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One-loop conformal invariance of the type II pure spinor superstring in a curved background

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One-loop conformal invariance of the type II pure spinor superstring in a curved background

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Abstract: We compute the one-loop beta functions for the Type II superstring using the pure spinor formalism in a generic supergravity background. It is known that the classical pure spinor BRST symmetry puts the background fields on-shell. In this paper we show that the one-loop beta functions vanish as a consequence of the classical BRST symmetry of the action.

Keywords: Conformal Field Models in String Theory, Anomalies in Field and String Theories.

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1. Introduction

It is well known that superstrings are not consistent on any background field. In the simplest case of a closed bosonic string coupled to a curved background, the quantum preservation of the Weyl symmetry implies that the background metric satisfies the Einstein equations plus α'-corrections. The Weyl symmetry preservation can be seen as the absence of ultra-violet (UV) divergences in the quantum effective action [1].

In the case of superstrings in curved background, the preservation of Weyl symmetry at the quantum level is much more involved, although the case where only Neveu Schwarz-Neveu Schwarz background fields are turned on is very similar to the bosonic string case [2]. Once we allow the full supersymmetric multiplet to be turned on, we need a manifestly supersymmetric space-time sigma model in order to study the corresponding quantum regime. The Green-Schwarz formalism provides us with a sigma model action which is manifestly space-time supersymmetric, nevertheless, one does not manifestly preserve this symmetry if one tries to quantize.

There exists another formalism for the superstring which does not suffer of this disadvantage, known as the pure spinor formalism [3]. In this formalism the space-time supersymmetry is manifest and the quantization is straightforward by requiring a BRST-like symmetry. We will briefly discuss the basics of this model in section 2, now let us mention what have been done to check the consistency of this formalism. The spectrum of the model is equivalent to the Green-Schwarz spectrum in the light-cone gauge [4] and it also allows to find the physical spectrum in a manifestly super-Poincaré covariant manner [5]. The pure spinor formalism is suitable to describe strings in background fields with Ramond-Ramond fields strengths turned on, as it happens on anti-de Sitter geometries. In this case, it has been checked the classical [6] and the quantum BRST invariance of the model [7] as well as its quantum conformal invariance [8, 7] (see also [9] and [10]).

The superstring in the pure spinor formalism can be coupled to a generic supergravity background field, as it was shown in [11], where a sigma model action was written for the Heterotic and Type II superstrings. Here, the classical BRST invariance puts the background fields on-shell. In the heterotic string case, the background fields satisfy the ten-dimensional N=1 supergravity equations plus the super Yang-Mills equations in a curved background. In the type II case, the background fields satisfy the ten-dimensional type II supergravity equations. Of course, it could be very interesting to obtain α'-corrections to these equations in this formalism by requiring the quantum preservation of some symmetries of the classical sigma-model action. Since the lowest order in α' BRST symmetry puts the background fields on shell, one expects that the equations of motion for the background fields derived from the beta function calculation are implied by the BRST symmetry. This property was verified for the heterotic string, in whose case, the classical BRST symmetry implies one-loop conformal invariance [12]. In this paper we show that the same property is also valid for the Type II superstring[1].

In section 2 we review the type II sigma-model construction of [11]. In section 3 we expand the action by using a covariant background field expansion. In section 4 we determine the UV divergent part of the effective action at the one-loop level and finally, in section 5, we use the expanded action of section 3, to write from the UV divergent part found in section 4, the beta functions for the Type II superstring. Then we show that beta functions vanish after using the constraints on the background fields implied by the classical BRST invariance of the sigma-model action[2].

2. Classical BRST constraints

The pure spinor closed string action in flat space-time is defined by the superspace coordinates X^m with $m = 0, \dots, 9$ and the conjugate pairs $(p_\alpha, \theta^\alpha), (\widetilde{p_{\overline{\alpha}}}, \widetilde{\theta^{\overline{\alpha}}})$ with $(\alpha, \overline{\alpha}) = 1, \dots, 16$. For the type IIA superstring the spinor indices α and $\overline{\alpha}$ have the opposite chirality while for the type IIB superstring they have the same chirality. In order to define a conformal invariant system we need to include a pair of pure spinor ghost variables $(\lambda^\alpha, \omega_\alpha)$ and $(\widetilde{\lambda^{\overline{\alpha}}}, \widetilde{\omega}_{\overline{\alpha}})$. These ghosts are constrained to satisfy the pure spinor conditions $(\lambda\gamma^m\lambda) = (\widetilde{\lambda}\gamma^m\widetilde{\lambda}) = 0$, where $\gamma^m_{\alpha\beta}$ and $\gamma^m_{\overline{\alpha}\overline{\beta}}$ are the 16×16 symmetric ten dimensional gamma matrices. Because of the pure spinor conditions, ω and $\widetilde{\omega}$ are defined up to $\delta\omega = (\lambda\gamma^m)\Lambda_m$ and $\delta\widetilde{\omega} = (\widetilde{\lambda}\gamma^m)\widetilde{\Lambda}_m$. The quantization of the model is performed after

[1] There is a more recent development [13] with an even richer world-sheet structure, called non-minimal pure spinor formalism, which is interpreted as a critical topological string. Nevertheless, in this paper we restrict to the so called minimal pure spinor formalism of [3].

[2] In a similar way, using the he hybrid formalism [14], in [15] and [16] were derived the type II 4D supergravity equations of motion in superspace by requiring superconformal invariance.

the construction of the BRST-like charges $Q = \oint \lambda^\alpha d_\alpha, \widetilde{Q} = \oint \widetilde{\lambda}^{\overline{\alpha}} \widetilde{d}_{\overline{\alpha}}$, here d_α and $\widetilde{d}_{\overline{\alpha}}$ are the world-sheet variables corresponding to the $N = 2$ $D = 10$ space-time supersymmetric derivatives and are supersymmetric combinations of the space-time superspace coordinates of conformal weights $(1, 0)$ and $(0, 1)$ respectively. The action in flat space is a free action involving the above fields, that is

$$S = \frac{1}{2\pi\alpha'} \int d^2z \; (\frac{1}{2}\partial X^m \overline{\partial} X_m + p_\alpha \overline{\partial}\theta^\alpha + \widetilde{p}_{\overline{\alpha}} \partial \widetilde{\theta}^{\overline{\alpha}}) + S_{pure}, \tag{2.1}$$

where S_{pure} is the action for the pure spinor ghosts.

In a curved background, the pure spinor sigma model action for the type II superstring is obtained by adding to the flat action of (2.1) the integrated vertex operator for supergravity massless states and then covariantizing respect to ten dimensional $N = 2$ super-reparameterization invariance. The result of doing this is

$$\begin{aligned} S = \frac{1}{2\pi\alpha'} \int d^2z \; (&\frac{1}{2}\Pi^a \overline{\Pi}^b \eta_{ab} + \frac{1}{2}\Pi^A \overline{\Pi}^B B_{BA} + d_\alpha \overline{\Pi}^\alpha + \widetilde{d}_{\overline{\alpha}} \Pi^{\overline{\alpha}} + (\lambda^\alpha \omega_\beta) \overline{\Omega}_\alpha{}^\beta + (\widetilde{\lambda}^{\overline{\alpha}} \widetilde{\omega}_{\overline{\beta}}) \widetilde{\Omega}_{\overline{\alpha}}{}^{\overline{\beta}} \\ &+ d_\alpha \widetilde{d}_{\overline{\beta}} P^{\alpha\overline{\beta}} + (\lambda^\alpha \omega_\beta) \widetilde{d}_{\overline{\gamma}} C_\alpha{}^{\beta\overline{\gamma}} + (\widetilde{\lambda}^{\overline{\alpha}} \widetilde{\omega}_{\overline{\beta}}) d_\gamma \widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + (\lambda^\alpha \omega_\beta)(\widetilde{\lambda}^{\overline{\alpha}} \widetilde{\omega}_{\overline{\beta}}) S_{\alpha\overline{\alpha}}{}^{\beta\overline{\beta}}) + S_{pure} + S_{FT} \, , \end{aligned} \tag{2.2}$$

where $\Pi^A = \partial Z^M E_M{}^A, \overline{\Pi}^A = \overline{\partial} Z^M E_M{}^A$ with $E_M{}^A$ the supervielbein and Z^M are the curved superspace coordinates, B_{BA} is the super two-form potential. The connections appears as $\overline{\Omega}_\alpha{}^\beta = \overline{\partial} Z^M \Omega_{M\alpha}{}^\beta = \overline{\Pi}^A \Omega_{A\alpha}{}^\beta$ and $\widetilde{\Omega}_{\overline{\alpha}}{}^{\overline{\beta}} = \partial Z^M \widetilde{\Omega}_{M\overline{\alpha}}{}^{\overline{\beta}} = \Pi^A \widetilde{\Omega}_{A\overline{\alpha}}{}^{\overline{\beta}}$. They are independent since the action of (2.2) has two independent Lorentz symmetry transformations. One acts on the α-type indices and the other acts on the $\overline{\alpha}$-type indices. S_{pure} is the action for the pure spinor ghosts and is the same as in the flat space case of (2.1).

As was shown in [11], the gravitini and the dilatini fields are described by the lowest θ-components of the superfields $C_\alpha{}^{\beta\overline{\gamma}}$ and $\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}$, while the Ramond-Ramond field strengths are in the superfield $P^{\alpha\overline{\beta}}$. The dilaton is the theta independent part of the superfield Φ which defines the Fradkin-Tseytlin term

$$S_{FT} = \frac{1}{2\pi} \int d^2z \; r \; \Phi, \tag{2.3}$$

where r is the world-sheet curvature. Because of the pure spinor constraints, the superfields in (2.2) cannot be arbitrary. In fact, it is necessary that

$$\begin{aligned} \Omega_{A\alpha}{}^\beta &= \Omega_A \delta_\alpha{}^\beta + \frac{1}{4}\Omega_{Aab}(\gamma^{ab})_\alpha{}^\beta, \quad \widetilde{\Omega}_{A\overline{\alpha}}{}^{\overline{\beta}} = \widetilde{\Omega}_A \delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}\widetilde{\Omega}_{Aab}(\gamma^{ab})_{\overline{\alpha}}{}^{\overline{\beta}}, \\ C_\alpha{}^{\beta\overline{\gamma}} &= C^{\overline{\gamma}} \delta_\alpha{}^\beta + \frac{1}{4}C_{ab}{}^{\overline{\gamma}}(\gamma^{ab})_\alpha{}^\beta, \quad \widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} = \widetilde{C}^\gamma \delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}\widetilde{C}_{ab}{}^\gamma(\gamma^{ab})_{\overline{\alpha}}{}^{\overline{\beta}}, \\ S_{\alpha\overline{\alpha}}{}^{\beta\overline{\beta}} &= S\delta_\alpha{}^\beta \delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}S_{ab}(\gamma^{ab})_\alpha{}^\beta \delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}\widetilde{S}_{ab}(\gamma^{ab})_{\overline{\alpha}}{}^{\overline{\beta}} \delta_\alpha{}^\beta + \frac{1}{16}S_{abcd}(\gamma^{ab})_\alpha{}^\beta (\gamma^{cd})_{\overline{\alpha}}{}^{\overline{\beta}}. \end{aligned} \tag{2.4}$$

The action of (2.2) is BRST invariant if the background fields satisfy suitable constraints. As was shown in [11], these constraints imply that the background field satisfy the type II supergravity equations. The BRST invariance is obtained by requiring that the BRST currents $j_B = \lambda^\alpha d_\alpha$ and $\widetilde{j}_B = \widetilde{\lambda}^{\overline{\alpha}} \widetilde{d}_{\overline{\alpha}}$ are conserved. Besides, the BRST charges $Q = \oint j_B$ and $\widetilde{Q} = \oint \widetilde{j}_B$ are nilpotent and anticommute. Let us review these properties now.

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2.1 Nilpotency

As was shown in [11] (see also [17]), nilpotency is obtained after defining momentum variables in (2.2) and then using the canonical Poisson brackets. The only momentum variable that does not appear in (2.2) is the conjugate momentum of Z^M which is defined as $P_M = (2\pi\alpha')\delta S/\delta(\partial_0 Z^M)$ where $\partial_0 = \frac{1}{2}(\partial + \overline{\partial})$. It is not difficult to see that ω_α is the conjugate momentum to λ^α and that $\widetilde{\omega}_{\overline{\alpha}}$ is the one for $\widetilde{\lambda}^{\overline{\alpha}}$. Nilpotence of Q determines the constraints

$$\lambda^\alpha\lambda^\beta H_{\alpha\beta A} = \lambda^\alpha\lambda^\beta\lambda^\gamma R_{\alpha\beta\gamma}{}^\delta = \lambda^\alpha\lambda^\beta\widetilde{R}_{\alpha\beta\overline{\gamma}}{}^{\overline{\delta}} = 0,$$
$$\lambda^\alpha\lambda^\beta T_{\alpha\beta}{}^a = \lambda^\alpha\lambda^\beta T_{\alpha\beta}{}^\gamma = \lambda^\alpha\lambda^\beta T_{\alpha\beta}{}^{\overline{\gamma}} = 0, \qquad (2.5)$$

where $H = dB$, the torsion $T_{AB}{}^\alpha$ and $R_{AB\gamma}{}^\delta$ are the torsion and the curvature constructed using $\Omega_{A\beta}{}^\gamma$ as connection. Similarly, $T_{AB}{}^{\overline{\gamma}}$ and $\widetilde{R}_{AB\overline{\gamma}}{}^{\overline{\delta}}$ are the torsion and the curvature using $\widetilde{\Omega}_{A\overline{\beta}}{}^{\overline{\gamma}}$ as connection.

The nilpotence of the BRST charge $\widetilde{Q}$ leads to the constraints

$$\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}} H_{\overline{\alpha}\overline{\beta}A} = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}} R_{\overline{\alpha}\overline{\beta}\gamma}{}^\delta = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}\widetilde{\lambda}^{\overline{\gamma}}\widetilde{R}_{\overline{\alpha}\overline{\beta}\overline{\gamma}}{}^{\overline{\delta}} = 0,$$
$$\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}} T_{\overline{\alpha}\overline{\beta}}{}^a = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}} T_{\overline{\alpha}\overline{\beta}}{}^\gamma = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}\widetilde{T}_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} = 0. \qquad (2.6)$$

Finally, the anticommutation between Q and $\widetilde{Q}$ determines

$$H_{\alpha\overline{\beta}A} = T_{\alpha\overline{\beta}}{}^a = T_{\alpha\overline{\beta}}{}^\gamma = T_{\alpha\overline{\beta}}{}^{\overline{\gamma}} = \lambda^\alpha\lambda^\beta R_{\overline{\gamma}\alpha\beta}{}^\delta = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}}\widetilde{R}_{\gamma\overline{\alpha}\overline{\beta}}{}^{\overline{\delta}} = 0. \qquad (2.7)$$

Note that given the decomposition (2.4) for the connections, we can respectively write

$$R_{DC\alpha}{}^\beta = R_{DC}\delta_\alpha{}^\beta + \frac{1}{4}R_{DCef}(\gamma^{ef})_\alpha{}^\beta,$$
$$\widetilde{R}_{DC\overline{\alpha}}{}^{\overline{\beta}} = \widetilde{R}_{DC}\delta_{\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{4}\widetilde{R}_{DCef}(\gamma^{ef})_{\overline{\alpha}}{}^{\overline{\beta}}. \qquad (2.8)$$

2.2 Holomorphicity

The holomorphicity of j_B and the antiholomorphicity of $\widetilde{j}_B$ constraints are determined after the use of the equations of motion derived from the action (2.2). The equation for the pure spinor ghosts are

$$\overline{\nabla}\lambda^\alpha + \lambda^\beta(\widetilde{d}_{\overline{\gamma}}C_\beta{}^{\alpha\overline{\gamma}} + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}S_{\beta\overline{\alpha}}{}^{\alpha\overline{\beta}}) = 0, \quad \overline{\nabla}\omega_\alpha - (\widetilde{d}_{\overline{\gamma}}C_\alpha{}^{\beta\overline{\gamma}} + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}S_{\alpha\overline{\alpha}}{}^{\beta\overline{\beta}})\omega_\beta = 0, \qquad (2.9)$$

and

$$\nabla\widetilde{\lambda}^{\overline{\alpha}} + \widetilde{\lambda}^{\overline{\beta}}(d_\gamma\widetilde{C}_{\overline{\beta}}{}^{\overline{\alpha}\gamma} + \lambda^\alpha\omega_\beta S_{\alpha\overline{\beta}}{}^{\beta\overline{\alpha}}) = 0, \quad \nabla\widetilde{\omega}_{\overline{\alpha}} - (d_\gamma\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + \lambda^\alpha\omega_\beta S_{\alpha\overline{\alpha}}{}^{\beta\overline{\beta}})\widetilde{\omega}_{\overline{\beta}} = 0, \qquad (2.10)$$

where ∇ is a covariant derivative which acts with Ω or $\widetilde{\Omega}$ connections according to the index structure of the fields it is acting on. For example,

$$\nabla P^{\alpha\overline{\beta}} = \partial P^{\alpha\overline{\beta}} + P^{\gamma\overline{\beta}}\Omega_\gamma{}^\alpha + P^{\alpha\overline{\gamma}}\widetilde{\Omega}_{\overline{\gamma}}{}^{\overline{\beta}}.$$

The variations respect to d_α and $\widetilde{d}_{\overline{\alpha}}$ provide the equations

$$\overline{\Pi}^{\alpha}+\widetilde{d}_{\overline{\beta}}P^{\alpha\overline{\beta}}+\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\alpha}=0,\quad \Pi^{\overline{\alpha}}-d_\beta P^{\beta\overline{\alpha}}+\lambda^\alpha\omega_\beta C_\alpha{}^{\beta\overline{\alpha}}=0. \tag{2.11}$$

The most difficult equations to obtain are those coming from the variation of the superspace coordinates. Let us define $\sigma^A=\delta Z^M E_M{}^A$, then it is not difficult to obtain

$$\delta\Pi^A=\partial\sigma^A-\sigma^B\Pi^C E_B{}^M E_C{}^N\partial_{[N}E_{M]}{}^A(-1)^{C(B+M)}.$$

Here we can express this variation in terms of the connection Ω . In fact,

$$\delta\Pi^A=\nabla\sigma^A-\sigma^B\Pi^C(T_{CB}{}^A+\Omega_{BC}{}^A(-1)^{BC}).$$

There is a point about our notation for the torsion that we should make clear. Using tangent superspace indices, the torsion can be written as

$$T_{BC}{}^A=-E_B{}^N(\partial_N E_C{}^M)E_M{}^A+(-)^{BC}E_C{}^N(\partial_N E_B{}^M)E_M{}^A+\Omega_{BC}{}^A-(-)^{BC}\Omega_{CB}{}^A. \tag{2.12}$$

In our notation, $T_{BC}{}^\alpha$ will mean that the connection in (2.12) is $\Omega_{C\beta}{}^\alpha$ while $T_{BC}{}^{\overline{\alpha}}$ means that the connection in (2.12) is $\widetilde{\Omega}_{C\overline{\beta}}{}^{\overline{\alpha}}$. Since we also have two connections with bosonic tangent space index $\Omega_{Cb}{}^a$ and $\widetilde{\Omega}_{Cb}{}^a$, we use $T_{BC}{}^a$ to denote the torsion when we use the first and $\widetilde{T}_{BC}{}^a$ to denote the torsion when we use the second.

We vary the action (2.2) under these transformations and, after using the equations (2.10), (2.11) and some of the nilpotence constraints, we obtain

$$\begin{aligned}
\overline{\nabla}d_\alpha = & -\frac{1}{2}\Pi^a\overline{\Pi}^b(T_{\alpha(ba)}+H_{\alpha ba})+\frac{1}{2}\Pi^\beta\overline{\Pi}^a(T_{\beta\alpha a}-H_{\beta\alpha a})-d_\beta\overline{\Pi}^a T_{a\alpha}{}^\beta\\
&-\widetilde{d}_{\overline{\beta}}\Pi^a(T_{a\alpha}{}^{\overline{\beta}}+\frac{1}{2}P^{\gamma\overline{\beta}}(T_{\gamma\alpha a}+H_{\gamma\alpha a}))\\
&+\lambda^\beta\omega_\gamma\overline{\Pi}^a R_{a\alpha\beta}{}^\gamma+\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\Pi^a(\widetilde{R}_{a\alpha\overline{\beta}}{}^{\overline{\gamma}}-\frac{1}{2}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta}(T_{\delta\alpha a}+H_{\delta\alpha a}))\\
&-\widetilde{d}_{\overline{\beta}}\Pi^\gamma(T_{\gamma\alpha}{}^{\overline{\beta}}+\frac{1}{2}P^{\delta\overline{\beta}}H_{\delta\gamma\alpha})+\lambda^\beta\omega_\gamma\overline{\Pi}^{\overline{\delta}}R_{\overline{\delta}\alpha\beta}{}^\gamma+\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\Pi^\delta(\widetilde{R}_{\delta\alpha\overline{\beta}}{}^{\overline{\gamma}}+\frac{1}{2}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\rho}H_{\rho\delta\alpha})\\
&+d_\beta\widetilde{d}_{\overline{\gamma}}(P^{\delta\overline{\gamma}}T_{\delta\alpha}{}^\beta-\nabla_\alpha P^{\beta\overline{\gamma}})+\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}d_\delta(\nabla_\alpha\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta}+\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\rho}T_{\rho\alpha}{}^\delta+P^{\delta\overline{\rho}}\widetilde{R}_{\overline{\rho}\alpha\overline{\beta}}{}^{\overline{\gamma}})\\
&+\lambda^\beta\omega_\gamma\widetilde{d}_{\overline{\delta}}(\nabla_\alpha C_\beta{}^{\gamma\overline{\delta}}-P^{\rho\overline{\delta}}R_{\rho\alpha\beta}{}^\gamma)-\lambda^\beta\omega_\gamma\widetilde{\lambda}^{\overline{\delta}}\widetilde{\omega}_{\overline{\rho}}(\nabla_\alpha S_{\beta\overline{\delta}}{}^{\gamma\overline{\rho}}+C_\beta{}^{\gamma\overline{\sigma}}\widetilde{R}_{\overline{\sigma}\alpha\overline{\delta}}{}^{\overline{\rho}}+\widetilde{C}_{\overline{\delta}}{}^{\overline{\rho}\sigma}R_{\sigma\alpha\beta}{}^\gamma),
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
\nabla\widetilde{d}_{\overline{\alpha}} = & -\frac{1}{2}\Pi^a\overline{\Pi}^b(T_{\overline{\alpha}(ba)}+H_{\overline{\alpha}ba})+\frac{1}{2}\Pi^a\overline{\Pi}^{\overline{\beta}}(T_{\overline{\beta}\overline{\alpha}a}+H_{\overline{\beta}\overline{\alpha}a})-\widetilde{d}_{\overline{\beta}}\Pi^a T_{a\overline{\alpha}}{}^{\overline{\beta}}\\
&-d_\beta\overline{\Pi}^a(T_{a\overline{\alpha}}{}^\beta-\frac{1}{2}P^{\beta\overline{\gamma}}(T_{\overline{\gamma}\overline{\alpha}a}-H_{\overline{\gamma}\overline{\alpha}a}))\\
&+\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\Pi^a\widetilde{R}_{a\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}+\lambda^\beta\omega_\gamma\overline{\Pi}^a(R_{a\overline{\alpha}\beta}{}^\gamma-\frac{1}{2}C_\beta{}^{\gamma\overline{\delta}}(T_{\overline{\delta}\overline{\alpha}a}-H_{\overline{\delta}\overline{\alpha}a}))\\
&-d_\beta\overline{\Pi}^{\overline{\gamma}}(T_{\overline{\gamma}\overline{\alpha}}{}^\beta+\frac{1}{2}P^{\beta\overline{\delta}}H_{\overline{\gamma}\overline{\delta}\overline{\alpha}})+\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\Pi^\delta\widetilde{R}_{\delta\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}+\lambda^\beta\omega_\gamma\overline{\Pi}^{\overline{\delta}}(R_{\overline{\delta}\overline{\alpha}\beta}{}^\gamma+\frac{1}{2}C_\beta{}^{\gamma\overline{\rho}}H_{\overline{\delta}\overline{\rho}\overline{\alpha}})\\
&+d_\beta\widetilde{d}_{\overline{\gamma}}(P^{\beta\overline{\delta}}T_{\overline{\delta}\overline{\alpha}}{}^{\overline{\gamma}}-\nabla_{\overline{\alpha}}P^{\beta\overline{\gamma}})+\lambda^\beta\omega_\gamma\widetilde{d}_{\overline{\delta}}(\nabla_{\overline{\alpha}}C_\beta{}^{\gamma\overline{\delta}}+C_\beta{}^{\gamma\overline{\rho}}\widetilde{T}_{\overline{\rho}\overline{\alpha}}{}^{\overline{\delta}}-P^{\rho\overline{\delta}}R_{\rho\overline{\alpha}\beta}{}^\gamma)\\
&+\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}d_\delta(\nabla_{\overline{\alpha}}\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta}+P^{\delta\overline{\rho}}\widetilde{R}_{\overline{\rho}\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}})-\lambda^\beta\omega_\gamma\widetilde{\lambda}^{\overline{\delta}}\widetilde{\omega}_{\overline{\rho}}(\nabla_{\overline{\alpha}}S_{\beta\overline{\delta}}{}^{\gamma\overline{\rho}}+C_\beta{}^{\gamma\overline{\sigma}}\widetilde{R}_{\overline{\sigma}\overline{\alpha}\overline{\delta}}{}^{\overline{\rho}}+\widetilde{C}_{\overline{\delta}}{}^{\overline{\rho}\sigma}R_{\sigma\overline{\alpha}\beta}{}^\gamma).
\end{aligned} \tag{2.14}$$

From these equations, (2.9), (2.10) and also two equations in (2.7) we obtain the holomorphicity constraints. In fact, $\overline{\nabla} j_B = 0$ implies

$$
\begin{aligned}
T_{\alpha(ab)} &= H_{\alpha ab} = T_{\alpha\beta a} - H_{\alpha\beta a} = T_{a\alpha}{}^{\beta} \\
&= T_{a\alpha}{}^{\overline{\beta}} + P^{\gamma\overline{\beta}} T_{\gamma\alpha a} = \lambda^{\alpha}\lambda^{\beta} R_{a\alpha\beta}{}^{\gamma} = 0, \quad (2.15) \\
\widetilde{R}_{a\alpha\overline{\beta}}{}^{\overline{\gamma}} - \widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta} T_{\delta\alpha a} &= T_{\gamma\alpha}{}^{\overline{\beta}} + \frac{1}{2} P^{\delta\overline{\beta}} H_{\delta\gamma\alpha} = \widetilde{R}_{\delta\alpha\overline{\beta}}{}^{\overline{\gamma}} + \frac{1}{2} \widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\rho} H_{\rho\delta\alpha} \\
&= P^{\delta\overline{\gamma}} T_{\delta\alpha}{}^{\beta} - \nabla_{\alpha} P^{\beta\overline{\gamma}} - C_{\alpha}{}^{\beta\overline{\gamma}} = 0, \\
\nabla_{\alpha} \widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta} + \widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\rho} T_{\rho\alpha}{}^{\delta} + P^{\delta\overline{\rho}} \widetilde{R}_{\overline{\rho}\alpha\overline{\beta}}{}^{\overline{\gamma}} - S_{\alpha\overline{\beta}}{}^{\delta\overline{\gamma}} &= \lambda^{\alpha}\lambda^{\beta} (\nabla_{\alpha} C_{\beta}{}^{\gamma\overline{\delta}} - P^{\rho\overline{\delta}} R_{\rho\alpha\beta}{}^{\gamma}) = 0, \\
\lambda^{\alpha}\lambda^{\beta} (\nabla_{\alpha} S_{\beta\overline{\delta}}{}^{\gamma\overline{\rho}} + C_{\beta}{}^{\gamma\overline{\sigma}} \widetilde{R}_{\overline{\sigma}\alpha\overline{\delta}}{}^{\overline{\rho}} + \widetilde{C}_{\overline{\delta}}{}^{\overline{\rho}\sigma} R_{\sigma\alpha\beta}{}^{\gamma}) &= 0, \qquad (2.16)
\end{aligned}
$$

and $\nabla \widetilde{j}_B = 0$ implies

$$
\begin{aligned}
& T_{\overline{\alpha}(ab)} = H_{\overline{\alpha}ab} = T_{\overline{\alpha}\overline{\beta}a} + H_{\overline{\alpha}\overline{\beta}a} = T_{a\overline{\alpha}}{}^{\overline{\beta}} = T_{a\overline{\alpha}}{}^{\beta} - P^{\beta\overline{\gamma}} T_{\overline{\gamma}\overline{\alpha}a} = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}} \widetilde{R}_{a\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} = 0, \\
& R_{a\overline{\alpha}\beta}{}^{\gamma} - C_{\beta}{}^{\gamma\overline{\delta}} T_{\overline{\delta}\overline{\alpha}a} = T_{\overline{\gamma}\overline{\alpha}}{}^{\beta} + \frac{1}{2} P^{\beta\overline{\delta}} H_{\overline{\gamma}\overline{\delta}\overline{\alpha}} = R_{\overline{\delta}\overline{\alpha}\beta}{}^{\gamma} + \frac{1}{2} C_{\beta}{}^{\gamma\overline{\rho}} H_{\overline{\delta}\overline{\rho}\overline{\alpha}} \\
& = P^{\beta\overline{\delta}} T_{\overline{\delta}\overline{\alpha}}{}^{\overline{\gamma}} - \nabla_{\overline{\alpha}} P^{\beta\overline{\gamma}} + \widetilde{C}_{\overline{\alpha}}{}^{\overline{\gamma}\beta} = 0, \\
& \nabla_{\overline{\alpha}} C_{\beta}{}^{\gamma\overline{\delta}} + C_{\beta}{}^{\gamma\overline{\rho}} T_{\overline{\rho}\overline{\alpha}}{}^{\overline{\delta}} - P^{\rho\overline{\delta}} R_{\rho\overline{\alpha}\beta}{}^{\gamma} - S_{\beta\overline{\alpha}}{}^{\gamma\overline{\delta}} = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}} (\nabla_{\overline{\alpha}} \widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\delta} + P^{\delta\overline{\rho}} \widetilde{R}_{\overline{\rho}\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}) = 0, \\
& \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\lambda}^{\overline{\beta}} (\nabla_{\overline{\alpha}} S_{\delta\overline{\beta}}{}^{\rho\overline{\gamma}} + C_{\delta}{}^{\rho\overline{\sigma}} \widetilde{R}_{\overline{\sigma}\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} + \widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\sigma} R_{\sigma\overline{\alpha}\delta}{}^{\rho}) = 0. \qquad (2.17)
\end{aligned}
$$

2.3 Solving the Bianchi identities

We can gauge-fix some of the torsion components and determine others through the use of Bianchi identities. It is not necessary but it will simplify the computation of the one-loop beta functions. As in [11], we can set $H_{\alpha\beta\gamma} = H_{\alpha\beta\overline{\gamma}} = H_{\alpha\overline{\beta}\overline{\gamma}} = H_{\overline{\alpha}\overline{\beta}\overline{\gamma}} = 0$ since there is no such ten-dimensional superfields satisfying the constrains of (2.15) and (2.17). We can use the Lorentz rotations to gauge fix $T_{\alpha\beta}{}^{a} = \gamma^{a}_{\alpha\beta}$ and $T_{\overline{\alpha}\overline{\beta}}{}^{a} = \gamma^{a}_{\overline{\alpha}\overline{\beta}}$, therefore the above constraints imply $H_{\alpha\beta a} = (\gamma_a)_{\alpha\beta}$ and $H_{\overline{\alpha}\overline{\beta}a} = -(\gamma_a)_{\overline{\alpha}\overline{\beta}}$. We can use the shift symmetry of the action (2.2)

$$
\begin{aligned}
\delta d_{\alpha} &= \delta\Omega_{\alpha\beta}{}^{\gamma}\lambda^{\beta}\omega_{\gamma}, \quad \delta\widetilde{d}_{\overline{\alpha}} = \delta\widetilde{\Omega}_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}\widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}, \quad \delta C_{\alpha}{}^{\beta\overline{\gamma}} = P^{\delta\overline{\gamma}}\delta\Omega_{\delta\alpha}{}^{\beta}, \quad \delta\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} = -P^{\gamma\overline{\delta}}\delta\widetilde{\Omega}_{\overline{\delta}\overline{\alpha}}{}^{\overline{\beta}}, \\
\delta S_{\alpha\overline{\beta}}{}^{\gamma\overline{\delta}} &= C_{\alpha}{}^{\gamma\overline{\rho}}\delta\widetilde{\Omega}_{\overline{\rho}\overline{\beta}}{}^{\overline{\delta}} + \widetilde{C}_{\overline{\beta}}{}^{\overline{\delta}\rho}\delta\Omega_{\rho\alpha}{}^{\gamma},
\end{aligned}
$$

to gauge-fix $T_{\alpha\beta}{}^{\gamma} = T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} = 0$.

The Bianchi identity for the torsion is

$$
(\nabla T)_{ABC}{}^{D} \equiv \nabla_{[A} T_{BC]}{}^{D} + T_{[AB}{}^{E} T_{EC]}{}^{D} - R_{[ABC]}{}^{D} = 0, \qquad (2.18)
$$

where brackets in (2.18) mean (anti-)symmetrization respect to the ABC indices. The curvature will be R or $\widetilde{R}$ if the upper index D is δ or $\overline{\delta}$ respectively. When $D = d$, we use the notation $(\nabla T)_{ABC}{}^{d}$ or $(\nabla\widetilde{T})_{ABC}{}^{d}$, if we use the connection $\Omega_{Bc}{}^{a}$ or $\widetilde{\Omega}_{Bc}{}^{a}$; then the curvatures in each case will be R or $\widetilde{R}$.

The Bianchi identity $(\nabla T)_{\alpha\beta\gamma}{}^{a} = 0$ implies $T_{\alpha ab} = 2(\gamma_{ab})_{\alpha}{}^{\beta}\Omega_{\beta}$. Similarly, the Bianchi identity $(\nabla\widetilde{T})_{\overline{\alpha}\overline{\beta}\overline{\gamma}}{}^{a} = 0$ implies $\widetilde{T}_{\overline{\alpha}ab} = 2(\gamma_{ab})_{\overline{\alpha}}{}^{\overline{\beta}}\widetilde{\Omega}_{\overline{\beta}}$. The Bianchi identity $(\nabla T)_{\alpha\overline{\beta}\overline{\gamma}}{}^{a} = 0$

implies $\widetilde{\Omega}_\alpha = \widetilde{T}_{\alpha a}{}^b = 0$. Similarly, the Bianchi identity $(\nabla T)_{\overline{\alpha}\beta\gamma}{}^a = 0$ implies $\Omega_{\overline{\alpha}} = T_{\overline{\alpha}a}{}^b = 0$. It is not difficult to show that the constraints $T_{a\alpha}{}^\alpha = T_{a\overline{\alpha}}{}^{\overline{\alpha}} = 0$ imply $\Omega_a = \widetilde{\Omega}_a = 0$.

We can write two sets of Bianchi identities for H depending on what is the connection we choose in the covariant derivative. Note that the components of the superfield H do not depend on such choice. The Bianchi identities come from $\nabla H = 0$ and $\widetilde{\nabla} H = 0$ and it is not difficult to check that both sets are equivalent. Let us write only one of them

$$(\nabla H)_{ABCD} \equiv \nabla_{[A} H_{BCD]} + \frac{3}{2} T_{[AB}{}^E H_{ECD]} = 0. \tag{2.19}$$

There is one more Bianchi identity involving a derivative of the curvature

$$(\nabla R)_{ABCD}{}^E \equiv \nabla R_{[ABC]D}{}^E + T_{[AB}{}^F R_{FC]D}{}^E = 0. \tag{2.20}$$

The identities $(\nabla H)_{\alpha\beta\gamma\delta}, (\nabla H)_{\alpha\beta\gamma\overline{\delta}}, (\nabla H)_{\alpha\beta\overline{\gamma}\overline{\delta}}, (\nabla H)_{\alpha\overline{\beta}\overline{\gamma}\overline{\delta}}, (\nabla H)_{\overline{\alpha}\overline{\beta}\overline{\gamma}\overline{\delta}}$ are easily satisfied if we recall the identities for gamma matrices $\gamma^a_{(\alpha\beta}(\gamma_a)_{\gamma)\delta} = \gamma^a_{(\overline{\alpha}\overline{\beta}}(\gamma_a)_{\overline{\gamma})\overline{\delta}} = 0$. The identities $(\nabla H)_{a\alpha\beta\gamma}, (\nabla H)_{a\alpha\beta\overline{\gamma}}, (\nabla H)_{a\alpha\overline{\beta}\overline{\gamma}}, (\nabla H)_{a\overline{\alpha}\beta\overline{\gamma}}$ are satisfied after using the dimension-$\frac{1}{2}$ constraints. The identity $(\nabla H)_{ab\alpha\beta} = 0$ implies $T_{abc} + H_{abc} = 0$ and the identity $(\widetilde{\nabla} H)_{ab\overline{\alpha}\overline{\beta}} = 0$ implies $\widetilde{T}_{abc} - H_{abc} = 0$. The identity $(\nabla H)_{ab\alpha\overline{\beta}} = 0$ is satisfied if we use the constraints involving the superfield $P^{\alpha\overline{\beta}}$ in the first lines of (2.15) and (2.17).

2.4 The remaining equation of motion

In the computation of the one-loop beta function we will need to know the equation of motion for Π^a and $\overline{\Pi}^a$. Since we know that the difference $\nabla\overline{\Pi}^a - \overline{\nabla}\Pi^a$ is given by the torsion components, then we only need to determine $\nabla\overline{\Pi}^a + \overline{\nabla}\Pi^a$ which is determined by the varying the action respect to $\sigma^a = \delta Z^M E_M{}^a$. To make life simpler we will write this equation using the above results for torsion and H components. The equation turns out to be

$$\begin{aligned}
&\frac{1}{2}(\widetilde{\nabla}\overline{\Pi}_a + \overline{\nabla}\Pi_a) = \frac{1}{2}\Pi^b\overline{\Pi}^c H_{cba} - \frac{1}{2}\Pi^\alpha\overline{\Pi}^b T_{\alpha ab} + d_\alpha \overline{\Pi}^b T_{ab}{}^\alpha + \lambda^\alpha\omega_\beta\overline{\Pi}^b R_{ab\alpha}{}^\beta \\
&+\widetilde{d}_{\overline{\alpha}}\Pi^b(T_{ab}{}^{\overline{\alpha}} + \frac{1}{2}P^{\beta\overline{\alpha}}T_{\beta ab}) + \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\Pi^b(\widetilde{R}_{ab\overline{\alpha}}{}^{\overline{\beta}} + \frac{1}{2}\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}T_{\gamma ab}) \\
&+\frac{1}{2}\widetilde{d}_{\overline{\alpha}}\Pi^\beta T_{a\beta}{}^{\overline{\alpha}} + \frac{1}{2}\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\Pi^\gamma \widetilde{R}_{a\gamma\overline{\alpha}}{}^{\overline{\beta}} \\
&+\frac{1}{2}d_\alpha\overline{\Pi}^{\overline{\beta}}T_{a\overline{\beta}}{}^\alpha + \frac{1}{2}\lambda^\alpha\omega_\beta\overline{\Pi}^{\overline{\gamma}}R_{a\overline{\gamma}\alpha}{}^\beta + d_\alpha\widetilde{d}_{\overline{\beta}}\nabla_a P^{\alpha\overline{\beta}} + \lambda^\alpha\omega_\beta\widetilde{d}_{\overline{\gamma}}(\nabla_a C_\alpha{}^{\beta\overline{\gamma}} - P^{\delta\overline{\gamma}}R_{a\delta\alpha}{}^\beta) \\
&+\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}d_\gamma(\nabla_a\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + P^{\gamma\overline{\delta}}\widetilde{R}_{a\overline{\delta}\overline{\alpha}}{}^{\overline{\beta}}) + \lambda^\alpha\omega_\beta\widetilde{\lambda}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}(\nabla_a S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}} - \widetilde{C}_{\overline{\gamma}}{}^{\overline{\delta}\rho}R_{a\rho\alpha}{}^\beta - C_\alpha{}^{\beta\overline{\rho}}\widetilde{R}_{a\overline{\rho}\overline{\gamma}}{}^{\overline{\delta}})
\end{aligned} \tag{2.21}$$

2.5 Ghost number conservation

As it was shown in [11], the vanishing of the ghost number anomaly determines that the spinorial derivatives of the dilaton superfield Φ are proportional to the connection Ω. This relation is crucial to cancel the beta function in heterotic string case [12] and will be equally essential in our computation. Let us recall how this relation is obtained. Consider the coupling between ghost number currents and the connections in the action (2.2). Namely

$$\frac{1}{2\pi\alpha'}\int d^2z\ (J\overline{\Omega} + \widetilde{J}\Omega).$$

The BRST variation on this term contains the term

$$-\frac{1}{2\pi\alpha'}\int d^2z\;(\overline{\partial}J\lambda^\alpha\Omega_\alpha+\partial\widetilde{J}\lambda^\alpha\widetilde{\Omega}_\alpha).$$

The anomaly in the ghost number current conservation turns out to be proportional to the two dimensional Ricci scalar, as noted by dimensional grounds. The proportionality can be determined by performing a Weyl transformation, around the flat world-sheet, of the anomaly equation. In this way, the triple-pole in the OPE between the current and the corresponding stress tensor yields

$$\nabla_\alpha\Phi=4\Omega_\alpha,\quad \nabla_{\overline{\alpha}}\Phi=4\widetilde{\Omega}_{\overline{\alpha}}, \tag{2.22}$$

which will be used in section 5 to cancel the UV divergent part of the effective action.

3. Covariant background field expansion

We use the method explained in [18] and [12]. Here, we need to define a straight-line geodesic which joins a point in superspace to neighbor ones and allows us to perform an expansion in superspace. It is given by Y^A which satisfies the geodesic equation $\Delta Y^A = Y^B\nabla_B Y^A=0$. The connection we choose to define this covariant derivative has the non-vanishing components $\Omega_{Aa}{}^b, \Omega_{A\alpha}{}^\beta$ and $\widetilde{\Omega}_{A\overline{\alpha}}{}^{\overline{\beta}}$. These same connections are defined in the action (2.2). In this way, the covariant expansions of the different objects in (2.2) are determined by

$$\Delta\Pi^A=\nabla Y^A-Y^B\Pi^C T_{CB}{}^A,\quad \Delta\overline{\Omega}_\alpha{}^\beta=-Y^A\overline{\Pi}^B R_{BA\alpha}{}^\beta,\quad \Delta\widetilde{\Omega}_{\overline{\alpha}}{}^{\overline{\beta}}=-Y^A\Pi^B\widetilde{R}_{BA\overline{\alpha}}{}^{\overline{\beta}}. \tag{3.1}$$

Any superfield Ψ is expanded as $\Delta\Psi=Y^A\nabla_A\Psi$.

As in [12], we see that $d_\alpha,\widetilde{d_{\overline{\alpha}}}$ and the pure spinor ghosts are treated as fundamental fields, then we expand them according to

$$\begin{aligned}
&d_\alpha=d_{\alpha 0}+\widehat{d}_\alpha,\quad \lambda^\alpha=\lambda_0^\alpha+\widehat{\lambda}^\alpha,\quad \omega_\alpha=\omega_{\alpha 0}+\widehat{\omega}_\alpha,\\
&\widetilde{d}_{\overline{\alpha}}=\widetilde{d}_{\overline{\alpha}0}+\widehat{\widetilde{d}}_{\overline{\alpha}},\quad \widetilde{\lambda}^{\overline{\alpha}}=\widetilde{\lambda}_0^{\overline{\alpha}}+\widehat{\widetilde{\lambda}}^{\overline{\alpha}},\quad \widetilde{\omega}_{\overline{\alpha}}=\widetilde{\omega}_{\overline{\alpha}0}+\widehat{\widetilde{\omega}}_{\overline{\alpha}},
\end{aligned} \tag{3.2}$$

where the subindex 0 means the background value of the corresponding field which will dropped in the subsequent discussion.

The quadratic part of the expansion of (2.2), excluding the Fradkin-Tseytlin term, has the form

$$\begin{aligned}
S_2=S_p+\frac{1}{2\pi\alpha'}\int d^2z\;&(Y^AY^BE_{BA}+Y^A\overline{\nabla}Y^BC_{BA}+Y^A\nabla Y^B\overline{C}_{BA}+\widehat{d}_\alpha Y^A\overline{D}_A{}^\alpha\\
&+\widehat{\widetilde{d}}_{\overline{\alpha}}Y^AD_A{}^{\overline{\alpha}}+(\widehat{\lambda}^\alpha\widehat{\omega}_\beta)\overline{H}_\alpha{}^\beta+(\widehat{\widetilde{\lambda}}^{\overline{\alpha}}\widehat{\widetilde{\omega}}_{\overline{\beta}})H_{\overline{\alpha}}{}^{\overline{\beta}}+(\widehat{\lambda}^\alpha\omega_\beta+\lambda^\alpha\widehat{\omega}_\beta)Y^A\overline{I}_{A\alpha}{}^\beta\\
&+(\widehat{\widetilde{\lambda}}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}+\widetilde{\lambda}^{\overline{\alpha}}\widehat{\widetilde{\omega}}_{\overline{\beta}})Y^AI_{A\overline{\alpha}}{}^{\overline{\beta}}+\widehat{d}_\alpha\widehat{\widetilde{d}}_{\overline{\beta}}P^{\alpha\overline{\beta}}+(\widehat{\lambda}^\alpha\omega_\beta+\lambda^\alpha\widehat{\omega}_\beta)\widehat{\widetilde{d}}_{\overline{\gamma}}C_\alpha{}^{\beta\overline{\gamma}}\\
&+(\widehat{\widetilde{\lambda}}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}+\widetilde{\lambda}^{\overline{\alpha}}\widehat{\widetilde{\omega}}_{\overline{\beta}})\widehat{d}_\gamma\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}+(\widehat{\lambda}^\alpha\omega_\beta+\lambda^\alpha\widehat{\omega}_\beta)(\widehat{\widetilde{\lambda}}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}+\widetilde{\lambda}^{\overline{\gamma}}\widehat{\widetilde{\omega}}_{\overline{\delta}})S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}}),
\end{aligned} \tag{3.3}$$

where $E_{BA}, C_{BA}, \ldots$ are background superfields given by

$$
\begin{aligned}
E_{BA} = {} & \frac{1}{4}\Pi^C\overline{\Pi}^D(T_{CB}{}^E H_{EDA}(-1)^{D(C+B)} - T_{DB}{}^E H_{ECA}(-1)^{BC} \\
& +\nabla_B H_{DCA}(-1)^{B(C+D)} + 2T_{CB}{}^a T_{DAa}(-1)^{D(C+B)}) \\
& -\frac{1}{4}\Pi^{(a}\overline{\Pi}^{C)}(R_{CBAa} - T_{CB}{}^D T_{DAa} + \nabla_B T_{CAa}(-1)^{BC}) \\
& +\frac{1}{2}d_\alpha\overline{\Pi}^C(-1)^{A+B}(-R_{CBA}{}^\alpha + T_{CB}{}^D T_{DA}{}^\alpha - \nabla_B T_{CA}{}^\alpha(-1)^{BC}) \\
& +\frac{1}{2}\widetilde{d}_{\overline{\alpha}}\Pi^C(-1)^{A+B}(-R_{CBA}{}^{\overline{\alpha}} + T_{CB}{}^D T_{DA}{}^{\overline{\alpha}} - \nabla_B T_{CA}{}^{\overline{\alpha}}(-1)^{BC}) \\
& +\frac{1}{2}\lambda^\alpha\omega_\beta\overline{\Pi}^C(T_{CB}{}^D R_{DA\alpha}{}^\beta - \nabla_B R_{CA\alpha}{}^\beta(-1)^{BC}) \\
& +\frac{1}{2}\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\Pi^C(T_{CB}{}^D R_{DA\overline{\alpha}}{}^{\overline{\beta}} - \nabla_B R_{CA\overline{\alpha}}{}^{\overline{\beta}}(-1)^{BC}) + \frac{1}{2}d_\alpha\widetilde{d}_{\overline{\beta}}\nabla_B\nabla_A P^{\alpha\overline{\beta}} \\
& +\frac{1}{2}\lambda^\alpha\omega_\beta\widetilde{d}_{\overline{\gamma}}\nabla_B\nabla_A C_\alpha{}^{\beta\overline{\gamma}}(-1)^{A+B} + \frac{1}{2}\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}d_\gamma\nabla_B\nabla_A\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}(-1)^{A+B} \\
& +\frac{1}{2}\lambda^\alpha\omega_\beta\widetilde{\lambda}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}\nabla_B\nabla_A S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}},
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
C_{BA} = {} & -\frac{1}{4}\Pi^a T_{BAa} - \frac{1}{2}\Pi^C T_{CAa}\delta^a_B - \frac{1}{4}\Pi^A H_{CBA} - \frac{1}{2}d_\alpha T_{BA}{}^\alpha(-1)^{A+B} \\
& -\frac{1}{2}\lambda^\alpha\omega_\beta R_{BA\alpha}{}^\beta,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
\overline{C}_{BA} = {} & -\frac{1}{4}\Pi^b T_{BAa} - \frac{1}{2}\overline{\Pi}^C T_{CAa}\delta^a_B + \frac{1}{4}\Pi^A H_{CBA} - \frac{1}{2}\widetilde{d}_{\overline{\alpha}}\widetilde{T}_{BA}{}^{\overline{\alpha}}(-1)^{A+B} \\
& -\frac{1}{2}\widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}}\widetilde{R}_{BA\overline{\alpha}}{}^{\overline{\beta}},
\end{aligned} \tag{3.6}
$$

$$\overline{D}_A{}^\alpha = -\overline{\Pi}^B T_{BA}{}^\alpha + \widetilde{d}_{\overline{\beta}}\nabla_A P^{\alpha\overline{\beta}}(-1)^A + \widetilde{\lambda}^{\overline{\beta}}\widetilde{\omega}_{\overline{\gamma}}\nabla_A\widetilde{C}_{\overline{\beta}}{}^{\overline{\gamma}\alpha}, \tag{3.7}$$

$$D_A{}^{\overline{\alpha}} = -\Pi^B\widetilde{T}_{BA}{}^{\overline{\alpha}} - d_\beta\nabla_A P^{\beta\overline{\alpha}}(-1)^A + \lambda^\beta\omega_\gamma\nabla_A C_\beta{}^{\gamma\overline{\alpha}}, \tag{3.8}$$

$$\overline{H}_\alpha{}^\beta = \overline{\Omega}_\alpha{}^\beta + \widetilde{d}_{\overline{\gamma}}C_\alpha{}^{\beta\overline{\gamma}}\widetilde{\lambda}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}}, \tag{3.9}$$

$$H_{\overline{\alpha}}{}^{\overline{\beta}} = \widetilde{\Omega}_{\overline{\alpha}}{}^{\overline{\beta}} + d_\gamma\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma} + \lambda^\gamma\omega_\delta S_{\gamma\overline{\alpha}}{}^{\delta\overline{\beta}}, \tag{3.10}$$

$$\overline{I}_{A\alpha}{}^\beta = -\overline{\Pi}^A R_{BA\alpha}{}^\beta + \widetilde{d}_{\overline{\gamma}}\nabla_A C_\alpha{}^{\beta\overline{\gamma}}(-1)^A + \widetilde{\lambda}^{\overline{\gamma}}\widetilde{\omega}_{\overline{\delta}}\nabla_A S_{\alpha\overline{\gamma}}{}^{\beta\overline{\gamma}}, \tag{3.11}$$

$$I_{A\overline{\alpha}}{}^{\overline{\beta}} = -\Pi^B\widetilde{R}_{BA\overline{\alpha}}{}^{\overline{\beta}} + d_\gamma\nabla_A\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}(-1)^A + \lambda^\gamma\omega_\delta\nabla_A S_{\gamma\overline{\alpha}}{}^{\delta\overline{\beta}}. \tag{3.12}$$

In (3.3) S_p provides the propagators for the quantum fields and is given by

$$S_p = \frac{1}{2\pi\alpha'}\int d^2z\ (\frac{1}{2}\nabla Y^a\overline{\nabla}Y_a + \widehat{d}_\alpha\overline{\nabla}Y^\alpha + \widehat{\widetilde{d}}_{\overline{\alpha}}\nabla Y^{\overline{\alpha}}) + \mathcal{L}_{pure}, \tag{3.13}$$

where $\mathcal{L}_{pure}$ is the Lagrangian for the pure spinor ghosts.

4. The one-loop UV divergent part of the effective action

The effective action is given by

$$e^{-S_{eff}} = \int DQ\ e^{-S}, \tag{4.1}$$

where $\mathcal{Q}$ represents the quantum fluctuations.

To compute the one-loop beta functions we need to expand (2.2) up to second order in the quantum fields. In this way, we will obtain the UV divergent part of the effective action, S_Λ. Here Λ is UV scale. Note that the Fradkin-Tseytlin term is evaluated on a sphere with metric $\Lambda dz d\bar{z}$. Finally, the complete UV divergent part of the effective action becomes

$$S_\Lambda + \frac{1}{2\pi}\int d^2z\ (\nabla\overline{\Pi}^A\nabla_A\Phi + \overline{\Pi}^A\Pi^B\nabla_B\nabla_A\Phi)\log\Lambda. \tag{4.2}$$

The computation of S_Λ is performed by contracting the quantum fields. From (3.13) we read

$$\begin{aligned} &Y^a(z,\bar{z})Y^b(w,\bar{w}) \to -\alpha'\eta^{ab}\log|z-w|^2\,, \\ &\widehat{d}_\alpha(z)Y^\beta(w) \to \frac{\alpha'\delta_\alpha{}^\beta}{(z-w)}, \quad \widehat{\overline{d}}_{\overline{\alpha}}(\bar{z})Y^{\overline{\beta}}(\bar{w}) \to \frac{\alpha'\delta_{\overline{\alpha}}^{\overline{\beta}}}{(\bar{z}-\bar{w})}. \end{aligned} \tag{4.3}$$

For the pure spinor ghosts we note that, because of (2.4), they enter in the combinations

$$N^{ab} = \frac{1}{2}(\lambda\gamma^{ab}\omega), \quad J = \lambda^\alpha\omega_\alpha, \quad \widetilde{N}^{ab} = \frac{1}{2}(\widetilde{\lambda}\gamma^{ab}\widetilde{\omega}), \quad \widetilde{J} = \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\alpha}}.$$

We can expand each of these combinations as $J + J_1 + J_2$, similarly for $\widetilde{J}$, N^{ab} and $\widetilde{N}^{ab}$. As in [12], the only relevant OPE's involving the pure spinor ghosts and contributing to S_Λ are

$$N_1^{ab}(z)N_1^{cd}(w) \to \frac{1}{(z-w)}(-\eta^{a[c}N^{d]b}(w) + \eta^{b[c}N^{d]a}(w)), \tag{4.4}$$

$$\widetilde{N}_1^{ab}(\bar{z})\widetilde{N}_1^{cd}(\bar{w}) \to \frac{1}{(\bar{z}-\bar{w})}(-\eta^{a[c}\widetilde{N}^{d]b}(\bar{w}) + \eta^{b[c}\widetilde{N}^{d]a}(\bar{w})). \tag{4.5}$$

The one-loop contributions to S_Λ come from self-contraction of Y^A's in the term with E_{BA} in (3.3) and a series of double contractions in (3.3). These come from products between the term involving C_{BA} with the one involving $\overline{C}_{BA}$, C_{BA} with $\overline{D}_A{}^{\overline{\beta}}$, $\overline{C}_{BA}$ with $D_A{}^\beta$, $\overline{D}_A{}^{\overline{\beta}}$ with $D_A{}^\beta$, E_{BA} with $P^{\alpha\overline{\beta}}$, $\overline{I}_{C\alpha}{}^\beta$ with $C_\alpha{}^{\beta\overline{\gamma}}$, $I_{C\overline{\alpha}}{}^{\overline{\beta}}$ with $\widetilde{C}_{\overline{\alpha}}{}^{\overline{\beta}\gamma}$ and $S_{\alpha\overline{\gamma}}{}^{\beta\overline{\delta}}$ with itself. After adding up all these contributions, the one-loop UV divergent part of the effective action is proportional to

$$\begin{aligned} &\int d^2z[-\eta^{ab}E_{ba} + \eta^{a[c}\eta^{d]b}C_{ba}\overline{C}_{dc} + \eta^{ab}C_{[a\alpha]}\overline{D}_b{}^\alpha + \eta^{ab}\overline{C}_{[a\overline{\alpha}]}D_b{}^{\overline{\alpha}} + \overline{D}_{\overline{\alpha}}{}^\beta D_\beta{}^{\overline{\alpha}} + E_{[\alpha\overline{\beta}]}P^{\alpha\overline{\beta}} \\ &+N^{ab}\overline{I}_{\overline{\alpha}a}{}^cC_{cb}{}^{\overline{\alpha}} + \widetilde{N}^{ab}I_{\alpha a}{}^c\widetilde{C}_{cb}{}^\alpha + \frac{1}{2}N^{ab}\widetilde{N}^{cd}S_a{}^e{}_c{}^fS_{bedf} + \nabla\overline{\Pi}^A\nabla_A\Phi + \overline{\Pi}^A\Pi^B\nabla_B\nabla_A\Phi]\log\Lambda, \end{aligned} \tag{4.6}$$

where we used the expressions (2.4).

Now it will be shown that (4.6) vanishes as consequence of the classical BRST constraints.

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5. One-loop conformal invariance

To write the equations derived from the vanishing of (4.6), we need to determine $\nabla\overline{\Pi}^A$ from the classical equations of motion from (2.2). In order to do this, we need to know

$$\overline{\nabla}\Pi^A - \nabla\overline{\Pi}^A = \Pi^B\overline{\Pi}^C T_{CB}{}^A. \tag{5.1}$$

Note that we are using here the connection $\Omega_A{}^B$ to calculate the covariant derivatives and the torsion components.

The equation for $\nabla\overline{\Pi}_a$ is

$$\begin{aligned}\nabla\overline{\Pi}_a =& \Pi^b\overline{\Pi}^c T_{abc} - \Pi^\alpha\overline{\Pi}^b T_{\alpha ab} + \widetilde{d}_{\bar\alpha}\Pi^b T_{ab}{}^{\bar\alpha} + d_\alpha\overline{\Pi}^b T_{ab}{}^\alpha + \widetilde{\lambda}^{\bar\alpha}\widetilde{\omega}_{\bar\beta}\Pi^b\widetilde{R}_{ab\bar\alpha}{}^{\bar\beta} + \lambda^\alpha\omega_\beta\overline{\Pi}^b R_{ab\alpha}{}^\beta \\ &+ \widetilde{d}_{\bar\alpha}\Pi^\beta T_{a\beta}{}^{\bar\alpha} + \widetilde{\lambda}^{\bar\alpha}\widetilde{\omega}_{\bar\beta}\Pi^\gamma\widetilde{R}_{a\gamma\bar\alpha}{}^{\bar\beta} + d_\alpha\widetilde{d}_{\bar\beta}\nabla_a P^{\alpha\bar\beta} + \lambda^\alpha\omega_\beta\widetilde{d}_{\bar\gamma}(\nabla_a C_\alpha{}^{\beta\bar\gamma} - P^{\delta\bar\gamma}R_{a\delta\alpha}{}^\beta) \\ &+ \widetilde{\lambda}^{\bar\alpha}\widetilde{\omega}_{\bar\beta} d_\gamma(\nabla_a\widetilde{C}_{\bar\alpha}{}^{\bar\beta\gamma} + P^{\gamma\bar\delta}\widetilde{R}_{a\bar\delta\bar\alpha}{}^{\bar\beta}) + \lambda^\alpha\omega_\beta\widetilde{\lambda}^{\bar\gamma}\widetilde{\omega}_{\bar\delta}(\nabla_a S_{\alpha\bar\gamma}{}^{\beta\bar\delta} - \widetilde{C}_{\bar\gamma}{}^{\bar\delta\rho}R_{a\rho\alpha}{}^\beta - C_\alpha{}^{\beta\bar\rho}\widetilde{R}_{a\bar\rho\bar\gamma}{}^{\bar\delta}).\end{aligned} \tag{5.2}$$

Now we compute the equation for $\overline{\Pi}^\alpha$. We start by noting that this world-sheet field is determined from the equation of motion (2.11), then

$$\nabla\overline{\Pi}^\alpha = -\nabla(\widetilde{d}_{\bar\beta}P^{\alpha\bar\beta} + \widetilde{\lambda}^{\bar\beta}\widetilde{\omega}_{\bar\gamma}\widetilde{C}_{\bar\beta}{}^{\bar\gamma\alpha}).$$

Remember that the covariant derivative on $P^{\alpha\bar\beta}$ and $\widetilde{C}_{\bar\alpha}{}^{\bar\beta\gamma}$ acts with $\Omega_\alpha{}^\beta$ on α-indices and with $\widetilde{\Omega}_{\bar\alpha}{}^{\bar\beta}$ on $\bar\alpha$-indices. Now we can use the equations (2.10) and (2.14) to obtain

$$\begin{aligned}\nabla\overline{\Pi}^\alpha =& d_\beta\widetilde{d}_{\bar\gamma}(\widetilde{C}_{\bar\delta}{}^{\bar\gamma\beta}P^{\alpha\bar\delta} + P^{\beta\bar\delta}\nabla_{\bar\delta}P^{\alpha\bar\gamma}) + \lambda^\beta\omega_\gamma\widetilde{d}_{\bar\delta}(-S_{\beta\bar\rho}{}^{\gamma\bar\delta}P^{\alpha\bar\rho} + C_\beta{}^{\gamma\bar\rho}\nabla_{\bar\rho}P^{\alpha\bar\delta}) \\ &- \widetilde{d}_{\bar\beta}\Pi^a\nabla_a P^{\alpha\bar\beta} - \widetilde{d}_{\bar\beta}\Pi^\gamma\nabla_\gamma P^{\alpha\bar\beta} + \widetilde{\lambda}^{\bar\beta}\widetilde{\omega}_{\bar\gamma}d_\delta(\widetilde{C}_{\bar\beta}{}^{\bar\rho\delta}\widetilde{C}_{\bar\rho}{}^{\bar\gamma\alpha} - \widetilde{C}_{\bar\rho}{}^{\bar\gamma\delta}\widetilde{C}_{\bar\beta}{}^{\bar\rho\alpha} - P^{\delta\bar\rho}\nabla_{\bar\rho}\widetilde{C}_{\bar\beta}{}^{\bar\gamma\alpha} \\ &- P^{\alpha\bar\delta}(\nabla_{\bar\delta}\widetilde{C}_{\bar\beta}{}^{\bar\gamma\delta} + P^{\delta\bar\epsilon}\widetilde{R}_{\bar\epsilon\bar\delta\bar\beta}{}^{\bar\gamma})) + \lambda^\beta\omega_\gamma\widetilde{\lambda}^{\bar\delta}\widetilde{\omega}_{\bar\rho}(S_{\beta\bar\delta}{}^{\gamma\bar\sigma}\widetilde{C}_{\bar\sigma}{}^{\bar\rho\alpha} - S_{\beta\bar\sigma}{}^{\gamma\bar\rho}\widetilde{C}_{\bar\delta}{}^{\bar\sigma\alpha} + C_\beta{}^{\gamma\bar\sigma}\nabla_{\bar\sigma}\widetilde{C}_{\bar\delta}{}^{\bar\rho\alpha} \\ &+ P^{\alpha\bar\epsilon}(\nabla_{\bar\epsilon}S_{\beta\bar\delta}{}^{\gamma\bar\rho} + C_\beta{}^{\gamma\bar\sigma}\widetilde{R}_{\bar\sigma\bar\epsilon\bar\delta}{}^{\bar\rho} + \widetilde{C}_{\bar\delta}{}^{\bar\rho\sigma}R_{\sigma\bar\epsilon\beta}{}^\gamma)) - \widetilde{\lambda}^{\bar\beta}\widetilde{\omega}_{\bar\gamma}\Pi^a(\nabla_a\widetilde{C}_{\bar\beta}{}^{\bar\gamma\alpha} + \widetilde{R}_{a\bar\delta\bar\beta}{}^{\bar\gamma}P^{\alpha\bar\delta}) \\ &- \widetilde{\lambda}^{\bar\beta}\widetilde{\omega}_{\bar\gamma}\Pi^\delta S_{\delta\bar\beta}{}^{\alpha\bar\gamma}.\end{aligned} \tag{5.3}$$

To obtain the equation for $\overline{\Pi}^{\bar\alpha}$ we can use (5.1). After all this we get

$$\begin{aligned}\nabla\overline{\Pi}^{\bar\alpha} =& d_\beta\widetilde{d}_{\bar\gamma}(C_\delta{}^{\beta\bar\gamma}P^{\delta\bar\alpha} - P^{\delta\bar\gamma}\nabla_\delta P^{\beta\bar\alpha}) + \widetilde{\lambda}^{\bar\beta}\widetilde{\omega}_{\bar\gamma}d_\delta(S_{\rho\bar\beta}{}^{\delta\bar\gamma}P^{\rho\bar\alpha} - \widetilde{C}_{\bar\beta}{}^{\bar\gamma\rho}\nabla_\rho P^{\delta\bar\alpha}) \\ &+ d_\beta\overline{\Pi}^a\nabla_a P^{\beta\bar\alpha} + d_\beta\overline{\Pi}^{\bar\gamma}\nabla_{\bar\gamma}P^{\beta\bar\alpha} + \lambda^\beta\omega_\gamma\widetilde{d}_{\bar\delta}(C_\beta{}^{\rho\bar\delta}C_\rho{}^{\gamma\bar\alpha} - C_\rho{}^{\gamma\bar\delta}C_\beta{}^{\rho\bar\alpha} + P^{\rho\bar\delta}\nabla_\rho C_\beta{}^{\gamma\bar\alpha} \\ &+ P^{\delta\bar\alpha}(\nabla_\delta C_\beta{}^{\gamma\bar\delta} - P^{\epsilon\bar\delta}R_{\epsilon\delta\beta}{}^\gamma)) + \lambda^\beta\omega_\gamma\widetilde{\lambda}^{\bar\delta}\widetilde{\omega}_{\bar\rho}(S_{\beta\bar\delta}{}^{\sigma\bar\rho}C_\sigma{}^{\gamma\bar\alpha} - S_{\sigma\bar\delta}{}^{\gamma\bar\rho}C_\beta{}^{\sigma\bar\alpha} + \widetilde{C}_{\bar\delta}{}^{\bar\rho\sigma}\nabla_\sigma C_\beta{}^{\gamma\bar\alpha} \\ &+ P^{\alpha\bar\epsilon}(\nabla_{\bar\epsilon}S_{\beta\bar\delta}{}^{\gamma\bar\rho} + C_\beta{}^{\gamma\bar\gamma}\widetilde{R}_{\bar\gamma\bar\epsilon\bar\delta}{}^{\bar\rho} + \widetilde{C}_{\bar\delta}{}^{\bar\rho\sigma}R_{\sigma\bar\epsilon\beta}{}^\gamma)) - \lambda^\beta\omega_\gamma\overline{\Pi}^a(\nabla_a C_\beta{}^{\gamma\bar\alpha} - R_{a\epsilon\beta}{}^\gamma P^{\epsilon\bar\alpha}) \\ &- \lambda^\beta\omega_\gamma\overline{\Pi}^{\bar\delta}S_{\beta\bar\delta}{}^{\gamma\bar\alpha} + \Pi^a\overline{\Pi}^b T_{ab}{}^{\bar\alpha} - \Pi^\beta\overline{\Pi}^a T_{a\beta}{}^{\bar\alpha} - \widetilde{d}_{\bar\beta}\Pi^a P^{\gamma\bar\beta}T_{a\gamma}{}^{\bar\alpha} - \widetilde{\lambda}^{\bar\beta}\widetilde{\omega}_{\bar\gamma}\Pi^a\widetilde{C}_{\bar\beta}{}^{\bar\gamma\delta}T_{a\delta}{}^{\bar\alpha}.\end{aligned} \tag{5.4}$$

5.1 Beta functions

Now we can obtain the equations for the background fields implied by the vanishing of the beta functions. These are the background dependent expressions for the conformal weights

$(1,1)$ independent couplings in (4.6). That is, all the independent combinations formed from the products between $(\Pi^a, \Pi^\alpha, d_\alpha, \lambda^\alpha\omega_\beta)$ and $(\overline{\Pi}^a, \overline{\Pi}^{\overline{\alpha}}, \widetilde{d}_{\overline{\alpha}}, \widetilde{\lambda}^{\overline{\alpha}}\widetilde{\omega}_{\overline{\beta}})$ because $\Pi^{\overline{\alpha}}$ and $\overline{\Pi}^\alpha$ are determined from the equations of motion (2.11). Let us first concentrate on the beta functions coming from the couplings to $\Pi^A\overline{\Pi}^B$, $d_\alpha\overline{\Pi}^B$ and $\Pi^A\widetilde{d}_{\overline{\beta}}$ fields. After using the results for the expansion (3.4)-(3.12) and the equations (5.2)-(5.4) in (4.6), the couplings $\Pi^\alpha\overline{\Pi}^{\overline{\beta}}, \Pi^\alpha\overline{\Pi}^b, \Pi^a\overline{\Pi}^{\overline{\beta}}$ and $\Pi^a\overline{\Pi}^b$ lead respectively to a first set of equations

$$T_{c\overline{\beta}}{}^{\delta}T_{\delta\alpha}{}^{c} - T_{c\alpha}{}^{\overline{\delta}}T_{\overline{\delta}\overline{\beta}}{}^{c} + 4\nabla_\alpha\nabla_{\overline{\beta}}\Phi = 0, \tag{5.5}$$

$$\nabla^d T_{\alpha db} + R_{\alpha deb}\eta^{de} + T_{bc}{}^{\delta}T_{\delta\alpha}{}^{c} + 4\nabla_b\nabla_\alpha\Phi = 0, \tag{5.6}$$

$$R_{\overline{\beta}dea}\eta^{de} + T_{ac}{}^{\overline{\delta}}T_{\overline{\delta}\overline{\beta}}{}^{c} - T_{c\overline{\beta}}{}^{\delta}T_{\delta a}{}^{c} + 4\nabla_a\nabla_{\overline{\beta}}\Phi = 0, \tag{5.7}$$

$$\begin{aligned}\eta^{cd}(R_{acdb} + R_{bcda}) - \nabla^c T_{abc} + T_{c(a}{}^{\alpha}T_{b)\alpha}{}^{c} + 8T_{a\alpha}{}^{\overline{\beta}}T_{b\overline{\beta}}{}^{\alpha} \\ +4T_{ab}{}^{c}\nabla_c\Phi + 4T_{ab}{}^{\overline{\alpha}}\nabla_{\overline{\alpha}}\Phi + 4\nabla_a\nabla_b\Phi = 0.\end{aligned} \tag{5.8}$$

We wrote them by increasing their dimensions, that is, if X^a has dimension -1 and each θ^α, $\widetilde{\theta}^{\overline{\alpha}}$ have dimension $-\frac{1}{2}$, then the first has dimension 1, the second and third dimension $\frac{3}{2}$ and the fourth dimension 2. The couplings to $d_\alpha\overline{\Pi}^{\overline{\beta}}$, $\Pi^\alpha\widetilde{d}_{\overline{\beta}}$, $d_\alpha\overline{\Pi}^b$ and $\Pi^a\widetilde{d}_{\overline{\beta}}$ lead respectively to a second set of equations

$$\nabla^c T_{c\overline{\beta}}{}^{\alpha} - 2\nabla_{\overline{\beta}}P^{\alpha\overline{\gamma}}\nabla_{\overline{\gamma}}\Phi + 2P^{\alpha\overline{\gamma}}\nabla_{\overline{\gamma}}\nabla_{\overline{\beta}}\Phi = 0, \tag{5.9}$$

$$\nabla^c T_{c\alpha}{}^{\overline{\beta}} + 2\nabla_\alpha P^{\gamma\overline{\beta}}\nabla_\gamma\Phi - 2P^{\gamma\overline{\beta}}\nabla_\gamma\nabla_\alpha\Phi = 0, \tag{5.10}$$

$$\begin{aligned}&\nabla^c T_{cb}{}^{\alpha} - T_{cd}{}^{\alpha}T_b{}^{cd} + (T_{\delta b}{}^{c}T_{c\overline{\gamma}}{}^{\alpha} - R_{b\overline{\gamma}\delta}{}^{\alpha})P^{\delta\overline{\gamma}} + T_{b\overline{\gamma}}{}^{\delta}(3\nabla_\delta P^{\alpha\overline{\gamma}} - 2P^{\alpha\overline{\gamma}}\nabla_\delta\Phi) \\ &+2T_{bc}{}^{\alpha}\nabla^c\Phi - 2\nabla_b P^{\alpha\overline{\gamma}}\nabla_{\overline{\gamma}}\Phi = 0,\end{aligned} \tag{5.11}$$

$$\begin{aligned}&\nabla^c T_{ca}{}^{\overline{\beta}} - 2T_{cd}{}^{\overline{\beta}}T_a{}^{cd} + P^{\gamma\overline{\beta}}T_\alpha{}^{de}T_{ade} + \widetilde{R}_{a\gamma\overline{\delta}}{}^{\overline{\beta}}P^{\gamma\overline{\delta}} - T_{a\gamma}{}^{\overline{\delta}}(3\nabla_{\overline{\delta}}P^{\gamma\overline{\beta}} - 2P^{\gamma\overline{\beta}}\nabla_{\overline{\delta}}\Phi) \\ &+2T_{ac}{}^{\overline{\beta}}\nabla^c\Phi + 2\nabla_a P^{\gamma\overline{\beta}}\nabla_\gamma\Phi = 0.\end{aligned} \tag{5.12}$$

The first two have dimension 2 and the second two have dimension $\frac{5}{2}$. Now we will prove that these equations are implied by the classical BRST constraints, the Bianchi identities (2.18) and the relations (2.22).

Firstly, it is important to know the expression for the scale curvature in terms of the scale connection. This are found to be

$$\begin{aligned}&R_{\alpha\beta} = \nabla_{(\alpha}\Omega_{\beta)}, \quad R_{\alpha\overline{\beta}} = \nabla_{\overline{\beta}}\Omega_\alpha, \quad R_{\overline{\alpha}\overline{\beta}} = 0, \\ &R_{ab} = T_{ab}{}^{\gamma}\Omega_\gamma, \quad R_{a\beta} = \nabla_a\Omega_\beta, \quad R_{a\overline{\beta}} = T_{a\overline{\beta}}{}^{\gamma}\Omega_\gamma.\end{aligned} \tag{5.13}$$

$$\begin{aligned}&\widetilde{R}_{\overline{\alpha}\overline{\beta}} = \nabla_{(\overline{\alpha}}\widetilde{\Omega}_{\overline{\beta})}, \quad \widetilde{R}_{\alpha\overline{\beta}} = \nabla_\alpha\widetilde{\Omega}_{\overline{\beta}}, \quad \widetilde{R}_{\alpha\beta} = 0, \\ &\widetilde{R}_{ab} = T_{ab}{}^{\overline{\gamma}}\widetilde{\Omega}_{\overline{\gamma}}, \quad \widetilde{R}_{a\overline{\beta}} = \nabla_a\widetilde{\Omega}_{\overline{\beta}}, \quad \widetilde{R}_{a\beta} = T_{a\beta}{}^{\overline{\gamma}}\widetilde{\Omega}_{\overline{\gamma}}.\end{aligned} \tag{5.14}$$

Secondly, let us write some expressions useful for later use. We note that the Bianchi identity $(\nabla T)_{\alpha ab}{}^{c} = 0$, using (5.13) can be written as

$$R_{\alpha[ab]c} = \nabla_\alpha T_{abc} - 2(\gamma_{c[a})_\alpha{}^{\beta}R_{b]\beta} + (\gamma_c)_{\alpha\beta}T_{ab}{}^{\beta} - T_{\alpha dc}T_{ab}{}^{d} - T_{\alpha[a}{}^{d}T_{b]dc}, \tag{5.15}$$

now, we can use the identity

$$2R_{\alpha abc} = R_{\alpha[ab]c} + R_{\alpha[ca]b} - R_{\alpha[bc]a}, \tag{5.16}$$

and the Bianchi identity $(\nabla H)_{\alpha abc} = 0$ to write (5.15) as

$$R_{\alpha abc} = T_{a[b}{}^{\beta}(\gamma_{c]})_{\beta\alpha} - 2(\gamma_{bc})_{\alpha}{}^{\beta}R_{a\beta}. \tag{5.17}$$

An identical procedure starting with $(\nabla\widetilde{T})_{\overline{\alpha}ab}{}^{c} = 0$ allows us to find

$$\widetilde{R}_{\overline{\alpha}abc} = T_{a[b}{}^{\overline{\beta}}(\gamma_{c]})_{\overline{\alpha\beta}} - 2(\gamma_{bc})_{\overline{\alpha}}{}^{\overline{\beta}}\widetilde{R}_{a\overline{\beta}}. \tag{5.18}$$

Then, replacing (5.17) and (5.18) respectively in $(\nabla T)_{a\alpha\beta}{}^{\beta} = 0$ and $(\nabla T)_{a\overline{\alpha}\overline{\beta}}{}^{\overline{\beta}} = 0$, we find

$$\gamma^{b}_{\alpha\beta}T_{ba}{}^{\beta} = 8R_{a\alpha}, \quad \gamma^{b}_{\overline{\alpha\beta}}T_{ba}{}^{\overline{\beta}} = 8\widetilde{R}_{a\overline{\alpha}}. \tag{5.19}$$

We have enough information to show that the equations (5.5) , (5.6) and (5.7) are satisfied. From the Bianchi identity $(\nabla T)_{\alpha\beta\overline{\gamma}}{}^{\beta} = 0$ we obtain

$$T_{\alpha\beta}{}^{d}T_{d\overline{\gamma}}{}^{\beta} = 17R_{\alpha\overline{\gamma}} + \frac{1}{4}R_{\overline{\gamma}\beta cd}(\gamma^{cd})_{\alpha}{}^{\beta}. \tag{5.20}$$

Since we need an expression for $R_{\overline{\gamma}\beta cd}$, we can use $(\nabla T)_{\alpha\overline{\beta}a}{}^{b} = 0$, finding

$$R_{\overline{\gamma}\beta cd} = 2(\gamma_{cd})_{\beta}{}^{\delta}\nabla_{\overline{\gamma}}\Omega_{\delta} + T_{c\overline{\gamma}}{}^{\epsilon}(\gamma_{d})_{\epsilon\beta} + T_{c\beta}{}^{\overline{\epsilon}}(\gamma_{d})_{\overline{\epsilon\gamma}}. \tag{5.21}$$

Replacing (5.21) in (5.20) ,using the second equation in (5.13) , $\nabla_{\alpha}\Phi = 4\Omega_{\alpha}$ and the constraints coming from holomorphicity-antiholomorphicity of the BRST current $T_{a\beta}{}^{\overline{\gamma}} = -(\gamma_{a})_{\beta\delta}P^{\delta\overline{\gamma}}$, $T_{a\overline{\beta}}{}^{\gamma} = (\gamma_{a})_{\overline{\beta\delta}}P^{\gamma\overline{\delta}}$ we can verify the equation (5.5).

To verify (5.6) and (5.7), we must contract the a and b indices using η^{ab} in (5.17) and (5.18), and use (5.19) together with the relations (2.22).

For deriving the remaining equation of the first set, the coupling to $\Pi^{a}\overline{\Pi}^{b}$, it is useful to find an expression for R_{abcd}, which can be found from the Bianchi identity $(\nabla T)_{ab\alpha}{}^{\beta}$

$$R_{abcd} = -\frac{1}{8}(\gamma_{cd})_{\beta}{}^{\alpha}(\nabla_{\alpha}T_{ab}{}^{\beta} - T_{\alpha[a}{}^{e}T_{b]e}{}^{\beta} - T_{\alpha[a}{}^{\overline{\gamma}}T_{b]\overline{\gamma}}{}^{\beta}), \tag{5.22}$$

from this equation we construct $\eta^{cd}(R_{acdb} + R_{bcda})$:

$$\begin{aligned}\eta^{cd}(R_{acdb} + R_{bcda}) = &-\frac{1}{8}\eta^{cd}[(\gamma_{db})_{\beta}{}^{\alpha}\nabla_{\alpha}T_{ac}{}^{\beta} + (\gamma_{da})_{\beta}{}^{\alpha}\nabla_{\alpha}T_{bc}{}^{\beta}]\\ &+\frac{1}{8}\eta^{cd}[(\gamma_{db})_{\beta}{}^{\alpha}T_{\alpha[a}{}^{e}T_{c]e}{}^{\beta} + (\gamma_{da})_{\beta}{}^{\alpha}T_{\alpha[b}{}^{e}T_{c]e}{}^{\beta}]\\ &+\frac{1}{8}\eta^{cd}[(\gamma_{db})_{\beta}{}^{\alpha}T_{\alpha[a}{}^{\overline{\epsilon}}T_{c]\overline{\epsilon}}{}^{\beta} + (\gamma_{da})_{\beta}{}^{\alpha}T_{\alpha[b}{}^{\overline{\epsilon}}T_{c]\overline{\epsilon}}{}^{\beta}].\end{aligned} \tag{5.23}$$

Let us consider the right hand side of (5.23) line by line. We can use (5.19) , the Bianchi identity $(\nabla R)_{\alpha a\delta\beta}{}^{\gamma}$ to write

$$(\gamma_{b})^{\delta\alpha}\nabla_{\alpha}R_{a\delta} = -2\nabla_{a}\nabla_{b}\Phi - 2T_{ab}{}^{\gamma}\nabla_{\gamma}\Phi - 2(\gamma_{b}\gamma_{ae})^{\delta\beta}\Omega_{\beta}R^{e}{}_{\delta} - (\gamma_{a}\gamma_{b})_{\beta}{}^{\delta}P^{\beta\overline{\epsilon}}R_{\overline{\epsilon}\delta}, \tag{5.24}$$

and the beta function with dimension 1 (5.5) to find the following expression for the first line in the right hand side of (5.23)

$$-4\nabla_b\nabla_a\Phi+2T_{ab}{}^{C}\nabla_C\Phi-4\eta_{ab}(\gamma^e)^{\delta\beta}\Omega_\beta R_{e\delta}+4(\gamma_b)^{\delta\beta}\Omega_\beta R_{a\delta}+4(\gamma_a)^{\delta\beta}\Omega_\beta R_{b\delta}+\frac{1}{4}\eta_{ab}\eta^{cd}T_{c\beta}{}^{\overline{\delta}}T^{\beta}_{d\overline{\delta}}. \tag{5.25}$$

Finding an expression for the second line is a matter of gamma matrices algebra, once we use (5.13) . For this line we find $\frac{1}{4}\eta_{ab}T_{\beta cd}T^{cd\beta}-\frac{3}{4}T_{c(a}{}^{\beta}T_{b)\beta}{}^{c}$. Using $T_{a\beta}{}^{\overline{\gamma}}=-(\gamma_a)_{\beta\delta}P^{\delta\overline{\gamma}}$ and some gamma matrices algebra, it is straightforward to find $T_{\beta(a}{}^{\overline{\gamma}}T_{b)\overline{\gamma}}{}^{b}-\frac{1}{4}\eta_{ab}\eta^{cd}T_{d\beta}{}^{\overline{\gamma}}T_{c\overline{\gamma}}{}^{\beta}$ for the third line. So, adding the results for the three lines and using (5.19) we find

$$\eta^{cd}(R_{acdb}+R_{bcda})=-4\nabla_b\nabla_a\Phi-T_{c(a}{}^{\beta}T_{b)\beta}{}^{c}+2T_{ab}{}^{E}\nabla_E\Phi+T_{\beta(a}{}^{\overline{\gamma}}T_{b)\overline{\gamma}}{}^{\beta}, \tag{5.26}$$

which contains some of the terms in (5.8) . It is also needed to use $(\nabla T)_{abc}{}^{c}=0$ in order to generate the term $\nabla^c T_{abc}$. This Bianchi identity gives

$$\nabla^c T_{abc}-T_{c[a}{}^{e}T_{b]e}{}^{c}-T_{c[a}{}^{\epsilon}T_{b]\epsilon}{}^{c}-\eta^{cd}(R_{acdb}-R_{bcda})=0. \tag{5.27}$$

Finding an expression for $\eta^{cd}(R_{acdb}-R_{bcda})$ is not difficult following the description given to compute (5.26) . After we compute it and replace it in (5.27) we find

$$\nabla^c T_{abc}+T_{\beta[a}{}^{\overline{\delta}}T_{b]\overline{\delta}}{}^{\beta}-2T_{ab}{}^{c}\nabla_c\Phi+2T_{ab}{}^{\gamma}\nabla_\gamma\Phi-2T_{ab}{}^{\overline{\gamma}}\nabla_{\overline{\gamma}}\Phi=0. \tag{5.28}$$

Combining (5.26) and (5.28) gives the desired beta function equation (5.8).

A similar procedure, but with more steps, is performed to prove the equations of the second group. To probe (5.9) one can start by computing $\{\nabla_{\overline{\alpha}},\nabla_{\overline{\beta}}\}P^{\gamma\overline{\beta}}=-\gamma^{c}_{\overline{\gamma}\overline{\beta}}\nabla_c P^{\gamma\overline{\beta}}+\widetilde{R}_{\overline{\alpha}\overline{\beta}\overline{\delta}}{}^{\overline{\beta}}P^{\gamma\overline{\delta}}$. Then we split the curvature as a scale curvature plus a Lorentz curvature. For the latter, use $(\nabla\widetilde{T})_{\overline{\alpha}\overline{\beta}c}{}^{d}=0$ to obtain

$$\widetilde{R}_{\overline{\alpha}\overline{\beta}cd}(\gamma^{cd})_{\overline{\delta}}{}^{\overline{\beta}}=-180\nabla_{\overline{\alpha}}\widetilde{\Omega}_{\overline{\delta}}+(\gamma^{cd})_{\overline{\delta}}{}^{\overline{\beta}}\nabla_{\overline{\beta}}\widetilde{T}_{\overline{\alpha}cd}+16\widetilde{T}_{\overline{\delta}}{}^{cd}\widetilde{T}_{\overline{\alpha}cd}+(\gamma^{cd}\gamma^{e})_{\overline{\delta}\overline{\alpha}}\widetilde{T}_{ecd}, \tag{5.29}$$

so on one hand we will have

$$\begin{aligned}\{\nabla_{\overline{\alpha}},\nabla_{\overline{\beta}}\}P^{\gamma\overline{\beta}}=&-\nabla^c T_{c\overline{\alpha}}{}^{\gamma}+\widetilde{R}_{\overline{\alpha}\overline{\delta}}P^{\gamma\overline{\delta}}-45\nabla_{\overline{\alpha}}\widetilde{\Omega}_{\overline{\delta}}P^{\gamma\overline{\delta}}+\frac{1}{4}(\gamma^{cd})_{\overline{\delta}}{}^{\overline{\beta}}\nabla_{\overline{\beta}}\widetilde{T}_{\overline{\alpha}cd}P^{\gamma\overline{\delta}}\\&-4\widetilde{T}_{\overline{\alpha}cd}\widetilde{T}_{\overline{\delta}}{}^{cd}P^{\gamma\overline{\delta}}+\frac{1}{4}(\gamma^{cd}\gamma^{e})_{\overline{\delta}\overline{\alpha}}\widetilde{T}_{ecd}P^{\gamma\overline{\delta}}.\end{aligned} \tag{5.30}$$

On the other hand, we can use $\nabla_{\overline{\alpha}}P^{\beta\overline{\gamma}}=\widetilde{C}_{\overline{\alpha}}{}^{\overline{\gamma}\beta}$, $\widetilde{C}^{\gamma}=-P^{\gamma\overline{\delta}}\widetilde{\Omega}_{\overline{\delta}}$ and $\widetilde{C}_{cd}{}^{\gamma}=1/10(\gamma^a)^{\gamma\alpha}\widetilde{R}_{a\alpha cd}$, which come from antiholomorphicity of the BRST current, to write

$$\{\nabla_{\overline{\alpha}},\nabla_{\overline{\beta}}\}P^{\gamma\overline{\beta}}=-17\nabla_{\overline{\alpha}}P^{\gamma\overline{\delta}}\widetilde{\Omega}_{\overline{\delta}}-17P^{\gamma\overline{\delta}}\nabla_{\overline{\alpha}}\widetilde{\Omega}_{\overline{\delta}}+\frac{1}{40}(\gamma^a)^{\gamma\alpha}\nabla_{\overline{\beta}}(\widetilde{R}_{a\alpha cd}(\gamma^{cd})_{\overline{\alpha}}{}^{\overline{\beta}}). \tag{5.31}$$

Using $(\nabla\widetilde{T})_{\alpha bcd}=0$ and $(\widetilde{\nabla}H)_{\alpha bcd}=0$ it is straightforward to find

$$(\gamma^a)^{\gamma\alpha}\widetilde{R}_{a\alpha cd}=10T_{cd}{}^{\gamma}-10P^{\gamma\overline{\epsilon}}\widetilde{T}_{\overline{\epsilon}cd}. \tag{5.32}$$

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Since there is a derivative acting on this terms in (5.31) , we make use of $(\nabla\widetilde{T})_{\overline{\beta}cd}{}^{\gamma}=0$ to find

$$(\gamma^{cd})_{\overline{\alpha}}{}^{\overline{\beta}}\nabla_{\overline{\beta}}T_{cd}{}^{\gamma}=-18\nabla^{d}T_{d\overline{\alpha}}{}^{\gamma}+(\gamma^{cd}\gamma^{e})_{\overline{\alpha}\overline{\delta}}\widetilde{T}_{ecd}P^{\gamma\overline{\delta}}+16\widetilde{T}_{\overline{\alpha}cd}T_{dc}{}^{\gamma}. \tag{5.33}$$

We can now replace the last two equations in (5.31) and equate it to (5.1) . The identity

$$(\gamma^{ab})_{(\overline{\alpha}}{}^{\overline{\beta}}(\gamma_{ab})_{\overline{\gamma})}{}^{\overline{\delta}}=-10\delta_{(\overline{\alpha}}{}^{\overline{\beta}}\delta_{\overline{\gamma})}{}^{\overline{\delta}}+8(\gamma^{a})_{\overline{\alpha}\overline{\gamma}}(\gamma_{a})^{\overline{\beta}\overline{\delta}}, \tag{5.34}$$

which can be proved using $(\gamma^{a})_{(\overline{\alpha}\overline{\beta}}(\gamma_{a})_{\overline{\gamma})\overline{\delta}}=0$, will be of help to find (5.9). A completely analog procedure allows us to arrive to (5.10).

To prove (5.11) we make use of the Bianchi identities $(\nabla R)_{\alpha ab\beta}{}^{\gamma}=0$, $(\nabla T)_{c\alpha\beta}{}^{\gamma}=0$ and the identity $(\gamma_{a})^{\alpha\beta}R_{\alpha\beta\gamma}{}^{\delta}=-2(\gamma_{a})^{\alpha\beta}R_{\gamma\alpha\beta}{}^{\delta}$, which follows from $(\nabla T)_{\alpha\beta\gamma}{}^{\delta}=0$, to arrive to

$$\begin{aligned}&(\gamma)^{\alpha\beta}(\nabla_{\alpha}R_{ab\beta}{}^{\gamma}-2T_{\alpha[a}{}^{e}R_{b]e\beta}{}^{\gamma}-T_{\alpha[a}{}^{\overline{\epsilon}}R_{b]\overline{\epsilon}\beta}{}^{\gamma})-8T_{b}{}^{ac}T_{ac}{}^{\gamma}+8\nabla^{a}T_{ab}{}^{\gamma}+2T_{ab}{}^{\gamma}\nabla^{a}\Phi\\&-\frac{1}{8}(\gamma)^{\alpha\beta}(\gamma^{cd})_{\epsilon}{}^{\gamma}R_{\alpha\beta cd}T_{ab}{}^{\epsilon}+T_{ab}{}^{\overline{\epsilon}}(\gamma^{a})^{\alpha\beta}R_{\overline{\epsilon}\alpha\beta}{}^{\gamma}=0.\end{aligned} \tag{5.35}$$

The last term in this equation is zero as can easily seen using $(\nabla T)_{\overline{\epsilon}\alpha\beta}{}^{\gamma}=0$. The first term can be worked out using (5.22) and $(\nabla T)_{a\overline{\epsilon}\beta}{}^{\gamma}=0$, the curvature in the first term of the second line can be rewritten using $(\nabla T)_{\alpha\beta a}{}^{b}=0$. The use of $(\nabla T)_{cdb}{}^{\delta}=0$ will be also needed to generate (5.11). Again, an analog procedure will allow at arrive to (5.12).

So far, we concentrated on a specific set of beta functions. The remaining ones can be classified in a third and fourth sets. The third set involves first order derivatives of the curvatures. We present it again as the dimension increases.

At dimension 5/2 we find respectively from the couplings to $J\overline{\Pi}^{\overline{\beta}}$, $\Pi^{\alpha}\widetilde{J}$, $N^{ac}\overline{\Pi}^{\overline{\beta}}$ and $\Pi^{\alpha}\widetilde{N}^{bc}$

$$\nabla^{a}R_{a\overline{\beta}}+\nabla_{(\overline{\epsilon}}R_{\overline{\beta})\delta}P^{\delta\overline{\epsilon}}+2(\nabla_{\overline{\beta}}C^{\overline{\alpha}}-R_{\overline{\beta}\gamma}P^{\gamma\overline{\alpha}})\nabla_{\overline{\alpha}}\Phi+2C^{\overline{\alpha}}\nabla_{\overline{\alpha}}\nabla_{\overline{\beta}}\Phi=0, \tag{5.36}$$

$$\nabla^{b}\widetilde{R}_{b\alpha}-\nabla_{(\delta}\widetilde{R}_{\alpha)\overline{\epsilon}}P^{\delta\overline{\epsilon}}+2(\nabla_{\alpha}\widetilde{C}^{\beta}+\widetilde{R}_{\alpha\overline{\gamma}}P^{\beta\overline{\gamma}})\nabla_{\beta}\Phi+2\widetilde{C}^{\beta}\nabla_{\beta}\nabla_{\alpha}\Phi=0, \tag{5.37}$$

$$\nabla^{d}R_{d\overline{\beta}ac}+\nabla_{(\overline{\epsilon}}R_{\overline{\beta}\delta)ac}P^{\delta\overline{\epsilon}}+2(\nabla_{\overline{\beta}}C_{ac}{}^{\overline{\alpha}}-R_{\overline{\beta}\gamma ac}P^{\gamma\overline{\alpha}})\nabla_{\overline{\alpha}}\Phi+2C_{ac}{}^{\overline{\delta}}\nabla_{\overline{\delta}}\nabla_{\overline{\beta}}\Phi=0, \tag{5.38}$$

$$\nabla^{d}\widetilde{R}_{d\alpha bc}-\nabla_{(\delta}\widetilde{R}_{\alpha)\overline{\epsilon}bc}P^{\delta\overline{\epsilon}}+2(\nabla_{\alpha}\widetilde{C}_{bc}{}^{\gamma}+\widetilde{R}_{\alpha\overline{\delta}bc}P^{\gamma\overline{\delta}})\nabla_{\gamma}\Phi+2\widetilde{C}_{bc}{}^{\gamma}\nabla_{\gamma}\nabla_{\alpha}\Phi=0. \tag{5.39}$$

While at dimension 3 we find respectively from the couplings to $J\overline{\Pi}^{b}$, $\Pi^{a}\widetilde{J}$, $N^{ac}\overline{\Pi}^{b}$ and $\Pi^{a}\widetilde{N}^{bc}$

$$\begin{aligned}&\nabla^{a}R_{ab}-T_{ba}{}^{c}R^{a}{}_{c}+T_{ba}{}^{\gamma}R^{a}{}_{\gamma}+3T_{b\overline{\gamma}}{}^{\alpha}\nabla_{\alpha}C^{\overline{\gamma}}+2R_{bc}\nabla^{c}\Phi+2R_{b\overline{\alpha}}P^{\gamma\overline{\alpha}}\nabla_{\gamma}\Phi\\&+2(\nabla_{b}C^{\overline{\alpha}}-R_{b\gamma}P^{\gamma\overline{\alpha}})\nabla_{\overline{\alpha}}\Phi+P^{\delta\overline{\epsilon}}(\nabla_{\overline{\epsilon}}R_{\delta b}+T_{b\delta}{}^{c}R_{\overline{\epsilon}c}+T_{b\overline{\epsilon}}{}^{\gamma}R_{\delta\gamma})=0,\end{aligned} \tag{5.40}$$

$$\begin{aligned}&\nabla^{b}\widetilde{R}_{ba}+T_{ab}{}^{\overline{\gamma}}\widetilde{R}^{b}{}_{\overline{\gamma}}+T_{abc}\widetilde{C}^{\delta}T_{\delta}{}^{bc}+3T_{a\gamma}{}^{\overline{\beta}}\nabla_{\overline{\beta}}\widetilde{C}^{\gamma}+2\widetilde{R}_{ab}\nabla^{b}\Phi-2\widetilde{R}_{a\gamma}P^{\gamma\overline{\beta}}\nabla_{\overline{\beta}}\Phi\\&+2(\nabla_{a}\widetilde{C}^{\beta}+\widetilde{R}_{a\overline{\gamma}}P^{\beta\overline{\gamma}})\nabla_{\beta}\Phi-P^{\delta\overline{\epsilon}}(\nabla_{\delta}\widetilde{R}_{\overline{\epsilon}a}+T_{\delta a}{}^{c}\widetilde{R}_{\overline{\epsilon}c}+T_{a\delta}{}^{\overline{\gamma}}\widetilde{R}_{\overline{\epsilon}\overline{\gamma}})=0,\end{aligned} \tag{5.41}$$

$$\begin{aligned}&\nabla^{d}R_{dbac}-T_{b}{}^{de}R_{deac}+T_{b}{}^{d\epsilon}R_{d\epsilon ac}+3T_{b\overline{\delta}}{}^{\gamma}\nabla_{\gamma}C_{ac}{}^{\overline{\delta}}+2R_{bdac}\nabla^{d}\Phi+2R_{b\overline{\delta}ac}P^{\epsilon\overline{\delta}}\nabla_{\epsilon}\Phi\\&+2(\nabla_{b}C_{ac}{}^{\overline{\delta}}-R_{b\epsilon ac}P^{\epsilon\overline{\delta}})\nabla_{\overline{\delta}}\Phi+2R_{b\overline{\delta}ea}C_{c}{}^{e\overline{\delta}}+P^{\delta\overline{\epsilon}}(\nabla_{\overline{\epsilon}}R_{\delta bac}+T_{b\delta}{}^{f}R_{\overline{\epsilon}fac}+T_{b\overline{\epsilon}}{}^{\gamma}R_{\delta\gamma ac})=0,\end{aligned} \tag{5.42}$$

$$\nabla^d \widetilde{R}_{dabc} + T_a{}^{d\bar\epsilon} \widetilde{R}_{d\bar\epsilon bc} + T_{adf}\widetilde{C}_{bc}{}^{\epsilon} T_\epsilon{}^{df} + 3T_{a\delta}{}^{\bar\epsilon}\nabla_{\bar\epsilon}\widetilde{C}_{bc}{}^{\delta} + 2\widetilde{R}_{adbc}\nabla^d\Phi - 2\widetilde{R}_{a\delta bc}P^{\delta\bar\epsilon}\nabla_{\bar\epsilon}\Phi$$
$$+2(\nabla_a \widetilde{C}_{bc}{}^{\delta} + \widetilde{R}_{a\bar\epsilon bc}P^{\delta\bar\epsilon})\nabla_\delta\Phi + 2\widetilde{R}_{a\delta eb}\widetilde{C}_c{}^{e\delta} - P^{\delta\bar\epsilon}(\nabla_\delta \widetilde{R}_{\bar\epsilon abc} + T_{a\delta}{}^{f}\widetilde{R}_{\bar\epsilon fbc} + T_{a\delta}{}^{\bar\gamma}\widetilde{R}_{\bar\epsilon\bar\gamma bc}) = 0. \quad (5.43)$$

The fourth set involves second order derivatives of the background fields $P^{\alpha\bar\beta}$, $C_\alpha{}^{\bar\beta\bar\gamma}$, $\widetilde{C}_{\bar\alpha}{}^{\bar\beta\gamma}$ and $S_{\alpha\bar\beta}{}^{\gamma\bar\delta}$. There is an equation at dimension 3, coming from the coupling to $d_\alpha \widetilde{d}_{\bar\beta}$

$$\nabla^2 P^{\alpha\bar\beta} - 2P^{\gamma\bar\delta}S_{\gamma\bar\delta}{}^{\alpha\bar\beta} + T_{de}{}^{\alpha}T^{de\bar\beta} - 2\nabla_{\bar\gamma}P^{\delta\bar\beta}\nabla_\delta P^{\alpha\bar\gamma} - 2\nabla_c P^{\alpha\bar\beta}\nabla^c\Phi$$
$$-2(P^{\gamma\bar\delta}\nabla_{\bar\delta}P^{\alpha\bar\beta} + P^{\alpha\bar\delta}\nabla_{\bar\delta}P^{\gamma\bar\beta})\nabla_\gamma\Phi + 2(P^{\delta\bar\gamma}\nabla_\delta P^{\alpha\bar\beta} + P^{\delta\bar\beta}\nabla_\delta P^{\alpha\bar\gamma})\nabla_{\bar\gamma}\Phi = 0. \quad (5.44)$$

At dimension 7/2 we find respectively from the couplings to $J\widetilde{d}_{\bar\beta}$, $d_\alpha \widetilde{J}$, $N^{ac}\widetilde{d}_{\bar\beta}$ and $d_\alpha \widetilde{N}^{bc}$

$$\nabla^2 C^{\bar\beta} - P^{\alpha\bar\gamma}\nabla_{[\alpha}\nabla_{\bar\gamma]}C^{\bar\beta} - T_{ac}{}^{\bar\beta}R^{ac} + 2R_\gamma{}^a\nabla_a P^{\gamma\bar\beta} + 2\nabla_{\bar\gamma}P^{\alpha\bar\beta}\nabla_\alpha C^{\bar\gamma} - C^{\bar\alpha}\widetilde{R}_{\bar\alpha\delta\bar\epsilon}{}^{\bar\beta}P^{\delta\bar\epsilon}$$
$$+P^{\alpha\bar\beta}(\nabla_c R_\alpha{}^c - \nabla_{[\delta}R_{\bar\gamma]\alpha}P^{\delta\bar\gamma}) - 2(\nabla_a C^{\bar\beta} - P^{\gamma\bar\beta}R_{a\gamma})\nabla^a\Phi - 2(P^{\alpha\bar\beta}\nabla_\alpha C^{\bar\gamma} + P^{\alpha\bar\gamma}\nabla_\alpha C^{\bar\beta}$$
$$+P^{\alpha\bar\beta}R_{\alpha\gamma}P^{\gamma\bar\gamma})\nabla_{\bar\gamma}\Phi + 2(SP^{\alpha\bar\beta} + \frac{1}{4}\widetilde{S}_{cd}(\gamma^{cd})_{\bar\epsilon}{}^{\bar\beta}P^{\alpha\bar\epsilon} - C^{\bar\gamma}\nabla_{\bar\gamma}P^{\alpha\bar\beta})\nabla_\alpha\Phi = 0, \quad (5.45)$$
$$\nabla^2 \widetilde{C}^{\alpha} - P^{\beta\bar\gamma}\nabla_{[\beta}\nabla_{\bar\gamma]}\widetilde{C}^{\alpha} - T_{bc}{}^{\alpha}\widetilde{R}^{bc} - 2\widetilde{R}_{\bar\gamma}{}^{b}\nabla_b P^{\alpha\bar\gamma} - 2\nabla_{\bar\gamma}\widetilde{C}^{\beta}\nabla_\beta P^{\alpha\bar\gamma} + \widetilde{C}^{\beta}R_{\beta\bar\epsilon\delta}{}^{\alpha}P^{\delta\bar\epsilon}$$
$$-P^{\alpha\bar\beta}(\nabla_c \widetilde{R}_{\bar\beta}{}^{c} + \nabla_{[\delta}\widetilde{R}_{\bar\gamma]\bar\beta}P^{\delta\bar\gamma}) - 2(\nabla_b \widetilde{C}^{\alpha} + P^{\alpha\bar\gamma}\widetilde{R}_{b\bar\gamma})\nabla^b\Phi + 2(P^{\alpha\bar\beta}\nabla_{\bar\beta}\widetilde{C}^{\gamma} + P^{\gamma\bar\beta}\nabla_{\bar\beta}\widetilde{C}^{\alpha}$$
$$+P^{\alpha\bar\epsilon}\widetilde{R}_{\bar\epsilon\bar\gamma}P^{\gamma\bar\gamma})\nabla_\gamma\Phi - 2(SP^{\alpha\bar\beta} + \frac{1}{4}S_{cd}(\gamma^{cd})_{\epsilon}{}^{\alpha}P^{\epsilon\bar\beta} - \widetilde{C}^{\gamma}\nabla_\gamma P^{\alpha\bar\beta})\nabla_{\bar\beta}\Phi = 0, \quad (5.46)$$
$$\nabla^2 C_{ac}{}^{\bar\beta} - P^{\delta\bar\epsilon}\nabla_{[\delta}\nabla_{\bar\epsilon]}C_{ac}{}^{\bar\beta} - R_{deac}T^{de\bar\beta} - 2R_{d\epsilon ac}\nabla^d P^{\epsilon\bar\beta} + 2\nabla_{\bar\delta}P^{\epsilon\bar\beta}\nabla_\epsilon C_{ac}{}^{\bar\delta} - C_{ac}{}^{\bar\gamma}\widetilde{R}_{\bar\gamma\delta\bar\epsilon}{}^{\bar\beta}P^{\delta\bar\epsilon}$$
$$-P^{\beta\bar\beta}(\nabla^d R_{d\beta ac} - \nabla_{[\delta}R_{\bar\epsilon]\beta ac}P^{\delta\bar\epsilon} + 2R_{\beta\bar\delta ea}C_c{}^{e\bar\delta}) + 2\nabla_{\bar\delta}C_{ea}{}^{\bar\beta}C_c{}^{e\bar\delta}$$
$$-2(\nabla_d C_{ac}{}^{\bar\beta} - P^{\epsilon\bar\beta}R_{d\epsilon ac})\nabla^d\Phi - 2(C_{ac}{}^{\bar\delta}\nabla_{\bar\delta}P^{\gamma\bar\beta} - S_{ac}P^{\gamma\bar\beta} - \frac{1}{4}S_{acbd}(\gamma^{bd})_{\bar\delta}{}^{\bar\beta}P^{\gamma\bar\delta})\nabla_\gamma\Phi$$
$$-2(P^{\gamma\bar\beta}\nabla_\gamma C_{ac}{}^{\bar\delta} + P^{\gamma\bar\delta}\nabla_\gamma C_{ac}{}^{\bar\beta} - P^{\epsilon\bar\beta}R_{\epsilon\gamma ac}P^{\gamma\bar\delta})\nabla_{\bar\delta}\Phi = 0, \quad (5.47)$$
$$\nabla^2 \widetilde{C}_{bc}{}^{\alpha} - P^{\delta\bar\epsilon}\nabla_{[\delta}\nabla_{\bar\epsilon]}\widetilde{C}_{bc}{}^{\alpha} - \widetilde{R}_{debc}T^{de\alpha} + 2\widetilde{R}_{d\bar\epsilon bc}\nabla^d P^{\alpha\bar\epsilon} - 2\nabla_{\bar\delta}\widetilde{C}_{bc}{}^{\epsilon}\nabla_\epsilon P^{\alpha\bar\delta} + \widetilde{C}_{bc}{}^{\gamma}R_{\gamma\bar\epsilon\delta}{}^{\alpha}P^{\delta\bar\epsilon}$$
$$-P^{\alpha\bar\beta}(\nabla^d \widetilde{R}_{d\bar\beta bc} - \nabla_{[\delta}\widetilde{R}_{\bar\epsilon]\bar\beta bc}P^{\delta\bar\epsilon} + 2\widetilde{R}_{\bar\beta\delta eb}\widetilde{C}_c{}^{e\delta}) + 2\nabla_\delta \widetilde{C}_{eb}{}^{\alpha}\widetilde{C}_c{}^{e\delta} - 2(\nabla_d \widetilde{C}_{bc}{}^{\alpha} + P^{\epsilon\bar\beta}\widetilde{R}_{d\bar\beta bc})\nabla^d\Phi$$
$$+2(\widetilde{C}_{bc}{}^{\delta}\nabla_\delta P^{\alpha\bar\gamma} - \widetilde{S}_{bc}P^{\alpha\bar\gamma} - \frac{1}{4}S_{adbc}(\gamma^{ad})_{\delta}{}^{\alpha}P^{\delta\bar\gamma})\nabla_{\bar\gamma}\Phi$$
$$+2(P^{\alpha\bar\beta}\nabla_{\bar\beta}\widetilde{C}_{bc}{}^{\delta} + P^{\delta\bar\beta}\nabla_{\bar\beta}\widetilde{C}_{bc}{}^{\alpha} + P^{\alpha\bar\epsilon}\widetilde{R}_{\bar\epsilon\bar\gamma bc}P^{\delta\bar\gamma})\nabla_\delta\Phi = 0. \quad (5.48)$$

Finally, at dimension 4 we find from the couplings to $J\widetilde{J}$, $J\widetilde{N}^{ac}$, $N^{ab}\widetilde{J}$ and $N^{ab}\widetilde{N}^{cd}$ respectively

$$\nabla^2 S - P^{\delta\bar\epsilon}\nabla_{[\delta}\nabla_{\bar\epsilon]}S - R^{ab}\widetilde{R}_{ab} + 2\widetilde{R}_{a\bar\beta}\nabla^a C^{\bar\beta} + 2R_{a\beta}\nabla^a\widetilde{C}^{\beta} - 2\nabla_{\bar\alpha}\widetilde{C}^{\beta}\nabla_\beta C^{\bar\alpha}$$
$$-\widetilde{C}^{\beta}(\nabla^a R_{a\beta} - P^{\delta\bar\epsilon}\nabla_{[\delta}R_{\bar\epsilon]\beta}) - C^{\bar\beta}(\nabla^a\widetilde{R}_{a\bar\beta} - P^{\delta\bar\epsilon}\nabla_{[\delta}\widetilde{R}_{\bar\epsilon]\bar\beta}) + 2(\widetilde{C}^{\alpha}R_{b\alpha} + C^{\bar\alpha}\widetilde{R}_{b\bar\alpha})\nabla^b\Phi$$
$$-2(C^{\bar\alpha}\nabla_{\bar\alpha}\widetilde{C}^{\beta} + P^{\beta\bar\alpha}(\nabla_{\bar\alpha}S + C^{\bar\gamma}\widetilde{R}_{\bar\gamma\bar\alpha} + \widetilde{C}^{\gamma}R_{\gamma\bar\alpha}))\nabla_\beta\Phi - 2(\widetilde{C}^{\alpha}\nabla_\alpha C^{\bar\beta} - P^{\alpha\bar\beta}(\nabla_\alpha S$$
$$+C^{\bar\gamma}\widetilde{R}_{\bar\gamma\alpha} + \widetilde{C}^{\gamma}R_{\gamma\alpha}))\nabla_{\bar\beta}\Phi = 0, \quad (5.49)$$
$$\nabla^2 \widetilde{S}_{ac} - P^{\delta\bar\epsilon}\nabla_{[\delta}\nabla_{\bar\epsilon]}\widetilde{S}_{ac} - R^{ed}\widetilde{R}_{edac} + 2\widetilde{R}_{b\bar\delta ac}\nabla^b C^{\bar\delta} + 2R_{b\delta}\nabla^b\widetilde{C}_{ac}{}^{\delta}$$
$$-2\nabla_{\bar\beta}\widetilde{C}_{ac}{}^{\delta}\nabla_\delta C^{\bar\beta} - 2\nabla_\delta\widetilde{S}_{ba}\widetilde{C}_c{}^{b\delta}$$

$$
\begin{aligned}
&-C^{\overline{\beta}}(\nabla^d \widetilde{R}_{d\overline{\beta}ac} - P^{\delta\overline{\epsilon}}\nabla_{[\delta}\widetilde{R}_{\overline{\epsilon}]\overline{\beta}} + 2\widetilde{R}_{\overline{\beta}\delta ea}\widetilde{C}_c{}^{e\delta}) - \widetilde{C}_{ac}{}^{\beta}(\nabla^d R_{d\beta} - P^{\delta\overline{\epsilon}}\nabla_{[\delta}R_{\overline{\epsilon}]\beta})\\
&+2(\widetilde{C}_{ac}{}^{\beta}R_{d\beta} + C^{\overline{\beta}}\widetilde{R}_{d\overline{\beta}ac})\nabla^d\Phi - 2C^{\overline{\delta}}\nabla_{\overline{\delta}}\widetilde{C}_{ac}{}^{\gamma}\nabla_\gamma\Phi + 4\widetilde{S}_{ab}\widetilde{C}_c{}^{b\gamma}\nabla_\gamma\Phi\\
&-2\widetilde{C}_{ac}{}^{\beta}\nabla_\beta C^{\overline{\gamma}}\nabla_{\overline{\gamma}}\Phi = 0, \qquad (5.50)\\
&\nabla^2 S_{ab} - P^{\delta\overline{\epsilon}}\nabla_{[\delta}\nabla_{\overline{\epsilon}]}S_{ab} - \widetilde{R}^{cd}R_{cdab} + 2R_{c\delta ab}\nabla^c\widetilde{C}^{\delta} + 2\widetilde{R}_{c\overline{\delta}}\nabla^c C_{ab}{}^{\overline{\delta}}\\
&-2\nabla_{\overline{\gamma}}\widetilde{C}^{\delta}\nabla_\delta C_{ab}{}^{\overline{\gamma}} - 2\nabla_{\overline{\delta}}S_{ca}C_b{}^{c\overline{\delta}}\\
&-\widetilde{C}^{\gamma}(\nabla^d R_{d\gamma ab} - P^{\delta\overline{\epsilon}}\nabla_{[\delta}R_{\overline{\epsilon}]\gamma ab} + 2R_{\gamma\overline{\delta}ea}\widetilde{C}_b{}^{e\overline{\delta}}) - C_{ab}{}^{\overline{\gamma}}(\nabla^d\widetilde{R}_{d\overline{\gamma}} - P^{\delta\overline{\epsilon}}\nabla_{[\delta}R_{\overline{\epsilon}]\overline{\gamma}})\\
&+2(\widetilde{C}^{\gamma}R_{d\gamma ab} + C_{ab}{}^{\overline{\gamma}}\widetilde{R}_{d\overline{\gamma}ab})\nabla^d\Phi - 2C_{ab}{}^{\overline{\gamma}}\nabla_{\overline{\gamma}}\widetilde{C}^{\delta}\nabla_\delta\Phi + 4S_{ac}C_b{}^{c\overline{\delta}}\nabla_{\overline{\delta}}\Phi\\
&-2\widetilde{C}^{\gamma}\nabla_\gamma C_{ab}{}^{\overline{\delta}}\nabla_{\overline{\delta}}\Phi = 0, \qquad (5.51)\\
&\nabla^2 S_{abcd} - P^{\delta\overline{\epsilon}}\nabla_{[\delta}\nabla_{\overline{\epsilon}]}S_{abcd} - \widetilde{R}^{ef}{}_{cd}R_{efab} + 2\widetilde{R}_{f\overline{\epsilon}cd}\nabla^f C_{ab}{}^{\overline{\epsilon}} + 2R_{f\epsilon ab}\nabla^f\widetilde{C}_{cd}{}^{\epsilon}\\
&-2\nabla_{\overline{\epsilon}}\widetilde{C}_{cd}{}^{\gamma}\nabla_\gamma C_{ab}{}^{\overline{\epsilon}} + 2\nabla_{\overline{\epsilon}}S_{afcd}C_b{}^{f\overline{\epsilon}} + 2\nabla_\epsilon S_{abcd}\widetilde{C}_d{}^{f\epsilon}\\
&-C_{ab}{}^{\overline{\epsilon}}(\nabla^e\widetilde{R}_{e\overline{\epsilon}cd} - P^{\delta\overline{\gamma}}\nabla_{[\delta}\widetilde{R}_{\overline{\gamma}]\overline{\epsilon}cd} + 2\widetilde{R}_{\overline{\epsilon}\delta ec}\widetilde{C}_d{}^{e\delta}) - \widetilde{C}_{cd}{}^{\epsilon}(\nabla^e R_{e\epsilon ab}\\
&-P^{\delta\overline{\gamma}}\nabla_{[\delta}R_{\overline{\gamma}]\epsilon ab}) + 2(\widetilde{C}_{cd}{}^{\epsilon}R_{e\epsilon ab} + C_{ab}{}^{\overline{\epsilon}}\widetilde{R}_{e\overline{\epsilon}cd})\nabla^e\Phi - 2C_{ab}{}^{\overline{\gamma}}\nabla_{\overline{\gamma}}\widetilde{C}_{cd}{}^{\epsilon}\nabla_\epsilon\Phi + 4S_{abcf}\widetilde{C}_d{}^{f\epsilon}\nabla_\epsilon\Phi\\
&-2\widetilde{C}_{cd}{}^{\gamma}\nabla_\gamma C_{ab}{}^{\overline{\epsilon}}\nabla_{\overline{\epsilon}}\Phi + 4S_{afcd}C_b{}^{f\overline{\epsilon}}\nabla_{eb}\Phi = 0. \qquad (5.52)
\end{aligned}
$$

Since the Bianchi identities allow to write the curvature components in terms of the torsion components, we expect that the beta functions of the third set will be implied by the eight beta functions already proven, i.e first and second set. In the same way we expect that the beta functions of the fourth set will also be implied by the first two sets of beta functions since the constraints coming from holomorphicity and antiholomorphicity of the BRST current allows to relate the background fields to some components of the torsion. This is not too hard to check in the case of lower dimension, for example, at dimension 5/2 consider the beta functions coming from the coupling to $J\overline{\Pi}^{\overline{\beta}}$

$$
\nabla^a R_{a\overline{\beta}} + \nabla_{(\overline{\epsilon}}R_{\overline{\beta})\delta}P^{\delta\overline{\epsilon}} + 2(\nabla_{\overline{\beta}}C^{\overline{\alpha}} - R_{\overline{\beta}\gamma}P^{\gamma\overline{\alpha}})\nabla_{\overline{\alpha}}\Phi + 2C^{\overline{\alpha}}\nabla_{\overline{\alpha}}\nabla_{\overline{\beta}}\Phi = 0. \qquad (5.53)
$$

By using $R_{a\overline{\beta}} = T_{a\overline{\beta}}{}^{\gamma}\Omega_\gamma$ and $R_{\overline{\beta}\delta} = \nabla_{\overline{\beta}}\Omega_\delta$, which follow from the definition of the curvature, and $C^{\overline{\beta}} = P^{\alpha\overline{\beta}}\Omega_\alpha$, which follows from the antiholomorphicity constraints, we find that (5.53) can be written as

$$
(\nabla^c T_{c\overline{\beta}}{}^{\alpha} - 2\nabla_{\overline{\beta}}P^{\alpha\overline{\gamma}}\nabla_{\overline{\gamma}}\Phi + 2P^{\alpha\overline{\gamma}}\nabla_{\overline{\gamma}}\nabla_{\overline{\beta}}\Phi)\Omega_\alpha = 0, \qquad (5.54)
$$

so, the beta function (5.9) with dimension 2 implies (5.53) . Similarly we checked that (5.10) implies (5.37) and that the beta functions with dimension 5/2 (5.11) and (5.12) imply respectively the beta functions with dimension 3 (5.40) and (5.41) .

We have not found proofs for the vanishing of the remaining beta functions, since this task becomes clumsy as the dimension increases. Nevertheless, given the above explanation, we consider our work sufficient to assure that the beta functions vanish as a consequence of the classical BRST symmetry of the action for the Type II superstring in a generic supergravity background.

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