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Generation of Multiscroll Attractors by Controlling the Equilibria ⋆

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Abstract: This work shows the generation of multi-scroll attractors in $\mathbb{R}^3$ by controlling the equilibrium point of an unstable dissipative system. The switching control signal that governs the position of the equilibrium point changes according to the number of scrolls that is displayed in the attractor. Thus, if two systems display a different number of scrolls they have different control signals. The analysis of their Lyapunov exponents along with some bifurcation diagrams are presented. The possibility of hyper-chaos in $\mathbb{R}^3$ is considered.

Keywords: Chaos theory; Switching functions; System design; Piecewise linear analysis; Multi-scroll Attractors.

1. INTRODUCTION

Switched systems have acquired a great deal of attention recently and they have been considered for a wide range of applications mostly in electrical engineering. These systems consist on a set of subsystems and a switching control signal which is activated or fixed at some values through some intervals of time. Among all the uses they may present, the generation of multi-scrolls and chaos has been of great interest for the scientific community. Chaos has been an extremely studied area in last decades. One of the most remarkable developments is that simpler nonlinear deterministic equations can have unpredictable (chaotic) long-term solution.

Despite of the fact that there is no unique definition of chaos that all the international scientific community may adopt, there are several basis and theorems that we can seize in order to characterize the behavior of any system throughout nature. Characterization of dynamical behavior can be achieved by means of the Lyapunov exponents (LE). With the aid of their diagnostic, one can measure the average exponential rates of divergence or convergence of nearby orbits in the phase space, overall with their signs, a qualitative picture of the variety of dynamics that the systems may exhibit, ranging from fixed points via limit cycles and tori to more complex chaotic and hyperchaotic attractors.

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Whereas chaos can arise in discrete-time systems with only a single variable (which must be positive), at least three variables are required for chaos in continuous-time systems (Hirsch & Smale, 1974). Such systems are characterized by one positive (LE) in the Lyapunov spectrum. The behaviors described previously in Wolf, Swift, Swinney & Vastano (1985) can be defined with the sign of their LE as follows:

- In the presence of one positive LE, one negative and, one zero $(+, 0, -)$, the resulting attractor is "strange" or "chaotic".
- With a negative LE , and two zero $(-, 0, 0)$, the attractor is a two-torus.
- With a zero LE, and two negatives $(0, -, -)$, the attractor is a limit cycle.
- With three negative LE's $(-, -, -)$, is a fixed point.

A natural question is the following: Is there any system with the sign of their LE $(+, +, -)$ in $\mathbb{R}^3$?

A zero Lyapunov exponent indicates that the system is in some sort of steady state mode (Haken, 1983). A physical system with this exponent is conservative, so it is possible to construct a system that always presents stretching and folding. However, to obtain hyper-chaos, the system must be characterized by the presence of two or more positive LE's. The reason is that the trajectory has to be nonperiodic and bounded to some finite region, and yet it cannot intersect itself because every point has a unique direction of the flow. Hyperchaos in $\mathbb{R}^4$ has been reported in several papers for the last 40 years (Rössler, 1979; Chua, 1994; Matsumoto, Chua & Kobayashi, 1986; Baier & Klein, 1990), but it is posible to generate hyperchaos in $\mathbb{R}^3$.

There have been different approaches to yield multi-scroll chaotic attractors some of them modify the Chua's system (Chua, Komuro & Matsumoto, 1986; Madan, 1993) by replacing the nonlinear part with different nonlinear functions (Suykens & Vandewalle, 1993; Suykens , Huang & Chua, 1997). Some others are created by using nonsmooth nonlinear functions such as, hysteresis (Lü , Han, Yu. & Chen, 2004), saturation (Lü , Chen & Yu, 2004), step functions (Yalçin, Suykens, Vandewalle & Ozoguz, 2002) and inducing multi-scroll attractors by switching piecewise systems (Campos-Cantón, Campos-Cantón, González-Salas & Cruz-Ordaz, 2008) and controlling the stability of its equilibria.

Although multi-scroll attractors that have been generated by an autonomous hyperchaotic system were presented recently in Ahmad (2006). This approach comprises a system with canonical structure, one control parameter, and a switching-type nonlinearity. Besides the approaches mentioned above about Chua's system, there have been also some researches (Yu, Lü & Chen, 2007; Yalçin , Suykens & Vandewalle, 2000) that obtained multi-scrolls from the hyperchaotic Chua system.

In this work, we analyze a chaotic time series of a class of 3-D dynamical systems having multiple scrolls based on *unstable dissipative systems* (UDS) (Campos-Cantón, Barajas-Ramírez, Solís-Perales & Femat, 2010). This class of systems is constructed with a switching control signal to display various multi-scroll strange attractors. The multi-scroll strange attractors result from the combination of several unstable "one-spiral" trajectories by means of switching. The study of the LE is also added showing that the system presented here is hyperchaotic.

2. SWITCHING CONTROL SIGNAL FOR MULTI-SCROLL ATTRACTORS IN $\mathbb{R}^3$

In the same spirit that Campos-Cantón, Barajas-Ramírez, Solís-Perales & Femat (2010), we consider the class of linear system given by

$$\dot{\chi} = A\chi + B \tag{1}$$

where $\chi = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ is the state variable, $B = [\beta_1, \beta_2, \beta_3]^T \in \mathbb{R}^3$ stands for a real vector, $A = [\alpha_{ij}] \in \mathbb{R}^{3\times3}$ denotes a linear operator and the equilibrium point is located at $\chi^* = -A^{-1}B$.

Using the approach of the linear ordinary differential equation(ODE) written in the jerky form $\dddot{x}+\alpha_{33}\ddot{x}+\alpha_{32}\dot{x}+\alpha_{31}x+\beta_3 = 0$ we can represent the dynamics of the system in the state space as (1) where the matrix A and the vector B are found to be:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{pmatrix}; B = \begin{pmatrix} 0 \\ 0 \\ \beta_3 \end{pmatrix}, \tag{2}$$

Here the coefficients $\alpha_{31}, \alpha_{32}, \alpha_{33} \in \mathbb{R}$ may be any arbitrary scalar that assures the system as a UDS, this is that $\mathsf{Tr}(A) = \sum_{i=1}^{3} \lambda_i < 0$, and that the characteristic polinomial of $\mathbf{A}$ given by $g(\lambda) = \lambda^3 + \alpha_{33}\lambda^2 + \alpha_{32}\lambda + \alpha_{31}$ present one real negative root, and two complex roots with their real part positive. For this we are setting the coefficients as $\alpha_{31} = a,\ \alpha_{32} = 1,\ \alpha_{33} = 1$, where $a = 1.5$ unless told otherwise.

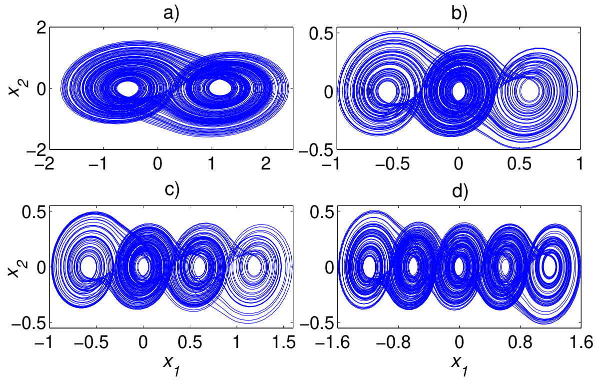

Fig. 1. The projection onto the plane (x_1, x_2) of the chaotic attractors generated by different switching control signal:(a) (3), (b) (4),(c) (5) and (d) (6), with $a = 1.5$

A switching control signal β_3 commutated in two values, S_1 and S_2, makes the system in the form (1) to present two equilibria and a double scroll is yielded. This signal is defined as a piecewise-linear function (PWL). Adding more PWL functions S_i to the control signal for β_3, it is possible to produce multi-scroll proportionally to the number of signals S_i. For simplicity, a switching control signal is given in terms of only one state, which defines the borders of domains as hyperplanes parallel to one axe.

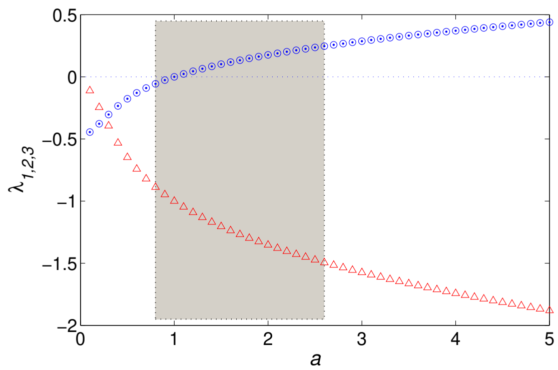

Fig. 2. Roots of the characteristic polynomial of $\mathbf{A}$ in system (2), with $0 < a < 5$. Mark with triangles the real root, with circle and dots the positive real part of the two complex roots. The gray area shows the values of $0.8 > a > 2.6$ for which the system (1) is stable.

The following switching control signal β_3 can generate a double-scroll and is given as follows:

$$\beta_3 = \begin{cases} S_1 = 1.8, & \text{if } x_1 \geq 0.3; \\ S_2 = -0.9 & \text{otherwise.} \end{cases} \tag{3}$$

The equilibrium points of the system (1) using the matrix A and vector B defined in (2) and the control signal (3) are $\chi_1^* = (0.6, 0, 0)^T$ and χ_2^* at the origin $(0, 0, 0)^T$.

Figure 1 (a) depicts the projection of the double-scroll chaotic attractor on the plane (x_1, x_2) generated by the β_3 switching control signal (3) under equations (1)-(2). Modifying the switching control signal then it is possible to generate an attractor with triple-scroll. Therefore the β_3 switching control signal is given as follows:

$$\beta_3 = \begin{cases} 0.9, & \text{if } x_1 \geq 0.3; \\ 0 & \text{if } -0.3 < x_1 < 0.3; \\ -0.9, & \text{if } x_1 \leq -0.3. \end{cases} \tag{4}$$

Notice that $\chi_3^* = -\chi_1^*$. This issue is intentionally defined to illustrate the symmetry in scrolls. Figure 1 (b) shows the projection of triple-scroll chaotic attractor on the plane (x_1, x_2) generated by the switching control signal (4) under equations(1)-(2).

So, quadtuple and quintuple scroll chaotic attractors are yielded by controlling β_3 switching control signal as follows:

$$\beta_3 = \begin{cases} 1.8, & \text{if } x_1 \geq 0.9; \\ 0.9, & \text{if } 0.3 \leq x_1 < 0.9; \\ 0, & \text{if } -0.3 < x_1 < 0.3; \\ -0.9, & \text{if } x_1 \leq -0.3. \end{cases} \tag{5}$$

$$\beta_3 = \begin{cases} 1.8, & \text{if } x_1 \geq 0.9; \\ 0.9, & \text{if } 0.3 \leq x_1 < 0.9; \\ 0, & \text{if } -0.3 < x_1 < 0.3; \\ -0.9, & \text{if } -0.9 < x_1 \leq -0.3; \\ -1.8, & \text{if } x_1 \leq -0.9;. \end{cases} \tag{6}$$

The β_3 switching control signals given by (5) and (6) introduce other equilibrium points located at $\chi_4^* = (1.2, 0, 0)^T$ and $\chi_5^* = (-1.2, 0, 0)^T$, respectively. Figures 1 (c) and 1 (d) show the projection of the quadtuple-scroll and quintuple-scroll chaotic attractors given by the β_3 switching control signals (5) and (6), respectively.

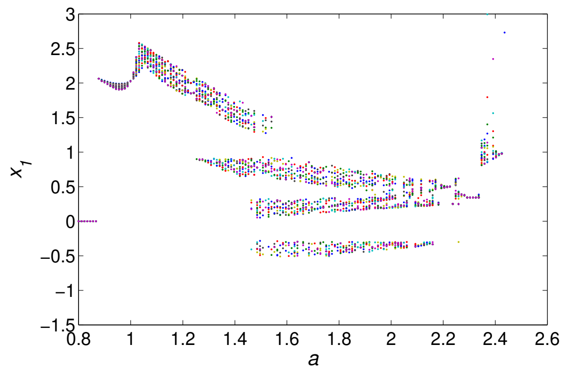

Fig. 3. Bifurcation diagram for system (2) with the switching control signal (5), for $0.8 < a < 2.6$.

Changing the value of the parameter a makes the system (1) in (2) to exit the UDS state, also the number of scrolls is affected. There are certain values for a in which the system may loose the number of scrolls without regarding the switching control signal applied. This may be shown in Figure 3 and 4, where bifurcation diagrams of the parameter $0.8 < a < 2.6$ are depicted for the system (2) with the switching control signals (5) and (6) which correspond to 4 and 5 scrolls accordingly. It can be seen that for some values the system change from a periodic orbit to $2, 4, 3, 2$ hyperchaotic scrolls in Figure 3, and from a periodic orbit to $1, 2, 5, 3$ and again $2, 1$ hyperchaotic scrolls. For greater or lower values of a, the system becomes unstable and the solution goes to infinity.

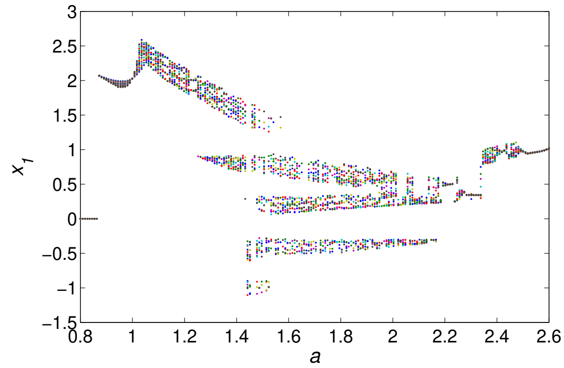

Fig. 4. Bifurcation diagram for system (2) with the switching control signal (6), for $0.8 < a < 2.6$.

The LE for the system (1) were calculated by the algorithm describe by Wolf, Swift, Swinney & Vastano (1985), and they are shown in Figure 5 depicting the following values $(0.10233, 0.10055, -1.2029)$. We can observe that there are two positive exponents along with one negative, this exponents are the same for any number of scrolls. The orders of magnitude between this two positive exponents is not enough to consider the asymptotic value attained by the mandatory zero exponent. In Haken (1983) it was prove, that there must be a zero exponent, or else the system will tend to collapse on a fixed point. In contrast with our system this is not true, there is no fixed point reached by the flow of the attractor.

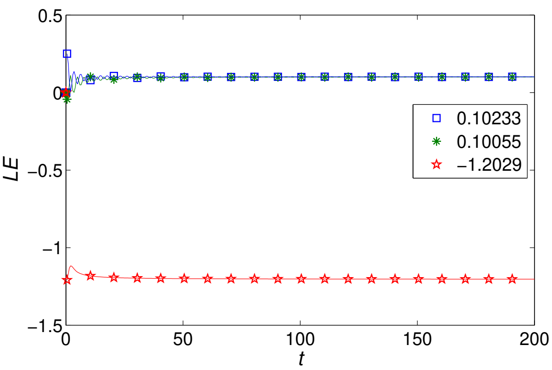

Fig. 5. Lyapunov Spectra for system (1) in (2), with $a = 1.5$

Calculating the exponents for different values of a as shown in Figure 6, outcome in no value for a where we may find a single zero exponent. We know that there is an exponential divergence of the trajectories because of the positive exponents therefore according to Wolf, Swift, Swinney & Vastano (1985) the system is hyperchaotic, and we can also check that the system is dissipative due to the sum of the three exponents.

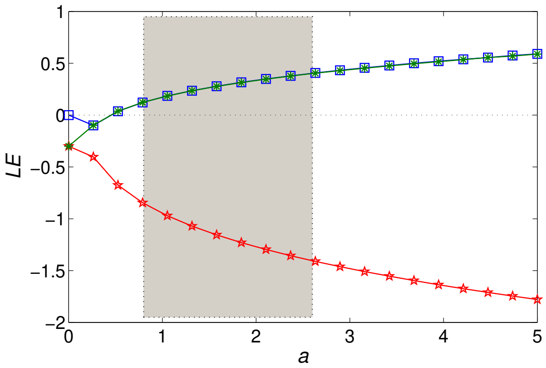

Fig. 6. Lyapunov Spectra for system (1) in (2), with $0 < a < 5$. The gray area shows the values of $0.8 > a > 2.6$ for which the system (1) is stable.

3. CONCLUSIONS

We have proposed an approach to generate an hyperchaotic system based on unstable dissipative systems and control their equilibrium points. The proposed systems have been demonstrated via numerical simulations to exhibit hyper-chaotic behavior for certain switching control signals. The hyperchaoticity was verified by checking the Lyapunov spectrum of the output data series, were we found that the system in $\mathbb{R}^3$, contains two positive LE.

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