

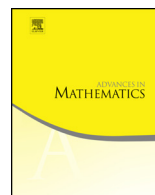


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Conformal geometry of embedded manifolds with boundary from universal holographic formulæ

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ABSTRACT

For an embedded conformal hypersurface with boundary, we construct critical order local invariants and their canonically associated differential operators. These are obtained holographically in a construction that uses a singular Yamabe problem and a corresponding minimal hypersurface with boundary. They include an extrinsic Q -curvature for the boundary of the embedded conformal manifold and, for its interior, the Q -curvature and accompanying boundary transgression curvatures. This gives universal formulæ for extrinsic analogs of Branson Q -curvatures that simultaneously generalize the Willmore energy density, including the boundary transgression terms required for conformal invariance. It also gives extrinsic conformal Laplacian power type operators associated with all these curvatures. The construction also gives formulæ for the divergent terms and anomalies in the volume and hyper-area asymptotics determined by minimal hypersurfaces having boundary at the conformal infinity. A main feature is the development of a universal, distribution-based,

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Minimal hypersurface asymptotics
 Yamabe problem
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boundary calculus for the treatment of these and related problems.

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1. Introduction

For closed, even dimensional conformal manifolds, the study of Branson’s Q -curvature invariant [7] has been a major focus in areas including conformal geometry, geometric analysis and physics, see for example [10,44,41,39,34,1,18,19,11,12,14,6,20]. Partly motivating these studies is the result that the integral of the Q -curvature over the manifold is a global conformal invariant. The analog for a dimension $d - 1 \geq 2$ manifold $\tilde{\Sigma}$ with boundary $\partial\tilde{\Sigma} = \Lambda$ is given by a Q -curvature–transgression pair (Q, T) , where

$$\int_{\tilde{\Sigma}} Q + (d - 2) \int_{\Lambda} T$$

is a global conformal invariant [10], even though Q and T separately have linear conformal transformation rules (see for example [12,47,9,25] for subsequent T -curvature

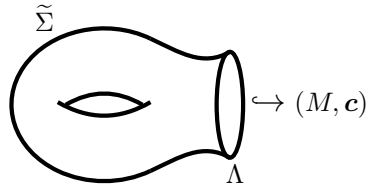
studies). Any attempt to handle these objects using standard Levi-Civita calculus is immediately frustrated by the facts that (i) the formulæ are extremely complicated and (ii) the definition of the Q -curvature is indirect. An aim of the current work is to solve this problem. In fact, we will treat a generalization.

It has recently emerged that there is a nice link between Q -curvature and the Willmore energy as well as its higher dimensional analogs. This is in the form of a hierarchy where for conformally embedded hypersurfaces there is a curvature quantity, also denoted Q , that gives a Willmore-type energy density for hypersurfaces which specializes to the usual Branson Q -curvature for suitable even-dimensional embeddings. When $\tilde{\Sigma}$ is the boundary of any conformal compactification, the Q -curvature generalizes canonically [29]. In the case when the conformal compactification obeys a suitable singular Yamabe problem [2], this Q -curvature is a distinguished, extrinsically coupled curvature invariant, determined by the conformal embedding and a choice of boundary metric with integral again a conformal invariant. (See also [46] for a general study of singular Yamabe problems.) When $\tilde{\Sigma}$ is a surface, the integral of this *extrinsic* Q -curvature is the Willmore energy functional; in general this provides a higher dimensional analog [28,30,32,35,31]. The Willmore functional plays an important rôle in both mathematics and physics (see *e.g.* [51,50,3]); recently the celebrated Willmore conjecture [61] concerning absolute minimizers of this energy was settled in [45]. The Willmore energy is also linked to physical observables [39] and in particular entanglement entropy [52,53]. We expect higher Willmore energies to be similarly important.

Recently the existence of a corresponding canonical transgression for the extrinsic Q -curvature was established [33]. Once again, the Levi-Civita and its accompanying Gauß–Codazzi–Ricci extrinsic calculus are inadequate for developing a general theory. We also treat this problem.

We attack all these problems *holographically*. This term originated in physics [58,56] where it was concretely realized in the AdS/CFT correspondence of [44]. This relates spacetime theories to theories living on the boundary of spacetime. Here this means that the geometry of $\tilde{\Sigma}$ will be studied by embedding it in a conformal manifold M of one dimension higher. The gain is simple formulæ on M that encode the complicated information of a (Q, T) -pair; upon restriction to $\tilde{\Sigma}$ these produce the quantities of interest.

The Branson Q -curvature was initially defined as an intrinsic quantity. However, the holographic approach is a natural setting to study a generalizing analog that includes extrinsic embedding data, and for this extrinsic Q -curvature, the corresponding transgression T . For a manifold $\tilde{\Sigma}$ with boundary Λ , we consider a conformal embedding, as depicted below:



Thus we study the sequence of submanifold embeddings

$$\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c}),$$

where M is a dimension d conformal manifold, Σ is a hypersurface (meaning dimension $d - 1$), Λ is a closed hypersurface in Σ (so $\dim(\Lambda) = d - 2$) with interior $\tilde{\Sigma}$. Throughout this paper we assume $d \geq 3$ (the cases $d = 1, 2$ are easily handled by more naïve methods). To facilitate the study of these embeddings we consider two classical PDE problems:

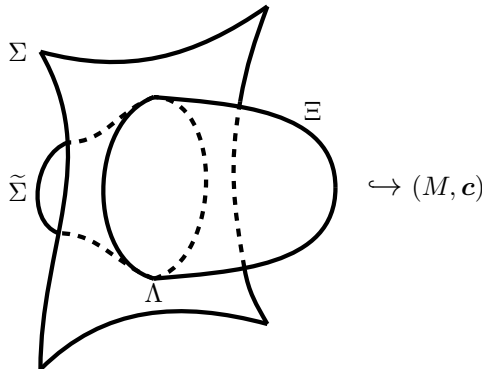
First, the data of the conformal embedding $\Sigma \hookrightarrow M$ determines an approximate metric g^σ that is singular along Σ . Given $g \in \mathbf{c}$, then $g^\sigma = \sigma^{-2}g$; this is determined canonically up to terms of order σ^d where σ is a defining function for Σ (meaning that Σ is the zero locus of σ and $d\sigma|_\Sigma \neq 0$). This is achieved via a singular Yamabe problem which requires the scalar curvature to obey

$$Sc^{g^\sigma} = -d(d - 1) + \mathcal{O}(\sigma^d).$$

Second, the singular metric g^σ can be used to determine the asymptotics of a hypersurface Ξ anchored along Λ (so $\partial\Xi = \Lambda$) that is minimal; that is, its mean curvature $H_\Xi^{g^\sigma}$ vanishes, again to some order in σ :

$$H_\Xi^{g^\sigma} = \mathcal{O}(\sigma^{d-1}).$$

We term Ξ an *asymptotically minimal hypersurface*. Furthermore, the defining function μ for Ξ can be canonically determined up to terms of order μ^d by the singular Yamabe problem for the embedding $\Xi \hookrightarrow M$. This geometry is depicted below:



This set-up allows us to study the embedding sequence $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ holographically.

The above geometry was partly inspired by seminal work of Graham and Witten [39] (see also [38]) who study the embedding of closed submanifolds in Σ by considering the volume asymptotics of a minimal submanifold in M whose boundary is the given submanifold in Σ . For this they considered the special case where the metric g^o obeys a Poincaré–Einstein condition; this effectively removes the information of the embedding $\Sigma \hookrightarrow (M, \mathbf{c})$. In Section 3 we further develop the theory of asymptotically minimal hypersurfaces, including results of independent interest.

Returning to our general setting, note that even in the case that Ξ and $\tilde{\Sigma}$ close off a compact region D , the volume of this is infinite with respect to g^o . Nevertheless, in the case of any suitably singular measure such as the one of g^o , there is a notion of a *regulated volume* Vol_ε . This is defined by computing, for $\varepsilon > 0$, the volumes of a one parameter family of regions $D_\varepsilon \subseteq D$, such that $D_0 = D$. (See Equation (4.1) and preceding text for the precise definition of the regulated volume.) The asymptotic behavior of Vol_ε is then given by

$$\text{Vol}_\varepsilon = \sum_{k=1}^{d-1} \frac{v_k}{\varepsilon^k} + \mathcal{V} \log \varepsilon + \text{Vol}_{\text{ren}} + \mathcal{O}(\varepsilon),$$

with details as follows: The coefficients of the poles in ε are local integrals (meaning their integrands are functions determined by finitely many jets of the given data) along $\tilde{\Sigma} = \Sigma \cap D$, these poles are termed *divergences*. These depend on the choice of regulating regions D_ε ; we give universal holographic formulæ for them in Theorem 6.1. The constant term Vol_{ren} is called the *renormalized volume*. The coefficient \mathcal{V} of the logarithm is the *volume anomaly*. It is the non-derivative term in an anomaly operator (see for example [33]) that measures the dependence of the renormalized volume on the choice of regulator and hence the metric in \mathbf{c}_Σ . Here \mathbf{c}_Σ is the conformal class of metrics induced on Σ by \mathbf{c} . The volume anomaly is conformally invariant and expressed as a sum of local integrals along $\tilde{\Sigma}$ and $\Lambda = \partial\tilde{\Sigma}$. In fact, distinguished integrands are canonically determined by our construction and give the extrinsic Q -curvature and corresponding transgression introduced above. As discussed above, explicit formulæ for Q -curvatures and their extrinsic generalizations explode in complexity beyond the simplest low dimensional examples (see [24]). However, our first main theorem establishes that these have remarkably simple and universal holographic formulæ in all dimensions.

In the following theorem we determine the regulated region D_ε by introducing a nowhere vanishing positive conformal density τ , termed a true scale or *regulator*. This additional data can be used to determine a metric in \mathbf{c} , and hence an induced metric along Σ . These notions are explained in Sections 2.1 and 4.

Theorem 1.1. *The anomaly in the regulated volume Vol_ε is determined by the embeddings $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ and is conformally invariant. For any regulator τ , it is given by*

$$\mathcal{V} = \frac{(-1)^{d-1}}{(d-1)!} \left[\frac{1}{(d-2)!} \int_{\tilde{\Sigma}} \mathbf{Q}_{\Sigma \hookrightarrow (M, c)}^\tau + \frac{1}{(d-3)!} \int_{\Lambda = \partial \tilde{\Sigma}} \mathbf{T}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, c)}^\tau \right],$$

where (\mathbf{Q}, \mathbf{T}) are local (density-valued) curvatures depending on the regulator τ and the conformal embeddings indicated. Given unit and minimal unit conformal defining densities σ and μ for Σ and Ξ , respectively, these have holographic formulæ

$$\mathbf{Q}_{\Sigma \hookrightarrow (M, c)}^\tau = \mathbf{L}_\sigma^{d-1} \log \tau \Big|_\Sigma, \quad \mathbf{T}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, c)}^\tau = \sum_{j=0}^{d-3} (\mathbf{L}_\sigma^\tau)^j \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-3} \log \tau \Big|_\Lambda.$$

In the above, the conformal densities σ and μ obey a unit and minimal unit property. This means that $\sigma = \sigma\tau$ and $\mu = \mu\tau$ with $\tau = [g; 1]$, where g, σ and μ obey the singular Yamabe and minimal hypersurface conditions that we just defined. The logarithm of a conformal density is defined in Sections 2.1. The operators $\mathbf{L}_\sigma, \mathbf{L}_\sigma^\tau$ and \mathbf{L}'_μ are all variants of the Laplace–Robin operator; these are distinguished Laplacian-type operators on the manifold M that are degenerate along boundaries Σ, Λ or Ξ respectively, where they become the conformally invariant Robin-type combination of Dirichlet and Neumann operators of Cherrier [13], see Sections 2.4 and 5.2. Theorem 6.1 gives analogous results for the divergences in the regulated volume.

A nice feature of this set-up is that it encodes a second geometric problem of independent interest (see [39]). Namely, the corresponding “area” of the minimal hypersurface Ξ (meaning its volume with respect to the induced singular metric; throughout we use “volume” to refer to volumes of codimension zero regions and “area” for volumes of codimension one regions) is also infinite, but again can be regulated using the same method as employed above for the volume problem. Here also the family of regulating regions $\Xi_\varepsilon \subset \Xi$ are described in terms of the true scale τ . The details of this construction may be found in Section 4. The asymptotic behavior of their areas is then given by

$$\text{Area}_\varepsilon = \sum_{k=d-2}^1 \frac{a_k}{\varepsilon^k} + \mathcal{A} \log \varepsilon + \text{Area}_{\text{ren}} + \mathcal{O}(\varepsilon),$$

where the coefficients of poles are regulator-dependent local integrals over $\Lambda = \Sigma \cap \Xi$; we give universal holographic formulæ for them in Theorem 6.2. The constant term Area_{ren} is the *renormalized area*. The coefficient \mathcal{A} of the logarithm is the *area anomaly*. It measures the response of the renormalized area to the choice of regulator and is itself conformally invariant. The integrand is another generalization of Branson’s Q -curvature to what we shall call a *submanifold Q -curvature*. Even for the case where Λ is a four-manifold and M is specialized to be Poincaré–Einstein, the classical treatment of this requires a computational *tour de force* [38]. Our next theorem gives a universal holographic formula for the area anomaly.

Theorem 1.2. *The anomaly in the regulated area Area_ε is determined by the embeddings $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ and is conformally invariant. For any regulator τ , it is given by*

$$\mathcal{A} = \frac{(-1)^{d-2}}{(d-2)!(d-3)!} \int_{\Lambda} \mathbf{Q}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{\tau},$$

where the holographic formula for the local (density-valued), τ -dependent, submanifold Q -curvature $\mathbf{Q}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{\tau}$, is

$$\mathbf{Q}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{\tau} = (\mathbf{L}_{\sigma}^{\mathbf{T}})^{d-2} \log \tau|_{\Lambda},$$

and where σ and μ are unit and minimal unit conformal defining densities for Σ and Ξ , respectively.

The proofs of Theorems 1.1 and 1.2 are given in Section 6.2. Key to these proofs is a calculus, both integral and differential, for conformal geometries coupled to multiple scales; this is described in Section 5.

A strong motivation for our work is that the volume and area problem not only leads to interesting global conformal invariants of the structure $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$, but also yields a rich local invariant theory surrounding the \mathbf{Q} and \mathbf{T} integrands. Of particular interest are conformally invariant differential operators that measure how Q and T curvatures depend on the choice of metric in the conformal class. For the special case of Branson’s Q -curvature, these are the (critical) conformally invariant Laplacian powers of [36]. Their generalizations follow immediately from our construction upon remembering that the choice of regulator τ also amounts to a choice of metric $g \in \mathbf{c}$ and that $\log(e^f \tau) = f + \log \tau$ for any smooth function f . For example, the extrinsic Q -curvature of $\Sigma \hookrightarrow (M, \mathbf{c})$ obeys

$$\mathbf{Q}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{e^f \tau} = \mathbf{Q}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{\tau} + \mathbf{P}_{\Sigma} f_{\Sigma},$$

where $f_{\Sigma} = f|_{\Sigma}$ and the conformally invariant operator $\mathbf{P}_{\Sigma} := \mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(d-1)}$ is the last member of a sequence of conformally invariant extrinsically coupled Laplacian powers, as described by simple holographic formulæ in the next theorem.

Theorem 1.3. *The operator defined by*

$$\begin{aligned} \mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(k)} : \Gamma(\mathcal{E}\Sigma[\frac{k-d+1}{2}]) &\rightarrow \Gamma(\mathcal{E}\Sigma[\frac{-k-d+1}{2}]), \\ \mathbf{f} &\mapsto (\mathbf{L}_{\sigma}^k \mathbf{f}_{\text{ext}})|_{\Sigma}, \end{aligned}$$

for $k \in \{1, \dots, d-1\}$ and \mathbf{f}_{ext} any extension of \mathbf{f} to M , is canonically determined by the embedding data $\Sigma \hookrightarrow (M, \mathbf{c})$. When k is even, this has leading derivative term

$((k - 1)!!)^2 (\Delta_\Sigma)^{\frac{k}{2}}$, where Δ_Σ is the intrinsic Laplacian of Σ as determined by a choice of $g_\Sigma \in \mathfrak{c}_\Sigma$.

Because the extrinsic Q -curvature pairs with a transgression \mathbf{T} for manifolds with boundary, the critical extrinsic Laplacian power \mathbf{P}_Σ pairs with a conformally invariant boundary operator $\mathbf{U}_{\Lambda \hookrightarrow \Sigma}$ that measures how the T -curvature responds to changing the choice of metric in the conformal class:

$$\mathbf{T}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathfrak{c})}^{ef\tau} = \mathbf{T}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathfrak{c})}^\tau + \mathbf{U}_{\Lambda \hookrightarrow \Sigma} f_\Sigma.$$

The sum of the integral of the extrinsic Q -curvature \mathbf{Q} along $\tilde{\Sigma}$ and $(d - 2)$ times that of the transgression \mathbf{T} along Λ is conformally invariant; this implies an integral identity for the $(\mathbf{P}_\Sigma, \mathbf{U}_{\Lambda \hookrightarrow \Sigma})$ pair. This and the holographic formula for $\mathbf{U}_{\Lambda \hookrightarrow \Sigma}$ are given in the next theorem.

Theorem 1.4. *The differential operator defined by*

$$\begin{aligned} \mathbf{U}_{\Lambda \hookrightarrow \Sigma} : \Gamma(\mathcal{E}\Sigma[0]) &\rightarrow \Gamma(\mathcal{E}\Sigma[2 - d])|_\Lambda, \\ f &\mapsto \left(\sum_{j=0}^{d-3} (\mathbf{L}_\sigma^\mathbf{T})^j \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-3} f_{\text{ext}} \right) \Big|_\Lambda, \end{aligned}$$

where f_{ext} is any extension of f to M , is canonically determined by the embedding data $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathfrak{c})$ and, for any $f \in C^\infty(\Sigma)$, obeys

$$\int_{\tilde{\Sigma}} \mathbf{P}_\Sigma f + (d - 2) \int_{\Lambda = \partial \tilde{\Sigma}} \mathbf{U}_{\Lambda \hookrightarrow \Sigma} f = 0.$$

Moreover, when d is odd, $\mathbf{U}_{\Lambda \hookrightarrow \Sigma}$ has leading derivative term

$$(d - 2)!!(d - 4)!! \nabla_{\hat{m}} (\Delta_\Sigma^g)^{\frac{d-3}{2}} + \text{LTOTs},$$

for any $g \in \mathfrak{c}$, where \hat{m} is the unit inward normal to Λ and “LTOTs” denotes terms of lower transverse order, meaning lower order in $\nabla_{\hat{m}}$.

The dependence on the choice of metric $g \in \mathfrak{c}$ for the submanifold Q -curvature is also controlled by a conformally invariant extrinsically coupled Laplacian power $\mathbf{P}_\Lambda := \mathbf{P}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathfrak{c})}^{(d-2)}$ (see Equation (6.6)). This operator is the last member of a sequence of invariant operators. These are described holographically in our next theorem:

Theorem 1.5. *The differential operator defined by*

$$\mathbf{P}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{(k)} : \Gamma(\mathcal{E}\Lambda[\frac{k-d+2}{2}]) \rightarrow \Gamma(\mathcal{E}\Lambda[\frac{-k-d+2}{2}]),$$

$$\mathbf{f} \mapsto ((\mathbf{L}_\sigma^T)^k \mathbf{f}_{\text{ext}})|_\Lambda,$$

for $k \in \{1, \dots, d - 2\}$ and \mathbf{f}_{ext} any extension of \mathbf{f} to M , is canonically determined by the embedding data $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$. When k is even, this has leading derivative term $((k - 1)!)^2 (\Delta_\Lambda^g)^{\frac{k}{2}}$ for any $g \in \mathbf{c}$. Moreover, when $\partial\Lambda = \emptyset$, for any $f \in C^\infty(\Lambda)$,

$$\int_\Lambda \mathbf{P}_\Lambda f = 0.$$

The proofs of Theorems 1.3, 1.4 and 1.5 are given in Section 7.

The simple holographic formulæ for the (\mathbf{Q}, \mathbf{T}) -curvature pair of Theorems 1.1 and 1.2 explode in complexity when expressed in terms of standard Riemannian invariants along Σ and Λ . For example, even the intrinsic Q -curvature for a conformal eight-manifold takes easily a page when expressed this way [24]. In the situation when one is given explicit data for the sequence of hypersurface embeddings $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ in higher dimensions where Riemannian formulæ are unwieldy, one can employ our holographic formulæ and computer software to compute these curvatures and operators. Examples of how to set up this kind of computation are given in [31,33]. When the bulk M is a three or four-manifold, explicit Riemannian formulæ are still relatively compact; these are given in following pair of theorems, the first of which gives the volume expansion, its anomaly and related curvatures and invariant operators. Any new notations appearing in these two theorems as well as their proofs are given in Section 8.

Theorem 1.6. *Given the sequence of hypersurface embeddings $\Lambda \hookrightarrow \Sigma \hookrightarrow (M^d, \mathbf{c})$ and $g \in \mathbf{c}$ (which induces $g_\Sigma \in \mathbf{c}_\Sigma$) the $(\mathbf{Q}_{\Sigma \hookrightarrow (M, \mathbf{c})}^g, \mathbf{T}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^g)$ curvature pair and associated $(\mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}, \mathbf{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})})$ operator pair are given by*

$$\mathbf{Q}_\Sigma^g = J^{g_\Sigma} - \frac{1}{2} \mathring{\mathbf{H}}_{ab}^\Sigma \mathring{\mathbf{H}}_\Sigma^{ab}, \quad \mathbf{T}_\Lambda^g = -H_{\Lambda \hookrightarrow \Sigma}^{g_\Sigma},$$

$$\mathbf{P}_\Sigma = \mathbf{\Delta}_\Sigma, \quad \mathbf{U}_\Lambda = \mathbf{\nabla}_{\hat{m}},$$

when $d = 3$, and

$$\mathbf{Q}_\Sigma^g = -4 \nabla_\Sigma^a \nabla_\Sigma^b \mathring{\mathbf{H}}_{ab}^\Sigma - 8 \mathring{\mathbf{H}}_{ab}^\Sigma \mathcal{F}_{ab}^\Sigma, \quad \mathbf{T}_\Lambda^g = -2 \hat{m}^a \nabla_\Sigma^b \mathring{\mathbf{H}}_{ab}^\Sigma + 2 \delta_R^{\Lambda \hookrightarrow \Sigma} \mathring{\mathbf{H}}_{\hat{m}\hat{m}}^\Sigma - 2 \mathring{\mathbf{H}}_{ab}^\Sigma \mathring{\mathbf{H}}_{\Lambda \hookrightarrow \Sigma}^{ab},$$

$$\mathbf{P}_\Sigma = 8 \nabla_a^\Sigma \circ \mathring{\mathbf{H}}_{ab}^\Sigma \circ \nabla_b^\Sigma, \quad \mathbf{U}_\Lambda = 4 \hat{m}^a \mathring{\mathbf{H}}_{ab}^\Sigma \nabla_\Sigma^b,$$

when $d = 4$.

Accordingly, the regulated volume expansion (4.3) is given by

$$\text{Vol}_\varepsilon^g = \frac{1}{2\varepsilon^2} \int_{\tilde{\Sigma}} dV_{g_\Sigma} + \frac{1}{\varepsilon} \int_{\tilde{\Sigma}} dV_{g_\Sigma} H_\Sigma^g + \log \varepsilon \left[\pi \chi_{\tilde{\Sigma}} - \frac{1}{4} \int_{\tilde{\Sigma}} \mathring{\mathbf{H}}_{ab}^\Sigma \mathring{\mathbf{H}}_{\Sigma}^{ab} \right] + \text{Vol}_{\text{ren}} + \varepsilon \mathcal{R}(\varepsilon),$$

when $d = 3$, and

$$\begin{aligned} \text{Vol}_\varepsilon^g = & \frac{1}{3\varepsilon^3} \int_{\tilde{\Sigma}} dV_{g_\Sigma} + \frac{1}{\varepsilon^2} \int_{\tilde{\Sigma}} dV_{g_\Sigma} H_\Sigma^g - \frac{1}{2\varepsilon} \int_{\tilde{\Sigma}} dV_{g_\Sigma} J^{g_\Sigma} + \frac{1}{2\varepsilon} \int_{\Lambda} dV_{g_\Lambda} H_{\Lambda \hookrightarrow \Sigma}^{g_\Sigma} \\ & + \frac{2}{3} \log \varepsilon \left[\int_{\tilde{\Sigma}} \mathring{\mathbf{H}}_{\Sigma}^{ab} \mathcal{F}_{ab}^\Sigma - \frac{1}{2} \int_{\Lambda} \left(\delta_{\mathring{m}\mathring{m}}^{\Lambda \hookrightarrow \Sigma} \mathring{\mathbf{H}}_{\mathring{m}\mathring{m}}^\Sigma - \mathring{\mathbf{H}}_{ab}^\Sigma \mathring{\mathbf{H}}_{\Lambda \hookrightarrow \Sigma}^{ab} \right) \right] + \text{Vol}_{\text{ren}} + \varepsilon \mathcal{R}(\varepsilon), \end{aligned}$$

when $d = 4$.

The area expansion, its anomaly and related curvatures and invariant operators are given in our final theorem.

Theorem 1.7. *Given the sequence of hypersurface embeddings $\Lambda \hookrightarrow \Sigma \hookrightarrow (M^d, \mathbf{c})$ and $g \in \mathbf{c}$ (which induces g_Λ along Λ), the $\mathbf{Q}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}$ curvature and associated $\mathbf{P}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}$ operator are given by*

$$\mathbf{Q}_{\Lambda \hookrightarrow \Sigma} \stackrel{\Lambda}{=} \mathbf{C} := -\mathring{\mathbf{H}}_{\mathring{m}\mathring{m}}^\Sigma, \quad \mathbf{P}_\Lambda = 0,$$

when $d = 3$ and

$$\mathbf{Q}_{\Lambda \hookrightarrow \Sigma} \stackrel{\Lambda}{=} J^{g_\Lambda} + \frac{1}{2} \mathbf{K}_{\Lambda \hookrightarrow \Sigma} + \frac{1}{4} \mathbf{K}_\Sigma - 2\mathcal{F}_{\mathring{m}\mathring{m}}^\Sigma - 2\mathring{\mathbf{H}}_{\mathring{m}a}^\Sigma g_\Lambda^{ab} \mathring{\mathbf{H}}_{\mathring{m}b}^\Sigma - \mathbf{C}^2, \quad \mathbf{P}_\Lambda = \mathbf{\Delta}_\Lambda,$$

when $d = 4$.

Accordingly, the regulated area expansion (4.7) is given by

$$\text{Area}_\varepsilon^g = \frac{1}{\varepsilon} \int_{\Lambda} dV_{g_\Lambda} + \log \varepsilon \int_{\Lambda} \mathring{\mathbf{H}}_{\mathring{m}\mathring{m}}^\Sigma + \text{Area}_{\text{ren}} + \varepsilon \mathcal{R}(\varepsilon),$$

when $d = 3$ and

$$\begin{aligned} \text{Area}_\varepsilon^g = & \frac{1}{2\varepsilon^2} \int_{\Lambda} dV_{g_\Lambda} + \frac{1}{\varepsilon} \int_{\Lambda} dV_{g_\Lambda} (H_\Sigma - \mathring{\mathbf{H}}_{\mathring{m}\mathring{m}}^\Sigma) \\ & + \log \varepsilon \left[\pi \chi_\Lambda + \frac{1}{4} \int_{\Lambda} \left(\mathbf{K}_{\Lambda \hookrightarrow \Sigma} - 4\mathcal{F}_{\mathring{m}\mathring{m}}^\Sigma - 4\mathring{\mathbf{H}}_{\mathring{m}a}^\Sigma g_\Lambda^{ab} \mathring{\mathbf{H}}_{\mathring{m}b}^\Sigma - 2\mathbf{C}^2 \right) \right] \\ & + \text{Area}_{\text{ren}} + \varepsilon \mathcal{R}(\varepsilon), \end{aligned}$$

when $d = 4$.

Many of our constructions and results have origins in tractor calculus techniques for hypersurfaces, see in particular [30,32]. They allow the Laplace–Robin technology to be extended to any conformally compact structure as well as to higher rank tensors. Moreover, these techniques suggest a natural generalization of our method to problems involving higher codimension embeddings; the key is to develop suitable analogs of the tangential Laplace–Robin operator \mathbf{L}_g^T . We plan to present the details of a higher codimension tractor calculus for conformally embedded submanifolds and related volume and area problems in a separate manuscript.

1.1. Conventions

We employ the notation M^d to indicate the dimension d of a smooth manifold M and drop the superscript when this is clear from context. The Euler characteristic of a manifold M is denoted by χ_M , and the exterior derivative is denoted by d . When M is equipped with a metric g , the corresponding volume element is denoted dV_g . We will often employ an abstract index notation for tensors on M , where for example, v^a denotes a section of TM but no choice of coordinates is implied (see, for example [49]). Canonical operations such as contraction of a vector field v^a and a one-form $\omega_a \in \Gamma(T^*M)$, are then given by expressions such as $v^a\omega_a$. In this notation the squared length $|v|_g^2$ of a vector v^a with respect to a metric g is $v^ag_{ab}v^b$, but we will often abbreviate this quantity by v^2 . In the same vein, $u.v$ denotes $g_{ab}u^av^b = g(u, v)$. Throughout we work with Euclidean signature metrics, but many of our results generalize directly to a pseudo-Riemannian setting. The Levi–Civita connection of a metric g is denoted ∇^g (again the superscript g will be dropped when context makes this clear), and the negative energy rough Laplacian of ∇^g is denoted $\Delta^g = \nabla^a\nabla_a$. The Riemann tensor R of ∇ is defined by $R(u, v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u, v]}$, where $[\cdot, \cdot]$ is the Lie bracket and u, v, w are arbitrary vector fields. In the abstract index notation R is denoted by $R_{ab}{}^c{}_d$ and the Ricci tensor is $R_{ab}{}^a{}_d =: Ric_{bd}$. In $d \geq 3$ dimensions, this is related to the Schouten tensor P by the trace-adjustment $Ric_{ab} = (d-2)P_{ab} + g_{ab}J$, where $J := P_a^a$ and the scalar curvature $Sc = Ric_a^a = 2(d-1)J$. When $d = 2$, we define $J := \frac{1}{2}Sc$. In dimensions $d \geq 4$, the Weyl tensor is defined by $W_{abcd} = R_{abcd} - P_{ac}g_{bd} + P_{bc}g_{ad} - P_{bd}g_{ac} + P_{ad}g_{bc}$. Moreover $W^{\Omega^2 g}_{ab}{}^c{}_d = W^g_{ab}{}^c{}_d$ for any smooth function Ω . As for the quantity zero, we denote $0! = 1$ and any operator A raised to the zeroth power is the identity map: $A^0 = \text{Id}$.

2. The calculus of scale

An effective method for handling geometric problems holographically is to treat the bulk as a conformal manifold. In this approach, asymptotically hyperbolic (or asymptotically anti de Sitter) metrics are treated as a coupling of conformal geometry to a singular scale. This allows us to utilize potent hidden symmetries because the bulk con-

formal structure extends smoothly to the boundary even when the hyperbolic metric does not.

2.1. Conformal geometry

A conformal manifold (M^d, \mathbf{c}) is a manifold equipped with a conformal class of metrics

$$\mathbf{c} := [g] = [\Omega^2 g],$$

where Ω is any smooth, strictly-positive function. Conformal densities are fundamental objects on conformal manifolds: A (conformal) density of weight $w \in \mathbb{R}$ is an equivalence class of metrics and smooth functions defined by

$$\nu := [g; \nu] = [\Omega^2 g; \Omega^w \nu].$$

Conformal densities may also be treated as sections of the line bundle $\mathcal{E}M[w]$ induced from the corresponding \mathbb{R}_+ representation, where one views the conformal structure as a ray subbundle of $\odot^2 T^*M$ and in turn as an \mathbb{R}_+ principal bundle over M [24,8]. Weight zero densities $\mathbf{f} = [g; f] = [\Omega^2 g; f]$ can be treated as functions and thus we often use the notation f for these. We will also be interested in operators between sections of line bundles $\mathcal{A} : \Gamma(\mathcal{E}M[w_1]) \rightarrow \Gamma(\mathcal{E}M[w_2])$. In this case we will refer to the difference $w_2 - w_1 =: w(\mathcal{A})$ as the operator weight of \mathcal{A} and also use the terminology “ \mathcal{A} is an operator of weight $w_2 - w_1$ ”. We also need the notion of a log density. These are (see [29]) equivalence classes of functions defined by an additive \mathbb{R}_+ representation

$$\lambda = [g; \lambda] = [\Omega^2 g; \lambda + \ell \log \Omega].$$

The number ℓ is called the (log)-weight of λ . A log density is a section of a log-density bundle $\mathcal{F}M[\ell]$ for some $\ell \in \mathbb{R}$. A more detailed description of the algebra and calculus of conformal densities and their tensor analogs may be found in [29,31]. For example, with the obvious adaptation of the abstract index notation to tensor-valued conformal densities, the equation $\mathbf{v}^a = [g; v^a] \in \Gamma(TM[w])$ denotes a vector-valued density of weight w and its corresponding section space.

When $\tau = [g; \tau]$ is a weight $w = 1$ density and the smooth function τ is strictly positive, we call τ a true scale. Given this, we can define a weight $\ell = 1$ log-density $\log \tau := [g; \log \tau]$, and any weight $\ell = 1$ log density can be written this way for some true scale.

A true scale canonically determines a Riemannian geometry (M, g_τ) via the equivalence class representative $\tau = [g_\tau; 1]$. Conversely, given a true scale τ and a density \mathbf{f} , this canonically determines a function f by expressing $\mathbf{f} = [g_\tau, f]$, or equivalently $f = \mathbf{f}/\tau^w$, where w is the weight of \mathbf{f} . Thus one may also regard a true scale as a choice of metric $g_\tau \in \mathbf{c}$. Alternatively, defining (tautologically) the symmetric cotensor-valued conformal

metric $g_{ab} = [g; g_{ab}] \in \Gamma(\odot^2 T^* M[2])$ (which we denote by g in an index-free notation), the metric given by the choice of scale τ is

$$g_\tau = \frac{g}{\tau^2}.$$

It is often propitious to perform computations involving densities by making such a choice, we shall term this operation *working in a scale* and label it either by a choice of metric or a true scale, and use unbolded symbols for the corresponding equivalence class representatives for densities. Similarly, an unbolded notation will be used for operators acting on densities when a choice of scale has been made. Once a scale has been chosen we may (or may not, according to convenience) use this to trivialize density bundles.

For submanifolds embedded in a conformal manifold, density bundles of the induced conformal structure agree with the restriction of ambient density bundles of the same weight. We shall use this frequently without further mention.

2.2. Defining densities

The data of a conformal manifold and a true scale is equivalent to a Riemannian geometry. The situation is far richer when one considers scales with a non-trivial zero locus.

Given an embedded hypersurface $\Sigma \hookrightarrow M$, a *defining density* σ is a weight $w = 1$ density $\sigma = [g; \sigma]$ with zero locus

$$\mathcal{Z}(\sigma) := \{p \in M \mid \sigma(p) = 0\} = \Sigma,$$

and such that $d\sigma(p) \neq 0$, for all $p \in \Sigma$ (so the function σ is a *defining function* for Σ). For a given hypersurface, a defining density always exists locally.

Given a defining density σ , we define the weight zero density

$$\mathcal{S}_\sigma := \left[g; \left| d\sigma \right|_g^2 - \frac{2\sigma}{d} (\Delta^g \sigma + \sigma J) \right], \tag{2.1}$$

termed the *S-curvature*. Away from the zero locus Σ of σ , the \mathcal{S} -curvature is the scalar curvature of the metric $g^\sigma = g/\sigma^2$,

$$Sc^{g^\sigma} = -d(d-1)\mathcal{S}_\sigma,$$

up to the given negative multiplicative constant. This shows that the scalar curvature Sc^{g^σ} of the metric g^σ extends smoothly to Σ , where g^σ is singular.

When working in a scale g , we will denote $n := d\sigma$ and

$$\rho_\sigma := -\frac{1}{d}(\Delta^g + J)\sigma.$$

The \mathcal{S} -curvature $[g; \mathcal{S}]$ is then given by the function

$$\mathcal{S} = n^2 + 2\rho\sigma. \tag{2.2}$$

Hence, along the hypersurface Σ , the \mathcal{S} -curvature determines the length of the conormal n and \mathcal{S}_σ is necessarily non-vanishing in a neighborhood of Σ . Since our considerations concern such a neighborhood, throughout and without loss of generality we assume that \mathcal{S}_σ is everywhere non-vanishing.

2.3. The singular Yamabe problem

Every Riemannian metric g on a closed manifold M can be conformally rescaled to one of constant scalar curvature [59,54,4], and the problem of finding $\Omega \in C^\infty(M)$ such that Sc^{Ω^2g} is constant is termed the *Yamabe problem* [62]. Compact, connected, manifolds with boundary and a singular metric $g^\circ = g/\sigma^2$, where the boundary and singularity of g° of M are given by the zero locus of the defining function $\sigma \in C^\infty(M)$, are termed *conformally compact*. (Note that, in this situation, the term conformally compact is often reserved for the complement of the zero locus of σ .) A Yamabe-type problem for the metric g° was considered, first in Euclidean space with g the standard flat metric by Loewner and Nirenberg [43], and subsequently for general compact Riemannian manifolds (M, g) with boundary in [2]. More general related problems were called the *singular Yamabe problem* and studied in [46]. When the metric g° is in addition Einstein on $M \setminus \partial M$, we say that (M, g°) is *Poincaré-Einstein*.

Actually it will be convenient to consider conformal manifolds (M, c) with defining density σ for an embedded, oriented, separating hypersurface Σ . On each side of Σ , the defining density gives a canonical singular metric g° via $\sigma = [g^\circ; 1]$; or equivalently

$$g^\circ = \frac{g}{\sigma^2}.$$

The hypersurface Σ is termed a *conformal infinity* for the metric g° , and we may again ask whether there exists a defining function for Σ such that its singular metric has constant scalar curvature. We shall also use the singular Yamabe problem moniker for this problem.

Whether Σ is a boundary component or hypersurface, the requirement that Sc^{g° is constant may be phrased in terms of the \mathcal{S} -curvature. In particular, for negative constant scalar curvature equaling $-d(d - 1)$ we must solve

$$\mathcal{S}_\sigma = 1, \tag{2.3}$$

for σ given $\Sigma \hookrightarrow (M, c)$. Of relevance to us here is that this problem is intimately related to the study of invariants of the conformal embedding $\Sigma \hookrightarrow (M, c)$ (see [28,30,32]). Indeed, there is an obstruction to the existence of a smooth solution of the singular

Yamabe problem [2] (see also [30]). However there does exist an asymptotic solution for σ such that

$$\mathcal{S}_\sigma = 1 + \sigma^d \mathcal{F}, \tag{2.4}$$

where \mathcal{F} is a smooth, weight $-d$ density known as the *obstruction density*. The quantity $\mathbf{F}_\Sigma = \mathcal{F}|_\Sigma$ is a non-trivial invariant of $\Sigma \hookrightarrow (M, \mathbf{c})$. When M is a conformal three-manifold this gives the *Willmore invariant*, which is the functional gradient of the Willmore energy functional [28]. A defining density solving Equation (2.4) is termed a (*conformal*) *unit defining density*. Unit defining densities always exist and are unique up to addition of terms $\sigma^{d+1} \mathbf{A} = \mathcal{O}(\sigma^{d+1})$ where \mathbf{A} is any smooth weight $-d$ density. In general, given any weight one density ν , we use the notation $\mathcal{O}(\nu^k)$ to denote $\nu^k \mathbf{A}$ where \mathbf{A} is any smooth density of the appropriate weight and tensor type determined by context.

2.4. The Laplace–Robin operator

Natural Laplace-type equations on conformal geometries coupled to scale enjoy a hidden solution generating algebra [29]. To uncover this, inspired by the tractor calculus based approach of [22,27,29], we introduce an operator that combines a bulk Laplace operator and a conformally invariant boundary operator:

Let $\sigma = [g; \sigma]$ be a weight 1 density. Then the corresponding *Laplace–Robin operator*

$$\mathbf{L}_\sigma : \Gamma(\mathcal{EM}[w]) \longrightarrow \Gamma(\mathcal{EM}[w - 1]),$$

maps weight w densities $\mathbf{f} = [g; f]$ to weight $w - 1$ densities according to

$$\mathbf{L}_\sigma \mathbf{f} := [g; (d + 2w - 2)(\nabla_n + w\rho)f - \sigma(\Delta + wJ)f]. \tag{2.5}$$

The Laplace–Robin operator also maps weight ℓ log-densities $\boldsymbol{\lambda}$ to weight -1 densities:

$$\mathbf{L}_\sigma \boldsymbol{\lambda} := [g; (d - 2)(\nabla_n \lambda + \ell\rho) - \sigma(\Delta \lambda + \ell J)]. \tag{2.6}$$

It is not difficult to verify that if the log density $\boldsymbol{\lambda} = \log(e^f \tau)$ where $\tau > 0$ is a density of any non-vanishing weight, and $f \in C^\infty(M)$, then

$$\mathbf{L}_\sigma \log(e^f \tau) = \mathbf{L}_\sigma \log \tau + \mathbf{L}_\sigma f,$$

where the right hand side is computed using the definitions given in Equations (2.5) and (2.6).

The operator \mathbf{L}_σ is a Laplacian-type operator that is degenerate along the zero locus of σ . It arises naturally in a tractor calculus description of (pseudo)Riemannian geometries and related physical wave equations from a conformal perspective [22,27]. In the

case that σ is a defining density for a hypersurface Σ , this operator restricted to Σ is proportional to a conformally invariant Robin-type (i.e., Dirichlet plus Neumann) operator along the corresponding hypersurface Σ . When the defining density additionally obeys $\mathcal{S}_\sigma = 1 + \mathcal{O}(\sigma^2)$ then, on weight $w \neq 1 - \frac{d}{2}$ densities $\mathbf{f} = [g; f]$, along Σ one has

$$\frac{1}{d + 2w - 2} \mathbf{L}_\sigma \mathbf{f} \Big|_\Sigma = [g; (\nabla_{\hat{n}} - wH^g)f] \Big|_\Sigma =: \delta_{\mathbb{R}}^\Sigma \mathbf{f}. \tag{2.7}$$

The operator $\delta_{\mathbb{R}}^\Sigma$ is well-defined by the above display for all weights w , and is termed the *conformal Robin operator* [13]. Here \hat{n} is the unit conormal to Σ , and H^g is the mean curvature of Σ with respect to the metric g . The quantity $\delta_{\mathbb{R}}^\Sigma \mathbf{f}$ is an example of a conformal hypersurface invariant. These are invariants of the conformal embedding $\Sigma \hookrightarrow (M, \mathbf{c})$. The precise definition is given in [30, Section 6.1]. We use a boldface notation for these invariants, other examples include the trace-free second fundamental form $\hat{\mathbf{H}}_{ab}^\Sigma \in \Gamma(\odot^2 T^* \Sigma[1])$ of the hypersurface Σ and its unit conormal $\hat{\mathbf{n}}_a \in \Gamma(T^* \Sigma[1])$.

Away from the hypersurface Σ , when computed in the scale g° corresponding to the scale σ , the Laplace–Robin operator gives the Laplacian-type operator

$$-\Delta^\circ - \frac{2J^\circ}{d} w(d + w - 1),$$

where Δ° is the Laplacian of the singular metric g° and similarly for J° . Moreover, acting on a weight $w = 1 - \frac{d}{2}$ density \mathbf{f} , the Laplace–Robin operator obeys

$$\mathbf{L}_\sigma \mathbf{f} = -\sigma \square \mathbf{f},$$

where \square is the conformally invariant *Yamabe operator*

$$\begin{aligned} \square : \Gamma(\mathcal{EM}[1 - \frac{d}{2}]) &\longrightarrow \Gamma(\mathcal{EM}[-1 - \frac{d}{2}]) \\ \cup &\qquad \qquad \qquad \cup \\ [g; f] &\longmapsto [g; \Delta f + (1 - \frac{d}{2})Jf]; \end{aligned} \tag{2.8}$$

because several definitions for the Yamabe operator \square appear in the literature, do recall that Δ denotes the negative Laplacian. We will refer to the distinguished weight $w = 1 - \frac{d}{2}$ as the *Yamabe weight*.

The utility of the Laplace–Robin operator is based on the following solution generating $\mathfrak{sl}(2)$ triple (see [29]): Define the operators

$$\begin{aligned} x : \Gamma(\mathcal{EM}[w]) &\longrightarrow \Gamma(\mathcal{EM}[w + 1]) \\ \cup &\qquad \qquad \qquad \cup \\ [g; f] &\longmapsto [g; \sigma f], \\ h : \Gamma(\mathcal{EM}[w]) &\longrightarrow \Gamma(\mathcal{EM}[w]) \\ \cup &\qquad \qquad \qquad \cup \\ [g; f] &\longmapsto [g; (d + 2w)f], \end{aligned}$$

and, supposing the \mathcal{S} -curvature is nowhere vanishing,

$$\begin{array}{ccc}
 y & : & \Gamma(\mathcal{EM}[w]) \longrightarrow \Gamma(\mathcal{EM}[w-1]) \\
 & & \Downarrow \qquad \qquad \qquad \Downarrow \\
 & & [g; f] \qquad \longmapsto \qquad [g; -L_\sigma f / \mathcal{S}],
 \end{array}$$

where $L_\sigma f := (d + 2w - 2)(\nabla_n + w\rho)f - \sigma(\Delta + wJ)f$. Note that the operator $y = -(1/\mathcal{S}\sigma)\mathbf{L}\sigma$, and x is multiplication by σ . Then, a key result of [29] is that for any conformal class of metrics \mathbf{c} , the operator triple (x, h, y) obeys the standard $\mathfrak{sl}(2)$ commutator relations

$$[x, y] = h, \quad [h, x] = 2x, \quad [y, h] = 2y. \tag{2.9}$$

In particular, we will rely heavily on the $\mathcal{U}(\mathfrak{sl}(2))$ enveloping algebra identities

$$[y, x^k] = -x^{k-1}k(h + k - 1), \quad [y^k, x] = -y^{k-1}k(h - k + 1). \tag{2.10}$$

Observe that, acting on densities of weight $w = \frac{1}{2}(k - d - 1)$, the right hand side of the second identity above vanishes, which implies that the operator y^k acting on densities \mathbf{f} of weight $\frac{1}{2}(k - d + 1)$ has the special property

$$y^k(\mathbf{f} + \sigma\mathbf{f}')|_\Sigma = y^k\mathbf{f}|_\Sigma,$$

where here \mathbf{f}' is any smooth density of weight $\frac{1}{2}(k - d - 1)$. So along Σ , the action of y^k is independent of the choice of extension \mathbf{f} of the boundary data $\mathbf{f}|_\Sigma$. We encode this notion in a general definition:

Definition 2.1. Let $\mathcal{A} : \Gamma(\mathcal{EM}[w]) \rightarrow \Gamma(\mathcal{EM}[w'])$. Then we call the operator \mathcal{A} *tangential to $\Sigma = \mathcal{Z}(\sigma)$* if it obeys

$$\mathcal{A}(\mathbf{f} + \sigma\mathbf{f}')|_\Sigma = \mathcal{A}\mathbf{f}|_\Sigma, \tag{2.11}$$

for any $\mathbf{f} \in \Gamma(\mathcal{EM}[w])$ and $\mathbf{f}' \in \Gamma(\mathcal{EM}[w-1])$.

For operators acting on log-densities, we say $\mathcal{A} : \Gamma(\mathcal{FM}[1]) \rightarrow \Gamma(\mathcal{EM}[w])$ is *tangential to Σ* when, for any true scale τ ,

$$\mathcal{A}(\log(f\tau))|_\Sigma = \mathcal{A}(\log(\tau))|_\Sigma,$$

for all $0 < f \in C^\infty(M)$ such that $f|_\Sigma = 1$. ■

Remark 2.2. Since \mathbf{f} and \mathbf{f}' are smooth and \mathcal{A} is linear, the requirement in Equation (2.11) can be restated as

$$\mathcal{A}(\sigma\mathbf{f}') = \sigma\mathbf{h},$$

for some smooth $\mathbf{h} \in \Gamma(\mathcal{EM}[w' - 1])$, and analogously for operators on log densities. Tangential operators are useful for expressing operators along Σ holographically because, as a consequence of the above definition, a tangential operator \mathcal{A} defines an operator

$$\mathbf{A} : \Gamma(\mathcal{E}\Sigma[w]) \rightarrow \Gamma(\mathcal{E}\Sigma[w'])$$

according to

$$\mathbf{A}\mathbf{f} := \mathcal{A}\mathbf{f}_{\text{ext}}|_{\Sigma},$$

where \mathbf{f}_{ext} is any smooth extension of $\mathbf{f} \in \Gamma(\mathcal{E}\Sigma[w])$ to M . ■

Definition 2.1 and its consequences, remarked upon above, extend naturally to higher codimension embedded submanifolds. In particular, for a codimension two embedded submanifold Λ defined as the zero locus of a pair of defining densities σ and μ , we require

$$\mathcal{A}(\mathbf{f} + \sigma\mathbf{f}' + \mu\mathbf{f}'')|_{\Lambda} = \mathcal{A}\mathbf{f}|_{\Lambda},$$

for any $\mathbf{f}', \mathbf{f}'' \in \Gamma(\mathcal{EM}[w - 1])$. We call such operators *tangential to $\Lambda = \mathcal{Z}(\sigma, \mu)$* ; they define operators $\Gamma(\mathcal{E}\Lambda[w]) \rightarrow \Gamma(\mathcal{E}\Lambda[w'])$. We will also encounter the intermediate case where tangentiality of an operator only holds along a hypersurface $\Lambda \hookrightarrow \Sigma$ with respect to a single defining function σ , so that

$$\mathcal{A}(\mathbf{f} + \sigma\mathbf{f}')|_{\Lambda} = \mathcal{A}\mathbf{f}|_{\Lambda},$$

for any $\mathbf{f}' \in \Gamma(\mathcal{EM}[w - 1])$. Here we will say that \mathcal{A} is *tangential to Σ along Λ* . This defines a conformally invariant operator along Λ that may take derivatives in directions normal to the embedding of Λ in Σ , and that outputs a density on Λ . When Λ is the intersection of two hypersurfaces Σ and Ξ , tangentiality to Σ along Λ and tangentiality to Ξ along Λ together imply tangentiality to Λ .

2.5. Leibniz rules

Because it is a scale-coupled conformal analog of the Laplace operator, the Laplace–Robin operator does not obey the Leibniz rule when acting on products of densities. To handle this feature we proceed as follows. For weight $w \neq 1 - \frac{d}{2}$ densities \mathbf{f} and weight ℓ log densities λ we define

$$\begin{cases} \widehat{\mathbf{L}}_{\sigma}\mathbf{f} := \frac{1}{d+2w-2}\mathbf{L}_{\sigma}\mathbf{f}, \\ \widehat{\mathbf{L}}_{\sigma}\lambda := \frac{1}{d-2}\mathbf{L}_{\sigma}\lambda. \end{cases} \tag{2.12}$$

The first of these operators obeys a generalized Leibniz rule:

$$\widehat{\mathbf{L}}_{\sigma}(ff') = (\widehat{\mathbf{L}}_{\sigma}f)f' + f\widehat{\mathbf{L}}_{\sigma}f' - \frac{2\sigma}{d+2(w+w')-2} \langle f, f' \rangle. \tag{2.13}$$

In the above f and f' are densities of weight $w \neq 1 - \frac{d}{2} \neq w'$ and $w + w' \neq 1 - \frac{d}{2}$. The bracket $\langle \cdot, \cdot \rangle$ is defined by

$$\begin{aligned} \langle f, f' \rangle &= \left[g; (\nabla f) \cdot \nabla f' - \frac{w}{d+2w'-2} f \Delta f' - \frac{w'}{d+2w-2} f' \Delta f - \frac{2ww'(d+w+w'-2)}{(d+2w-2)(d+2w'-2)} Jff' \right] \\ &\in \Gamma(\mathcal{EM}[w+w'-2]). \end{aligned} \tag{2.14}$$

In the case that $w = w' = 1$ we will often use an unbolded notation for the resulting function-valued symmetric bracket. Indeed, when σ and μ are defining densities for hypersurfaces Σ and Ξ intersecting along a codimension two submanifold $\Sigma \cap \Xi$, then

$$\frac{\langle \sigma, \mu \rangle}{\sqrt{\langle \sigma, \sigma \rangle \langle \mu, \mu \rangle}} \Big|_{\Sigma \cap \Xi}$$

computes the cosine of the angle between the respective conormals to Σ and Ξ . Note also that

$$\langle \sigma, \sigma \rangle = \mathcal{S}_{\sigma}.$$

Moreover, when f has weight $w \neq 1 - \frac{d}{2}$, it follows that

$$\langle \sigma, f \rangle = \widehat{\mathbf{L}}_{\sigma}f. \tag{2.15}$$

We will also need analogs of the generalized Leibniz rule (2.13) at certain critical weights. First, when $w + w' = 1 - \frac{d}{2}$ but $w \neq 1 - \frac{d}{2} \neq w'$ we have

$$\mathbf{L}_{\sigma}(ff') = -2\sigma \langle f, f' \rangle. \tag{2.16}$$

The case when $w = 1 - \frac{d}{2} \neq w', w + w'$ is more delicate. For that we focus on the case that σ is a defining density for a hypersurface Σ . Then we observe that the space of equivalence classes with respect to the equivalence $f \sim f + \sigma f'$ for any $f' \in \Gamma(\mathcal{EM}[w-1])$, denoted by

$$\Gamma_{\Sigma}(\mathcal{EM}[w]) := \{[f] : f \in \Gamma(\mathcal{EM}[w])\}, \tag{2.17}$$

is congruent to the space $\Gamma(\mathcal{E}\Sigma[w])$ of conformal densities along the hypersurface with conformal class of metrics c_{Σ} induced by c . Observe that operators that are tangential along Σ are canonically well-defined on the space $\Gamma_{\Sigma}(\mathcal{EM}[w])$.

Now we note that along the hypersurface Σ , the operator $\nabla_n + (1 - \frac{d}{2})\rho_{\sigma}$ is the conformal Robin operator $\delta_{\mathbb{R}}^{\Sigma}$ at the weight $1 - \frac{d}{2}$. Therefore we have the following well-defined operator

$$\begin{aligned} \widetilde{\mathbf{L}}_\sigma : \Gamma(\mathcal{EM}[1 - \frac{d}{2}]) &\longrightarrow \Gamma_\Sigma(\mathcal{EM}[-\frac{d}{2}]) \\ \cup &\cup \\ [g; f] &\longmapsto [g; \nabla_n f + (1 - \frac{d}{2})\rho_\sigma f]. \end{aligned} \tag{2.18}$$

In the last expression in the above display we have used square brackets to indicate equivalence classes with respect to both metric rescalings and the addition of smooth terms proportional to σ .

Given a defining density $\mu = [g; \mu]$ for a second hypersurface Ξ , it is possible to build invariant operators along Ξ from combinations and compositions of representatives of the operators $\widetilde{\mathbf{L}}_\sigma$ and $\widetilde{\mathbf{L}}_\mu$, as well as analogs for the bracket of Equation (2.14). A key example is the combination (which by a slight abuse of notation will be) denoted by $\widetilde{\mathbf{L}}_\sigma - \sigma \widehat{\mathbf{L}}_\mu \widetilde{\mathbf{L}}_\mu$ with domain $\Gamma(\mathcal{EM}[1 - \frac{d}{2}])$. This can be made well-defined with codomain $\Gamma_\Sigma(\mathcal{EM}[-\frac{d}{2}])$ by suitably interpreting the second term, which in any case does not contribute. In fact, one can also make it well-defined with codomain $\Gamma_\Xi(\mathcal{EM}[-\frac{d}{2}])$ via a continuation argument in the weight w by simultaneously using the corresponding representatives of the two tilded operators. Indeed, there is a pole $1/(d + 2w - 2)$ when extending the operator $\widehat{\mathbf{L}}_\cdot$ defined in Equation (2.12) (for any weight one density σ) to the critical weight $w = 1 - \frac{d}{2}$, but this pole cancels for the particular combination $\widetilde{\mathbf{L}}_\sigma - \sigma \widehat{\mathbf{L}}_\mu \widetilde{\mathbf{L}}_\mu$. Hence, along Ξ , we may invariantly define

$$\begin{aligned} \widetilde{\mathbf{L}}_\sigma - \sigma \widehat{\mathbf{L}}_\mu \widetilde{\mathbf{L}}_\mu : \Gamma(\mathcal{EM}[1 - \frac{d}{2}]) &\longrightarrow \Gamma_\Xi(\mathcal{EM}[-\frac{d}{2}]) \\ \cup &\cup \\ [g; f] &\longmapsto [g; (\nabla_n + (1 - \frac{d}{2})\rho_\sigma - \sigma(\nabla_m - \frac{d}{2}\rho_\mu)(\nabla_m + (1 - \frac{d}{2})\rho_\mu))f], \end{aligned} \tag{2.19}$$

where $m := d\mu$.

Along similar lines, we define a modified “bracket”, whose codomain depends on its second argument:

$$\langle \cdot, \cdot \rangle := \left[g; (\nabla f) \cdot \nabla f' + \frac{d-2}{2(d+2w'-2)} f(\Delta + w'J)f' \right] \in \text{coker } f'. \tag{2.20}$$

In the above, $\text{coker } f'$ is the cokernel of f' viewed as the linear operator mapping $\Gamma(\mathcal{EM}[-1 - \frac{d}{2}]) \rightarrow \Gamma(\mathcal{EM}[-1 - \frac{d}{2} + w'])$ that acts by multiplication. In these terms, we then have the Leibniz-type rule for the critical case $w = 1 - \frac{d}{2} \neq w', w + w'$:

$$\widehat{\mathbf{L}}_\sigma(f f') = ((\widetilde{\mathbf{L}}_\sigma + \frac{1}{2w'} \mathbf{L}_\sigma) f) f' + f \widehat{\mathbf{L}}_\sigma f' - \frac{\sigma}{w'} \langle f, f' \rangle \in \Gamma_\Xi(\mathcal{EM}[-\frac{d}{2} + w']). \tag{2.21}$$

It is not difficult to use a weight continuation argument similar to that discussed above (and employing a similar abuse of notation) that the operator $\widetilde{\mathbf{L}}_\cdot$ and the modified bracket $\langle \cdot, \cdot \rangle$ combine in this formula to produce a density-valued result.

In the doubly critical case, $w = 1 - \frac{d}{2} = w'$ one has

$$\widehat{\mathbf{L}}_\sigma(f f') = \left(\widetilde{\mathbf{L}}_\sigma f - \frac{1}{d-2} \mathbf{L}_\sigma f \right) f' + f \left(\widetilde{\mathbf{L}}_\sigma f' - \frac{1}{d-2} \mathbf{L}_\sigma f' \right) + \frac{2}{d-2} \sigma \langle f, f' \rangle \tag{2.22}$$

$$\in \Gamma_{\Xi}(\mathcal{E}M[1 - d]),$$

where the doubly-modified “bracket” is defined similarly to above by

$$\langle \mathbf{f}, \mathbf{f}' \rangle = [g; (\nabla f) \cdot \nabla f'] \in \text{coker}(\mathbf{f}, \mathbf{f}').$$

Here the coker notation above means we quotient by the linear span of the images of \mathbf{f} and \mathbf{f}' .

Given a pair of scales σ and τ we may also form the invariant differential operator $\mathbf{L}_{\sigma, \tau} : \Gamma(\mathcal{E}M[w]) \rightarrow \Gamma(\mathcal{E}M[w])$ defined by

$$\mathbf{L}_{\sigma, \tau} \mathbf{f} = [g; \tau(\nabla_n + w\rho_{\sigma})f - \sigma(\nabla_k + w\rho_{\tau})f], \tag{2.23}$$

where $k := d\tau$. At weight $w = 0$, in the τ scale, this operator maps functions f to $\nabla_n f$, and thus was denoted by $\nabla_n \tau$ and dubbed the coupled conformal gradient operator in [33]. Note also that at weight $w \neq 1 - \frac{d}{2}$ one has

$$\mathbf{L}_{\sigma, \tau} = \tau \widehat{\mathbf{L}}_{\sigma} - \sigma \widehat{\mathbf{L}}_{\tau}.$$

At the critical weight $w = 1 - \frac{d}{2}$, the above identity still holds (abusing notation as above) upon replacing $\widehat{\mathbf{L}}$ with $\widetilde{\mathbf{L}}$.

2.6. Distributions and integral theory

A weight $w = -d$ density $\mathbf{f} = [g; f]$ can be invariantly integrated over a conformal d -manifold (or some region $D \subset M$ thereof) since the volume element dV^g of $g \in \mathbf{c}$ defines a weight d measure-valued density $[g; dV^g]$. Thus we define the conformally invariant integral of \mathbf{f} over D by

$$\int_D \mathbf{f} := \int_D dV^g f.$$

Given a hypersurface $\Sigma \hookrightarrow M$ and a function $f_{\Sigma} \in C^{\infty}(\Sigma)$, it is propitious to treat the integral of f_{Σ} over Σ in terms of a defining function σ for Σ . In particular, given $g \in \mathbf{c}$, we have (see, for example, [48] or [21])

$$\int_{\Sigma} dV^{g_{\Sigma}} f_{\Sigma} = \int_M dV^g |d\sigma|_g \delta(\sigma) f. \tag{2.24}$$

In the above display, g_{Σ} is the metric along Σ induced by g , f denotes any (smooth) extension of f_{Σ} to M , and $\delta(\sigma)$ is the Dirac-delta distribution.

The distributional identity $\delta(\Omega\sigma) = \Omega^{-1}\delta(\sigma)$ (valid for any $0 < \Omega \in C^{\infty}(M)$) implies that if $\sigma = [g; \sigma]$ is any weight $w = 1$ density, then

$$\delta(\boldsymbol{\sigma}) := [g; \delta(\sigma)]$$

is a weight $w = -1$ distribution-valued conformal density (see [31] for details). Thus $\delta(\boldsymbol{\sigma} - \varepsilon\boldsymbol{\tau})$, where $\varepsilon \in \mathbb{R}$ and $\boldsymbol{\tau} \in \Gamma(\mathcal{EM}[1])$, gives a one parameter family of weight $w = -1$ densities. Successively differentiating this k times with respect to ε and subsequently setting ε to zero establishes that $\delta^{(k)}(\boldsymbol{\sigma}) = [g; \delta^{(k)}(\sigma)]$ is a weight $w = -k-1$ distribution-valued density. Moreover, the conformally invariant integral of a weight $1 - d$ density $\mathbf{f}_\Sigma = [g_\Sigma; f|_\Sigma]$ along Σ may be expressed in terms of any smooth extension \mathbf{f} of this density via

$$\int_\Sigma \mathbf{f}_\Sigma = \int_M \delta(\boldsymbol{\sigma}) \sqrt{\mathcal{S}} \mathbf{f}. \tag{2.25}$$

This relation reduces to Equation (2.24) upon expressing it in a choice of metric.

We will employ standard distributional identities (on \mathbb{R}) for the Dirac delta $\delta(x)$ and Heaviside step function $\theta(x)$, such as

$$\theta'(x) = \delta(x), \quad x\delta(x) = 0, \quad x\delta'(x) = -\delta(x),$$

and

$$x\delta^{(n)}(x) := x \frac{d^n \delta(x)}{dx^n} = -n\delta^{(n-1)}(x), \quad n \in \mathbb{Z}_{\geq 1},$$

where $n \in \mathbb{Z}_{\geq 1}$. These are valid when integrating against suitable test functions. In particular we will need to consider the situation where the coordinate x is replaced by a defining function σ . Again, this is discussed in detail in [31,33], the key maneuver is to assume that in a neighborhood of Σ , the bulk manifold M can be treated as a product $\Sigma \times I$ where I is an open interval about 0 and the defining function σ pulls back to a coordinate x on I . Thus, integrals over such neighborhoods can be handled using Fubini’s theorem.

Remark 2.3. The distributional calculus is also well adapted to hypersurface computations. For example, when $\boldsymbol{\sigma}$ is a defining density for a hypersurface Σ , we have the operator identity relating the Laplace–Robin and Robin operators

$$\delta(\boldsymbol{\sigma}) \widehat{\mathbf{L}}_\boldsymbol{\sigma} = \delta(\boldsymbol{\sigma}) \delta_{\mathbb{R}}^\Sigma,$$

valid acting on any density of weight $w \neq 1 - \frac{d}{2}$. At the critical weight we may use the operator $\widetilde{\mathbf{L}}_\boldsymbol{\sigma}$ of Equation (2.18) to write the identity

$$\delta(\boldsymbol{\sigma}) \widetilde{\mathbf{L}}_\boldsymbol{\sigma} = \delta(\boldsymbol{\sigma}) \delta_{\mathbb{R}}^\Sigma,$$

because $\delta(\boldsymbol{\sigma})\Gamma(\mathcal{EM}[w]) = \delta(\boldsymbol{\sigma})\Gamma_\Sigma(\mathcal{EM}[w])$. ■

One integral result will play a key role, namely that the Laplace–Robin operator is formally self-adjoint [31]. Hence, if M is a closed conformal manifold, \mathbf{f} a weight $1-d-w$ density and \mathbf{g} a weight w density, then

$$\int_M \mathbf{f} \mathbf{L}_\sigma \mathbf{g} = \int_M \mathbf{g} \mathbf{L}_\sigma \mathbf{f}. \tag{2.26}$$

The same conclusion holds if \mathbf{f} or \mathbf{g} have compact support. We will use the notation \dagger for the formal adjoint along M , which ignores boundary terms, so that the above equation reads

$$\mathbf{L}_\sigma = \mathbf{L}_\sigma^\dagger.$$

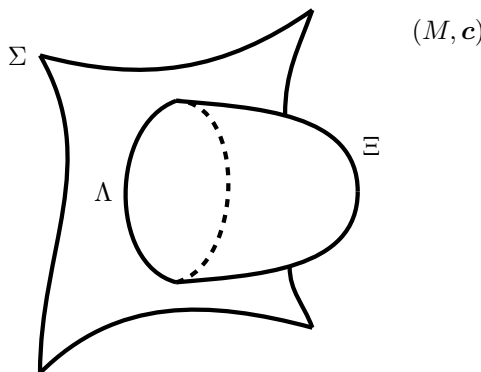
The boundary terms are given in [32, Theorem 4.3].

3. Minimal hypersurfaces for singular metrics

Here we begin with the data of a sequence of conformal hypersurface embeddings

$$\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c}), \tag{3.1}$$

meaning that Λ is a hypersurface in Σ and in turn Σ is a hypersurface in M . Then, given a choice of defining density σ for Σ and thus a metric g^σ that is singular along Σ , we consider the problem of determining, at least asymptotically, an oriented hypersurface Ξ (with boundary) that meets Σ transversely with intersection $\Lambda = \partial\Xi$, and such that Ξ is minimal with respect to g^σ . We will often refer to Λ as the anchoring hypersurface/sub-manifold. This situation is depicted below:



As discussed earlier, in the case that the singular metric g^σ is Poincaré–Einstein, this problem has been studied by Graham and Witten [39] (see also [38]) using a different approach.

A minimal hypersurface Ξ is characterized by the vanishing of its mean curvature H_{Ξ}^g with respect to the ambient metric g . Our treatment of minimal surfaces relies on formulating this condition in terms of defining densities. This is achieved by the next proposition. In what follows we use the notation $\mathbf{A} \stackrel{\mu}{\sim} \mathbf{B}$ when smooth densities \mathbf{A} , \mathbf{B} and μ obey $\mathbf{A} = \mathbf{B} + \mu\mathbf{C}$ for some smooth density \mathbf{C} . Also recall that we universally, and without loss of generality, assume that \mathcal{S} -curvatures depending on the defining density of a hypersurface are everywhere non-vanishing.

Proposition 3.1. *Let μ and σ be defining densities for embedded hypersurfaces Ξ and Σ , respectively. Then, away from Λ , the mean curvature of Ξ with respect to the metric $g^\circ = g/\sigma^2$ is given by*

$$H_{\Xi}^{g^\circ} = - \left. \frac{\mathcal{H}_{\mu}^{\sigma}}{\sqrt{\mathcal{S}_{\mu}}} \right|_{\Xi}, \tag{3.2}$$

where

$$\mathcal{H}_{\mu}^{\sigma} = \langle \sigma, \mu \rangle + \frac{1}{2(d-1)(d-2)} \sigma \mathbf{L}_{\mu} \log \mathcal{S}_{\mu}, \tag{3.3}$$

and hence $H_{\Xi}^{g^\circ}$ is extended smoothly to Λ by the right hand side of Equation (3.2). Moreover, if f is any smooth, strictly positive function, then

$$\mathcal{H}_{f\mu}^{\sigma} \stackrel{\mu}{\sim} f\mathcal{H}_{\mu}^{\sigma} \text{ and } \mathcal{S}_{f\mu} \stackrel{\mu}{\sim} f^2\mathcal{S}_{\mu}.$$

Proof. Given any defining function μ and metric g° , the mean curvature of $\Xi \hookrightarrow (M, g^\circ)$ is given by

$$H_{\Xi}^{g^\circ} = \left. \frac{\nabla^{g^\circ} \hat{m}}{d-1} \right|_{\Xi}, \tag{3.4}$$

where \hat{m} is the extension of the unit conormal to Ξ given by

$$\hat{m} = \frac{d\mu}{|d\mu|_{g^\circ}}.$$

Now, the divergence of \hat{m} with respect to the Levi-Civita connection of $g^\circ = g/\sigma^2$ is related to that of the metric g via

$$g_o^{ab} \nabla_a^{g^\circ} \hat{m}_b = \sigma^2 g^{ab} \nabla_a \hat{m}_b - (d-2) \sigma g^{ab} n_a \hat{m}_b.$$

Calling $m = d\mu$, we have $\hat{m} = \sigma^{-1}m/|m|_g$ so that

$$g_o^{ab} \nabla_a^{g^\circ} \hat{m}_b = \frac{1}{|m|_g} \left(\sigma g^{ab} \nabla_a m_b - (d-1) g^{ab} n_a m_b - \frac{1}{2} \sigma g^{ab} m_a \nabla_b \log |m|_g^2 \right).$$

Hence we have (using an index-free notation now that all explicit metric dependence is through g)

$$H_{\Xi}^{g^{\circ}} \stackrel{\mu}{\sim} - \frac{m \cdot n + \sigma \rho_{\mu} + \mu \rho_{\sigma} + \frac{1}{2(d-1)(d-2)} \sigma((d-2)\nabla_m - \mu\Delta) \log(|m|_g^2 + 2\rho_{\mu}\mu)}{\sqrt{|m|_g^2 + 2\rho_{\mu}\mu}}.$$

Comparing the above display with the definition of the bracket $\langle \cdot, \cdot \rangle$ in Equation (2.14), the \mathcal{S} -curvature in Equation (2.1) and the Laplace–Robin operator in Equation (2.5), gives the result claimed in the first two displays of the proposition.

To prove the second claim, we could rely on the fact that Equation (3.4) gives the mean curvature for *any* defining function μ and only compute the homogeneity of the \mathcal{S} -curvature. Here, we give a detailed proof to further develop our hypersurface calculus. First we note that for any weight $w = 1$ densities μ and σ , given a smooth function f , it is not difficult (using, away from critical weights, the identity $\langle \sigma, \cdot \rangle = \widehat{\mathbf{L}}_{\sigma} \cdot$ and the Leibniz rule (2.13)) to verify that

$$\langle \sigma, f\mu \rangle = f\langle \sigma, \mu \rangle + \mu\langle \sigma, f \rangle - \frac{2}{d}\sigma\langle f, \mu \rangle \stackrel{\mu}{\sim} f(\langle \sigma, \mu \rangle - \frac{2}{d}\sigma\widehat{\mathbf{L}}_{\mu}\log f).$$

Thus, since $\mathcal{S}_{\mu} = \langle \mu, \mu \rangle$, it follows upon applying the above display twice that

$$\mathcal{S}_{f\mu} := \langle f\mu, f\mu \rangle = f^2\langle \mu, \mu \rangle + \frac{2(d-2)}{d}\mu f\langle \mu, f \rangle + \mathcal{O}(\mu^2) \stackrel{\mu}{\sim} f^2\mathcal{S}_{\mu},$$

which gives the claimed result for the \mathcal{S} -curvature. The above display and the additive property for logarithms of products of densities allows us to compute

$$\begin{aligned} \mathbf{L}_{f\mu} \log \mathcal{S}_{f\mu} &\stackrel{\mu}{\sim} f\mathbf{L}_{\mu} \left[\log \mathcal{S}_{\mu} + 2\log f + \log \left(1 + \frac{2(d-2)}{d} \frac{\mu}{\mathcal{S}_{\mu}} \widehat{\mathbf{L}}_{\mu} \log f \right) \right] \\ &\stackrel{\mu}{\sim} f\mathbf{L}_{\mu} \log \mathcal{S}_{\mu} + \frac{4(d-1)(d-2)}{d} f\widehat{\mathbf{L}}_{\mu} \log f. \end{aligned}$$

Here we have used that $\log \mathcal{S}_{f\mu}$ is a weight zero density and that \mathbf{L}_{μ} is a derivation along the zero locus of μ . Combining the last and next to last displays gives the required result. \square

Given a hypersurface $\Sigma \hookrightarrow (M, \mathbf{c})$, we may always find a conformal unit defining density μ solving the singular Yamabe problem $\mathcal{S}_{\mu} = 1 + \mathcal{O}(\mu^d)$. In that case the minimal surface condition

$$\mathcal{H}_{\mu}^{\sigma} \stackrel{\mu}{\sim} 0 \tag{3.5}$$

simplifies to

$$\langle \sigma, \mu \rangle \stackrel{\mu}{\sim} 0.$$

This implies that if a minimal hypersurface Ξ for a singular metric $g^\circ = \mathbf{g}/\sigma^2$ intersects the zero locus Σ of $\sigma = [g; \sigma]$, it does so at right angles.

In general the minimal surface condition (3.5) for a singular metric g° determined by σ cannot be solved smoothly, so instead we solve this problem asymptotically, in the following sense:

Problem 3.2. Let σ be a defining function for $\Sigma = \mathcal{Z}(\sigma) \hookrightarrow (M, \mathbf{c})$ and let $\Lambda \hookrightarrow \Sigma$ be an embedded hypersurface. Find a hypersurface Ξ such that

$$H_{\Xi}^{g^\circ} = \sigma^k \mathbf{A}|_{\Xi},$$

where g° is the singular metric determined by σ , $\Lambda = \Xi \cap \Sigma$, the density \mathbf{A} is smooth, and the order $k \in \mathbb{Z}_{\geq 1}$ is as high as possible.

The key to solving this problem is the following lemma:

Lemma 3.3. Let $k \in \mathbb{Z}_{\geq 1}$ and $\mu' = \mu + \sigma^{k+1} \mu_{k+1}$, for any $\mu_{k+1} \in \Gamma(\mathcal{E}M[-k])$. Then

$$\mathcal{H}_{\mu'}^\sigma \stackrel{\mu'}{\sim} \mathcal{H}_\mu^\sigma (1 + \mathcal{O}(\sigma^k)) + \frac{(k+1)(d-1-k)}{d-1} \mathcal{S}_\sigma \sigma^k \mu_{k+1} + \mathcal{O}(\sigma^{k+1}).$$

Proof. The proof is an elementary application of the Leibniz rules developed in Section 2.5. The details are as follows: First we use the $\mathfrak{sl}(2)$ identity obeyed by the Laplace–Robin operator in (2.10) to compute

$$\langle \sigma, \mu' \rangle - \langle \sigma, \mu \rangle = \frac{1}{d} \mathbf{L}_\sigma (\sigma^{k+1} \mu_{k+1}) = \frac{(k+1)(d-k)}{d} \sigma^k \mathcal{S}_\sigma \mu_{k+1} + \mathcal{O}(\sigma^{k+1}). \tag{3.6}$$

Then we note that acting on a weight $w = 0$ density \mathbf{f} , we have

$$\mathbf{L}_{\mu'} \mathbf{f} - \mathbf{L}_\mu \mathbf{f} = (k+1) \sigma^k \mu_{k+1} \mathbf{L}_\sigma \mathbf{f} + \mathcal{O}(\sigma^{k+1}).$$

This identity is easily established by examining the definition of the Laplace–Robin operator in Equation (2.5). Thus, because $\sigma \mathbf{L}_\mu \log \mathcal{S}_\mu$ is itself a weight $w = 0$ density and $\log \mathcal{S}_{\mu'} = \log \mathcal{S}_\mu + \log \left(\frac{\mathcal{S}_{\mu'}}{\mathcal{S}_\mu} \right)$,

$$\sigma \mathbf{L}_{\mu'} \log \mathcal{S}_{\mu'} - \sigma \mathbf{L}_\mu \log \mathcal{S}_\mu = \sigma \mathbf{L}_\mu \log \left(\frac{\mathcal{S}_{\mu'}}{\mathcal{S}_\mu} \right) + \mathcal{O}(\sigma^{k+1}). \tag{3.7}$$

Moreover, using

$$\widehat{\mathbf{L}}_\mu \sigma^{k+1} = (k+1) \sigma^k \langle \sigma, \mu \rangle - \frac{k(k+1)}{d+2k} \sigma^{k-1} \mathcal{S}_\sigma \mu,$$

and

$$\langle \sigma^{k+1}, \mu_{k+1} \rangle = (k + 1) \sigma^k \widehat{\mathbf{L}}_\sigma \mu_{k+1} + \frac{k^2(k+1)}{d+2k} \sigma^{k-1} \mathcal{S}_\sigma \mu_{k+1},$$

we have

$$\begin{aligned} \mathcal{S}_{\mu'} - \mathcal{S}_\mu &= 2\langle \mu, \sigma^{k+1} \mu_{k+1} \rangle + \langle \sigma^{k+1} \mu_{k+1}, \sigma^{k+1} \mu_{k+1} \rangle \\ &= 2\mu_{k+1} \left((k + 1) \sigma^k \langle \sigma, \mu \rangle - \frac{k(k+1)}{d+2k} \sigma^{k-1} \mathcal{S}_\sigma \mu \right) + 2\sigma^{k+1} \widehat{\mathbf{L}}_\mu \mu_{k+1} \\ &\quad - \frac{4}{d} \mu \left((k + 1) \sigma^k \widehat{\mathbf{L}}_\sigma \mu_{k+1} + \frac{k^2(k+1)}{d+2k} \sigma^{k-1} \mathcal{S}_\sigma \mu_{k+1} \right) + \mathcal{O}(\sigma^{2k}) \\ &= -\mu \left(\frac{2k(k+1)}{d} \sigma^{k-1} \mathcal{S}_\sigma \mu_{k+1} + \mathcal{O}(\sigma^k) \right) + \mathcal{H}_\mu^\sigma \mathcal{O}(\sigma^k) + \mathcal{O}(\sigma^{k+1}). \end{aligned} \tag{3.8}$$

Now, acting on weight w densities—again thanks to the Laplace–Robin $\mathfrak{sl}(2)$ algebra—we have the operator identity

$$\mathbf{L}_\mu \circ \mu \overset{\mu'}{\sim} (d + 2w) \mathcal{S}_\mu + \mathcal{O}(\sigma^{k+1}),$$

so that Equations (3.7) and (3.8) imply

$$\sigma \mathbf{L}_{\mu'} \log \mathcal{S}_{\mu'} \overset{\mu'}{\sim} \sigma \mathbf{L}_\mu \log \mathcal{S}_\mu - \frac{2k(k+1)(d-2)}{d} \sigma^k \mathcal{S}_\sigma \mu_{k+1} + \mathcal{H}_\mu^\sigma \mathcal{O}(\sigma^k) + \mathcal{O}(\sigma^{k+1}).$$

Here we used that $\mathbf{L}_\mu \circ \sigma^k \overset{\mu'}{\sim} \mathcal{O}(\sigma^{k-1})$ and the identity $\log(A/B) = \log(1 + \frac{A-B}{B})$. Employing Equation (3.6) and the above display to compute $\mathcal{H}_{\mu'}^\sigma$ as defined by Equation (3.3) gives the quoted result. \square

The above lemma is the basis for an iterative solution to the minimal surface condition (3.5). First consider a defining density for a hypersurface that meets Σ transversely along Λ . Working locally, it is straightforward to improve this to a defining function μ_0 for a hypersurface Ξ_0 such that Σ and Ξ_0 intersect along Λ at right angles. Moreover, without loss of generality, assume that μ_0 is a conformal unit defining density, so that $\mathcal{S}_{\mu_0} = 1 + \mathcal{O}(\mu_0^d)$. Then

$$\mathcal{H}_{\mu_0}^\sigma \overset{\mu_0}{\sim} \langle \sigma, \mu_0 \rangle = \frac{1}{d} \mathbf{L}_\sigma \mu_0 \overset{\mu_0}{\sim} \mathcal{O}(\sigma),$$

since $\langle \sigma, \mu_0 \rangle$ vanishes along Λ . Hence, we consider an improved defining density

$$\mu = \mu_0 + \sigma^2 \mu_2.$$

By the above lemma we should choose μ_2 that solves

$$\frac{1}{d} \mathbf{L}_\sigma \mu_0 + \frac{2(d-2)}{d-1} \mathcal{S}_\sigma \sigma \mu_2 \overset{\mu}{\sim} \mathcal{O}(\sigma^2).$$

It is not difficult to verify that $\mathbf{L}_\sigma(\mu f) \overset{\mu}{\sim} \mathcal{O}(\sigma)$ for any density f . Thus, dividing the above display by \mathcal{S}_σ (as remarked earlier this is well defined, at least in some neighbor-

hood of Σ) and then acting with \mathbf{L}_σ , with the help of the $\mathfrak{sl}(2)$ algebra (see Section 2.4), we find

$$\frac{1}{d} \mathbf{L}_\sigma \circ \mathcal{S}_\sigma^{-1} \circ \mathbf{L}_\sigma \mu_0 + \frac{2(d-2)^2}{d-1} \mathcal{S}_\sigma \mu_2 \stackrel{\mu}{\sim} \mathcal{O}(\sigma).$$

Hence we have proved the following Lemma:

Lemma 3.4. *Let μ_0 be a defining density for a hypersurface Ξ_0 that intersects $\Sigma = \mathcal{Z}(\sigma)$ at right angles. Then the density*

$$\mu = \mu_0 - \frac{d-1}{2d(d-2)^2} \sigma^2 (\mathcal{S}_\sigma^{-1} \circ \mathbf{L}_\sigma)^2 \mu_0$$

obeys

$$\mathcal{H}_\mu^\sigma \stackrel{\mu}{\sim} \mathcal{O}(\sigma^2).$$

The preceding two lemmas are the induction and base steps that establish the following theorem:

Theorem 3.5. *Given the conformal embedding data $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ and a defining density σ for Σ , there exists a conformal unit defining density μ such that*

$$\mathcal{H}_\mu^\sigma \stackrel{\mu}{\sim} \sigma^{d-1} \mathcal{B}. \tag{3.9}$$

Moreover, the weight $w = 1 - d$ density along

$$\mathbf{B}_\Lambda := \mathcal{B}|_\Lambda$$

is uniquely determined by the above data.

The quantity \mathbf{B}_Λ obstructs smooth solutions to the singular minimal hypersurface problem and is therefore termed the *minimal obstruction density*. It is an invariant of the conformal embedding data $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ and the defining density σ . Then if σ is determined, to sufficient order, in terms of these embeddings via a suitable problem it follows that \mathbf{B}_Λ is determined in terms of the conformal embeddings. For example, the singular Yamabe problem determines σ modulo terms of order σ^{d+1} , and so achieves this. In the special case that (M, g°) is Poincaré–Einstein, \mathbf{B}_Λ is an invariant of $\Lambda \hookrightarrow (\Sigma, \mathbf{c}_\Sigma)$. This is another hierarchy in the same vein as that discussed in the introduction. Moreover, for Poincaré–Einstein structures, it is known that \mathbf{B}_Λ is the functional gradient of the area anomaly for the minimal surface Ξ [38].

Remark 3.6. A useful tool for computations with Poincaré–Einstein structures is to choose a canonical metric $g \in \mathbf{c}$. Writing σ to denote the function in σ determined

by g , which may be taken to equal the arc length x for a suitable geodesic shooting problem, one obtains the Graham–Lee normal form [37] for the singular metric

$$g^o = \frac{dx^2 + h(x)}{x^2}.$$

The Poincaré–Einstein condition then gives that $h(x)$ has an even expansion in x . This facilitates a simple proof of vanishing theorems for anomalies and obstructions in appropriate dimension parities. For example, this implies the area anomaly (in the sense used in this article) and hence the minimal obstruction density \mathbf{B}_Λ vanishes in this case when d is odd [39,38]. Noting that the Laplace–Robin operator is odd under the interchange $x \leftrightarrow -x$, when expressed in the Graham–Lee normal form, it is not difficult to check that Lemmas 3.3 and 3.4 also lead to an even expansion in x for the function μ corresponding to $\boldsymbol{\mu}$, and in turn vanishing obstruction for d odd. ■

We will employ the term *minimal defining density* for densities $\boldsymbol{\mu}$ obeying the minimal condition (3.9) given the data $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, g^o)$, where the singular metric g^o is determined by the unit defining density $\boldsymbol{\sigma}$ (given the conformal class). When we need to emphasize the choice of $\boldsymbol{\sigma}$, we will use the term $\boldsymbol{\sigma}$ -*minimal*. This condition restricts the zero locus $\mathcal{Z}(\boldsymbol{\mu})$ to be a hypersurface Ξ that solves Problem 3.2 to order $k = d - 1$. We will term such a hypersurface an *asymptotically minimal hypersurface*. When $\boldsymbol{\mu}$ is also chosen to further obey the singular Yamabe condition $\mathcal{S}_\mu = 1 + \mathcal{O}(\mu^d)$, we term $\boldsymbol{\mu}$ a *minimal unit defining density*. A minimal defining density may always be improved to a minimal unit one (while keeping Ξ in the zero locus) with the same zero locus Ξ (see Section 2.3 and [30,32]).

Equation (3.9) implies that (generically) the failure of \mathcal{H}_μ^σ to vanish along Σ is proportional to the minimal defining density $\boldsymbol{\mu}$; this leads to another invariant of the embedding data $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$, which we record in the following lemma.

Lemma 3.7. *Let the conformal data $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ be given and let $\boldsymbol{\sigma}$ be a corresponding unit defining density for Σ . Moreover let $\boldsymbol{\mu}$ be a $\boldsymbol{\sigma}$ -minimal unit defining density for Ξ with $\partial\Xi = \Lambda$. Then*

$$\mathcal{H}_\mu^\sigma = \boldsymbol{\mu}\mathcal{C} + \boldsymbol{\sigma}^{d-1}\mathcal{B}$$

for some $\mathcal{C} \in \Gamma(\mathcal{EM}[-1])$, and $\mathcal{C}_\Lambda := \mathcal{C}|_\Lambda$ is a uniquely defined invariant of the embedding data $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ given by

$$\mathcal{C}_\Lambda = \hat{\mathbf{H}}_{\hat{\mathbf{m}}\hat{\mathbf{m}}}^\Sigma|_\Lambda,$$

where $\hat{\mathbf{m}}$ is the unit conormal for the hypersurface embedding $\Lambda \hookrightarrow \Sigma$.

Proof. Together, the minimal unit and unit defining density definitions for $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ imply

$$\mathcal{H}_\mu^\sigma = \langle \sigma, \mu \rangle + \sigma \mu^{d-1} \mathcal{C}' = \mu \mathcal{C} + \sigma^{d-1} \mathcal{B}, \tag{3.10}$$

for some smooth, weight $w = -1$ density \mathcal{C} and some weight $-d$ density \mathcal{C}' (coming from the log term in Equation (3.3)). Uniqueness of $\mathcal{C}_\Lambda = \mathcal{C}|_\Lambda$ is guaranteed by Theorem 3.5 and the uniqueness property of unit defining density solutions to the singular Yamabe problem. In particular this determines μ uniquely to $\mathcal{O}(\mu^d)$, which suffices for uniqueness of $\mathcal{C}|_\Lambda$.

Now let us denote $\mu = [g; \mu]$, $\sigma = [g; \sigma]$, $m := d\mu$, $n := d\sigma$. From the minimal unit and unit defining density definitions (see Equations (2.1), (2.14) and (3.3)) we then have

$$m^2 + 2\mu\rho_\mu = 1 + \mathcal{O}(\mu^d), \quad n^2 + 2\sigma\rho_\sigma = 1 + \mathcal{O}(\sigma^d), \tag{3.11}$$

and

$$m.n + \mu\rho_\sigma + \rho_\mu\sigma = \mu\mathcal{C} + \mathcal{O}(\sigma\mu^{d-1}) + \mathcal{O}(\sigma^{d-1}). \tag{3.12}$$

Here \mathcal{C} denotes \mathcal{C} in the scale g . Differentiating the last display along the conormal m and restricting to Λ (which we indicate by a superscript Λ above the equals sign) we have

$$\nabla_m(m.n) + \rho_\sigma \stackrel{\Lambda}{=} \mathcal{C}.$$

Here we used $\nabla_m\mu = m^2 \stackrel{\Lambda}{=} 1$ and $\nabla_m\sigma = m.n \stackrel{\Lambda}{=} 0$. Now in general for unit defining densities (see [32, Lemma 3.3]) one has

$$\rho_\sigma \stackrel{\Sigma}{=} -H^\Sigma, \quad \nabla_a n_b \stackrel{\Sigma}{=} \mathring{H}^\Sigma_{ab} + H^\Sigma g_{ab}, \tag{3.13}$$

and thus $\nabla_m m_a \stackrel{\Xi}{=} m_a H^\Xi$. Hence, since $m.n \stackrel{\Lambda}{=} 0$ we have

$$\mathcal{C} \stackrel{\Lambda}{=} \mathring{H}^\Sigma_{mm}.$$

Along Λ we have that $m = \hat{m}$; this gives the quoted result for $\mathcal{C}|_\Lambda$. \square

There are various conformally invariant relations obeyed by extrinsic quantities associated to an asymptotically minimal hypersurface Ξ along its boundary Λ . The first of these was discussed above, namely that unit conormals \hat{m} and \hat{n} of Ξ and Σ obey $\hat{m}.\hat{n} \stackrel{\Lambda}{=} 0$. Another example is given in the following Lemma:

Lemma 3.8. *Let Ξ be an asymptotically minimal hypersurface determined by the sequence of embeddings (3.1) where \hat{n} is the unit conormal to Σ . Then the trace-free second fundamental form of Ξ obeys*

$$\mathring{H}^\Xi_{\hat{n}\hat{n}}|_\Lambda = 0. \tag{3.14}$$

Proof. This is a corollary of Equation (3.20) below, but a direct proof can also be given: Without loss of generality we can take Ξ to be the zero locus of a minimal unit defining density μ . This means that we can use Equations (3.11) and (3.12) of the previous proof. So now we compute

$$n^a n^b (\nabla_a m_b + \rho_\mu g_{ab}) = \nabla_n(m.n) - n^a m^b \nabla_a n_b + n^2 \rho_\mu.$$

Along Λ , the left hand side above equals $\mathring{H}_{\hat{n}\hat{n}}^\Xi$ while for the right hand side we find

$$\nabla_n(-\sigma\rho_\mu - \mu\rho_\sigma + \mu\mathcal{C} + \mathcal{O}(\sigma\mu^{d-1}) + \mathcal{O}(\sigma^{d-1})) + \rho_\mu \stackrel{\Delta}{=} 0.$$

Here we have used Equation (3.13) and $m.n|_\Lambda = 0$ to show that $n^a m^b \nabla_a n_b|_\Lambda = 0$. Similarly $\nabla_n \mu|_\Lambda = 0$. This completes the proof. \square

Remark 3.9. When the singular metric $g^\circ = g/\sigma^2$ is Poincaré–Einstein, the hypersurface Σ is necessarily umbilic [42,23], so the invariant $\mathcal{C}|_\Lambda$ then vanishes. Also note that Equation (3.14) implies that $\hat{n}^a \mathring{H}_{ab}^\Xi|_\Lambda$ is a covector tangent to Λ . \blacksquare

The following Lemma explains the significance of the result in Equation (3.14), and in particular demonstrates that along Λ , the mean curvatures of Ξ equals that of Λ when Ξ is an asymptotically minimal hypersurface determined by the embedding sequence (3.1).

Lemma 3.10. *Let Ξ and Σ be hypersurfaces in a Riemannian d -manifold (M, g) that intersect at right angles along a submanifold Λ . Then the following relation on mean curvatures holds along Λ ,*

$$H_{\Lambda \hookrightarrow (\Sigma, g_\Sigma)} = H_{\Xi \hookrightarrow (M, g)} - \frac{1}{d-2} \mathring{H}_{\hat{n}\hat{n}}^{\Xi \hookrightarrow (M, g)},$$

where g_Σ is the induced metric on Σ , and \hat{n} is the unit conormal to Σ .

Proof. Let m_a be any extension to M of the unit conormal \hat{m}_a of Ξ . Then the mean curvature of $\Lambda \hookrightarrow (\Sigma, g_\Sigma)$ is given along by

$$\frac{1}{d-2} \nabla_a^\Sigma \left(\frac{m^a}{|m|} \Big|_\Sigma \right) = \frac{1}{d-2} \left((\nabla_a - n_a \nabla_n) \left(\frac{m^a}{|m|} \right) \right) \Big|_\Sigma,$$

where n is any extension of the unit conormal of Σ . Without loss of generality (see [30, Proposition 2.5]) we may choose $m = d\mu$ where $|m|_g = 1$ in M (i.e., μ is a unit defining function for Ξ). This further implies that \mathring{H}_{ab}^Ξ equals $\nabla_a m_b$ along Ξ . Hence the above display becomes

$$\frac{1}{d-2} \left((d-1) H_{\Xi \hookrightarrow (M, g)} - \mathring{H}_{\hat{n}\hat{n}}^{\Xi \hookrightarrow (M, g)} \right).$$

The proof is completed by using that $\mathring{H}_{ab}^{\Xi \hookrightarrow (M, g)} = \mathring{H}_{ab}^{\Xi \hookrightarrow (M, g)} + g_{ab}^\Xi H_{\Xi \hookrightarrow (M, g)}$. \square

We now turn to the problem of computing the minimal obstruction density. This may be achieved using a variant the idea behind the recursion developed in [30] for calculating the obstruction to smoothly solving the singular Yamabe problem. The key is to compute $d - 1$ derivatives normal to Σ of the canonical extension \mathcal{H}_μ^σ of the mean curvature of the singular metric defined in Equation (3.2). When μ is chosen to solve the singular Yamabe problem we may instead study $\langle \sigma, \mu \rangle$ (see Equation (3.10)). The following result underlies a recursion for that computation.

Lemma 3.11. *Let $\sigma = [g; \sigma]$ be a unit defining density, $\mu = [g; \mu]$ be a σ -minimal unit defining density, and suppose $k \in \{1, \dots, d - 1\}$. Then for a given $g \in \mathbf{c}$ and $k > 1$,*

$$(\nabla_n^\top)^k \langle \sigma, \mu \rangle + (d - k - 1)(\nabla_n^\top)^{k-1} \rho_\mu \stackrel{\Delta}{=} -(\nabla_n^\top)^{k-1} (I^{ab} \nabla_a m_b) - (k - 1)(\nabla_n^\top)^{k-2} A + \text{LOT},$$

where $m = d\mu$, $n = d\sigma$, $I_{ab} := g_{ab} - m_a m_b - n_a n_b$ smoothly extends the first fundamental form $I_{ab}^\Delta = g_{ab}^\Delta$ to M , $\nabla^\top = \nabla - m \nabla_m$, $A = \nabla_m \rho_\sigma - \rho_\mu \rho_\sigma - \rho_\mu m^a \nabla_m n_a$, and “LOT” denotes terms involving at most $k - 2$ powers of the operator ∇_n^\top acting on ρ_μ . When $k = 1$, the result is $(d - 2)\rho_\mu \stackrel{\Delta}{=} -I^{ab} \nabla_a m_b$.

Proof. The weight zero density $\langle \sigma, \mu \rangle$ is given in any scale $g \in \mathbf{c}$ by

$$m.n + \sigma \rho_\mu + \mu \rho_\sigma.$$

However, along Ξ we have $\nabla_n^\top \mu|_\Xi = 0$. Hence we only need study powers of ∇_n^\top acting on $m.n + \sigma \rho_\mu$. Moreover, by assumption, Equations (3.11) and (3.12) hold. It therefore follows that

$$\begin{aligned} \nabla_n^\top (m.n) - n^a n^b \nabla_a^\top m_b &= m^a \nabla_n^\top n_a \stackrel{\mu}{\sim} \frac{1}{2} \nabla_m n^2 + \sigma \rho_\mu m^a \nabla_m n_a + \mathcal{O}(\sigma^{d-1}) \\ &\stackrel{\mu}{\sim} -\sigma (\nabla_m \rho_\sigma - \rho_\mu \rho_\sigma - \rho_\mu m^a \nabla_m n_a) + \mathcal{O}(\sigma^{d-1}). \end{aligned} \tag{3.15}$$

The second term on the left hand side of the previous display gives

$$-n^a n^b \nabla_a^\top m_b = I^{ab} \nabla_a^\top m_b - \nabla_a^\top m^a + m^a m^b \nabla_a^\top m_b.$$

Observe that $\nabla_a^\top m^a = \nabla_a m^a - \frac{1}{2} \nabla_m m^2 \stackrel{\mu}{\sim} -(d - 1)\rho_\mu$ and $m^a m^b \nabla_a^\top m_b \stackrel{\mu}{\sim} 0$, because $m^a \nabla_a^\top = \nabla_m - m^2 \nabla_m$ and $m^2 = 1 - 2\mu \rho_\mu + \mathcal{O}(\mu^d) \stackrel{\mu}{\sim} 1$. Thus we have

$$\nabla_n^\top (m.n) \stackrel{\mu}{\sim} -I^{ab} \nabla_a^\top m_b - (d - 1)\rho_\mu - \sigma (\nabla_m \rho_\sigma - \rho_\mu \rho_\sigma - \rho_\mu m^a \nabla_m n_a) + \mathcal{O}(\sigma^{d-1}).$$

So far we have found that

$$(\nabla_n^\top)^k \langle \sigma, \mu \rangle \stackrel{\Delta}{=} (\nabla_n^\top)^{k-1} (-I^{ab} \nabla_a^\top m_b - (d - 1)\rho_\mu - \sigma A) + (\nabla_n^\top)^k (\sigma \rho_\mu). \tag{3.16}$$

Now $\nabla_n^\top \sigma = n^2 - (m.n)^2 \overset{\mu}{\sim} 1 - 2\sigma\rho_\sigma - \sigma^2\rho_\mu^2 + \mathcal{O}(\sigma^d)$. Hence along Λ , $(\nabla_n^\top)^{k-1}(\sigma A)$ can be replaced with $(k-1)(\nabla_n^\top)^{k-2}A$ modulo terms involving fewer than $k-2$ powers of ∇_n^\top acting on ρ_μ . Similarly, $(\nabla_n^\top)^k(\sigma\rho_\mu)$ can be replaced with $k(\nabla_n^\top)^{k-1}\rho_\mu$ modulo terms involving at most $k-2$ powers of ∇_n^\top acting on ρ_μ . The $k=1$ case follows directly from the above display and explanation. \square

Remark 3.12. Observe that when $k < d-1$, the first term displayed in the above lemma $(\nabla_n^\top)^k \langle \sigma, \mu \rangle \overset{\Lambda}{=} 0$, because μ is a minimal unit defining density. Hence, for these k , the lemma then determines $(\nabla_n^\top)^{k-1}\rho_\mu$ in terms of terms involving lower powers of ∇_n^\top acting on ρ_μ . When $k = d-1$, we have

$$(\nabla_n^\top)^k \langle \sigma, \mu \rangle \overset{\Lambda}{=} (d-1)! B_\Lambda,$$

so the lemma instead determines the minimal obstruction density. \blacksquare

The above recursion can be used to compute the minimal obstruction density for three-manifolds, as given in the following proposition.

Proposition 3.13. *Given the data $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$, where M is a three-manifold, the minimal obstruction density B_Λ is given by*

$$B_\Lambda = \left[g; \left(\nabla^{b\Lambda} (\hat{m}^a \overset{\circ}{\Pi}_{ac}^\Sigma I_c^b) + \frac{1}{2} (\hat{m}^a \hat{m}^b \nabla_m^\Sigma \overset{\circ}{\Pi}_{ab}^\Sigma + H_{\Lambda \hookrightarrow \Sigma} \overset{\circ}{\Pi}_{\hat{m}\hat{m}}^\Sigma) \right) \Big|_\Lambda \right].$$

Proof. Without loss of generality we may always choose μ to be a minimal unit defining density. When $k=1$, Lemma 3.11 simply says that

$$\rho_\mu \overset{\Lambda}{=} -I^{ab} \nabla_a m_b \overset{\Lambda}{=} -H_\Xi,$$

where the last equality uses Equations (3.14) and (3.13). This is a particular case of the more general result $\rho_\mu|_\Xi = -H_\Xi$ (see [23, Proposition 3.5]). Now we proceed to the case $k=2$ which determines the obstruction. We must now be careful to record the terms labeled ‘‘LOT’’ in Lemma 3.11. For that we return to Equation (3.16) which for $k=2$ gives

$$\begin{aligned} 2\mathcal{B} \overset{\Lambda}{=} (\nabla_n^\top)^2 \langle \sigma, \mu \rangle \overset{\Lambda}{=} & -\nabla_n^\top (I^{ab} \nabla_a^\top m_b) - \nabla_m \rho_\sigma + \rho_\mu \rho_\sigma + \rho_\mu m^a \nabla_m n_a + \rho_\mu (\nabla_n^\top)^2 \sigma \\ & \overset{\Lambda}{=} -\nabla_n (I^{ab} \nabla_a m_b) - \nabla_m \rho_\sigma - H_\Xi \overset{\circ}{\Pi}_{\hat{m}\hat{m}}^\Sigma - 2H_\Sigma H_\Xi, \end{aligned}$$

where in the last line we used that $\nabla_n(n^2 - (m.n)^2) \overset{\Lambda}{=} -2\rho_\sigma$ as well as Equation (3.13).

Now we note that

$$\nabla_n n_a \overset{\Sigma}{=} n_a H_\Sigma, \quad \nabla_n m_a \overset{\Xi}{=} n^b \overset{\circ}{\Pi}_{ab}^\Xi + n_a H_\Xi, \quad \nabla_m m_a \overset{\Xi}{=} m_a H_\Xi. \tag{3.17}$$

These identities follow from Equation (3.13). Applying them and remembering that $\mathring{H}_{\hat{n}\hat{n}}^\Xi|_\Lambda = 0$, we find that along Λ , the quantity $-\nabla_n(I^{ab}\nabla_a m_b)$ equals

$$-I^{ab}\nabla_n\nabla_a m_b + 2H_\Sigma n_a\nabla_n m^a \stackrel{\Lambda}{=} -I^{ab}\nabla_n\nabla_a m_b + 2H_\Sigma H_\Xi.$$

So far we have found

$$2\mathcal{B} \stackrel{\Lambda}{=} -I^{ab}\nabla_n\nabla_a m_b - \nabla_m\rho_\sigma - H_\Xi\mathring{H}_{\hat{m}\hat{m}}^\Sigma.$$

We now attack the first term on the right hand side along Λ and find

$$-I^{ab}\nabla_a\nabla_n m_b - I^{ab}n^c R_{cabd}m^d + I^{ab}(\nabla_a n^c)\nabla_c m_b.$$

Each term above is easily handled. For the first we note that, because the operator $I^{ab}\nabla_a$ is tangential along Λ , by virtue of Equation (3.17) we may replace $\nabla_n m_b$ in the first term by any extension of $\hat{n}^a\mathring{H}_{ab}^\Xi + \hat{n}_b H_\Xi$. Then remembering that $\hat{n}^a\mathring{H}_{ab}^\Xi$ gives a tangent covector to Λ , and using that for any smooth extension v_a of a covector $v_a^\Lambda \in T^*\Lambda$ it holds that $I^{ab}\nabla_a v_b \stackrel{\Lambda}{=} \nabla^b v_b^\Lambda$, we have $-I^{ab}\nabla_a\nabla_n m_b \stackrel{\Lambda}{=} -\nabla^b\mathring{H}_{ab}^\Xi(\hat{n}^a\mathring{H}_{ab}^\Xi) + (\mathring{H}_{\hat{m}\hat{m}}^\Sigma - H_\Sigma)H_\Xi$. Along Λ , the second term of the above display equals $Ric_{\hat{n}\hat{m}} = P_{\hat{n}\hat{m}}$. For the third term, using Equation (3.13), we find $g^{ac}g^{bd}\mathring{H}_{ab}^\Sigma\mathring{H}_{cd}^\Xi - H_\Sigma\mathring{H}_{\hat{n}\hat{n}}^\Xi - H_\Xi\mathring{H}_{\hat{m}\hat{m}}^\Sigma + H_\Sigma H_\Xi$. Using Equation (3.14), the identity

$$\nabla_m\rho_\sigma \stackrel{\Lambda}{=} -\nabla_m^\Sigma H_\Sigma, \tag{3.18}$$

and orchestrating these results gives

$$2\mathcal{B} \stackrel{\Lambda}{=} -\nabla^b\mathring{H}_{ab}^\Xi(\hat{n}^a\mathring{H}_{ab}^\Xi) - H_\Xi\mathring{H}_{\hat{m}\hat{m}}^\Sigma + \mathring{H}_{ab}^\Sigma\mathring{H}^{ab\Xi} + \nabla_m^\Sigma H_\Sigma + P_{\hat{n}\hat{m}}.$$

By the Codazzi–Mainardi equation specialized to $d = 3$ (see for example [30, Equation (2.9)]) the last two terms obey the identity

$$\nabla_m^\Sigma H_\Sigma + P_{\hat{n}\hat{m}} \stackrel{\Sigma}{=} \hat{m}^a\nabla^b\mathring{H}_{ab}^\Sigma. \tag{3.19}$$

We now want to rewrite the above in a form similar to the first term in $2\mathcal{B}$, this is done as follows: First we note that it equals the restriction to Λ of

$$m^a(\nabla^b - n^b\nabla_n)\mathring{H}_{ab}^{\Sigma,\text{ext}} = (\nabla^b - n^b\nabla_n)(m^a\mathring{H}_{ab}^{\Sigma,\text{ext}}) - \mathring{H}_{ab}^\Sigma\nabla^b m^a,$$

where $\mathring{H}_{ab}^{\Sigma,\text{ext}}$ is any smooth extension to M of the trace-free second fundamental form of Σ . Using Equation (3.13), the derivatives of m in the last term can be replaced by \mathring{H}^Ξ while for first term we convert the leading derivative to one tangential to Λ . Then the above display becomes

$$I_c^b \nabla^c (m^a \mathring{H}_{ab}^{\Sigma, \text{ext}}) + m^b \nabla_m (m^a \mathring{H}_{ab}^{\Sigma, \text{ext}}) - \mathring{H}_{ab}^{\Xi} \mathring{H}^{\Sigma ab} \\ \stackrel{\Lambda}{=} \nabla^b \Lambda (\hat{m}^a \mathring{H}_{ac}^{\Sigma} I_b^c) + I^{ab} (\nabla_a m_b) \mathring{H}_{\hat{m}\hat{m}}^{\Sigma} + \hat{m}^a \hat{m}^b \nabla_{\hat{m}}^{\Sigma} \mathring{H}_{ab}^{\Sigma} + H_{\Xi} \mathring{H}_{\hat{m}\hat{m}}^{\Sigma} - \mathring{H}_{ab}^{\Xi} \mathring{H}^{\Sigma ab}.$$

Noting that along Λ we have $I^{ab} \nabla_a m_b = H_{\Xi}$, collecting terms gives

$$2\mathcal{B} \stackrel{\Lambda}{=} -\nabla^b \Lambda (\hat{n}^a \mathring{H}_{ab}^{\Xi}) + \nabla^b \Lambda (\hat{n}^a \mathring{H}_{ac}^{\Sigma} I_b^c) + \hat{n}^a \hat{m}^b \nabla_{\hat{m}}^{\Sigma} \mathring{H}_{ab}^{\Sigma} + H_{\Xi} \mathring{H}_{\hat{m}\hat{m}}^{\Sigma}.$$

The proof is completed upon using Lemmas 3.8 and 3.10, and then demonstrating the following identity (which in fact holds in any dimension d) along Λ

$$\hat{n}^a \mathring{H}_{ab}^{\Xi} + \hat{n}^a \mathring{H}_{ac}^{\Sigma} I_b^c = 0. \tag{3.20}$$

This is established by computing $\nabla_a^{\Lambda} (\hat{m} \cdot \hat{n})$ and noting that this quantity must vanish since \hat{m} and \hat{n} are perpendicular along Λ . \square

Remark 3.14. The above proposition shows that the three-manifold minimal obstruction vanishes when Σ is umbilic. In particular this is the case when (M, g^o) is Poincaré–Einstein. Also, we note that the quantity $\nabla^b \Lambda (\hat{n}^a \mathring{H}_{ac}^{\Sigma} I_b^c) + \frac{1}{2} (\hat{m}^a \hat{m}^b \nabla_{\hat{m}}^{\Sigma} \mathring{H}_{ab}^{\Sigma} + H_{\Lambda \hookrightarrow \Sigma} \mathring{H}_{\hat{m}\hat{m}}^{\Sigma})$ is a weight -2 boundary invariant for a conformally embedded surface Σ , which is new to the best of our knowledge. A straightforward computation shows that under the transformation $g \mapsto \Omega^2 g$, invariance requires all three terms to be present. That this invariant is non-trivial can be verified by checking that the obstruction is non-vanishing for a sample geometry; an example is given below. \blacksquare

Example 3.15. Let $M = \mathbb{R}^3$ with conformal class of metrics \mathbf{c} containing the metric

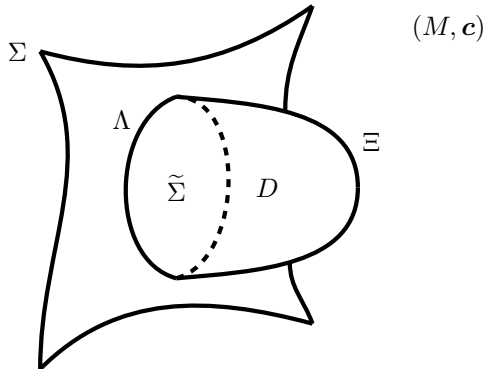
$$ds^2 = dx^2 + (1 + \alpha x)dr^2 + (1 + \beta x)r^2 d\theta^2,$$

where (x, r, θ) are cylindrical coordinates and α, β are constant parameters inserted to deform the geometry away from Euclidean 3-space. Take Σ to be the plane $x = 0$ and Λ to be the circle $x = 0 = r - 1$. Then for the asymptotically singular Yamabe metric $g^o = ds^2 / (x[1 + \frac{1}{8}(\alpha + \beta)x - \frac{1}{24}(2\alpha^2 - \alpha\beta + 2\beta^2)x^2])^2$, the asymptotically minimal surface Ξ is given by $r = 1 - \frac{1}{2}x^2$ and the obstruction $B_{\Lambda} = \frac{1}{8}(\alpha - \beta)$. The trace-free second fundamental form $\mathring{H} = \frac{1}{4}(\alpha - \beta)(dr^2 - r^2 d\theta^2)$, the mean curvature $H_{\Lambda \hookrightarrow \Sigma} = 1$, and the unit conormal to Λ in Σ is $\hat{m} = dr$. Only the last term in the result for B_{Λ} in Proposition 3.13 is non-vanishing for this simple example; it produces the quoted result for the obstruction. \blacksquare

4. Renormalized volumes and areas

Here we consider the sequence of conformal embeddings $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ where Σ is an orientable hypersurface in M and Λ is a closed orientable hypersurface in Σ . The

closure of the interior of Λ in Σ is denoted by $\tilde{\Sigma}$. We assume that $\tilde{\Sigma}$ is compact and that $\tilde{\Sigma} \cap \partial M = \emptyset$, this may involve a choice of interior for Λ . In addition we consider a second orientable hypersurface $\Xi \hookrightarrow (M, \mathbf{c})$ anchored along $\Lambda = \partial \Xi$ such that the union $\Xi \cup \tilde{\Sigma}$ is closed with interior denoted by D . For technical reasons we require some subset of D to be a collar neighborhood of $\tilde{\Sigma}$. At this point we do not require any of these hypersurfaces to be minimal. This geometry is depicted below:



We now input the additional data of a defining density $\sigma = [g; \sigma]$ for Σ which determines a metric g^σ on $M \setminus \Sigma$ that is singular along Σ . We assume σ is positive on the side of Σ occupied by D . We wish to study the “volume” of D and the “area” of Ξ with respect to this singular metric. These diverge so we must regulate them suitably.

To begin with, we introduce a unit defining density $\mu = [g; \mu]$ for Ξ where

$$D = \{p \in M \mid 0 \leq \mu(p), 0 \leq \sigma(p)\},$$

and

$$\Xi = \mathcal{Z}(\mu) \cap \{p \in M \mid \sigma(p) \geq 0\}.$$

Then we pick any true scale $\tau = [g; \tau]$ and define a *regulating hypersurface*

$$\Sigma_\varepsilon = \mathcal{Z}(\sigma/\tau - \varepsilon),$$

where $1 \gg \varepsilon \in \mathbb{R}_+$. Then the metric g^σ is non-singular in the region

$$D_\varepsilon := \{p \in M \mid 0 < \mu(p), 0 < \tau(p)\varepsilon < \sigma(p)\}.$$

The choice of τ determines the regulating hypersurface, hence we call it the *regulator*. Of key interest are quantities that are independent of the choice of regulator.

The *regulated volume* Vol_ε of D is defined as the volume of the region D_ε with respect to the singular metric g^σ . This is regulator-dependent and given by

$$\text{Vol}_\varepsilon = \int_{\tilde{D}_\varepsilon} \frac{1}{\sigma^d}. \tag{4.1}$$

It turns out to be advantageous to write this result in terms of Heaviside step functions and an integral over the manifold M :

$$\text{Vol}_\varepsilon = \int_M \frac{\theta(\boldsymbol{\mu})\theta(\boldsymbol{\sigma}/\boldsymbol{\tau} - \varepsilon)}{\sigma^d}.$$

In what follows we will assume that $\partial M = \emptyset$ (alternatively one can multiply the measure of integration by a smooth unit cut-off function that vanishes far from the region D). This leads to no loss of generality for our purposes.

The regulated area of Ξ is computed by integrating the volume element of the pull-back of g° along the hypersurface $\Xi_\varepsilon := \{p \in \Xi \mid \sigma(p) > \varepsilon\tau(p)\}$. This could be computed (inefficiently) by recycling the renormalized volume computations of [31] for the conformal manifold (Ξ, \mathbf{c}_Ξ) , where \mathbf{c}_Ξ is the conformal class of metrics induced by \mathbf{c} . Instead it is propitious to also write the regulated area in terms of distributions; indeed using Equation (2.25) and that $\boldsymbol{\mu}$ is a unit defining density, we have

$$\text{Area}_\varepsilon = \int_M \frac{\delta(\boldsymbol{\mu})\theta(\boldsymbol{\sigma}/\boldsymbol{\tau} - \varepsilon)}{\sigma^{d-1}}. \tag{4.2}$$

4.1. The ε expansion

By using the distributional identity

$$\frac{d\theta(\boldsymbol{\sigma}/\boldsymbol{\tau} - \varepsilon)}{d\varepsilon} = -\delta(\boldsymbol{\sigma}/\boldsymbol{\tau} - \varepsilon),$$

in [31,33] it was proved that

$$\text{Vol}_\varepsilon = \sum_{k=d-1}^1 \frac{v_k}{\varepsilon^k} + \mathcal{V} \log \varepsilon + \text{Vol}_{\text{ren}} + \varepsilon \mathcal{R}(\varepsilon), \tag{4.3}$$

where $\mathcal{R}(\varepsilon)$ is smooth. The ε -independent term Vol_{ren} defines the *renormalized volume*. The poles in ε (which we denote $\text{Poles}(\text{Vol}_\varepsilon)$) are called *divergences* and have coefficients given by local (along Σ), but regulator-dependent, formulæ

$$v_k = \frac{(-1)^{d-1-k}}{(d-1-k)!k} \int_M \frac{\theta(\boldsymbol{\mu}) \delta^{(d-1-k)}(\boldsymbol{\sigma})}{\tau^k}. \tag{4.4}$$

The coefficient of the $\log \varepsilon$ term is called the *anomaly*. It is again local but in addition regulator-independent and given by

$$\mathcal{V} = \frac{(-1)^d}{(d-1)!} \int_M \theta(\boldsymbol{\mu}) \delta^{(d-1)}(\boldsymbol{\sigma}). \tag{4.5}$$

Analogous results may be derived for the regulated area using the methods developed in [31,33], a sketch goes as follows: First we differentiate Equation (4.2) with respect to ε to obtain

$$-\varepsilon^{d-1} \frac{d\text{Area}_\varepsilon}{d\varepsilon} = \int_M \frac{\delta(\boldsymbol{\mu})\delta(\boldsymbol{\sigma} - \varepsilon\boldsymbol{\tau})}{\boldsymbol{\tau}^{d-2}}.$$

The right hand side is manifestly an analytic function of ε , and hence we may compute its Taylor series coefficients by taking successive ε derivatives and setting ε to zero. This gives

$$-\varepsilon^{d-1} \frac{d\text{Area}_\varepsilon}{d\varepsilon} = \int_M \frac{\delta(\boldsymbol{\mu})\delta(\boldsymbol{\sigma})}{\boldsymbol{\tau}^{d-2}} - \varepsilon \int_M \frac{\delta(\boldsymbol{\mu})\delta'(\boldsymbol{\sigma})}{\boldsymbol{\tau}^{d-3}} + \dots + \frac{(-\varepsilon)^\ell}{\ell!} \int_M \frac{\delta(\boldsymbol{\mu})\delta^{(\ell)}(\boldsymbol{\sigma})}{\boldsymbol{\tau}^{d-\ell-2}} + \dots, \tag{4.6}$$

whence we have

$$\text{Area}_\varepsilon = \sum_{k=d-2}^1 \frac{a_k}{\varepsilon^k} + \mathcal{A} \log \varepsilon + \text{Area}_{\text{ren}} + \varepsilon \mathcal{R}(\varepsilon), \tag{4.7}$$

where $\mathcal{R}(\varepsilon)$ is smooth and Area_{ren} is the constant of integration. The coefficients of the divergences are given by

$$a_k = \frac{(-1)^{d-2-k}}{(d-2-k)!k} \int_M \frac{\delta(\boldsymbol{\mu})\delta^{(d-2-k)}(\boldsymbol{\sigma})}{\boldsymbol{\tau}^k}, \tag{4.8}$$

while the anomaly is

$$\mathcal{A} = \frac{(-1)^{d-1}}{(d-2)!} \int_M \delta(\boldsymbol{\mu})\delta^{(d-2)}(\boldsymbol{\sigma}). \tag{4.9}$$

The renormalized area Area_{ren} is not determined by Equation (4.6) and thus is not forced to be local along Σ . Our next task is to develop machinery to compute holographic formulæ for the area and volume anomalies \mathcal{A} and \mathcal{V} as well as the divergence coefficients a_k, v_k .

5. Distributional calculus

In this section we develop the general calculus required for handling products of distributions involving multiple defining densities such as those appearing in (4.4), (4.5), (4.8)

and (4.9). Just as for single scale problems, main ingredients are the \mathfrak{sl}_2 algebra involving the Laplace–Robin operator described in Section 2.4, and formal self-adjointness of the Laplace–Robin operator (2.26).

5.1. Multiscale calculus

In the previous section, we showed that the divergences and anomalies we are interested in computing are written in terms of integrals of the form

$$\int_M \theta(\boldsymbol{\mu})\delta^{(k+1)}(\boldsymbol{\sigma})\mathbf{f} \quad \text{and} \quad \int_M \delta(\boldsymbol{\mu})\delta^{(k)}(\boldsymbol{\sigma})\mathbf{f}, \tag{5.1}$$

where k is a non-negative integer and \mathbf{f} is some weight $-d + k + 2$ density, or possibly log density when $k = d - 2$. Our aim is to rewrite integrals of the above form as

$$\int_M \theta(\boldsymbol{\mu})\delta(\boldsymbol{\sigma})\mathbf{F} \quad \text{and} \quad \int_M \delta(\boldsymbol{\mu})\delta(\boldsymbol{\sigma})\mathbf{G}, \tag{5.2}$$

where \mathbf{F} and \mathbf{G} have weight $1 - d$ and $2 - d$, respectively. The integrands in Equation (5.1) have support along the compact hypersurface and submanifold $\tilde{\Sigma}$ and Λ , respectively. This will allow us to discard boundary terms when integrating by parts because we uniformly assume either that $\partial M \cap \tilde{\Sigma} = \emptyset$ or that we are dealing with integrals where a cut-off function has been inserted to ensure vanishing support there. Moreover, when $\boldsymbol{\sigma}$ is a unit defining density, the first integral in Equation (5.2) becomes $\int_{\tilde{\Sigma}} \mathbf{F}$, and when in addition the density $\boldsymbol{\mu}$ is unit defining, the second integral is $\int_{\Lambda} \mathbf{G}$. Hence \mathbf{F} and \mathbf{G} are holographic formulæ in the sense discussed in the introduction.

We will mainly focus, partly because it is canonical and simplifying, on the case that $\boldsymbol{\sigma} = [g; \sigma]$ is a unit defining density; the method to handle to general $\boldsymbol{\sigma}$ is described in [31]. Our main interest is in the case where Ξ is the zero locus of a minimal unit defining density $\boldsymbol{\mu}$. In other words the singular metric g^o has asymptotically constant scalar curvature and Ξ is an asymptotically minimal surface:

$$Sc^{g^o} = 1 + \mathcal{O}(\sigma^d), \quad H_{\Xi}^{g^o} \stackrel{\Xi}{\equiv} \mathcal{O}(\sigma^{d-1}).$$

A key tool is the following (single scale) recursive identity for derivatives of delta functions of unit defining densities (see [31, Proposition 3.3])

$$(\mathbf{L}_{\boldsymbol{\sigma}})^k \delta(\boldsymbol{\sigma}) = (d - k - 1)(d - k - 2) \cdots (d - 2)\delta^{(k)}(\boldsymbol{\sigma}), \quad k \in \{1, \dots, d - 1\}. \tag{5.3}$$

Turning to the multiscale setting, our starting point is the following identity for Laplace–Robin operators acting on the Heaviside function:

Lemma 5.1. *Let μ and σ be arbitrary weight $w = 1$ densities, then*

$$\mathbf{L}_\sigma \theta(\mu) = -\sigma \mathcal{S}_\mu \delta'(\mu) + (d - 2)\langle \sigma, \mu \rangle \delta(\mu).$$

Proof. This is a simple application of the distributional identity $d\theta(x)/dx = \delta(x)$ and the Definition (2.5) of the Laplace–Robin operator. Calling $\sigma = [g; \sigma]$ and $\mu = [g; \mu]$ one has

$$\begin{aligned} \mathbf{L}_\sigma \theta(\mu) &= \mathbf{L}_\sigma [g; \theta(\mu)] = [g; (d - 2)\nabla_n \theta(\mu) - \sigma \Delta \theta(\mu)] \\ &= [g; (d - 2)m.n \delta(\mu) - \sigma(m^2 \delta'(\mu) + \delta(\mu)\Delta \mu)]. \end{aligned}$$

Here, as usual, we denoted $n = d\sigma$ and $m = d\mu$. Now, using that $\mu \delta(\mu) = 0 = \mu^2 \delta'(\mu)$ and $\mu \delta'(\mu) = -\delta(\mu)$, we have the identities

$$\begin{aligned} m.n \delta(\mu) &= (m.n - \frac{1}{d}(\sigma \Delta \mu + \mu \Delta \sigma + 2J\sigma\mu))\delta(\mu) + \frac{1}{d}\delta(\mu)\sigma \Delta \mu, \\ \sigma m^2 \delta'(\mu) &= \sigma(m^2 - \frac{2}{d}\mu(\Delta \mu + J\mu))\delta'(\mu) + \frac{2}{d}\delta(\mu)\sigma \Delta \mu. \end{aligned}$$

Using these, all terms proportional to $\delta(\mu)\sigma \Delta \mu$ cancel and, thanks to Equations (2.14) and (2.1), the coefficients of $\delta(\mu)$ and $\delta'(\mu)$ are respectively $(d - 2)\langle \sigma, \mu \rangle$ and $-\sigma \mathcal{S}_\mu$. \square

The above lemma has the following corollary:

Corollary 5.2. *Let σ be a weight $w = 1$ density, then*

$$\mathbf{L}_\sigma \theta(\sigma) = (d - 1)\mathcal{S}_\sigma \delta(\sigma).$$

Exactly the same proof technique as used for the above lemma extends to the Laplace–Robin operator acting on differentiated delta functions:

Lemma 5.3. *Let μ and σ be arbitrary weight $w = 1$ densities. Then, if $k \in \mathbb{Z}_{\geq 1}$,*

$$\mathbf{L}_\mu \delta^{(k-1)}(\sigma) = -\mu \mathcal{S}_\sigma \delta^{(k+1)}(\sigma) + (d - 2k - 2)\langle \sigma, \mu \rangle \delta^{(k)}(\sigma).$$

Remark 5.4. The central result (5.3) is a corollary to Lemma 5.3, since replacing μ by σ and using $\sigma \delta^{(k+1)}(\sigma) = -(k + 1)\delta^{(k)}(\sigma)$ gives (for $k \in \mathbb{Z}_{\geq 0}$)

$$\mathbf{L}_\sigma \delta^{(k-1)}(\sigma) = (d - k - 1)\mathcal{S}_\sigma \delta^{(k)}(\sigma). \tag{5.4}$$

In the above, we have employed the notation $\delta^{(0)}(\sigma) := \delta(\sigma)$. Also, calling $\delta^{(-1)}(\sigma) := \theta(\sigma)$, Lemma 5.1 extends Lemma 5.3 to the case $k = 0$. \blacksquare

5.2. Multiple distributions

Lemmas 5.1 and 5.3 provide the basic identities required to handle integrals of the type (5.1), which include distributions with codimension two support that are built from distributions with support along hypersurfaces. The central relations we need to derive are Equations (5.9) and (5.17) below; these underlie the key Proposition 5.18.

To begin with we observe that generically the product of a differentiated delta function multiplied by a Heaviside function lies in the span of images of Laplace–Robin operators:

Proposition 5.5. *Let σ be a unit defining density and μ a minimal unit defining density. If $\frac{d}{2} \neq k \in \{1, \dots, d - 2\}$, then*

$$\theta(\mu)\delta^{(k)}(\sigma) = \frac{1}{d - k - 1} \left[\mathbf{L}_\sigma(\theta(\mu)\delta^{(k-1)}(\sigma)) - \frac{k - 1}{d - 2k} \mathbf{L}_\mu(\delta(\mu)\delta^{(k-2)}(\sigma)) \right],$$

where the second term is absent when $k = 1$.

Proof. Since our strategy relies on the Leibniz rules of Section 2.5 we first assume $k \neq \frac{d}{2} - 1$ and use Equation (2.14) to compute (again much along the lines of the proof of Lemma 5.1)

$$\langle \theta(\mu), \delta^{(k-1)}(\sigma) \rangle = \langle \sigma, \mu \rangle \delta(\mu)\delta^{(k)}(\sigma) + \frac{k}{d-2} \mathcal{S}_\mu \delta'(\mu)\delta^{(k-1)}(\sigma) = \frac{k}{d-2} \delta'(\mu)\delta^{(k-1)}(\sigma).$$

Here we used that the unit and minimal unit defining properties imply $\langle \sigma, \mu \rangle \stackrel{\mu}{\sim} \mathcal{O}(\sigma^{d-1})$. Combining this display with Lemma 5.1, Equation (5.4) and the Leibniz rule of Section 2.5 we find

$$\mathbf{L}_\sigma(\theta(\mu)\delta^{(k-1)}(\sigma)) = (k - 1)\delta'(\mu)\delta^{(k-2)}(\sigma) + (d - k - 1)\theta(\mu)\delta^{(k)}(\sigma),$$

where the first term is absent when $k = 1$. Although the above display was computed when $k \neq \frac{d}{2} - 1$, it is not difficult to choose a scale $g \in \mathfrak{c}$ and verify that it holds for any $k \in \{1, \dots, d - 2\}$.

Next we address the other term on the right hand side of the result. Because $\delta(\mu)$ has Yamabe weight in four dimensions, we first consider $d \neq 4$ and compute (again using Equation (2.14) and taking $k \geq 2$)

$$\langle \delta(\mu), \delta^{(k-2)}(\sigma) \rangle = \langle \sigma, \mu \rangle \delta'(\mu)\delta^{(k-1)}(\sigma) + \frac{k-1}{d-4} \delta''(\mu)\delta^{(k-2)}(\sigma) + \frac{1}{d-2k} \delta(\mu)\delta^{(k)}(\sigma), \tag{5.5}$$

in order to use the generalized Leibniz rule (2.13) (away from $k = \frac{d}{2} - 1$ and $d = 4$), which then gives

$$\mathbf{L}_\mu(\delta(\mu)\delta^{(k-2)}(\sigma)) = (d - 2k)\delta'(\mu)\delta^{(k-2)}(\sigma). \tag{5.6}$$

In fact the last display holds for any integer $k \in \{2, \dots, d - 2\}$ and for $d = 4$; this can be checked by direct computation in any choice of scale $g \in \mathfrak{c}$. Combining the above displays completes the proof. \square

Corollary 5.6. *Let σ be a unit defining density and μ a minimal unit defining density. If \mathbf{f} is a weight $w = k + 1 - d \neq 1 - \frac{d}{2}$ density with $k \in \{1, \dots, d - 2\}$, then*

$$\int_M \theta(\mu)\delta^{(k)}(\sigma)\mathbf{f} = \frac{1}{d - k - 1} \left[\int_M \theta(\mu)\delta^{(k-1)}(\sigma)\mathbf{L}_\sigma\mathbf{f} + (k-1) \int_M \delta(\mu)\delta^{(k-2)}(\sigma)\widehat{\mathbf{L}}_\mu\mathbf{f} \right].$$

The second term on the right hand side is absent when $k = 1$.

Proof. The above result is a direct corollary of Proposition 5.5, the vanishing of $\partial M \cap \widetilde{\Sigma}$ (or the support of the integrands there) and the formal self-adjointness of the Laplace–Robin operator (see Equation (2.26)). \square

There is a critical value of k excluded from the above proposition and corollary, at which the density \mathbf{f} has bulk Yamabe weight $w = 1 - \frac{d}{2}$. At this weight the operator $\widehat{\mathbf{L}}_\mu\mathbf{f}$ becomes ill-defined. However, observe that in the above display, the product $\delta(\mu)\widehat{\mathbf{L}}_\mu$ appears; this suggests the replacement

$$\widehat{\mathbf{L}}_\mu \rightarrow \widetilde{\mathbf{L}}_\mu,$$

when $k = \frac{d}{2}$, where the operator $\widetilde{\mathbf{L}}$ is defined in (2.18). This is executed as follows:

Lemma 5.7. *Let σ be a unit defining density and μ a minimal unit defining density. Suppose $d \geq 4$ is even, and let \mathbf{f} be a weight $w = 1 - \frac{d}{2}$ density, then*

$$\int_M \theta(\mu)\delta^{(\frac{d}{2})}(\sigma)\mathbf{f} = \frac{2}{d - 2} \left[\int_M \theta(\mu)\delta^{(\frac{d}{2}-1)}(\sigma)\mathbf{L}_\sigma\mathbf{f} + \frac{d - 2}{2} \int_M \delta(\mu)\delta^{(\frac{d}{2}-2)}(\sigma)\widetilde{\mathbf{L}}_\mu\mathbf{f} \right].$$

Proof. We begin by using $\delta^{(\frac{d}{2})}(\sigma) = \frac{2}{d-2}\mathbf{L}_\sigma\delta^{(\frac{d}{2}-1)}$ (see Equation (5.4)) and formal self-adjointness of the Laplace–Robin operator to obtain

$$\int_M \theta(\mu)\delta^{(\frac{d}{2})}(\sigma)\mathbf{f} = \frac{2}{d - 2} \int_M \delta^{(\frac{d}{2}-1)}(\sigma)\mathbf{L}_\sigma(\theta(\mu)\mathbf{f}).$$

Because \mathbf{f} has critical weight we encounter the bulk Yamabe operator

$$\mathbf{L}_\sigma(\theta(\mu)\mathbf{f}) = -\sigma\Box(\theta(\mu)\mathbf{f}).$$

Now we pick a $g \in \mathfrak{c}$ and compute

$$\begin{aligned}
 (\Delta^g + [1 - \frac{d}{2}]J)(\theta(\mu)f) &= (|m|_g^2 \delta'(\mu) + (\nabla^g.m)\delta(\mu))f + 2\delta(\mu)\nabla_m f + \theta(\mu)\square f \\
 &= \delta'(\mu)f + 2\delta(\mu)(\nabla_m - \frac{d-2}{2}\rho_\mu)f + \theta(\mu)\square f.
 \end{aligned}$$

Here $m := d\mu$ and the second line used that μ is a unit conformal density so that $|m|_g^2 + 2\rho_\mu\mu = 1 + \mathcal{O}(\mu^d)$ with $\rho_\mu := -\frac{1}{d}(\nabla^g.m + J\mu)$. Using $\mu\delta(\mu) = 0$, it follows that

$$\mathbf{L}_\sigma(\theta(\mu)\mathbf{f}) = -\sigma\delta'(\mu)\mathbf{f} - 2\sigma\delta(\mu)\tilde{\mathbf{L}}_\mu\mathbf{f} + \theta(\mu)\mathbf{L}_\sigma\mathbf{f},$$

and hence

$$\begin{aligned}
 \int_M \theta(\mu)\delta^{(\frac{d}{2})}(\sigma)\mathbf{f} &= \frac{2}{d-2} \int_M \left[\theta(\mu)\delta^{(\frac{d}{2}-1)}(\sigma)\mathbf{L}_\sigma\mathbf{f} + (d-2)\delta(\mu)\delta^{(\frac{d}{2}-2)}(\sigma)\tilde{\mathbf{L}}_\mu\mathbf{f} \right. \\
 &\quad \left. + \frac{d-2}{2}\delta'(\mu)\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f} \right].
 \end{aligned}$$

We still need to deal with the last term above and find ourselves in the situation where Equation (5.6) cannot be used. Therefore we use the same method just employed to compute

$$\int_M \delta'(\mu)\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f} = \frac{1}{d-2} \int_M \delta(\mu)\mathbf{L}_\mu(\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f}) = - \int_M \delta(\mu)\widehat{\mathbf{L}}_\mu(\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f}).$$

The last integrand is easily calculated in some scale $g \in \mathcal{c}$ and gives

$$\delta(\mu)(\nabla_m - (d-2)\rho_\mu)(\delta^{(\frac{d}{2}-2)}(\sigma)f) = \delta(\mu)\delta^{(\frac{d}{2}-2)}(\sigma)(\nabla_m - \frac{d-2}{2}\rho_\mu)f.$$

Here we used that $m.n + \sigma\rho_\mu$ (where $n := d\sigma$) vanishes along Ξ to order $\mathcal{O}(\sigma^{d-1})$ and $\sigma\delta^{(\frac{d}{2}-1)}(\sigma) = -\frac{d-2}{2}\delta^{(\frac{d}{2}-2)}(\sigma)$. The above display corresponds to $\delta(\mu)\delta^{(\frac{d}{2}-2)}(\sigma)\tilde{\mathbf{L}}_\mu\mathbf{f}$ so that

$$\int_M \delta'(\mu)\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f} = - \int_M \delta(\mu)\delta^{(\frac{d}{2}-2)}(\sigma)\tilde{\mathbf{L}}_\mu\mathbf{f}, \tag{5.7}$$

and this completes the proof. \square

Corollary 5.6 and its critical extension, Lemma 5.7 are the two basic integrated recursive relations required to handle integrals involving the product of distributions $\theta(\mu)\delta^{(k)}(\sigma)$. Adopting a unified notation by introducing the operator family

$$\mathbf{L}'_\mu\mathbf{f} := \begin{cases} (d+w-2)\widehat{\mathbf{L}}_\mu\mathbf{f}, & w \neq 1 - \frac{d}{2}, \\ \frac{d-2}{2}\tilde{\mathbf{L}}_\mu\mathbf{f}, & w = 1 - \frac{d}{2}, \end{cases} \tag{5.8}$$

both cases can be described by a single expression

$$\int_M \theta(\boldsymbol{\mu}) \delta^{(k)}(\boldsymbol{\sigma}) \mathbf{f} = \frac{1}{d-k-1} \int_M \left[\theta(\boldsymbol{\mu}) \delta^{(k-1)}(\boldsymbol{\sigma}) \mathbf{L}_\sigma + \delta(\boldsymbol{\mu}) \delta^{(k-2)}(\boldsymbol{\sigma}) \mathbf{L}'_\mu \right] \mathbf{f}, \quad (5.9)$$

where $k \in \{1, \dots, d-2\}$ and the second term on the right hand side is absent when $k = 1$.

We will later need the following technical lemma giving the algebra obeyed by \mathbf{L}'_μ and $\boldsymbol{\sigma}$ along Ξ .

Lemma 5.8. *Let \mathbf{f} be a weight $w - 1 \neq 2 - d$ density, $\boldsymbol{\sigma}$ be a unit defining density and $\boldsymbol{\mu}$ a minimal unit defining density. Then along Ξ , the operator \mathbf{L}'_μ obeys*

$$(d + w - 3) \mathbf{L}'_\mu(\boldsymbol{\sigma} \mathbf{f}) \stackrel{\Xi}{=} (d + w - 2) \boldsymbol{\sigma} \mathbf{L}'_\mu \mathbf{f} + \mathcal{O}(\boldsymbol{\sigma}^{d-1}).$$

Moreover, when $w = 3 - d$ one has $\mathbf{L}'_\mu(\boldsymbol{\sigma} \mathbf{f}) \stackrel{\Xi}{=} \mathcal{O}(\boldsymbol{\sigma})$.

Proof. The simplest proof is based on a choice of scale $g \in \mathfrak{c}$ where $\mathbf{f} = [g; f]$. In that case, for all weights w (including $w = 1 - \frac{d}{2}$ and $w = 3 - d$) we have that $\mathbf{L}'_\mu(\boldsymbol{\sigma} \mathbf{f})$ along Ξ is given by

$$\begin{aligned} (d + w - 2)(\nabla_m + w\rho_\mu)(\boldsymbol{\sigma} f) &= (d + w - 2)(\boldsymbol{\sigma} \nabla_m f + m.n f + w\rho_\mu \boldsymbol{\sigma} f) \\ &= (d + w - 2)\boldsymbol{\sigma}(\nabla_m + (w - 1)\rho_\mu)f + \mathcal{O}(\boldsymbol{\sigma}^{d-1}). \end{aligned}$$

In the above, we used that μ vanishes along Ξ , as well as the minimal condition (3.12). It is not difficult to assemble the quoted results from this display. \square

For the particular case of the volume anomaly in three dimensions (see Section 8.1), we shall also need the action of the operator \mathbf{L}' on the log of a true scale $\boldsymbol{\tau}$. Hence in that case we extend Equation (5.8) to read

$$\mathbf{L}'_\mu \log \boldsymbol{\tau} := (d - 2) \widehat{\mathbf{L}}_\mu \log \boldsymbol{\tau} = \mathbf{L}_\mu \log \boldsymbol{\tau}. \quad (5.10)$$

We now proceed to develop the analogous machinery needed for handling densities integrated against the product distribution $\delta(\boldsymbol{\mu})\delta^{(k)}(\boldsymbol{\sigma})$. Performing the $\delta(\boldsymbol{\mu})$ integration gives integrals along the hypersurface Ξ . Therefore, to deal with the operators that appear in this case, we rely on the notion of tangentiality developed in Section 2.4, see in particular Definition 2.1.

Definition 5.9. Let $\boldsymbol{\sigma}$ be a unit defining density and $\boldsymbol{\mu}$ a minimal unit defining density. The *tangential Laplace–Robin operator* for the hypersurface $\Xi = \mathcal{Z}(\boldsymbol{\mu})$ is defined as follows:

In the case where \mathbf{f} is a weight $w \neq 1 - \frac{d}{2}, 2 - \frac{d}{2}$ density, we define the map

$$\mathbf{L}_\sigma^T : \Gamma(\mathcal{EM}[w]) \rightarrow \Gamma(\mathcal{EM}[w - 1])$$

according to

$$\mathbf{L}_\sigma^T \mathbf{f} := \left[\frac{d + 2w - 3}{d + 2w - 2} \mathbf{L}_\sigma - w(\widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle) + \frac{\sigma \mathbf{L}_\mu^2}{(d + 2w - 4)(d + 2w - 2)} \right] \mathbf{f}. \tag{5.11}$$

When the density \mathbf{f} has bulk Yamabe weight $w = 1 - \frac{d}{2}$, we define the map

$$\mathbf{L}_\sigma^T : \Gamma(\mathcal{EM}[1 - \frac{d}{2}]) \rightarrow \Gamma_\Xi(\mathcal{EM}[-\frac{d}{2}])$$

by (see Display (2.17) and Equation (2.19))

$$\mathbf{L}_\sigma^T \mathbf{f} := \left[\mathbf{L}_\sigma + \frac{d-2}{2}(\widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle) + \sigma \widehat{\mathbf{L}}_\mu \widetilde{\mathbf{L}}_\mu - \widetilde{\mathbf{L}}_\sigma \right] \mathbf{f}. \tag{5.12}$$

For \mathbf{f} of weight $w = 2 - \frac{d}{2}$ we define the map

$$\mathbf{L}_\sigma^T : \Gamma(\mathcal{EM}[2 - \frac{d}{2}]) \rightarrow \Gamma_\Xi(\mathcal{EM}[1 - \frac{d}{2}])$$

by

$$\mathbf{L}_\sigma^T \mathbf{f} := \left[\widehat{\mathbf{L}}_\sigma + \frac{d-4}{2}(\widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle) + \sigma \widetilde{\mathbf{L}}_\mu \widehat{\mathbf{L}}_\mu \right] \mathbf{f}. \tag{5.13}$$

■

Remark 5.10. Note that although the codomain of the map \mathbf{L}_σ^T depends on whether acting at a critical weight or not, we are mostly interested in computing $\mathbf{L}_\sigma^T \mathbf{f}|_\Xi$. ■

The tangential Laplace–Robin operator can also be defined acting on log densities:

Definition 5.11. Let σ and μ be unit defining and minimal unit defining densities respectively, and let τ be an arbitrary true scale. Then, if $d \neq 4$, we define the *tangential Laplace–Robin operator* acting on log densities

$$\mathbf{L}_\sigma^T : \Gamma(\mathcal{FM}[1]) \rightarrow \Gamma(\mathcal{EM}[-1])$$

by

$$\mathbf{L}_\sigma^T \log \tau := (d - 3) \widehat{\mathbf{L}}_\sigma \log \tau - (\widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle) + \sigma \widehat{\mathbf{L}}_\mu^2 \log \tau.$$

In the case when $d = 4$, the *tangential Laplace–Robin operator acting on log densities*

$$\mathbf{L}_\sigma^T : \Gamma(\mathcal{FM}[1]) \rightarrow \Gamma_\Xi(\mathcal{EM}[-1])$$

is defined as

$$\mathbf{L}_\sigma^T \log \tau := \widehat{\mathbf{L}}_\sigma \log \tau - (\widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle) + \sigma \widetilde{\mathbf{L}}_\mu \widehat{\mathbf{L}}_\mu \log \tau .$$

■

Remark 5.12. Observe that, just as in the case of Definition 5.9 for the tangential Laplace–Robin operator acting on standard densities, the codomain of the map \mathbf{L}_σ^T acting on $\log \tau$ is different in dimension four, since in this case $\widehat{\mathbf{L}}_\mu \log \tau$ has critical Yamabe weight $w = 1 - \frac{d}{2} = -1$ (see Remark 5.10). ■

The key property of the Laplace–Robin operator \mathbf{L}_σ^T is that it is tangential along Ξ in the sense of Definition 2.1 and the obvious variant for codomains $\Gamma_\Xi(\mathcal{E}M[w])$:

Lemma 5.13. *Let σ be a unit defining density and μ a minimal unit defining density. Then the operators \mathbf{L}_σ^T of Definitions 5.9 and 5.11 are tangential along Ξ .*

Proof. Tangentiality of the operator \mathbf{L}_σ^T acting at weights $w \neq 1 - \frac{d}{2}$ and $2 - \frac{d}{2}$ is proved in [32, Proposition 4.7] (see also [26]). Hence only the two critical weights remain. For the case $w = 1 - \frac{d}{2}$, consider f' to be an arbitrary, weight $w = -\frac{d}{2}$ density. Then, the non-vanishing contributions to $\mathbf{L}_\sigma^T(\mu f')$ along Ξ are given by

$$\mathbf{L}_\sigma^T(\mu f')|_\Xi = \left(\mathbf{L}_\sigma - \widetilde{\mathbf{L}}_\sigma + \sigma \widehat{\mathbf{L}}_\mu \widetilde{\mathbf{L}}_\mu \right) (\mu f')|_\Xi .$$

We need to show that the quantity in the above display in fact vanishes. For that, picking $g \in \mathfrak{c}$, a direct calculation using the various definitions of the Laplace–Robin operator and its decorated extensions introduced in Sections 2.4 and 2.5, yields

$$\begin{aligned} \mathbf{L}_\sigma(\mu f')|_\Xi &= [g; -\sigma f' \Delta \mu - 2\sigma \nabla_m f'] , \\ -\widetilde{\mathbf{L}}_\sigma(\mu f')|_\Xi &= [g; \rho_\mu \sigma f'] , \\ \sigma \widehat{\mathbf{L}}_\mu \widetilde{\mathbf{L}}_\mu(\mu f')|_\Xi &= [g; 2\sigma \nabla_m f' - (d + 1)\rho_\mu \sigma f'] , \end{aligned}$$

with $m := d\mu$, and where we have used Equation (2.2) to express $m^2 = 1 - 2\rho_\mu \mu + \mathcal{O}(\mu^d)$. Adding the three pieces above (remembering that $\Delta \mu|_\Xi = -d\rho_\mu$) we obtain $\mathbf{L}_\sigma^T(\mu f')|_\Xi = 0$, as required from Definition 2.1. A completely analogous computation, now using a density f' of weight $w = 1 - \frac{d}{2}$, proves tangentiality of \mathbf{L}_σ^T at weight $w = 2 - \frac{d}{2}$. The proof of tangentiality in the log-density case follows from the previous cases specialized to weight $w = 0$, because if $0 < f \in C^\infty(M)$ and $f|_\Sigma = 1$ we may write $f = e^{\mu\omega}$ where $\omega \in \Gamma(\mathcal{E}M[-1])$ and use $\log(f\tau) = \log \tau + \mu\omega$. This reduces the required computations to the previous cases. □

Because the operator \mathbf{L}_σ^T is tangential, we may define an analog of the Laplace–Robin operator along the hypersurface Ξ by the following holographic formula:

$$\mathcal{L}_\sigma^\Xi : \Gamma(\mathcal{E}\Xi[w]) \rightarrow \Gamma(\mathcal{E}\Xi[w - 1])$$

with

$$\mathcal{L}_\sigma^\Xi \mathbf{f}_\Xi = \mathbf{L}_\sigma^\mathbb{T} \mathbf{f}|_\Xi, \tag{5.14}$$

where $\mathbf{f}_\Xi \in \Gamma(\mathcal{E}\Xi[w])$ and $\mathbf{f} \in \Gamma(\mathcal{E}M[w])$ is any extension of \mathbf{f}_Ξ to M . The operator \mathcal{L}_σ^Ξ underlies an $\mathfrak{sl}(2)$ algebra analogous to that constructed for the Laplace–Robin operator in Section 2.4. The key relation for this is given below:

Lemma 5.14. *Let $\mathbf{f}_\Xi \in \Gamma(\mathcal{E}\Xi[w])$. Then*

$$\mathcal{L}_\sigma^\Xi(\sigma_\Xi \mathbf{f}_\Xi) - \sigma_\Xi \mathcal{L}_\sigma^\Xi \mathbf{f}_\Xi = (d + 2w - 1)\mathbf{f}_\Xi + \mathcal{O}(\sigma_\Xi^{d-1}),$$

where σ_Ξ is the restriction of σ to Ξ .

Proof. There many ways to prove this result, an expedient method is to first establish the result along the interior of Ξ because this allows us to calculate using the scale determined by the singular metric g° for which (away from Σ) $\sigma = [g^\circ; 1]$. In particular, in this scale, when $w \neq 1 - \frac{d}{2}, 2 - \frac{d}{2}$, using Equation (5.11) we find

$$\begin{aligned} \mathbf{L}_\sigma^\mathbb{T} \mathbf{f} \stackrel{\Xi}{=} & (d + 2w - 3) \left(-\frac{w}{d} J^{g^\circ} \right) f - \frac{d+2w-3}{d+2w-2} (\Delta + wJ^{g^\circ}) f - wf \nabla_m (\rho_\mu - \frac{\mu}{d} J^{g^\circ}) \\ & + (\nabla_m + (w - 1)\rho_\mu) \left(\nabla_m + w\rho_\mu - \frac{\mu}{d+2w-2} (\Delta + wJ^{g^\circ}) \right) f \\ = & -(\Delta - \nabla_m^2 + (2w - 1)H_\Xi^{g^\circ} \nabla_m) f + \left(w(w - 1)(H_\Xi^{g^\circ})^2 - \frac{2w(d+w-2)}{d} J^{g^\circ} \right) f. \end{aligned}$$

Here we used that in this scale $n = \nabla \sigma = \nabla 1 = 0$ and $\rho_\sigma = \rho_1 = -\frac{1}{d} J^{g^\circ}$. We also used that $\rho_\mu|_\Xi = -H_\Xi^{g^\circ}$. The second equality of the above display shows that the singularities at weights $1 - \frac{d}{2}, 2 - \frac{d}{2}$ of $\mathbf{L}_\sigma^\mathbb{T}$ as defined by Equation (5.11) are removable. Indeed, it is straightforward to use Equations (5.12) and (5.13) to establish that $\mathbf{L}_\sigma^\mathbb{T} \mathbf{f}$ along Ξ is as stated above for these critical weights. In the g° scale, the above display also gives the result for $\sigma \mathbf{L}_\sigma^\mathbb{T} \mathbf{f}$ along Ξ . Moreover, to compute $\mathbf{L}_\sigma^\mathbb{T}(\sigma \mathbf{f})$ we only need write out the above display replacing w by $w + 1$ because $\sigma \mathbf{f} \in \Gamma(\mathcal{E}M[w + 1])$. Hence for the difference of these we find

$$\sigma \mathbf{L}_\sigma^\mathbb{T} \mathbf{f} - \mathbf{L}_\sigma^\mathbb{T}(\sigma \mathbf{f}) \stackrel{\Xi}{=} \frac{2J^{g^\circ}}{d} (d + 2w - 1)f + 2H_\Xi^{g^\circ} (\nabla_{\hat{m}} - wH_\Xi^{g^\circ}) f.$$

In the interior $-\frac{2}{d} J^{g^\circ} = \mathcal{S}_\sigma$ and $-H_\Xi^{g^\circ} = \langle \sigma, \mu \rangle$. The former of these equals $1 + \mathcal{O}(\sigma^d)$ while along Ξ , the latter is $\mathcal{O}(\sigma_\Xi^{d-1})$. This establishes the quoted result on the interior of Ξ . Since both the left and right hand sides of this result are smoothly defined along Ξ , the above interior computation suffices to establish their equality along all of Ξ . \square

The tangential Laplace–Robin operator $\mathbf{L}_\sigma^\mathbb{T}$ introduced in Definition 5.9 is built from formally self adjoint Laplace–Robin operators; see Equation (2.26). When integrating

over manifolds without boundary (or integrands with no support there), we encounter the formal adjoint $(\mathbf{L}_\sigma^T)^\dagger$ of the tangential operator \mathbf{L}_σ^T . This is easily computed using formal self-adjointness of the \mathbf{L} -operator:

$$(\mathbf{L}_\sigma^T)^\dagger \mathbf{f} = \left[\frac{d + 2w + 1}{d + 2w} \mathbf{L}_\sigma + (d + w - 1) \langle \widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle \rangle + \frac{\mathbf{L}_\mu^2 \circ \sigma}{(d + 2w + 2)(d + 2w)} \right] \mathbf{f}, \tag{5.15}$$

when \mathbf{f} is a density of weight $w \neq -1 - \frac{d}{2}, -\frac{d}{2}$.

Remarkably, the product $\delta(\mu)\delta^{(k)}(\sigma)$ can be expressed in terms of the formal adjoint given in (5.15) acting on a product of delta functions with one fewer derivatives. This result, given in the lemma below, may also be viewed as an generalization of the key recursive relation (5.4) to products of distributions.

Proposition 5.15. *Let σ be a unit defining density and μ a minimal unit defining density. If $\frac{d}{2}, \frac{d}{2} - 1 \neq k \in \{1, \dots, d - 3\}$, then*

$$\delta(\mu)\delta^{(k)}(\sigma) = \frac{1}{d - k - 2} (\mathbf{L}_\sigma^T)^\dagger (\delta(\mu)\delta^{(k-1)}(\sigma)).$$

Proof. We proceed along similar lines to the proof of Proposition 5.5. First we use the generalized Leibniz rule (2.13), Lemma 5.3 and Equation (5.5), avoiding special dimensions and values of k , to compute

$$\begin{aligned} \mathbf{L}_\sigma (\delta(\mu)\delta^{(k-1)}(\sigma)) &= (d - k - 3)\delta(\mu)\delta^{(k)}(\sigma) \\ &\quad + (d - 4)\langle \sigma, \mu \rangle \delta'(\mu)\delta^{(k-1)}(\sigma) + (k - 1)\delta''(\mu)\delta^{(k-2)}(\sigma). \end{aligned} \tag{5.16}$$

The above display in fact holds in all dimensions and for $k \in \{2, \dots, d - 3\}$. When $k = 1$, the last term is absent. Equation (5.6) gives $\mathbf{L}_\mu (\delta(\mu)\delta^{(k-1)}(\sigma)) = (d - 2k - 2)\delta'(\mu)\delta^{(k-1)}(\sigma)$, so it remains to employ our Leibniz identities to calculate (when $k \neq 1$)

$$\begin{aligned} \mathbf{L}_\mu^2 (\delta(\mu)\delta^{(k-2)}(\sigma)) &= (d - 2k)\mathbf{L}_\mu (\delta'(\mu)\delta^{(k-2)}(\sigma)) \\ &= (d - 2k) [\delta(\mu)\delta^{(k)}(\sigma) + (d - 2k)\langle \sigma, \mu \rangle \delta'(\mu)\delta^{(k-1)}(\sigma) \\ &\quad + (d - 2k - 1)\delta''(\mu)\delta^{(k-2)}(\sigma)], \end{aligned}$$

where again the last term is absent when $k = 1$. In the above we used the identity

$$\mu \langle \delta'(\mu), \delta^{(k-2)}(\sigma) \rangle = -2\langle \sigma, \mu \rangle \delta'(\mu)\delta^{(k-1)}(\sigma) - \frac{2}{d-2k} \delta(\mu)\delta^{(k)}(\sigma) - \frac{3(k-1)}{d-6} \delta''(\mu)\delta^{(k-2)}(\sigma),$$

which can be derived from Equation (2.14). However once again, the display before last in fact holds for $k \in \{2, \dots, d - 3\}$.

Simple algebra now gives

$$\delta(\boldsymbol{\mu})\delta^{(k)}(\boldsymbol{\sigma}) = \frac{1}{d-2k-2} \left(\left[\frac{d-2k-1}{d-k-2} \mathbf{L}\boldsymbol{\sigma} - \langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle \mathbf{L}\boldsymbol{\mu} \right] (\delta(\boldsymbol{\mu})\delta^{(k-1)}(\boldsymbol{\sigma})) - \frac{k-1}{(d-k-2)(d-2k)} \mathbf{L}\boldsymbol{\mu}^2 (\delta(\boldsymbol{\mu})\delta^{(k-2)}(\boldsymbol{\sigma})) \right).$$

Here, using that $\mathbf{L}\boldsymbol{\mu}(\langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle \delta(\boldsymbol{\mu})\delta^{(k-1)}(\boldsymbol{\sigma})) = 0$ together with the Leibniz rule (2.13) and the identity $\boldsymbol{\mu}\langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle, \delta(\boldsymbol{\mu})\delta^{(k-1)}(\boldsymbol{\sigma}) = -(\widehat{\mathbf{L}}\boldsymbol{\mu}\langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle) \delta(\boldsymbol{\mu})\delta^{(k-1)}(\boldsymbol{\sigma})$, we can then rewrite the second term in the above expression as

$$\langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle \mathbf{L}\boldsymbol{\mu} (\delta(\boldsymbol{\mu})\delta^{(k-1)}(\boldsymbol{\sigma})) = -(d-2k-2)(\widehat{\mathbf{L}}\boldsymbol{\mu}\langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle) \delta(\boldsymbol{\mu})\delta^{(k-1)}(\boldsymbol{\sigma}),$$

while (for $k \neq 1$) the third term can be combined with the first two by means of $\boldsymbol{\sigma}\delta^{(k-1)}(\boldsymbol{\sigma}) = -(k-1)\delta^{(k-2)}(\boldsymbol{\sigma})$, producing

$$\delta(\boldsymbol{\mu})\delta^{(k)}(\boldsymbol{\sigma}) = \frac{1}{d-k-2} \left[\frac{d-2k-1}{d-2k-2} \mathbf{L}\boldsymbol{\sigma} + (d-k-2)(\widehat{\mathbf{L}}\boldsymbol{\mu}\langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle) + \frac{\mathbf{L}\boldsymbol{\mu}^2 \circ \boldsymbol{\sigma}}{(d-2k)(d-2k-2)} \right] (\delta(\boldsymbol{\mu})\delta^{(k-1)}(\boldsymbol{\sigma})).$$

Expressing the factors within the square bracket in terms of the weight $w = -k - 1$, one identifies the formal adjoint operator (5.15), and thus completes the proof. \square

Remark 5.16. Proposition 5.15 may also be read as saying

$$\frac{1}{d-k-2} (\mathbf{L}\boldsymbol{\sigma}^\top)^\dagger (\delta(\boldsymbol{\mu})\delta^{(k-1)}(\boldsymbol{\sigma})) = \frac{1}{d-k-1} \delta(\boldsymbol{\mu}) \mathbf{L}\boldsymbol{\sigma} \delta^{(k-1)}(\boldsymbol{\sigma}).$$

Notice that the right hand side integrated against a density-valued test function \mathbf{f} is independent of the extension of \mathbf{f} off of the zero locus of $\boldsymbol{\mu}$. This explains why the formal adjoint of a tangential operator must appear on the left hand side. \blacksquare

We now develop the integrated analog of Proposition 5.15 and extend this to include critical weights.

Proposition 5.17. *Let $\boldsymbol{\sigma}$ be a unit defining density and $\boldsymbol{\mu}$ a minimal unit defining density. Then if \mathbf{f} is a weight $w = k + 2 - d$ density with $k \in \{1, \dots, d - 3\}$ and $d \geq 4$,*

$$\int_M \delta(\boldsymbol{\mu})\delta^{(k)}(\boldsymbol{\sigma}) \mathbf{f} = \frac{1}{d-k-2} \int_M \delta(\boldsymbol{\mu})\delta^{(k-1)}(\boldsymbol{\sigma}) \mathbf{L}\boldsymbol{\sigma}^\top \mathbf{f}. \tag{5.17}$$

Proof. The proof for non-critical weights $w \neq 1 - \frac{d}{2}, 2 - \frac{d}{2}$ follows directly from Proposition 5.15. For critical weights, first observe that, by virtue of Equation (5.4) and formal self-adjointness of \mathbf{L}_σ , we have

$$\int_M \delta(\boldsymbol{\mu})\delta^{(k)}(\boldsymbol{\sigma})\mathbf{f} = \frac{1}{d-k-1} \int_M \delta^{(k-1)}(\boldsymbol{\sigma})\mathbf{L}_\sigma(\delta(\boldsymbol{\mu})\mathbf{f}). \tag{5.18}$$

We now break the computation of $\mathbf{L}_\sigma(\delta(\boldsymbol{\mu})\mathbf{f})$ into the two cases of interest.

Case (i) $w = 1 - \frac{d}{2}$ and thus $k = \frac{d}{2} - 1$: We begin by computing $\mathbf{L}_\sigma(\delta(\boldsymbol{\mu})\mathbf{f})$ using the method of proof of Lemma 5.7. We pick a scale $g \in \mathbf{c}$ and find

$$\begin{aligned} & -2f(\nabla_n - \rho_\sigma)\delta(\boldsymbol{\mu}) - 2\delta(\boldsymbol{\mu})(\nabla_n + [1 - \frac{d}{2}]\rho_\sigma)f - \sigma(\Delta^g - \frac{d}{2}J)(\delta(\boldsymbol{\mu})f) \\ & = -2f(m.n + \mu\rho_\sigma + \sigma\rho_\mu)\delta'(\boldsymbol{\mu}) - 2\delta(\boldsymbol{\mu})(\nabla_n + [1 - \frac{d}{2}]\rho_\sigma)f - \sigma(m^2 + 2\mu\rho_\mu)\delta''(\boldsymbol{\mu})f \\ & \quad - 2\sigma\delta'(\boldsymbol{\mu})(\nabla_m + [1 - \frac{d}{2}]\rho_\mu)f - \sigma\delta(\boldsymbol{\mu})(\Delta^g + [1 - \frac{d}{2}]J)f. \end{aligned} \tag{5.19}$$

Thus, using that $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are unit and minimal unit defining densities respectively, we have

$$\mathbf{L}_\sigma(\delta(\boldsymbol{\mu})\mathbf{f}) = \delta(\boldsymbol{\mu})(\mathbf{L}_\sigma - 2\tilde{\mathbf{L}}_\sigma)\mathbf{f} - 2\delta'(\boldsymbol{\mu})(\boldsymbol{\sigma}\tilde{\mathbf{L}}_\mu + \langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle)\mathbf{f} - \sigma\delta''(\boldsymbol{\mu})\mathbf{f}.$$

Using the identities $\boldsymbol{\sigma}\delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma}) = -\frac{1}{2}(d-4)\delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma})$ and $\mathbf{L}_\sigma\mathbf{f} = -\sigma\Box\mathbf{f}$, we so far have

$$\begin{aligned} \int_M \delta(\boldsymbol{\mu})\delta^{(\frac{d}{2}-1)}(\boldsymbol{\sigma})\mathbf{f} &= \frac{2}{d} \int_M \left(-2\delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma})[\delta(\boldsymbol{\mu})\tilde{\mathbf{L}}_\sigma + \delta'(\boldsymbol{\mu})\langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle]\mathbf{f} \right. \\ & \quad \left. + \frac{1}{2}(d-4)\delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma})[\delta(\boldsymbol{\mu})\Box + 2\delta'(\boldsymbol{\mu})\tilde{\mathbf{L}}_\mu + \delta''(\boldsymbol{\mu})]\mathbf{f} \right). \end{aligned}$$

We can handle the terms involving $\delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma})\delta'(\boldsymbol{\mu})$ and $\delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma})\delta'(\boldsymbol{\mu})$ in the above expression using Equations (5.7) and (5.6), respectively; remembering that $\langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle|_\Xi = \mathcal{O}(\boldsymbol{\sigma}^{d-1})$ and that \mathbf{L}_μ is formally self-adjoint. We obtain

$$\begin{aligned} \int_M \delta(\boldsymbol{\mu})\delta^{(\frac{d}{2}-1)}(\boldsymbol{\sigma})\mathbf{f} &= \frac{2}{d} \int_M \left(-2\delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma})\delta(\boldsymbol{\mu})\left[\tilde{\mathbf{L}}_\sigma - (\widehat{\mathbf{L}}_\mu\langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle)\right]\mathbf{f} \right. \\ & \quad \left. + \frac{1}{2}(d-4)\delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma})\left[\delta(\boldsymbol{\mu})(\Box - 2\widehat{\mathbf{L}}_\mu\tilde{\mathbf{L}}_\mu) + \delta''(\boldsymbol{\mu})\right]\mathbf{f} \right). \end{aligned}$$

To handle the last term above, we recall that $\mathbf{L}_\mu\delta'(\boldsymbol{\mu}) = (d-3)\delta''(\boldsymbol{\mu})$ and so perform a computation similar to that of Display (5.19) to obtain

$$\begin{aligned} \mathbf{L}_\mu(\delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma})\mathbf{f}) &= \delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma})(\mathbf{L}_\mu - (d-4)\tilde{\mathbf{L}}_\mu)\mathbf{f} \\ &\quad - \delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma})(2\mu\tilde{\mathbf{L}}_\sigma + (d-4)\langle\boldsymbol{\sigma}, \mu\rangle)\mathbf{f} - \mu\delta^{(\frac{d}{2}-1)}(\boldsymbol{\sigma})\mathbf{f}. \end{aligned}$$

Using Equations (5.4), (5.7) and (5.6) again, as well as $\mu\delta'(\mu) = -\delta(\mu)$ and $\delta(\mu)\widehat{\mathbf{L}}_\mu\mathbf{L}_\mu\mathbf{f} = -\delta(\mu)\square\mathbf{f}$, gives

$$\begin{aligned} \int_M \delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma})\delta''(\mu)\mathbf{f} &= \frac{1}{d-3} \int_M \delta(\mu)\left(\delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma})(\square + (d-4)\widehat{\mathbf{L}}_\mu\tilde{\mathbf{L}}_\mu)\mathbf{f}\right. \\ &\quad \left.+ \delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma})(2\tilde{\mathbf{L}}_\sigma\mathbf{f} + (d-4)\mathbf{f}\widehat{\mathbf{L}}_\mu\langle\boldsymbol{\sigma}, \mu\rangle) + \delta^{(\frac{d}{2}-1)}(\boldsymbol{\sigma})\mathbf{f}\right). \end{aligned}$$

Collating the above calculations yields

$$\begin{aligned} \int_M \delta(\mu)\delta^{(\frac{d}{2}-1)}(\boldsymbol{\sigma})\mathbf{f} &= \frac{1}{d-2} \int_M \delta(\mu)\left(\delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma})\left[-2\tilde{\mathbf{L}}_\sigma + (d-2)(\widehat{\mathbf{L}}_\mu\langle\boldsymbol{\sigma}, \mu\rangle)\right]\right. \\ &\quad \left.- (d-4)\delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma})\left[\widehat{\mathbf{L}}_\mu\tilde{\mathbf{L}}_\mu - \square\right]\right)\mathbf{f}. \end{aligned}$$

Using the identity $\delta^{(\frac{d}{2}-3)}(\boldsymbol{\sigma}) = -\frac{2}{d-4}\boldsymbol{\sigma}\delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma})$ and the fact that acting on a critical density $\mathbf{L}_\sigma\mathbf{f} = -\boldsymbol{\sigma}\square\mathbf{f}$, we can rewrite the previous display as

$$\int_M \delta(\mu)\delta^{(\frac{d}{2}-1)}(\boldsymbol{\sigma})\mathbf{f} = \frac{2}{d-2} \int_M \delta(\mu)\delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma})\left[\mathbf{L}_\sigma + \frac{d-2}{2}(\widehat{\mathbf{L}}_\mu\langle\boldsymbol{\sigma}, \mu\rangle) + (\boldsymbol{\sigma}\widehat{\mathbf{L}}_\mu\tilde{\mathbf{L}}_\mu - \tilde{\mathbf{L}}_\sigma)\right]\mathbf{f}.$$

Then the quoted result follows by virtue of Equation (5.12) in Definition 5.9.

Case (ii) $w = 2 - \frac{d}{2}$ and $k = \frac{d}{2}$: Note that this case only pertains to $d > 4$. We begin by first noting that

$$\mathbf{L}_\sigma(\delta(\mu)\mathbf{f}) = -\boldsymbol{\sigma}\square(\delta(\mu)\mathbf{f}).$$

Now the proof again proceeds along the lines of Lemma 5.7. Choosing a scale $g \in \mathbf{c}$ we compute $\square(\delta(\mu)\mathbf{f})$ and find (using $\delta(\mu) = -\mu\delta'(\mu)$)

$$\begin{aligned} (\Delta + [1 - \frac{d}{2}]J)(\delta(\mu)\mathbf{f}) &= (m^2\delta''(\mu) + \nabla \cdot m\delta'(\mu))\mathbf{f} + 2\delta'(\mu)\nabla_m\mathbf{f} + \delta(\mu)(\Delta + [1 - \frac{d}{2}]J)\mathbf{f}. \\ &= \delta''(\mu)(m^2 + 2\rho_\mu\mu)\mathbf{f} + \delta'(\mu)[2(\nabla_m + [2 - \frac{d}{2}]\rho_\mu)\mathbf{f} - \mu(\Delta + [2 - \frac{d}{2}]J)\mathbf{f}]. \end{aligned}$$

Thus, using the unit conformal property of μ and Equation (5.4), it follows that

$$\mathbf{L}_\sigma(\delta(\mu)\mathbf{f}) = -\boldsymbol{\sigma}\delta''(\mu)\mathbf{f} - 2\boldsymbol{\sigma}\delta'(\mu)\widehat{\mathbf{L}}_\mu\mathbf{f} = -\frac{\boldsymbol{\sigma}}{d-2}\left(\frac{1}{d-3}\mathbf{f}\mathbf{L}_\mu + 2(\widehat{\mathbf{L}}_\mu\mathbf{f})\right)\mathbf{L}_\mu\delta(\mu).$$

In turn via formal self-adjointness of \mathbf{L}_μ and the identity $\sigma\delta^{(\frac{d}{2}-1)}(\sigma) = -\frac{1}{2}(d-2)\delta^{(\frac{d}{2}-2)}(\sigma)$, it follows from Equation (5.18) that

$$\int_M \delta(\mu)\delta^{(\frac{d}{2})}(\sigma)\mathbf{f} = \frac{1}{d-2} \int_M \delta(\mu) \left(\frac{1}{d-3} \mathbf{L}_\mu^2(\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f}) + 2\mathbf{L}_\mu(\delta^{(\frac{d}{2}-2)}(\sigma)\widehat{\mathbf{L}}_\mu\mathbf{f}) \right). \tag{5.20}$$

Care is now needed because the delta distribution $\delta^{(\frac{d}{2}-2)}(\sigma)$ has Yamabe weight $1 - \frac{d}{2}$. A simple computation in a choice of scale shows that because the defining density μ is minimal unit, we have

$$\delta(\mu)\delta_{\mathbb{R}}^{\equiv}\delta^{(k)}(\sigma) = 0, \tag{5.21}$$

for any positive integer $k \leq d - 3$. Here $\delta_{\mathbb{R}}^{\equiv}$ is the Robin operator defined in Equation (2.7). In particular this implies that $\delta(\mu)\widetilde{\mathbf{L}}_\mu\delta^{(\frac{d}{2}-2)}(\sigma) = 0$. Then using this fact in Equation (2.22) applied to the last term in Equation (5.20) we have

$$\begin{aligned} \widehat{\mathbf{L}}_\mu(\delta^{(\frac{d}{2}-2)}(\sigma)\widehat{\mathbf{L}}_\mu\mathbf{f}) &= \left(\widetilde{\mathbf{L}}_\mu\delta^{(\frac{d}{2}-2)}(\sigma) - \frac{1}{2}\mu\Box\delta^{(\frac{d}{2}-2)}(\sigma) \right) \widehat{\mathbf{L}}_\mu\mathbf{f} \\ &\quad + \delta^{(\frac{d}{2}-2)}(\sigma) \left(\widetilde{\mathbf{L}}_\mu\widehat{\mathbf{L}}_\mu\mathbf{f} - \frac{1}{2}\mu\Box\widehat{\mathbf{L}}_\mu\mathbf{f} \right) + \mu \langle \delta^{(\frac{d}{2}-2)}(\sigma), \widehat{\mathbf{L}}_\mu\mathbf{f} \rangle, \end{aligned}$$

so

$$\delta(\mu)\mathbf{L}_\mu(\delta^{(\frac{d}{2}-2)}(\sigma)\widehat{\mathbf{L}}_\mu\mathbf{f}) = -(d-2)\delta(\mu)\delta^{(\frac{d}{2}-2)}(\sigma)\widetilde{\mathbf{L}}_\mu\widehat{\mathbf{L}}_\mu\mathbf{f}.$$

This handles the last term in Equation (5.20) so it now follows that

$$\int_M \delta(\mu)\delta^{(\frac{d}{2})}(\sigma)\mathbf{f} = \frac{1}{d-2} \int_M \delta(\mu) \left(\frac{1}{d-3} \mathbf{L}_\mu^2(\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f}) - 2(d-2)\delta^{(\frac{d}{2}-2)}(\sigma)\widetilde{\mathbf{L}}_\mu\widehat{\mathbf{L}}_\mu\mathbf{f} \right).$$

To treat the first term on the right hand side of the above display, we first note that $\mathbf{L}_\mu^2(\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f}) = (d-2)(d-4)\widehat{\mathbf{L}}_\mu^2(\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f})$, so

$$\delta(\mu)\mathbf{L}_\mu^2(\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f}) = (d-2)(d-4)\delta(\mu)\delta_{\mathbb{R}}^{\equiv}\widehat{\mathbf{L}}_\mu(\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f}). \tag{5.22}$$

Note that because the Robin operator $\delta_{\mathbb{R}}^{\equiv}$ is first order, it obeys the standard Leibniz rule. Then we use Equation (2.21) to compute

$$\begin{aligned} \widehat{\mathbf{L}}_\mu(\delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f}) &= \left(\widetilde{\mathbf{L}}_\mu - \frac{1}{d-4}\mathbf{L}_\mu \right) \delta^{(\frac{d}{2}-2)}(\sigma)\mathbf{f} + \delta^{(\frac{d}{2}-2)}(\sigma)\widehat{\mathbf{L}}_\mu\mathbf{f} + \frac{2\mu}{d-4} \langle \delta^{(\frac{d}{2}-2)}(\sigma), \mathbf{f} \rangle \\ &= \frac{1}{d-4} \mu\delta^{(\frac{d}{2})}(\sigma)\mathbf{f} + \delta^{(\frac{d}{2}-2)}(\sigma)\widehat{\mathbf{L}}_\mu\mathbf{f} + \delta^{(\frac{d}{2}-1)}(\sigma) \left[\frac{2\mu}{d-4} \widehat{\mathbf{L}}_\sigma\mathbf{f} + \langle \sigma, \mu \rangle \mathbf{f} \right]. \end{aligned}$$

The terms in square brackets in the last line above come from the first and last terms on the right hand side of the previous line. This requires a computation in a choice of scale of the type performed above, relying on Equation (2.20). The remaining terms on the last line above used $\mathbf{L}_\mu \delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma}) = -\boldsymbol{\mu} \square \delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma})$, and that $\boldsymbol{\sigma}$ is unit defining, so

$$\square \delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma}) = \delta^{(\frac{d}{2})}(\boldsymbol{\sigma}).$$

Now using Equations (5.22) and (5.21), and $\delta(\boldsymbol{\mu}) \delta_{\mathbb{R}}^{\Xi} \boldsymbol{\mu} = \delta(\boldsymbol{\mu})$ it follows that

$$\begin{aligned} \delta(\boldsymbol{\mu}) \mathbf{L}_\mu^2 (\delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma}) \mathbf{f}) &= (d-2) \delta(\boldsymbol{\mu}) \delta^{(\frac{d}{2})}(\boldsymbol{\sigma}) \mathbf{f} + (d-2)(d-4) \delta(\boldsymbol{\mu}) \delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma}) \tilde{\mathbf{L}}_\mu \widehat{\mathbf{L}}_\mu \mathbf{f} \\ &\quad + (d-2) \delta(\boldsymbol{\mu}) \delta^{(\frac{d}{2}-1)}(\boldsymbol{\sigma}) \left[2 \widehat{\mathbf{L}}_\sigma \mathbf{f} + (d-4) \mathbf{f} \widehat{\mathbf{L}}_\mu \langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle \right]. \end{aligned}$$

Orchestrating the above gives

$$\begin{aligned} \int_M \delta(\boldsymbol{\mu}) \delta^{(\frac{d}{2})}(\boldsymbol{\sigma}) \mathbf{f} &= \int_M \delta(\boldsymbol{\mu}) \left(-\frac{d-2}{d-4} \delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma}) \tilde{\mathbf{L}}_\mu \widehat{\mathbf{L}}_\mu \mathbf{f} \right. \\ &\quad \left. + \frac{1}{d-4} \delta^{(\frac{d}{2}-1)}(\boldsymbol{\sigma}) \left[2 \widehat{\mathbf{L}}_\sigma \mathbf{f} + (d-4) \mathbf{f} \widehat{\mathbf{L}}_\mu \langle \boldsymbol{\sigma}, \boldsymbol{\mu} \rangle \right] \right). \end{aligned}$$

Using the identity $\delta^{(\frac{d}{2}-2)}(\boldsymbol{\sigma}) = -\frac{2}{d-2} \boldsymbol{\sigma} \delta^{(\frac{d}{2}-1)}(\boldsymbol{\sigma})$ one recognizes (from Equation (5.13)) the action of the tangential operator on \mathbf{f} . The final result then follows. \square

We have now established the two main recursive relations needed to compute integrals of the type (5.1) and in turn the regulated volume and area expansions in Equations (4.3) and (4.7). The proposition below gives our result for such integrals. Its proof follows simply by iterating the recursions given in Equations (5.9) and (5.17).

Proposition 5.18. *Let $\boldsymbol{\sigma}$ be a unit defining density and $\boldsymbol{\mu}$ a minimal unit defining density. Then, if \mathbf{f} is a weight $w \neq 0$ density and $1 \leq k \leq d-3, d-2$, respectively, the following relations hold*

$$\begin{aligned} \int_M \delta(\boldsymbol{\mu}) \delta^{(k)}(\boldsymbol{\sigma}) \mathbf{f} &= \frac{(d-k-3)!}{(d-3)!} \int_M \delta(\boldsymbol{\mu}) \delta(\boldsymbol{\sigma}) (\mathbf{L}_\sigma^T)^k \mathbf{f}, \\ \int_M \theta(\boldsymbol{\mu}) \delta^{(k)}(\boldsymbol{\sigma}) \mathbf{f} &= \frac{(d-k-2)!}{(d-2)!} \left[\int_M \theta(\boldsymbol{\mu}) \delta(\boldsymbol{\sigma}) \mathbf{L}_\sigma^k \mathbf{f} \right. \\ &\quad \left. + (d-2) \sum_{j=0}^{k-2} \int_M \delta(\boldsymbol{\mu}) \delta(\boldsymbol{\sigma}) (\mathbf{L}_\sigma^T)^j \mathbf{L}'_\mu \mathbf{L}_\sigma^{k-2-j} \mathbf{f} \right], \end{aligned}$$

where the second term on the right hand side of the last displayed equality is absent when $k = 1$.

6. Holographic formulæ

6.1. Divergences

We now deduce results in arbitrary dimensions for the volume and area divergences of the regulated volume Vol_ε and area Area_ε defined in Section 4. Recall that in distributional form their coefficients are given (from Equations (4.4) and (4.8)) by:

$$v_k = \frac{(-1)^{d-1-k}}{(d-1-k)!k} \int_M \frac{\theta(\boldsymbol{\mu}) \delta^{(d-1-k)}(\boldsymbol{\sigma})}{\boldsymbol{\tau}^k}, \quad a_k = \frac{(-1)^{d-2-k}}{(d-2-k)!k} \int_M \frac{\delta(\boldsymbol{\mu}) \delta^{(d-2-k)}(\boldsymbol{\sigma})}{\boldsymbol{\tau}^k}.$$

The following pair of theorems express the above expressions as integrals over $\tilde{\Sigma}$ and Λ , with local holographic formulæ for integrands. These follow immediately from Proposition 5.18 by setting $\mathbf{f} = \boldsymbol{\tau}^{-k}$, where the regulator $\boldsymbol{\tau}$ is an arbitrary true scale.

Theorem 6.1. *The divergences in the regulated volume Vol_ε determined by the sequence of embeddings $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ and the regulating hypersurface Σ_ε are local integrals given by*

$$\text{Poles}(\text{Vol}_\varepsilon) = \sum_{k=d-1}^1 \frac{(-1)^{d-k-1}}{(d-k-1)!k\varepsilon^k} \frac{(k-1)!}{(d-2)!} \left[\int_{\tilde{\Sigma}} \mathbf{v}_k + (d-2) \int_{\Lambda} \mathbf{v}'_k \right].$$

The integrands are given in terms of the unit and minimal unit defining densities $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ for Σ and Ξ , respectively, and the regulator $\boldsymbol{\tau}$, by the holographic formulæ

$$\mathbf{v}_k = \mathbf{L}_\sigma^{d-1-k} \boldsymbol{\tau}^{-k} \Big|_{\tilde{\Sigma}}, \quad \mathbf{v}'_{k \leq d-3} = \sum_{j=0}^{d-k-3} (\mathbf{L}_\sigma^T)^j \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-k-3-j} \boldsymbol{\tau}^{-k} \Big|_{\Lambda}, \quad \mathbf{v}'_{d-2} = 0 = \mathbf{v}'_{d-1}.$$

Theorem 6.2. *The divergences in the regulated area Area_ε determined by the sequence of embeddings $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$ and the regulating hypersurface Σ_ε , are the local integrals*

$$\text{Poles}(\text{Area}_\varepsilon) = \sum_{k=d-2}^1 \frac{(-1)^{d-k-2}}{(d-k-2)!k\varepsilon^k} \frac{(k-1)!}{(d-3)!} \int_{\Lambda} \mathbf{a}_k.$$

The integrands are given holographically in terms of the unit and minimal unit defining densities $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ for Σ and Ξ , respectively, and the regulator $\boldsymbol{\tau}$, by

$$\mathbf{a}_k = (\mathbf{L}_\sigma^T)^{d-k-2} \boldsymbol{\tau}^{-k} \Big|_{\Lambda}.$$

Remark 6.3. Note that the results stated in the two previous theorems hold upon performing delta integrations according to Equation (2.25); the measure factors $\sqrt{\mathbf{S}}$ usually

incurred when integrating along submanifolds, with respect to their induced conformal structures, are both one because the defining densities σ and μ are conformal unit and minimal conformal unit, respectively. ■

6.2. Anomalies

We now study the critical $\log \varepsilon$ divergences appearing in the regulated volume and area expansions. In particular, in this section we prove Theorems 1.1 and 1.2. In arbitrary dimensions, the volume and area anomalies are given, respectively, by the integrals

$$\mathcal{V} = \frac{(-1)^d}{(d-1)!} \int_M \theta(\mu) \delta^{(d-1)}(\sigma), \quad \mathcal{A} = \frac{(-1)^{d-1}}{(d-2)!} \int_M \delta(\mu) \delta^{(d-2)}(\sigma);$$

see Section 4. In the volume case, the recursion (5.4) gives

$$\mathbf{L}_\sigma \delta^{(d-2)}(\sigma) = 0,$$

and so cannot be used to handle $\delta^{(d-1)}(\sigma)$. Instead one must introduce a log density into the problem (see [31] and [33]). This relies on the following proposition from [33, Proposition 4.4], specialized to assume that σ is a unit defining density and written in the notation of Section 2.5.

Proposition 6.4. *Let f be a weight zero density, where $f \delta^{(d-1)}(\sigma)$ is compactly supported, and let τ be any true scale, then*

$$\int_M f \delta^{(d-1)}(\sigma) = - \int_M \delta^{(d-2)}(\sigma) (f \mathbf{L}_\sigma \log \tau + \tau^{-1} \mathbf{L}_{\sigma, \tau} f).$$

Note that the proof of [33] was for the case that f is a smooth function, but it is easily seen to extend to the particular instance where $f = \theta(\mu)$, because $\tilde{\Sigma}$ is compact. Thus the proposition can be applied to recast the \mathcal{V} -integral as

$$\mathcal{V} = - \frac{(-1)^d}{(d-1)!} \int_M \delta^{(d-2)}(\sigma) \left(\theta(\mu) \mathbf{L}_\sigma \log \tau + \tau^{-1} \mathbf{L}_{\sigma, \tau} \theta(\mu) \right). \tag{6.1}$$

The differentiated Dirac delta distribution $\delta^{(d-2)}(\sigma)$ in the first term on the right hand side above may now be handled using Equation (5.9) and Proposition 5.18, while the second term is discussed below.

As for the area anomaly \mathcal{A} , the distributional identity $\delta^{(d-2)}(\sigma) = \frac{1}{1-d} \sigma \delta^{(d-1)}(\sigma)$ implies that

$$\mathcal{A} = \frac{(-1)^d}{(d-1)!} \int_M \sigma \delta(\mu) \delta^{(d-1)}(\sigma),$$

which in turn can be rewritten using Proposition 6.4 for the case where $f = \sigma\delta(\mu)$, as

$$\mathcal{A} = \frac{(-1)^{d-1}}{(d-1)!} \int_M \delta^{(d-2)}(\sigma) \left(\sigma\delta(\mu) \mathbf{L}_\sigma \log \tau + \tau^{-1} \mathbf{L}_{\sigma,\tau}(\sigma\delta(\mu)) \right). \tag{6.2}$$

Here we have again extended the scope of Proposition 6.4 to the case where f is a weight zero, distribution-valued density. The proof applies *mutatis mutandis* to this case because $\Lambda = \partial\tilde{\Sigma}$ is closed. Again, the first term on the right hand side can be treated using Proposition 5.18. For the remaining terms involving $\mathbf{L}_{\sigma,\tau}$ operators in Equations (6.1) and (6.2), we need the following lemma:

Lemma 6.5. *Let σ be a weight one density, μ a hypersurface defining density, and τ a true scale. Then*

$$\begin{aligned} \tau^{-1} \mathbf{L}_{\sigma,\tau} \theta(\mu) &= \delta(\mu) \left(\langle \sigma, \mu \rangle - \frac{\sigma}{d-2} \mathbf{L}_\mu \log \tau \right), \\ \tau^{-1} \mathbf{L}_{\sigma,\tau}(\sigma\delta(\mu)) &= \delta(\mu) \left(\mathcal{S}_\sigma - \frac{\sigma}{d-2} \mathbf{L}_\sigma \log \tau \right) + \sigma\delta'(\mu) \left(\langle \sigma, \mu \rangle - \frac{\sigma}{d-2} \mathbf{L}_\mu \log \tau \right). \end{aligned}$$

Proof. The first equality can be demonstrated directly from the definition of the operator $\mathbf{L}_{\sigma,\tau}$ acting on the weight zero density $\theta(\mu)$. Indeed, (2.23) implies that

$$\tau^{-1} \mathbf{L}_{\sigma,\tau} \theta(\mu) = [g; \nabla_n \theta(\mu) - \sigma\tau^{-1} \nabla_k \theta(\mu)] = [g; \delta(\mu)(n.m - \sigma\tau^{-1} m.k)],$$

where we have denoted the triple of one-forms $(m, n, k) := (d\mu, d\sigma, d\tau)$ and used the distributional identity $\theta'(x) = \delta(x)$. Remembering the distributional identity $x\delta(x) = 0$, and that $\langle \sigma, \mu \rangle := [g; \langle \sigma, \mu \rangle] = [g; \sigma\rho_\mu + n.m + \mu\rho_\sigma]$ as well as the definition of $\widehat{\mathbf{L}}_\mu \log \tau$ in Equation (2.6) and (2.12), we can write the expression above as

$$\tau^{-1} \mathbf{L}_{\sigma,\tau} \theta(\mu) = [g; \delta(\mu)(\langle \sigma, \mu \rangle - \sigma(\nabla_m \log \tau + \rho_\mu))] = \delta(\mu) \left(\langle \sigma, \mu \rangle - \sigma \widehat{\mathbf{L}}_\mu \log \tau \right),$$

as required.

The second equality of the lemma can be obtained via a similar route: From (2.23) and recalling that the product $\sigma\delta(\mu)$ has weight zero, it follows that

$$\tau^{-1} \mathbf{L}_{\sigma,\tau}(\sigma\delta(\mu)) = [g; \nabla_n(\sigma\delta(\mu)) - \sigma\tau^{-1} \nabla_k(\sigma\delta(\mu))].$$

The above can be further developed using our distributional calculus:

$$\begin{aligned} \nabla_n(\sigma\delta(\mu)) - \sigma\tau^{-1} \nabla_k(\sigma\delta(\mu)) &= \delta(\mu)(\nabla_n \sigma - \sigma\tau^{-1} \nabla_k \sigma) + \sigma(\nabla_n \delta(\mu) - \sigma\tau^{-1} \nabla_k \delta(\mu)) \\ &= \delta(\mu)(n^2 - \sigma \nabla_n \log \tau) + \sigma\delta'(\mu)(n.m - \sigma \nabla_m \log \tau) \\ &= \delta(\mu)(\mathcal{S}_\sigma - \sigma\rho_\sigma - \sigma(\nabla_n \log \tau + \rho_\sigma)) + \sigma\delta'(\mu)(\langle \sigma, \mu \rangle - \mu\rho_\sigma - \sigma(\nabla_m \log \tau + \rho_\mu)). \end{aligned}$$

Observe that the second and sixth terms above cancel each other. Next, using the action of the Laplace–Robin operator on log-densities (2.5), the last expression of the previous display may be written as

$$\delta(\mu) \left[\mathcal{S}_\sigma - \sigma \left(\widehat{\mathbf{L}}_\sigma \log \tau - \frac{\sigma}{d} (\Delta \log \tau + J) \right) \right] + \sigma \delta'(\mu) \left[\langle \sigma, \mu \rangle - \sigma \left(\widehat{\mathbf{L}}_\mu \log \tau - \frac{\mu}{d} (\Delta \log \tau + J) \right) \right].$$

Once again, thanks to the identity $x\delta'(x) = -\delta(x)$, the terms with coefficient $1/d$ cancel, so we obtain

$$\tau^{-1} \mathbf{L}_{\sigma, \tau} (\sigma \delta(\mu)) = \delta(\mu) \left(\mathcal{S}_\sigma - \sigma \widehat{\mathbf{L}}_\sigma \log \tau \right) + \sigma \delta'(\mu) \left(\langle \sigma, \mu \rangle - \sigma \widehat{\mathbf{L}}_\mu \log \tau \right),$$

again as required. \square

We are now ready to complete the proofs of our first two main results.

Proof of Theorems 1.1 and 1.2. We begin with the volume anomaly. By virtue of Lemma 6.5, the volume anomaly integral (6.1) may be written as

$$\mathcal{V} = \frac{(-1)^{d-1}}{(d-1)!} \int_M \delta^{(d-2)}(\sigma) \left(\theta(\mu) \mathbf{L}_\sigma \log \tau + \delta(\mu) \langle \sigma, \mu \rangle - \frac{\sigma}{d-2} \delta(\mu) \mathbf{L}_\mu \log \tau \right).$$

Here the second term under the integral vanishes due to the minimal surface condition (as formulated in Equation (3.10)) while the last one can be rewritten by means of the identity $\sigma \delta^{(d-2)}(\sigma) = -(d-2) \delta^{(d-3)}(\sigma)$; this gives

$$\mathcal{V} = \frac{(-1)^{d-1}}{(d-1)!} \int_M \left(\theta(\mu) \delta^{(d-2)}(\sigma) \mathbf{L}_\sigma + \delta(\mu) \delta^{(d-3)}(\sigma) \mathbf{L}_\mu \right) \log \tau. \tag{6.3}$$

Likewise, Lemma 6.5 applied to the surface anomaly integral (6.2) yields

$$\begin{aligned} \mathcal{A} = \frac{(-1)^{d-1}}{(d-1)!} \int_M \delta^{(d-2)}(\sigma) \left[\sigma \delta(\mu) \mathbf{L}_\sigma \log \tau + \sigma \delta'(\mu) \left(\langle \sigma, \mu \rangle - \frac{\sigma}{d-2} \mathbf{L}_\mu \log \tau \right) \right. \\ \left. + \delta(\mu) \left(1 - \frac{\sigma}{d-2} \mathbf{L}_\sigma \log \tau \right) \right], \end{aligned}$$

where we used that $\mathcal{S}_\sigma = 1 + \mathcal{O}(\sigma^d)$. Applying the same set of distributional identities as used earlier, the integral above can be re-expressed as

$$\begin{aligned} \mathcal{A} = \frac{(-1)^d}{(d-1)!} \int_M \left[(d-3) \delta(\mu) \delta^{(d-3)}(\sigma) \mathbf{L}_\sigma \log \tau + (d-2) \delta'(\mu) \delta^{(d-3)}(\sigma) \langle \sigma, \mu \rangle \right. \\ \left. - \delta(\mu) \delta^{(d-2)}(\sigma) + (d-3) \delta'(\mu) \delta^{(d-4)}(\sigma) \mathbf{L}_\mu \log \tau \right]. \end{aligned} \tag{6.4}$$

Observe that the third integrand is proportional to the original anomaly (4.9), subtracting this from the left hand side yields $(1 - \frac{1}{d-1})\mathcal{A}$ there. The second term on the right can be handled using (5.6), which implies

$$\delta'(\boldsymbol{\mu})\delta^{(d-3)}(\boldsymbol{\sigma}) = -\frac{1}{d-2}\mathbf{L}_{\boldsymbol{\mu}}(\delta(\boldsymbol{\mu})\delta^{(d-3)}(\boldsymbol{\sigma})) .$$

When $d \neq 4$, the very last term can also be treated using (5.6). When $d = 4$, however, the density $\delta(\boldsymbol{\sigma})$ has critical weight and the integrated formula (5.7) is needed. This distinction between critical ($d = 4$) and non-critical ($d \neq 4$) weights fits with that for the tangential Laplace–Robin operator $\mathbf{L}_{\boldsymbol{\sigma}}^T$ acting on log-densities as given in Definition 5.11. Indeed, by using this definition directly in Equation (6.4) and performing computations that rely on formal self-adjointness of $\mathbf{L}_{\boldsymbol{\mu}}$ and the identity $\boldsymbol{\sigma}\delta^{(d-3)}(\boldsymbol{\sigma}) = -(d-3)\delta^{(d-4)}(\boldsymbol{\sigma})$, we obtain

$$\mathcal{A} = \frac{(-1)^{d-2}}{(d-2)!} \int_M \delta(\boldsymbol{\mu})\delta^{(d-3)}(\boldsymbol{\sigma})\mathbf{L}_{\boldsymbol{\sigma}}^T \log \tau . \tag{6.5}$$

Equation (6.5) is a nice analog of the key Lemma 3.8 of [31].

Applying Proposition 5.18 to Equations (6.3) and (6.5) completes the proof of our results for the volume and area anomalies. \square

The regulated volume and area anomalies \mathcal{V} and \mathcal{A} do not depend on the choice of regulator τ , even though their corresponding extrinsic Q and T curvature integrands do. Remember that the choice of a true scale $\tau = [g; 1]$ is equivalent to a choice of metric g in the conformal class \mathbf{c} . (In what follows we sometimes abbreviate our notation for the dependence of Q and T curvatures and related operators on the underlying data that determines them, for example writing \mathbf{T}_{Λ}^g for $\mathbf{T}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{\tau}$.) Changing the regulator τ to a new true scale $\hat{\tau} = e^f \tau$ where $f \in C^{\infty}(M)$, the identity $\log(e^f \tau) = \log \tau + f$ implies

$$\begin{aligned} \mathbf{Q}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{\hat{\tau}} - \mathbf{Q}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{\tau} &= \mathbf{L}_{\boldsymbol{\sigma}}^{d-1} f|_{\Sigma} , \\ \mathbf{T}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{\hat{\tau}} - \mathbf{T}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{\tau} &= \sum_{j=0}^{d-3} (\mathbf{L}_{\boldsymbol{\sigma}}^T)^j \mathbf{L}'_{\boldsymbol{\mu}} \mathbf{L}_{\boldsymbol{\sigma}}^{d-j-3} f|_{\Lambda} , \\ \mathbf{Q}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{\hat{\tau}} - \mathbf{Q}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{\tau} &= (\mathbf{L}_{\boldsymbol{\sigma}}^T)^{d-2} f|_{\Lambda} . \end{aligned} \tag{6.6}$$

The above formulæ encode how extrinsic (and submanifold) Q and T curvatures transform when moving to a conformally related metric $e^{-2f}g$. Necessarily the sum of the integral over $\tilde{\Sigma}$ of the right hand side of the first equation displayed above plus $(d-2)$ times the integral over $\Lambda = \partial\tilde{\Sigma}$ of the right hand side of the second equation must vanish because \mathcal{V} does not depend on f . Similarly, the integral of the right hand side of the third equation over Λ vanishes because \mathcal{A} is also independent of f . The three operators

acting on f above are conformally invariant and canonically determined by the structure $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$, and are therefore of independent interest. They are the subject of the following section.

7. Extrinsic conformal Laplacian powers and associated boundary operators

Theorems 1.1 and 1.2 suggest that, given the data of a minimal defining density σ and unit minimal defining density μ determined by the sequence of embeddings $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$, we define the following operators

$$\mathcal{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(k)} := \mathbf{L}_{\sigma}^k : \Gamma(\mathcal{E}M[\frac{k-d+1}{2}]) \rightarrow \Gamma(\mathcal{E}M[\frac{-k-d+1}{2}]), \quad k \in \{1, \dots, d-1\},$$

$$\mathcal{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})} := \sum_{j=0}^{d-3} (\mathbf{L}_{\sigma}^T)^j \mathbf{L}'_{\mu} \mathbf{L}_{\sigma}^{d-j-3} : \Gamma(\mathcal{E}M[0]) \rightarrow \Gamma(\mathcal{E}M[2-d]),$$

$$\mathcal{P}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{(k)} := (\mathbf{L}_{\sigma}^T)^k : \Gamma(\mathcal{E}M[\frac{k-d+2}{2}]) \rightarrow \Gamma(\mathcal{E}M[\frac{-k-d+2}{2}]), \quad k \in \{1, \dots, d-2\}.$$

The first important property of these operators is that they are tangential (see Definition 2.1 and the text that follows there). This means that they give holographic formulæ for operators along Σ or Λ . These tangentiality properties are stated in the next proposition.

Proposition 7.1. *The operator $\mathcal{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(k)}$ is tangential along $\Sigma = \mathcal{Z}(\sigma)$, the operator $\mathcal{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}$ is tangential with respect to Σ along $\Lambda = \mathcal{Z}(\sigma, \mu)$, and the operator $\mathcal{P}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{(k)}$ is tangential along $\Lambda = \mathcal{Z}(\sigma, \mu)$.*

Proof. Tangentiality of the operator $\mathcal{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(k)}$ is a special case of a result proved in [29, Theorem 4.1] using the $\mathfrak{sl}(2)$ algebra in Equation (2.9) and its enveloping algebra identities (2.10). The result there pertains to general tractor bundles and so applies to conformal densities. The quantity I^2 in that article is the \mathcal{S} -curvature, which equals one to order $\mathcal{O}(\sigma^d)$ in the present context because σ is a unit defining density (see Equation (2.4)). It is not difficult to verify (again using the $\mathfrak{sl}(2)$ algebra) that the \mathcal{S} -curvature can be replaced by unity without destroying tangentiality so long as the power k of the Laplace–Robin operator is in $\{1, \dots, d-1\}$.

Tangentiality of the operator $\mathcal{P}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{(k)}$ along Λ follows from a similar argument to that for $\mathcal{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(k)}$. The codimension two submanifold Λ is the intersection of Ξ and Σ so it suffices to check tangentiality to each of these hypersurfaces along Λ separately. Tangentiality to Ξ is guaranteed because already the operator \mathbf{L}_{σ}^T is tangential to Ξ . Then Lemma 5.14 provides the analog of the $\mathfrak{sl}(2)$ algebra used in the previous argument.

Only the proof of tangentiality of the operator $\mathcal{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}$ to Σ along Λ remains. For that we need to show that

$$\sum_{j=0}^{d-3} (\mathbf{L}_\sigma^T)^j \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-3}(\sigma \mathbf{f})$$

vanishes along Λ for any smooth weight -1 density \mathbf{f} . We first use the $\mathfrak{sl}(2)$ algebra identities (2.10) to show

$$\mathbf{L}_\sigma^{d-j-3}(\sigma \mathbf{f}) = \sigma \mathbf{L}_\sigma^{d-j-3} \mathbf{f} + (d-j-3)(j+2) \mathbf{L}_\sigma^{d-j-4} \mathbf{f} + \mathcal{O}(\sigma^{j+4}).$$

The term $\mathcal{O}(\sigma^{j+4})$ appears because $\mathcal{S}_\sigma = 1 + \mathcal{O}(\sigma^d)$. Now, along Ξ when $j \neq 0$, Lemma 5.8 gives

$$\mathbf{L}'_\mu(\sigma \mathbf{L}_\sigma^{d-j-3} \mathbf{f}) \stackrel{\Xi}{=} \frac{j+1}{j} \sigma \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-3} \mathbf{f} + \mathcal{O}(\sigma^{d-1}).$$

When $j = 0$, Lemma 5.8 says $\mathbf{L}'_\mu(\sigma \mathbf{L}_\sigma^{d-3} \mathbf{f}) \stackrel{\Xi}{=} \mathcal{O}(\sigma)$, so this term in the sum does not contribute along Λ .

To compute $(\mathbf{L}_\sigma^T)^j(\sigma \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-3} \mathbf{f})$ along Ξ for $j \geq 1$, we note that along Ξ the operator \mathbf{L}_σ^T equals \mathcal{L}_σ^Ξ (see Equation (5.14)), which obeys Lemma 5.14. Thus, using again the $\mathfrak{sl}(2)$ algebra identities (2.10) adjusted to account for the factor $d + 2w - 1$ (rather than $d + 2w$) appearing in Lemma 5.14, along Λ we have

$$(\mathbf{L}_\sigma^T)^j(\sigma \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-3} \mathbf{f}) \stackrel{\Lambda}{=} -j(d-j-2)(\mathbf{L}_\sigma^T)^{j-1} \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-3} \mathbf{f}.$$

Assembling the above three displays gives

$$\begin{aligned} \sum_{j=0}^{d-3} (\mathbf{L}_\sigma^T)^j \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-4}(\sigma \mathbf{f}) &\stackrel{\Lambda}{=} - \sum_{j=1}^{d-3} (j+1)(d-j-2)(\mathbf{L}_\sigma^T)^{j-1} \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-3} \mathbf{f} \\ &\quad + \sum_{j=0}^{d-4} (d-j-3)(j+2)(\mathbf{L}_\sigma^T)^j \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-4} \mathbf{f}. \end{aligned}$$

Reindexing one of the summations shows that the right hand side above vanishes as required. \square

As a consequence of this proposition, we can define the following differential operators holographically:

$$\begin{aligned}
 \mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(k)} : \Gamma(\mathcal{E}\Sigma[\frac{k-d+1}{2}]) &\rightarrow \Gamma(\mathcal{E}\Sigma[\frac{-k-d+1}{2}]), & k \in \{1, \dots, d-1\}, \\
 \Psi & \\
 \mathbf{f} &\mapsto (\mathbf{L}_{\sigma}^k \mathbf{f}_{\text{ext}})|_{\Sigma}, \\
 \\
 \mathbf{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})} : \Gamma(\mathcal{E}\Sigma[0]) &\rightarrow \Gamma(\mathcal{E}\Sigma[2-d])|_{\Lambda}, & (7.1) \\
 \Psi & \\
 \mathbf{f} &\mapsto \left(\sum_{j=0}^{d-3} (\mathbf{L}_{\sigma}^T)^j \mathbf{L}'_{\mu} \mathbf{L}_{\sigma}^{d-j-3} \mathbf{f}_{\text{ext}} \right)|_{\Lambda}, \\
 \\
 \mathbf{P}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{(k)} : \Gamma(\mathcal{E}\Lambda[\frac{k-d+2}{2}]) &\rightarrow \Gamma(\mathcal{E}\Lambda[\frac{-k-d+2}{2}]), & k \in \{1, \dots, d-2\}, \\
 \Psi & \\
 \mathbf{f} &\mapsto ((\mathbf{L}_{\sigma}^T)^k \mathbf{f}_{\text{ext}})|_{\Lambda}.
 \end{aligned}$$

In the above the subscript “ext” denotes an arbitrary smooth extension to the bulk manifold M . The operators $\mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(k)}$ are the extrinsically coupled, conformal Laplacian powers introduced in [30] (in that work these operators are also generalized to act not only on densities but also on general tractors, moreover the integer k can be extended to include any positive even k). Also here we have defined the operators $\mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(k)}$ with an additional factor of $(-1)^k$ as compared to [29] in order to simplify later expressions. For the case when the interior conformal class of metrics includes a formal Poincaré–Einstein metric (in the sense of [16,17]), it was proved in [29] that the operator $\mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(k)}$ reduces to the Laplacian powers of [36]. Those results were summarized in Theorem 1.3.

The operator $\mathbf{P}_{\Sigma} := \mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}^{(d-1)}$ is of particular significance because it determines the conformal transformation property of the $\mathbf{Q}_{\Sigma \hookrightarrow (M, \mathbf{c})}$ curvature. It naturally pairs with the operator $\mathbf{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}$ because together they satisfy Theorem 1.4, whose proof we are now ready to give.

Proof of Theorem 1.4. That $\mathbf{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}$ is canonically determined by the stated embedding data, follows by uniqueness of the unit and minimal unit defining densities σ and μ up to higher order terms in σ , which cannot contribute by virtue of the $\mathfrak{sl}(2)$ algebra of Equation (2.9) and its analog along Ξ in Lemma 5.14.

As explained at the end of Section 6.2, because the volume anomaly \mathcal{V} is independent of the choice of regulator τ , it follows from Theorem 1.1, upon replacing τ by $e^f \tau$ where $f \in C^\infty(M)$, and the identity $\log(e^f \tau) = f + \log \tau$, that

$$\int_{\tilde{\Sigma}} \mathbf{L}_{\sigma}^{d-1} f + (d-2) \int_{\Lambda} \sum_{j=0}^{d-3} (\mathbf{L}_{\sigma}^T)^j \mathbf{L}'_{\mu} \mathbf{L}_{\sigma}^{d-j-3} f = 0.$$

The integral formula of the theorem now follows directly by using the definition of the $(\mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}, \mathbf{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})})$ -pair given in Equation (7.1) (and text directly thereafter).

The first of the leading derivative terms can be deduced from a counting argument and the divergence theorem which implies that

$$\int_{\tilde{\Sigma}} \Delta_{\Sigma} \tilde{f} + \int_{\Lambda} \nabla_{\hat{m}} \tilde{f} = 0,$$

for any smooth function \tilde{f} on $\tilde{\Sigma}$ where \hat{m} is the inward unit normal to Λ . Since the smooth function f is arbitrary and $\mathbf{P}_{\Sigma \hookrightarrow (M, \mathbf{c})}$ has leading derivative term $((d - 2)!!)^2 (\Delta_{\Sigma})^{\frac{d-1}{2}}$, this implies that the operator $\mathbf{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}$ has the leading derivative term stated in the theorem, modulo terms that are total derivatives. To establish that the latter are of lower transverse order, it suffices to write out the holographic formula for $\mathbf{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}$ in a flat limit. More precisely for $M = \mathbb{R}^d$ with its standard conformally flat structure δ , take Σ to be a hyperplane. Moreover, since we are considering the leading derivative terms, we take Λ to be a hyperplane in Σ . It is then easy to verify that the minimal surface Ξ is the hyperplane in M intersecting Σ along Λ at right angles. We leave it to the reader to compute $\mathbf{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (\mathbb{R}^d, \delta)}$, and thus complete the proof. \square

Remark 7.2. Observe that the operator $\mathbf{U}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}$ is order $d - 2$ in derivatives $\nabla_{\hat{m}}$ normal to Λ . Choosing (M, g^o) Poincaré–Einstein, Theorem 1.4 solves holographically the problem of finding a canonical, conformally invariant operator of this order acting on functions, determined by the embedding $\Lambda \hookrightarrow (\Sigma, \mathbf{c}_{\Sigma})$ when $\dim(\Sigma)$ is even. \blacksquare

We can also now give the proof of Theorem 1.5.

Proof of Theorem 1.5. A somewhat tedious direct computation in a choice of scale shows that along Ξ , the operator \mathbf{L}_{σ}^T differs only by terms involving multiplication by extrinsic curvatures from the Laplace–Robin operator intrinsic to Ξ depending on the defining density $\sigma|_{\Xi}$. (Alternatively, this result is an immediate consequence of the fact that the Laplace–Robin operator can be constructed from the Thomas D -operator of [57,5] as well as the results for the Thomas D -operator along hypersurfaces given in [32, Section 4.2].) Canonical determinedness of $\mathbf{P}_{\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})}^{(k)}$ and its leading derivative behavior can then be proved following *mutatis mutandis* the proof method of Theorem 1.3. The displayed integral formula follows via the reasoning given below Equation (6.6). (This logic is also the same as that used to prove the analogous statement with boundary in Theorem (1.4).) \square

8. Examples

We now give the proofs of Theorems 1.6 and 1.7, which give examples of our results when the bulk M is a three- or four-manifold. The proofs are direct computations

of the holographic formulæ of Theorems 6.1 and 6.2 determining the divergences in the regulated volume and area expansions, as well as Theorems 1.1 and 1.2 which give the extrinsic (Q, T) -curvature pair. The critical extrinsic Laplacian powers and the corresponding conformally invariant boundary operators are directly computed from Definition (7.1). In general, Q and T curvatures yield integrated conformal invariants so involve a mixture of terms that are either dependent or independent of the choice of $g \in \mathfrak{c}$ (or equivalently a true scale τ). We will use a mixed bolded and unbolded notation to keep track of these dependencies. We also suppress the full extrinsic embedding data dependence appearing as subscripts on curvatures and operators. We organize these calculations by dimensionality and start with dimension three where we encounter at most two powers of the Laplace–Robin \mathbf{L}_\bullet operator. We will conclude by computing four-dimensional quantities that involve the action of no more than three \mathbf{L}_\bullet operators.

8.1. Three bulk dimensions

8.1.1. Divergences

In this case $\dim(\Lambda) = 1$. The non-critical divergences in the regulated volume expansion (4.3) are determined by Theorem 6.1. These are given by

$$\text{Poles}(\text{Vol}_\varepsilon) = \frac{1}{2\varepsilon^2} \left(\int_{\tilde{\Sigma}} \mathbf{v}_2 + \int_{\Lambda} \mathbf{v}'_2 \right) - \frac{1}{\varepsilon} \left(\int_{\tilde{\Sigma}} \mathbf{v}_1 + \int_{\Lambda} \mathbf{v}'_1 \right) .$$

Here, the two integrals along Λ vanish because $\mathbf{v}'_2 = 0 = \mathbf{v}'_1$. Next, computing in the scale $\tau = [g; 1]$ we find

$$\mathbf{v}_2 = 1 , \quad \mathbf{v}_1 = \mathbf{L}_\sigma \tau^{-1} \Big|_{\tilde{\Sigma}} = -H_\Sigma .$$

Thus

$$\text{Poles}(\text{Vol}_\varepsilon) = \frac{1}{2\varepsilon^2} \int_{\tilde{\Sigma}} dV_{g_\Sigma} + \frac{1}{\varepsilon} \int_{\tilde{\Sigma}} dV_{g_\Sigma} H_\Sigma^g .$$

For the divergences in the regulated area expansion (4.7), Theorem 6.2 gives

$$\text{Poles}(\text{Area}_\varepsilon) = \frac{1}{\varepsilon} \int_{\Lambda} dV_{g_\Lambda} ,$$

where g_Λ is the metric induced along Λ by g .

8.1.2. $(\mathbf{Q}_\Sigma^g, \mathbf{T}_\Lambda^g)$ pair

The corresponding extrinsically coupled Q -curvature along Σ involves two powers of the Laplace–Robin operator and was calculated in [31] and is given by

$$Q_\Sigma^g = J^{g\Sigma} - \frac{1}{2} K_\Sigma,$$

where K_Σ denotes the *rigidity density* of a conformally embedded hypersurface Σ —so-called because it is the Lagrangian density for a rigid string [50]. This is defined (in any dimension $d \geq 3$) by

$$K_\Sigma := \mathring{\mathbf{H}}_{ab}^\Sigma \mathring{\mathbf{H}}_\Sigma^{ab} \in \Gamma(\mathcal{E}\Sigma[-2]).$$

From Theorem 1.1 and using Equations (5.10), (2.6), the transgression is

$$T_\Lambda = \mathbf{L}'_\mu \log \tau|_\Lambda = \mathbf{L}_\mu \log \tau|_\Lambda \stackrel{g}{=} \rho_\mu|_\Lambda = -H_\Xi|_\Lambda.$$

In the above, g is the metric determined by τ . The Chern–Gauß–Bonnet theorem states that the Euler characteristic $\chi_{\tilde{\Sigma}}$ of a surface $\tilde{\Sigma}$ is given by

$$2\pi\chi_{\tilde{\Sigma}} = \int_{\tilde{\Sigma}} J^{g\Sigma} + \int_{\partial\tilde{\Sigma}} (-H_{\partial\tilde{\Sigma} \hookrightarrow \tilde{\Sigma}}),$$

where the mean curvature $H_{\partial\tilde{\Sigma} \hookrightarrow \tilde{\Sigma}}$ is given by the divergence of any extension of the inward unit conormal to $\partial\tilde{\Sigma}$ (this quantity is minus the geodesic curvature of $\partial\tilde{\Sigma}$). Thus, using that Lemmas 3.8 and 3.10 give that $H_{\Lambda \hookrightarrow \Sigma} = H_\Xi|_\Lambda$ (note that the defining density μ is positive on the interior of Ξ so that $d\mu$ is the inward conormal), Theorem 1.1 then yields the volume anomaly

$$\mathcal{V} = \pi\chi_{\tilde{\Sigma}} - \frac{1}{4} \int_{\tilde{\Sigma}} \mathring{\mathbf{H}}_{ab}^\Sigma \mathring{\mathbf{H}}_\Sigma^{ab}. \tag{8.1}$$

This establishes the regulated volume expansion and (Q, T) curvatures of Theorem 1.6 when $d = 3$.

8.1.3. $(\mathbf{P}_\Sigma, \mathbf{U}_\Lambda)$ pair

The extrinsic conformal Laplacian \mathbf{P}_Σ was computed in [32] and equals the Laplacian intrinsic to Σ ,

$$\mathbf{P}_\Sigma = \Delta_\Sigma.$$

We have written the Laplace operator in bold because, for surfaces, it is conformally invariant acting on weight zero densities f_Σ . When f is any smooth extension of f_Σ to M , the corresponding \mathbf{U} operator has holographic formula

$$\mathbf{U}_\Lambda f_\Sigma = \mathbf{L}'_\mu f|_\Lambda.$$

Using Equation (5.8), and choosing a scale $g \in \mathcal{C}$, the above equals $\nabla_m f$. Hence we have

$$\mathbf{U}_\Lambda = \nabla \hat{\mathbf{m}}.$$

This is manifestly conformally invariant and, because $\hat{\mathbf{m}}$ is the inward pointing unit normal, the divergence theorem gives $\int_{\tilde{\Sigma}} \mathbf{P}_\Sigma f + \int_\Lambda \mathbf{U}_\Lambda f = 0$ in concordance with Theorem 1.4. This establishes the $(\mathbf{P}_\Sigma, \mathbf{U}_\Lambda)$ operator pair when $d = 3$ in Theorem 1.6.

8.1.4. Q_Λ and \mathbf{P}_Λ

Our holographic formula for the Q -curvature of Λ in three bulk dimensions is given by (see Theorem 1.2)

$$Q_\Lambda = \mathbf{L}_\sigma^T \log \tau|_\Lambda = -\hat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle|_\Lambda = -\hat{\mathbf{H}}^{\Sigma}_{\hat{\mathbf{m}}\hat{\mathbf{m}}}|_\Lambda.$$

In the above we used Definition (5.11) and Lemma (3.7). Hence the area anomaly is given by

$$\mathcal{A} = \int_\Lambda \hat{\mathbf{H}}^{\Sigma}_{\hat{\mathbf{m}}\hat{\mathbf{m}}}.$$

Observe that this vanishes when the embedding of Σ is umbilic, and thus also when the singular metric g^ρ is (asymptotically) Poincaré–Einstein. Moreover, the above integrand is manifestly conformally invariant, and hence it is possible that the corresponding extrinsic Laplacian power may vanish; indeed this is the case:

$$\mathbf{P}_\Lambda = 0.$$

To see this, it is easy to verify using Definition 5.9 that $\mathbf{L}_\sigma^T f|_\Lambda = 0$ for $f \in C^\infty(M)$, and hence that Theorem 1.5 gives a vanishing result for \mathbf{P}_Λ .

8.2. Four bulk dimensions

8.2.1. Divergences

Theorem 6.1 indicates that the divergences in the regulated volume expansion are

$$\text{Poles}(\text{Vol}_\varepsilon) = \frac{1}{3\varepsilon^3} \int_{\tilde{\Sigma}} \mathbf{v}_3 - \frac{1}{4\varepsilon^2} \int_{\tilde{\Sigma}} \mathbf{v}_2 + \frac{1}{4\varepsilon} \left(\int_{\tilde{\Sigma}} \mathbf{v}_1 + 2 \int_\Lambda \mathbf{v}'_1 \right),$$

where $\mathbf{v}'_3 = 0 = \mathbf{v}'_2$. In the τ scale, the expansion coefficients are

$$\mathbf{v}_3 = 1, \quad \mathbf{v}_2 = \mathbf{L}_\sigma \tau^{-2}|_{\tilde{\Sigma}} = -4H_\Sigma^g, \quad \mathbf{v}_1 = \mathbf{L}_\sigma^2 \tau^{-1}|_{\tilde{\Sigma}} = -2J_\Sigma^g.$$

(These have been calculated in [31].) The boundary contribution is computed as follows:

$$\mathbf{v}'_1 = \mathbf{L}'_\mu \tau^{-1}|_\Lambda = \tilde{\mathbf{L}}_\mu \tau^{-1}|_\Lambda = -\rho_\mu|_\Lambda = H_{\Lambda \hookrightarrow \Sigma}^{g_\Sigma},$$

where we have made use of Definitions (5.8) and (2.18) as well as the identity $H_{\Lambda \leftrightarrow \Sigma} = H_{\Xi}|_{\Lambda}$ (which in turn follows from Lemmas 3.8 and 3.10) to compute the action of the critical operator \mathbf{L}'_{μ} on the bulk Yamabe weight -1 density τ^{-1} . Altogether this gives

$$\text{Poles}(\text{Vol}_{\varepsilon}) = \frac{1}{3\varepsilon^3} \int_{\tilde{\Sigma}} dV^{g_{\Sigma}} + \frac{1}{\varepsilon^2} \int_{\tilde{\Sigma}} dV^{g_{\Sigma}} H_{\Sigma}^g - \frac{1}{2\varepsilon} \int_{\tilde{\Sigma}} dV^{g_{\Sigma}} J_{\Sigma}^g + \frac{1}{2\varepsilon} \int_{\Lambda} dV^{g_{\Lambda}} H_{\Lambda \leftrightarrow \Sigma}^{g_{\Sigma}} .$$

Turning to the divergences in the $d = 4$ area expansion, Theorem 6.2 says that

$$\text{Poles}(\text{Area}_{\varepsilon}) = \frac{1}{2\varepsilon^2} \int_{\Lambda} \mathbf{a}_2 - \frac{1}{\varepsilon} \int_{\Lambda} \mathbf{a}_1 ,$$

where the expansion coefficients

$$\mathbf{a}_2 = 1 , \quad \mathbf{a}_1 = \mathbf{L}_{\sigma}^T \tau^{-1}|_{\Lambda} = (\widehat{\mathbf{L}}_{\mu} \langle \sigma, \mu \rangle - \tilde{\mathbf{L}}_{\sigma}) \tau^{-1}|_{\Lambda} = \mathring{\mathbf{H}}_{\hat{m}\hat{n}}^{\Sigma} - H_{\Sigma} .$$

In the computation of \mathbf{a}_1 we used the definition of the critical tangential operator \mathbf{L}_{σ}^T given in Equation (5.12) as well as Lemma 3.7. Thus we have

$$\text{Poles}(\text{Area}_{\varepsilon}) = \frac{1}{2\varepsilon^2} \int_{\Lambda} dV_{g_{\Lambda}} + \frac{1}{\varepsilon} \int_{\Lambda} dV_{g_{\Lambda}} (H_{\Sigma} - \mathring{\mathbf{H}}_{\hat{m}\hat{n}}^{\Sigma}) .$$

8.2.2. $(\mathbf{Q}_{\Sigma}, \mathbf{T}_{\Lambda})$ pair

Here Λ is a surface without boundary. In four dimensions, the extrinsic Q -curvature involves three powers of the Laplace–Robin operator and was calculated in [21,31]. It is given for some choice of $g \in \mathbf{c}$ by

$$\mathbf{Q}_{\Sigma} = -4\nabla_{\Sigma}^a \nabla_{\Sigma}^b \mathring{\mathbf{H}}_{ab}^{\Sigma} - 8\mathring{\mathbf{H}}_{\Sigma}^{ab} \mathcal{F}_{ab}^{\Sigma} ,$$

where $\mathcal{F}_{ab}^{\Sigma} \in \Gamma(\odot^2 T^* M[0])$ is the *Fialkow tensor* [15] (see also [60,40,55,30]) and equals (in any dimension $d \geq 4$)

$$\mathcal{F}_{ab}^{\Sigma} := \frac{1}{d-3} \left(\mathring{\mathbf{H}}_{ac}^{\Sigma} g_{\Sigma}^{cd} \mathring{\mathbf{H}}_{bd}^{\Sigma} - \frac{1}{2(d-2)} g_{ab}^{\Sigma} \mathbf{K}_{\Sigma} - \mathbf{W}_{cabd} \hat{n}^c \hat{n}^d \right) .$$

The transgression again follows from Theorem 1.1 and Equations (5.8), (5.10):

$$\mathbf{T}_{\Lambda \leftrightarrow \Sigma} = (\mathbf{L}'_{\mu} \mathbf{L}_{\sigma} + \mathbf{L}_{\sigma}^T \mathbf{L}'_{\mu}) \log \tau \Big|_{\Lambda} = (\tilde{\mathbf{L}}_{\mu} \mathbf{L}_{\sigma} + 2\mathbf{L}_{\sigma}^T \widehat{\mathbf{L}}_{\mu}) \log \tau \Big|_{\Lambda} .$$

Using (2.6), the first term on the right is

$$\begin{aligned} \tilde{\mathbf{L}}_{\mu} \mathbf{L}_{\sigma} \log \tau \Big|_{\Lambda} &\stackrel{g}{=} (\nabla_m - \rho_{\mu})(2\rho_{\sigma} - \sigma J) \Big|_{\Lambda} \\ &= (2\nabla_m \rho_{\sigma} - 2\rho_{\mu} \rho_{\sigma}) \Big|_{\Lambda} = -\hat{n}^a \nabla_{\Sigma}^b \mathring{\mathbf{H}}_{ab}^{\Sigma} + 2P_{\hat{m}\hat{n}} - 2H_{\Sigma} H_{\Xi} . \end{aligned}$$

In the above g is the metric determined by τ . We also used $\nabla_m \sigma|_\Lambda = m.n|_\Lambda = 0$, and that $\nabla_m \rho_\sigma \stackrel{\Delta}{=} -\nabla_{\hat{m}}^\Sigma H_\Sigma$ as well as the trace of the $d = 4$ Codazzi–Mainardi equation (see [30, Equation (2.9)]) along Λ :

$$\hat{m}^a \nabla_\Sigma^b \mathring{H}_{ab}^\Sigma = 2(\nabla_{\hat{m}}^\Sigma H_\Sigma + P_{\hat{m}\hat{n}}).$$

The remaining term in T_Λ can be similarly handled using Equation (5.12):

$$\begin{aligned} 2\mathbf{L}_\sigma^T \widehat{\mathbf{L}}_\mu \log \tau \Big|_\Lambda &= 2\left((\widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle) - \widetilde{\mathbf{L}}_\sigma \right) \widehat{\mathbf{L}}_\mu \log \tau \Big|_\Lambda \\ &\stackrel{g}{=} 2\left(\mathring{H}_{\hat{m}\hat{m}}^\Sigma - \nabla_{\hat{n}} + \rho_\sigma \right) \left(\rho_\mu - \frac{1}{2} \mu J \right) \Big|_\Lambda \\ &= -2\mathring{H}_{\hat{m}\hat{m}}^\Sigma H_\Xi + \hat{n}^a \nabla_\Xi^b \mathring{H}_{ab}^\Xi - 2P_{\hat{m}\hat{n}} + 2H_\Sigma H_\Xi. \end{aligned}$$

Orchestration gives

$$T_\Lambda \stackrel{g}{=} -\hat{m}^a \nabla_\Sigma^b \mathring{H}_{ab}^\Sigma + \hat{n}^a \nabla_\Xi^b \mathring{H}_{ab}^\Xi - 2\mathring{H}_{\hat{m}\hat{m}}^\Sigma H_\Xi.$$

We want to write this expression in terms of the embedding sequence $\Lambda \hookrightarrow \Sigma \hookrightarrow (M, \mathbf{c})$. Recalling that Lemmas 3.8 and 3.10 imply that $H_{\Lambda \hookrightarrow \Sigma} = H_\Xi|_\Lambda$, so we only need deal with the second term in the above (note that along Λ , the unit conormal \hat{m} is determined by the embedding $\Lambda \hookrightarrow \Sigma$). For this we first note that the trace of the $d = 4$ Codazzi–Mainardi equation (see for example [30, Equation (2.9)]) along Λ gives

$$\hat{n}^a \nabla_\Xi^b \mathring{H}_{ab}^\Xi = 2(\nabla_{\hat{n}}^\Xi H_\Xi + P_{\hat{n}\hat{m}}).$$

Therefore, we need to study $\nabla_{\hat{n}}^\Xi H_\Xi$ which we can write along Λ in terms of our canonical extensions n and m of \hat{n} and \hat{m} , as (using that $\rho_\mu = -\frac{1}{4}(\nabla \cdot m + \mu J)$ and $m.n|_\Lambda = 0$)

$$\begin{aligned} \frac{1}{4} \nabla_n (\nabla \cdot m + \mu J) &\stackrel{\Delta}{=} \frac{1}{4} \left(\nabla^b \nabla_n m_b - Ric_{mn} - (\nabla_b n^a) \nabla_a m^b \right) \\ &\stackrel{\Delta}{=} \frac{1}{4} \left(\Delta(m.n) - m^a \Delta n_a - 2P_{\hat{m}\hat{n}} - 2(\mathring{H}_{ab}^\Sigma + g_{ab} H_\Sigma)(\mathring{H}_\Xi^{ab} + g^{ab} H_\Xi) \right). \end{aligned}$$

Hence

$$\nabla_{\hat{n}}^\Xi H_\Xi + P_{\hat{n}\hat{m}} \stackrel{\Delta}{=} \frac{1}{2} P_{\hat{m}\hat{n}} - \frac{1}{2} \mathring{H}_{ab}^\Sigma \mathring{H}_\Xi^{ab} - 2H_\Sigma H_{\Lambda \hookrightarrow \Sigma} + \frac{1}{4} \Delta(m.n) - \frac{1}{4} m^a \Delta n_a.$$

The last term above is easily computed using similar techniques:

$$m^a \Delta n_a = m^a \nabla_b \nabla_a n^b = \nabla_m \nabla \cdot n + Ric_{mn} \stackrel{\Delta}{=} 4\nabla_{\hat{m}}^\Sigma H_\Sigma + 2P_{\hat{m}\hat{n}}.$$

Moreover,

$$\begin{aligned} \Delta(m.n) &= \Delta(-\mu\rho_\sigma - \sigma\rho_\mu + \mu\mathcal{C} + \mathcal{O}(\sigma^3)) \\ &\stackrel{\Lambda}{=} 8H_\Sigma H_\Xi + 2\nabla_{\hat{m}}^\Sigma H_\Sigma + 2\nabla_{\hat{n}}^\Xi H_\Xi + 4H_\Xi\mathcal{C} + 2\nabla_{\hat{m}}^\Sigma \mathcal{C}. \end{aligned}$$

Here we used $\Delta\mu \stackrel{\Xi}{=} 4H_\Xi$ and $\Delta\sigma \stackrel{\Sigma}{=} 4H_\Sigma$. It follows, using the traced Codazzi–Mainardi equation along Σ , that

$$\begin{aligned} \nabla_{\hat{n}}^\Xi H_\Xi + P_{\hat{m}\hat{n}} &\stackrel{\Lambda}{=} -\nabla_{\hat{m}}^\Sigma H_\Sigma - P_{\hat{m}\hat{n}} + (\nabla_{\hat{m}}^\Sigma + 2H_{\Lambda\leftrightarrow\Sigma})\hat{\mathbb{I}}_{\hat{m}\hat{m}}^\Sigma - \hat{\mathbb{I}}_{ab}^\Sigma \hat{\mathbb{I}}_{\Xi}^{ab} \\ &\stackrel{\Lambda}{=} -\frac{1}{2}\hat{m}^a \nabla_\Sigma^b \hat{\mathbb{I}}_{ab}^\Sigma + (\nabla_{\hat{m}}^\Sigma + 2H_{\Lambda\leftrightarrow\Sigma})\hat{\mathbb{I}}_{\hat{m}\hat{m}}^\Sigma - \hat{\mathbb{I}}_{ab}^\Sigma \hat{\mathbb{I}}_{\Xi}^{ab}. \end{aligned}$$

We still need to develop the last term above. For that we compute using the same methodology as above as follows:

$$\begin{aligned} \hat{\mathbb{I}}_{\Sigma}^{ab} \hat{\mathbb{I}}_{ab}^\Xi &\stackrel{\Lambda}{=} \hat{\mathbb{I}}_{\Sigma}^{ab} (\nabla_a m_b + g_{ab}\rho_\mu) \stackrel{\Lambda}{=} \hat{\mathbb{I}}_{\Sigma}^{ab} \nabla_a m_b \stackrel{\Lambda}{=} \hat{\mathbb{I}}_{\Sigma}^{ab} ([\nabla_a - m_a \nabla_m] m_b + \frac{1}{2} m_a \nabla_b m^2) \\ &\stackrel{\Lambda}{=} \hat{\mathbb{I}}_{\Sigma}^{ab} ([\nabla_a^\Sigma - m_a \nabla_m^\Sigma] m_b + \frac{1}{2} m_a \nabla_b m^2) \stackrel{\Lambda}{=} \hat{\mathbb{I}}_{\Sigma}^{ab} (\mathbb{I}_{ab}^{\Lambda\leftrightarrow\Sigma} + \hat{m}_a \hat{m}_b H_\Xi) = \hat{\mathbb{I}}_{\Sigma}^{ab} \hat{\mathbb{I}}_{ab}^{\Lambda\leftrightarrow\Sigma}. \end{aligned}$$

Thus, noting that the definition of the Robin operator in Equation (2.7) gives $\nabla_{\hat{m}}^\Sigma \hat{\mathbb{I}}_{\hat{m}\hat{m}}^\Sigma + H_{\Lambda\leftrightarrow\Sigma} \hat{\mathbb{I}}_{\hat{m}\hat{m}}^\Sigma \stackrel{\Lambda}{=} \delta_{\mathbb{R}}^{\Lambda\leftrightarrow\Sigma} \hat{\mathbb{I}}_{\hat{m}\hat{m}}^\Sigma$, we finally obtain

$$T_\Lambda \stackrel{g}{=} -2\hat{m}^a \nabla_\Sigma^b \hat{\mathbb{I}}_{ab}^\Sigma + 2\delta_{\mathbb{R}}^{\Lambda\leftrightarrow\Sigma} \hat{\mathbb{I}}_{\hat{m}\hat{m}}^\Sigma - 2\hat{\mathbb{I}}_{ab}^\Sigma \hat{\mathbb{I}}_{\Lambda\leftrightarrow\Sigma}^{ab}.$$

Remembering that \hat{m} is the inward normal to Λ , then the divergence theorem together with Theorem 1.1 yields the volume anomaly

$$\mathcal{V} = \frac{2}{3} \int_{\hat{\Sigma}} \hat{\mathbb{I}}_{\Sigma}^{ab} \mathcal{F}_{ab}^\Sigma - \frac{1}{3} \int_{\Lambda} \left(\delta_{\mathbb{R}}^{\Lambda\leftrightarrow\Sigma} \hat{\mathbb{I}}_{\hat{m}\hat{m}}^\Sigma - \hat{\mathbb{I}}_{ab}^\Sigma \hat{\mathbb{I}}_{\Lambda\leftrightarrow\Sigma}^{ab} \right). \tag{8.2}$$

Here the integrands of the above result are manifestly conformally invariant because the scale dependent terms in the T -curvature conspire to precisely cancel those in the extrinsic Q -curvature. Notice that this anomaly vanishes when Σ is umbilic and in particular when the bulk is Poincaré–Einstein.

8.2.3. $(\mathbf{P}_\Sigma, \mathbf{U}_\Lambda)$ pair

The extrinsic conformal Laplacian power was calculated in [32] (see also [21]) and, acting on weight zero densities, is given by (not forgetting a factor $(-1)^{d-1} = -1$ accounting for differing sign conventions)

$$\mathbf{P}_\Sigma = 8\nabla_a^\Sigma \circ \hat{\mathbb{I}}_{\Sigma}^{ab} \circ \nabla_b^\Sigma.$$

Observe that although Σ has dimension three so that there is no intrinsic conformally invariant Laplacian power at this weight [36], the above operator is Laplacian-like with

the trace-free second fundamental form appearing in place of the inverse metric. We have used a bold notation for both gradient operators because the gradient of a function and the divergence of a weight -3 vector are conformally invariant operations in three dimensions. It follows from the above display and the divergence theorem that if $f \in C^\infty(M)$, then

$$\int_{\tilde{\Sigma}} \mathbf{P}_\Sigma f = -8 \int_\Lambda \hat{m}_a \hat{\mathbb{H}}_\Sigma^{ab} \nabla_b^\Sigma f,$$

where \hat{m} is the inward unit normal to Λ .

The \mathbf{U}_Λ operator associated to \mathbf{P}_Σ can be computed from Equation (7.1). First note that when $d = 4$ we have

$$\sum_{j=0}^{d-3} (\mathbf{L}_\sigma^T)^j \mathbf{L}'_\mu \mathbf{L}_\sigma^{d-j-3} = \mathbf{L}_\sigma^T \mathbf{L}'_\mu + \mathbf{L}'_\mu \mathbf{L}_\sigma \stackrel{\Delta}{=} 2(\mathcal{C} - \tilde{\mathbf{L}}_\sigma) \hat{\mathbf{L}}_\mu + \tilde{\mathbf{L}}_\mu \mathbf{L}_\sigma.$$

To achieve the last equality we used the definitions of \mathbf{L}'_μ and \mathbf{L}_σ^T in Equations (5.8) and (5.12) of \mathbf{L}_σ^T , as well as Equation (3.10) in conjunction with the identity $\hat{\mathbf{L}}_\mu \mu \stackrel{\Xi}{=} 1$. We now choose a scale $g \in \mathfrak{c}$ and write out the operators appearing on the right hand side above. Working along Λ this gives

$$2(\mathcal{C} - \nabla_n + \rho_\sigma) \nabla_m + (\nabla_m - \rho_\mu)(2\nabla_n - \sigma\Delta) \stackrel{\Delta}{=} 2[\nabla_m, \nabla_n] - 2H_\Sigma \nabla_m + 2H_\Xi \nabla_n + 2\mathcal{C} \nabla_m.$$

Recalling that $\nabla_m n_a \stackrel{\Delta}{=} m^b \hat{\mathbb{H}}_{ba}^\Sigma + m_a H_\Sigma$ and $\nabla_n m_a \stackrel{\Delta}{=} n^b \hat{\mathbb{H}}_{ba}^\Xi + n_a H_\Xi$, the commutator term yields $[\nabla_m, \nabla_n] \stackrel{\Delta}{=} m^b \hat{\mathbb{H}}_{ba}^\Sigma \nabla_\Sigma^a + H_\Sigma \nabla_m - n^b \hat{\mathbb{H}}_{ba}^\Xi \nabla_\Xi^a - H_\Xi \nabla_n$, where along Λ and acting on scalars we have used that $\nabla_\Xi^a = \nabla^a - m^a \nabla_m$ and $\nabla_\Sigma^a = \nabla^a - n^a \nabla_n$. In concordance with the tangentiality result of Proposition 7.1, applying this commutator result to the above display, all instances of the gradient operator ∇_n along the conormal to Σ cancel, and we are left with the operator

$$2m^b \hat{\mathbb{H}}_{ba}^\Sigma \nabla_\Sigma^a - 2n^b \hat{\mathbb{H}}_{ba}^\Xi \nabla_\Xi^a + 2\mathcal{C} \nabla_m.$$

Using the identity of Equation (3.20) and Lemma 3.7, we have the conformally invariant result for the operator \mathbf{U}_Λ

$$\mathbf{U}_\Lambda = 4\hat{m}^a \hat{\mathbb{H}}_{ab}^\Sigma \nabla_\Sigma^b.$$

Indeed, as proved in Theorem 1.4

$$\int_{\tilde{\Sigma}} \mathbf{P}_\Sigma f + 2 \int_\Lambda \mathbf{U}_\Lambda f = 0.$$

We observe that just as for the \mathbf{P}_Σ operator, the \mathbf{U}_Λ operator is obtained from its $d = 3$ counterpart $\nabla_{\hat{m}}$ by replacing the inverse metric with the trace-free second fundamental form.

8.2.4. \mathbf{Q}_Λ and \mathbf{P}_Λ

The holographic formula for the \mathbf{Q} curvature of Λ in four bulk dimensions is given by (see Theorem 1.2)

$$\mathbf{Q}_\Lambda = (\mathbf{L}_\sigma^T)^2 \log \tau|_\Lambda.$$

In four dimensions $\mathbf{L}_\sigma^T \log \tau$ has bulk Yamabe weight $w = -1$, so we must use Equation (5.12) of Definition 5.9 as well as Definition 5.11. This yields

$$\begin{aligned} \mathbf{L}_\sigma^T{}^2 \log \tau &\stackrel{\Delta}{=} \left((\widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle) - \widetilde{\mathbf{L}}_\sigma \right) \left(\widehat{\mathbf{L}}_\sigma \log \tau - \widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle + \sigma \widetilde{\mathbf{L}}_\mu \widehat{\mathbf{L}}_\mu \log \tau \right) \\ &\stackrel{\Delta}{=} -\widetilde{\mathbf{L}}_\mu \widehat{\mathbf{L}}_\mu \log \tau - \widetilde{\mathbf{L}}_\sigma \widehat{\mathbf{L}}_\sigma \log \tau + \widetilde{\mathbf{L}}_\sigma \widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle + \mathcal{C} (\widehat{\mathbf{L}}_\sigma \log \tau - \mathcal{C}). \end{aligned}$$

Here \mathcal{C} is as given in Lemma 3.7 and we have also used $\widetilde{\mathbf{L}}_\sigma \sigma|_\Sigma = 1$. We now calculate each of the four terms above for the choice of $g \in \mathbf{c}$ determined by τ . First, using Equations (2.12) and (2.6) and $\nabla_m \mu|_\Xi = 1$,

$$-\widetilde{\mathbf{L}}_\mu \widehat{\mathbf{L}}_\mu \log \tau|_\Lambda = -(\nabla_m - \rho_\mu)(\rho_\mu - \frac{1}{2}\mu J) \stackrel{\Delta}{=} -\nabla_m \rho_\mu + \rho_\mu^2 + \frac{1}{2}J.$$

Similarly

$$-\widetilde{\mathbf{L}}_\sigma \widehat{\mathbf{L}}_\sigma \log \tau \stackrel{\Delta}{=} -\nabla_n \rho_\sigma + \rho_\sigma^2 + \frac{1}{2}J.$$

For the second last of the abovementioned four terms we use that $\widehat{\mathbf{L}}_\sigma \log \tau \stackrel{\Delta}{=} \rho_\sigma$. That leaves the term

$$\begin{aligned} \widetilde{\mathbf{L}}_\sigma \widehat{\mathbf{L}}_\mu \langle \sigma, \mu \rangle &\stackrel{\Delta}{=} \nabla_n \nabla_m (m.n + \sigma \rho_\mu + \mu \rho_\sigma) - \rho_\sigma \mathcal{C} \\ &\stackrel{\Delta}{=} \nabla_n \nabla_m m.n + \rho_\mu \nabla_n m.n + \nabla_m \rho_\mu + \nabla_n \rho_\sigma - \rho_\sigma \mathcal{C}. \end{aligned}$$

Here we used $\nabla_n \mu = m.n \stackrel{\Delta}{=} 0$. Using that $\nabla_m m = \frac{1}{2} \nabla m^2 = -\nabla(\mu \rho_\mu) + \mathcal{O}(\mu^3)$ we have

$$\nabla_n \nabla_m m.n \stackrel{\Delta}{=} \nabla_n (-\nabla_n(\mu \rho_\mu) + m^a \nabla_m n_a) \stackrel{\Delta}{=} -\rho_\mu \nabla_n m.n + \nabla_n(m^a \nabla_m n_a).$$

Collating the above computations we find

$$(\mathbf{L}_\sigma^T)^2 \log \tau \stackrel{\Delta}{=} J + H_\Sigma^2 + H_\Xi^2 - \mathcal{C}^2 + \nabla_n(m^a \nabla_m n_a).$$

We now focus on the last term on the line above:

$$\begin{aligned} \nabla_n(m^a \nabla_m n_a) &\stackrel{\Delta}{=} \mathring{I}\mathring{I}_{\hat{n}a}^{\Xi} g^{ab} \mathring{I}\mathring{I}_{\hat{m}b}^{\Sigma} + m^a [\nabla_n, \nabla_m] n_a - m^a \nabla_m \nabla_a (\sigma \rho_\sigma) \\ &\stackrel{\Delta}{=} \mathring{I}\mathring{I}_{\hat{n}a}^{\Xi} g^{ab} \mathring{I}\mathring{I}_{\hat{m}b}^{\Sigma} + m^a (\nabla_n m^c) \nabla_c n_a - m^a (\nabla_n m^b) \nabla_b n_a \\ &\quad + R_{\hat{n}\hat{m}\hat{m}\hat{n}} + (\mathcal{C} + H_\Sigma) H_\Sigma \\ &\stackrel{\Delta}{=} 2\mathring{I}\mathring{I}_{\hat{n}a}^{\Xi} g^{ab} \mathring{I}\mathring{I}_{\hat{m}b}^{\Sigma} - \mathring{I}\mathring{I}_{\hat{m}a}^{\Sigma} g^{ab} \mathring{I}\mathring{I}_{\hat{m}b}^{\Sigma} - H_\Sigma \mathcal{C} - W_{\hat{n}\hat{m}\hat{n}\hat{m}} - P_{\hat{m}\hat{m}} - P_{\hat{n}\hat{n}}. \end{aligned}$$

In the above we used $\nabla n \stackrel{\Sigma}{=} \mathring{I}\mathring{I}^{\Sigma} + gH_\Sigma$ as well as the analogous relation and ∇m . Also $\nabla_n n = -\nabla(\sigma \rho_\sigma) + \mathcal{O}(\sigma^3)$. This implies that $m^a \nabla_m n_a \stackrel{\Sigma}{=} \mathring{I}\mathring{I}_{\hat{m}\hat{m}}^{\Sigma} + H_\Sigma \stackrel{\Delta}{=} \mathcal{C} + H_\Sigma$, which was used to reach the second equality. Moreover $n|_\Sigma = \hat{n}$, $m|_\Sigma = \hat{m}$, $\rho_\sigma|_\Sigma = -H_\Sigma$ and $\rho_\mu|_\Sigma = -H_\Sigma$. To achieve the last line we additionally employed Equation (3.17). Altogether we now get

$$\begin{aligned} (\mathbf{L}_\sigma^T)^2 \log \tau &\stackrel{\Delta}{=} J - P_{\hat{m}\hat{m}} - P_{\hat{n}\hat{n}} + H_\Sigma^2 + H_\Xi^2 \\ &\quad + 2\mathring{I}\mathring{I}_{\hat{n}a}^{\Xi} g^{ab} \mathring{I}\mathring{I}_{\hat{m}b}^{\Sigma} - \mathring{I}\mathring{I}_{\hat{m}a}^{\Sigma} g^{ab} \mathring{I}\mathring{I}_{\hat{m}b}^{\Sigma} - W_{\hat{n}\hat{m}\hat{n}\hat{m}} - \mathcal{C}(\mathcal{C} + H_\Sigma). \end{aligned}$$

Employing Equation (3.20) and that $H_{\Lambda \hookrightarrow \Sigma} = H_\Xi|_\Lambda$ (see Lemma 3.10), we then have

$$Q_\Lambda = H_{\Lambda \hookrightarrow \Sigma}^2 + P_{ab} g_\Lambda^{ab} + H_\Sigma^2 - H_\Sigma \mathcal{C} - 2\mathcal{C}^2 - 3\mathring{I}\mathring{I}_{\hat{m}a}^{\Sigma} g_\Lambda^{ab} \mathring{I}\mathring{I}_{\hat{m}b}^{\Sigma} - W_{\hat{n}\hat{m}\hat{n}\hat{m}}.$$

Along Σ , the Fialkow–Gauß Equation [30, Equation 2.7] gives

$$P_{ab} - \hat{n}_a P_{\hat{n}b} - \hat{n}_b P_{\hat{n}a} + \hat{n}_a \hat{n}_b P_{\hat{n}\hat{n}} = P_{ab}^\Sigma - H_\Sigma \mathring{I}\mathring{I}_{ab}^\Sigma - \frac{1}{2} (g_{ab} - \hat{n}_a \hat{n}_b) H_\Sigma^2 + \mathcal{F}_{ab}^\Sigma,$$

where the Fialkow tensor in $d = 4$ is given by $\mathcal{F}_{ab}^\Sigma = \mathring{I}\mathring{I}_{ac}^\Sigma g_\Sigma^{cd} \mathring{I}\mathring{I}_{bd}^\Sigma - \frac{1}{4} g_{ab}^\Sigma K_\Sigma + W_{\hat{n}a\hat{n}b}$. Note that $g_\Sigma^{ab} \mathcal{F}_{ab}^\Sigma = \frac{1}{4} K_\Sigma$. Thus, along Λ ,

$$P_{ab} g_\Lambda^{ab} = J_\Sigma - P_{\hat{m}\hat{m}}^\Sigma + H_\Sigma \mathcal{C} - H_\Sigma^2 + \frac{1}{4} K_\Sigma - \mathcal{F}_{\hat{m}\hat{m}}^\Sigma.$$

Using $W_{\hat{n}\hat{m}\hat{n}\hat{m}} = \mathcal{F}_{\hat{m}\hat{m}}^\Sigma - \mathring{I}\mathring{I}_{\hat{m}a}^\Sigma g_\Lambda^{ab} \mathring{I}\mathring{I}_{\hat{m}b}^\Sigma - \mathcal{C}^2 + \frac{1}{4} K_\Sigma$, this gives

$$Q_\Lambda = H_{\Lambda \hookrightarrow \Sigma}^2 + J_\Sigma - P_{\hat{m}\hat{m}}^\Sigma - 2\mathcal{F}_{\hat{m}\hat{m}}^\Sigma - 2\mathring{I}\mathring{I}_{\hat{m}a}^\Sigma g_\Lambda^{ab} \mathring{I}\mathring{I}_{\hat{m}b}^\Sigma - \mathcal{C}^2.$$

A necessary condition for the singular metric g° to be Poincaré–Einstein is that the embedding $\Sigma \hookrightarrow (M, \mathbf{c})$ is umbilic so that $\mathring{I}\mathring{I}_{ab}^\Sigma = 0$. Moreover the Fialkow tensor vanishes, this is easily verified by demonstrating that $W_{\hat{n}a\hat{n}b} \stackrel{\Sigma}{=} 0$ for Poincaré–Einstein structures. In that case only the first three terms on the right hand side of the above display survive and the area anomaly (see Theorem 1.2) is given by $\mathcal{A} = \frac{1}{2} \int_\Lambda (H_{\Lambda \hookrightarrow \Sigma}^2 + J_\Sigma - P_{\hat{m}\hat{m}}^\Sigma)$. This is in concordance with the original expression for the log coefficient of Graham and

Witten [39] (noting that their mean curvature is the sum, not average of the eigenvalues of the second fundamental form).

To see that Q_Λ^g is an extrinsically coupled Q -curvature type invariant for the submanifold Λ , we recall that the Gauß equations imply

$$J_\Sigma - P_{\hat{m}\hat{m}}^\Sigma = J_\Lambda - H_{\Lambda \hookrightarrow \Sigma}^2 + \frac{1}{2}K_{\Lambda \hookrightarrow \Sigma},$$

where $J_\Lambda := \frac{1}{2}Sc_\Lambda$. Thus we have

$$Q_\Lambda^g = J_\Lambda + \frac{1}{2}K_{\Lambda \hookrightarrow \Sigma} - 2\mathcal{F}_{\hat{m}\hat{m}}^\Sigma - 2\hat{\mathbf{I}}_{\hat{m}a}^\Sigma g^{ab} \hat{\mathbf{I}}_{\hat{m}b}^\Sigma - \mathcal{C}^2.$$

Hence, using the Gauß–Bonnet theorem, the area anomaly is given by

$$\mathcal{A} = \pi\chi_\Lambda + \frac{1}{4} \int_\Lambda \left[K_{\Lambda \hookrightarrow \Sigma} - 4\mathcal{F}_{\hat{m}\hat{m}}^\Sigma - 4\hat{\mathbf{I}}_{\hat{m}a}^\Sigma g^{ab} \hat{\mathbf{I}}_{\hat{m}b}^\Sigma - 2\mathcal{C}^2 \right].$$

The first term above is proportional to the integral over the intrinsic Q -curvature of the manifold Λ and gives the three manifold anomaly in the regulated volume when the bulk is Poincaré–Einstein.

Finally we turn to the operator \mathbf{P}_Λ . From Theorem 1.5 and Equations (5.12), (5.13) of Definition 5.9, the extrinsic Laplacian power associated to the submanifold Q -curvature Q_Λ has a holographic formula

$$\mathbf{P}_\Lambda \stackrel{\Lambda}{=} (\mathbf{L}_\sigma^\top)^2 = (\mathcal{C} - \tilde{\mathbf{L}}_\sigma) \circ (\hat{\mathbf{L}}_\sigma + \sigma \tilde{\mathbf{L}}_\mu \hat{\mathbf{L}}_\mu) = \mathcal{C} \hat{\mathbf{L}}_\sigma - \tilde{\mathbf{L}}_\sigma \hat{\mathbf{L}}_\sigma - \tilde{\mathbf{L}}_\mu \hat{\mathbf{L}}_\mu.$$

Proceeding in a choice of $g \in \mathfrak{c}$ the above operator becomes (along Λ)

$$\begin{aligned} \mathcal{C} \nabla_n - (\nabla_n - \rho_\sigma) \circ (\nabla_n - \frac{1}{2} \sigma \Delta) - (\nabla_m - \rho_\mu) \circ (\nabla_m - \frac{1}{2} \mu \Delta) \\ = \Delta - \nabla_n^2 - \nabla_m^2 + (\mathcal{C} - H_\Sigma) \nabla_n - H_\Xi \nabla_m. \end{aligned}$$

It is not difficult to verify that the Laplacian Δ_Λ along Λ acting on scalars has holographic formula

$$\begin{aligned} g^{ab} (\nabla_a - m_a \nabla_m - n_a \nabla_n) (\nabla_b - m_b \nabla_m - n_b \nabla_n) \\ \stackrel{\Lambda}{=} \Delta - \nabla_m^2 - \nabla \cdot m \nabla_m - \nabla_n^2 - \nabla \cdot n \nabla_n \\ + (\nabla_m m^a) \nabla_a + m^a (\nabla_m m_a) \nabla_m + m^a (\nabla_m n_a) \nabla_n \\ + (\nabla_n n^a) \nabla_a + n^a (\nabla_n m_a) \nabla_m + n^a (\nabla_n n_a) \nabla_n. \end{aligned}$$

Note that $m^a \nabla_m m_a = \frac{1}{2} \nabla_m m^2 \stackrel{\Xi}{=} H_\Xi \stackrel{\Xi}{=} \frac{1}{4} \nabla_m m$ and $\nabla_m m_a = \frac{1}{2} \nabla_a m^2 \stackrel{\Xi}{=} \hat{m}_a H_\Xi$. Analogous identities hold replacing m with n . Also $m^a m^b \nabla_a n_b \stackrel{\Lambda}{=} \mathcal{C} + H_\Sigma$ and $n^a n^b \nabla_a m_b \stackrel{\Lambda}{=} H_\Xi$. This establishes that

$$\mathbf{P}_\Lambda = \Delta_\Lambda.$$

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