_3}(G)$. Hence, by the minimality of G, we have the following observation.

Observation 4.15 *If $e \in E$, then either e is a bridge of G or $\delta(G - e) = 1$.*

Since the F_3-domination number of a graph cannot decrease if edges are removed, the next result is a consequence of the inductive hypothesis.

Observation 4.16 *If G' is a connected subgraph of G of order $n' < n$ with $\delta(G') \geq 2$, then either $G' \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(G') < (n' - 1)/2$.*

The following observation will prove useful.

Observation 4.17 *Let G' be a graph and let v be a vertex of G' all of whose neighbors have degree at most 2 in G'. Then in any F_3-coloring of G', at least one vertex in $N[v]$ is colored red.*

Proof. If every vertex in $N[v]$ is colored blue in some F_3-coloring of G', then there must be a red vertex at distance 2 from v. But then the neighbor of v that is adjacent to such a red vertex is not rooted at a copy of F_3, a contradiction. □

We now return to the proof of Theorem 4.1. If $G = C_n$, then, by Corollary 4.6, $G \in \mathcal{C}$. If G is a daisy, then by Corollary 4.8, $G \in \mathcal{D}$. If G is a dumbbell, then, by

Corollary 4.10, $G \in \{A_1(4), A_1(5)\}$. So we may assume that G **is neither a cycle, nor a daisy, nor a dumbbell.** Hence, G contains at least two vertices of degree at least 3. Let $S = \{v \in V \mid \deg v \geq 3\}$. Then, $|S| \geq 2$ and each vertex of $V - S$ has degree 2 in G.

For each $v \in S$, we define the **2-graph of** v to be the component of $G - (S - \{v\})$ that contains v. So each vertex of the 2-graph of v has degree 2 in G, except for v. Furthermore, the 2-graph of v consists of edge-disjoint cycles through v, which we call **2-graph cycles**, and paths emanating from v, which we call **2-graph paths.**

Using the inductive hypothesis, we shall prove the following lemma, a proof of which is given in Subsection 4.4.1.

Lemma 4.18 *If S is not an independent set, then $G = A_k(4)$ for some $k \geq 2$.*

By Lemma 4.18, we may assume that **S is an independent set**, for otherwise $G \in \mathcal{A}$. Let u and v be two vertices of S that are joined by a path $u, u_1, \ldots, u_m, v$ every internal vertex of which has degree 2 in G. By assumption, $d(u, v) \geq 2$, whence $m \geq 1$. If m is large, then the following result, a proof of which is presented in Subsection 4.4.2, shows that $G = B_4$.

Lemma 4.19 *If $m \geq 4$, then $G = B_4$.*

By Lemma 4.19, we may assume that **every 2-graph path has length at most 3**. In particular, $m \leq 3$. Let P denote the path $u_1, \ldots, u_m$. Let $H = G - V(P)$. Then, H has order $n' = n - m$ and $\delta(H) \geq 2$. Possibly, H is disconnected in which case H has two components, one containing u and the other v. Since S is an independent set, we observe that each neighbor of a vertex of S has degree 2 in G. In particular each neighbor of u and v in H has degree 2. Thus any F_3-coloring of H that colors u (respectively, v) blue must color at least one neighbor of u (respectively, v) red. A proof of the following lemma is given in Subsection 4.4.3.

Lemma 4.20 *If H is disconnected, then $G = A_k(5)$ for some $k \geq 2$.*

By Lemma 4.20, we may assume that removing the vertices in $V - S$ from any 2-graph path in G produces a connected graph, for otherwise $G \in \mathcal{A}$. In particular, **H is connected.**

In what follows, for each vertex $u \in S$, let $G_u = G - N[u]$. A proof of Lemma 4.21 is given in Subsections 4.4.4, 4.4.5 and 4.4.6.

Lemma 4.21 *If every 2-graph path has length exactly 1, then:*

(a) There is no 2-graph cycle in G.

(b) If $u, v \in S$, then $N(u) \not\subseteq N(v)$.

(c) $\delta(G_u) = 1$ for every $u \in S$.

A proof of Lemma 4.22 is given in Subsection 4.4.7.

Lemma 4.22 *At least one 2-graph path has length 2 or 3.*

By Lemma 4.22, we may assume that $m \in \{2, 3\}$. By Observation 4.16, $H \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(H) < (n' - 1)/2$. Let $\mathcal{C}_H$ be a minimum F_3-coloring of H. A proof of the following two lemmas are given in Subsections 4.4.8 and 4.4.9.

Lemma 4.23 *If $m = 3$, then $G = B_5$.*

Lemma 4.24 *If $m = 2$, then $G = B_3$.*

This completes the proof of Theorem 4.1.

4.4.1 Proof of Lemma 4.18

Let $e = uv$ be an edge, where $u, v \in S$. By Observation 4.15, e must be a bridge of G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components of $G - e$ where $u \in V_1$. For $i = 1, 2$, let $|V_i| = n_i$. Each G_i satisfies $\delta(G_i) \geq 2$ and is connected. Hence by Observation 4.16, $G_i \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(G_i) < (n_i - 1)/2$ for $i = 1, 2$. If $\gamma_{F_3}(G_i) < (n_i - 1)/2$ for $i = 1, 2$, then $\gamma_{F_3}(G) \leq \gamma_{F_3}(G_1) + \gamma_{F_3}(G_2) < (n-1)/2$, a contradiction. Hence at least one of G_1 and G_2, say G_1, must belong to $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$.

Claim 4.25 $G_1 \neq C_5$.

Proof. Suppose $G_1 = C_5$. By assumption, G is not a dumbbell, and so G_2 is not a cycle. Thus if $\gamma_{F_3}(G_2) \geq n_2/2$, then, by Observation 4.11, $G_2 = B_2$. But then G is not an F_3-minimal graph (either we contradict Observation 4.15 or $\gamma_{F_3}(G) < (n-1)/2$), a contradiction. Thus, $\gamma_{F_3}(G_2) \leq (n_2 - 1)/2$.

Suppose $\gamma_{F_3}(G_2) = (n_2 - 1)/2$ (and still G_2 is not a cycle). Then, $G_2 \in (\mathcal{A} \cup \mathcal{B} \cup \mathcal{D}) - \{B_2\}$. Suppose $G_2 = D_k(5)$ for some $k \geq 2$ and v is the central vertex of G_2. Then a minimum F_3-coloring of G_2 can be extended to an F_3-coloring of G by coloring the two neighbors of u in G_1 red and the three remaining vertices of G_1 blue. Thus, $\gamma_{F_3}(G) \leq (n_2 - 1)/2 + 2 = (n-2)/2$, a contradiction. If v is not the central vertex of a daisy $D_k(5)$, then by Observation 4.13, there is a minimum F_3-coloring of G_2 in which v is colored red. Such an F_3-coloring of G_2 can be extended to an F_3-coloring of G by coloring u and its two neighbors in G_1 blue and coloring the remaining two vertices of G_1 red. Thus, $\gamma_{F_3}(G) \leq (n_2 - 1)/2 + 2 = (n-2)/2$, a contradiction. Hence, $\gamma_{F_3}(G_2) \leq (n_2 - 2)/2$.

If $\gamma_{F_3}(G_2) \leq (n_2 - 3)/2$, then $\gamma_{F_3}(G) \leq \gamma_{F_3}(G_1) + \gamma_{F_3}(G_2) \leq 3 + (n_2 - 3)/2 = (n-2)/2$, a contradiction. Hence, $\gamma_{F_3}(G_2) = (n_2 - 2)/2$. If there exists a minimum F_3-coloring of G_2 in which v or a neighbor of v is colored red, then such an F_3-coloring of G_2 can be extended to an F_3-coloring of G by coloring exactly two vertices of G_1

red, and so $\gamma_{F_3}(G) \leq (n_2 - 2)/2 + 2 = (n - 3)/2$, a contradiction. On the other hand, suppose that every minimum F_3-coloring of G_2 colors v and all its neighbors blue. Then at least one neighbor w of v in G_2 must have degree at least 3, and so $w \in S$. By Observation 4.15, the edge vw must be a bridge of G. Let H_1 and H_2 be the two components of $G - vw$ where $v \in V(H_1)$. For $i = 1, 2$, let H_i have order n'_i. Each H_i satisfies $\delta(H_i) \geq 2$ and is connected. Hence by Observation 4.16, $H_i \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(H_i) < (n'_i - 1)/2$ for $i = 1, 2$. Since $H_1 \notin \{B_2, C_4, C_5, C_8\}$, $\gamma_{F_3}(H_1) \leq (n'_1 - 1)/2$. If $H_2 \in \{B_2, C_4, C_5, C_8\}$, then there would exists a minimum F_3-coloring of G_2 in which w is colored red, contrary to our earlier assumption that there is no such minimum F_3-coloring of G_2. Hence, $\gamma_{F_3}(H_2) \leq (n'_2 - 1)/2$. Thus, $\gamma_{F_3}(G) \leq \gamma_{F_3}(H_1) + \gamma_{F_3}(H_2) < (n - 1)/2$, a contradiction. □

By Claim 4.25, $G_1 \neq C_5$. Similarly, $G_2 \neq C_5$.

Claim 4.26 $G_1 \notin \{B_2, C_8\}$.

Proof. Suppose $G_1 \in \{B_2, C_8\}$. If $G_2 \in \{B_2, C_4, C_8\}$, then G is not an F_3-minimal graph (either we contradict Observation 4.15 or $\gamma_{F_3}(G) < (n-1)/2$), a contradiction. If $\gamma_{F_3}(G_2) \leq (n_2-2)/2$, then $\gamma_{F_3}(G) \leq n_1/2 + (n_2-2)/2 < (n-1)/2$, a contradiction. Hence, $\gamma_{F_3}(G_2) = (n_2 - 1)/2$, and so $G_2 \in (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}) - \{B_2, C_4, C_5, C_8\}$. By Observation 4.13, there is a minimum F_3-coloring $\mathcal{C}_1$ of G_2 in which v is colored blue and is adjacent to a vertex colored red.

Suppose $G_1 = B_2$. If u is a vertex of degree 2 in G_1, then at least one of the two edges incident with u in G_1 joins two vertices of S but is not a bridge of G, contradicting Observation 4.15. Hence the vertex u must be a vertex of degree 3 in G_1. The F_3-coloring $\mathcal{C}_1$ of G_2 can be extended to an F_3-coloring of G as follows: Color one neighbor of u on the 4-cycle in G_1 red, color the neighbor of u in G_1 that does not belong to the 4-cycle red, and color the remaining four vertices of G_1 blue. Thus, $\gamma_{F_3}(G) \leq \gamma_{F_3}(G_2) + 2 = (n_2 - 1)/2 + 2 = (n - 3)/2$, a contradiction.

Suppose $G_1 = C_8$. The F_3-coloring C_1 of G_2 can be extended to an F_3-coloring of G by coloring the two neighbors of u in G_1 red, coloring the vertex at maximum distance 4 from u in G_1 red, and coloring the remaining five vertices of G_1 (including u) blue. Thus, $\gamma_{F_3}(G) \leq (n_2 - 1)/2 + 3 = (n - 3)/2$, a contradiction. □

By Claim 4.26, $G_1 \notin \{B_2, C_8\}$. Similarly, $G_2 \notin \{B_2, C_8\}$. If neither G_1 nor G_2 is a 4-cycle, then for $i = 1, 2$, $\gamma_{F_3}(G_i) \leq (n_i - 1)/2$, and so $\gamma_{F_3}(G) \leq \gamma_{F_3}(G_1) + \gamma_{F_3}(G_2) < (n - 1)/2$, a contradiction. Hence at least one of G_1 and G_2, say G_1, is a 4-cycle.

If $G_2 = C_4$, then $\gamma_{F_3}(G) = (n - 2)/2$, contradicting the fact that G is an F_3-minimal graph. If $\gamma_{F_3}(G_2) \leq (n_2 - 2)/2$, then $\gamma_{F_3}(G) \leq n_1/2 + (n_2 - 2)/2 < (n - 1)/2$, a contradiction. Hence, $\gamma_{F_3}(G_2) = (n_2 - 1)/2$, and so $G_2 \in (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}) - \{B_2, C_4, C_5, C_8\}$. If v is not the central vertex of a daisy $D_k(5)$, then by Observation 4.13, there is a minimum F_3-coloring of G_2 in which v is colored red. Such an F_3-coloring of G_2 can be extended to an F_3-coloring of G by coloring the vertex in G_1 at distance 2 from u with the color red and coloring the remaining three vertices of G_1 blue. Thus, $\gamma_{F_3}(G) \leq (n_2 - 1)/2 + 1 = (n - 3)/2$, a contradiction. Thus, v must be the central vertex of a daisy $D_k(5)$ for some $k \geq 2$, whence $G = A_k(4)$.

4.4.2 Proof of Lemma 4.19

Let G' be the graph obtained from G by removing the vertices u_1, u_2, u_3, u_4, and either adding the edge uu_5 if $m \geq 5$ or adding the edge uv if $m = 4$. Then, G' is a connected graph of order $n' = n - 4$ with $\delta(G') \geq 2$. By Observation 4.4, $\gamma_{F_3}(G) \leq \gamma_{F_3}(G') + 2$. If $\gamma_{F_3}(G') < (n' - 1)/2$, then $\gamma_{F_3}(G) < (n - 1)/2$, a contradiction. Hence, $\gamma_{F_3}(G') \geq (n' - 1)/2$. Since G is an F_3-minimal graph, it follows that G' is an F_3-minimal graph. Since G is neither a cycle nor a dumbbell, G' is not a cycle or a dumbbell. Further the degree of each vertex of S is unchanged in G and G', and so G' has at least two vertices of degree at least 3. Hence applying the inductive hypothesis to G', $G' \in \mathcal{A} \cup \{B_1, B_2, \ldots, B_5\}$. A straightforward check confirms that if $G' \neq B_1$, then $\gamma_{F_3}(G) < (n-1)/2$. Therefore, $G' = B_1$, whence $G = B_4$.

4.4.3 Proof of Lemma 4.20

Let H_1 and H_2 be the two components of H, where $u \in V(H_1)$. For $i = 1, 2$, let H_i have order n_i. Each H_i is a connected graph with $\delta(H_i) \geq 2$. By Observation 4.16, $H_i \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(H_i) < (n_i - 1)/2$.

Claim 4.27 $\gamma_{F_3}(H_i) \geq n_i/2$ *for* $i = 1$ *or* $i = 2$.

Proof. Suppose $\gamma_{F_3}(H_i) \leq (n_i - 1)/2$ for $i = 1, 2$. Let $\mathcal{C}_{12}$ be a minimum F_3-coloring of $H_1 \cup H_2$. Then the restriction of $\mathcal{C}_{12}$ to $V(H_i)$ is a minimum F_3-coloring of H_i for $i = 1, 2$, and so $\mathcal{C}_{12}$ colors at most $(n_1 + n_2)/2 - 1$ vertices of H red.

Suppose $m = 3$. Then, $n_1 + n_2 = n - 3$. If at least one of u and v, say u, is colored red in $\mathcal{C}_{12}$, then $\mathcal{C}_{12}$ can be extended to an F_3-coloring of G by coloring the vertex u_3 red and the vertices u_1 and u_2 blue. On the other hand, if both u and v are colored blue in $\mathcal{C}_{12}$, then $\mathcal{C}_{12}$ can be extended to an F_3-coloring of G by coloring the vertex u_2 red and the vertices u_1 and u_3 blue. Hence, $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction.

Suppose $m = 2$. Then, $n_1 + n_2 = n - 2$. If u and v are colored with the same color in $\mathcal{C}_{12}$, then $\mathcal{C}_{12}$ can be extended to an F_3-coloring of G by coloring both u_1 and u_2 blue, whence $\gamma_{F_3}(G) \leq (n-4)/2$, a contradiction. If u and v are colored with different colors in $\mathcal{C}_{12}$, say u is colored red and v blue, then $\mathcal{C}_{12}$ can be extended to an F_3-coloring of G by coloring u_1 red and u_2 blue, and so $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction. Hence, $m = 1$ and $n_1 + n_2 = n - 1$.

If u or v is colored blue in $\mathcal{C}_{12}$, then $\mathcal{C}_{12}$ can be extended to an F_3-coloring of G by coloring u_1 blue, whence $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction. Hence in every minimum F_3-coloring of H, the vertices u and v are colored red. There is therefore no minimum F_3-coloring of H_1 that colors u blue, and so it follows from Observation 4.13 that $\gamma_{F_3}(H_1) \le (n_1-2)/2$. Similarly, $\gamma_{F_3}(H_2) \le (n_2-2)/2$. Thus, $\mathcal{C}_{12}$ colors at most $(n_1+n_2-4)/2 = (n-5)/2$ vertices of H red. The coloring $\mathcal{C}_{12}$ can be extended to an F_3-coloring of G by coloring u_1 red, and so $\gamma_{F_3}(G) \le (n-3)/2$, a contradiction. □

By Claim 4.27 and Observation 4.11, we may assume that $H_1 \in \{B_2, C_4, C_5, C_8\}$. We consider each possibility in turn.

Claim 4.28 *If $H_1 = C_5$, then $G = A_k(5)$ for some $k \ge 2$.*

Proof. Since G is not a dumbbell, H_2 is not a cycle. If $H_2 = B_2$, then v must be one of the two vertices of degree 3 in B_2 and it is easy to check that for each value of $m \in \{1, 2, 3\}$, $\gamma_{F_3}(G) < (n-1)/2$, a contradiction. Hence, by Observation 4.16, $H_2 \in \mathcal{A} \cup (\mathcal{B} - \{B_2\}) \cup \mathcal{D}$ or $\gamma_{F_3}(H_2) < (n_2-1)/2$. In particular, $\gamma_{F_3}(H_2) \le (n_2-1)/2$. Let $\mathcal{C}_2$ be a minimum F_3-coloring of H_2.

Suppose $m = 3$. If v is colored red in the coloring $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring u_1 and the two vertices in H_1 not adjacent to u with the color red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring u_2 and the two neighbors of u in H_1 with the color red and coloring all remaining uncolored vertices of G blue. In both cases we color at most $(n-3)/2$ vertices red, and so $\gamma_{F_3}(G) \le (n-3)/2$, a contradiction.

Suppose $m = 2$. If v is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring u and its two neighbors in H_1 with the color red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring u_1 and the two vertices in H_1 not adjacent to u with the color red and coloring all remaining uncolored vertices of G blue. In both cases we color at most $(n-2)/2$ vertices red, and so $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction. Hence, $m = 1$.

If v is colored red in C_2, then C_2 can be extended to an F_3-coloring of G by coloring the two neighbors of u in H_1 with the color red and coloring all remaining uncolored vertices of G blue, whence $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction. Hence, by Observation 4.13, either $H_2 = D_k(5)$ for some $k \geq 2$ with v the central vertex of this daisy, or $\gamma_{F_3}(H_2) \leq (n_2-2)/2$ and v is colored blue in C_2. In the latter case, C_2 can be extended to an F_3-coloring of G by coloring u and its two neighbors in H_1 with the color red and coloring all remaining uncolored vertices of G blue, whence $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction. Hence, $H_2 = D_k(5)$ for some $k \geq 2$ with v the central vertex of this daisy. Thus, $G = A_k(5)$ for some $k \geq 2$. □

By Claim 4.28, we may assume that neither H_1 nor H_2 is a 5-cycle, for otherwise the desired result follows. Hence, $\gamma_{F_3}(H_2) \leq n_2/2$.

Claim 4.29 $H_1 \neq B_2$.

Proof. Suppose $H_1 = B_2$. Let C_2 be a minimum F_3-coloring of H_2. Suppose $m = 3$. If v is colored red in the coloring C_2, then C_2 can be extended to an F_3-coloring of G by coloring u_1 and the two vertices in H_1 not adjacent to u with the color red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in C_2, then C_2 can be extended to an F_3-coloring of G by coloring u_2 and two neighbors of u in H_1 that lie on a common 5-cycle with the color red and coloring all remaining uncolored vertices of G blue. In both cases we color at most $(n-3)/2$ vertices red, and so $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction.

Suppose $m = 2$. If v is colored red in C_2, then C_2 can be extended to an F_3-coloring of G by coloring u and two neighbors of u in H_1 that lie on a common 5-cycle with the color red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring u_1 and the two vertices in H_1 not adjacent to u with the color red and coloring all remaining uncolored vertices of G blue. In both cases we color at most $(n-2)/2$ vertices red, and so $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction.

Suppose $m = 1$. If v is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring two neighbors of u in H_1 that lie on a common 5-cycle with the color red and coloring all remaining uncolored vertices of G blue, and so $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction. Hence we may assume that every minimum F_3-coloring of H_2 colors v blue, for otherwise we reach a contradiction. Thus by Observations 4.11 and 4.13, $\gamma_{F_3}(H_2) \leq (n_2-1)/2$. Since v is colored blue in $\mathcal{C}_2$, the coloring $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring u_1 and the two vertices in H_1 not adjacent to u with the color red and coloring all remaining uncolored vertices of G blue, whence $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction. □

By Claim 4.29, $H_1 \neq B_2$. Similarly, $H_2 \neq B_2$.

Claim 4.30 $H_1 \neq C_4$.

Proof. Suppose $H_1 = C_4$. Let H_1 be the 4-cycle u, w, x, y, u. Since G is not a dumbbell, H_2 is not a cycle. Hence, $\gamma_{F_3}(H_2) \leq (n_2-1)/2$. Let $\mathcal{C}_2$ be a minimum F_3-coloring of H_2.

Suppose $m = 3$. If v is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring u_1 and x red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring the vertices u_2, x and y red, and coloring all remaining uncolored vertices of G blue. Hence, $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction.

Suppose $m = 2$. If v is colored red in C_2, then C_2 can be extended to an F_3-coloring of G by coloring u and w red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in C_2, then C_2 can be extended to an F_3-coloring of G by coloring the vertices u_1 and x red, and coloring all remaining uncolored vertices of G blue. Hence, $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction.

Suppose $m = 1$. If v is colored red in C_2, then C_2 can be extended to an F_3-coloring of G by coloring x and y red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in C_2, then C_2 can be extended to an F_3-coloring of G by coloring the vertices u and w red, and coloring all remaining uncolored vertices of G blue. Hence, $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction. □

We now return to the proof of Lemma 4.20 By Claim 4.30, $H_1 \neq C_4$. Similarly, $H_2 \neq C_4$. Hence, $H_1 = C_8$. Let H_1 be the 8-cycle $u = w_1, w_2, \ldots, w_8, u$. Since G is not a dumbbell, H_2 is not a cycle. Hence, $\gamma_{F_3}(H_2) \leq (n_2 - 1)/2$. Let C_2 be a minimum F_3-coloring of H_2.

Suppose $m = 3$. If v is colored red in C_2, then C_2 can be extended to an F_3-coloring of G by coloring the vertices in the set $\{u_1, w_3, w_4, w_7\}$ red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in C_2, then C_2 can be extended to an F_3-coloring of G by coloring the vertices in the set $\{u_2, w_2, w_5, w_8\}$ red, and coloring all remaining uncolored vertices of G blue. Hence, $\gamma_{F_3}(G) \leq (n-4)/2$, a contradiction.

Suppose $m = 2$. If v is colored red in C_2, then C_2 can be extended to an F_3-coloring of G by coloring the vertices in the set $\{w_1, w_2, w_5, w_8\}$ red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in C_2, then C_2 can be extended to an F_3-coloring of G by coloring the vertices in the set $\{u_1, w_3, w_4, w_7\}$ red, and coloring all remaining uncolored vertices of G blue. Hence, $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction.

Suppose $m = 1$. If v is colored red in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring the vertices in the set $\{w_2, w_5, w_8\}$ red and coloring all remaining uncolored vertices of G blue. On the other hand, if v is colored blue in $\mathcal{C}_2$, then $\mathcal{C}_2$ can be extended to an F_3-coloring of G by coloring the vertices in the set $\{w_1, w_2, w_5, w_8\}$ red, and coloring all remaining uncolored vertices of G blue. Hence, $\gamma_{F_3}(G) \le (n-2)/2$, a contradiction. This completes the proof of Lemma 4.20.

4.4.4 Proof of Lemma 4.21(a)

Suppose that there is a 2-graph cycle in G. Since $|S| \ge 2$, each vertex of S that has a 2-graph cycle also has a 2-graph path. Hence we may assume that the vertex u (defined earlier) has a 2-graph cycle C_u of order $n_1 + 1$. Let $H_u = G - (V(C_u) - \{u\})$ have order n_2. Then, H_u is a connected graph of order $n_2 = n - n_1$. If $\deg_G(u) = 3$, then the graph $H = G - u_1$ (defined earlier) is disconnected, contrary to assumption. Hence, $\deg_G(u) \ge 4$, and so $\delta(H_u) \ge 2$. By Observation 4.16, $H_u \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ or $\gamma_{F_3}(H_u) < (n_2 - 1)/2$.

Since v is a vertex of degree at least 3 in H_u, the graph H_u is not a cycle. Further by our earlier assumptions (that every 2-graph path has length exactly 1; that the set S is an independent set with $|S| \ge 2$; that removing the vertices not in S of any 2-graph path from G produces a connected graph), it follows that $H_u \notin \mathcal{A} \cup (\mathcal{B} - \{B_1\}) \cup \mathcal{C} \cup (\mathcal{D} - D(4,4)\}$. Hence by Observation 4.11, either $H_u \in \{B_1, D(4,4)\}$ or $\gamma_{F_3}(H_u) \le (n_2 - 2)/2$.

Let C^* be a minimum F_3-coloring of H_u. If $n_1 \ne 4$ (i.e., if C_u is not a 5-cycle), then irrespective of whether u is colored is red or blue in C^*, the coloring C^* can be extended to an F_3-coloring of G by coloring at most $(n_1 - 1)/2$ additional vertices in C_u red, and so $\gamma_{F_3}(G) \le (n-2)/2$, a contradiction. If $n_1 = 4$, then C^* can be extended to an F_3-coloring of G by coloring $n_1/2$ additional vertices red. Thus, if $\gamma_{F_3}(H_u) \le (n_2 - 2)/2$, then $\gamma_{F_3}(G) \le (n-2)/2$, a contradiction. Hence, $n_1 = 4$ and

$H_u \in \{B_1, D(4,4)\}$. But once again, $\gamma_{F_3}(G) \leq (n-3)/2$, a contradiction.

4.4.5 Proof of Lemma 4.21(b)

By Lemma 4.21(a), G is a bipartite graph with partite sets S and $V - S$. Every vertex in $V - S$ has degree exactly 2, while every vertex in S has degree at least 3.

Suppose that $N(u) \subseteq N(v)$ for some pair of vertices $u, v \in S$. If $|S| = 2$ (and still $n \geq 7$), then coloring u and one neighbor of u red and coloring all remaining vertices blue produces an F_3-coloring of G, and so $\gamma_{F_3}(G) = 2 \leq (n-3)/2$, a contradiction. Hence, $|S| \geq 3$, and so at least one neighbor of v is not a neighbor of u.

Suppose $\deg_G(v) = \deg_G(u) + 1$. Let $G' = G - N[v] - u$ have order n'. Then, $n' \leq n - 6$ and G' is an induced subgraph of G with $\delta(G') \geq 2$. Since G' is a bipartite graph, G' has no 5-cycles, and so, by the inductive hypothesis, $\gamma_{F_3}(G') \leq n'/2 \leq (n-6)/2$. Any minimum F_3-coloring of G' can be extended to an F_3-coloring of G by coloring u and one neighbor of u red and coloring all remaining uncolored vertices blue. Thus, $\gamma_{F_3}(G) \leq 2 + \gamma_{F_3}(G') \leq (n-2)/2$, a contradiction.

On the other hand, suppose $\deg_G(v) \geq \deg_G(u) + 2$. Let $G^* = G - N[u]$ have order n^*. Then, $n^* \leq n - 4$ and G^* is an induced subgraph of G with $\delta(G^*) \geq 2$. Since G^* is a bipartite graph, G^* has no 5-cycles, and so, by the inductive hypothesis, $\gamma_{F_3}(G^*) \leq n^*/2 \leq (n-4)/2$. Any minimum F_3-coloring of G^* that colors v red can be extended to an F_3-coloring of G by coloring one neighbor of u red (and coloring all remaining uncolored vertices blue), while any minimum F_3-coloring of G^* that colors v blue can be extended to an F_3-coloring of G by coloring the vertex u red. Hence, $\gamma_{F_3}(G) \leq 1 + \gamma_{F_3}(G^*) \leq (n-2)/2$, a contradiction. We deduce, therefore, that for any pair of vertices $u, v \in S$, $N(u) \not\subseteq N(v)$.

4.4.6 Proof of Lemma 4.21(c)

Suppose $\delta(G_u) \geq 2$ for some vertex $u \in S$. Then, G_u is an induced subgraph (possible disconnected) of G. Since G_u is a bipartite graph, G_u is C_5-free. Hence by the inductive hypothesis, each component of G_u has F_3-domination number at most one-half its order, and so $\gamma_{F_3}(G_u) \leq |V(G_u)|/2 \leq (n-4)/2$.

Let $S_u = \{w \in S \mid d_G(u,w) = 2\}$. By Lemma 4.21(b), $|S_u| \geq 2$. Let $\mathcal{C}_u$ be a minimum F_3-coloring of G_u. Suppose $\mathcal{C}_u$ colors a vertex x in S_u red. Let x' be a common neighbor of u and x. Then, $\mathcal{C}_u$ can be extended to an F_3-coloring of G by coloring one vertex in $N(u) - \{x'\}$ red (and coloring all other vertices in $N[u]$ blue). On the other hand, if $\mathcal{C}_u$ colors no vertex in S_u red, then it follows by Observation 4.17 that $\mathcal{C}_u$ can be extended to an F_3-coloring of G by coloring the vertex u red (and coloring all remaining uncolored vertices blue). Hence, $\gamma_{F_3}(G) \leq 1 + \gamma_{F_3}(G_u) \leq (n-2)/2$, a contradiction.

4.4.7 Proof of Lemma 4.22

Suppose that every 2-graph path has length exactly 1. Let $u \in S$. By Lemma 4.21, $\delta(G_u) = 1$ and G is a bipartite graph with partite sets S and $V - S$. We may assume that v has degree 1 in G_u. Thus, u and v have at least two common neighbors. Let c be one such common neighbor of u and v. Let v' be the neighbor of v in G_u, and let $N(v') = \{v, w\}$. Then, $w \in S - \{u, v\}$ and w has degree at least 2 in G_v. Since $\delta(G_v) = 1$ by Lemma 4.21(c), the vertex u must have degree 1 in G_v. Let u' be the neighbor of u in G_v, and let $N(u') = \{u, z\}$. Then, $z \in S - \{u, v\}$ and z has degree at least 2 in G_v (possibly, $z = w$). Let $G' = G - N[u] - N[v]$ have order n'. Then, $n' \leq n - 6$ and G' is an induced subgraph of G. In particular, G' has no 5-cycles. Let $\mathcal{C}'$ be a minimum F_3-coloring of G'. We now consider two possibilities.

Suppose $z \neq w$. Then, $\delta(G') \geq 2$. It follows from the inductive hypothesis that $\gamma_{F_3}(G') \leq n'/2 \leq (n-6)/2$. If C' colors both w and z red, then C' can be extended to an F_3-coloring of G by coloring the vertex c red and coloring all remaining uncolored vertices blue. If C' colors both w and z blue, then it follows from Observation 4.17 that C' can be extended to an F_3-coloring of G by coloring the vertices c and u red and coloring all remaining uncolored vertices blue. Suppose, finally, that C' colors exactly one of w and z red. By symmetry, we may assume that w is colored red. Then C' can be extended to an F_3-coloring of G by coloring the vertices c and u red and coloring all remaining uncolored vertices blue. Thus, $\gamma_{F_3}(G) \leq 2 + \gamma_{F_3}(G') \leq (n-2)/2$.

Suppose, on the other hand, that $z = w$. If $\deg_G(w) = 3$, then $\delta(G_w) \geq 2$, which contradicts Lemma 4.21(c). Hence, $\deg_G(w) \geq 4$. Then, $\delta(G') \geq 2$. It follows from the inductive hypothesis that $\gamma_{F_3}(G') \leq n'/2 \leq (n-6)/2$. If C' colors w red, then C' can be extended to an F_3-coloring of G by coloring the vertex c red and coloring all remaining uncolored vertices blue. If C' colors w blue, then it follows from Observation 4.17 that C' can be extended to an F_3-coloring of G by coloring the vertices c and u red and coloring all remaining uncolored vertices blue. Thus, $\gamma_{F_3}(G) \leq 2 + \gamma_{F_3}(G') \leq (n-2)/2$.

4.4.8 Proof of Lemma 4.23

Suppose $\gamma_{F_3}(H) \leq (n'-1)/2 = (n-4)/2$. If at least one of u and v, say u, is colored red in C_H, then C_H can be extended to an F_3-coloring of G by coloring the vertex u_3 red and the vertices u_1 and u_2 blue. On the other hand, if both u and v are colored blue in C_H, then C_H can be extended to an F_3-coloring of G by coloring the vertex u_2 red and the vertices u_1 and u_3 blue. Hence, $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction. Thus, $\gamma_{F_3}(H) \geq n'/2$, whence $H \in \{B_2, C_4, C_5, C_8\}$. If $H \in \{B_2, C_4, C_5\}$, then it is easily checked that $\gamma_{F_3}(G) < (n-1)/2$, a contradiction. If $H = C_8$ and if u and v are at distance 2 or 3 apart in H, then $\gamma_{F_3}(G) < (n-1)/2$, a contradiction. Thus,

$H = C_8$ and the vertices u and v are at distance 4 apart in H, and so $G = B_5$.

4.4.9 Proof of Lemma 4.24

Note that $n' = n - 2 \geq 5$. If $H \in \{B_2, C_5, C_8\}$, then it is easily checked that $\gamma_{F_3}(G) < (n-1)/2$, a contradiction. Hence, $\gamma_{F_3}(H) \leq (n'-1)/2 = (n-3)/2$. If u and v are both colored with the same color in $\mathcal{C}_H$, then $\mathcal{C}_H$ can be extended to an F_3-coloring of G by coloring both u_1 and u_2 blue, and so $\gamma_{F_3}(G) < (n-1)/2$, a contradiction. Hence every minimum F_3-coloring of H colors u and v with different colors. We may assume that u is colored red in $\mathcal{C}_H$. If $\gamma_{F_3}(H) \leq (n-4)/2$, then $\mathcal{C}_H$ can be extended to an F_3-coloring of G by coloring u_1 red and u_2 blue, and so $\gamma_{F_3}(G) \leq (n-2)/2$, a contradiction. Hence, $\gamma_{F_3}(H) = (n'-1)/2$. Since the set S is independent and since u and v must receive different colors in every minimum F_3-coloring of H, it therefore follows that $H \in \{B_1, B_2, B_3, A_1(5), D(4,4), D_2(5)\}$. If $H \neq B_1$, then it is easily checked that $\gamma_{F_3}(G) < (n-1)/2$, a contradiction. Hence, $H = B_1$, and so $G = B_3$.

4.5 Proof of Theorem 4.2

The proof of Theorem 4.2 follows readily from Theorem 4.1. Since the F_3-domination number of a graph cannot decrease if edges are removed, it follows from Theorem 4.1 and Observation 4.11 that the F_3-domination number of G is at most $(n+1)/2$. Further suppose $\gamma_{F_3}(G) \geq n/2$. Then by removing edges of G, if necessary, we produce an F_3-minimal graph G'. By Theorem 4.1 and Observation 4.11, $G' \in \{B_2, C_4, C_5, C_8\}$. In all cases it can be readily checked that $G = G'$ unless possibly if $G' = C_8$ in which case $G \in \{C_8, H_1\}$.

4.6 Proof of Theorem 4.3

The proof of Theorem 4.3 follows readily from Theorem 4.1. Since the F_3-domination number of a graph cannot decrease if edges are removed, and since $n \geq 9$, it follows from Theorem 4.1 and Observation 4.11 that the F_3-domination number of G is at most $(n-1)/2$. Further suppose $\gamma_{F_3}(G) = (n-1)/2$. Then by removing edges of G, if necessary, we produce an F_3-minimal graph G'. By Theorem 4.1 and Observation 4.11, $G' \in \mathcal{A} \cup (\mathcal{D} - \{D(3,5), D(4,4)\})$ or $G' \in \{B_4, B_5, C_{11}\}$. In all cases it is straightforward to check that $G = G'$ unless possibly if $G' = C_{11}$ in which case $G \in \{C_{11}, H_2\}$.

Chapter 5

SIMULTANEOUS STRATIFICATION IN GRAPHS

5.1 Introduction

In this chapter we focus on two variations on the domination theme that are well studied in graph theory called total domination and restrained domination. Let $G = (V, E)$ be a graph. Recall, a set $S \subseteq V$ is a total dominating set (or TDS) in G if every vertex of G is adjacent to a vertex of S and S is a restrained dominating set (or RDS) in G if every vertex not in S is adjacent to a vertex in S and to a vertex in $V \setminus S$.

5.2 Simultaneous stratification

The concepts of stratification and domination in graphs explored by Chartrand et al. [10] and others (see for example, [9, 37]) may be extended in a number of ways. In [10], the following extension is considered.

Let $\mathcal{F} = \{F_1, \cdots, F_m\}$, where F_i, $1 \leq i \leq m$, is a 2-stratified graph rooted at some blue vertex v. We define an $\mathcal{F}$-coloring of a graph G to be a red-blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of F_i rooted at v for every $i = 1, \ldots, m$. We define the *$\mathcal{F}$-domination number* $\gamma_{\mathcal{F}}(G)$ of G as the minimum number of red vertices of G in an $\mathcal{F}$-coloring of G, and we define a *$\gamma_{\mathcal{F}}$-coloring* of G as an $\mathcal{F}$-coloring of G that colors $\gamma_{\mathcal{F}}(G)$ vertices red.

Throughout the rest of this chapter we take $\mathcal{F} = \{F_1, F_4\}$, where F_1 and F_4 are the 2-stratified graphs shown in Figure 1.2. Hence in our $\mathcal{F}$-coloring of a graph G, every blue vertex v is rooted at both a copy of F_1 and a copy of F_4.

We remark that our $\mathcal{F}$-coloring can be thought of as a 2-stratified P_4 coloring: If F is a 2-stratified P_4 given by v_1, v_2, v_3, v_4 where v_1 and v_2 are colored blue and v_3 and v_4 are colored red that is rooted at the blue vertex $v = v_2$, then our $\mathcal{F}$-coloring is precisely an F-coloring.

5.2.1 $\mathcal{F}$-domination versus total restrained domination

The following observation shows that the $\mathcal{F}$-domination number is bounded below by the restrained domination number and by the total domination number, and is bounded above by the total restrained domination number.

Observation 5.1 *For every graph G without isolated vertices,*

$$\max\{\gamma_r(G), \gamma_t(G)\} \leq \gamma_{\mathcal{F}}(G) \leq \gamma_{tr}(G).$$

Proof. Coloring the vertices of a minimum TRDS of G red and the remaining vertices blue produces an $\mathcal{F}$-coloring of G, and so $\gamma_{\mathcal{F}}(G) \leq \gamma_{tr}(G)$. This establishes the upper bound. To prove the lower bound, notice that the set of red vertices in an $\mathcal{F}$-coloring of a graph G is a RDS of G, whence $\gamma_r(G) \leq \gamma_{\mathcal{F}}(G)$. Observe that the set of red vertices in an $\mathcal{F}$-coloring of a graph G without isolated vertices is not necessarily a TDS of G, since there may be isolated vertices in the subgraph induced by the red vertices. However, every $\mathcal{F}$-coloring of G is by definition an F_1-coloring of G. Hence there exists an F_1-coloring of G with $\gamma_{\mathcal{F}}(G)$ red vertices. Among all F_1-colorings of G with $\gamma_{\mathcal{F}}(G)$ red vertices, choose one to minimize the number of isolated vertices in the subgraph induced by its red vertices. Then, as shown in the proof of Proposition 1 in [10], every red vertex in such an F_1-coloring is adjacent to some other red vertex, and so the red vertices form a TDS of G. This implies that $\gamma_t(G) \leq \gamma_{\mathcal{F}}(G)$. □

Our next example illustrates that the bounds in Observation 5.1 can be strict even for the family of trees.

Example 1. Let T be the tree obtained from two disjoint paths P_6 by joining a vertex at distance 2 from a leaf on one path to a vertex at distance 2 from a leaf on the other path, and then subdividing the resulting edge once. Then, $\gamma_{tr}(T) = 10$, $\gamma_{\mathcal{F}}(T) = 9$, $\gamma_t(T) = 8$, and $\gamma_r(T) = 7$ as illustrated in Figure 5.1.

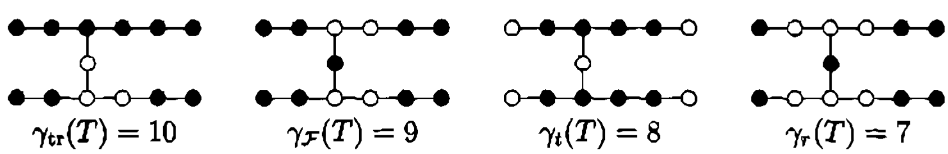


Figure 5.1:

Example 2. For $k \geq 2$, an integer, let $T_1, T_2, \ldots, T_k$ be k disjoint copies of the tree T defined in Example 1 and let v_i be a vertex of degree 3 in T_i. Let G be the tree obtained from the disjoint union of the trees T_i, $1 \leq i \leq k$, by adding a new vertex v and the edges vv_i for $i = 1, 2, \ldots, k$. Then, $\gamma_{tr}(G) = 10k$ and $\gamma_{\mathcal{F}}(G) = 9k + 1$. This example serves to illustrate that there exists trees G such that $\gamma_{tr}(G) - \gamma_{\mathcal{F}}(G)$

can be made arbitrarily large.

5.2.2 Cycles

In this section, we compute the $\mathcal{F}$-domination number of a cycle.

Proposition 5.2 *For $n \geq 3$ and for $i = 0,1,2,3$, $\gamma_{\mathcal{F}}(C_n) = (n+i)/2$ where $n \equiv i \pmod 4$.*

Proof. We proceed by induction on the order $n \geq 3$ of a cycle C_n. The result is straightforward to verify for $n \in \{3,4,5,6\}$. Assume then that $n \geq 7$ and that the result of the proposition is true for all cycles on fewer than n vertices. Consider a cycle $C: v_1, v_2, \ldots, v_n, v_1$. Let $\mathcal{C}$ be a $\gamma_{\mathcal{F}}$-coloring of C. Then every blue vertex is adjacent to a blue vertex and to a red vertex, with the red vertex itself adjacent to some other red vertex. Renaming vertices if necessary, we may assume that $\mathcal{C}$ colors v_1 and v_2 red, and v_3 and v_4 blue. Hence, v_5 and v_6 are colored red under $\mathcal{C}$. Let C' be the cycle obtained from C by deleting the vertices v_1, v_2, v_3 and v_4 and adding the edge v_5v_n, i.e., $C' = (C - \{v_1, v_2, v_3, v_4\}) \cup \{v_5v_n\}$. Then, C' is a cycle of order $n-4 \geq 3$ and the restriction of $\mathcal{C}$ to C' is an $\mathcal{F}$-coloring of C' that colors $\gamma_{\mathcal{F}}(C) - 2$ vertices red. Hence, $\gamma_{\mathcal{F}}(C') \leq \gamma_{\mathcal{F}}(C) - 2$. On the other hand, let $\mathcal{C}'$ be a $\gamma_{\mathcal{F}}$-coloring of C'. If $\mathcal{C}'$ colors v_5 blue and v_n red, then color v_1 and v_2 blue and v_3 and v_4 red. If $\mathcal{C}'$ colors v_5 red and v_n blue, then color v_1 and v_2 red and v_3 and v_4 blue. If $\mathcal{C}'$ color both v_5 and v_n red, then color v_1 and v_4 red and v_2 and v_3 blue. If $\mathcal{C}'$ color both v_5 and v_n blue, then color v_1 and v_4 blue and color v_2 and v_3 red. In all four cases, $\mathcal{C}'$ can be extended to an $\mathcal{F}$-coloring of C by coloring an additional two vertices red. Thus, $\gamma_{\mathcal{F}}(C) \leq \gamma_{\mathcal{F}}(C') + 2$. Consequently, $\gamma_{\mathcal{F}}(C) = \gamma_{\mathcal{F}}(C') + 2$. The result now follows by applying the inductive hypothesis to the cycle $C' \cong C_{n-4}$. □

5.2.3 Bounds involving maximum degree

Let G be a connected graph of order n and maximum degree Δ. Berge [2] was the first to observe that $\gamma(G) \leq n - \Delta$, and graphs achieving this bound were characterized in [23]. Cockayne, Dawes and Hedetniemi [14] observed that if $n \geq 3$ and $\Delta \leq n-2$, then $\gamma_t(G) \leq n - \Delta$. Recently it was shown in [15] that if $\delta(G) \geq 2$, then $\gamma_r(G) \leq n - \Delta$. Hence if $\delta(G) \geq 2$, then both the total domination and the restrained domination numbers are bounded above by $n-\Delta$. Our aim in this section is to show that if $\Delta \leq n - 2$ and the minimum degree of G is at least two, then $\gamma_{\mathcal{F}}(G) \leq n - \Delta + 1$.

Recall, by a proper subgraph of a graph G we mean a subgraph of G different from G. We also defined a vertex as ***small*** if it has degree 2, and ***large*** if it has degree more than 2 and we defined a ***ray*** as a path (not necessarily induced) of length 3 the two internal vertices of which are small vertices. Let G be a graph with minimum degree at least two, and let $\mathcal{L}$ be set of all large vertices of G. Recall also that if $|\mathcal{L}| \geq 1$ and C is any component of $G - \mathcal{L}$; it is a path. Then, if C has only one vertex, or has at least two vertices but the two ends of C are adjacent in G to different large vertices, we say that C is a 2-***path***. Otherwise we say that C is a 2-***handle***.

Theorem 5.3 *If G is a connected graph of order n, size m, maximum degree Δ where $\Delta \leq n-2$, and minimum degree at least 2, then*

$$\gamma_{\mathcal{F}}(G) \leq n - \Delta + 1,$$

and this bound is sharp.

Proof. We proceed by induction on $\ell = n + m$. For notational convenience, we let $\phi(n, \Delta) = n - \Delta + 1$. We wish to show that $\gamma_{\mathcal{F}}(G) \leq \phi(n, \Delta)$. Note that $n \geq 4$ and $m \geq 4$, and so $\ell \geq 8$. When $\ell = 8$, the graph G is a 4-cycle and so

$\gamma_{\mathcal{F}}(G) = 2 < \phi(4, 2) = \phi(n, \Delta)$. This establishes the base case. For the inductive hypothesis, let $\ell \geq 9$ and assume that for all connected graphs G' of order n' and size m' with $n' + m' < \ell$ that have maximum degree Δ' where $\Delta' \leq n' - 2$ and minimum degree at least 2 that $\gamma_{\mathcal{F}}(G') \leq \phi(n', \Delta')$. Let G be a connected graph of order n and size m with $\ell = n + m$, maximum degree Δ where $\Delta \leq n - 2$ and minimum degree at least 2.

Observation 5.4 *We may assume that G contains no ray.*

Proof. Suppose that G contains a ray $P: v, v_1, v_2, w$. Thus, both v_1 and v_2 are small vertices of G. If $\Delta = n - 2$, then v or w, say v, is a vertex of maximum degree Δ in G. Coloring v and v_1 red and every other vertex blue produces an $\mathcal{F}$-coloring of G, and so $\gamma_{\mathcal{F}}(G) = 2 = \phi(n, n-2) - 1 < \phi(n, \Delta)$. Hence we may assume that $\Delta \leq n - 3$. Let G' be the graph obtained from G by removing the vertex v_1 and adding the edge vv_2. Then, G' is a connected graph of order $n' = n - 1$ and size $m' = m - 1$, with maximum degree $\Delta' = \Delta$ with $\Delta' \leq n' - 2$, and with minimum degree at least 2. Applying the inductive hypothesis to G', we have that $\gamma_{\mathcal{F}}(G') \leq \phi(n', \Delta') = \phi(n - 1, \Delta) = \phi(n, \Delta) - 1$. Any $\gamma_{\mathcal{F}}$-coloring of G' can be extended to an $\mathcal{F}$-coloring of G by coloring the vertex v_1 red, unless v and v_2 are both colored blue, in which case recolor v_2 red and color v_1 blue. Hence, $\gamma_{\mathcal{F}}(G) \leq \gamma_{\mathcal{F}}(G') + 1 \leq \phi(n, \Delta)$, as desired. □

By Observation 5.4, every 2-path in G has order 1, while every 2-handle of G has order 2. Hence every large vertex in G is either adjacent to a large vertex or at distance 2 from some large vertex.

Observation 5.5 *If G' is a connected proper subgraph of G of order n', with maximum degree $\Delta' = \Delta$ where $\Delta \leq n' - 2$, and minimum degree at least 2, then $\gamma_{\mathcal{F}}(G) \leq \phi(n, \Delta)$.*

Proof. Let $n' = n - k$, where $k \geq 0$, and let G' have size m'. Then, $n' + m' < \ell$. Applying the inductive hypothesis to G', we have that $\gamma_{\mathcal{F}}(G') \leq \phi(n', \Delta') = \phi(n-k, \Delta) = \phi(n, \Delta) - k$. Any $\gamma_{\mathcal{F}}$-coloring of G' can be extended to an $\mathcal{F}$-coloring of G by coloring every vertex in $V(G) \setminus V(G')$ with the color red. Hence, $\gamma_{\mathcal{F}}(G) \leq \gamma_{\mathcal{F}}(G') + k \leq \phi(n, \Delta)$. □

Let v be a vertex of maximum degree Δ.

Observation 5.6 *We may assume that no vertex of $\mathcal{L} \setminus \{v\}$ is adjacent with an end of a 2-handle.*

Proof. Suppose that $w \in \mathcal{L} \setminus \{v\}$ is adjacent with an end of a 2-handle x, y. Thus, w, x, y, w is a cycle in G. By definition, x and y are small vertices. If $\deg_G(w) \geq 4$, then $G - \{x, y\}$ is a connected subgraph of G that satisfies the statement of Observation 5.5, and so $\gamma_{\mathcal{F}}(G) \leq \phi(n, \Delta)$. Hence we may assume that $\deg_G(w) = 3$. Let u be the neighbor of w different from x and y.

Let G' be obtained from G by removing the vertex w and adding the edges ux and uy. Then, G' is a connected graph of order $n' = n - 1$ and size $m' = m - 1$, with maximum degree Δ' where $\Delta' \in \{\Delta, \Delta + 1\}$, and with minimum degree at least 2. If $\Delta' = n' - 1$, then $u = v$ and v is adjacent in G' to every other vertex. But then coloring v and w red, and coloring all other vertices blue, produces an $\mathcal{F}$-coloring of G, and so $\gamma_{\mathcal{F}}(G) = 2 < \phi(n, \Delta)$. Thus, we may assume that $\Delta' \leq n' - 2$. Applying the inductive hypothesis to G', we have that $\gamma_{\mathcal{F}}(G') \leq \phi(n', \Delta') = \phi(n-1, \Delta') = \phi(n, \Delta') - 1$. Any $\gamma_{\mathcal{F}}$-coloring of G' can be extended to an $\mathcal{F}$-coloring of G by coloring the vertex w red, and so $\gamma_{\mathcal{F}}(G) \leq \gamma_{\mathcal{F}}(G') + 1 \leq \phi(n, \Delta')$. Now either $\Delta' = \Delta$, in which case $\phi(n, \Delta') = \phi(n, \Delta)$, or $\Delta' = \Delta + 1$, in which case $\phi(n, \Delta') = \phi(n, \Delta + 1) = \phi(n, \Delta) - 1$. In both cases, $\gamma_{\mathcal{F}}(G) \leq \phi(n, \Delta)$ (with strict inequality if $\Delta' = \Delta + 1$), as desired. □

Observation 5.7 *We may assume that the subgraph $G[\mathcal{L}]$ induced by the large vertices contains no cycle.*

Proof. Suppose that the subgraph $G[\mathcal{L}]$ induced by the large vertices contains a cycle. Such a cycle necessarily contains an edge whose removal produces a connected (spanning) subgraph of G that satisfies the statement of Observation 5.5, and so $\gamma_{\mathcal{F}}(G) \leq \phi(n, \Delta)$. □

Observation 5.8 *We may assume that $\mathcal{L} \setminus \{v\}$ is an independent set.*

Proof. Suppose $e = uw$ is an edge of G joining two vertices u and w of $\mathcal{L} \setminus \{v\}$. If e is a cycle edge, then $G - e$ is a connected subgraph of G that satisfies the statement of Observation 5.5, and so $\gamma_{\mathcal{F}}(G) \leq \phi(n, \Delta)$. Hence we may assume that e is a bridge of G. Let G_u be the component of $G - e$ containing u, and G_w the component containing w. We may assume that $v \in V(G_u)$. Then, G_u is a connected subgraph of G of order n' with maximum degree Δ and minimum degree at least 2. If $\Delta \leq n' - 2$, then G_u satisfies the statement of Observation 5.5, and so $\gamma_{\mathcal{F}}(G) \leq \phi(n, \Delta)$. Hence we may assume that v dominates G_u. Let $x \in N(u) \setminus \{v, w\}$. Then, $x \in V(G_u)$ and, since $G[\mathcal{L}]$ contains no cycles, x is a small vertex. Coloring the vertices in $(V(G_w) \setminus \{w\}) \cup \{v, x\}$ red, and coloring all other vertices blue produces an $\mathcal{F}$-coloring of G, and so $\gamma_{\mathcal{F}}(G) \leq n - \Delta = \phi(n, \Delta) - 1$. □

By Observation 5.8, the only edges in $G[\mathcal{L}]$, if any, are incident with v.

Observation 5.9 *We may assume that no two vertices in $\mathcal{L} \setminus \{v\}$ have a common neighbor that is a small vertex.*

Proof. Suppose two vertices u and w in $\mathcal{L} \setminus \{v\}$ have a common neighbor y that is a small vertex. If $G - y$ is connected, then $G - y$ satisfies the statement of

Observation 5.5, and so $\gamma_{\mathcal{F}}(G) \leq \phi(n, \Delta)$. Hence we may assume that y is a cut-vertex of G. Let G_u be the component of $G-y$ containing u, and G_w the component containing w. We may assume that $v \in V(G_u)$. Then, G_u is a connected subgraph of G of order n' with maximum degree Δ and minimum degree at least 2. If $\Delta \leq n'-2$, then G_u satisfies the statement of Observation 5.5, and so $\gamma_{\mathcal{F}}(G) \leq \phi(n, \Delta)$. Hence we may assume that v dominates G_u. Let $x \in N(u) \setminus \{v, w\}$. Then, $x \in V(G_u)$ and x is a small vertex. Coloring the vertices in $V(G_w) \cup \{v, x\}$ red, and coloring all other vertices blue produces an $\mathcal{F}$-coloring of G, and so $\gamma_{\mathcal{F}}(G) \leq n - \Delta = \phi(n, \Delta) - 1$. □

By Observation 5.9, we may assume that every neighbor of a large vertex, different from v, is a small vertex adjacent with v. If v is the only large vertex, then G can be constructed from disjoint triangles by identifying one vertex from each triangle into one vertex v. But then $\Delta = n-1$, contrary to assumption. Hence, there are least two large vertices. If every large vertex different from v is adjacent to v, then $\Delta = n-1$, a contradiction. Hence there exists at least one large vertex u that is not adjacent to v. Suppose there exists $k \geq 1$ such large vertices. Then $\Delta = n-k-1$. Coloring v and exactly one neighbor of every large vertex red and coloring all remaining vertices blue produces an $\mathcal{F}$-coloring of G. Hence, $\gamma_{\mathcal{F}}(G) = k + 1 = n - \Delta = \phi(n, \Delta) - 1$. This establishes the proof of the upper bound.

It remains for us to establish that the upper bound is sharp. For $t \geq 2$ an integer, let G be the graph constructed from t disjoint 6-cycles by identifying a set of t vertices, one from each cycle, into one vertex v and then joining v to every vertex at distance 2 from it in the resulting graph. Then, G has order $n = 5t + 1$, maximum degree $\Delta = 4t$, and $\gamma_{\mathcal{F}}(G) = t + 2 = n - \Delta + 1$. This completes the proof of the theorem. □

5.2.4 Bounds involving the order

Let G be a connected graph of order n and minimum degree $\delta(G) \geq 2$. It is shown in [34] that $\gamma_t(G) \leq 4n/7$, unless $G \in \{C_3, C_5, C_6, C_{10}\}$. Domke et al. [19] showed that $\gamma_r(G) \leq (n-1)/2$, apart for eight exceptional graphs (one of orders four, five and six, and five of order eight). Our aim in this section is to establish an upper bound on the $\mathcal{F}$-domination number of a connected graph with minimum degree at least two in terms of only the order of the graph. We shall prove:

Theorem 5.10 *If $G \neq C_7$ is a connected graph of order $n \geq 4$ with minimum degree at least 2, then $\gamma_{\mathcal{F}}(G) \leq 2n/3$.*

Proof. We proceed by induction on the order $n \geq 4$ of a connected graph G with minimum degree at least 2. For $n \in \{4,5,6\}$ the result is straightforward to verify. This establishes the base case. For the inductive hypothesis, let $n \geq 7$ and assume that for all connected graphs $G' \neq C_7$ of order n', where $4 \leq n' < n$, that have minimum degree at least 2 that $\gamma_{\mathcal{F}}(G') \leq 2n'/3$.

Let $G \neq C_7$ be a connected graph of order $n \geq 4$ with $\delta(G) \geq 2$. Since the $\mathcal{F}$-domination number of a graph cannot increase if edges are added, $\gamma_{\mathcal{F}}(G) \leq \gamma_{\mathcal{F}}(G-e)$ for every edge $e \in E(G)$. Hence we may assume that G is edge-minimal with respect to satisfying the conditions that $\delta(G) \geq 2$ and G is connected.

Let $\mathcal{L}$ denote the set of large vertices of G. If G is a cycle, then the desired result follows from Proposition 5.2. Hence we may assume that $|\mathcal{L}| \geq 1$.

Observation 5.11 *We may assume that G cannot be obtained from a graph H by subdividing an edge four times.*

Proof. Suppose that G is obtained from a graph H by subdividing an edge four times. By assumption, G is not a cycle, and so $H \neq C_7$. Let uv be an edge of

H that is subdivided four times to obtain the graph G, and let u, u_1, u_2, u_3, u_4, v be the resulting path in G. Any minimum $\mathcal{F}$-coloring of H can be extended to an $\mathcal{F}$-coloring of G as follows. If both u and v are colored red, then color u_1 and u_4 red and u_2 and u_3 blue. If both u and v are colored blue, then color u_2 and u_3 red and u_1 and u_4 blue. If exactly one of u and v, say u, is colored red and has a red neighbor, then color u_3 and u_4 red and u_1 and u_2 blue. If exactly one of u and v, say u, is colored red and has no red neighbor, then recolor u blue and color u_1, u_2 and u_3 red and u_4 blue. In all cases, we produce an $\mathcal{F}$-coloring of G that colors exactly two more vertices red than does the original $\mathcal{F}$-coloring of H. It follows that $\gamma_{\mathcal{F}}(G) \leq \gamma_{\mathcal{F}}(H) + 2$. Applying the inductive hypothesis to H, we have $\gamma_{\mathcal{F}}(H) \leq 2(n-4)/3$, whence $\gamma_{\mathcal{F}}(G) < 2n/3$. □

By Observation 5.11, G contains no induced path on six vertices, every internal vertex of which has degree 2 in G. Hence, every 2-path has at most three vertices and every 2-handle has at most five vertices.

Observation 5.12 *We may assume that the set $\mathcal{L}$ is an independent set in G.*

Proof. Suppose G contains an edge $e = uv$ joining two large vertices. By the edge-minimality of G, e is a bridge of G. Let G_u and G_v be the components of $G - e$ containing u and v, respectively. Then, $\delta(G_u) \geq 2$ and $\delta(G_v) \geq 2$. If both G_u and G_v are cycles, then a simple check shows that $\gamma_{\mathcal{F}}(G) \leq 2n/3$. Hence at least one of G_u and G_v, say G_v, is not a cycle. Then, $|V(G_v)| \geq 4$. By the inductive hypothesis, $\gamma_{\mathcal{F}}(G_v) \leq 2|V(G_v)|/3$. If $G_u \notin \{C_3, C_7\}$, then by the inductive hypothesis, $\gamma_{\mathcal{F}}(G_u) \leq 2|V(G_u)|/3$, and so $\gamma_{\mathcal{F}}(G) \leq \gamma_{\mathcal{F}}(G_u) + \gamma_{\mathcal{F}}(G_v) \leq 2n/3$. Hence we may assume that $G_u = C_3$.

Suppose that $G_u = C_3$. We extend a $\gamma_{\mathcal{F}}$-set of G_v to an $\mathcal{F}$-coloring of G by coloring at most two vertices of G_u red: If v is colored red (resp., blue), then color

u red (resp., blue) and the remaining two vertices of G_u blue (resp., red). Hence, $\gamma_{\mathcal{F}}(G) \leq \gamma_{\mathcal{F}}(G_v) + 2 \leq 2(n-3)/3 + 2 = 2n/3$, as desired. □

Observation 5.13 *We may assume that every 2-path has at most two vertices.*

Proof. By our earlier assumptions, every 2-path has at most three vertices. Assume that there is a 2-path $P: v_1, v_2, v_3$. Let u be the large vertex adjacent to v_1 and v the large vertex adjacent to v_3. Let $G' = G - V(P)$. Then, $\delta(G') \geq 2$. By Observation 5.12, u and v are not adjacent, and so G' has order at least 4. If $G' = C_7$, then G consists of two vertices u and v joined by three 2-paths (one on two vertices and two on three vertices), and so $\gamma_{\mathcal{F}}(G) = 4 < 2n/3$. Hence we may assume that $G' \neq C_7$. If G' is disconnected, let G_u and G_v be the components of G' containing u and v, respectively. Then, $\delta(G_u) \geq 2$ and $\delta(G_v) \geq 2$. By assumption every 2-handle has at most five vertices, and so neither G_u nor G_v is a 7-cycle.

If G' is connected, or if G' is disconnected and both G_u and G_v have order at least 4, then applying the inductive hypothesis to G' or to each component of G', $\gamma_{\mathcal{F}}(G') \leq 2(n-3)/3$. Every $\gamma_{\mathcal{F}}$-set of G' can be extended to an $\mathcal{F}$-coloring of G by coloring at most two vertices of P red: If u and v are both colored red (resp., blue), then color v_1 and v_2 blue (resp., red) and v_3 red (resp., blue), while if u is colored red and v blue, then color v_1 and v_2 red and v_3 blue. Hence, $\gamma_{\mathcal{F}}(G) \leq \gamma_{\mathcal{F}}(G') + 2 \leq 2(n-3)/3 + 2 = 2n/3$.

Hence we may assume that G' is disconnected and that at least one of G_u and G_v is a 3-cycle, say G_u. If $G_v = C_3$, then $n = 9$ and $\gamma_{\mathcal{F}}(G) \leq 5 < 2n/3$. Hence we may assume that $|V(G_v)| \geq 4$. By the inductive hypothesis, $\gamma_{\mathcal{F}}(G_v) \leq 2|V(G_v)|/3 = 2(n-6)/3$. Every $\gamma_{\mathcal{F}}$-set of G_v can be extended to an $\mathcal{F}$-coloring of G by coloring at most three vertices in $V(G) \setminus V(G_v)$ red: If v is colored red, then color u and v_1 red and the remaining four uncolored vertices blue, while if v is colored blue, then color u, v_1 and v_2 red and the remaining three uncolored vertices blue. Hence,

$\gamma_{\mathcal{F}}(G) \leq \gamma_{\mathcal{F}}(G_v) + 3 \leq 2n/3 - 1$. □

Observation 5.14 *We may assume that every 2-handle has two vertices, and that the large vertex adjacent to these two vertices has degree at least 4.*

Proof. Let C be a 2-handle of G. Let $v \in \mathcal{L}$ be the common neighbor of its ends. By our earlier assumptions, $|V(C)| \leq 5$.

Suppose first that $\deg v = 3$. Let P be the 2-path with an end adjacent to v and that contains a small vertex not on C. By Observation 5.13, $|V(P)| \leq 2$. Let u be the large vertex, different from v, adjacent to an end of P. Let e be the edge joining P and u, and let G_u and G_v be the components of $G - e$ containing u and v, respectively. If G_u is a cycle, then it is straightforward to check that $\gamma_{\mathcal{F}}(G) < 2n/3$. Hence we may assume that G_u is not a cycle. By the inductive hypothesis, $\gamma_{\mathcal{F}}(G_u) \leq 2|V(G_u)|/3 = 2(n - |V(G_v)|/3$. Every $\gamma_{\mathcal{F}}$-set of G_u can be extended to an $\mathcal{F}$-coloring of G by coloring less than two-thirds of the vertices of G_v red, and so $\gamma_{\mathcal{F}}(G) < 2n/3$. Hence we may assume that $\deg v \geq 4$.

Suppose that C has at least three vertices. Let $G' = G - V(C)$. Then, $\delta(G') \geq 2$. If $G' = C_3$, then G can be constructed from two disjoint cycles by identifying two vertices, one from each cycle, into one vertex v. But then $\gamma_{\mathcal{F}}(G) < 2n/3$. Hence we may assume that $|V(G')| \geq 4$. By the inductive hypothesis, $\gamma_{\mathcal{F}}(G') \leq 2|V(G')|/3 = 2(n - |V(C)|)/3$. Since $|V(C)| \geq 3$, every $\gamma_{\mathcal{F}}$-set of G' can be extended to an $\mathcal{F}$-coloring of G by coloring at most two-thirds of the vertices of C red, and so $\gamma_{\mathcal{F}}(G) \leq 2n/3$. □

Before proceeding further, we introduce some additional notation. For each $v \in \mathcal{L}$, let H_v denote the graph obtained from G by deleting v and all 2-paths and 2-handles that have an end adjacent with v, and let $n_v = |V(H_v)|$.

Observation 5.15 *For every vertex $v \in \mathcal{L}$, we may assume that $\delta(H_v) \leq 1$ or H_v has a C_3-component or H_v has a C_7-component.*

Proof. Let $v \in \mathcal{L}$ and suppose that $\delta(H_v) \geq 2$ and that H_v has neither a C_3-component nor a C_7-component. Applying the inductive hypothesis to H_v, $\gamma_{\mathcal{F}}(H_v) \leq 2n_v/3$. Let $\mathcal{C}$ be a $\gamma_{\mathcal{F}}$-coloring of H_v. Let R_v (respectively, B_v) denote the set of vertices of H_v that are colored red (respectively, blue) under $\mathcal{C}$ and are adjacent in G to a vertex of $V(G) \setminus V(H_v)$. Observe that $R_v \subset \mathcal{L}$, and so every neighbor of a vertex in R_v has degree 2, implying that at least one neighbor of every vertex in R_v in H_v must be colored red.

We now write the set $V(G) \setminus (V(H_v) \cup \{v\})$ as the disjoint union of five sets (some of which may possibly be empty), B_1, B_2, R_1, R_2 and H as follows. Let H be the set of vertices that belong to a 2-handle that has an end adjacent with v. Let R_1 (respectively, B_1) denote the set of vertices that belong to a 2-path on one vertex that is adjacent in G with v and a vertex of R_v (respectively, B_v). Let R_2 (respectively, B_2) denote the set of vertices that belong to a 2-path on two vertices that has one end adjacent with v and the other end adjacent with a vertex of R_v (respectively, B_v). For $i = 1, 2$, let $|R_i| = r_i$ and let $|B_i| = b_i$. Let $|H| = h$. Then, $n - n_v = b_1 + b_2 + h + r_1 + r_2 + 1$. We now extend $\mathcal{C}$ to an $\mathcal{F}$-coloring of G that colors at most $2n/3$ vertices red, implying that $\gamma_{\mathcal{F}}(G) \leq 2n/3$, as desired.

Case 1. $r_1 + 1 \leq 2(b_1 + h + r_2) + b_2/2$. We observe then that $r_1 + 1 + b_2/2 \leq 2(n - n_v)/3$. We now color v red. On the one hand, suppose that $r_1 + b_2 = 0$. If $b_1 = 0$, then every 2-path that has an end adjacent with v has order 2. In this case, color both vertices on one 2-path red and color all remaining (at least four) uncolored vertices blue. In this way we extend $\mathcal{C}$ to an $\mathcal{F}$-coloring of G that colors at most $2n_v/3 + 3 \leq 2(n - 7)/3 + 3 < 2n/3$ vertices red. If $b_1 \geq 1$, then color one vertex in B_1 red and color all remaining (at least two) uncolored vertices blue. In this way we extend $\mathcal{C}$ to an $\mathcal{F}$-coloring of G that colors at most $2n_v/3+2 \leq 2(n-4)/3+2 < 2n/3$

vertices red. On the other hand, suppose that $r_1 + b_2 \geq 1$. Then color every vertex in R_1 red, color every neighbor of v in B_2 red, and color all remaining uncolored vertices blue. In this way we extend $\mathcal{C}$ to an $\mathcal{F}$-coloring of G that colors at most $2n_v/3 + r_1 + 1 + b_2/2 \leq 2n_v/3 + 2(n - n_v)/3 = 2n/3$ vertices red, as desired.

Case 2. $r_1+1 \geq 2(b_1+h+r_2)+b_2/2$. Then, $r_1+1 \geq (b_1+b_2+h)/2-r_2/4$, and so $4(r_1+1)+r_2 \geq 2(b_1+b_2+h)$. Hence, $4(b_1+b_2+h+r_1+r_2+1) \geq 6(b_1+b_2+h)+3r_2$, implying that $2(n - n_v)/3 \geq b_1 + b_2 + h + r_2/2$. We now color v blue. On the one hand, suppose that $b_1 + b_2 + h = 0$. Then, $r_1 + 1 \geq 2r_2$. We color every vertex of R_2 that is at distance 2 from v red and color one neighbor of v red. In this way we extend $\mathcal{C}$ to an $\mathcal{F}$-coloring of G that colors at most $2n_v/3+r_2/2+1 \leq 2n_v/3+2(n-n_v)/3 = 2n/3$ vertices red, as desired. On the other hand, suppose that $b_1 + b_2 + h \geq 1$. Then we color every vertex in $B_1 \cup B_2 \cup H$ red and every vertex of R_2 that is at distance 2 from v red and we color all remaining uncolored vertices blue. In this way we extend $\mathcal{C}$ to an $\mathcal{F}$-coloring of G that colors at most $2n_v/3 + b_1 + b_2 + h + r_2/2 \leq 2n_v/3 + 2(n - n_v)/3 = 2n/3$ vertices red, as desired. □

Observation 5.16 *For every vertex $v \in \mathcal{L}$, we may assume that H_v has no C_7-component.*

Proof. Let $v \in \mathcal{L}$ and suppose that H_v has a C_7-component, say $C : v_1, v_2, \ldots, v_7$. Renaming vertices if necessary, we may assume that there exist 2-paths joining v with each of v_1, v_3 and v_6 (i.e., we have a 2-path with one end adjacent to v and the other end adjacent to v_i for each $i \in \{1, 3, 6\}$). Hence, v_1, v_3 and v_6 are large vertices in G. Let $i \in \{1, 3, 6\}$. Then, H_{v_i} is connected and has minimum degree at least 2. By Observation 5.15, H_{v_i} must therefore be a 7-cycle. Hence, there are exactly three 2-paths with an end adjacent to v in G, one of order 2 and the other two each of order 1. Thus, G is the graph shown in Figure 5.2 of order 12, and so $\gamma_{\mathcal{F}}(G) \leq 6 < 2n/3$. □

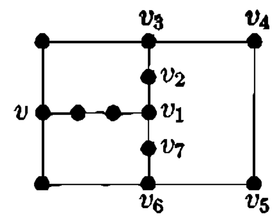


Figure 5.2:

Observation 5.17 *For every vertex $v \in \mathcal{L}$, we may assume that H_v has no C_3-component.*

Proof. Let $v \in \mathcal{L}$ and suppose that H_v has a C_3-component $C\colon u, u_1, u_2, u$. We may assume that $u \in \mathcal{L}$ (and so, u_1 and u_2 are small vertices). By Observation 5.14, $\deg_G u \geq 4$. Every neighbor of u not on C is on a 2-path (of order 1 or 2) that has an end adjacent to u and an end adjacent to v.

Suppose that v, v_1, v_2, u is a path (and so, v_1, v_2 is a 2-path). Let $G' = G - \{v_1, v_2\}$. Then, G' is a connected graph of order at least 4 with $\delta(G') \geq 2$ and $G' \neq C_7$. Applying the inductive hypothesis to G', $\gamma_{\mathcal{F}}(G') \leq 2(n-2)/3$. Let C' be a minimum $\mathcal{F}$-coloring of G'. Since $\deg_G u \geq 4$, we observe that there is a 2-path distinct from v_1, v_2. If C' colors both u and v blue, then u_1 and u_2 are colored red and all vertices on a 2-path that has an end adjacent with u are colored red. But then u has no blue neighbor, contradicting the fact that C' is an $\mathcal{F}$-coloring of G'. Hence at least one of u and v is colored red under C'. We now extend C' to an $\mathcal{F}$-coloring of G as follows: If both u and v are colored red, color v_1 and v_2 blue; if u is colored red and v blue, color v_1 blue and v_2 red; if u is colored blue and v red, color v_1 red and v_2 blue. In this way we extend C' to an $\mathcal{F}$-coloring of G that colors at most $2(n-2)/3 + 1 < 2n/3$ vertices red. Thus we may assume that every 2-path that has an end adjacent with u has order 1.

Suppose that there are three or more 2-paths that have an end adjacent with u. Let x be a vertex of such a 2-path (and so, v, x, u is a path in G) and let $G' = G - x$. Then, G' is a connected graph of order at least 4 with $\delta(G') \geq 2$ and $G' \neq C_7$. Applying the inductive hypothesis to G', $\gamma_{\mathcal{F}}(G') \leq 2(n-1)/3$. Let C' be a minimum $\mathcal{F}$-coloring of G'. If C' colors both u and v red, then every common neighbor of u and v in G' is colored red, while u_1 and u_2 are colored blue. Recoloring u and all but one of its neighbors on a 2-path blue and recoloring u_1 and u_2 red produces a new $\mathcal{F}$-coloring of G' that colors at most as many vertices red as does C'. Hence we may assume that exactly one of u and v is colored red under C'. But then C' can be extended to an $\mathcal{F}$-coloring of G by coloring x blue. In this way we extend C' to an $\mathcal{F}$-coloring of G that colors at most $2(n-1)/3$ vertices red. Thus we may assume that there are exactly two 2-paths (of order 1) that have an end adjacent with u. Let u_3 and u_4 be the vertices of these 2-paths adjacent with u.

Suppose that $\deg_G v \geq 4$. Let $G' = G - N[u]$. Then, G' is a connected graph with $\delta(G') \geq 2$ and $G' \neq C_7$. If $G' = C_3$, then $n = 8$ and $\gamma_{\mathcal{F}}(G) \leq 4 < 2n/3$. Hence we may assume that $G' \neq C_3$. Applying the inductive hypothesis to G', $\gamma_{\mathcal{F}}(G') \leq 2(n-5)/3$. Let C' be a minimum $\mathcal{F}$-coloring of G'. We now extend C' to an $\mathcal{F}$-coloring of G as follows: If v is colored red, color u_1 and u_2 red and the remaining three uncolored vertices blue; if v is colored blue, color u, u_3 and u_4 red and color u_1 and u_2 blue. In this way we extend C' to an $\mathcal{F}$-coloring of G that colors at most $2(n-5)/3 + 3 < 2n/3$ vertices red. Hence we may assume that $\deg_G v = 3$.

Let Q be the 2-path that has an end adjacent with v and an end adjacent with a large vertex w different from u and v. Let $G' = G - N[u] - \{v\} - V(Q)$. Then, G' is a connected graph with $\delta(G') \geq 2$ and $G' \neq C_7$. If $G' = C_3$, then $n \in \{10, 11\}$ and $\gamma_{\mathcal{F}}(G) = 4 < 2n/3$. Hence we may assume that $G' \neq C_3$. Applying the inductive hypothesis to G', $\gamma_{\mathcal{F}}(G') \leq 2(n-6-|V(Q)|)/3 \leq 2(n-7)/3$. A minimum $\mathcal{F}$-coloring of G' can be extended to an $\mathcal{F}$-coloring of G by coloring u and u_3 red, coloring the vertices on Q red, and coloring the remaining four vertices blue. This produces an $\mathcal{F}$-coloring of G that colors at most $2(n-7)/3+4 < 2n/3$ vertices red. □

As a consequence of Observation 5.15, 5.16 and 5.17, for every vertex $v \in \mathcal{L}$ we have that $\delta(H_v) \leq 1$ and H_v has neither a C_3-component nor a C_7-component.

Observation 5.18 *There is a unique partition of the set $\mathcal{L}$ into 2-elements subsets $\{u,v\}$ such that $\deg_{H_u} v \leq 1$ and $\deg_{H_v} u \leq 1$.*

Proof. Let $v \in \mathcal{L}$ and let u be a vertex of minimum degree in H_v. Since $\delta(H_v) \leq 1$, we have that $\deg_{H_v} u \leq 1$ and $u \in \mathcal{L}$. However, $\delta(H_u) \leq 1$, implying that v is the unique vertex in $\mathcal{L}$ such that $\deg_{H_u} v \leq 1$. This in turn implies that u is the unique vertex in $\mathcal{L}$ such that $\deg_{H_v} u \leq 1$. □

Observation 5.19 *There is no 2-handle.*

Proof. Suppose that C is a 2-handle of G. Let $v \in \mathcal{L}$ be the common neighbor of its ends. Then, $\deg_{H_u} v \geq 2$ for every vertex $u \in \mathcal{L} \setminus \{v\}$, contradicting Observation 5.18. □

Observation 5.20 *We may assume that $|\mathcal{L}| \geq 4$.*

Proof. By Observation 5.18, $|\mathcal{L}|$ is even. Suppose $|\mathcal{L}| = 2$. Let $\mathcal{L} = \{u,v\}$. Then for some integers r and s where $r+s \geq 3$, G is the graph of order $n = 2+2s+r$ obtained from $K_{2,r+s}$ by subdividing s edges incident with a large vertex once (and so, there are r 2-paths of order 1 and s 2-paths of order 2). If $s \geq 1$, then color v red, color both vertices on one 2-path red, color every end of a 2-path of order 2 that is adjacent with v red, and color all remaining uncolored vertices blue. In this way we produce an $\mathcal{F}$-coloring of G that colors $s+2 < 2n/3$ vertices red. Hence we may assume that $s = 0$ (and so, $G = K_{2,r}$). We now color v and one of its neighbors red and color all remaining uncolored vertices blue. In this way we produce an $\mathcal{F}$-coloring of G that colors $2 < 2n/3$ vertices red. □

Observation 5.21 *We may assume that* $\delta(H_v) = 1$ *for every vertex* $v \in \mathcal{L}$.

Proof. Let $v \in \mathcal{L}$ and suppose that $\delta(H_v) = 0$. Let u be the unique vertex such that $\deg_{H_v} u \leq 1$ and $\deg_{H_u} v \leq 1$. Then, $\deg_{H_v} u = 0$. Let P be a 2-path that has one end adjacent with u and the other end adjacent with v. Let $G' = G - V(P)$. Then, $\delta(G') \geq 2$ and $G' \neq C_7$. Applying the inductive hypothesis to G', $\gamma_{\mathcal{F}}(G') \leq 2(n - |V(P)|)/3$. Let C' be a minimum $\mathcal{F}$-coloring of G'.

Suppose first that P has order 2. Since C' is an $\mathcal{F}$-coloring of G', at least one of u and v is colored red under C'. Hence we can extend C' to an $\mathcal{F}$-coloring of G by coloring at most one additional vertex of P red; that is, by coloring at most $2(n-2)/3 + 1 < 2n/3$ vertices red. Hence we may assume that every 2-path that has one end adjacent with u is of order 1. In particular, P has order 1.

If both u and v are colored red under C', then every vertex on a 2-path in G' that has an end adjacent with u is colored red. But then recoloring u and one of its neighbors in G' blue, produces an $\mathcal{F}$-coloring of G' that colors fewer vertices red than does C', a contradiction. Hence at most one of u and v is colored red. If both u and v are colored blue, then u would have no blue neighbor (and no red neighbor that is adjacent to a red vertex), contradicting the fact that C' is an $\mathcal{F}$-coloring of G'. Hence exactly one of u and v is colored red. But then C' can be extended to an $\mathcal{F}$-coloring of G by coloring the vertex of P blue, and so $\gamma_{\mathcal{F}}(G) \leq 2(n-1)/3 < 2n/3$. □

Let $v \in \mathcal{L}$ and let u be the unique vertex such that $\deg_{H_u} v \leq 1$ and $\deg_{H_v} u \leq 1$. By Observation 5.21, $\deg_{H_v} u = 1$ and $\deg_{H_u} v = 1$. Let P_v (resp., P_u) be the 2-path that has an end adjacent with v (resp., u) and an end adjacent with a large vertex

v' (resp., u') different from u and v. Let $|V(P_v)| = a$ and $|V(P_u)| = b$ (and so, $1 \leq a, b \leq 2$). We may assume that $a \leq b$. Let R (resp., S) be the set of vertices that belong to a 2-path of order 1 (resp., 2) that has an end adjacent with v and an end adjacent with u. Let $|R| = r$ and $|S| = 2s$.

Let G' be the graph of order n' obtained from G by deleting u and v and all 2-paths with an end adjacent to u or v. If $u' = v'$, then we contradict Observation 5.18. Hence, $u' \neq v'$, and so $\delta(G') \geq 2$. Further, it follows from Observation 5.21 that G' is connected. Applying the inductive hypothesis to G', $\gamma_{\mathcal{F}}(G') \leq 2n'/3$. Let C' be a minimum $\mathcal{F}$-coloring of G'.

Observation 5.22 *We may assume that $s = 0$.*

Proof. Suppose that $s \geq 1$. If $r = 0$, then $s \geq 2$ and $n' = n - 2s - 2 - a - b$. We now extend C' to an $\mathcal{F}$-coloring of G as follows: Color u and v red, color every vertex on P_v and P_u red, and color all remaining uncolored vertices blue. In this way we produce an $\mathcal{F}$-coloring of G that colors at most $2n'/3 + 2 + a + b \leq 2(n - 2s - 2 - a - b)/3 + 2 + a + b < 2n/3$ vertices red. Hence we may assume that $r \geq 1$ (and still $s \geq 1$).

Suppose $b = 2$. Then, $n' = n - 2s - r - a - 4 \leq n - 2s - 6$. We now extend C' to an $\mathcal{F}$-coloring of G as follows: Color u red, color every neighbor of u in S red, and color the vertices of P_v red. If u' is colored blue under C', then color the neighbor of u on P_u red. If $a = 1$, then color one neighbor of u in R red. Color all remaining uncolored vertices blue. In this way (irrespective of whether $a = 1$ or $a = 2$) we color at most $s + 4$ additional vertices red and produce an $\mathcal{F}$-coloring of G that colors at most $2n'/3 + s + 4 \leq 2(n - 2s - 6)/3 + s + 4 < 2n/3$ vertices red. Hence we may assume that $b = 1$ (and so, $a = 1$). Then, $n' = n - 2s - r - 4$.

Suppose at least one of u' and v' is colored red under C', say v'. Since every neighbor of v' has degree 2, at least one neighbor of v' in G' must be colored

red. We now extend C' to an $\mathcal{F}$-coloring of G as follows: Color u red, color every neighbor of u in S red, color one neighbor of u in R red, color the vertex of P_u red, and color all remaining uncolored vertices blue. In this way we color an additional $s+3$ vertices red and produce an $\mathcal{F}$-coloring of G that colors at most $2n'/3+s+3 \leq 2(n-2s-5)/3+s+3 < 2n/3$ vertices red. Hence we may assume that both u' and v' are colored blue under C'.

If $r \geq 2$, then $n' \leq n-2s-6$ and we extend C' to an $\mathcal{F}$-coloring of G as follows: Color u red, color every neighbor of u in S red, color one neighbor of u in R red, color the vertex on P_v red, and color all remaining uncolored vertices blue. In this way we color an additional $s+3$ vertices red and produce an $\mathcal{F}$-coloring of G that colors at most $2n'/3+s+3 < 2n/3$ vertices red. Hence we may assume that $r=1$, and so $n' = n-2s-5 \leq n-7$. We now extend C' to an $\mathcal{F}$-coloring of G as follows: Color u and v red, color the neighbor of u in R red, and color all remaining uncolored vertices blue. In this way we color an additional three vertices red and produce an $\mathcal{F}$-coloring of G that colors at most $2n'/3+3 \leq 2(n-7)/3+3 < 2n/3$ vertices red. □

By Observation 5.22, $s=0$, and so $r \geq 2$.

Suppose $b=2$ (and $1 \leq a \leq 2$). Then, $n' = n-r-a-4 \leq n-a-6$. Suppose u' is colored blue under C'. We now extend C' to an $\mathcal{F}$-coloring of G as follows: Color u red, color a neighbor of u in R red, color the neighbor of u on P_u red, color the vertices on P_v red, and color all remaining uncolored vertices blue. In this way we color an additional $a+3$ vertices red and produce an $\mathcal{F}$-coloring of G that colors at most $2n'/3+a+3 \leq 2(n-a-6)/3+a+3 < 2n/3$ vertices red. Hence we may assume that u' is colored red under C'. We now extend C' to an $\mathcal{F}$-coloring of G as follows: Color the neighbor of u' on P_u red, color v red, color the vertices on P_v red, color a neighbor of v in R red, and color all remaining uncolored vertices blue. In this way we color an additional $3 + a$ vertices red and produce an $\mathcal{F}$-coloring of G that colors at most $2n'/3 + a + 3 \leq 2(n - a - 6)/3 + a + 3 < 2n/3$ vertices red.

Hence we may assume that $a = b = 1$, and so $n' = n - r - 4 \leq n - 6$. We now extend C' to an $\mathcal{F}$-coloring of G as follows: If u' is colored blue under C', then color u and a neighbor of u in R red, color the vertex on P_v red, and color all remaining uncolored vertices blue. If u' is colored red under C', then color v and the vertices on P_u and P_v red, and color all remaining uncolored vertices blue. In this way we color an additional three vertices red and produce an $\mathcal{F}$-coloring of G that colors at most $2n'/3 + 3 \leq 2(n - 6)/3 + 3 < 2n/3$ vertices red. □

We remark that the bound of Theorem 5.10 is attainable as can be seen, for example, with the cycle C_6. However we do not know of any infinite family of graphs which achieves this upper bound.

Chapter 6

TOTAL RESTRAINED DOMINATION IN GRAPHS

6.1 Introduction

In this chapter, we continue our investigation on total domination and restrained domination.

Recall, if S is simultaneously a TDS and a RDS, then S is a *total restrained dominating set* (TRDS) of G. The minimum cardinality of a TRDS of G is the *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$.

A TRDS can be interpreted as a red-blue coloring of the vertices, with the red vertices forming the TRDS. We call a red-blue coloring of vertices such that every blue vertex has both a red and a blue neighbor and every red vertex has a red neighbor a tr-coloring (total restrained coloring) of G. The total restrained domination number $\gamma_{tr}(G)$ of G is the minimum number of red vertices of G in a tr-coloring of G. We call a tr-coloring of G that colors $\gamma_{tr}(G)$ vertices red a γ_{tr}-coloring of G.

6.2 Main Results

Let G be a connected graph of order n and maximum degree Δ. Our aim in this chapter is to investigate a bound on the total restrained domination number in terms of the order and maximum degree of the graph. We shall show:

Theorem 6.1 *If G is a connected graph of order $n \geq 4$, maximum degree Δ where $\Delta \leq n-2$, and minimum degree at least 2, then*

$$\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1,$$

and this bound is sharp.

If we restrict our attention to bipartite graphs, then we show that the bound of Theorem 6.1 can be improved.

Theorem 6.2 *If G is a connected bipartite graph of order $n \geq 5$, maximum degree Δ where $3 \leq \Delta \leq n-2$, and minimum degree at least 2, then*

$$\gamma_{tr}(G) \leq n - \frac{2}{3}\Delta - \frac{2}{9}\sqrt{3\Delta - 8} - \frac{7}{9},$$

and this bound is sharp.

6.3 Notation

For notational convenience, we let

$$\begin{aligned} \varphi(n, \Delta) &= n - \frac{\Delta}{2} - 1, \quad \text{and} \\ \psi(n, \Delta) &= n - \frac{2}{3}\Delta - \frac{2}{9}\sqrt{3\Delta - 8} - \frac{7}{9}. \end{aligned}$$

6.4 Proof of Theorem 6.1

We proceed by induction on $\ell = n + m$, where m denotes the size of G. We wish to show that $\gamma_{tr}(G) \leq \varphi(n, \Delta)$. Note that $n \geq 4$ and $m \geq 4$, and so $\ell \geq 8$. When $\ell = 8$, the graph G is a 4-cycle, and so $\gamma_{tr}(G) = 2 = \varphi(4, 2) = \varphi(n, \Delta)$. This establishes the base case. For the inductive hypothesis, let $\ell \geq 9$ and assume that for all connected graphs G' of order n' and size m' with $n' + m' < \ell$ that have maximum degree Δ' where $\Delta' \leq n' - 2$ and minimum degree at least 2, that $\gamma_{tr}(G') \leq \varphi(n', \Delta')$. Let $G = (V, E)$ be a connected graph of order n and size m with $\ell = n + m$, maximum degree Δ where $\Delta \leq n - 2$ and minimum degree at least 2.

We begin with the following observation.

Observation 6.3 *If a connected proper subgraph G' of G of order n' has maximum degree Δ where $\Delta \leq n' - 2$ and minimum degree at least 2, and if the subgraph $G - V(G')$ contains no isolated vertices, then $\gamma_{tr}(G) \leq \varphi(n, \Delta)$.*

Proof. Let G' have size m'. Then, $n' + m' < \ell$, and so G' satisfies the inductive hypothesis. Let $n' = n - k$ where $k \geq 0$. Then by the inductive hypothesis, $\gamma_{tr}(G') \leq \varphi(n', \Delta) = \varphi(n - k, \Delta) = \varphi(n, \Delta) - k$. Any γ_{tr}-coloring of G' can be extended to a tr-coloring of G by coloring every vertex in $V(G) \setminus V(G')$ with the color red. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + k \leq \varphi(n, \Delta)$, as desired. □

Let v be a vertex of maximum degree Δ in G, and let $\mathcal{L}$ be the set of all large vertices of G.

Observation 6.4 *We may assume that the set $\mathcal{L} \setminus \{v\}$ is an independent set in G.*

Proof. Suppose $e = uw$ is an edge of G joining two vertices u and w of $\mathcal{L} \setminus \{v\}$. If e is a cycle edge, then $G - e$ is a connected subgraph of G that satisfies the statement of Observation 6.3, and so $\gamma_{tr}(G) \leq \varphi(n, \Delta)$. Hence we may assume that e is a bridge of G. Let G_u be the component of $G - e$ containing u, and G_w the component containing w. We may assume that $v \in V(G_u)$. Then, G_u is a connected subgraph of G of order n' with maximum degree Δ and minimum degree at least 2. If $\Delta \leq n' - 2$, then G_u satisfies the statement of Observation 6.3, and so $\gamma_{tr}(G) \leq \varphi(n, \Delta)$. Hence we may assume that v dominates $V(G_u)$. Let $x \in N(u) \setminus \{v, w\}$. Then, $x \in V(G_u)$ and, since $G[\mathcal{L}]$ contains no cycles, x is a small vertex. Coloring the vertices in $(V(G_w) \setminus \{w\}) \cup \{v, x\}$ red and coloring all other vertices blue produces a tr-coloring of G, and so $\gamma_{tr}(G) \leq n - \Delta < \varphi(n, \Delta)$, as desired. □

By Observation 6.4, the only edges in $G[\mathcal{L}]$, if any, are incident with v.

Observation 6.5 *We may assume that G contains no ray.*

Proof. Suppose that G contains a ray $P\colon u, u_1, u_2, w$. Thus both u_1 and u_2 are small vertices of G. If $\Delta = n - 2$, then u or w, say u, is a vertex of maximum degree Δ in G. Coloring u and u_1 red and every other vertex blue produces a tr-coloring of G, and so $\gamma_{tr}(G) = 2 = n - \Delta < \varphi(n, \Delta)$. Hence we may assume that $\Delta \leq n - 3$. Let G' be the graph obtained from G by removing the vertex u_1 and adding the edge uu_2. Then, G' is a connected graph of order $n' = n - 1$ and size $m' = m - 1$, with maximum degree Δ where $\Delta \leq n' - 2$, and minimum degree at least 2. Applying the inductive hypothesis to G', we have that $\gamma_{tr}(G') \leq \varphi(n', \Delta) = \varphi(n-1, \Delta) = \varphi(n, \Delta) - 1$. Any γ_{tr}-coloring of G' can be extended to a tr-coloring of G by coloring the vertex u_1 red, unless u and u_2 are both colored blue, in which case we recolor u_2 red and color u_1 blue. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 1 \leq \varphi(n, \Delta)$, as desired. □

By Observation 6.5, every 2-path in G has order 1, while every 2-handle of G has order 2. Thus every large vertex in G is either adjacent to v or at distance 2 from some large vertex.

Observation 6.6 *We may assume that every two vertices in $\mathcal{L} \setminus \{v\}$ have at most one common small neighbor.*

Proof. Suppose $\mathcal{L} \setminus \{v\}$ contains two vertices u and w that have at least two common small neighbors. Let x be a small vertex that is a common neighbor of u and w. Then, $G' = G - x$ has order $n' = n - 1$, size $m' = m - 2$, maximum degree $\Delta \leq n' - 2$ and minimum degree at least 2. By the inductive hypothesis, $\gamma_{\rm tr}(G') \leq \varphi(n', \Delta) = \varphi(n-1, \Delta) = \varphi(n, \Delta) - 1$. Any $\gamma_{\rm tr}$-coloring of G' colors u or w red, and can therefore be extended to a tr-coloring of G by coloring x red. Hence, $\gamma_{\rm tr}(G) \leq \gamma_{\rm tr}(G') + 1 = \varphi(n, \Delta)$, as desired. □

Before proceeding further, we recall some additional notation. For each $u \in \mathcal{L}$, let H_u denote the graph obtained from G by deleting u and all 2-paths and 2-handles that have an end adjacent with u, and let $n_u = |V(H_u)|$.

Observation 6.7 *If $\gamma_{\rm tr}(H_u) \leq \varphi(n_u, \Delta) + 1$ for some $u \in \mathcal{L} \setminus \{v\}$, then $\gamma_{\rm tr}(G) \leq \varphi(n, \Delta)$.*

Proof. Let $u \in \mathcal{L}\setminus\{v\}$ and suppose that $\gamma_{\rm tr}(H_u) \leq \varphi(n_u, \Delta)+1$. By Observation 6.4, every neighbor of u is either a small vertex or the vertex v. By Observations 6.5 and 6.6, every small neighbor of u is either on a 2-path of order 1 or on a 2-handle of order 2 (with both ends adjacent to u).

Suppose first that u is adjacent to the ends of a 2-handle x, y. If $\deg u = 3$, let w be a neighbor of u different from x and y (possibly, $w = v$). Let G' be the graph obtained from G by deleting u and joining x and y to w. Then, G' has order $n' = n - 1$, size $m' = m - 1$, maximum degree $\Delta' \geq \Delta$ and minimum degree at least 2. If $\Delta' = n' - 1$, then coloring u and w red and all remaining uncolored vertices of G blue produces a tr-coloring of G. Hence, $\gamma_{tr}(G) = 2 \leq n - \Delta < \varphi(n, \Delta)$. Thus we may assume $\Delta' \leq n' - 2$. By the inductive hypothesis, $\gamma_{tr}(G') \leq \varphi(n', \Delta') \leq \varphi(n-1, \Delta) = \varphi(n, \Delta) - 1$. Any γ_{tr}-coloring of G' can be extended to a tr-coloring of G by coloring u red, and so $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 1 \leq \varphi(n, \Delta)$. Hence we may assume that $\deg u \geq 4$. Then, $G - \{x, y\}$ satisfies the statement of Observation 6.3, and so $\gamma_{tr}(G) \leq \varphi(n, \Delta)$.

Thus we may assume that every small neighbor of u is on a 2-path of order 1. Let C' be a γ_{tr}-coloring of H_u. We extend C' to a tr-coloring of G as follows. If C' colors a vertex in H_u that has a common small neighbor with u blue, then we color this common small neighbor blue and color all remaining uncolored vertices of G red. Otherwise, we color u and a small neighbor of u blue and color all remaining uncolored vertices of G red. In this way we extend C' to a tr-coloring of G that colors at most $n - n_u - 1$ additional vertices red, and so $\gamma_{tr}(G) \leq \gamma_{tr}(H_u) + n - n_u - 1 \leq (\varphi(n_u, \Delta) + 1) + n - n_u - 1 = \varphi(n, \Delta)$, as desired. □

Observation 6.8 *We may assume that G has no 2-handle.*

Proof. Suppose that G has a 2-handle x, y. Let u be the large vertex adjacent to x and y. If $\deg u = 3$ or if $\deg u \geq 4$ and $\deg u < \Delta$, then by using a similar argument as in the proof of Observation 6.7, the result follows. Hence we may assume that $\deg u = \Delta \geq 4$. Renaming vertices, if necessary, we may assume that $u = v$. We consider two cases.

Case 1. There is a neighbor w of v that has no common neighbor with v.

Suppose that w is a small vertex. Let z be the (large) neighbor of w different from v. Then, vz is not an edge of G. Suppose v and z have at least two common small neighbors. Let G' be the graph obtained from G by deleting w and adding the edge vz. Then, G' has order $n' = n - 1$, size $m' < m$, maximum degree Δ and minimum degree at least 2. If $\Delta = n' - 1$, then coloring v and w red and all remaining uncolored vertices of G blue produces a tr-coloring of G, and so $\gamma_{tr}(G) = 2 \le n - \Delta < \varphi(n, \Delta)$. Thus we may assume $\Delta \le n' - 2$. By the inductive hypothesis, $\gamma_{tr}(G') \le \varphi(n', \Delta') \le \varphi(n, \Delta) - 1$. Any γ_{tr}-coloring of G' colors v or z red, and can therefore be extended to a tr-coloring of G by coloring w red. Hence, $\gamma_{tr}(G) \le \gamma_{tr}(G') + 1 \le \varphi(n, \Delta)$. Thus we may assume v and z have exactly one common neighbor, namely w. Then, H_z has order $n_z \le n - 4$, size $m' < m$, maximum degree $\Delta' = \Delta - 1$ and minimum degree at least 2. If $\Delta' = n_z - 1$, then coloring v, w and z red and all remaining uncolored vertices of G blue produces a tr-coloring of G. Hence, $\gamma_{tr}(G) = 3 < n - \Delta < \varphi(n, \Delta)$. Thus we may assume $\Delta' \le n_z - 2$. By the inductive hypothesis, $\gamma_{tr}(H_z) \le \varphi(n_z, \Delta') \le \varphi(n_z, \Delta) + \frac{1}{2}$. Hence, H_z satisfies the statement of Observation 6.7, and so $\gamma_{tr}(G) \le \varphi(n, \Delta)$.

Thus we may assume that w is a large vertex. Then, H_w has order $n_w \le n-3$, size $m' < m$, maximum degree $\Delta' = \Delta - 1$ and minimum degree at least 2. If $\Delta' = n_w - 1$, then coloring v and w red and all remaining uncolored vertices of G blue produces a tr-coloring of G. Hence, $\gamma_{tr}(H_w) = 2 \le n - \Delta < \varphi(n, \Delta)$. Thus we may assume $\Delta' \le n_w - 2$. By the inductive hypothesis, $\gamma_{tr}(H_w) \le \varphi(n_w, \Delta') = \varphi(n_w, \Delta) + \frac{1}{2}$. Hence, H_w satisfies the statement of Observation 6.7, and so $\gamma_{tr}(G) \le \varphi(n, \Delta)$.

Case 2. Every neighbor of v lies in a common triangle with v. Since $\Delta \le n-2$, at least one vertex of G is not a neighbor of v. Let S denote the set of all those vertices that are isolated in the subgraph induced by $V(G) \setminus N[v]$. Let H be the subgraph of G induced by $N[v] \cup S$. Suppose first that $S = \emptyset$. Then $V(H) = N[v]$. Hence every vertex in $V(G) \setminus N[v]$ has degree at least 1 in $G - V(H)$. By Observation 6.4, there exist a vertex w of degree 1 in $G - V(H)$. But then w is adjacent to a vertex $v_1 \in N(v)$ and v_1 lies in a common triangle with v and a small vertex v'. We obtain a tr-coloring of G that colors $n - \Delta + 1$ vertices red by coloring v, x and y red, coloring every vertex in $V(G)\setminus(N[v]\cup\{w\})$ red and coloring all remaining uncolored vertices of G blue. Thus, since $\Delta \geq 4$, $\gamma_{tr}(G) \leq n-\Delta+1 \leq \varphi(n,\Delta)$. Hence we may assume that $S \neq \emptyset$, and so H satisfies the statement of Observation 6.3. This implies that $H = G$, and therefore $S = V(G)\setminus N[v]$.

By Observation 6.4, every vertex of S is a small vertex of G and $N(S) \subseteq \mathcal{L}\setminus\{v\}$. Let $\mathcal{L}_S = \mathcal{L}\cap N(S)$. Then, $\mathcal{L}_S$ is an independent set and every vertex of $\mathcal{L}$ lies in a common triangle with v. By Observation 6.6, every two vertices in $\mathcal{L}_S$ have at most one common neighbor.

Suppose every vertex in $\mathcal{L}_S$ has at least two common neighbors with v. Observe that every vertex in S is adjacent to exactly two vertices in $\mathcal{L}_S$ and every vertex in $\mathcal{L}_S$ lies in a common triangle with v. Let v_1 and v_2 be two vertices in $\mathcal{L}_S$. Then v_1 (resp., v_2) has at least two common neighbors with v. Let w_1 be a common neighbor of v_1 and v and let w_2 be a common neighbor of v_2 and v. Let G' be the graph obtained from G by deleting the vertices x, y, w_1 and w_2. Then, G' has order $n' = n-4$, size $m' < m$, maximum degree Δ' where $\Delta' = \Delta-4 \leq n'-2$ and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \leq \varphi(n',\Delta') = \varphi(n-4,\Delta-4) = \varphi(n,\Delta)-2$. If a γ_{tr}-coloring of G' colors v red, then we color w_1 and w_2 red and color x and y blue. Otherwise, we color w_1 and w_2 blue and color x and y red. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G')+2 \leq \varphi(n,\Delta)$. Thus we may assume there is a vertex in $\mathcal{L}_S$, say z, that has exactly one common neighbor with v.

Then, H_z has order $n_z \leq n-3$, size $m' < m$, maximum degree $\Delta' = \Delta - 2$ and minimum degree at least 2. If $\Delta' = n_z - 1$, then coloring v red, coloring z and its common neighbor with v red and coloring the remaining uncolored vertices of G blue, produces a tr-coloring of G. Therefore, since $\Delta > 4$, $\gamma_{tr}(G) = 3 < \varphi(n,\Delta)$. Thus we may assume that $\Delta' \leq n_z - 2$. Then, H_z satisfies the inductive hypothesis, and so $\gamma_{tr}(H_z) \leq \varphi(n_z,\Delta') = \varphi(n_z,\Delta)+1$. Hence, H_z satisfies the statement of Observation 6.7, and so $\gamma_{tr}(G) \leq \varphi(n,\Delta)$, as desired. □.

Observation 6.9 *We may assume that every vertex in $\mathcal{L} \setminus \{v\}$ has a neighbor that is not a neighbor of v.*

Proof. Suppose $\mathcal{L} \setminus \{v\}$ contains a vertex w such that every neighbor of w is a neighbor of v. Then v and w contain at least two common neighbors. Let x be a common neighbor of v and w and let $G' = G - x$. Then G' has order $n' = n - 1$, size $m' = m - 2$, maximum degree $\Delta' = \Delta - 1 \leq n' - 2$ and minimum degree at least 2. Hence, G' satisfies the inductive hypothesis, and so $\gamma_{\rm tr}(G') \leq \varphi(n', \Delta') = \varphi(n-1, \Delta-1) = \varphi(n, \Delta) - \frac{1}{2}$. Every $\gamma_{\rm tr}$-coloring C' of G' colors v or w red. If C' color both v and w red, then we recolor w blue and color x blue; otherwise, we color x blue. Hence, $\gamma_{\rm tr}(G) \leq \gamma_{\rm tr}(G') < \varphi(n, \Delta)$, as desired. □

Observation 6.10 *We may assume that every vertex in $\mathcal{L} \setminus \{v\}$ is adjacent to v.*

Proof. Suppose $\mathcal{L} \setminus \{v\}$ contains a vertex w that is not adjacent to v. Suppose first that v and w have no common neighbors. Then, H_w satisfies the statement of Observation 6.7, and so $\gamma_{\rm tr}(G) \leq \varphi(n, \Delta)$. Hence we may assume that v and w have at least one common neighbor. By Observation 6.4, every common neighbor of v and w is a small vertex. Let x be a common neighbor of v and w.

Suppose v and w have at least two common neighbors. Let G' be the graph obtained from G by deleting x and adding the edge vw. Then, G' has order $n' = n-1$, size $m' < m$, maximum degree Δ and minimum degree at least 2. If $\Delta = n' - 1$, then coloring u and x red and all remaining uncolored vertices of G blue produces a tr-coloring of G. Hence $\gamma_{\rm tr}(G) = 2 \leq n - \Delta < \varphi(n, \Delta)$. Thus we may assume $\Delta \leq n' - 2$. By the inductive hypothesis, $\gamma_{\rm tr}(G') \leq \varphi(n', \Delta') \leq \varphi(n, \Delta) - 1$. Any $\gamma_{\rm tr}$-coloring of G' colors v or w red, and can therefore be extended to a tr-coloring of G by coloring x red. Hence, $\gamma_{\rm tr}(G) \leq \gamma_{\rm tr}(G') + 1 \leq \varphi(n, \Delta)$. Thus we may assume that v and w have at most one common neighbor. But then H_w satisfies the statement of Observation 6.7, and so $\gamma_{\rm tr}(G) \leq \varphi(n, \Delta)$, as desired. □

Observation 6.11 *We may assume that every vertex in $\mathcal{L} \setminus \{v\}$ has a common neighbor with v.*

Proof. Suppose $\mathcal{L} \setminus \{v\}$ contains a vertex w that has no common neighbor with v. By Observation 6.10, v and w are adjacent in G. Then, H_w satisfies the statement of Observation 6.7, and so $\gamma_{tr}(G) \leq \varphi(n, \Delta)$, as desired. □

With our earlier assumptions, we have that $\mathcal{L} \setminus \{v\}$ is an independent set and that any two vertices in $\mathcal{L} \setminus \{v\}$ have at most one common neighbor. Furthermore, each vertex in $\mathcal{L} \setminus \{v\}$ is adjacent to v, has at least one common neighbor with v and has at least one neighbor that is a small vertex not adjacent to v. Let $|\mathcal{L} \setminus \{v\}| = k$. Then $k \leq \Delta/2$. We now color v and every neighbor of v that is small vertex blue and color the remaining uncolored vertices of G red. Hence, $\gamma_{tr}(G) \leq n - (\Delta - k + 1) = n - \Delta + k - 1 \leq n - \Delta + \Delta/2 - 1 = \varphi(n, \Delta)$. This establishes the upper bound of the theorem.

It remains for us to show that this upper bound is sharp. Let G be the graph obtained from a complete graph on t vertices in which every edge is subdivided exactly once and identifying one vertex v that is a large vertex and joining v to every other large vertex of the resulting graph. Then, $n = t + \binom{t}{2} = 1 + 2(t-1) + \binom{t-1}{2}$ and $\Delta = 2(t-1)$. Every tr-coloring of G colors at least $t-1$ large vertices red (in order to totally dominate all the small vertices) and therefore colors at least $\binom{t-1}{2}$ small vertices red. Thus, $\gamma_{tr}(G) \geq (t-1) + \binom{t-1}{2} = n - (t-1) - 1 = n - \Delta/2 - 1 = \varphi(n, \Delta)$. Since the graph G satisfies the conditions of the theorem, we have already established that $\gamma_{tr}(G) \leq \varphi(n, \Delta)$. Consequently, $\gamma_{tr}(G) = \varphi(n, \Delta)$. This concludes the proof of Theorem 6.1. □

6.5 Proof of Theorem 6.2

Before presenting a proof of Theorem 6.2, we first prove a key lemma that will be very useful in proving our main result.

Lemma 6.12 *Let G be a connected bipartite graph of order $n \geq 5$, maximum degree $\Delta \geq 3$ and minimum degree at least 2. If G has a vertex of maximum degree Δ that is adjacent only to degree-2 vertices, then $\gamma_{tr}(G) \leq \psi(n, \Delta)$, and this bound is sharp.*

Proof. Let v be a vertex of maximum degree Δ in G. By assumption, every vertex adjacent to v has degree 2. Let A and B be the set of vertices u at distance 2 from v such that $N(u) \subseteq N(v)$ and $N(u) \not\subseteq N(v)$, respectively. Hence every vertex in B has at least one neighbor that is not a neighbor of v. Let $|A| = a$ and $|B| = b$. If $a \geq 1$, let $A = \{v_1, \ldots, v_a\}$ and if $b \geq 1$, let $B = \{w_1, \ldots, w_b\}$. If $a \geq 1$, then for $i = 1, \ldots, a$, let $\deg v_i = |N(v_i)| = \ell_i$ and let

$$\ell = \sum_{i=1}^{a} \ell_i.$$

By assumption $\delta(G) \geq 2$, and so $\ell_i \geq 2$ for $i = 1, \ldots, a$. Thus, $\ell \geq 2a$. If $b \geq 1$, then for $i = 1, \ldots, b$, let $N_i = N(v) \cap N(w_i)$, let $|N_i| = r_i$ and let

$$r = \sum_{i=1}^{b} r_i.$$

Then, $\Delta = \ell + r$. Let

$$\alpha = n - \frac{2}{3}(\ell + r) - \frac{2}{9}\sqrt{3(\ell + r) - 8} - \frac{7}{9}.$$

Hence we wish to show that

$$\gamma_{tr}(G) \leq \alpha.$$

For this purpose, we consider two tr-colorings of G, one in which v is colored red and the other in which v is colored blue.

We begin with the following observation.

Observation 6.13 *There exists a* tr-*coloring of* G *that colors the vertex* v *red and colors at least* $\ell + 2\sqrt{r} - 1$ *vertices blue.*

Proof. We begin by coloring v red. Then for each $i = 1, \ldots, a$, color one vertex in $N(v_i)$ red. Amongst all the sets, N_i, $1 \le i \le b$, we choose one of maximum cardinality, say N_1. By the Pigeonhole Principle, $|N_1| = r_1 \ge r/b$.

Let $F = G - N[v] - A$. Since G is a bipartite graph, the set B is an independent set in G. Let S be a packing in F that contains the vertex w_1 and as many other vertices from the set B. Hence the vertices of S are pairwise at distance greater than 2 apart in F (and therefore in G). Each vertex in $B \setminus S$ is at distance 2 from some vertex of S (by the maximality of S).

We now color each vertex in the set S blue. For each vertex of S, we color all but one of its common neighbors with v blue. For each vertex in $B \setminus S$, we select one of its neighbors that is also a neighbor of some vertex of S and we color this common neighbor with the color blue. We color all remaining uncolored vertices with the color red. In the resulting red-blue coloring of the vertices of G, if some vertex of S has no blue neighbor (such a vertex would have exactly one common neighbor with v), then we recolor one of its neighbors that is not common with v with the color blue. In this way we produces a tr-coloring of G in which (i) at least b vertices, including the vertex w_1, of F are colored blue, (ii) each vertex in A and all but one neighbor of each vertex of A is colored blue, and (iii) for each vertex of $S \subseteq B$, all but one of its common neighbors with v are colored blue. Hence in this tr-coloring of G we have colored at least $\ell + (r_1 - 1) + b \ge \ell + r/b + b - 1$ vertices blue. Since the function $r/b + b$ (for r fixed) is minimized when $b = \sqrt{r}$, it follows that our tr-coloring of G colors at least $\ell + 2\sqrt{r} - 1$ vertices blue, as desired. □

Observation 6.14 *There exists a* tr-*coloring of* G *that colors* v *blue and colors at least* $\ell/2 + r + 1$ *vertices blue.*

Proof. We begin by coloring v blue. For each $i = 1, \ldots, a$, we color v_i and one neighbor of v_i red. We then color all remaining uncolored vertices in $N(v)$ blue. Thereafter, we color all remaining uncolored vertices in G red. In this way we produce a tr-coloring of G that colors v blue and colors all but a neighbors of v blue. Since $\ell \geq 2a$, this tr-coloring of G colors $\ell + r - a + 1 \geq \ell/2 + r + 1$ vertices blue, as desired. □

Let $\mathcal{C}$ be a $\gamma_{\rm tr}$-coloring of G. Hence among all tr-colorings of G, the coloring $\mathcal{C}$ maximizes the number of vertices that can be colored blue. If $\mathcal{C}$ colors v red, then, by Observation 6.13, $\mathcal{C}$ colors at least $\ell + 2\sqrt{r} - 1$ vertices blue. On the other hand, if $\mathcal{C}$ colors v blue, then, by Observation 6.14, $\mathcal{C}$ colors at least $\ell/2 + r + 1$ vertices blue. Hence letting

$$\begin{aligned} \alpha_1 &= n - \tfrac{\ell}{2} - r - 1, \qquad \text{and} \\ \alpha_2 &= n - \ell - 2\sqrt{r} + 1, \end{aligned}$$

we have that

$$\gamma_{\rm tr}(G) \leq \min\{\alpha_1, \alpha_2\}.$$

We consider two possibilities.

Case 1. $\boldsymbol{\ell \leq 2(r - 2\sqrt{r} + 2)}$. Then, $\alpha_1 \leq \alpha_2$, and so $\gamma_{tr}(G) \leq \alpha_1$. Hence it suffices for us to show that $\alpha_1 \leq \alpha$. Now,

$$
\begin{aligned}
& & \alpha_1 &\leq \alpha \\
&\Leftrightarrow & n - \tfrac{\ell}{2} - r - 1 &\leq n - \frac{2}{3}(\ell + r) - \frac{2}{9}\sqrt{3(\ell + r) - 8} - \frac{7}{9} \\
&\Leftrightarrow & 4\sqrt{3(\ell + r) - 8} &\leq -3\ell + 6r + 4 \\
&\Leftrightarrow & 0 &\leq 9\ell^2 - 36(r + 2)\ell + (36r^2 + 144) \\
&\Leftrightarrow & \ell &\leq 2(r - 2\sqrt{r} + 2) \text{ or } \ell \geq 2(r + 2\sqrt{r} + 2).
\end{aligned}
$$

By assumption, $\ell \leq 2(r - 2\sqrt{r} + 2)$, implying that $\alpha_1 \leq \alpha$, whence $\gamma_{tr}(G) \leq \alpha$, as desired.

Case 2. $\boldsymbol{\ell \geq 2(r - 2\sqrt{r} + 2)}$. Then, $\alpha_2 \leq \alpha_1$, and so $\gamma_{tr}(G) \leq \alpha_2$. Hence it suffices for us to show that $\alpha_2 \leq \alpha$. Now,

$$
\begin{aligned}
& & \alpha_2 &\leq \alpha \\
&\Leftrightarrow & n - \ell - 2\sqrt{r} + 1 &\leq n - \frac{2}{3}(\ell + r) - \frac{2}{9}\sqrt{3(\ell + r) - 8} - \frac{7}{9} \\
&\Leftrightarrow & 2\sqrt{3(l + r) - 8} &\leq 3\ell - 6r + 18\sqrt{r} - 6 \\
&\Leftrightarrow & 0 &\leq 9\ell^2 - (36r - 108\sqrt{r} + 108)\ell + (36r^2 - 216r^{\frac{3}{2}} + 504r - 576\sqrt{r} + 288) \\
&\Leftrightarrow & \ell &\leq 2(r - 4\sqrt{r} + 2) \text{ or } \ell \geq 2(r - 2\sqrt{r} + 2).
\end{aligned}
$$

By assumption, $\ell \geq 2(r - 2\sqrt{r} + 2)$, implying that $\alpha_2 \leq \alpha$, whence $\gamma_{tr}(G) \leq \alpha$, as desired.

In both cases, the desired upper bound follows. It remains for us to establish that the upper bound is sharp. Let $t \geq 2$ be an integer, and let $a = t^2 - 2t + 2$, $b = t$, $\ell = 2(t^2 - 2t + 2)$ and $r = t^2$. Let $H = aP_3 \cup bK_{1,t}$. Let A be the set of a central vertices of the paths P_3, and let B be the set of b central vertices of the stars $K_{1,t}$. Let G be the graph obtained from H by forming a clique on the set B, subdividing each edge of the resulting complete graph on these b vertices exactly once, and adding a new vertex v and joining it to every vertex of degree 1 in H. Then, v has maximum degree in G, namely $2a + bt = \ell + r$. By construction, G is a connected bipartite graph of order n, maximum degree Δ and minimum degree at least 2, where $\Delta = \ell + r$ and $n = 1 + a + b + \binom{b}{2} + \ell + r$. Thus,

$$\begin{aligned} \Delta &= 3t^2 - 4t + 4, \qquad \text{and} \\ n &= \frac{9}{2}t^2 - \frac{11}{2}t + 7. \end{aligned}$$

Further, the vertex v is a vertex of maximum degree Δ in G that is only adjacent to degree-2 vertices. Thus the conditions of the lemma are satisfied. We show that the graph G achieves the upper bound of the lemma. Let $\mathcal{C}$ be a $\gamma_{\rm tr}$-coloring of G. We consider two possibilities.

Suppose $\mathcal{C}$ colors the vertex v blue. Then every vertex of B is red (since each common neighbor of v and a vertex of B must have a red neighbor), whence every degree-2 vertex joining two vertices of B is red. Further, each vertex of A is colored red (since each common neighbor of v and a vertex of A must have a red neighbor). Since each vertex of A must have a red neighbor, one neighbor of each vertex of A is colored red. Thus at least $n - (r + \ell/2 + 1) = n - 2t^2 + 2t - 3$ vertices are colored red.

On the other hand, suppose $\mathcal{C}$ colors the vertex v red. If two vertices of B are colored blue, then the common neighbor of these two vertices has no red neighbor, a contradiction. Hence at least $b - 1$ vertices of B are colored red. Let B' be a subset of $b-1$ vertices of B that are colored red. Every degree-2 vertex joining two vertices of B' is red. Every degree-2 vertex joining v and a vertex of B' is red. At least one neighbor of the vertex in $B \setminus B'$ is colored red. Since each vertex of A must have a red neighbor, one neighbor of each vertex of A is colored red. Thus at least $n-(\ell+2t-1)=n-2t^2+2t-3$ vertices are colored red.

In both cases, the $\gamma_{\rm tr}$-coloring $\mathcal{C}$ of G colors at least $n-2t^2+2t-3$ vertices red. Hence, $\gamma_{\rm tr}(G) \geq n-2t^2+2t-3 = \psi(n,\Delta)$. Since the upper bound of the lemma has been established, we know that $\gamma_{\rm tr}(G) \leq \psi(n,\Delta)$. Consequently, $\gamma_{\rm tr}(G) = \psi(n,\Delta)$. This completes the proof of the lemma. □

We are now ready to prove the main result of this section. Recall Theorem 6.2.

Theorem 6.2. *If G is a connected bipartite graph of order $n \geq 5$, maximum degree Δ where $3 \leq \Delta \leq n-2$, and minimum degree at least 2, then $\gamma_{\rm tr}(G) \leq \psi(n,\Delta)$, and this bound is sharp.*

Proof. We proceed by induction on $\ell = n+m$, where m denotes the size of G. Note that $n \geq 5$ and $m \geq 6$, and so $\ell \geq 11$. When $\ell = 11$, $G = K_{2,3}$ and $\gamma_{\rm tr}(G) = 2 = \psi(5,3) = \psi(n,\Delta)$. This establishes the base case. For the inductive hypothesis, let $\ell \geq 12$ and assume that for all connected bipartite graphs G' of order $n' \geq 5$ and size m' with $n'+m' < \ell$ that have maximum degree Δ' where $3 \leq \Delta' \leq n'-2$ and minimum degree at least 2 that $\gamma_{\rm tr}(G') \leq \psi(n',\Delta')$. Let G be a connected bipartite graph of order $n \geq 5$ and size m with $\ell = n+m$, maximum degree Δ where $3 \leq \Delta \leq n-2$ and minimum degree at least 2.

The proof of the following observation is almost identical to the proof of Observation 6.3, and is therefore omitted.

Observation 6.15 *If a connected proper subgraph G' of G of order n' has maximum degree Δ where $3 \leq \Delta \leq n'-2$ and minimum degree at least 2, and if the subgraph*

$G - V(G')$ contains no isolated vertices, then $\gamma_{tr}(G) \le \psi(n, \Delta)$.

Let v be a vertex of maximum degree Δ in G. Recall that $\mathcal{L}$ is the set of all large vertices of G.

Observation 6.16 *We may assume that the set $\mathcal{L} \setminus \{v\}$ is an independent set in G.*

Proof. Suppose $e = uw$ is an edge of G joining two vertices u and w of $\mathcal{L} \setminus \{v\}$. If e is a cycle edge, then $G - e$ is a connected proper subgraph of G that satisfies the statement of Observation 6.15, and so $\gamma_{tr}(G) \le \psi(n, \Delta)$. Hence we may assume that e is a bridge of G. Let G_u be the component of $G - e$ containing u, and G_w the component containing w. We may assume that $v \in V(G_u)$. Hence, G_u is a connected proper subgraph of G of order n' with maximum degree Δ where $\Delta \ge 3$ and minimum degree at least 2. If v dominates G_u, then u and v have a common neighbor. But this contradicts our assumption that G is bipartite. Hence, $\Delta \le n' - 2$. Thus, G_u satisfies the statement of Observation 6.15, and so $\gamma_{tr}(G) \le \psi(n, \Delta)$, as desired. □

By Observation 6.16, the only edges in $G[\mathcal{L}]$, if any, are incident with v.

Observation 6.17 *We may assume that every two vertices in $\mathcal{L} \setminus \{v\}$ have at most one common neighbor different from v.*

Proof. Suppose $\mathcal{L} \setminus \{v\}$ contains two vertices u and w that have at least two common neighbors different from v. Then both these common neighbors are small. Let x be a small vertex that is a common neighbor of u and w. Then, $G' = G - x$ is a connected bipartite graph of order $n' = n - 1$, size $m' = m - 2$, maximum degree Δ

where $3 \le \Delta \le n' - 2$ and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \le \psi(n', \Delta) = \psi(n-1, \Delta) = \psi(n, \Delta) - 1$. Any γ_{tr}-coloring of G' colors u or w red, and can therefore be extended to a tr-coloring of G by coloring x red. Hence, $\gamma_{tr}(G) \le \gamma_{tr}(G') + 1 = \psi(n, \Delta)$, as desired. □

Observation 6.18 *We may assume that there is no 2-handle whose ends are adjacent with a vertex in $\mathcal{L} \setminus \{v\}$ that is adjacent with v.*

Proof. Suppose that there is a 2-handle C whose ends are adjacent with a vertex $u \in \mathcal{L} \setminus \{v\}$ that is adjacent with v. Since G is bipartite, the $V(C)$ consists of an odd number of vertices. Let C be the 2-handle $u_1, u_2, \ldots, u_k$, for some $k \ge 3$.

Suppose that $k \ge 5$. Let G' be the graph obtained from G by deleting the vertices u_1 and u_2 and adding the edge uu_3. Then, G' is a connected bipartite graph of order $n' = n - 2$, size $m' < m$, maximum degree Δ where $3 \le \Delta \le n' - 2$, and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \le \psi(n', \Delta) = \psi(n, \Delta) - 2$. If a γ_{tr}-coloring of G' colors u or u_3 red, then we color u_1 and u_2 red. Otherwise, we recolor u_3 red, color u_2 red and color u_1 blue. Hence, $\gamma_{tr}(G) \le \gamma_{tr}(G') + 2 \le \psi(n, \Delta)$. Thus we may assume that $k = 3$.

If $\deg u \ge 4$, then $G - V(C)$ satisfies the statement of Observation 6.15, and so $\gamma_{tr}(G) \le \psi(n, \Delta)$. Hence we may assume $\deg u = 3$. Let G' be the graph obtained from G by deleting u and adding the edges $u_1 v$ and $u_3 v$. Then, G' is a connected bipartite graph of order $n' = n - 1$, size $m' = m - 1$, maximum degree $\Delta' = \Delta + 1$ where $3 \le \Delta' \le n' - 2$, and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \le \psi(n', \Delta') = \psi(n - 1, \Delta + 1) = \psi(n, \Delta + 1) - 1$. For $\Delta \ge 3$, $\psi(n, \Delta) - \psi(n, \Delta + 1) = 2/3 + 2(\sqrt{3\Delta - 5} - \sqrt{3\Delta - 8})/9$, and so $2/3 < \psi(n, \Delta) - \psi(n, \Delta+1) \le 8/9$. Thus, $\gamma_{tr}(G') \le \psi(n, \Delta+1) - 1 < \psi(n, \Delta) - 5/3$.

If a γ_{tr}-coloring of G' colors v red, then we color u red. If a γ_{tr}-coloring of G' colors v blue and colors a neighbor of v different from u_1 and u_3 red, then we color

u blue. If a γ_{tr}-coloring of G' colors v blue and colors every neighbor of v different from u_1 and u_3 blue, then exactly one of u_1 and u_3, say u_1, is colored red, and we recolor u_2 blue and color u red. In this way, we produce a tr-coloring of G from a γ_{tr}-coloring of G' that colors at most $\gamma_{tr}(G')+1$ vertices in G red. Hence, $\gamma_{tr}(G) \le \gamma_{tr}(G')+1 < \psi(n,\Delta) - 2/3 < \psi(n,\Delta)$, as desired. □

Observation 6.19 *If v has a large neighbor u, then we may assume that every vertex at distance 2 from u in $G-v$ is a large vertex in G.*

Proof. Suppose there is a vertex at distance 2 from u in $G-v$ that is a small vertex in G. We consider two cases.

Case 1. *There is a small vertex at distance 2 from u in $G-v$ that is not adjacent to v in G.* Let y be a small vertex at distance 2 from u in $G-v$ that is not adjacent to v in G, and let x be the common (small) neighbor of u and y. Let w be the neighbor of y different from x. Since G is bipartite, v and w are not adjacent. If w is large, then $G' = G - \{x,y\}$ satisfies the statement of Observation 6.15, and so $\gamma_{tr}(G) \le \psi(n,\Delta)$. Hence we may assume w is a small vertex. By Observation 6.18, u and w are not adjacent vertices. Let $N(w) = \{y,z\}$.

Let G' be the graph obtained from G by deleting x and y and adding the edge uw. Then, G' is a connected bipartite graph of order $n' = n-2$, size $m' < m$, maximum degree Δ where $3 \le \Delta \le n'-2$, and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \le \psi(n',\Delta) \le \psi(n,\Delta) - 2$. If a γ_{tr}-coloring of G' colors u or w red, then we color x and y red. Otherwise, if a γ_{tr}-coloring of G' colors both u and w blue, then it colors z red and we can therefore recolor w red, color y red and color x blue. Hence, $\gamma_{tr}(G) \le \gamma_{tr}(G') + 2 \le \psi(n,\Delta)$.

Case 2. *Every small vertex at distance 2 from u in $G - v$ is adjacent to v in G.* Let y be a small vertex at distance 2 from u in $G - v$ and let x be the common (small) neighbor of u and y. Then, v and y are adjacent vertices in G. Let S be the set of vertices that belong to a 2-path where one end is adjacent to u and the other end is adjacent to a large vertex different from v (possibly, $S = \emptyset$). Then, every vertex of S is the common small neighbor of u and a large vertex different from v. Further, by Observation 6.17, every two vertices in S have only the vertex u as a common neighbor. Let T denote the set of vertices that lie on a 2-path with one end adjacent to u and the other end adjacent to v, and let T_u and T_v be the vertices in T adjacent with u and v, respectively. Then, $x \in T_u$ and $y \in T_v$, and $N(u) = S \cup T_u \cup \{v\}$.

Case 2.1. $S \neq \emptyset$. Let G' be the component of $G - S$ that contains v (possibly, $G' = G - S$). Then, G' is a connected bipartite graph of order $n' = n - k$ where $k \geq |S|$, size $m' < m$, maximum degree Δ where $3 \leq \Delta \leq n' - 2$, and minimum degree at least 2. By the inductive hypothesis, $\gamma_{\mathrm{tr}}(G') \leq \psi(n', \Delta) = \psi(n, \Delta) - k$. Note that in G', $N(u) = T_u \cup \{v\}$.

Consider a γ_{tr}-coloring C' of G'. If C' colors both u and v blue, then it colors every vertex in T red, and so we recolor u red and every vertex of T_v blue. If C' colors u blue and v red, then it colors every vertex in T_v red and every vertex in T_u blue, and so we recolor u red and recolor every vertex in T_y blue. In both cases, we must have that $T = \{x, y\}$, for otherwise, we produce a new tr-coloring of G' that colors fewer vertices red than does C', which is impossible. Hence in both cases we produce a new γ_{tr}-coloring of G' that colors u red. Therefore we may assume that C' colors u red. But then we can extend C' to a tr-coloring of G by coloring all remaining k uncolored vertices red. Hence, $\gamma_{\mathrm{tr}}(G) \leq \gamma_{\mathrm{tr}}(G') + k \leq \psi(n, \Delta)$.

Case 2.2. $S = \emptyset$. Then, $N(u) = T_u \cup \{v\}$. Since $\deg u \geq 3$, $|T_u| \geq 2$. Let $G' = G - \{x, y\}$. Then, G' is a connected bipartite graph of order $n' = n - 2$, size $m' < m$, maximum degree Δ' where $\Delta - 1 \leq \Delta' \leq \Delta$ and $\Delta' \leq n' - 2$, and minimum degree at least 2. If $\Delta' = \Delta$, then G' satisfies the statement of Observation 6.15, and the desired result follows. Hence we may assume that $\Delta' = \Delta - 1$.

If $\Delta' = 2$, then $G' = C_4$ and G is obtained from a 6-cycle by adding an edge between two vertices at distance 3 apart on the cycle. Coloring u and v red and coloring every other vertex of G blue produces a tr-coloring of G, and so $\gamma_{\rm tr}(G) = 2 < 3 = \psi(6, 3) = \psi(n, \Delta)$. Hence we may assume $\Delta' \geq 3$ (and so, $\Delta \geq 4$). Applying the inductive hypothesis to G', $\gamma_{\rm tr}(G') \leq \psi(n', \Delta') = \psi(n-2, \Delta-1) = \psi(n, \Delta-1) - 2$. For $\Delta \geq 4$, $\psi(n, \Delta-1) - \psi(n, \Delta) \leq 8/9$, and so $\gamma_{\rm tr}(G') \leq \psi(n, \Delta) - 10/9$.

Consider a $\gamma_{\rm tr}$-coloring C' of G'. If C' colors both u and v blue, then it colors every vertex in $T \setminus \{x, y\}$ red, and so we recolor u red and recolor every vertex in T_v blue to produce a new $\gamma_{\rm tr}$-coloring of G'. Such a $\gamma_{\rm tr}$-coloring of G' can be extended to a tr-coloring of G by coloring x red and y blue. If C' colors both u and v red, then it can be extended to a tr-coloring of G by coloring both x and y blue. If C' colors u red and v blue (resp., u blue and v red), then it can be extended to a tr-coloring of G by coloring x red and y blue (resp., x blue and y red). Hence, $\gamma_{\rm tr}(G) \leq \gamma_{\rm tr}(G') + 1 \leq \gamma_{\rm tr}(G') - 1/9 < \psi(n, \Delta)$, as desired. □

Observation 6.20 *We may assume that v has no large neighbor.*

Proof. Suppose that v has a large neighbor u. Since G is bipartite, u and v have no common neighbors. By Observation 6.16, every neighbor of u different from v is small. By Observation 6.17, every two small neighbors of u have only the vertex u as a common neighbor. By Observation 6.19, every vertex at distance 2 from u in $G - v$ is large in G. Let $U = N[u] \setminus \{v\}$ and let $|U| = k$. We now consider the graph $G' = G - U$. Then, $\delta(G') \geq 2$.

Suppose G' is disconnected. Let F be a component of G' that does not contain the vertex v. Then the component of $G - V(F)$ that contains v contains the vertices in U and satisfies the statement of Observation 6.15, and the desired result follows. Hence we may assume that G' is connected. Let G' have maximum degree Δ'.

Suppose $\Delta' = 2$. Then G can be obtained from a 4-cycle $v = v_1, v_2, v_3, v_4, v$ by adding a new vertex u, joining it to each of v, v_2 and v_4, and then subdividing the edges uv_2 and uv_4 exactly once. Thus, $\gamma_{\rm tr}(G) \leq 4 = \psi(7,3) = \psi(n,\Delta)$. Hence we may assume that $\Delta' \geq 3$ (and so, $\Delta \geq 4$).

Thus, G' is a connected graph of order $n' = n - k$, size $m' < m$, maximum degree Δ' where $\Delta - 1 \leq \Delta' \leq \Delta$ and $\Delta' \leq n' - 2$, and minimum degree at least 2. If $\Delta' = \Delta$, then the desired result follows readily from Observation 6.15. Hence we may assume that $\Delta' = \Delta - 1$. Applying the inductive hypothesis to G', $\gamma_{\rm tr}(G') \leq \psi(n', \Delta') = \psi(n - k, \Delta - 1) = \psi(n, \Delta - 1) - k$. For $\Delta \geq 4$, $\psi(n, \Delta - 1) - \psi(n, \Delta) \leq 8/9$, and so $\gamma_{\rm tr}(G') \leq \psi(n, \Delta) - k + 8/9$.

Let C' be a $\gamma_{\rm tr}$-coloring of G'. If C' colors a vertex in G' that has a common small neighbor with u blue, then we extend C' to a tr-coloring of G by coloring this common small neighbor blue and coloring all remaining uncolored vertices of G red. Otherwise, we extend C' by coloring u and a small neighbor of u blue and coloring all remaining uncolored vertices of G red. In this way, we extend C' to a tr-coloring of G by coloring at most $k - 1$ additional vertices red, and so $\gamma_{\rm tr}(G) \leq \gamma_{\rm tr}(G') + k - 1 \leq \psi(n, \Delta) - 1/9 < \psi(n, \Delta)$. □

By Observation 6.20, the vertex v of maximum degree in G is adjacent only to degree-2 vertices. Hence by Lemma 6.12, $\gamma_{\rm tr}(G) \leq \psi(n, \Delta)$ and this bound is sharp. This completes the proof of the Theorem 6.2. □

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