TESIS DE DOCTORADO

Restricted Lie (super)algebras, central extensions of non-associative algebras and some tapas

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by

MARÍA PILAR PÁEZ GUILLÁN

DISSERTATION

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Contents

Abstract

The general framework of this dissertation is the theory of non-associative algebras. We tackle diverse problems regarding restricted Lie algebras and superalgebras, central extensions of different classes of algebras and crossed modules of Lie superalgebras. Namely, we study the relationships between the structural properties of a restricted Lie algebra and those of its lattice of restricted subalgebras; we define a non-abelian tensor product for restricted Lie superalgebras and for graded ideal crossed submodules of a crossed module of Lie superalgebras, and explore their properties from structural, categorical and homological points of view; we employ central extensions to classify nilpotent bicommutative algebras; and we compute central extensions of the associative null-filiform algebras and of axial algebras. Also, we include a final chapter devoted to compare the two main methods (Rabinowitsch's trick and saturation) to introduce negative conditions in the standard procedures of the theory of automated proving and discovery.

Introduction

The theory of non-associative algebras has experienced a huge development in the XXth century, partially due to its connections with geometry, physics, biology and other areas of mathematics and science. By an *algebra* $(\mathrm{A}, \cdot)$ over a field F, we understand a vector space A over F together with a bilinear operation $\cdot : \mathrm{A} \times \mathrm{A} \to \mathrm{A}$, which is usually called *product* and denoted by juxtaposition rather than by $\cdot$. If a basis $\{e_i\}_{i \in I}$ of A is fixed, the coefficients c_{ij}^k of the products $e_i e_j$ with respect to such basis are called *structural constants*. Algebras can also be considered over unital commutative rings.

This general notion of an algebra turns out to be too wide to lead to interesting structural results. Therefore, it is customary to focus on different classes, or *varieties*, defined by imposing different polynomial conditions on the bilinear operation. Indeed, if we identify a finite-dimensional algebra structure over $\mathbb{C}$ with its n^3 structural constants with respect to a given basis, the varieties of non-associative algebras form actual geometric varieties in $\mathbb{C}^{n^3}$ with respect to the Zariski topology.

Some of the varieties which have been studied the most throughout this last century are associative algebras, Lie algebras, Jordan algebras, alternative algebras and Malcev algebras, but there are many more, such as binary Lie, non-commutative Jordan, left- or right-alternative, Leibniz, Novikov or bicommutative algebras, among others. These varieties share some common properties that allow to pose similar problems about them; however, the answers to these questions can be rather different, depending on the diverse behaviours of the varieties under the same general conditions. One of these challenges is the classification problem: to give a list of algebras or of families of algebras, pairwise non-isomorphic, such that any algebra in the variety is isomorphic to one of them. Complete classifications are hard to obtain, so it is typical to focus on algebras of small dimensions or satisfying additional properties. Varieties can also be classified from a geometric point of view, namely finding the irreducible components of the corresponding geometric varieties.

The theory of non-associative algebras also studies algebras with some additional structure such as restricted Lie algebras over fields of positive characteristic, Clifford algebras, vertex operator algebras, superalgebras or crossed modules of algebras; or which are not defined by polynomial identities, such as genetic algebras or axial algebras. These types of algebras do not form a proper variety in the sense introduced beforehand, but many of the typical methods and problems from the varieties context are still applicable. Apart from the classification problem (from an algebraic approach) mentioned above, they admit a representation theory and provide a suitable framework to study other problems also shared by different algebraic structures, such as developing homological and cohomological theories or relating the properties of an object to those of the lattice of its subobjects.

Structure of the thesis

This memory is structured in three blocks or parts, each of them dealing with different types of non-associative algebras. The first Part I is reserved to restricted Lie algebras and superalgebras, and the second Part II is concerned with central extensions of algebras. The last Part III is partially devoted to crossed modules of Lie superalgebras, although we have included some other contents which lack a direct relation with the rest of the manuscript. We also include a short preliminary chapter revising some basic concepts of category theory which will be needed further on.

On the other hand, regarding the problems addressed, this dissertation is organised around three cornerstones, namely, the classification problem of different types of algebras, the relationship between restricted Lie algebras and their lattice of restricted subalgebras, and diverse generalisations of the non-abelian tensor product of Lie algebras. The contents regarding the classification problem lie entirely in Part II, and the study of the lattice of restricted subalgebras of restricted Lie algebras, in Part I. Regarding the tensor products, they are divided into Part I and Part III.

The structure of each part will be specified in its own introduction.

We include here a short comment about the notation. While for algebras in general we will be employing the notation A (and variants like B), and denoting the product by juxtaposition, for restricted Lie (super)algebras and Lie superalgebras we will prefer to use the notation L (and variants like M or T), and to denote the product by a bracket [,] as it is usual in the literature. Furthermore, when we will be dealing with Lie superalgebras defined over rings instead of fields, we will employ M (and variants like N) to denote them.

Although we have tried to keep them as uniform as possible, there may be other differences in the notations through the different chapters.

Hypotheses and objectives

This thesis has the following hypotheses and objectives.

H1 The lattice of subalgebras of a Lie algebra was extensively studied in the last decades of the past century, but interest then waned because only few Lie algebras satisfied the prescribed properties under study. One of the reasons of this situation is that every one-dimensional subspace is a subalgebra. However, a one-dimensional subspace of a restricted Lie algebra is not necessarily a restricted subalgebra, what makes the study of lattices of restricted subalgebras of restricted Lie algebras potentially more interesting.

O1 To study restricted Lie algebras whose lattice of restricted subalgebras satisfies some prescribed properties, namely, it is distributive, Boolean, atomistic, dually atomistic, upper semimodular or lower semimodular.

H2 Non-abelian tensor products have been introduced in the categories of Lie algebras, restricted Lie algebras and Lie superalgebras, and have been used to characterise universal central extensions of perfect objects in such categories.

O2 To introduce a non-abelian tensor product in the category of restricted Lie superalgebras and to relate it to universal central extensions of restricted Lie superalgebras.

H3 The algebraic and geometric classifications of a number of varieties of non-associative algebras, such as Lie, associative, Jordan, Malcev or Leibniz, has been an interesting and active research area in the last years.

O3 To determine all the four-dimensional nilpotent bicommutative algebras over $\mathbb{C}$ up to isomorphism, as well as the one-generated nilpotent bicommutative algebras of dimensions 5 and 6 over $\mathbb{C}$. Also, to find the irreducible components of the variety of four-dimensional nilpotent bicommutative algebras over $\mathbb{C}$.

H4 There are not non-trivial associative central extensions of the associative null-filiform algebra μ_0^n, but it does admit non-trivial bicommutative central extensions.

O4 To determine the central extensions of the null-filiform associative algebra μ_0^n in the varieties of left-commutative (and right-commutative), bicommutative, left-alternative (and right-alternative), left-symmetric (and right-symmetric), assosymmetric, Jordan and Novikov algebras.

H5 Axial algebras were introduced in 2015 and have been extensively studied since, partially due to their connection to vertex operator algebras and the Monster. The classification of such algebras is an open problem.

O5 To describe a method for constructing new axial algebras as central extensions of another given axial algebra.

H6 The non-abelian tensor products in the context of crossed modules of Lie algebras have been defined and studied for two arbitrary abelian crossed modules and two ideal crossed submodules of a given crossed module.

O6 To introduce the non-abelian tensor and exterior products of two graded ideal crossed submodules of a crossed module of Lie superalgebras, and to relate them with the homology of crossed modules of Lie superalgebras.

H7 Whitehead's quadratic functor for modules proved to be a useful tool to study the non-abelian tensor and exterior products of Lie algebras and of ideal crossed submodules of a crossed module of Lie algebras. We introduced a version for supermodules in the context of the study of the non-abelian tensor and exterior products of graded ideal crossed submodules of a crossed module of Lie superalgebras.

O7 To study the properties of Whitehead's quadratic functor for supermodules and for abelian crossed modules of Lie superalgebras.

H8 In the algebraic-geometry-based theory of automated proving and discovery, there exist two main procedures for including negative conditions, namely Rabinowitsch's trick and saturation.

O8 To compare the methods of Rabinowitsch's trick and saturation for introducing negative theses and negative hypotheses in the standard procedures for automated proving of geometric theorems.

Methodology

This thesis has followed the classic methodology in basic research in mathematics. Some standard tasks in this type of research are proposals for definitions, conjectures of results that generalise others already known, or which can be compared with them, and the search for new examples that are significant enough or have important applications in other areas of mathematics. To do so, it is necessary to carry out a preliminary and comprehensive study of the topics to be addressed, and it is also very convenient to get in contact with experts of other universities. Finally, the use of computers to perform symbolic calculations was an essential tool in different parts of the thesis.

Categorical preliminaries

In this preliminary chapter, we include some categorical notions which require some knowledge on the subject and will be useful for Chapter 2 and Chapter 6.

Let $\mathbf{C}$ be a semiabelian category [130]. First of all, we recall two absolute concepts: the centre of an object, in the sense of [118], and the Higgins commutator of two normal subobjects, from [113, 169].

Let A, B and C be objects in $\mathbf{C}$. Given two coterminal morphisms $f\colon A \to C$ and $g\colon B \to C$, we say that they *commute* when there exists another morphism $h\colon A \times B \to C$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "f"'] & A \times B \arrow[d, "h"] & B. \arrow[l, "i_B"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

A subobject Z of A is said to be central when there exists a monomorphism $f\colon Z \to A$ commuting with the identity id_A, and the *centre* $Z(A)$ of A is the maximal central subobject.

Suppose now that B and C are subobjects of A, and consider $k\colon B \diamond C \to B + C$ the kernel of the canonical morphism from the coproduct into the product $B + C \to B \times C$. The *Higgins commutator* $[B, C]$ is defined to be the subobject of A making commutative the diagram

$$\begin{tikzcd} B \diamond C \arrow[r, "k"] \arrow[d] & B + C \arrow[d] \\ {[B, C]} \arrow[r] & A. \end{tikzcd}$$

Now, we recall the definition of a Birkhoff subcategory of $\mathbf{C}$, and present some other notions which are relative to a particular Birkhoff subcategory.

A *Birkhoff subcategory* $\mathbf{B}$ of $\mathbf{C}$ is a full and reflective subcategory closed under subobjects and quotient objects.

Fix a Birkhoff subcategory $\mathbf{B}$, and denote the inclusion functor by ι, and the reflector functor by $\mathbf{b}$. We say that an object A is *perfect* if $\mathbf{b}(A) = 0$, where 0 denotes the zero object in $\mathbf{C}$. Also, let us recall from [77, 78] the concept of the one-dimensional *relative commutator* $[A, A]_{\mathbf{B}}$ of A, defined to be the kernel of the unit of the adjunction $\iota \vdash \mathbf{b}$.

The following definitions will be also necessary for the development of Chapter 2 and Chapter 6. An *extension* in $\mathbf{C}$ is a regular epimorphism. Following the theory in [129], we distinguish trivial, normal and central extensions. An extension $\psi \colon B \to A$ is *trivial* if the induced square

$$\begin{tikzcd}
B \arrow[r, "\psi"] \arrow[d] & A \arrow[d] \\
\mathbf{b}(B) \arrow[r, "\mathbf{b}(\psi)"'] & \mathbf{b}(A)
\end{tikzcd}$$

is a pullback; it is *normal* if one of the projections in the kernel pair is trivial; and it is *central* if there exists another extension $\phi \colon C \to A$ such that the pullback $\phi^*(\psi)$ of ψ along ϕ is trivial. In our semiabelian context, the concepts of normal and central extension are equivalent.

The most usual Birkhoff subcategory is that of the *abelian* objects, i.e. those objects which can be endowed with the structure of an internal abelian group. It is commonly denoted by $\mathbf{Ab}$. The reflector functor $\mathbf{b}$ receives the name of *abelianisation* functor, and is denoted by $(\ )_{\mathrm{ab}}$. In this case, the relative commutator $[A, A]_{\mathbf{Ab}}$ coincides with the Higgins commutator $[A, A]$, and we find also a practical characterisation of central extensions: an extension $\psi \colon B \to A$ is central if and only if $\ker \psi \subseteq Z(B)$.

Finally, let us comment that the central extensions in $\mathbf{C}$ (relative to any Birkhoff subcategory) form another category, whose morphisms are exactly the morphisms in $\mathbf{C}$, $\theta \colon B \to C$, making commutative the diagram

$$\begin{tikzcd}
B \arrow[rr, "\theta"] \arrow[dr, "\psi"'] & & C . \arrow[dl, "\phi"] \\
& A &
\end{tikzcd}$$

A central extension ψ is said to be *universal* if it is the initial object in the category of central extensions, i.e. if given another central extension ϕ, there exists one and only one morphism θ between them. From the definition, it is clear that the universal central extension is unique up to isomorphisms.

Part I

Restricted Lie (super)algebras

Introduction

The theory of Lie algebras was initiated by Sophus Lie at the end of the XIXth century in connection with the study of Lie groups. For this reason, the first ground field considered were the complex numbers, and the first techniques employed were mostly analytical. At some point, a turn into algebraic methods led to the extension of the theory to Lie algebras over arbitrary fields of characteristic zero. One of the most celebrated results in this theory is the complete classification of real and complex finite-dimensional semisimple Lie algebras, given by Killing and E. Cartan. Also, Lie algebras over fields of characteristic zero present an important connection with group theory, not only by means of Lie groups but also through free groups [162] and also algebraic groups [55], and with other areas of science, especially with physics.

The origins of the theory of Lie algebras over fields of positive characteristic, or modular Lie algebras, can be traced back to some time before 1937, with the discovery by Witt of a simple modular Lie algebra which behaved very differently from the known algebras in characteristic zero. In 1939, Zassenhaus [236] generalised Witt's example, giving rise to the class of modular Lie algebras today known as Zassenhaus algebras. Since then, many mathematicians worked towards the obtention of a classification of simple algebras, and also on a more general study of this new field.

However, few of the classical methods for characteristic 0, both structural and from representation theory, are transferable to the modular case. One of the most powerful tools, the Killing form, is no longer useful in characteristic $p > 0$; most of the results on semisimple algebras are not true for the modular theory; there is no analogue of Lie's theorem on solvable Lie algebras (except in particular cases); Weyl's theorem of complete reducibility of semisimple algebras does not hold; there is no suitable Jordan-Chevalley decomposition of elements, etc. Moreover, and from a perhaps more conceptual point of view, the lack of a connection with group theory makes harder the understanding of modular Lie algebras.

A partial solution for these problems is given by restricted Lie algebras, introduced by Jacobson as early as in 1937 [124]. Namely, a *restricted Lie algebra* L over a field F of characteristic $p > 0$ is a Lie algebra endowed with a *p-map* ${}^{[p]} : L \to L$ which abstracts the properties of the p-power in associative algebras, i.e. such that the following conditions are satisfied:

$$(\lambda x)^{[p]} = \lambda^p x^{[p]},$$
$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y),$$
$$\mathrm{ad}(x^{[p]})(y) = \mathrm{ad}^p(x)(y)$$

for all $x, y \in L$, $\lambda \in F$, and where $\mathrm{ad}^{p-1}(x\otimes t + y \otimes 1)(x \otimes 1) = \sum_{i=1}^{p-1} i s_i(x,y) \otimes t^{i-1}$ in $L \otimes_F F[t]$.

In fact, Jacobson introduced first the term "*restrictable*" for those Lie algebras admitting a p-map; later, he preferred to employ the term "restricted" for such algebras together with a fixed p-map.

Restricted Lie algebras appear naturally from associative algebras and in the algebras of derivations. They do present a straightforward connection with linear algebraic groups; indeed, the Lie algebras of those groups are canonically endowed with a p-map. Also, restricted Lie algebras allow to recover, in a certain way, results from the non-modular theory as the Jordan-Chevalley decomposition of their elements [204], and they provide a nicer frame than ordinary Lie algebras for developing new techniques and subtle arguments over fields of positive characteristic.

Finite-dimensional simple restricted Lie algebras over algebraically closed fields of characteristic $p > 7$ were completely classified by Block and Wilson [27], giving a (partial) affirmative answer to the Kostrikin-Šafarevič conjecture [155] on simple restricted Lie algebras over algebraically closed fields of characteristic $p > 5$. This classification proved to be a fundamental tool in Strade and Wilson's generalisation [214] to simple Lie algebras over algebraically closed fields of characteristic $p > 7$, not necessarily restricted. For the sake of completeness, we indicate that the classification of finite-dimensional modular simple Lie algebras over algebraically closed fields has been extended to characteristic $p > 3$ by Premet and Strade [190].

This first part includes two chapters. The first Chapter 1 combines the contents of the article [168], a joint work with Nicola Maletesta and Salvatore Siciliano, with some work in progress with Salvatore Siciliano and David Towers. In particular, restricted Lie algebras having a distributive and Boolean lattice of restricted subalgebras are characterised, and other properties of this lattice, such as being atomistic and dually atomistic, lower semimodular and upper semimodular, are studied for restricted Lie algebras over algebraically closed fields. Additionally, restricted Lie algebras over algebraically closed fields whose restricted subalgebras are restricted quasi-ideals are also investigated. On the other hand, in the second Chapter 2 we turn our attention to restricted Lie superalgebras, and combine the algebraic techniques with some category theory to define a non-abelian tensor product of restricted Lie superalgebras, as well as to study its relation with the universal central extensions in this category. This chapter collects some work in progress with Manuel Ladra.

The structure of each chapter will be described in its own introduction.

CHAPTER 1

On the subalgebra lattice of a restricted Lie algebra

In this chapter, we study the lattice of restricted subalgebras of a restricted Lie algebra. In particular, we consider those algebras in which this lattice is distributive, Boolean, dually atomistic, lower or upper semimodular, or in which every restricted subalgebra is a restricted quasi-ideal. The fact that there are one-dimensional subalgebras which are not restricted results in some of these conditions being weaker than for the corresponding conditions in the non-restricted case.

Introduction

The relationship between the structure of a group and that of its lattice of subgroups is highly developed and has attracted the attention of many leading algebraists (see e.g. the monograph [202] or the survey [185]). According to Schmidt [202], the origin of the subject can be traced back to Dedekind, who studied the lattice of ideals in a ring of algebraic integers; he discovered and used the modular identity, which is also called the Dedekind law, in his calculation of ideals. However, the actual beginnings of the study of subgroup lattices date from around 1930. One of the first remarkable achievements in this context was Ore's characterisation of groups with distributive subgroup lattices as the locally cyclic groups [183]. Since then modularity, distributivity and lattice conditions related to them have been studied in a number of contexts. The lattice of submodules of a module over a ring is modular, and hence so is the lattice of subgroups of an abelian group. The lattice of normal subgroups of a group is also modular, but the lattice of all subgroups is not in general [122, 123]. The lattice of ideals of a ring is also modular. The distributivity of the lattice of submodules of a module has been investigated in [46, 212, 218], and of the lattice of (right) ideals of a ring, or of different types of non-associative algebras, in [37, 131, 170, 218].

The study of the subalgebra and ideal lattices of a finite-dimensional Lie algebra was popular in the second half of the last century, especially in the 1980's and in the 90's (see, for example, [13, 18, 24, 33, 92–95, 100, 154, 160, 216, 217, 220–223]), but interest then waned. The likely reason is that most of the conditions under investigation were too strong and so few algebras satisfied them; a paradigmatic example of this is the characterisation of Lie algebras having a distributive lattice of subalgebras in [154, Theorem 2.1]. However, the lattice of restricted subalgebras of a restricted Lie algebra is fundamentally different; for example, not every element spans a one-dimensional restricted subalgebra. Thus, one could expect more interesting results to hold for restricted algebras and, as we shall see, this is indeed the case.

This chapter is organised as follows. In Section 1.1 we fix some notation and terminology and introduce some results that are needed later. In Section 1.2 we first establish when L has exactly two restricted subalgebras. Next, we provide a characterisation of restricted Lie algebras that have a distributive or Boolean lattice of restricted subalgebras, and also study some particular cases. Throughout most of the rest of the chapter, we will be assuming that the algebras have finite dimension and that the ground field is algebraically closed. In Section 1.3 we study restricted Lie algebras that are atomistic and dually atomistic. It turns out that they are more abundant than in the non-restricted case. We then investigate those restricted Lie algebras all of whose subalgebras (not necessarily restricted) are intersections of maximals. The objective in Section 1.4 is to study restricted Lie algebras L in which every restricted subalgebra is a restricted quasi-ideal. We characterise nilpotent restricted Lie algebras satisfying this condition over perfect fields of characteristic different from 2. Section 1.5 then goes on to consider J-algebras and lower semimodular restricted Lie algebras, concepts which turn out to be equivalent if the algebra is solvable. The final Section 1.6 is devoted to studying upper semimodular restricted Lie algebras. It is shown that these algebras are either almost abelian or nilpotent of class at most 2 and that every restricted subalgebra is a restricted quasi-ideal.

Throughout this chapter, F will denote a field of characteristic $p > 0$, unless otherwise stated. All the Lie algebras and restricted Lie algebras will be assumed to be over F. Unless in Section 1.2, throughout the rest of the paper all algebras are supposed to be finite-dimensional. Unless in Section 1.2, throughout the rest of the paper all algebras are supposed finite-dimensional. We will denote algebra direct sums by $\oplus$, and direct sums of the vector space structure alone, by $\dot{+}$.

1.1 Preliminaries

Here we fix some notation and terminology and introduce some results that will be needed later. Let L be a Lie algebra over a field F of arbitrary characteristic. We denote by $[L, L]$ the derived subalgebra of L. As usual, the *derived series* for L is defined inductively by $L^{(0)} = L$, $L^{(k+1)} = [L^{(k)}, L^{(k)}]$ for $k \geq 0$, $L^{(\infty)} = \cap_{k\geq 0} L^{(k)}$; L is *solvable* if $L^{(\infty)} = 0$. Similarly, the *lower central series* is defined inductively by $L^1 = L$, $L^{k+1} = [L^k, L]$ for $k \geq 1$; L is *nilpotent* if $L^k = 0$ for some $k \geq 1$. Also, L is said to be *supersolvable* if it admits a complete flag made up of ideals of L, that is, there exists a chain

$$0 = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n = L$$

of ideals of L such that $\dim L_j = j$ for every $0 \leq j \leq n$. The centre of L is denoted by $Z(L)$, and $C_B(A) = \{x \in B : [x, A] = 0\}$ denotes the *centraliser* in a subalgebra B of another subalgebra A. Also, the *ascending central series* is defined inductively by $C_1(L) = Z(L)$, $C_{n+1}(L) = \{x \in L : [x, L] \subseteq C_n(L)\}$. The *nilradical* $N(L)$ is defined to be the maximal nilpotent ideal, and the *solvable radical*, denoted by $R(L)$, is defined to be the maximal solvable ideal. For every $x \in L$, the *adjoint map* of x is defined by $\mathrm{ad}x : L \to L$, $a \mapsto [x, a]$. If X is a subalgebra of L, then the largest ideal of L contained in X is called the *core* of L and is denoted by X_L. The *Frattini subalgebra* $F(L)$ of L is the intersection of all maximal subalgebras of L; the *Frattini ideal* of L is $\phi(L) = F(L)_L$. The *abelian socle*, $Asoc(L)$, is the sum of the minimal abelian ideals of L.

We say that L is *dually atomistic* if every proper subalgebra of L is an intersection of maximal subalgebras of L. It is called *almost abelian* if it contains an abelian ideal of codimension 1, on which it acts by scalar multiplications. Scheiderer proved in [201] that, over a field of characteristic zero, every dually atomistic Lie algebra is abelian, almost abelian or simple. In his proof, he used some easy results, stated for Lie algebras over a field of characteristic zero but which are valid over any field. We state them here for the sake of convenience.

Lemma 1.1.1. *Let L be a dually atomistic Lie algebra over any field. Then:*

(i) For any maximal subalgebra M of L, $M \cap N(L)$ is an ideal of L;

(ii) $N(L)$ is abelian and every subspace of $N(L)$ is an ideal of L; and

(iii) $R(L)$ is abelian or almost abelian.

Now we establish a slightly weaker version of Scheiderer's result which is valid over any field.

Proposition 1.1.2. *If L is a dually atomistic Lie algebra over any field then L is either abelian, almost abelian or semisimple.*

Proof. Let L be dually atomistic and suppose that L is not semisimple. Then, it holds that $Asoc(L) \neq 0$ and L splits over $Asoc(L)$, by [215, Theorem 7.3]. Furthermore, the minimal abelian ideals of L are one-dimensional by Lemma 1.1.1(ii), so we can write $L = (Fa_1 \oplus \ldots \oplus Fa_n) \dot{+} B$, where Fa_i is a minimal ideal of L for each $1 \leq i \leq n$, B is a subalgebra of L, and $n \geq 1$.

Let M be a maximal subalgebra of L with $a_1 \notin M$. We shall show that $L^{(\infty)} \subseteq M$. Now $L = Fa_1 + M$, so M has codimension 1 in L. It follows that L/M_L is as described in [12, Theorems 3.1 and 3.2]. Also, $[Fa_1, M_L] \subseteq Fa_1 \cap M = 0$. We consider the three cases given in [12, Theorem 3.1] separately.

Case (a): Here L/M_L is one-dimensional, so $M = M_L$ and $L^2 \subseteq M$.

Case (b): Here L/M_L is two-dimensional, so $L = Fa_1 + Fm + M_L$ where $m \in M \setminus M_L$. Now $L^2 \subseteq Fa_1 + M_L$ and $L^{(2)} \subseteq M_L \subseteq M$.

Case (c): Here $L/M_L \simeq L_m(\Gamma)$. If m is odd then $L_m(\Gamma)$ is simple. But $(Fa_1 + M_L)/M_L$ is a one-dimensional ideal of L/M_L, which is a contradiction. Hence m is even, in which case $L_m(\Gamma) = Fx + L_m(\Gamma)^2$, where $L_m(\Gamma)^2$ is simple. Put $L/M_L = \bar{L}$, and so on. Then $\bar{L} = F\bar{a}_1 \oplus \bar{L}^2$ and $[\bar{L}, \bar{a}_1] = \bar{0}$; that is, $[L, a_1] \subseteq M_L$, whence $L^2 \subseteq M$.

In any case we have established that, for any maximal subalgebra M of L, either $a_1 \in M$ or $L^{(\infty)} \subseteq M$. Suppose that $L^{(\infty)} \neq 0$. Then $L^{(\infty)} \neq Fa_1$ (since $(Fa_1)^2 = 0$), so there is an element $x \in L^{(\infty)} \setminus Fa_1$. Let M be a maximal subalgebra containing $x + a_1$. Then either $a_1 \in M$ or $L^{(\infty)} \subseteq M$. In each case, $Fx + Fa_1 \subseteq M$. It follows that $F(x+a_1)$ cannot be an intersection of maximal subalgebras of L, a contradiction. Hence, $L^{(\infty)} = 0$ and L is solvable. The result now follows from Lemma 1.1.1(iii). □

We shall need the following result due to Grunewald, Kunyavskii, Nikolova and Plotkin for $p > 5$. However, the same proof works for $p > 3$ by using the Corollary in page 180 of [210]. Alexander Premet has pointed out that the result is also valid for $p = 3$, but that it relies on results that have not yet been published, so we omit this case.

Lemma 1.1.3. *Every simple Lie algebra L over an algebraically closed field F of characteristic $p > 3$ contains a subalgebra X with a quotient isomorphic to $\mathfrak{sl}(2, F)$.*

Proof. The proof is the same as for [106, Lemma 3.2] with the reference to [228, Part II, Corollary 1.4] replaced by [210, page 180, Corollary]. □

In what follows we shall be studying the lattice of restricted subalgebras of a restricted Lie algebra. From now on, the characteristic of the ground field F will be assumed to be $p > 0$.

If L is a restricted Lie algebra, we introduce as "restricted analogues" of earlier concepts, $F_p(L)$, the *Frattini p-subalgebra* of L, to be the intersection of the maximal restricted subalgebras of L, and $\phi_p(L)$, the *Frattini p-ideal* of L, to be the largest restricted ideal of L which is contained in $F_p(L)$. The *abelian p-socle*, $A_p soc(L)$, is the sum of the minimal abelian restricted ideals of L.

Moreover, if S is a subset of L, then we use the symbol $\langle S\rangle_p$ for the restricted subalgebra generated by S, and we put $S^{[p]} = \langle x^{[p]} | \, x \in S\rangle_p$. Following Hochschild [115], we say that L is *strongly abelian* if L is abelian and $L^{[p]} = 0$.

An element x of L is said to be *p-algebraic* if $\langle x^{[p]}\rangle_p$ is finite-dimensional. Moreover, x is said to be *semisimple* if $x \in \langle x^{[p]}\rangle_p$ and *toral* if $x^{[p]} = x$. An abelian restricted Lie algebra consisting of semisimple elements is called a *torus*. An element x of L is said to be *p-nilpotent* if $x^{[p]^n} = 0$ for some $n > 0$, and L is said to be *p-nil* if it consists of p-nilpotent elements. In particular, L is said to be *p-nilpotent* if there exists $n > 0$ such that $x^{[p]^n} = 0$ for every $x \in L$. By Engel's Theorem, it is easy to see that every finite-dimensional p-nil restricted Lie algebra is p-nilpotent but, in general, this can fail in the infinite-dimensional case.

We say that L is *cyclic* if $L = \langle x\rangle_p$ for some $x \in L$. In particular, if the generator x is p-nilpotent, then L is called *nilcyclic*. Moreover, L is called *locally finite-dimensional* (*locally cyclic*, respectively) if every finitely generated restricted subalgebra of L is finite-dimensional (cyclic, respectively).

Note that, up to isomorphism, for every non-negative integer n there exists a unique nilcyclic restricted Lie algebra C_n of dimension n over F. Every C_n embeds into C_{n+1}, and $\{C_n\}_{n\geq 0}$ with these inclusions forms a direct system. We will consider the direct limit

$$C_\infty = \varinjlim C_n.$$

Note that C_∞ has an F-basis consisting of elements $x_0, x_1, \dots$ such that $x_0^{[p]} = 0$ and $x_i^{[p]} = x_{i-1}$ for every $i > 0$. Clearly, C_∞ is the analogue of a Prüfer p-group in the context of restricted Lie algebras.

For a field F of characteristic $p > 0$ we will denote by $F[t, \sigma]$ the skew polynomial ring over F in the indeterminate t with respect to the Frobenius endomorphism σ of F. Thus $F[t, \sigma]$ is the ring consisting of all polynomials $f = \sum_{i \geq 0} \lambda_i t^i$ with respect to the usual sum and multiplication defined by the condition $t \cdot \lambda = \lambda^p t$ for every $\lambda \in F$. We recall that this ring is a principal left ideal domain, and it is also a principal right ideal domain in the case when F is perfect (see e.g. [126, Section 3.1]). As observed by Jacobson in [127, Section V.8], the study of $F[t, \sigma]$ and its modules turns out to be a natural tool for several questions concerning abelian restricted Lie algebras.

For a restricted Lie algebra L, we denote by $(\mathcal{S}(L), \vee, \wedge)$ its lattice of restricted subalgebras. Of course, for every $X, Y \in \mathcal{S}(L)$ one has that $X \wedge Y = X \cap Y$ and $X \vee Y$ is the restricted subalgebra generated by X and Y, $\langle X, Y \rangle_p$. We will say that the restricted Lie algebra L satisfies a certain lattice-theoretic property whenever such property is satisfied by the lattice $\mathcal{S}(L)$.

1.2 Distributive and Boolean restricted Lie algebras

Let L be a restricted Lie algebra. In our first result, which will be used later, we establish when $\mathcal{S}(L)$ is the two-elements lattice (in other words, L has no non-zero proper restricted subalgebras).

Proposition 1.2.1. *A restricted Lie algebra $L \neq 0$ has no non-zero proper restricted subalgebras if and only if $L \simeq \mathcal{L}/\langle \bar{f} \rangle_p$, where $\mathcal{L} = \langle x \rangle_p$ is a free cyclic restricted Lie algebra and $\bar{f} = \sum_{i \geq 0} \lambda_i x^{[p]^i}$ is an element of $\mathcal{L}$ such that $f = \sum_{i \geq 0} \lambda_i t^i$ is an irreducible element of the ring $F[t, \sigma]$.*

Proof. Assume first that L does not have any non-zero proper restricted subalgebras. Then, for every non-zero element y of L we must have $L = \langle y \rangle_p$, hence L is cyclic. As a consequence, L is isomorphic to $\mathcal{L}/\langle \bar{f} \rangle_p$ for some $\bar{f} = \sum_{i \geq 0} \lambda_i x^{[p]^i} \in \mathcal{L}$. Note that $\bar{f} \neq 0$, as the free cyclic restricted Lie algebra obviously contains non-zero proper restricted subalgebras. Now, as L is abelian we can regard it as a left module over $F[t, \sigma]$ with respect to the action defined by the condition $t * a = a^{[p]}$ for every $a \in L$. Note that $\mathcal{S}(L)$ coincides with the lattice of $F[t, \sigma]$-submodules of L. Moreover, the annihilator $\mathrm{ann}(x)$ of the element x is given by the left ideal of $F[t,\sigma]$ generated by $f = \sum_{i\geq 0} \lambda_i t^i$, and L is isomorphic as a left $F[t,\sigma]$-module to $F[t,\sigma]/\mathrm{ann}(x)$. Therefore, as such a module cannot be simple if $f = gh$ for some non-constant elements g, h of $F[t,\sigma]$, we conclude that f is an irreducible skew polynomial.

For the converse observe that, as $F[t,\sigma]$ is a left principal ideal domain and f is irreducible, $\mathcal{L}/\langle \bar{f} \rangle_p$ is a simple $F[t,\sigma]$-module. In view of the correspondence between the $F[t,\sigma]$-submodules and restricted subalgebras of L, this yields the result. □

Remark 1.2.2. It is worth observing that Proposition 1.2.1 provides a description of the composition factors of finite-dimensional solvable restricted Lie algebras. Moreover, in the statement of Proposition 1.2.1 one has that L is either one-dimensional strongly abelian or a cyclic torus corresponding to whether $\lambda_0 = 0$ or $\lambda_0 \neq 0$, respectively. However, again by Proposition 1.2.1, the converse is not true, because it is possible for a cyclic torus to contain non-zero proper restricted subalgebras.

In some cases, the statement of Proposition 1.2.1 takes a particularly simple form. For instance, if the ground field of L is the field F_p with p elements, then $F[t,\sigma]$ is nothing more than an ordinary polynomial ring. Furthermore, if F is algebraically closed, then by combining Proposition 1.2.1, Remark 1.2.2 and [213, Chapter 2, Theorem 3.6(2)] we conclude that L has no non-zero proper restricted subalgebras if and only if $\dim L = 1$.

Let L be an ordinary Lie algebra over a field of characteristic $p > 0$. For the definition of the *minimal p-envelope* of L we refer the reader to [213, Section 2.5]. Another consequence of Proposition 1.2.1 is the following result.

Corollary 1.2.3. *A restricted Lie algebra $L \neq 0$ has no non-zero proper restricted ideals if and only if one of the following conditions hold:*

1. *L is the minimal p-envelope of a simple Lie algebra;*

2. *$L \simeq \mathcal{L}/\langle \bar{f} \rangle_p$, where $\mathcal{L} = \langle x \rangle_p$ is a free cyclic restricted Lie algebra and $\bar{f} = \sum_{i\geq 0} \lambda_i x^{[p]^i}$ is an element of $\mathcal{L}$ such that $f = \sum_{i\geq 0} \lambda_i t^i$ is an irreducible element of the ring $F[t,\sigma]$.*

Proof. The claim follows from [27, p. 116] (see also [82, Lemma 6.3]) in the non-abelian case and from Proposition 1.2.1 in the abelian one. □

We say that a restricted Lie algebra L is *distributive* if, for every $X, Y, Z \subseteq L$ restricted subalgebras, one has

$$\langle X, Y \cap Z\rangle_p = \langle X, Y\rangle_p \cap \langle X, Z\rangle_p,$$

or, equivalently,

$$X \cap \langle Y, Z\rangle_p = \langle X \cap Y, X \cap Z\rangle_p.$$

Recall that this just means that the lattice $(\mathcal{L}, \vee, \wedge)$ is distributive in the usual sense. By a well-known theorem of Birkhoff (cf. [105, Section II.1, Theorem 1]), a lattice turns out to be distributive if and only if it does not contain the pentagon or the diamond as a sublattice.

Now we study when a restricted Lie algebra is distributive. By a classical theorem of Ore [183], a group G has a distributive lattice of subgroups if and only if G is locally cyclic. One might expect that the natural analogue of this result also holds in the setting of restricted Lie algebras. However, the next example shows that this is not the case.

Example 1.2.4. Let $L = Fx \oplus Fy$ be the abelian restricted Lie algebra with p-map defined by $x^{[p]} = y$ and $y^{[p]} = x$. Obviously, L is cyclic. On the other hand, L has $p+1$ distinct restricted subalgebras of dimension 1 given by $\langle x + \zeta_i y\rangle_p$, where ζ_i is a $(p+1)$th root of unity for every $i = 1, 2, \dots, p+1$. In particular, $\mathcal{S}(L)$ contains the diamond as a sublattice, and therefore $\mathcal{S}(L)$ is not distributive by [105, Section II.1, Theorem 1].

Distributive restricted Lie algebras are characterised in the following result:

Theorem 1.2.5. *A restricted Lie algebra L is distributive if and only if L is abelian and, for every restricted subalgebra H of L, L/H does not contain distinct isomorphic minimal restricted subalgebras.*

Proof. Assume first that L is distributive and let $x, y \in L$. One has $\langle x + y\rangle_p \vee \langle x\rangle_p = \langle x, y\rangle_p = \langle x + y\rangle_p \vee \langle y\rangle_p$ and so the distributivity of L implies that $\langle x + y\rangle_p \vee$

$\big(\langle x\rangle_p \cap \langle y\rangle_p\big) = \langle x, y\rangle_p$. It follows that

$$\frac{\langle x, y\rangle_p}{\langle x\rangle_p \cap \langle y\rangle_p} \simeq \frac{\langle x + y\rangle_p}{\langle x\rangle_p \cap \langle y\rangle_p \cap \langle x + y\rangle_p}$$

is cyclic. Moreover, as $\langle x\rangle_p \cap \langle y\rangle_p$ is clearly contained in the centre of $\langle x, y\rangle_p$, we conclude that $\langle x, y\rangle_p$ is abelian. Therefore, as in the proof of Proposition 1.2.1 we can regard L as a left $F[t, \sigma]$-module of L. Recall that the restricted subalgebras of L coincide with the $F[t, \sigma]$-submodules of L. Moreover, it is easily seen that two restricted subalgebras H_1 and H_2 of L are isomorphic if and only if they are isomorphic as $F[t, \sigma]$-submodules. By [46, Theorem 1], the $F[t, \sigma]$-module L has a distributive lattice of submodules if and only if, for every submodule H, the quotient L/H does not have contain two isomorphic simple submodules. From this the necessity part follows at once.

Conversely, as L is abelian we can regard it as a left $F[t, \sigma]$-module. For what was observed in the first part, for every $F[t, \sigma]$-module H of L we have that L/H does not contain distinct isomorphic simple submodules. Therefore [46, Theorem 1] ensures that the lattice of $F[t, \sigma]$-submodules of L is distributive, which is equivalent to say that $\mathcal{S}(L)$ is distributive, which completes the proof. □

Note that Theorem 1.2.5 together with [83, Theorem 2.4] yield that not every artinian restricted Lie algebra of finite representation type has a distributive lattice of subalgebras, contrary to the situation of the lattice of ideals of artinian associative algebras (cf. [188, Section 6.7]).

Corollary 1.2.6. *Let L be a distributive restricted Lie algebra over a perfect field. Then, L is locally cyclic. Furthermore, if $F = F_p$, then the converse is also true.*

Proof. By Theorem 1.2.5, L is abelian. If H is a finitely generated restricted subalgebra of L, then in view of [15, Section 4.3, Theorem 3.1] we have $H = \langle x_1\rangle_p \oplus \cdots \oplus \langle x_n\rangle_p$ for some elements $x_1, \dots, x_n \in L$. Let us show that H is cyclic by induction on n. Let $n > 1$, as otherwise the claim is trivial. By the induction hypothesis we have $\langle x_1\rangle_p \oplus \cdots \oplus \langle x_{n-1}\rangle_p = \langle y\rangle_p$ for a suitable $y \in L$. Since L is distributive, the same argument used in the first part of the proof of Theorem 1.2.5 yields $H \simeq \langle y + x_n\rangle_p$, which completes the inductive step.

Now, assume that $F = F_p$ and L is locally cyclic. Then L is abelian and so it can be regarded L as a left $F[t,\sigma]$-module. Notice that in this case $F[t,\sigma]$ is just the ordinary polynomial ring $F_p[t]$. Let $A, B, C \in \mathcal{S}(L)$. We need only to show that $(A + B) \cap (A + C) \subseteq A + (B \cap C)$, as the other inclusion is trivially true. Let $x \in (A + B) \cap (A + C)$. Then $x = a_1 + b = a_2 + c$ for some $a_1, a_2 \in A$, $b \in B$ and $c \in C$. Since L is locally cyclic, there exists $y \in L$ such that $\langle a_1, a_2, b, c\rangle_p = \langle y\rangle_p$. It follows that

$$\langle y\rangle_p = \left(A \cap \langle y\rangle_p\right) + \left(B \cap \langle y\rangle_p\right) = \left(A \cap \langle y\rangle_p\right) + \left(C \cap \langle y\rangle_p\right). \tag{1.2.1}$$

If $A \cap \langle y\rangle_p = 0$, then $a_1 = a_2 = 0$ and $x \in B \cap C$. On the other hand, if $B \cap \langle y\rangle_p = 0$, then $x = a_1 \in A$ and, analogously, if $C \cap \langle y\rangle_p = 0$, then $x = a_2 \in A$. Thus we may assume that A, B and C meet $\langle y\rangle_p$ non-trivially. Consider $\langle y\rangle_p$ as a left module over the ring $F_p[t]/\mathrm{ann}(y)$. As the class of cyclic restricted subalgebras is closed under restricted subalgebras, there exist $f, g, h \in F_p[t]/\mathrm{ann}(y)$ such that

$$A \cap \langle y\rangle_p = \langle f * y\rangle_p, \quad B \cap \langle y\rangle_p = \langle g * y\rangle_p, \quad C \cap \langle y\rangle_p = \langle h * y\rangle_p.$$

Since $F_p[t]/\mathrm{ann}(y)$ is a commutative principal ideal domain, by relation (1.2.1) we see that $\gcd(f, g) = \gcd(f, h) = 1$ and so $\gcd(f, gh) = 1$. Therefore there exist $\xi, \zeta \in F_p[t]/\mathrm{ann}(y)$ such that $\xi f + \zeta gh = 1$. Note that $(gh) * y = (hg) * y \in B \cap C \cap \langle y\rangle_p$. As a consequence, we get

$$y = (\xi f + \zeta gh) * y \in \left(A \cap \langle y\rangle_p\right) + \left(B \cap C \cap \langle y\rangle_p\right) \subseteq A + (B \cap C).$$

Therefore $x \in \langle y\rangle_p \subseteq A + (B \cap C)$, which completes the proof. □

Remark 1.2.7. In view of Example 1.2.4, the second part of Corollary 1.2.4 does not hold when F is an arbitrary perfect field of characteristic $p > 0$.

We now deal with finite-dimensional p-nilpotent restricted Lie algebras. We have the following result:

Theorem 1.2.8. *Let L be a finite-dimensional p-nilpotent restricted Lie algebra. Then the following conditions are equivalent:*

1. L is distributive;

2. the lattice of restricted subalgebras of L is a chain;

3. L is nilcyclic.

Proof. Suppose first that the lattice of restricted subalgebras of L is distributive. By [164, Corollary 5.2(ii)] we have $\phi_p(L) = [L, L] + L^{[p]}$. It follows that $L/\phi_p(L)$ is strongly abelian and so every vector subspace of it is a restricted subalgebra. Consequently, as the lattice of subalgebras of $L/\phi_p(L)$ is distributive, we must have $\dim L/\phi_p(L) \leq 1$. Let $s \in L$, $s \notin \phi_p(L)$. As the image of s in $L/\phi_p(L)$ spans $L/\phi_p(L)$, it follows from [199, Lemma 3.1(2)] that s generates L as a restricted ideal. As L is abelian (by Theorem 1.2.5), we conclude that L is nilcyclic.

Now suppose that L is nilcyclic and let x be a generator of L. If n is the minimal integer such that $x^{[p]^n} = 0$, then the restricted subalgebras of L are of the form $\langle x^{[p]^i} \rangle_p$ for $i = 0, 1, \dots, n$. In particular, the lattice of restricted subalgebras of L is a chain and so distributive, which completes the proof. □

In the next result we consider p-nil restricted Lie algebras of arbitrary dimension. We will prove that, up to isomorphism, the only infinite-dimensional p-nil restricted Lie algebra with a distributive lattice of restricted subalgebras is C_∞.

Corollary 1.2.9. *Let L be a p-nil restricted Lie algebra. Then L is distributive if and only if it is either nilcyclic or isomorphic to C_∞.*

Proof. By Theorem 1.2.8 the claim holds when L is finite-dimensional. Suppose then that L has infinite dimension. The condition is clearly sufficient, as the restricted subalgebras are given by the chain $0 = C_0 \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_\infty$.

Conversely, suppose that L is distributive. By Theorem 1.2.5 we have that L is abelian and so, as it is p-nil, if H is a finitely generated restricted subalgebra of L, then H is p-nilpotent and has finite dimension n. Now, by Theorem 1.2.8, H is nilcyclic, and moreover, by Theorem 1.2.5, H is the unique restricted subalgebra of L of dimension n. This implies that $L \simeq C_\infty$, and the claim follows at once. □

For restricted Lie algebras defined over algebraically closed fields we have the following result:

Theorem 1.2.10. *Let L be a restricted Lie algebra over an algebraically closed field. Then L is distributive if and only it is isomorphic to a restricted subalgebra of $T \oplus C_\infty$, where T is a one-dimensional torus.*

Proof. Assume first that L is distributive. We can suppose that L is not p-nil, otherwise the assertion immediately follows from Corollary 1.2.9. By Theorem 1.2.5 we know that L is abelian, therefore the semisimple elements of L form a torus T (see [213, Chapter 2, Proposition 3.3(3)]) and the p-nilpotent elements form a p-nil restricted subalgebra P. We claim that every element of L is p-algebraic. Suppose to the contrary that there is an element $h \in L$ that is not p-algebraic. Then h generates a free restricted subalgebra H. Since $H/\langle h^{[p]^2} - h\rangle_p$ is isomorphic to the restricted Lie algebra of Example 1.2.4, we deduce that its lattice of restricted subalgebra is not distributive, a contradiction. As a consequence, by [213, Chapter 2, Theorem 3.5], for every $x \in L$ we can consider its Jordan-Chevalley decomposition, i.e., $x = x_s + x_n$, where x_s is semisimple, x_n is p-nilpotent, and $[x_s, x_n] = 0$. This shows that $L = T \oplus P$. Suppose, if possible, that there exist two F-linearly independent elements x_1 and x_2 of T. Since F is algebraically closed, by [213, Chapter 2, Theorem 3.6] the finite-dimensional torus $\langle x_1, x_2\rangle_p$ has a basis consisting of toral elements. In particular, L contains two distinct isomorphic minimal restricted subalgebras. This contradicts Theorem 1.2.5, and thus T is one-dimensional. Moreover, by Corollary 1.2.9 we have that P is either nilcyclic or isomorphic to C_∞. In any case, L is isomorphic to a restricted subalgebra of $T \oplus C_\infty$.

Let us prove sufficiency. Since sublattices of a distributive lattice are again distributive, it is enough to prove that $\mathcal{S}\left(T \oplus C_\infty\right)$ is distributive. Also, as $T \oplus C_\infty$ is abelian, in view of Theorem 1.2.5 it suffices to show that for every restricted subalgebra H of $T \oplus C_\infty$, $\left(T \oplus C_\infty\right)/H$ does not contain distinct isomorphic minimal restricted subalgebras. Suppose first that $H \nsubseteq C_\infty$. As the ground field is algebraically closed, by [213, Chapter 2, Theorem 3.6(2)] we have $T = Fy$ with $y^{[p]} = y$. Let $h \in H$, $h \notin C_\infty$. Then we can write $h = \lambda y + \mu c$ for some $c \in C_\infty$ and $\lambda, \mu \in F$ with $\lambda \neq 0$. For a sufficiently large n we have $h^{[p]^n} = \lambda^{p^n} y$, which shows that both y and c are in H. This entails that H is of the form $H = T \oplus P$ for some restricted subalgebra P of C_∞. It follows that $\left(T \oplus C_\infty\right)/H$ is either zero or isomorphic to C_∞, so it does not contain distinct isomorphic (minimal) restricted subalgebras. Now suppose $H \subseteq C_\infty$. Then $\left(T \oplus C_\infty\right)/H$ is either isomorphic to T or to $T \oplus C_\infty$. In the former case, the only restricted subalgebras are zero and T. On the other hand, for what we have proved above, the restricted subalgebras of $T \oplus C_\infty$ are either restricted subalgebras of C_∞ or of the form $T \oplus P$ for some restricted subalgebra P of C_∞. Since these restricted subalgebras are pairwise non-isomorphic, this completes the proof. □

A restricted Lie algebra L is called *complemented* if for every restricted subalgebra X there exists another restricted subalgebra Y such that $X \cap Y = 0$ and $\langle X, Y\rangle_p = L$. The algebra L is said to be *Boolean* if it is both distributive and complemented.

Another consequence of Theorem 1.2.5 yields a characterisation of Boolean restricted Lie algebras.

Corollary 1.2.11. *A restricted Lie algebra L is Boolean if and only $L \simeq \bigoplus_{I\in\mathcal{F}} I$, where $\mathcal{F}$ is a family of pairwise non-isomorphic restricted Lie algebras each having no non-zero proper restricted subalgebras.*

Proof. Let us suppose first that L is Boolean. By Theorem 1.2.5 we know that L is abelian, and therefore L can be regarded as a left $F[t,\sigma]$-module. As the restricted subalgebras of L are exactly the $F[t,\sigma]$-submodules and L is complemented, it follows that L is a semisimple left $F[t,\sigma]$-module. Hence L is isomorphic to a direct sum of restricted Lie algebras having no non-zero proper restricted subalgebras, and by Theorem 1.2.5 these are pairwise non-isomorphic.

Now let us prove the converse. Notice that L is abelian, and thus it can be regarded as a left $F[t,\sigma]$-module. The hypothesis forces that such a module is semisimple. Hence L is complemented. Moreover, as the restricted Lie algebras in $\mathcal{F}$ are pairwise non-isomorphic, every restricted subalgebra H of $\bigoplus_{I\in\mathcal{F}} I$ is of the form $H = \bigoplus_{I\in\mathcal{G}} I$ for some $\mathcal{G} \subseteq \mathcal{F}$. Therefore L/H does not contain distinct isomorphic minimal restricted subalgebras and Theorem 1.2.5 allows to conclude that L is Boolean, which completes the proof. □

Remark 1.2.12. Let L be a restricted Lie algebra. By the previous result and Proposition 1.2.1 we deduce that L is Boolean if and only if it is a direct sum of a family of restricted Lie algebras of the form $\mathcal{L}/\langle f_i * x\rangle_p$, where $\mathcal{L}$ is a free restricted Lie algebra generated by the element x and $\{f_i | \, i \in I\}$ is a family of irreducible skew polynomials that are pairwise non-similar in the sense of [126, Section 3.4].

1.3 Dually atomistic restricted Lie algebras

We say that a restricted Lie algebra L is *dually atomistic* if every restricted subalgebra of L is an intersection of maximal restricted subalgebras of L. It is easy to see that if L is dually atomistic then so is every factor algebra of L, and if L is dually atomistic then it is ϕ_p-free.

Lemma 1.3.1. *Let L be a dually atomistic restricted Lie algebra. Then:*

(i) $N(L)$ is abelian;

(ii) $M \cap N(L)$ is a restricted ideal of L for every maximal restricted subalgebra M of L; and

(iii) for every subspace S of $N(L)$, $\langle S\rangle_p$ is a restricted ideal of L.

Proof. (i) $N(L)^2 \subseteq \phi_p(L) = 0$ by [215, Theorem 6.5] and [164, Theorem 3.5].

(ii) The result is clear if $N(L) \subseteq M$, so suppose that $N(L) \not\subseteq M$. Then $L = N(L) + M$ and

$$\begin{aligned}[L, N(L) \cap M] &= [N(L) + M, N(L) \cap M] \\ &\subseteq N(L)^2 + N(L) \cap M^2 \subseteq N(L) \cap M,\end{aligned}$$

using (i).

(iii) By (i), every subspace of $N(L)$ is a subalgebra of L. Let S be any subspace of $N(L)$. Then

$$\langle S\rangle_p = \langle S\rangle_p \cap N(L) = \left(\bigcap_{M\in\mathcal{M}} M\right) \cap N(L) = \bigcap_{M\in\mathcal{M}} (M \cap N(L)),$$

where $\mathcal{M}$ is the set consisting of all maximal restricted subalgebras of L containing $\langle S\rangle_p$. Therefore, $\langle S\rangle_p$ is an intersection of restricted ideals of L, by (ii), and so is itself a restricted ideal of L. □

Proposition 1.3.2. *Let L be a dually atomistic restricted Lie algebra over an algebraically closed field F. Then L is solvable or semisimple.*

Proof. Suppose that L is not semisimple. Then $L = N(L)\dotplus B = A_1 \oplus \cdots \oplus A_n \dotplus B$, where B is a restricted subalgebra of L and $A_1 \oplus \cdots \oplus A_n = A_p soc(L) \neq 0$, by [164, Theorems 3.4 and 4.2]. If $B = 0$, L is nilpotent and we are done. Assume therefore that $B \neq 0$.

Suppose first that $N(L) = Z(L)$. Then, $L = Z(L) \oplus B$ and $L^2 \subseteq B$. Then we must have that $N(L) = R(L)$, for, otherwise, there is a minimal ideal $A/N(L)$ of $L/N(L)$ with $A \subseteq R(L)$. But A is nilpotent, which is a contradiction. Thus, B is semisimple and $Z(L) \neq 0$. Let M be a maximal restricted subalgebra of L. If $Z(L)$ is not contained in M then $M + Z(L)$ is a restricted subalgebra properly containing M, so $L = M + Z(L)$ and $\langle B^2\rangle_p = \langle L^2\rangle_p \subseteq M$, since $L^2 \subseteq M$ and M is restricted. Hence, either $Z(L)$ or $\langle B^2\rangle_p$ is inside M.

Let $z \in Z(L)$ and $b \in \langle B^2\rangle_p$, and let M be a maximal restricted subalgebra containing $\langle z + b\rangle_p$. Then $z, b \in M$, so we must have $\langle z\rangle_p + \langle b\rangle_p = \langle z + b\rangle_p$. But then $b = \sum_{i=0}^n \lambda_i (b^{[p]^i} + z^{[p]^i})$, so $b = \sum_{i=0}^n \lambda_i b^{[p]^i}$ and $\sum_{i=0}^n \lambda_i z^{[p]^i} = 0$. If b is not semisimple, then $\lambda_0 = 1$ which implies that z is semisimple, from the second sum. This must hold for every choice of $z \in Z(L)$, so $Z(L)$ is a toral subalgebra of L, by [213, Chapter 2, Theorem 3.10]. A similar argument shows that if z is not semisimple then every b must be, in which case $\langle B^2\rangle_p$ is a torus of L. Hence, either $Z(L)$ or $\langle B^2\rangle_p$ is toral. In the latter case, $\langle B^2\rangle_p$ is abelian, contradicting the fact that B is semisimple. In the former case, both $Z(L)$ and $\langle B^2\rangle_p$ have a toral element: z and b, say. But then $\langle z\rangle_p + \langle b\rangle_p = Fz + Fb \neq F(z + b) = \langle z + b\rangle_p$, a contradiction.

Therefore, suppose that $N(L) \neq Z(L)$. Then there is a minimal restricted ideal A with $A \subseteq N(L)$ and $A \cap Z(L) = 0$. Moreover, if $a \in A$, we have that $a^{[p]} \in A \cap Z(L)$, so $A = Fa$ with $a^{[p]} = 0$, by Lemma 1.3.1(iii). Let M be a maximal restricted subalgebra of L such that $a \notin M$. We have $L = M \dotplus A$, by [164, Lemma 2.1], so

M has codimension 1 in L, and, as in Proposition 1.1.2, $\langle L^{(\infty)}\rangle_p \subseteq M$. It follows that $\langle L^{(\infty)}\rangle_p \cap A = 0$. Choose $x \in \langle L^{(\infty)}\rangle_p$. Then $[x, a] \in L^{(\infty)} \cap A = 0$. If $\langle x + a\rangle_p = \langle x\rangle_p + \langle a\rangle_p$, then we have $a = \sum_{i=0}^{n} \lambda_i (x + a)^{[p]^i} = \sum_{i=0}^{n} \lambda_i x^{[p]^i} + \lambda_0 a$. Hence $\lambda_0 = 1$ and x is semisimple. It follows from [213, Chapter 2, Theorem 3.10] that $\langle L^{(\infty)}\rangle_p$ is abelian. But this means that L is solvable. □

Proposition 1.3.3. *Let L be a solvable restricted Lie algebra over any field F. If L is dually atomistic then*

$$L \simeq \left(\mathcal{L}/\langle \bar{f}_1\rangle_p \oplus \cdots \oplus \mathcal{L}/\langle \bar{f}_r\rangle_p \oplus Fx_{r+1} \oplus \cdots \oplus Fx_n\right) \dot{+} Fb,$$

where $r \geq 0$, but $r \neq n$, b is toral, $\mathcal{L} = \langle x\rangle_p$ is a free cyclic restricted Lie algebra and $\bar{f}_i = \sum_{k=0}^{m_i} \lambda_k x^{[p]k}$ is an element of $\mathcal{L}$ such that $f_i = \sum_{k=0}^{m_i} \lambda_k t^k$ is an irreducible element of the ring $F[t, \sigma]$.

Proof. The nilradical $N(L)$ is non-zero and abelian by Lemma 1.3.1(i). As L is ϕ_p-free, $L = N(L) \dot{+} B$ for some restricted subalgebra B of L, and $N(L) = A_p soc(L) =: A$, by [164, Theorems 3.4 and 4.2]. Let $a \in A$. Then $C_B(A)$ is a restricted ideal of L and $C_B(A) \cap A = 0$, so $C_B(A) = 0$ and B acts faithfully on A. Also $\mathrm{ad}^2(a) = 0$ and so $\mathrm{ad}\left(a^{[p]}\right) = 0$, whence $a^{[p]} \in Z(L)$ for all $a \in A$.

We can write $A = A_1 \oplus \cdots \oplus A_n$, where A_i is an minimal abelian restricted ideal of L for $1 \leq i \leq n$. Moreover, $A_i \simeq \mathcal{L}/\langle \bar{f}_i\rangle_p$, where $\bar{f}_i = \sum_{k\geq 0}^{m_i} \lambda_k x^{[p]k}$ is an element of $\mathcal{L}$ such that $f_i = \sum_{k\geq 0}^{m_i} \lambda_k t^k$ is an irreducible element of the ring $F[t, \sigma]$, by Lemma 1.3.1(iii) and Proposition 1.2.1. Let $A_1 \oplus \cdots \oplus A_r = Z(L)$, where $r \geq 0$. Since B acts faithfully on A, we cannot have $r = n$. Then $[B, A] = A_{r+1} \oplus \cdots \oplus A_n = Fx_{r+1} \oplus \cdots \oplus Fx_n$. Now $C_B(x_i)$ is a restricted ideal of L, so $C_B(x_i) = 0$ for each $r + 1 \leq i \leq n$. Let $b_1, b_2 \in B$. Then $[b_i, x_n] = \lambda_i x_n$ for some $0 \neq \lambda_i \in F$, $i = 1, 2$. But then $[\lambda_2 b_1 - \lambda_1 b_2, x_n] = 0$, whence b_1 and b_2 are linearly dependent and B is one-dimensional. Choose $B = Fb$ such that $[b, x_n] = x_n$. Let $b^{[p]} = \mu b$. Then

$$x_n = [b^{[p]}, x_n] = \mu[b, x_n] = \mu x_n,$$

so $\mu = 1$ and b is toral. □

We introduce another piece of notation before presenting the following results. We say that a Lie algebra is *restricted dually atomistic* if it is restricted and every subalgebra is an intersection of maximal subalgebras.

Proposition 1.3.4. *Let L be a perfect restricted dually atomistic Lie algebra. Then every subalgebra of L is restricted.*

Proof. Arguing as in [164, Lemma 3.7], it is immediate to prove that every maximal subalgebra of L is self-idealising. It follows from [164, Lemma 3.9] that every maximal subalgebra of L is restricted. The result now follows from the fact that L is dually atomistic. □

Theorem 1.3.5. *There are no perfect restricted dually atomistic Lie algebras over an algebraically closed field.*

Proof. Suppose that L is a counterexample of minimal dimension. Proposition 1.3.4 yields that L is simple as a Lie algebra, and hence its absolute toral rank is just the dimension of a maximal torus T. Given two linearly independent elements $x, y \in T$, Proposition 1.3.4 forces $0 \neq (x + \lambda y)^{[p]} \in F(x + \lambda y)$ for all $\lambda \in F$, but this cannot happen since F is algebraically closed. Hence, L has absolute toral rank 1.

Now, if F has characteristic $p = 2, 3$, then [211, Theorem 6.5] yields that L is solvable or isomorphic to $\mathfrak{sl}(2, F)$ or to $\mathfrak{psl}(3, F)$. Otherwise, L has a restricted subalgebra with a quotient isomorphic to $\mathfrak{sl}(2, F)$, by Lemma 1.1.3 and Proposition 1.3.4. But both $\mathfrak{sl}(2, F)$ and $\mathfrak{psl}(3, F)$ have elements which are neither semisimple nor p-nilpotent, which clearly contradicts Proposition 1.3.4. □

As well as the three-dimensional non-split simple Lie algebra, which is dually atomistic in the characteristic zero case, there exist other perfect dually atomistic simple restricted Lie algebras over a perfect field which is not algebraically closed. For example, let L be the seven-dimensional simple Lie algebra over a perfect field of characteristic 3 constructed by Gein in [93, Example 2]. This algebra L can be endowed with a $[p]$-map such that every element is semisimple. Any two linearly independent elements of L generate a three-dimensional non-split restricted subalgebra which is maximal in L. Any second-maximal restricted subalgebra is then one-dimensional, and every one-dimensional restricted subalgebra X is inside more than one maximal restricted subalgebra whose intersection is X.

We finish this section by studying the so-called *atomistic* restricted Lie algebras, those in which every restricted subalgebra is generated by minimal restricted subalgebras.

Proposition 1.3.6. *A restricted Lie algebra L over an algebraically closed field is atomistic if and only if every p-nilpotent cyclic restricted subalgebra is one-dimensional.*

Proof. Note that L is atomistic if and only if all its cyclic restricted subalgebras are atomistic. Consider the cyclic restricted subalgebra C, whose semisimple elements form a torus T, and whose p-nilpotent elements form a p-nilpotent restricted subalgebra P. By [213, Chapter 2, Theorem 3.6], T is atomistic. From [213, Chapter 2, Theorem 3.5], it follows that $C = T \oplus P$, so C is atomistic if and only if P is atomistic. But this is equivalent to requiring that $\dim P = 1$ by Theorem 1.2.8. The result follows. □

1.4 Restricted quasi-ideals

A restricted subalgebra X of L is called a *restricted quasi-ideal* of L if $[X, Y] \subseteq X+Y$ for all restricted subalgebras Y of L. Clearly, every restricted subalgebra that is a quasi-ideal is also a restricted quasi-ideal.

Lemma 1.4.1. *If X is a restricted subalgebra of L, then X_L is a restricted ideal of L.*

Proof. Simply note that $\left(X_L\right)^{[p]}$ is an ideal of L inside X. □

Proposition 1.4.2. *Let L be a restricted Lie algebra over a perfect field. Then, $L^{[p]}$ is a restricted quasi-ideal if and only if it is an ideal of L.*

Proof. Suppose that $L^{[p]}$ is a restricted quasi-ideal of L. Then, for all $x \in L$

$$[L^{[p]}, x] \subseteq L^{[p]} + \langle x \rangle_p = L^{[p]} + Fx,$$

so $L^{[p]}$ is a quasi-ideal. Suppose that $L^{[p]}$ is not an ideal of L, and factor out $\left(L^{[p]}\right)_L$, so we can assume that $L^{[p]}$ is core-free. Then, by [11, Theorem 3.6], there are three possibilities which we will consider in turn.

Suppose first that $L^{[p]}$ has codimension 1 in L. Define $\left(L^{[p]}\right)_i$ as in [12, (5)]. Then every element $x \in L$ can be written as $x = x_s + x_n$, where x_s is semisimple and x_n is p-nilpotent, by [213, Theorem 3.5]. Moreover, all semisimple elements belong to $L^{[p]}$, so $L = L^{[p]} + Fx$ for some p-nilpotent element x. Suppose that $x^{[p]^k} = 0$. Now $\left(L^{[p]}\right)_i = \left\{y \in L^{[p]} \mid [y,_i x] \in L^{[p]}\right\}$ for $i \geq 0$, by [12, Lemma 2.1(b)]. Hence $[y,_{p^h} x] = [y, x^{[p]^h}] = 0$ for $h \geq k$. Also, $\left(L^{[p]}\right)_0 = L^{[p]}$ and $\left(L^{[p]}\right)_{i+1} \subseteq \left(L^{[p]}\right)_i$ for $i \geq 0$, so $0 = \left(L^{[p]}\right)_L = \cap_{i=0}^{\infty}\left(L^{[p]}\right)_i = L^{[p]}$, by [12, Lemma 2.1], contradicting the fact that $L^{[p]}$ is not an ideal of L.

On the other hand, [11, Theorem 3.6(c)] cannot hold, as the three-dimensional simple Lie algebra $W(1,2)^2$ over a field of characteristic 2 is not restrictable. To see this simply note that the derivation $\mathrm{ad}^2(x)$ is not inner.

Finally, suppose that [11, Theorem 3.6(d)] holds. Then $L = L^2 + Fy$ where $\mathrm{ad}(y)$ acts as the identity map on L^2 and $L^{[p]} = Fy$. Let $x \in L^2$. We have $\mathrm{ad}^p(y) = \mathrm{ad}(y)$ and $\mathrm{ad}^p(x) = 0$ for every $x \in L^2$. Therefore, as L is centreless, the p-map of L is determined by the conditions $y^{[p]} = y$ and $x^{[p]} = 0$. This implies $L^{[p]} = L$, a contradiction.

The converse is straightforward. □

Proposition 1.4.3. *Let L be a restricted Lie algebra such that every restricted subalgebra of L is a restricted quasi-ideal. Then $L^2 \subseteq L^{[p]}$. It follows that $L^3 = L^{p+1}$; in particular, if L is nilpotent, then L has nilpotency class at most* 2.

Proof. By Proposition 1.4.2, $L^{[p]}$ is a restricted ideal. Factor out $L^{[p]}$, so we can assume that $L^{[p]} = 0$. Then every subalgebra of L is a quasi-ideal. If L is not abelian then it is almost abelian, by [11, Theorem 3.8], so $L = L^2 + Fy$, where $\mathrm{ad}(y)$ acts as the identity map on L^2. But then, if $0 \neq x \in L^2$, $0 = [y^{[p]}, x] = x$, a contradiction. It follows that $L^2 = 0$. If $p \neq 2$, we are done. Assume then that $p = 2$, and suppose, by contradiction, that L has nilpotency class $n > 2$. Set $H = L/C_{n-3}(L)$, which has nilpotency class 3. By [14, Chapter 16, Proposition 1.1], H does not satisfy the second Engel condition, and therefore there exist $x, y \in H$ such that $[x, y^{[2]}] = [[x, y], y] \neq 0$. Set $\tilde{x}, \tilde{y}$ to be preimages of x, y in L, and note that $\tilde{x}^{[2]^2}, \tilde{y}^{[2]^2}, [\tilde{x}^{[2]}, \tilde{y}^{[2]}] \in C_{n-3}(L)$. Then, by hypothesis we can write $[x, y^{[2]}] = \lambda_1 x + \lambda_2 x^{[2]} + \lambda_3 y^{[2]}$ for some $\lambda_i \in F$,

$i = 1, 2, 3$. Also we have that $[[x, y^{[2]}], z] = 0$ for any $z \in H$. Taking $z = y^{[2]}$, we obtain that $\lambda_1 = 0$; taking $z = x$, that $\lambda_3 = 0$; and taking $z = y$, that $[x^{[2]}, y] = 0$. Now, write $[x, y] = \lambda_4 x + \lambda_5 x^{[2]} + \lambda_6 y + \lambda_7 y^{[2]}$, for some $\lambda_i \in F$, $i = 4, \ldots, 7$. But then $[x, y^{[2]}] = [[x, y], y] = \lambda_4 [x, y]$, and $0 = [[x, y], y^{[2]}] = \lambda_4 [x, y^{[2]}]$. Consequently, $\lambda_4 = 0$, a contradiction. □

Lemma 1.4.4. *Let L be a restricted Lie algebra over an algebraically closed field in which every restricted subalgebra is a restricted quasi-ideal. If H is a Cartan subalgebra of L, then L has root space decomposition*

$$L = H \dotplus \left(\oplus_{\alpha \in \Phi} \left(L_\alpha \dotplus L_{-\alpha} \right) \oplus_{\beta \in \Psi} L_\beta \right),$$

where Φ is the set of roots α for which $-\alpha$ is also a root, and Ψ is the remaining set of roots.

Proof. Let T be a maximal torus, $H = C_L(T)$ and let $L = H \dotplus_{\alpha \in \Pi} L_\alpha$ be the corresponding root space decomposition. Then

$$[x_\alpha, x_\beta] = \lambda x_\alpha + \mu x_\beta + h \text{ for some } h \in H,$$

since $L_\alpha^{[p]} \subseteq H$ for all $\alpha \in \Pi$, by [213, Corollary 4.3]. But $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$, so, either $[L_\alpha, L_\beta] = 0$ or $[L_\alpha, L_\beta] \subseteq H$ and $\alpha + \beta = 0$. If $[L_\alpha, L_\beta] = 0$ for $\alpha \neq \beta$ then $[L_{-\alpha}, L_\beta] = 0$ also, giving the root space decomposition claimed. □

From now on assume that every restricted subalgebra of L is a restricted quasi-ideal of L. Let S be the subspace spanned by the semisimple elements of L and let P be the subspace spanned by the p-nilpotent elements of L. Then S and P are subalgebras of L, since $[x, y] \in \langle x \rangle_p \dotplus \langle y \rangle_p$, and, if F is perfect, $L = S \dotplus P$. Moreover, both are restricted, since

$$(\lambda x + \mu y)^{[p]} = \lambda^p x^{[p]} + \mu^p y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y),$$

and $x^{[p]}, y^{[p]}$ are semisimple/p-nilpotent if x, y are, and $s_i(x, y) \in \langle x, y \rangle^p$.

Restricted Lie algebras over perfect fields all of whose restricted subalgebras are ideals were characterised in [207]. The next proposition addresses a similar issue for restricted quasi-ideals, with the additional condition of L been nilpotent.

Proposition 1.4.5. *Let L be a nilpotent restricted Lie algebra over a perfect field. Then every restricted subalgebra of L is a restricted quasi-ideal of L if and only if $L = T \oplus P$, where T is a torus and P is a p-nilpotent ideal in which every restricted subalgebra is a restricted quasi-ideal.*

Proof. Suppose that every restricted subalgebra of L is a restricted quasi-ideal of L. By Proposition 1.4.3, $L^3 = 0$ and $L^{[p]} \subseteq Z(L)$. Then, for all $x, y \in L$, $(x+y)^{[p]} = x^{[p]} + y^{[p]}$, so S, P are just the sets of semisimple and p-nilpotent elements of L respectively. Take $T = S$. Then $T \cap P = 0$ and $T \subseteq Z(L)$. It follows that $L = T \oplus P$ and that T is a torus.

The converse is straightforward. □

Corollary 1.4.6. *Let L be a restricted Lie algebra over an algebraically closed field of characteristic different from 2 in which every restricted subalgebra of L is a restricted quasi-ideal of L. Then L has a Cartan subalgebra H such that $H = T \oplus P$ where T is a torus and P is the set of p-nilpotent elements in H, and $L = T \dot{+} N$ where N is an ideal, $N^3 = 0$ and $N^{[p]} \subseteq Z(H)$.*

Proof. We have that L has the form given in Lemma 1.4.4 and $H = T \oplus P$, by Proposition 1.4.5. Now $L_\alpha^2 = L_{-\alpha}^2 = L_\beta^2 = 0$ since 2α, -2α and 2β are not roots. For every $h \in H, \alpha \in \Pi = \Phi \cup \Psi$, we have that $[h, x_\alpha] \in \left(\langle h\rangle_p + \langle x_\alpha\rangle_p\right) \cap L_\alpha$, so $[h, x_\alpha] = \lambda x_\alpha$ for some $\lambda \in F$; that is, h acts semisimply on L_α. Also $\alpha\left(x_\alpha^{[p]}\right) = 0$, by [213, Chapter 2, Corollary 4.3 (4)]. It follows that $[x_\alpha^{[p]}, x_{-\alpha}] = 0$. Similarly, $[x_{-\alpha}^{[p]}, x_\alpha] = 0$. Now $[x_\alpha, x_{-\alpha}] \in \langle x_\alpha^{[p]}\rangle_p + \langle x_{-\alpha}^{[p]}\rangle_p$, so, if $N = P + \sum_{\alpha\in\Phi}\left(L_\alpha + L_{-\alpha}\right) + \sum_{\beta\in\Psi} L_\beta$ we have $N^3 = 0$ and $N^{[p]} \subseteq Z(H)$. □

1.5 J-algebras and lower semimodular restricted Lie algebras

For this section, it will be useful to handle the following result.

Lemma 1.5.1. *Let L be a restricted Lie algebra over an algebraically closed field. If L is supersolvable, then L admits a complete flag made up of restricted ideals of L.*

Proof. Plainly, it is enough to show that L has a one-dimensional restricted ideal, from which the conclusion will follow by induction. Suppose $\dim L > 1$, the claim being trivial otherwise. Consider a complete flag

$$0 = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n = L$$

of ideals of L. If the ideal L_1 is restricted, then we are done. Thus we can suppose that there exists $x \in L_1$ such that $x^{[p]} \notin L_1$. As L_1 is an abelian ideal, the restricted subalgebra H generated by $x^{[p]}$ is contained in the centre of L. Since the ground field is algebraically closed, by [213, Chapter 2, Theorem 3.6] we see that H contains a toral element t. We conclude that $I = Ft$ is a one-dimensional restricted ideal of L, as desired. □

Note that the assumption that the ground field is algebraically closed is essential for the validity of Lemma 1.5.1. In fact, over arbitrary fields of positive characteristic, there can be cyclic restricted Lie algebras of arbitrary dimension with no non-zero proper restricted subalgebras, as Proposition 1.2.1 shows.

Let L be a restricted Lie algebra. A restricted subalgebra X of L is called *lower semimodular* in L if $X \cap Y$ is maximal in Y for every restricted subalgebra Y of L such that X is maximal in $\langle X, Y\rangle_p$. We say that L is *lower semimodular* if every restricted subalgebra of L is lower semimodular in L.

If X, Y are restricted subalgebras of L with $X \subseteq Y$, a *J-series* (or *Jordan-Dedekind series*) for (X, Y) is a series

$$X = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = Y$$

of restricted subalgebras such that X_i is a maximal subalgebra of X_{i+1} for $0 \le i \le r-1$. This series has *length* equal to r. We shall call L a *J-algebra* if, whenever X and Y are restricted subalgebras of L with $X \subseteq Y$, all J-series for (X, Y) have the same finite length, $d(X, Y)$. Put $d(L) = d(0, L)$.

Proposition 1.5.2. *For a solvable restricted Lie algebra L over an algebraically closed field, the following are equivalent:*

(i) L is lower semimodular;

(ii) L is a J-algebra; and

(iii) L is supersolvable.

Proof. (i)⇒(ii): This is just a lattice theoretic result (see [26, Theorem V3]).
(ii)⇒(iii): We first show by induction on $\dim L$ that there exists a series of restricted subalgebras from zero to L having length $\dim L$. Suppose $L \neq 0$. As L is solvable, it follows from [213, Section 2.1, Exercise 2] that $\langle L^{(1)}\rangle_p \neq L$, so the inductive hypothesis ensures the existence of a series of restricted subalgebras

$$X = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = \langle L^{(1)}\rangle_p$$

with $\dim X_i = i$ for all $0 \leq i \leq r$. Moreover, as $L/\langle L^{(1)}\rangle_p$ is abelian, Lemma 1.5.1 yields the claim.

Now, by hypothesis, all J-series of restricted subalgebras from zero to L have length $\dim L$, and consequently all maximal restricted subalgebras have codimension 1 in L. On the other hand, if M is a maximal subalgebra of L which is not restricted, then pick an element x of M such that $x^{[p]} \notin M$. Then $M + Fx^{[p]}$ is a subalgebra of L properly containing M, so $M + Fx^{[p]} = L$ by the maximality of M. Therefore, every maximal subalgebra has codimension 1 in L, which allows to conclude that L is supersolvable, by [19, Theorem 7].

(iii)⇒(i): Let X, Y be restricted subalgebras of L such that X is maximal in $\langle X, Y\rangle_p$. By Lemma 1.5.1, X has codimension 1 in $\langle X, Y\rangle_p$, which forces $\langle X, Y\rangle_p = X + Y$. It follows that $\dim(Y/(X \cap Y)) = \dim((X + Y)/X) = 1$, whence $X \cap Y$ is maximal in Y, completing the proof. □

Note that the assumption of solvability is actually needed in the previous result. In fact, consider the restricted Lie algebra $L = \mathfrak{sl}(2, F)$ over an algebraically closed field F of characteristic $p > 2$. Then all J-series of restricted subalgebras of L have length 3, despite the fact that L is simple.

1.6 Upper semimodular restricted Lie algebras

Let L be a restricted Lie algebra. We say that a restricted subalgebra X of a restricted Lie algebra L is *upper semimodular* in L if X is maximal in $\langle X, Y\rangle_p$ for every restricted subalgebra Y of L such that $X \cap Y$ is maximal in Y. The restricted Lie algebra L is called *upper semimodular* if all of its restricted subalgebras are upper semimodular in L.

This section is devoted to study the structure of upper semimodular restricted Lie algebras over algebraically closed fields. In particular, our main aim of this section is to prove the following result:

Theorem 1.6.1. *Let L be a restricted Lie algebra over an algebraically closed field. The following conditions are equivalent:*

(i) L is upper semimodular;

(ii) L is modular;

(iii) every restricted subalgebra of L is a restricted quasi-ideal.

Moreover, if one of the previous statements holds, then L is either almost abelian or nilpotent of class at most 2*.*

We start with some preliminary results.

Let L be an almost abelian Lie algebra over a field F of characteristic $p > 0$. Suppose that $L = A \dot{+} Fx$, where A is an abelian ideal and x acts as the identity map on A. It is immediate to check that L is restrictable and also centreless, so it admits a unique p-map by [213, Chapter 2, Corollary 2.2]. Explicitly, this p-map is given by $a^{[p]} = 0$ for all $a \in A$ and $x^{[p]} = x$.

Lemma 1.6.2. *Let L be an upper semimodular restricted Lie algebra over an algebraically closed field. If L is generated by two distinct one-dimensional restricted subalgebras X and Y, then L is two-dimensional.*

Proof. Let Z be a non-zero proper restricted subalgebra of L. Assume first that $X \subseteq Z, Y \nsubseteq Z$. As $X \cap Y = 0$ is maximal in Y, X must be maximal in L, yielding $Z = X$. Assume now that $X, Y \nsubseteq Z$ and take a one-dimensional restricted subalgebra Z' of Z. By the previous case, $\langle X, Z' \rangle_p = L$. Since $X \cap Z' = 0$ is maximal in X, Z' is maximal in L and $Z = Z'$. Thus, all non-zero proper restricted subalgebras of L are one-dimensional, and it follows from [240, Lemma 1.6] that L is two-dimensional. □

Lemma 1.6.3. *Let L be a non-abelian upper semimodular restricted Lie algebra over an algebraically closed field generated by three one-dimensional restricted subalgebras. Then, L is centreless.*

Proof. Let Fx, Fy, Fz be three distinct one-dimensional restricted subalgebras generating L and suppose, by contradiction, that $Z(L) \neq 0$. Note that we can take x to be either toral or such that $x^{[p]} = 0$. By Lemma 1.6.2 and without loss of generality, we may also assume $x \in Z(L)$ and that $\langle y, z\rangle_p$ is almost abelian, with $[y, z] = z$, $y^{[p]} = y$ and $z^{[p]} = 0$. If $x^{[p]} = 0$, then $\langle x + z\rangle_p \cap Fy = 0$ is maximal in $\langle x + z\rangle_p$, but Fy is not maximal in $\langle x + z, y\rangle_p = L$, a contradiction. On the other hand, if x is toral, then

$$x \in \langle x - z\rangle_p \subseteq \langle x + y, y + z\rangle_p,$$

so $\langle x + y, y + z\rangle_p = L$. Now $\langle x + y\rangle_p \cap \langle y + z\rangle_p = 0$ is maximal in $\langle x + y\rangle_p$, but $\langle y + z\rangle_p$ is not maximal in L, a contradiction. □

Proposition 1.6.4. *Any upper semimodular restricted Lie algebra L over an algebraically closed field generated by its one-dimensional restricted subalgebras is either abelian or almost abelian.*

Proof. By Lemma 1.6.2, all the restricted subalgebras of L generated by two one-dimensional restricted subalgebras are abelian or almost abelian. Suppose that $\langle y, x_1\rangle_p$ is almost abelian, where Fy, Fx_1 are restricted subalgebras of L with $[y, x_1] = x_1$, $y^{[p]} = y$ and $x_1^{[p]} = 0$. Write $L = \langle y, x_1, \dots, x_s\rangle_p$, where $y, x_1, \dots, x_s$ are linearly independent.We claim that $\langle y, x_i\rangle_p$ is almost abelian for $i = 2, \dots, s$. Suppose otherwise that $[y, x_i] = 0$ for some $i \neq 1$. By Lemma 1.6.3, we must have $[x_1, x_i] \neq 0$. Then $\langle x_1, x_i\rangle_p$ would be almost abelian and $[x_1, x_i] = \lambda x_1$ for some $\lambda \in F$, $\lambda \neq 0$. But then $y + \lambda^{-1}x_i \in Z\left(\langle y, x_1, x_i\rangle_p\right) = 0$ by Lemma 1.6.3, a contradiction. Note also that $[y, x_i] \notin Fy$, as otherwise $y^{[p]} = 0$. Therefore, we can clearly assume that $[y, x_i] = x_i$. For $i \neq j$ write $[x_i, x_j] = \lambda_{ij}x_i + \mu_{ij}x_j$. We have

$$\begin{aligned} 0 &= [[y, x_i], x_j] + [[x_i, x_j], y] + [[x_j, y], x_i] \\ &= \lambda_{ij}x_i + \mu_{ij}x_j - \lambda_{ij}x_i - \mu_{ij}x_j + \lambda_{ij}x_i + \mu_{ij}x_j \\ &= \lambda_{ij}x_i + \mu_{ij}x_j, \end{aligned}$$

hence $\lambda_{ij} = \mu_{ij} = 0$.

Therefore, $L = \langle x_1, \dots, x_s\rangle_p \dot{+} Fy$ is an almost abelian restricted Lie algebra of dimension $s + 1$, as desired. □

Note that the hypothesis of F being algebraically closed is essential for our results. Indeed, the Lie algebra L over a perfect field of characteristic 3 given by Gein in [93, Example 2], with the p-map indicated in Section 1.3, is upper semimodular, generated by its minimal restricted subalgebras and semisimple. The reader could ask if, ruling out the hypothesis of F being algebraically closed, any upper semimodular restricted Lie algebra generated by its minimal restricted subalgebras would be abelian, almost abelian or semisimple, in a way somehow similar to the situation in the ordinary Lie algebra setting (see [92]). However, this is not the case either: the restricted Lie algebra $Fx \oplus L$, with $x^{[p]} = 0$, is generated by its minimal restricted subalgebras and it is upper semimodular, but it is neither abelian, nor almost abelian, nor semisimple. Furthermore, it is even possible to pick a modular restricted subalgebra of $Fx \oplus L$ which does not lie in any of these three cases.

Proposition 1.6.5. *Let L be an upper semimodular restricted Lie algebra over an algebraically closed field. Let B be the restricted subalgebra generated by the one-dimensional restricted subalgebras of L. If B is almost abelian, then $L = B$.*

Proof. Assume $L \neq B$. By Proposition 1.3.6, there exists a p-nilpotent element $x \in L$ with order of p-nilpotency 2. Write $B = A \dot{+} Fy$, where A is a strongly abelian restricted ideal of B, and y is a toral element which acts as the identity map on A. Since $x^{[p]} \in A$, we have $\mathrm{ad}^p(x)(y) = [x^{[p]}, y] = -x^{[p]}$. Set $w = \mathrm{ad}^{p-1}(x)(y)$, and note that $[x, w] = -x^{[p]}$ and $[x^{[p]}, w] = [x, w^{[p]}] = 0$.

As $\langle x\rangle_p \cap \langle x^{[p]}, y\rangle_p = Fx^{[p]}$ is maximal in $\langle x^{[p]}, y\rangle_p = Fx^{[p]} + Fy$, one has that $\langle x\rangle_p$ must be maximal in $\langle x, x^{[p]}, y\rangle_p = \langle x, y\rangle_p$. We have

$$\langle x\rangle_p \subsetneq \langle x, w\rangle_p \subseteq \langle x, y\rangle_p.$$

It follows that $y \in \langle x, w\rangle_p = \langle x\rangle_p + \langle w\rangle_p$, from which $[x, y] = \lambda[x, w] = -\lambda x^{[p]}$, for some $\lambda \in F$. But then

$$-x^{[p]} = \mathrm{ad}^p(x)(y) = -\lambda\, \mathrm{ad}^{p-1}(x)(x^{[p]}) = 0,$$

a contradiction. Therefore, $L = B$ and L is almost abelian. □

Theorem 1.6.6. *Any upper semimodular restricted Lie algebra L over an algebraically closed field is either abelian, almost abelian or of the form*

$$L = \langle x_1, \dots, x_r, B\rangle_p,$$

where x_i is p-nilpotent of nilpotency order $n_i > 1$ for all $i = 1, \dots, r$, B is an abelian restricted subalgebra and $[L, L] \subseteq \langle x_1, \dots, x_r\rangle_p$.

Proof. Let B be the restricted subalgebra generated by the one-dimensional restricted subalgebras of L. By Proposition 1.6.4, B is either abelian or almost abelian. If $L \neq B$, then B is abelian by Proposition 1.6.5, and every $x_i \notin B$ is p-nilpotent of p-nilpotency order $n_i > 1$ by Proposition 1.3.6.

To prove that $[L, L] \subseteq \langle x_1, \dots, x_r\rangle_p$, it suffices to see that $[x_i, b] \in \langle x_i\rangle_p$, for $i = 1, \dots, r$ and $b \in B$ such that $\langle b\rangle_p$ is one-dimensional. Take such a $b \in B$. If $b \in \langle x_i\rangle_p$, then we are done. Otherwise, $\langle x_i\rangle_p \cap \langle b\rangle_p = 0$ is maximal in $\langle b\rangle_p = Fb$, and then $\langle x_i\rangle_p$ must be maximal in $\langle x_i, b\rangle_p$. Write $w = \mathrm{ad}^{r-1}(x_i)(b) \neq 0$, where r is such that $\mathrm{ad}^r(x_i)(b) = 0$. We have the following chain of inclusions

$$\langle x_i\rangle_p \subseteq \langle x_i, w\rangle_p \subsetneq \langle x_i, b\rangle_p.$$

Then, $w \in \langle x_i\rangle_p$. Assume now that $\mathrm{ad}^{r-k}(x_i)(b) \in \langle x_i\rangle_p$ for some $k > 1$, and set $w' = \mathrm{ad}^{r-k-1}(x_i)(b)$. Again, it is clear that

$$\langle x_i\rangle_p \subseteq \langle x_i, w'\rangle_p \subseteq \langle x_i, b\rangle_p,$$

where one inclusion has to be an equality. By assumption, if $b \in \langle x_i, w'\rangle_p = \langle x_i\rangle_p + \langle w'\rangle_p$, then $[x_i, b] \in \langle x_i\rangle_p$. Therefore $w' \in \langle x_i\rangle_p$, and by induction we have that $[x_i, b] \in \langle x_i\rangle_p$. □

Note that, although any abelian or almost abelian restricted Lie algebra is upper semimodular, the converse of Theorem 1.6.6 does not hold, as the following example shows.

Example 1.6.7. Let $L = \langle x, y, z\rangle_p$ with $x^{[p]^2} = y^{[p]} = z^{[p]} = 0$ and $[x, y] = z$ as the only non-zero product. Then the restricted subalgebra $B = Fx^{[p]} \oplus Fy \oplus Fz$

generated by all the one-dimensional restricted subalgebras is abelian. However, L is not upper semimodular, as $\langle x\rangle_p \cap Fy = 0$ is maximal in Fy, but $\langle x\rangle_p$ is not maximal in $\langle x, y\rangle_p = L$.

Proposition 1.6.8. *Let L be an upper semimodular restricted Lie algebra over an algebraically closed field. Then, L is almost abelian or nilpotent.*

Proof. Assume that L is not almost abelian. Let T be a torus of L. By [213, Chapter 2, Theorem 3.6], T has a basis consisting of toral elements and therefore $T \subseteq B$, in the notation of Theorem 1.6.6. Then, the restricted subalgebra S formed by the semisimple elements of B is the unique maximal torus of L, and [213, Section 2.4, Exercise 5] yields that L is nilpotent. □

Corollary 1.6.9. *Let L be an upper semimodular restricted Lie algebra over an algebraically closed field. Then, L is also lower semimodular and a J-algebra.*

Proof. It follows from Proposition 1.6.8 and Proposition 1.5.2. □

Proposition 1.6.10. *Let L be an upper semimodular restricted Lie algebra over an algebraically closed field. Then, every restricted subalgebra of L is a restricted quasi-ideal.*

Proof. By Proposition 1.6.8, L is either almost abelian or nilpotent. If L is almost abelian, then we are done, so suppose that it is nilpotent. Let $x, y \in L$. If x, y are semisimple, then we have that $x, y \in B$ and $[x, y] = 0$. If x is semisimple and y is p-nilpotent, then $x \in B$ and we get that $[x, y] \in \langle y\rangle_p$ as in Theorem 1.6.6. If x, y are p-nilpotent, we claim that $[x, y] \in \langle x\rangle_p + \langle y\rangle_p$. Indeed, let s be the sum of their orders of p-nilpotency. We will proceed by induction on s. If $s = 2$, then $x, y \in B$ and therefore $\langle x, y\rangle_p \subseteq \langle x\rangle_p + \langle y\rangle_p$. Fix now $s > 2$, and assume that $x^{[p]} \neq 0$. If $x \in \langle x^{[p]}, y\rangle_p$, it holds that $\langle x, y\rangle_p = \langle x^{[p]}, y\rangle_p$ is contained in $\langle x^{[p]}\rangle_p + \langle y\rangle_p$ by induction. Otherwise, $\langle x^{[p]}\rangle_p = \langle x\rangle_p \cap \langle x^{[p]}, y\rangle_p$ is maximal in $\langle x\rangle_p$, so $\langle x^{[p]}, y\rangle_p$ is maximal in $\langle x, y\rangle_p$. Then $\langle x^{[p]}, y\rangle_p$ has codimension 1 in $\langle x, y\rangle_p$ and $\langle x, y\rangle_p = \langle x\rangle_p + \langle x^{[p]}, y\rangle_p$. But by induction, $\langle x^{[p]}, y\rangle_p \subseteq \langle x^{[p]}\rangle_p + \langle y\rangle_p$.

Now take x, y two arbitrary elements in L and consider their Jordan-Chevalley decompositions, $x = x_s + x_n$ and $y = y_s + y_n$. The above arguments show that $[x, y] \in \langle x_n \rangle_p + \langle y_n \rangle_p$. Since $x_s^{[p]^r} \in \langle x \rangle_p$ and $y_s^{[p]^t} \in \langle y \rangle_p$ for r and t large enough and x_s, y_s are semisimple, we get that $x_n \in \langle x \rangle_p$ and $y_n \in \langle y \rangle_p$. It follows that $[x, y] \in \langle x \rangle_p + \langle y \rangle_p$. □

The following easy lemma is all what is left to prove Theorem 1.6.1. We need a simple consideration first.

Let X be a restricted quasi-ideal of a restricted Lie algebra L. Then, for every restricted subalgebra Y of L, it holds that $X + Y = \langle X, Y \rangle_p$ is a restricted subalgebra of L.

Lemma 1.6.11. *Let L be a restricted Lie algebra in which every restricted subalgebra is a restricted quasi-ideal. Then, L is modular, and consequently, upper semimodular and lower semimodular.*

Proof. Let X, Y and Z be restricted subalgebras of L such that $X \subseteq Z$. Take $z \in \langle X, Y \rangle_p \cap Z = (X + Y) \cap Z$, and write $z = x + y$ for some $x \in X$, $y \in Y$. Then $x \in Z$, yielding that $y \in Y \cap Z$. Therefore, $z \in X + (Y \cap Z) = \langle X, Y \cap Z \rangle_p$. Then, L is modular. □

Proof of Theorem 1.6.1. It follows from the combination of Proposition 1.6.8, Proposition 1.6.10, Proposition 1.4.3 and Lemma 1.6.11. □

CHAPTER 2

The non-abelian tensor product of restricted Lie superalgebras

In this chapter, we define a non-abelian tensor product for restricted Lie superalgebras and study some of its properties, especially its relation with universal central extensions.

Introduction

Lie superalgebras originally appeared associated to certain generalised groups, today known as formal Lie supergoups, in the decade of 1930. However, it was not until forty years later when these objects achieved real importance in the mathematical and physical communities, due to their connection with the theory of supersymmetry (see [21, 192], for example). This theory intended to provide a unified treatment for bosons and fermions, the two classes of elemental particles composing the universe, and to model the transitions between them. Lie superalgebras are a key object in this framework, and this motivated a deep study not only from the perspective of mathematical physics, but also from a purely algebraic approach. Examples of this can be the celebrated classification of finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero by Kac [133], its real counterpart [206] or the partial results towards a classification of simple Lie superalgebras of infinite dimension (for instance [134]).

As it happens in the non-graded case, to deal with modular Lie superalgebras it is convenient to handle restricted Lie superalgebras. Since its introduction in 1988 by Mikhalëv [175], they have proved to be useful to obtain new results about the representation theory or the classification of modular Lie superalgebras (see [15, 31, 32, 174, 186, 225, 226], among others). They have been studied in many other references such as [25, 81, 165, 219, 224], for example.

On the other hand, non-abelian tensor products have a long history in the literature. The first occurrence was in the context of groups: indeed, Loday and Brown [36] defined a tensor product between not necessarily abelian groups acting on each other, which they called non-abelian in order to avoid confusion with the well-known tensor product of $\mathbb{Z}$-modules. They also introduced a quotient of this object, the so-called non-abelian exterior product. After that, Ellis constructed his non-abelian tensor and exterior products of Lie algebras in [76]. These products have been generalised in different directions, such as for restricted Lie algebras [157], for crossed modules of Lie algebras [75, 193], or for Lie superalgebras [91]. In all cases, they have been used to obtain results about the (co)homology in low dimensions of the respective algebraic structures, and also to find an explicit construction of the universal central extensions of the perfect objects, delving in this way in the results stating than an object is perfect if and only if it admits a universal central extension [142, 156, 180].

In this chapter, we extend diverse results from [76, 91, 142, 156, 157, 180] by introducing a non-abelian tensor product for restricted Lie superalgebras, studying its basic properties and relating it to central extensions relative to the Birkhoff subcategory of abelian objects **Ab**. We highlight that our construction generalises that of the short note [157] and therefore yields some new results in the ambit of restricted Lie algebras. However, we do not define a non-abelian exterior product of restricted Lie superalgebras, nor deal with any (co)homological applications of our conclusions.

Note also that, although we focus mostly on the Birkhoff subcategory **Ab**, there exist other Birkhoff subcategories which would be worthy studying, namely the subcategory **0pSLie** of restricted Lie superalgebras where the p-map is identically zero, or the intersection **sAb** of **Ab** and **0pSLie**, i.e. the subcategory formed by the abelian restricted Lie superalgebras with zero p-map.

$$\begin{tikzcd}
& \mathbf{pSLie} & \\
\mathbf{Ab} \arrow[ur] & & \mathbf{0pSLie} \arrow[ul] \\
& \mathbf{sAb} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Therefore, this work needs to be understood as a first step towards a comprehensive study of the relationship between the different types of central extensions of restricted Lie superalgebras and their corresponding (co)homology theories.

The structure of this chapter is as follows. In Section 2.1, we review some basic theory about the category of restricted Lie superalgebras. Section 2.2 contains the main definition of this chapter, namely the non-abelian tensor product of two restricted Lie superalgebras acting on each other, and studies some of its algebraic and categorical properties. Finally, in Section 2.3 we explore the connection of the non-abelian tensor product with the universal central extensions of restricted Lie superalgebras, with respect to the Birkhoff subcategory of abelian objects.

Through this chapter, F will denote a field of positive characteristic $p > 2$. This assumption is not necessary for all the results of the chapter, but it is essential, for example, for endowing the Lie superalgebra of derivations of a restricted Lie algebra L, $\mathrm{Der}^{[p]}(L)$, with a p-map (and therefore having a natural definition of actions of restricted Lie superalgebras), or for constructing the universal enveloping algebra of L. Unless otherwise stated, all the vector (super)spaces and (super)algebras in this paper will be considered over F. The symbol $\dotplus$ will denote the direct sum as vector (super)spaces, while the symbol $\oplus$ will be reserved for direct sums of (super)algebras.

2.1 Preliminaries on restricted Lie superalgebras

This section intends to give a basic review on the category **pSLie** of restricted Lie superalgebras. Most of the results were obtained by combining the existing theories on Lie algebras, Lie superalgebras and restricted Lie algebras (see [47, 91, 127, 157, 178, 213], for example); a careful exposition can be found in [184, Chapter 2].

We define a *superspace* as a vector space V endowed with a grading in $\mathbb{Z}_2$. We write $V = V_{\bar{0}} \oplus V_{\bar{1}}$; the elements in $V_{\bar{0}}$ will be called *even* or of degree $\bar{0}$, and the elements in $V_{\bar{1}}$, *odd* or of degree $\bar{1}$. The element zero will be assumed to have both degrees. We denote the degree of an element v with $|v|$. Non-zero elements of $V_{\bar{0}} \cup V_{\bar{1}}$ will be called *homogeneous*. The direct sum $V \dotplus W$ of two superspaces has the following induced grading: $(V \dotplus W)_{\bar{0}} = V_{\bar{0}} \dotplus W_{\bar{0}}$ and $(V \dotplus W)_{\bar{1}} = V_{\bar{1}} \dotplus W_{\bar{1}}$. The *homomorphisms* of superspaces are just homomorphisms of vector spaces. They form a superspace with the following grading: a homomorphism is even if it preserves the degree of the elements, and it is odd if it changes such degree.

If we worked over a ring R instead of the field F, we would talk about *supermodules* over R.

A *Lie superalgebra* is a superspace $L = L_{\bar{0}} \oplus L_{\bar{1}}$ endowed with a bilinear operation $[\,,]$ such that $|[x,y]| = |x| + |y|$, and satisfying

$$[x,y] = -(-1)^{|x||y|}[y,x],$$
$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]], \tag{2.1.1}$$
$$[x_{\bar{1}},[x_{\bar{1}},x_{\bar{1}}]] = 0, \tag{2.1.2}$$

for $x, y, z \in L$ homogeneous elements, $x_{\bar{0}} \in L_{\bar{0}}$, $x_{\bar{1}} \in L_{\bar{1}}$ and where we consider $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. Note that equation (2.1.2) follows from equation (2.1.1) when F has characteristic $p > 3$. Henceforth, we will assume that we are dealing with homogeneous elements when their degrees appear in any formula.

Note that Lie superalgebras are algebras in the sense of this manuscript, with some additional structure. Also, the even part $L_{\bar{0}}$ of a Lie superalgebra L is a Lie algebra. On the other hand, the odd part $L_{\bar{1}}$ is an $L_{\bar{0}}$-module.

A Lie superalgebra is said to be *restricted* if it is a Lie superalgebra L whose even part $L_{\bar{0}}$ is endowed with the structure of a restricted Lie algebra and such that $\operatorname{ad}\left(x_{\bar{0}}^{[p]}\right)(x) = \operatorname{ad}^p\left(x_{\bar{0}}\right)(x)$ for all $x_{\bar{0}} \in L_{\bar{0}}$ and all $x \in L$. Restricted Lie superalgebras were defined for the first time by Mikhalëv in [175].

Note that $L_{\bar{1}}$ is also a restricted module over $L_{\bar{0}}$.

A *homomorphism* of restricted Lie superalgebras f is a homomorphism of superspaces such that $f([x,y]) = [f(x), f(y)]$ and $f\left(x_{\bar{0}}^{[p]}\right) = f\left(x_{\bar{0}}\right)^{[p]}$, for all $x, y \in L$ and all $x_{\bar{0}} \in L_{\bar{0}}$. Note that, to that purpose, it is necessary that f has even degree as a homomorphism of superspaces. We denote the category with restricted Lie superalgebras as objects, and homomorphisms between them as morphisms, by **pSLie**. It is a semiabelian category.

The subobjects in **pSLie** are the *graded restricted subalgebras*, subalgebras H of L with the grading $H_{\bar{0}} = L_{\bar{0}} \cap H$ and $H_{\bar{1}} = L_{\bar{1}} \cap H$ such that $H_{\bar{0}}$ is also a restricted subalgebra. Furthermore, the normal subobjects are graded restricted subalgebras which are also ideals in the usual sense; they are called *graded restricted ideals*. Note that a subalgebra is graded if and only if it is generated by homogeneous elements. Also, the quotient objects in **pSLie** are just the quotients of Lie superalgebra by a graded restricted ideal, with the induced grading.

Noticing that the product of two restricted Lie superalgebras is just their direct sum, it is easy to characterise the centre $Z(L)$ of a restricted Lie superalgebra L as the graded ideal $Z(L) = \{x \in L \mid [x,y] = 0 \text{ for all } y \in L\}$. Also, the kernels, cokernels and images in **pSLie** coincide with their non-categorical equivalents.

Note that the subcategory **Ab** of **pSLie** is formed plainly by the abelian restricted Lie superalgebras, and that the abelianisation L_{ab} of a restricted Lie superalgebra L is just the quotient $\frac{L}{\langle L \rangle_p}$. If we fix this Birkhoff subcategory, we find that the perfect objects are those L such that $L = \langle [L, L] \rangle_p$, the relative commutator is $[L, L]_{\mathbf{Ab}} = \langle [L, L] \rangle_p$, and that an extension $\varphi\colon M \to L$ is central if and only if $\ker \varphi \subseteq Z(M)$.

At the end of the chapter, we will also consider shortly the nested Birkhoff subcategory **sAb** of restricted Lie superalgebras L with $L_{\bar{0}}$ strongly abelian. The reflector functor for **sAb** will be denoted by $(\ )_{\mathrm{sab}}$; explicitly, $L_{\mathrm{sab}} = \frac{L}{\left\langle [L,L], L_{\bar{0}}^{[p]} \right\rangle_p}$. Then, the **sAb**-perfect objects are the restricted Lie superalgebras L with $L = \left\langle [L, L], L_{\bar{0}}^{[p]} \right\rangle_p$, the relative commutator is $[L, L]_{\mathbf{sAb}} = \left\langle [L, L], L_{\bar{0}}^{[p]} \right\rangle_p$, and the **sAb**-central extensions are the surjective morphisms $\varphi\colon M \to L$ satisfying that $\ker \varphi \subseteq Z(M)$ and $\ker \varphi^{[p]} = 0$.

Unless otherwise stated, we will assume that we are working with respect to the Birkhoff subcategory **Ab**.

Now we recall a relationship between restricted Lie superalgebras and associative superalgebras, defined in the obvious way. First, let A be an associative superalgebra. Then A can be given structure of restricted Lie superalgebra by setting $[x, y] = xy - (-1)^{|x||y|} yx$ and $x_{\bar{0}}^{[p]} = x_{\bar{0}}^p$, for all $x, y \in A$ and all $x_{\bar{0}} \in A_{\bar{0}}$. We denote this new restricted Lie superalgebra by A_{pL}. The correspondence $(-)_{pL}$ induces a functor between the category of associative superalgebras **SAss** and **pSLie**.

We will find that $(-)_{pL}$ is a right adjoint of the universal enveloping functor. To construct the universal enveloping superalgebra of a restricted Lie superalgebra L, we need to introduce first the so-called tensor superalgebra $T(V)$ of a superspace V. The tensor product $V \otimes_F V$ admits a structure of superspace with the grading

$$\begin{aligned} \left(V \otimes_F V\right)_{\bar{0}} &= \left(V_{\bar{0}} \otimes_F V_{\bar{0}}\right) \oplus \left(V_{\bar{1}} \otimes_F V_{\bar{1}}\right); \\ \left(V \otimes_F V\right)_{\bar{1}} &= \left(V_{\bar{0}} \otimes_F V_{\bar{1}}\right). \end{aligned}$$

Recursively, we obtain a superspace structure for $V^{\otimes n} = V \otimes_F \overset{(n)}{\cdots} \otimes_F V$ for any $n \in \mathbb{N}$, and we can define the *tensor superspace* $T(V) := \bigoplus_{n \geq 0} V^{\otimes n}$, setting $V^{\otimes 0} = F$. The homogeneous components of $T(V)$ are given by the sum of those of the terms, assuming that $\left(V^{\otimes 0}\right)_{\bar{0}} = V^{\otimes 0}$. This superspace $T(V)$ can be endowed with structure of unital associative superalgebra by considering the product given by juxtaposition; it is called the *tensor superalgebra* of V. Note that, for this construction, it was only needed the underlying superspace structure of V. Note also that V can be naturally embedded in $T(V)$.

Now we can introduce the *universal enveloping superalgebra* of a restricted Lie superalgebra L [186]: it is the unital associative superalgebra $u(L)$ defined as the quotient of $T(L)$ by the graded restricted ideal generated by the homogeneous elements

$$\begin{aligned} &x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y], \\ &x_{\bar{0}}^{[p]} - x_{\bar{0}}^{p}, \end{aligned}$$

for all $x, y \in L$, $x_{\bar{0}} \in L_{\bar{0}}$. Note that the canonical embedding of L in $T(L)$ gives rise to an injective morphism $\rho\colon L \to u(L)_{pL}$. The pair $(u(L), \rho)$ satisfies the following universal property: for any other pair (A, κ) of an associative superalgebra A and a morphism in **pSLie** $\kappa\colon L \to A_{pL}$, there exists a unique morphism in **pSLie** $\theta\colon u(L)_{pL} \to A_{pL}$ such that $\theta\rho = \kappa$.

$$\begin{tikzcd} L \arrow[r, "\rho"] \arrow[dr, "\kappa"'] & u(L)_{pL} \arrow[d, "\theta"] \\ & A_{pL} \end{tikzcd}$$

A homomorphism of restricted Lie superalgebras $f\colon L \to M$ can be extended to a homomorphism of associative superalgebras $u(f)\colon u(L) \to u(M)$ in a functorial way. Then $u(-)$ is a functor with right adjoint $(-)_{pL}$.

We also recall the definition of the *augmentation ideal* of L, commonly denoted by $\Omega(L)$: it is the kernel of the homomorphism of associative superalgebras $\epsilon\colon u(L) \to F$ (where F is considered as an associative superalgebra without odd component) induced by the zero map from L to F.

Lemma 2.1.1. *Let L be a restricted Lie superalgebra. Then, there exists an isomorphism of superspaces*

$$\Omega(L) \simeq \frac{L \otimes_F u(L)}{Z},$$

where Z is the graded ideal of $L \otimes_F u(L)$ generated by the elements

$$\begin{aligned} &[x, y] \otimes 1 - x \otimes y + (-1)^{|x||y|} y \otimes x, \\ &x_{\bar{0}}^{[p]} \otimes 1 - x_{\bar{0}} \otimes x_{\bar{0}}^{p-1}, \end{aligned}$$

for all $x, y \in L$, $x_{\bar{0}} \in L_{\bar{0}}$.

We also need to mention the existence of free objects on $\mathbb{Z}_2$-graded sets in the category **pSLie**. Namely, given a $\mathbb{Z}_2$-graded set X and the free superspace F^X on X, the *free restricted Lie superalgebra* $L(X)$ on X is the graded restricted subalgebra of $T\left(F^X\right)_{pL}$ generated by X.

Another concept which will be fundamental for the rest of the chapter is that of an action of restricted Lie superalgebras. Recall that a *homogeneous derivation* D of degree $\bar{\alpha}$ of a restricted Lie superalgebra L is a linear map $D\colon L \to L$ of degree $\bar{\alpha}$ satisfying

$$D(xy) = D(x)y + (-1)^{|D||x|}xD(y)$$

for all $x, y \in L$. We denote this set of linear maps by $\mathrm{Der}_{\bar{\alpha}}(L)$, and define a *derivation* of L as an element of

$$\mathrm{Der}(L) = \mathrm{Der}_{\bar{0}}(L) \oplus \mathrm{Der}_{\bar{1}}(L).$$

Furthermore, a derivation is said to be *restricted* if

$$D\left(x_{\bar{0}}^{[p]}\right) = \mathrm{ad}^{p-1}\left(x_{\bar{0}}\right)\left(D\left(x_{\bar{0}}\right)\right)$$

for all $x_{\bar{0}} \in L_{\bar{0}}$. The set of the restricted derivations of L is denoted by $\mathrm{Der}^{[p]}(L)$. It can be proved that $\mathrm{Der}^{[p]}(L)$ is a restricted Lie superalgebra, in a similar fashion as in [125] in the restricted Lie algebra setting.

Given L, M two restricted Lie superalgebras, we can define now an *action* of M on L as a map

$$\begin{aligned} M \times L &\to L \\ (m, x) &\mapsto {}^m x := \phi(m)(x), \end{aligned}$$

where $\phi\colon M \to \mathrm{Der}^{[p]}(L)$ is a homomorphism of restricted Lie superalgebras. Equivalently, an action of M on L is bilinear map $M \times L \to L$ of even degree (i.e. $|{}^m x| = |m| + |x|$) satisfying

$$\begin{aligned} {}^{[m,n]}x &= {}^m\left({}^n x\right) - (-1)^{|m||n|}\left({}^n\left({}^m x\right)\right); \\ {}^m[x, y] &= [{}^m x, y] + (-1)^{|m||x|}\,[x, {}^m y]; \\ {}^{m_{\bar{0}}^{[p]}}x &= {}^{m_{\bar{0}}^{p}}x; \\ {}^m x_{\bar{0}}^{[p]} &= \mathrm{ad}^{p-1}\left(x_{\bar{0}}\right)\left({}^m x_{\bar{0}}\right), \end{aligned}$$

for all $x, y \in L$, $m, n \in M$, $x_{\bar{0}} \in L_{\bar{0}}$ and $m_{\bar{0}} \in M_{\bar{0}}$, and where ${}^{m_{\bar{0}}^{p}}x := {}^{m_{\bar{0}}}(\cdots({}^{m_{\bar{0}}}x)\cdots)$. Note that the definition by identities could be extended to restricted Lie superalgebras over field of characteristic $p = 2$, but the definition would lose its naturalness.

Weakening the concept of an action, we find representations of restricted Lie superalgebras and restricted supermodules. The endomorphisms of a superspace V, $\mathrm{End}(V)$, have structure of an associative superalgebra with the composition. Given a restricted Lie superalgebra L and a superspace V, we say that V is a *restricted supermodule* over L if there exists a homomorphism of restricted Lie superalgebras $\phi\colon L \to \mathrm{End}(V)_{pL}$; we will denote $x \cdot v := \phi(x)(v)$. This morphism is called a *representation* of M on V. Equivalently, V is a restricted supermodule over L if and only if there exists a linear map of even degree

$$\begin{aligned} L \times V &\to V \\ (x, v) &\mapsto x \cdot v \end{aligned}$$

satisfying

$$\begin{aligned} [x, y] \cdot v &= x \cdot (y \cdot (v)) - (-1)^{|x||y|}\, y \cdot (x \cdot (v))\,; \\ x_{\bar{0}}^{[p]} \cdot v &= x_{\bar{0}} \cdot \left(\overset{(p)}{\cdots} \left(x_{\bar{0}} \cdot v \right) \cdots \right), \end{aligned}$$

for all $x, y \in L$, $x_{\bar{0}} \in L_{\bar{0}}$ and $v \in V$. Note that V is a restricted supermodule over L if and only if it is a supermodule over $u(L)$ in the classic sense.

If L and M are two restricted Lie superalgebras acting on each other, we say that they act *compatibly* if

$$\begin{aligned} {}^{({}^{x}m)}y &= -\,(-1)^{|m||x|}\,[{}^{m}x, y]; \\ {}^{({}^{m}x)}n &= -\,(-1)^{|m||x|}\,[{}^{x}m, n], \end{aligned}$$

are satisfied for all $x, y \in L$ and all $m, n \in M$. For example, the bracket between elements of two graded restricted ideals I and J of a restricted Lie superalgebra L defines compatible actions of I and J on each other.

The pairs of restricted Lie superalgebras (L, M) acting compatibly on each other are the objects of a new semiabelian category $\mathbf{pSLie^2}$, whose morphisms are pairs of homomorphisms of restricted Lie superalgebras $(\varphi, \psi) : (L, M) \to (L', M')$ preserving the actions; i.e. satisfying

$$\varphi\left({}^{m}x\right) = {}^{\psi(m)}\varphi\left(x\right), \qquad \psi\left({}^{x}m\right) = {}^{\varphi(x)}\psi\left(m\right),$$

for all $x \in L$, $m \in M$.

The concept of actions of restricted Lie superalgebras allows us to define *crossed modules* of restricted Lie superalgebras: triples (L, M, ∂) such that L and M are restricted Lie superalgebras with M acting on L and $\partial : L \to M$ is a morphism in **pSLie**, satisfying the conditions

$$\partial\left({}^{m}x\right) = [m, \partial(x)];$$
$${}^{\partial(x)}y = [x, y],$$

for all $x, y \in L$, $m \in M$. For example, the inclusion $\iota : I \to L$ of a graded restricted ideal I in L is a crossed module with respect to the bracket. The following straightforward property of the crossed modules will be necessary later in this chapter.

Lemma 2.1.2. *Let $\partial : L \to M$ be a crossed module of restricted Lie superalgebras. Then,* $\ker \partial$ *is contained in the centre $Z(L)$.*

Finally, we define the *semidirect product $L \rtimes M$* of two restricted Lie superalgebras L and M, with M acting on L, as the superspace $L \dotplus M$ with bracket

$$[x + m, y + n] = \left([x, y] + {}^{m}y - (-1)^{|x||n|}\left({}^{n}x\right) + [m, n]\right),$$

for all $x, y \in L$, $m, n \in M$, and p-map

$$\left(x_{\bar{0}} + m_{\bar{0}}\right)^{[p]} = x_{\bar{0}}^{[p]} + m_{\bar{0}}^{[p]} + \sum_{i=1}^{p-1} s_i\left(x_{\bar{0}}, m_{\bar{0}}\right),$$

for all $x_{\bar{0}} \in L_{\bar{0}}$ and all $m_{\bar{0}} \in M_{\bar{0}}$, where $s_i\left(x_{\bar{0}}, m_{\bar{0}}\right)$ has the usual meaning explained in the Introduction of this Part I.

2.2 Non-abelian tensor product of restricted Lie superalgebras

In this section, we introduce the definition of the non-abelian tensor product of restricted Lie superalgebras, and study some of their basic properties.

Definition 2.2.1. Let L and M be two restricted Lie superalgebras acting on each other, and let $X_{L,M}$ be the set of symbols $x \otimes m$, for x and m homogeneous elements of L and M, respectively. Endow $X_{L,M}$ with the $\mathbb{Z}_2$-grading $|x \otimes m| = |x| + |m|$. The *non-abelian tensor product $L \otimes M$* is defined as the restricted Lie superalgebra generated by $X_{L,M}$ and subject to the following relations:

$$\lambda(x \otimes m) = \lambda x \otimes m = x \otimes \lambda m;$$
$$(x+y) \otimes m = x \otimes m + y \otimes m;$$
$$x \otimes (m+n) = x \otimes m + x \otimes n;$$
$$[x,y] \otimes m = x \otimes {}^{y}m - (-1)^{|x||y|}\, y \otimes {}^{x}m;$$
$$x \otimes [m,n] = (-1)^{|x||n|}(-1)^{|m||n|}\left({}^{n}x \otimes m\right) - (-1)^{|x||m|}\left({}^{m}x \otimes n\right);$$
$${}^{m}x \otimes {}^{y}n = -(-1)^{|x||m|}\,[x \otimes m, y \otimes n];$$
$$x_{\bar{0}}^{[p]} \otimes m = x_{\bar{0}} \otimes \left({}^{x_{\bar{0}}^{p-1}}m\right);$$
$$x \otimes m_{\bar{0}}^{[p]} = \left({}^{m_{\bar{0}}^{p-1}}x\right) \otimes m_{\bar{0}},$$

for all $\lambda \in F$, $x, y \in L$, $m, n \in M$, $x_{\bar{0}} \in L_{\bar{0}}$ and $m_{\bar{0}} \in M_{\bar{0}}$.

Given x, m two non-homogeneous elements of L and M, respectively, we denote $x \otimes m := x_{\bar{0}} \otimes m_{\bar{0}} + x_{\bar{0}} \otimes m_{\bar{1}} + x_{\bar{1}} \otimes m_{\bar{0}} + x_{\bar{1}} \otimes m_{\bar{1}}$.

Note that, given L and M two arbitrary restricted Lie superalgebras, we can always construct their non-abelian tensor product with respect to the trivial actions. In this case, $L \otimes M$ is abelian.

We will characterise $L \otimes M$ as a universal object in the category **pSLie**. To that purpose, we introduce the following definition.

Definition 2.2.2. Let L and M be two restricted Lie superalgebras acting on each other. A *restricted Lie superpair* with respect to L and M is a restricted Lie superalgebra H together with a bilinear map of even degree $\xi : L \times M \to H$ satisfying

$$\xi([x,y],m) = \xi(x, {}^{y}m) - (-1)^{|x||y|}\,\xi(y, {}^{x}m);$$
$$\xi(x,[m,n]) = (-1)^{|x||n|}(-1)^{|m||n|}\,\xi({}^{n}x, m) - (-1)^{|x||m|}\,\xi({}^{m}x, n);$$
$$\xi({}^{m}x, {}^{y}n) = -(-1)^{|x||m|}\,[\xi(x,m), \xi(y,n)];$$
$$\xi\left(x_{\bar{0}}^{[p]}, m\right) = \xi\left(x_{\bar{0}}, {}^{x_{\bar{0}}^{p-1}}m\right);$$
$$\xi\left(x, m_{\bar{0}}^{[p]}\right) = \xi\left({}^{m_{\bar{0}}^{p-1}}x, m_{\bar{0}}\right),$$

for all $x, y \in L$, $m, n \in M$, $x_{\bar{0}} \in L_{\bar{0}}$ and $m_{\bar{0}} \in M_{\bar{0}}$.

Furthermore, a restricted Lie superpair is called *universal* if for any other pair $\left(H', \xi'\right)$ in the same conditions, there exists a unique morphism in **pSLie** $\theta\colon\ H \to H'$ such that $\theta\xi = \xi'$.

$$\begin{tikzcd} L \times M \arrow[r, "\xi"] \arrow[dr, "\xi'"'] & H \arrow[d, "\theta"] \\ & H' \end{tikzcd}$$

Clearly, the universal restricted Lie superpair is unique up to isomorphism.

Proposition 2.2.3. *Let L and M be two restricted Lie superalgebras acting on each other. The pair $(L \otimes M, \chi)$ with $\chi\colon\ L \times M \to L \otimes M$ defined by $\chi(x, m) = x \otimes m$ is the universal restricted Lie superpair with respect to L and M.*

Proof. It is obvious that $(L \otimes M, \chi)$ is a restricted Lie superpair with respect to L and M. Also, given another restricted Lie superpair (H, ξ), there exist a unique homomorphism $\theta\colon\ L \otimes M \to H$ such that $\theta\chi = \xi$, given by $\theta(x \otimes m) = \xi(x, m)$. Note that θ is well-defined precisely because (H, ξ) is a restricted Lie superpair. □

If the actions of L and M on each other are compatible, L and M together with the following maps provide easy examples of restricted Lie superpairs:

$$\begin{aligned} \xi_\mu\colon\ & L \times M \to L \\ & (x, m) \mapsto -(-1)^{|x||m|}\left({}^m x\right), \end{aligned} \qquad \begin{aligned} \xi_\nu\colon\ & L \times M \to L \\ & (x, m) \mapsto {}^x m. \end{aligned}$$

By Proposition 2.2.3, there exist homomorphisms of restricted Lie superalgebras $\mu\colon\ L \otimes M \to L$ and $\nu\colon\ L \otimes M \to M$ defined by $\mu(x \otimes m) = -(-1)^{|x||m|}\left({}^m x\right)$ and $\nu(x \otimes m) = {}^x m$, respectively.

In particular, taking the graded restricted ideals I and J of F, acting on each other through the bracket of L, the homomorphisms μ and ν reduce to the bracket. This consideration leads to the interpretation of the non-abelian tensor product $I \otimes J$ as a certain "universalisation" of the bracket.

The next technical lemma will be useful later on.

Lemma 2.2.4. *Let L and M be two restricted Lie superalgebras acting compatibly on each other. Then, there exist actions of L and M on $L \otimes M$ determined respectively by*

$$
\begin{aligned}
{}^{z}(x \otimes m) &= [z, x] \otimes m + (-1)^{|z||x|}\, x \otimes {}^{z}m, \\
{}^{q}(x \otimes m) &= {}^{q}x \otimes m + (-1)^{|q||x|}\, x \otimes [q, m],
\end{aligned}
$$

for all $x, z \in L$ and $m, q \in M$. Furthermore, the homomorphisms μ and ν are crossed modules of restricted Lie superalgebras with respect to these actions.

Proof. The relative complexity of the direct calculations makes useful to follow this sketch of proof. First, fix an homogeneous $z \in L$ and prove that the assignment

$$
\begin{aligned}
\phi_z \colon L \otimes M &\to L \otimes M \\
x \otimes m &\mapsto {}^{z}(x \otimes m)
\end{aligned}
$$

can be extended to a restricted derivation. To do so, it suffices to check that ϕ_z respects the relations of $L \otimes M$. Later, we need to ensure that

$$
\begin{aligned}
\phi \colon L &\to \mathrm{Der}^{[p]}(L \otimes M) \\
z &\mapsto \phi_z
\end{aligned}
$$

is a homomorphism of restricted Lie superalgebras. To see that ϕ preserves the p-map, it is convenient to handle the equality

$$
{}^{x^k}(x \otimes m) = \sum_{i=0}^{k} \binom{k}{i} \mathrm{ad}^{p-i}(x)(m) \otimes \left({}^{x^i}m\right),
$$

which can be proved by induction in $k \in \mathbb{N}$, and to apply it for $k = p$. Checking that μ is a crossed module with respect to this action is just routine.

The computations for the action of M on $L \otimes M$ and ν are completely analogous, and therefore we omit them. □

The following results provide basic properties of the non-abelian tensor product.

Proposition 2.2.5. *Let L and M be two restricted Lie superalgebras acting on each other. The non-abelian tensor products $L \otimes M$ and $M \otimes L$ are isomorphic.*

Proof. Simply note that $\theta(x \otimes m) = -(-1)^{|x||m|}\, m \otimes x$ induces an isomorphism between $L \otimes M$ and $M \otimes L$. □

Proposition 2.2.6. *Let L, M and H be restricted Lie superalgebras such that both L and M, and M and H, act on each other. Suppose further that the following hypotheses are satisfied:*

1. ${}^{h}({}^{x}m) = (-1)^{|x||h|}\left({}^{x}\left({}^{h}m\right)\right)$ *for all* $x \in L$, $m \in M$ *and* $h \in H$.

2. $x \otimes {}^{y}m = h \otimes {}^{g}m = 0$ *for all* $x, y \in L$, $h, g \in H$ *and* $m \in M$.

Then there are actions of $H \oplus L$ on M and reciprocally, and

$$(H \oplus L) \otimes M \simeq (H \otimes M) \oplus (L \otimes M).$$

Proof. The action of $H \oplus L$ on M is given by the map

$$\begin{aligned} (H \oplus L) \times M &\to M \\ (h + x, m) &\mapsto {}^{h+x}m := {}^{h}m + {}^{x}m, \end{aligned}$$

and the action of M on $H \oplus L$, by

$$\begin{aligned} M \times (H \oplus L) &\to H \oplus L \\ (m, h + x) &\mapsto {}^{m}(h + x) := {}^{m}h + {}^{n}x. \end{aligned}$$

Proving that these maps are indeed actions is routine, so we omit the explicit computations. It is also routine to prove that $(L \otimes N) \oplus (M \otimes N)$ together with

$$\begin{aligned} \xi\colon\ (H \oplus L) \times M &\to (H \otimes M) \oplus (L \otimes M) \\ (h + x, m) &\mapsto h \otimes m + x \otimes m, \end{aligned}$$

is a restricted Lie superpair with respect to $H \oplus L$ and M. Then there exists an induced homomorphism of restricted Lie superalgebras $\alpha\colon\ (H \oplus L) \otimes M \to (H \otimes M) \oplus (L \otimes M)$, with inverse $\beta\colon\ (H \otimes M) \oplus (L \otimes M) \to (H \oplus L) \otimes M$ defined by $\beta(h \otimes m + x \otimes n) = h \otimes m + x \otimes n$. $\square$

An example of a triple of restricted Lie superalgebras satisfying the hypotheses of Proposition 2.2.6 can be H, L and $H \oplus L$, where H and L are identified with their canonical inclusions in $H \oplus L$, together with the actions induced by the brackets.

Now, we focus on the relationship between the non-abelian tensor product of restricted Lie superalgebras and the tensor product (in the classic sense) of their underlying vector spaces. The skew polynomial ring $F[\sigma, t]$ with respect to the Frobenius endomorphism σ of F will play a key role.

One first easy relationship is the following: if L and M are strongly abelian and we consider the trivial actions between them, then the non-abelian tensor product $L \otimes M$ and the tensor product of the underlying spaces coincide.

Proposition 2.2.7. *Let L and M be two restricted Lie superalgebras acting compatibly on each other. Then, there exists a surjective homomorphism of superspaces*

$$f: F[\sigma, t] \otimes_F \left(L \otimes_F M\right) \to L \otimes M.$$

Proof. First of all, note that $F[\sigma, t] \otimes_F \left(L \otimes_F M\right)$ has a structure of left F-superspace determined by $\lambda\,(g \otimes x \otimes m) = \lambda g \otimes x \otimes m$. Note that

$$\lambda^p\,(g \otimes x \otimes m) = \lambda^p g \otimes x \otimes m = g\lambda \otimes x \otimes m = g \otimes \lambda x \otimes m = g \otimes x \otimes \lambda m,$$

where $\lambda \in F$, $g \in F[\sigma, t]$, $x \in L$ and $m \in M$. If $\left\{x_\alpha\right\}_{\alpha \in A}$ and $\left\{m_\beta\right\}_{\beta \in B}$ are bases for L and M, respectively, $\left\{t^r \otimes x_\alpha \otimes m_\beta\right\}_{\alpha \in A, \beta \in B, r \in \mathbb{N}}$ is a basis for $F[\sigma, t] \otimes_F \left(L \otimes_F M\right)$. Setting $(x \otimes m)^{[p]^0} := x \otimes m$, we can take

$$f\left(t^r \otimes x_\alpha \otimes m_\beta\right) = \left(x_\alpha \otimes m_\beta\right)^{[p]^r},$$

which gives rise to a well-defined homomorphism

$$f: F[\sigma, t] \otimes_F \left(L \otimes_F M\right) \to L \otimes M$$

of even degree. To prove that f is surjective, it suffices to find a preimage for the elements $\left(x_\alpha \otimes m_\beta\right)^{[p]^r}$, thanks to the relations of $L \otimes M$. Such a preimage can be $t^r \otimes x_\alpha \otimes m_\beta$. □

Note that the homomorphism f constructed in Proposition 2.2.7 is not necessarily injective. However, if we consider trivial actions between L and M, and work with $F[\sigma, t] \otimes_F \left(L_{\mathrm{sab}} \otimes_F M_{\mathrm{sab}}\right)$ instead of $F[\sigma, t] \otimes_F \left(L \otimes_F M\right)$, we get that $\ker f = 0$, and f is an isomorphism of superspaces. Then, the structure of abelian restricted Lie superalgebra of $L \otimes M$ can be transferred to $F[\sigma, t] \otimes_F \left(L_{\mathrm{sab}} \otimes_F M_{\mathrm{sab}}\right)$.

Proposition 2.2.8. *Let L and M be two restricted Lie superalgebras acting compatibly on each other. Then, there exists a subspace W of $L \otimes_F M$ such that $F[\sigma, t] \otimes_F \frac{L \otimes_F M}{W}$ and $L \otimes M$ are isomorphic as superspaces.*

Proof. By Proposition 2.2.7, we have the surjective homomorphism of superspaces $f : F[\sigma, t] \otimes_F \left(L \otimes_F M\right) \to L \otimes M$. Construct the subspace W_1 of $L \otimes_F M$ generated by the elements

$$
\begin{aligned}
&[x, y] \otimes m - x \otimes {}^{y}m + (-1)^{|x||y|} y \otimes {}^{x}m,\\
&x \otimes [m, n] - (-1)^{|x||n|}(-1)^{|m||n|} \left({}^{n}x \otimes m\right) + (-1)^{|x||m|} \left({}^{m}x \otimes n\right),\\
&x_{\bar{0}}^{[p]} \otimes m - x_{\bar{0}} \otimes \left({}^{x_{\bar{0}}^{p-1}} m\right),\\
&x \otimes m_{\bar{0}}^{[p]} - \left({}^{m_{\bar{0}}^{p-1}} x\right) \otimes m_{\bar{0}},
\end{aligned}
$$

for all $x, y \in L$, $m, n \in M$, $x_{\bar{0}} \in L_{\bar{0}}$ and $m_{\bar{0}} \in M_{\bar{0}}$. Denote by $\underset{x\otimes m, y\otimes n, z\otimes q}{\circlearrowleft}$ the circular sum with respect to $x \otimes m$, $y \otimes n$ and $z \otimes q$, and consider also the elements of $L \otimes M$

$$(-1)^{|x||m|} \left({}^{m}x \otimes {}^{y}n\right) + (-1)^{(|x|+|m|)(|y|+|n|)} (-1)^{|y||n|} \left({}^{n}y \otimes {}^{x}m\right),$$

$$\underset{x\otimes m, y\otimes n, z\otimes q}{\circlearrowleft} (-1)^{(|x|+|m|)(|z|+|q|)} (-1)^{|x||m|} \left({}^{m}x \otimes [{}^{y}n, {}^{z}q]\right),$$

$${}^{m}x \otimes [{}^{x}m, {}^{x}m] \quad \text{for } |x| \neq |m|,$$

for all $x, y, z \in L$ and $m, n, q \in M$, which are identically zero due to the anticommutativity and the graded Jacobi identity. Define W_2 to be the subspace of $L \otimes_F M$ generated by all the preimages by f of the previous elements, and take $W = W_1 + W_2$.

By construction, f induces another homomorphism $\overline{f} : F[\sigma, t] \otimes_F \frac{L \otimes_F M}{W} \to L \otimes M$, which is bijective. □

Once again, the isomorphism of Proposition 2.2.8 allows to transfer the structure of restricted Lie superalgebra from $L \otimes M$ to $F[\sigma, t] \otimes_F \frac{L \otimes_F M}{W}$.

We present another isomorphism involving the augmentation ideal $\Omega(L)$ (cf. [50]).

Proposition 2.2.9. *Let L be a restricted Lie superalgebra and M a restricted supermodule over L. Then, there exists an isomorphism of superspaces*

$$L \otimes M \simeq F[\sigma, t] \otimes_F \left(\Omega(L) \otimes_{u(L)} M\right).$$

Proof. Note first that M can be identified with a strongly abelian restricted Lie superalgebra, with L acting on it. Then, Proposition 2.2.8 yields the isomorphism of superspaces

$$L \otimes M \simeq F[\sigma, t] \otimes_F \left(\frac{L \otimes_F M}{W}\right);$$

under the current hypotheses, W is generated by the elements

$$[x, y] \otimes m - x \otimes {}^{y}m + (-1)^{|x||y|} y \otimes {}^{x}m;$$
$$x_{\bar{0}}^{[p]} \otimes m - x_{\bar{0}} \otimes \left({}^{x_{\bar{0}}^{p-1}} m\right),$$

for all $x, y \in L$, $m \in M$ and $x_{\bar{0}} \in L_{\bar{0}}$.

On the other hand, $\Omega(L)$ is a right $u(L)$-supermodule in the classic sense, and M a left $u(L)$-supermodule; we can then construct the tensor product $\Omega(L) \otimes_{u(L)} M$. Lemma 2.1.1 gives the isomorphism

$$\Omega(L) \otimes_{u(L)} M \simeq \frac{L \otimes_F u(L)}{Z} \otimes_{u(L)} M;$$

also, by the right exactness of the tensor product,

$$\frac{L \otimes_F u(L)}{Z} \otimes_{u(L)} M \simeq \frac{\left(L \otimes_F u(L)\right) \otimes_{u(L)} M}{Z \otimes_{u(L)} M}.$$

We claim that the right-hand supermodule is isomorphic to $\frac{L \otimes_F M}{W}$. Indeed,

$$\left(L \otimes_F u(L)\right) \otimes_{u(L)} M \simeq L \otimes_F \left(u(L) \otimes_{u(L)} M\right) \simeq L \otimes_F M,$$

and this isomorphism transforms the generators of $Z \otimes_{u(L)} M$,

$$([x, y] \otimes 1) \otimes m - (x \otimes y) \otimes m + (-1)^{|x||y|} (y \otimes x) \otimes m,$$
$$\left(x_{\bar{0}}^{[p]} \otimes 1\right) \otimes m - \left(x_{\bar{0}} \otimes x_{\bar{0}}^{p-1}\right) \otimes m,$$

with $x, y \in L$, $m \in M$ and $x_{\bar{0}} \in L_{\bar{0}}$, into the generators of W. The result follows. □

The final part of this section is devoted to study the functoriality of the non-abelian tensor product between the categories of $\mathbf{pSLie^2}$ and **pSLie**. The following lemma shows how the non-abelian tensor product behaves with respect to the morphisms in $\mathbf{pSLie^2}$.

Lemma 2.2.10. *Let (φ, ψ) be a morphism in* $\mathbf{pSLie^2}$ *between the objects (L, M) and (S, T). Then, there exists a morphism in* **pSLie**

$$\begin{aligned} \varphi \otimes \psi \colon\ L \otimes M &\to S \otimes T \\ x \otimes m &\mapsto \varphi(x) \otimes \psi(m). \end{aligned}$$

Proof. It suffices to check that the map $\xi \colon\ L \times M \ \to\ S \otimes T$ defined by $\xi(x, m) = \varphi(x) \otimes \psi(m)$ is a restricted Lie superpair with respect to L and M. □

Proposition 2.2.11. *The non-abelian tensor product is a functor between the categories* $\mathbf{pSLie^2}$ *and* **pSLie***.*

Proof. Firstly, note that $\mathrm{id}_{(L,M)} = \left(\mathrm{id}_L, \mathrm{id}_M\right)$ is the identity morphism in $\mathbf{pSLie^2}$ associated to (L, M), and that $\mathrm{id}_M \otimes \mathrm{id}_N = \mathrm{id}_{M \otimes N}$. Secondly, let $(\varphi, \psi) \ :\ (L, M) \to (S, T)$ and $\left(\varphi', \psi'\right) \ :\ (S, T) \to (U, V)$ be two morphisms in $\mathbf{pSLie^2}$. We have that $\left(\varphi', \psi'\right)(\varphi, \psi) = \left(\varphi'\varphi, \psi'\psi\right)$, and also

$$\left(\left(\varphi' \otimes \psi'\right)(\varphi \otimes \psi)\right)(x \otimes m) = \varphi'\left(\varphi(x)\right) \otimes \psi'\left(\psi(m)\right) = \left(\left(\varphi'\varphi\right) \otimes \left(\psi'\psi\right)\right)(x \otimes m)$$

for all $x \in L$, $m \in M$. It follows that $\left(\varphi' \otimes \psi'\right)(\varphi \otimes \psi) = \left(\varphi'\varphi\right) \otimes \left(\psi'\psi\right)$, and we are done. □

Our last result deals with a certain exactness of the non-abelian tensor product. Note that the kernels, cokernels and images in $\mathbf{pSLie^2}$ are just the pairs formed, respectively, by the kernels, cokernels and images of the two homomorphisms of restricted Lie superalgebras composing the morphisms in $\mathbf{pSLie^2}$.

Proposition 2.2.12. *Let*

$$(0,0) \to (K, H) \xrightarrow{(i,j)} (L, M) \xrightarrow{(\varphi,\psi)} (S, T) \to (0,0)$$

be an exact sequence in **pSLie²**. *Then, the sequence*

$$K \otimes M \rtimes L \otimes H \xrightarrow{\alpha} L \otimes M \xrightarrow{\varphi \otimes \psi} S \otimes T \to 0 \tag{2.2.1}$$

is exact in **pSLie**, *being* $\alpha(u+v) = \left(i \otimes \mathrm{id}_M\right)(u) + \left(\mathrm{id}_L \otimes j\right)(v)$, *where* $u+v$ *denotes an arbitrary element of* $K \otimes M \rtimes L \otimes H$.

Proof. Note first that the actions of L and M on each other induce actions of K and M, and of H and L, on each other, which will be compatible. So we can construct the non-abelian tensor products $K \otimes M$ and $L \otimes H$.

Recall the action of M on $L \otimes M$ constructed in Proposition 2.2.4: ${}^{q}(x \otimes m) = {}^{q}x \otimes m + (-1)^{|q||m|} x \otimes [q, m]$, for all $x \in L$, $q, m \in M$. This action also induces another one of M on $K \otimes M$, i.e. a morphism $\phi\colon M \to \mathrm{Der}^{[p]}(K \otimes M)$.

Consider also the morphism

$$\begin{aligned} \eta\colon L \otimes H &\to H \\ x \otimes h &\mapsto {}^{x}h. \end{aligned}$$

The composition $\left(\phi|_H\right)\eta$ gives an action of $L \otimes H$ on $K \otimes M$, which allows to construct the semidirect product $K \otimes M \rtimes L \otimes H$.

As η is nothing but the restriction $\nu|_{L \otimes H}\colon L \otimes H \to M$, Lemma 2.2.4 and the properties of crossed modules ensure that

$$\left(i \otimes \mathrm{id}_M\right)({}^{v}u) = \left(i \otimes \mathrm{id}_M\right)\left({}^{\eta(v)}u\right) = [\left(\mathrm{id}_L \otimes j\right)(v), \left(i \otimes \mathrm{id}_M\right)(u)],$$

for all $u \in K \otimes M$ and $v \in L \otimes H$.

Taking into account this last equality, it is routine to prove that α is a morphism in **pSLie**. To prove the exactness of the sequence (2.2.1), it suffices to work with the elements $k \otimes m + x \otimes h \in K \otimes M \rtimes L \otimes H$ and $x \otimes m \in L \otimes M$.

The surjectivity of $\varphi \otimes \psi$ and the inclusion $\mathrm{Im}\,\alpha \subseteq \ker(\varphi \otimes \psi)$ are immediate. The properties of the crossed module ν yield that $\mathrm{Im}\,\alpha$ is a graded restricted ideal of $L \otimes M$, so we are able to construct the quotient $\frac{L \otimes M}{\mathrm{Im}\,\alpha}$. The morphism $\varphi \otimes \psi$ induces another one $\overline{\varphi \otimes \psi}\colon \frac{L \otimes M}{\mathrm{Im}\,\alpha} \to S \otimes T$. The restricted Lie superpair with respect to S

and T

$$\xi : S \times T \to \frac{L \otimes M}{\operatorname{Im} \alpha}$$
$$(s,t) \mapsto x \otimes m + \operatorname{Im} \alpha,$$

with $x \in L$ and $m \in M$ satisfying $\varphi(x) = s$ and $\psi(m) = t$, induces a homomorphism of restricted Lie superalgebras, $\beta : S \otimes T \to \frac{L \otimes M}{\operatorname{Im} \alpha}$, which is the inverse of $\overline{\varphi \otimes \psi}$. Therefore, $\ker(\varphi \otimes \psi) \subseteq \operatorname{Im} \alpha$, and the sequence (2.2.1) is exact. □

Note that the morphism α defined in Proposition 2.2.12 is not necessarily injective. Indeed, take an element $k \otimes h \in (K \otimes M) \cap (L \otimes H)$, and check that $\alpha(-k \otimes h + k \otimes h) = 0$ although $-k \otimes h + k \otimes h$ is non-zero in $K \otimes M \rtimes L \otimes H$.

In particular, Proposition 2.2.12 yields that the sequence of restricted Lie superalgebras

$$K \otimes L \rtimes L \otimes K \xrightarrow{\alpha} L \otimes L \xrightarrow{\pi \otimes \pi} L/K \otimes L/K \to 0,$$

is exact for every graded restricted ideal K of L.

2.3 Central extensions of restricted Lie superalgebras

In this short section, we apply the results obtained in Section 2.2 to the theory of central extensions of restricted Lie superalgebras. In the first place, we are going to work with central extensions relative to the Birkhoff subcategory **Ab**.

We begin with some simple results (cf. [180]) which will be necessary for the proof of the main theorem of this section. Their proofs follow directly from the definitions, so we omit them.

Lemma 2.3.1. *Let L and M be two restricted Lie superalgebras, with M perfect, and let $\varphi : M \to L$ be a central extension. Then, L is also perfect.*

Lemma 2.3.2. *Let L and M be two restricted Lie superalgebras, and let $\varphi : M \to L$ be a central extension. Then:*

1. *If $\varphi(m) = \varphi(n)$ and $\varphi(m') = \varphi(n')$ for some $m, m', n, n' \in M$, then $[n, n'] = [y, y']$.*

2. *If H is another restricted Lie superalgebra and $\phi, \phi' : H \to M$ are morphisms satisfying $\varphi\phi = \varphi\phi'$, then $\phi|_{\langle [H,H] \rangle_p} = \phi'|_{\langle [H,H] \rangle_p}$.*

Corollary 2.3.3. *Let L, M and H be restricted Lie superalgebras, with M perfect, and let $\varphi : M \to L$ and $\psi : H \to L$ be central extensions. Then, there exists at most one homomorphism between the central extensions φ and ψ.*

Now we are prepared to present the main result of this section. The first part of the proof uses the same argument as in [52, Theorem 3.7].

Theorem 2.3.4. *A restricted Lie superalgebra L admits a universal central extension if and only if it is perfect.*

Proof. Assume first that L admits a universal central extension $\varphi : M \to L$. Denote $\overline{M} := M \oplus M_{\mathrm{ab}}$, and define the morphism $\psi : \overline{M} \to L$ by $\psi\left(m + n + \langle [M,M] \rangle_p\right) = \varphi(m)$, which is another central extension of L. Define now the homomorphisms of central extensions $\phi, \phi' : M \to \overline{M}$ by $\phi(m) = m + m + \langle [M,M] \rangle_p$ and $\phi'(m) = m$. Since φ is universal, we have that $\phi = \phi'$ and consequently M is perfect. By Lemma 2.3.1, L is also perfect.

$$\begin{tikzcd}
N \arrow[rr, bend left, "\phi"] \arrow[rr, bend right, "\phi'"'] \arrow[dr, "\varphi"'] & & \overline{N} \arrow[dl, "\psi"] \\
& M &
\end{tikzcd}$$

Assume now that L is perfect. By Lemma 2.2.4, the morphism of restricted Lie superalgebras

$$\begin{aligned}
\mu : L \otimes L &\to L \\
(x, y) &\mapsto [x, y]
\end{aligned}$$

is a crossed module. Since L is perfect, μ is surjective, and Lemma 2.1.2 ensures that it is a central extension for L.

Suppose that $\varphi\colon M \to L$ is another central extension, and consider the restricted Lie superpair given by

$$\begin{aligned} \xi\colon L\times L &\to M \\ (x,y) &\mapsto [m,n], \end{aligned}$$

where $\varphi(m) = x$ and $\varphi(n) = y$. Then, it is induced a morphism $\phi\colon L\otimes L \to M$ with $\phi(x\otimes y) = [m,n]$. By construction, it holds that $\varphi\phi = \mu$, so ϕ can be seen as a homomorphism of central extensions from μ to φ. It is easy to check that $[x,y]^{[p]^r}\otimes[x',y']^{[p]^s} \in \langle [L\otimes L, L\otimes L]\rangle_p$ for all $x,x',y,y' \in L$ such that $|x| = |y|$ and $|x'| = |y'|$. As L is perfect, it follows that $L\otimes L$ is also perfect, and Corollary 2.3.3 yields the uniqueness of ϕ. Therefore, μ is the universal central extension of L. □

We offer also a partial result for central extensions relative to the Birkhoff subcategory **sAb**.

Lemma 2.3.5. *Let L and M be two restricted Lie superalgebras, with M* **sAb***-perfect, and let $\varphi\colon M \to L$ be a* **sAb***-central extension. Then, L is also* **sAb***-perfect.*

Proposition 2.3.6. *Let L be a restricted Lie superalgebra admitting a universal* **sAb***-central extension. Then, L is* **sAb***-perfect.*

Proof. Let $\varphi\colon M \to L$ be the universal **sAb**-central extension. Substitute $\overline{M}$ in the proof of Theorem 2.3.4 by $\widetilde{M} := M \oplus M_{\mathrm{sab}}$, and define the morphism $\psi\colon \widetilde{M} \to L$ by $\psi\left(m+n+[M,M]_{\mathbf{sAb}}\right) = \varphi(m)$. Since φ is **sAb**-central, if follows that ψ is **sAb**-central too. The proof finishes as in Theorem 2.3.4. □

Part II

Central extensions of non-associative algebras

Introduction

The subject of algebra extensions has long been a focal point of interest in mathematics and physics. In short, an algebra A is an extension of another algebra B by K if there exists a short exact sequence $0 \to \mathrm{K} \to \mathrm{A} \to \mathrm{B} \to 0$. The easiest example is the direct sum $\mathrm{B} \oplus \mathrm{K}$, along with the inclusion and projection maps. Imposing new conditions on the extensions, we find important special types, such as the split extensions, the HNN-extensions, which have been used to prove numerous theorems of embeddability in different varieties of non-associative algebras [159, 163, 227], and many others. In this memory, we will be concerned with central extensions, i.e. extensions in which the centre (or, as we will be calling it throughout this Part II, the annihilator) of A contains K. Several important algebras can be constructed as central extensions; for example, the Virasoro algebra is the universal central extension of the Witt algebra, and the Heisenberg algebra is a central extension of a commutative Lie algebra.

The algebraic study of central extensions of different varieties of non-associative algebras has an interest on its own [8, 138, 239], but also plays an important role in the classification problem in such varieties. Since Skjelbred and Sund in 1978 devised a method for classifying nilpotent Lie algebras [209], making crucial use of central extensions and the second cohomology space with trivial coefficients, it has been profusely used [57, 63, 99] and adapted to many other varieties of algebras, including associative [64], Jordan [6, 7], Malcev [3, 4], Novikov [137] or anticommutative [42, 143], among others (see [5, 101, 120], for instance). It has also been employed to classify types of non-associative algebras not defined by polynomial identities, such as p-nilpotent restricted Lie algebras [62] or n-dimensional algebras with annihilator of dimension $n-2$ [44].

The key idea in this method for classifying nilpotent algebras of a certain variety is to regard them as central extensions of algebras of smaller dimension, defined by means of the second cohomology space. The isomorphism problem is then solved by considering the orbits of the automorphism group on the set of subspaces of this cohomology space, together with certain technical nuances.

A nilpotent algebra has maximum index of nilpotency if and only if it is one-generated. This fact highlights the importance of studying one-generated objects in the classification of nilpotent algebras. Also, the description of one-generated, or cyclic, groups is well known: there exists a unique one-generated group of order n, up to isomorphism. One could wonder if the situation would be similar in the case of some varieties of algebras. In fact, it has been proved that there exists only one n-dimensional one-generated nilpotent algebra in the varieties of associative [68], non-commutative Jordan [132] or Leibniz [171]. However, this circumstance does not hold for every variety of non-associative algebras. For example, the classifications of four-dimensional Novikov [137], assosymmetric [120] or terminal [144] nilpotent algebras show that there exist several one-generated algebras of dimension 4 from these varieties. Recently, one-generated nilpotent Novikov and assosymmetric algebras in dimensions 5 and 6, and one-generated nilpotent terminal algebras in dimension 5 were classified in [45, 102, 145].

Once the algebraic classification of the algebras of fixed dimension from a given variety is known, one can aim to classify them also geometrically in the sense explained in the sense of this manuscript. In particular, it is interesting to describe the so-called rigid algebras, since the closures of their orbits under the action of the generalised linear group form irreducible components.

We can find numerous examples of this approach in the literature, too. To cite some of them, we have [41, 107, 108, 203] for Lie algebras, [60, 61, 90, 172] for associative algebras, [103, 139–141] for Jordan algebras, [150] for Malcev algebras, [22, 23, 137] for Novikov algebras, and many others [5, 43, 84, 121, 151].

This second part of the manuscript is divided into three chapters. Chapter 3 amalgamates the contents of the articles [147] and [148] (joint works with Ivan Kaygorodov and Vasily Voronin), presenting the algebraic and geometric classifications of the four-dimensional nilpotent bicommutative algebras over $\mathbb{C}$, as well as the algebraic classifications of five- and six-dimensional one-generated nilpotent bicommutative algebras. The finding of two central extensions of the one-generated algebra $\mathcal{B}^3_{02}(1)$ will be the starting point of Chapter 4. In this chapter, we study different central extensions of the n-dimensional null-filiform associative algebra μ_0^n, which in dimension 3 coincides with $\mathcal{B}^3_{02}(1)$. The chapter corresponds to the article [146], a joint work with Ivan Kaygorodov and Samuel Lopes. Finally, in Chapter 5 we deal with a class of non-associative algebra which is not a variety, namely axial algebras. We base on the method of Skjelbred-Sund to describe a technique for constructing central extensions of axial algebras.This chapter corresponds to some work in progress with Ivan Kaygorodov and Cándido Martín González.

The structure of each chapter will be described in its own introduction.

CHAPTER 3

Algebraic and geometric classifications of nilpotent bicommutative algebras

In this chapter, we give a classification of the four-dimensional and the one-generated five- and six-dimensional nilpotent bicommutative algebras over $\mathbb{C}$. We also classify the four-dimensional complex nilpotent bicommutative algebras from a geometric approach.

Introduction

The variety of *bicommutative* algebras, also called *LR-algebras*, is defined by the polynomial identities of left- and right-commutativity. Explicitly, an algebra $(\mathrm{A}, \cdot)$ is bicommutative if

$$x\,(yz) \;=\; y\,(xz)\,, \qquad (xy)\,z \;=\; (xz)\,y,$$

for all $x, y, z \in \mathrm{A}$.

The first known example of one-sided commutative algebras is the right-symmetric Witt algebra in one variable, and dates back to 1857 [54]. This algebra satisfies the identity of left-commutativity but not of right-commutativity, so it is not bicommutative. The simpler examples of bicommutative algebras are the commutative and associative algebras. Note that bicommutative algebras are Lie admissible: the commutator $[x, y] = xy - yx$ defines an associated Lie algebra structure on A.

Bicommutative algebras over $\mathbb{R}$ naturally appear in connection with geometry, in particular with simply transitive affine actions of nilpotent Lie groups [38]. These bicommutative algebras are complete (i.e. the operators of left multiplication L_x are nilpotent for all $x \in A$) and their associated Lie algebra structures are nilpotent. This fact motivates the classification in [39] for dimension $n \leq 4$, where the authors limit to consider real complete bicommutative algebras with nilpotent associated Lie structure. Bicommutative algebras have also been studied in [40, 70–74]. We highlight the algebraic and geometric classifications of two-dimensional bicommutative algebras over an algebraically closed field in [151].

In this chapter, we also offer a partial classification of bicommutative algebras of dimension $n \leq 4$, but with a different approach to that of [39]. On the one hand, we work over $\mathbb{C}$, instead of $\mathbb{R}$. On the other hand, we classify nilpotent bicommutative algebras, which is a larger class than that of complete bicommutative algebras with nilpotent associated Lie structure (see [40, Proposition 2.2]). Also, the classification in [39] depends heavily on the classification of nilpotent Lie algebras, while ours lies entirely within the variety of bicommutative algebras, after a preliminary selection of the two- and three-dimensional nilpotent complex algebras satisfying the identities of left- and right-commutativity.

This chapter is organised as follows. Section 3.1 summarises the methods for classifying nilpotent bicommutative algebras from both algebraic and geometric approaches, including also the current classifications of nilpotent bicommutative algebras in dimensions 2 and 3. In Sections 3.2, 3.3 and 3.4, we apply the analogue of the Skjelbred and Sund method [209] to obtain complete classifications of, respectively, four-dimensional, one-generated five-dimensional and one-generated six-dimensional nilpotent bicommutative algebras over $\mathbb{C}$. Finally, in Section 3.5 we apply the methods for computing degenerations to obtain the irreducible components of the variety of four-dimensional nilpotent bicommutative algebras over $\mathbb{C}$. We show that there exist two irreducible components in this variety.

Throughout this chapter, all algebras will be assumed to be over the field $\mathbb{C}$ of complex numbers.

3.1 Preliminaries on the classification of nilpotent bicommutative algebras

3.1.1 Method of the algebraic classification of nilpotent bicommutative algebras

In this section, we offer an analogue of the Skjelbred-Sund method for classifying nilpotent bicommutative algebras. As other analogues of this method were carefully explained in, for instance, [3, 44], we will limit to expose its general lines, and refer the interested reader to the previous sources.

We would like to highlight that this method will be generalised for any variety of non-associative algebras in Section 4.1. However, we decided to include it here too to facilitate the reading of this chapter.

Let $(\mathrm{A}, \cdot)$ be a bicommutative algebra of dimension n and V a vector space of dimension s . We define the $\mathbb{C}$-linear space $\mathrm{Z}^2(\mathrm{A},\mathrm{V})$ as the set of all bilinear maps $\theta\colon \mathrm{A}\times\mathrm{A}\to\mathrm{V}$ such that

$$\theta(xy,z)=\theta(xz,y),\ \ \theta(x,yz)=\theta(y,xz).$$

These maps will be called *cocycles*. Consider a linear map f from A to V, and set $\delta f\colon \mathrm{A}\times\mathrm{A}\to\mathrm{V}$ with $\delta f(x,y)=f(xy)$. Then, δf is a cocycle, and we define $\mathrm{B}^2(\mathrm{A},\mathrm{V})=\{\theta=\delta f \mid f\in\mathrm{Hom}(\mathrm{A},\mathrm{V})\}$, which is a linear subspace of $\mathrm{Z}^2(\mathrm{A},\mathrm{V})$. Its elements are called *coboundaries*. The *second cohomology space* $\mathrm{H}^2(\mathrm{A},\mathrm{V})$ is defined to be the quotient space $\mathrm{Z}^2(\mathrm{A},\mathrm{V})\big/\mathrm{B}^2(\mathrm{A},\mathrm{V})$.

Let $\phi\in\mathrm{Aut}(\mathrm{A})$ be an automorphism of A. Every $\theta\in\mathrm{Z}^2(\mathrm{A},\mathrm{V})$ defines another cocycle $\phi\theta$ by $\phi\theta(x,y)=\theta(\phi(x),\phi(y))$. This construction induces a right action of $\mathrm{Aut}(\mathrm{A})$ on $\mathrm{Z}^2(\mathrm{A},\mathrm{V})$, which leaves $\mathrm{B}^2(\mathrm{A},\mathrm{V})$ invariant. So, it follows that $\mathrm{Aut}(\mathrm{A})$ also acts on $\mathrm{H}^2(\mathrm{A},\mathrm{V})$.

Given a bilinear map $\theta\colon \mathrm{A}\times\mathrm{A}\to\mathrm{V}$, we can construct the direct sum $\mathrm{A}_\theta=\mathrm{A}\oplus\mathrm{V}$ and endow it with the bilinear product $[-,-]_{\mathrm{A}_\theta}$ defined by $\left[x+x',y+y'\right]_{\mathrm{A}_\theta}=xy+\theta(x,y)$ for all $x,y\in\mathrm{A}, x',y'\in\mathrm{V}$. It is clear that the algebra A_θ is bicommutative if and only if $\theta\in\mathrm{Z}^2(\mathrm{A},\mathrm{V})$. Note that it is a s- dimensional central extension of A by V; it is not difficult to prove that every central extension of A is of this type.

An important object for the development of the method is the so-called *annihilator* of θ, namely $\mathrm{Ann}(\theta)=\{x\in\mathrm{A}\mid\theta(x,\mathrm{A})+\theta(\mathrm{A},x)=0\}$. We also recall that the *annihilator* (or *centre*) of an algebra A is the ideal $\mathrm{Ann}(\mathrm{A})=\{x\in\mathrm{A}\mid x\mathrm{A}+\mathrm{A}x=0\}$. It holds that $\mathrm{Ann}\left(\mathrm{A}_\theta\right)=(\mathrm{Ann}(\theta)\cap\mathrm{Ann}(\mathrm{A}))\oplus\mathrm{V}$. If $\mathrm{A}=\mathrm{A}_0\oplus I$ for any subspace

I of $\mathrm{Ann}(\mathrm{A})$, then I is called an *annihilator component* of A. A central extension of an algebra A with annihilator component is called a *split* central extension; without annihilator component, *non-split*.

The following result is fundamental for the classification method. For an idea of the proof, we refer the reader to [3, Lemma 5].

Lemma 3.1.1. *Let* A *be an n-dimensional bicommutative algebra such that satisfies* $\dim(\mathrm{Ann}(\mathrm{A})) = s \neq 0$*. Then there exists, up to isomorphism, a unique* $(n-s)$*-dimensional bicommutative algebra* A' *and a bilinear map* $\theta \in \mathrm{Z}^2(\mathrm{A}, \mathrm{V})$ *such that* $\mathrm{Ann}\left(\mathrm{A}'\right) \cap \mathrm{Ann}(\theta) = 0$*, where* V *is a vector space of dimension s, such that* $\mathrm{A} \simeq \mathrm{A}'_{\theta}$ *and* $\mathrm{A}/\mathrm{Ann}(\mathrm{A}) \simeq \mathrm{A}'$.

In view of this lemma, we can solve the isomorphism problem for bicommutative algebras with non-zero annihilator just working with central extensions.

Let us fix a basis $\left\{e_1, \ldots, e_s\right\}$ of V. Given a cocycle θ, it can be uniquely written as $\theta(x, y) = \displaystyle\sum_{i=1}^{s} \theta_i(x, y) e_i$, where $\theta_i \in \mathrm{Z}^2(\mathrm{A}, \mathbb{C})$. The main relations between θ and θ_i are that $\theta \in \mathrm{B}^2(\mathrm{A}, \mathrm{V})$ if and only if all $\theta_i \in \mathrm{B}^2(\mathrm{A}, \mathbb{C})$, and that $\mathrm{Ann}(\theta) = \mathrm{Ann}\left(\theta_1\right) \cap \ldots \cap \mathrm{Ann}\left(\theta_s\right)$. Furthermore, we have the following lemma (see [3, Lemma 13]).

Lemma 3.1.2. *With the previous notations, if* $\mathrm{Ann}(\theta) \cap \mathrm{Ann}(\mathrm{A}) = 0$*, then* A_θ *has an annihilator component if and only if* $\left[\theta_1\right], \ldots, \left[\theta_s\right]$ *are linearly dependent in* $\mathrm{H}^2(\mathrm{A}, \mathbb{C})$.

Recall that, given a finite-dimensional vector space V over $\mathbb{C}$, the *Grassmannian* $G_k(\mathrm{V})$ is the set of all k-dimensional linear subspaces of V. Given

$$W = \left\langle \left[\theta_1\right], \ldots, \left[\theta_s\right] \right\rangle \in G_s\left(\mathrm{H}^2(\mathrm{A}, \mathbb{C})\right)$$

and $\phi \in \mathrm{Aut}(\mathrm{A})$, we define ϕW as $\phi W = \left\langle \left[\phi\theta_1\right], \ldots, \left[\phi\theta_s\right] \right\rangle$, which again belongs to $G_s\left(\mathrm{H}^2(\mathrm{A}, \mathbb{C})\right)$. This induces an action of $\mathrm{Aut}(\mathrm{A})$ on $G_s\left(\mathrm{H}^2(\mathrm{A}, \mathbb{C})\right)$; the orbit of $W \in G_s\left(\mathrm{H}^2(\mathrm{A}, \mathbb{C})\right)$ under this action will be denoted by $\mathrm{Orb}(W)$.

Consider the set

$$T_s(\mathrm{A}) = \left\{ W = \left\langle \left[\theta_1\right], \ldots, \left[\theta_s\right] \right\rangle \in G_s\left(\mathrm{H}^2(\mathrm{A}, \mathbb{C})\right) \mid \bigcap_{i=1}^{s} \mathrm{Ann}\left(\theta_i\right) \cap \mathrm{Ann}(\mathrm{A}) = 0 \right\};$$

it is well defined, as whenever $\left\langle \left[\theta_1\right], \ldots, \left[\theta_s\right] \right\rangle = \left\langle \left[\vartheta_1\right], \ldots, \left[\vartheta_s\right] \right\rangle \in G_s\left(\mathrm{H}^2\left(\mathrm{A}, \mathbb{C}\right)\right)$, it holds that

$$\bigcap_{i=1}^{s} \mathrm{Ann}\left(\theta_i\right) \cap \mathrm{Ann}\left(\mathrm{A}\right) = \bigcap_{i=1}^{s} \mathrm{Ann}\left(\vartheta_i\right) \cap \mathrm{Ann}\left(\mathrm{A}\right).$$

In addition, $T_s\left(\mathrm{A}\right)$ is stable under the action of $\mathrm{Aut}\left(\mathrm{A}\right)$.

Let us denote by $\mathrm{E}\left(\mathrm{A}, \mathrm{V}\right)$ the set of all non-split s-dimensional central extensions of A by V:

$$\mathrm{E}\left(\mathrm{A}, \mathrm{V}\right) = \left\{ \mathrm{A}_\theta \mid \theta\left(x, y\right) = \sum_{i=1}^{s} \theta_i\left(x, y\right) e_i \text{ and } \left\langle \left[\theta_1\right], \ldots, \left[\theta_s\right] \right\rangle \in T_s\left(\mathrm{A}\right) \right\}.$$

The following lemma (cf. [3, Lemma 17]) solves the isomorphism problem for non-split central extensions of bicommutative algebras.

Lemma 3.1.3. *Let* $\mathrm{A}_\theta, \mathrm{A}_\vartheta \in \mathrm{E}\left(\mathrm{A}, \mathrm{V}\right)$. *Suppose that* $\theta\left(x, y\right) = \sum_{i=1}^{s} \theta_i\left(x, y\right) e_i$ *and* $\vartheta\left(x, y\right) = \sum_{i=1}^{s} \vartheta_i\left(x, y\right) e_i$. *Then the bicommutative algebras* A_θ *and* A_ϑ *are isomorphic if and only if*

$$\mathrm{Orb}\left\langle \left[\theta_1\right], \ldots, \left[\theta_s\right] \right\rangle = \mathrm{Orb}\left\langle \left[\vartheta_1\right], \ldots, \left[\vartheta_s\right] \right\rangle.$$

In conclusion, we can construct all the non-split central extensions of a bicommutative algebra A of dimension $n - s$ by following this procedure:

1. Determine $\mathrm{H}^2\left(\mathrm{A}, \mathbb{C}\right)$, $\mathrm{Ann}\left(\mathrm{A}\right)$ and $\mathrm{Aut}\left(\mathrm{A}\right)$.

2. Determine the set of $\mathrm{Aut}\left(\mathrm{A}\right)$-orbits on $T_s\left(\mathrm{A}\right)$.

3. For each orbit, construct the bicommutative algebra associated with a representative of it.

Any nilpotent bicommutative algebra has non-zero annihilator, and hence can be regarded as a central extension of a bicommutative algebra of smaller dimension, by Lemma 3.1.1. Note that this algebra has to be nilpotent, too. Then, Lemma 3.1.3 yields that, provided that the classifications of nilpotent bicommutative algebras of dimension up to $n-1$ are known, we can also classify all the nilpotent bicommutative algebras of dimension n.

Note also that if we want to stick to the one-generated case, it suffices to consider the non-split central extensions of one-generated nilpotent bicommutative algebras of lower dimension. Indeed, the central extensions of an algebra which is not one-generated cannot be one-generated; on the other hand, considering the definition of $\mathrm{B}^2(\mathrm{A},\mathrm{V})$ and Lemma 3.1.2, it is not difficult to see that the non-split extensions of a one-generated algebra are again one-generated.

3.1.2 Method of the description of degenerations of bicommutative algebras

Let W be an n-dimensional vector space over $\mathbb{C}$. The set $\mathrm{Hom}(\mathrm{W}\otimes\mathrm{W},\mathrm{W})\simeq\mathrm{W}^*\otimes\mathrm{W}^*\otimes\mathrm{W}$ is a vector space of dimension n^3, and it has the structure of the affine variety $\mathbb{C}^{n^3}$. Indeed, if we fix a basis $\left\{e_1,\ldots,e_n\right\}$ of W, then any $\eta\in\mathrm{Hom}(\mathrm{W}\otimes\mathrm{W},\mathrm{W})$ is determined by n^3 structure constants $c_{ij}^k\in\mathbb{C}$ such that $\eta\left(e_i\otimes e_j\right)=\sum\limits_{k=1}^{n}c_{ij}^k e_k$. A subset of $\mathrm{Hom}(\mathrm{W}\otimes\mathrm{W},\mathrm{W})$ is called *Zariski-closed* if it can be defined by a set of polynomial equations in the variables c_{ij}^k $(1\leq i,j,k\leq n)$.

Let T be the set of the polynomial identities of left- and right-commutativity. It holds that the algebra structures on W satisfying the polynomial identities from T form a Zariski-closed subset of the variety $\mathrm{Hom}(\mathrm{W}\otimes\mathrm{W},\mathrm{W})$; it is denoted by $\mathbb{L}(T)$. There exists a natural action of the general linear group $GL(\mathrm{W})$ on $\mathbb{L}(T)$ defined by

$$(g*\eta)(x\otimes y)=g\eta\left(g^{-1}x\otimes g^{-1}y\right)$$

for $x,y\in\mathrm{W}$, $\eta\in\mathbb{L}(T)$ and $g\in GL(\mathrm{W})$. Then, $\mathbb{L}(T)$ can be decomposed into $GL(\mathrm{W})$-orbits corresponding to the isomorphism classes of the algebras. We will denote by $O(\eta)$ the orbit of $\eta\in\mathbb{L}(T)$ under the action of $GL(\mathrm{W})$, and by $\overline{O(\eta)}$ the Zariski closure of $O(\eta)$.

Let A and B be two n-dimensional bicommutative algebras, and let $\eta,\nu\in\mathbb{L}(T)$ represent A and B, respectively. We say that A *degenerates* to B, and write $\mathrm{A}\to\mathrm{B}$, if $\nu\in\overline{O(\eta)}$. Note that, in particular, it holds that $\overline{O(\nu)}\subseteq\overline{O(\eta)}$. Hence, the definition of a degeneration does not depend on the choice of η and ν. If $\mathrm{A}\not\simeq\mathrm{B}$, then the assertion $\mathrm{A}\to\mathrm{B}$ is called a *proper degeneration*. Also, we write $\mathrm{A}\not\to\mathrm{B}$ if $\nu\notin\overline{O(\eta)}$.

Now consider $\mathrm{A}(*):=\{\mathrm{A}(\lambda)\}_{\lambda\in\mathrm{I}}$ and $\mathrm{B}(*):=\{\mathrm{B}(\mu)\}_{\mu\in\mathrm{J}}$ two infinite families of algebras parameterised by λ and μ, respectively, and let $\mathrm{A}(\lambda)$, for $\lambda\in I$, be represented by the structure $\eta(\lambda)\in\mathbb{L}(T)$, and $\mathrm{B}(\mu)$, for $\mu\in J$, by the structure $\nu(\mu)\in\mathbb{L}(T)$. Then $\mathrm{A}(*)\to\mathrm{B}$ means $\nu\in\overline{\{O(\eta(\lambda))\}_{\lambda\in I}}$, and $\mathrm{A}(*)\not\to\mathrm{B}$ means

$\nu \notin \overline{\{O(\eta(\lambda))\}_{\lambda\in I}}$. On the other hand, A $\to$ B $(*)$ means that $\nu(\mu) \in \overline{O(\eta)}$ for all $\mu \in B$ except a finite number of instances, and A $\not\to$ B $(*)$ means that $\nu(\mu) \notin \overline{O(\eta)}$ for infinite $\mu \in B$.

Moreover, we call A *rigid* in $\mathbb{L}(T)$ if $O(\eta)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called *irreducible* if it cannot be represented as a union of two non-trivial closed subsets, and that a maximal irreducible closed subset of a variety is called an *irreducible component*. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. Then, we have the following characterization of rigidity: A is rigid in $\mathbb{L}(T)$ if and only if $\overline{O(\eta)}$ is an irreducible component of $\mathbb{L}(T)$.

To describe the degenerations between bicommutative algebras, we use the methods applied to Lie algebras in [41, 107, 108, 203]. Let Der (A) denote the Lie algebra of derivations of A. Our first and useful consideration is that if A $\to$ B and A $\not\cong$ B, then $\dim \operatorname{Der}(\mathrm{A}) < \dim \operatorname{Der}(\mathrm{B})$ and $\dim \mathrm{A}^2 \geq \dim \mathrm{B}^2$. Then, we will compute the dimensions of algebras of derivations and will check the assertion A $\to$ B only for A and B such that $\dim \operatorname{Der}(\mathrm{A}) < \dim \operatorname{Der}(\mathrm{B})$. Among them, we will calculate the dimension of the squares of the algebras and check A $\to$ B only for A and B such that $\dim \mathrm{A}^2 \geq \dim \mathrm{B}^2$.

Now, we explain our method for proving degenerations. Let A, A $(*)$, B and B $(*)$ be as above. Fixed a basis $\{e_1, \dots, e_n\}$ of W, let c_{ij}^k $(1 \leq i, j, k \leq n)$ be the structure constants of ν in this basis, and $c_{ij}^k(\mu)$ $(1 \leq i, j, k \leq n)$ be the structure constants of $\nu(\mu)$. On the one hand, if there exist $a_{ij} : \mathbb{C}^* \to \mathbb{C}$ $(1 \leq i, j \leq n)$ such that $E_i^t = \sum_{j=1}^{n} a_{ij}(t) e_j$, for $1 \leq i \leq n$, form a basis of W for any $t \in \mathbb{C}^*$, and the structure constants of η in the basis $\{E_1^t, \dots, E_n^t\}$ are such polynomials $c_{ij}^k(t) \in \mathbb{C}[t]$ that $c_{ij}^k(0) = c_{ij}^k$, then A $\to$ B. In this case $\{E_1^t, \dots, E_n^t\}$ is called a *parameterised basis* for A $\to$ B.

Also, if there exist some $a_{ij} : J \times \mathbb{C}^* \to \mathbb{C}$ $(1 \leq i, j \leq n)$ such that $E_i^t(\mu) = \sum_{j=1}^{n} a_{ij}(\mu, t) e_j$, for $1 \leq i \leq n$, form a basis of W for any $t \in \mathbb{C}^*$ and all $\mu \in J$ except for a finite number of instances, and the structure constants of η in the basis $\{E_1^t(\mu), \dots, E_n^t(\mu)\}$ are such polynomials $c_{ij}^k(\mu, t) \in \mathbb{C}[\mu, t]$ that $c_{ij}^k(\mu, 0) = c_{ij}^k(\mu)$, then A $\to$ B $(*)$. The basis $\{E_1^t(\mu), \dots, E_n^t(\mu)\}$ is called a *parameterised basis* for A $\to$ B $(*)$.

On the other hand, if we construct $a_{ij} : \mathbb{C}^* \to \mathbb{C}$ $(1 \le i, j \le n)$ and $f : \mathbb{C}^* \to I$ such that $E_i^t = \sum_{j=1}^{n} a_{ij}(t) e_j$, for $1 \le i \le n$, form a basis of W for any $t \in \mathbb{C}^*$, and the structure constants of $\eta(f(t))$ in the basis $\{E_1^t, \dots, E_n^t\}$ are such polynomials $c_{ij}^k(t) \in \mathbb{C}[t]$ that $c_{ij}^k(0) = c_{ij}^k$, then $\mathrm{A}(*) \to \mathrm{B}$. In this case $\{E_1^t, \dots, E_n^t\}$ and $f(t)$ are called a *parameterised basis* and a *parameterised index* for $\mathrm{A}(*) \to \mathrm{B}$, respectively.

3.1.3 Notations

Let A be a bicommutative algebra over $\mathbb{C}$ with basis $\{e_1, \dots, e_n\}$, and let V be a vector space over $\mathbb{C}$. We will denote by Δ_{ij} the bicommutative bilinear form $\Delta_{ij} : \mathrm{A} \times \mathrm{A} \to \mathbb{C}$ with $\Delta_{ij}(e_l, e_m) = \delta_{il}\delta_{jm}$. Then, the set $\{\Delta_{ij} \mid 1 \le i, j \le n\}$ is a basis for the linear space of the bilinear forms on A, and every $\theta \in \mathrm{Z}^2(\mathrm{A}, \mathrm{V})$ can be uniquely written as $\theta = \sum_{1 \le i, j \le n} c_{ij}\Delta_{ij}$, where $c_{ij} \in \mathbb{C}$. Henceforth, we will denote:

$\mathcal{B}_j^{i*}$ — jth i-dimensional nilpotent *non-pure* bicommutative algebra (with identity $xyz = 0$);
$\mathcal{B}_j^{i}$ — jth i-dimensional nilpotent *pure* bicommutative algebra (without identity $xyz = 0$);
$\mathcal{C}_j^{i}$ — jth i-dimensional nilpotent one-generated bicommutative algebra;
$\mathfrak{N}_i$ — i-dimensional algebra with zero product;
$(\mathrm{A})_{i,j}$ — jth i-dimensional central extension of A.

Also, all algebras and vector spaces will be assumed to be over $\mathbb{C}$.

3.1.4 Low dimensional nilpotent bicommutative algebras

There are no non-trivial one-dimensional nilpotent bicommutative algebras, and there is only one non-trivial two-dimensional nilpotent bicommutative algebra, which is exactly the non-split central extension of the one-dimensional algebra with zero product $\mathfrak{N}_1$:

$$\mathcal{B}_{01}^{2*} \quad : \quad (\mathfrak{N}_1)_{2,1} \quad : \quad e_1e_1 = e_2.$$

From this algebra, we construct the three-dimensional nilpotent bicommutative algebra $\mathcal{B}_{01}^{3*} = \mathcal{B}_{01}^{2*} \oplus \mathbb{C}e_3$.

Also, [44] gives the description of all central extensions of $\mathcal{B}_{01}^{2*}$ and $\mathfrak{N}_2$. Choosing the bicommutative algebras between them, we have the classification of all non-split three-dimensional nilpotent bicommutative algebras:

$$\begin{array}{llllllll}
\mathcal{B}_{02}^{3*} & : & (\mathfrak{N}_2)_{3,1} & : & e_1e_1 = e_3 & e_2e_2 = e_3 & \\
\mathcal{B}_{03}^{3*} & : & (\mathfrak{N}_2)_{3,2} & : & e_1e_2 = e_3 & e_2e_1 = -e_3 & \\
\mathcal{B}_{04}^{3*}(\lambda)_{\lambda\neq 0} & : & (\mathfrak{N}_2)_{3,3} & : & e_1e_1 = \lambda e_3 & e_2e_1 = e_3 & e_2e_2 = e_3 \\
\mathcal{B}_{04}^{3*}(0) & : & (\mathfrak{N}_2)_{3,3} & : & e_1e_2 = e_3 & & \\
\mathcal{B}_{01}^{3} & : & (\mathcal{B}_{01}^{2*})_{3,1} & : & e_1e_1 = e_2 & e_2e_1 = e_3 & \\
\mathcal{B}_{02}^{3}(\lambda) & : & (\mathcal{B}_{01}^{2*})_{3,2} & : & e_1e_1 = e_2 & e_1e_2 = e_3 & e_2e_1 = \lambda e_3.
\end{array}$$

Remark 3.1.4. The reasons for considering separately the cases $\mathcal{B}_{04}^{3*}(\lambda)$ with $\lambda \neq 0$ and $\mathcal{B}_{04}^{3*}(0)$ will become apparent in next section, as their cohomology spaces are rather different.

From the previous list, we can select the one-generated algebras in dimension up to 3:

$$\begin{array}{lllllll}
\mathcal{C}_{01}^{2} & : & \mathcal{B}_{01}^{2*} & : & e_1e_1 = e_2 & & \\
\mathcal{C}_{01}^{3} & : & \mathcal{B}_{01}^{3} & : & e_1e_1 = e_2 & e_2e_1 = e_3 & \\
\mathcal{C}_{02}^{3}(\lambda) & : & \mathcal{B}_{02}^{3}(\lambda) & : & e_1e_1 = e_2 & e_1e_2 = e_3 & e_2e_1 = \lambda e_3.
\end{array}$$

Let us focus now on the geometric classification of bicommutative algebras of low dimensions. The variety of two-dimensional nilpotent bicommutative algebras is trivially irreducible; for the general case of two-dimensional bicommutative algebras, see [151]. Regarding dimension 3, the classification of the nilpotent algebras can be extracted from the graph of degenerations of all the nilpotent algebras of dimension 3 [84]. There are two irreducible components, defined by the rigid algebras $\mathcal{B}_{01}^{3}$ and $\mathcal{B}_{02}^{3}(\lambda)$.

3.2 Algebraic classification of four-dimensional nilpotent bicommutative algebras

Note that the seven three-dimensional algebras from Subsection 3.1.4 provide seven bicommutative algebras of dimension 4 by considering $\mathbb{C}^4$ as the underlying vector space. They will be denoted by $\mathcal{B}_{01}^{4*}$, $\mathcal{B}_{02}^{4*}$, $\mathcal{B}_{03}^{4*}$, $\mathcal{B}_{04}^{4*}(\lambda)_{\lambda\neq 0}$, $\mathcal{B}_{04}^{4*}(0)$, $\mathcal{B}_{01}^{4}$ and $\mathcal{B}_{02}^{4}(\lambda)$, respectively.

3.2.1 Second cohomology space of three-dimensional nilpotent bicommutative algebras

In the following Table 3.1 we give the description of the second cohomology space of three-dimensional nilpotent bicommutative algebras described in Subsection 3.1.4.

$\mathrm{Z}^2\left(\mathcal{B}_{01}^{3*}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, \Delta_{31}, \Delta_{33} \right\rangle$
$\mathrm{B}^2\left(\mathcal{B}_{01}^{3*}\right)$	$=$	$\left\langle \Delta_{11} \right\rangle$
$\mathrm{H}^2\left(\mathcal{B}_{01}^{3*}\right)$	$=$	$\left\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{B}_{02}^{3*}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \right\rangle$
$\mathrm{B}^2\left(\mathcal{B}_{02}^{3*}\right)$	$=$	$\left\langle \Delta_{11} + \Delta_{22} \right\rangle$
$\mathrm{H}^2\left(\mathcal{B}_{02}^{3*}\right)$	$=$	$\left\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{B}_{03}^{3*}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \right\rangle$
$\mathrm{B}^2\left(\mathcal{B}_{03}^{3*}\right)$	$=$	$\left\langle \Delta_{12} - \Delta_{21} \right\rangle$
$\mathrm{H}^2\left(\mathcal{B}_{03}^{3*}\right)$	$=$	$\left\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{B}_{04}^{3*}(\lambda)_{\lambda\neq 0}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \right\rangle$
$\mathrm{B}^2\left(\mathcal{B}_{04}^{3*}(\lambda)_{\lambda\neq 0}\right)$	$=$	$\left\langle \lambda\Delta_{11} + \Delta_{21} + \Delta_{22} \right\rangle$
$\mathrm{H}^2\left(\mathcal{B}_{04}^{3*}(\lambda)_{\lambda\neq 0}\right)$	$=$	$\left\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{B}_{04}^{3*}(0)\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, \Delta_{22}, \Delta_{32} \right\rangle$
$\mathrm{B}^2\left(\mathcal{B}_{04}^{3*}(0)\right)$	$=$	$\left\langle \Delta_{12} \right\rangle$
$\mathrm{H}^2\left(\mathcal{B}_{04}^{3*}(0)\right)$	$=$	$\left\langle [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], [\Delta_{22}], [\Delta_{32}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{B}_{01}^{3}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{B}^2\left(\mathcal{B}_{01}^{3}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{21} \right\rangle$
$\mathrm{H}^2\left(\mathcal{B}_{01}^{3}\right)$	$=$	$\left\langle [\Delta_{12}], [\Delta_{31}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{B}_{02}^{3}(\lambda)\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \lambda\Delta_{22} + \lambda\Delta_{31} \right\rangle$
$\mathrm{B}^2\left(\mathcal{B}_{02}^{3}(\lambda)\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12} + \lambda\Delta_{21} \right\rangle$
$\mathrm{H}^2\left(\mathcal{B}_{02}^{3}(\lambda)\right)$	$=$	$\left\langle [\Delta_{21}], \alpha[\Delta_{22}] + [\Delta_{13}] + \alpha[\Delta_{31}] \right\rangle$

Table 3.1: Second cohomology space of three-dimensional nilpotent bicommutative algebras.

Remark 3.2.1. From the description of the cocycles of the algebras $\mathcal{B}_{02}^{3*}$, $\mathcal{B}_{03}^{3*}$ and $\mathcal{B}_{04}^{3*}(\lambda)_{\lambda\neq 0}$, it follows that the one-dimensional central extensions of these algebras are two-dimensional central extensions of two-dimensional nilpotent bicommutative algebras. Thanks to [44] we have the description of the unique non-split two-dimensional central extension of nilpotent bicommutative algebras of dimension 2: $\mathcal{B}_{03}^{4} = \left(\mathcal{B}_{01}^{2*}\right)_{4,1}$. The multiplication table can be found in Table 3.2 (Subsection 3.2.6).

Then, in the following subsections we limit to study the central extensions of the other algebras.

3.2.2 Central extensions of $\mathcal{B}_{01}^{3*}$

Since the second cohomology spaces and automorphism groups of $\mathcal{B}_{01}^{3*}$ and $\mathcal{N}_{01}^{3*}$ (from [137]) coincide, these algebras have the same central extensions. Therefore, thanks to [137] we have all the new four-dimensional nilpotent bicommutative algebras constructed from $\mathcal{B}_{01}^{3*}$: $\mathcal{B}_{04}^{4}(\lambda), \ldots, \mathcal{B}_{09}^{4}$. The multiplication tables of these algebras can be found in Table 3.2 (Section 3.2.6).

3.2.3 Central extensions of $\mathcal{B}_{04}^{3*}(0)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \nabla_2 = [\Delta_{13}], \nabla_3 = [\Delta_{21}], \nabla_4 = [\Delta_{22}], \nabla_5 = [\Delta_{32}].$$

The automorphism group of $\mathcal{B}_{04}^{3*}(0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & t & xy \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_2 \\ \alpha_3 & \alpha_4 & 0 \\ 0 & \alpha_5 & 0 \end{pmatrix} \phi = \begin{pmatrix} x\left(x\alpha_1 + z\alpha_2\right) & \alpha^* & x^2y\alpha_2 \\ xy\alpha_3 & y\left(y\alpha_4 + t\alpha_5\right) & 0 \\ 0 & xy^2\alpha_5 & 0 \end{pmatrix},$$

we have that the orbit of the action of $\mathrm{Aut}\left(\mathcal{B}_{04}^{3*}(0)\right)$ on the subspace $\left\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \right\rangle$, where

$$\begin{aligned} \alpha_1^* &= x\left(x\alpha_1 + z\alpha_2\right), & \alpha_2^* &= x^2y\alpha_2, & \alpha_3^* &= xy\alpha_3, \\ \alpha_4^* &= y\left(y\alpha_4 + t\alpha_5\right), & \alpha_5^* &= xy^2\alpha_5. \end{aligned}$$

It is easy to see that the elements $\alpha_1\nabla_1 + \alpha_3\nabla_3 + \alpha_4\nabla_4$ give algebras which are central extensions of two-dimensional algebras. We find the following new cases:

1. $\alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_5 \neq 0$. Choosing $x = \frac{\alpha_3}{\alpha_2}$, $y = \frac{\alpha_3}{\alpha_5}$, $z = -\frac{x\alpha_1}{\alpha_2}$ and $t = -\frac{y\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_5 \rangle$.
2. $\alpha_2 \neq 0, \alpha_3 = 0, \alpha_5 \neq 0$. Choosing $y = \frac{x\alpha_2}{\alpha_5}$, $z = -\frac{x\alpha_1}{\alpha_2}$ and $t = -\frac{y\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_2 + \nabla_5 \rangle$.
3. $\alpha_2 = 0, \alpha_3 \neq 0, \alpha_5 \neq 0$:
   (a) if $\alpha_1 \neq 0$, then choosing $y = \frac{\alpha_3}{\alpha_5}$, $x = \frac{y\alpha_3}{\alpha_1}$ and $t = -\frac{y\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_5 \rangle$.
   (b) if $\alpha_1 = 0$, then choosing $y = \frac{\alpha_3}{\alpha_5}$ and $t = -\frac{y\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle$.
4. $\alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_5 = 0$:
   (a) if $\alpha_4 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_2}$, $y = \frac{x\alpha_3}{\alpha_4}$ and $z = -\frac{x\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle$.
   (b) if $\alpha_4 = 0$, then choosing $x = \frac{\alpha_3}{\alpha_2}$ and $z = -\frac{x\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_3 \rangle$.
5. $\alpha_2 \neq 0, \alpha_3 = 0, \alpha_5 = 0$:
   (a) if $\alpha_4 \neq 0$, then choosing $y = \frac{x^2\alpha_2}{\alpha_4}$ and $z = -\frac{x\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_4 \rangle$.
   (b) if $\alpha_4 = 0$, then choosing $z = -\frac{x\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$.
6. $\alpha_2 = 0, \alpha_3 = 0, \alpha_5 \neq 0$:
   (a) if $\alpha_1 \neq 0$, then choosing $x = \frac{y^2\alpha_5}{\alpha_1}$ and $t = -\frac{y\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_1 + \nabla_5 \rangle$.
   (b) if $\alpha_1 = 0$ then choosing $t = -\frac{y\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_5 \rangle$.

Now we have all the new four-dimensional nilpotent bicommutative algebras constructed from $\mathcal{B}^{3*}_{04}(0)$: $\mathcal{B}^4_{10}, \dots, \mathcal{B}^4_{19}$. The multiplication tables of these algebras can be found in Table 3.2 (Section 3.2.6).

3.2.4 Central extensions of $\mathcal{B}^3_{01}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \nabla_2 = [\Delta_{31}].$$

The automorphism group of $\mathcal{B}^3_{01}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & xy & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & 0 \\ 0 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & x^3\alpha_1 & 0 \\ \alpha^{**} & 0 & 0 \\ x^4\alpha_2 & 0 & 0 \end{pmatrix},$$

we have that the orbit of the action of Aut $\left(\mathcal{B}^3_{01}\right)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^3\alpha_1, \quad \alpha_2^* = x^4\alpha_2.$$

It is straightforward that the elements $\alpha_1\nabla_1$ lead to central extensions of two-dimensional algebras. The new cases are the following:

1. $\alpha_1 \neq 0, \alpha_2 \neq 0$. Choosing $x = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.

2. $\alpha_1 = 0, \alpha_2 \neq 0$. Choosing $x = \frac{1}{\sqrt[4]{\alpha_2}}$, we have the representative $\langle \nabla_2 \rangle$.

Now we have all the new four-dimensional nilpotent bicommutative algebras constructed from $\mathcal{B}^3_{01}$: $\mathcal{B}^4_{20}$ and $\mathcal{B}^4_{21}$. The multiplication tables of these algebras can be found in Table 3.2 (Section 3.2.6).

3.2.5 Central extensions of $\mathcal{B}^3_{02}(\lambda)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \nabla_2 = [\Delta_{13}] + \lambda[\Delta_{22}] + \lambda[\Delta_{31}].$$

The automorphism group of $\mathcal{B}^3_{02}(\lambda)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & (\alpha+1)\,xy & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_2 \\ \alpha_1 & \lambda\alpha_2 & 0 \\ \lambda\alpha_2 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_2^* \\ \alpha_1^* + \lambda\alpha^{**} & \lambda\alpha_2^* & 0 \\ \lambda\alpha_2^* & 0 & 0 \end{pmatrix},$$

we have that the orbit of the action of $\operatorname{Aut}\left(\mathcal{B}^3_{02}(\lambda)\right)$ on the subspace $\left\langle \sum\limits_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum\limits_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^2\left(x\alpha_1 + \alpha(1-\lambda)\,y\alpha_2\right), \quad \alpha_2^* = x^4\alpha_2.$$

The elements $\alpha_1\nabla_1$ give central extensions of two-dimensional algebras, so we will consider only cases with $\alpha_2 \neq 0$. We find the following new cases:

1. $\lambda = 0$ or $\lambda = 1$:

   (a) if $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.

   (b) if $\alpha_1 = 0$, then choosing $x = \frac{1}{\sqrt[4]{\alpha_2}}$, we have the representative $\langle \nabla_2 \rangle$.

2. $\lambda \neq 0, 1$, then choosing $x = \frac{1}{\sqrt[4]{\alpha_2}}$ and $y = \frac{-x\alpha_1}{\alpha_2\alpha(1-\lambda)}$, we have the representative $\langle \nabla_2 \rangle$.

Now we have all the new four-dimensional nilpotent bicommutative algebras constructed from $\mathcal{B}^3_{02}(\lambda)$: $\mathcal{B}^4_{22}$, $\mathcal{B}^4_{23}$ and $\mathcal{B}^4_{24}(\lambda)$. The multiplication tables of these algebras can be found in Table 3.2 (Section 3.2.6).

3.2.6 Classification theorem

We distinguish two main classes of bicommutative algebras: the *pure* and the *non-pure* or *trivial* ones. By the non-pure ones, we mean those algebras A satisfying the identities $(xy)\,z = 0$ and $x\,(yz) = 0$ for all $x, y, z \in$ A; the pure ones are the rest.

These trivial algebras can be considered in many varieties of algebras defined by polynomial identities of degree three (associative, Leibniz, Zinbiel...), and they can be expressed as central extensions of suitable algebras with zero product. Those with dimension 4 are already classified: the list of the non-anticommutative ones can be found in [69], and there is only one nilpotent and anticommutative.

Regarding the pure four-dimensional nilpotent bicommutative algebras, we have the following theorem, whose proof is based on the classification of three-dimensional nilpotent bicommutative algebras and on the results of Subsections 3.2.2 to 3.2.5.

Theorem 3.2.2. *Let* A *be a four-dimensional nilpotent pure bicommutative algebra. Then,* A *is isomorphic to one of the algebras in the following Table 3.2:*

$\mathcal{B}^4_{01}$	$e_1e_1 = e_2$	$e_2e_1 = e_3$			
$\mathcal{B}^4_{02}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_2e_1 = \lambda e_3$		
$\mathcal{B}^4_{03}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$		
$\mathcal{B}^4_{04}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = \lambda e_4$	$e_3e_3 = e_4$	
$\mathcal{B}^4_{05}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_3e_3 = e_4$
$\mathcal{B}^4_{06}(\lambda)_{\lambda\neq 0}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	$e_2e_1 = \lambda e_4$	
$\mathcal{B}^4_{07}$	$e_1e_1 = e_2$	$e_2e_1 = e_4$	$e_3e_3 = e_4$		
$\mathcal{B}^4_{08}$	$e_1e_1 = e_2$	$e_1e_3 = e_4$	$e_2e_1 = e_4$		
$\mathcal{B}^4_{09}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_3e_1 = e_4$		
$\mathcal{B}^4_{10}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_3e_2 = e_4$	
$\mathcal{B}^4_{11}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_3e_2 = e_4$		
$\mathcal{B}^4_{12}$	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_3e_2 = e_4$	
$\mathcal{B}^4_{13}$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_3e_2 = e_4$		
$\mathcal{B}^4_{14}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_2e_2 = e_4$	
$\mathcal{B}^4_{15}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$		
$\mathcal{B}^4_{16}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_4$		

$\mathcal{B}^4_{17}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$		
$\mathcal{B}^4_{18}$	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_3e_2 = e_4$	
$\mathcal{B}^4_{19}$	$e_1e_2 = e_3$	$e_3e_2 = e_4$		
$\mathcal{B}^4_{20}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$	$e_3e_1 = e_4$
$\mathcal{B}^4_{21}$	$e_1e_1 = e_2$	$e_2e_1 = e_3$	$e_3e_1 = e_4$	
$\mathcal{B}^4_{22}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$
$\mathcal{B}^4_{23}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_4$	$e_3e_1 = e_4$	
$\mathcal{B}^4_{24}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
	$e_2e_1 = \lambda e_3$	$e_2e_2 = \lambda e_4$	$e_3e_1 = \lambda e_4$	

Table 3.2: Four-dimensional nilpotent *pure* bicommutative algebras.

From the previous list, we can select the one-generated algebras. Note that these exhaust the four-dimensional one-generated nilpotent bicommutative algebras, as the *non-pure* bicommutative algebras cannot be one-generated.

$$
\begin{array}{lllllll}
\mathcal{C}^4_{01} & : & \mathcal{B}^4_{03} & : & e_1e_1 = e_2 & e_1e_2 = e_4 & e_2e_1 = e_3 & \\
\mathcal{C}^4_{02} & : & \mathcal{B}^4_{20} & : & e_1e_1 = e_2 & e_1e_2 = e_4 & e_2e_1 = e_3 & e_3e_1 = e_4 \\
\mathcal{C}^4_{03} & : & \mathcal{B}^4_{21} & : & e_1e_1 = e_2 & e_2e_1 = e_3 & e_3e_1 = e_4 & \\
\mathcal{C}^4_{04} & : & \mathcal{B}^4_{22} & : & e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = e_4 & e_2e_1 = e_4 \\
\mathcal{C}^4_{05} & : & \mathcal{B}^4_{23} & : & e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = e_4 & \\
 & & & & e_2e_1 = e_3 + e_4 & e_2e_2 = e_4 & e_3e_1 = e_4 & \\
\mathcal{C}^4_{06}(\lambda) & : & \mathcal{B}^4_{24}(\lambda) & : & e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = e_4 & \\
 & & & & e_2e_1 = \lambda e_3 & e_2e_2 = \lambda e_4 & e_3e_1 = \lambda e_4. &
\end{array}
$$

3.3 Algebraic classification of five-dimensional one-generated nilpotent bicommutative algebras

In this section, we will base on the lists of one-generated nilpotent bicommutative algebras of dimension up to 4 from Subsection 3.1.4 and Subsection 3.2.6. Note that there are not three-dimensional central extensions of the two-dimensional one-generated bicommutative algebra $\mathcal{C}^2_{01}$.

3.3.1 Two-dimensional central extensions of three-dimensional one-generated algebras

Considering the two-dimensional central extensions of the three-dimensional one-generated nilpotent bicommutative algebras $\mathcal{C}_{01}^{3}$ and $\mathcal{C}_{02}^{3}(\lambda)$ we get the algebras $\mathcal{C}_{01}^{5}$ and $\mathcal{C}_{02}^{5}(\lambda)$, whose multiplication tables can be consulted in Table 3.4 (Subsection 3.3.9).

3.3.2 Second cohomology space of four-dimensional one-generated nilpotent bicommutative algebras

In the following Table 3.3 we give the description of the second cohomology space of four-dimensional one-generated nilpotent bicommutative algebras from Subsection 3.2.6.

$\mathrm{Z}^2\left(\mathcal{B}_{01}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13}+\Delta_{22}+\Delta_{41}, \Delta_{14}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{B}^2\left(\mathcal{C}_{01}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{21} \right\rangle$
$\mathrm{H}^2\left(\mathcal{C}_{01}^4\right)$	$=$	$\left\langle [\Delta_{14}], [\Delta_{13}]+[\Delta_{22}]+[\Delta_{41}], [\Delta_{31}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{C}_{02}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13}+\Delta_{22}+\Delta_{41}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{B}^2\left(\mathcal{C}_{02}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}+\Delta_{31}, \Delta_{21} \right\rangle$
$\mathrm{H}^2\left(\mathcal{C}_{02}^4\right)$	$=$	$\left\langle [\Delta_{13}]+[\Delta_{22}]+[\Delta_{41}], [\Delta_{31}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{C}_{03}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{31}, \Delta_{41} \right\rangle$
$\mathrm{B}^2\left(\mathcal{C}_{03}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{H}^2\left(\mathcal{C}_{03}^4\right)$	$=$	$\left\langle [\Delta_{12}], [\Delta_{41}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{C}_{04}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}+\Delta_{22}+\Delta_{31}, \Delta_{21} \right\rangle$
$\mathrm{B}^2\left(\mathcal{C}_{04}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13}+\Delta_{21} \right\rangle$
$\mathrm{H}^2\left(\mathcal{C}_{04}^4\right)$	$=$	$\left\langle [\Delta_{14}]+[\Delta_{22}]+[\Delta_{31}], [\Delta_{21}] \right\rangle$
$\mathrm{Z}^2\left(\mathcal{C}_{05}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13}+\Delta_{22}+\Delta_{31}, \Delta_{14}+\Delta_{22}+\Delta_{23}+\Delta_{31}+\Delta_{32}+\Delta_{41}, \Delta_{21} \right\rangle$
$\mathrm{B}^2\left(\mathcal{C}_{05}^4\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}+\Delta_{21}, \Delta_{13}+\Delta_{21}+\Delta_{22}+\Delta_{31} \right\rangle$
$\mathrm{H}^2\left(\mathcal{C}_{05}^4\right)$	$=$	$\left\langle [\Delta_{14}]+[\Delta_{22}]+[\Delta_{23}]+[\Delta_{31}]+[\Delta_{32}]+[\Delta_{41}], [\Delta_{21}] \right\rangle$

$\mathrm{Z}^2\left(C_{06}^4(\lambda)\right)$	$=$	$\langle\Delta_{11},\Delta_{12},\Delta_{13}+\lambda\Delta_{22}+\lambda\Delta_{31},\Delta_{14}+\lambda\Delta_{23}+\lambda\Delta_{32}+\lambda\Delta_{41},\Delta_{21}\rangle$
$\mathrm{B}^2\left(C_{06}^4(\lambda)\right)$	$=$	$\langle\Delta_{11},\Delta_{12}+\lambda\Delta_{21},\Delta_{13}+\lambda\Delta_{22}+\lambda\Delta_{31}\rangle$
$\mathrm{H}^2\left(C_{06}^4(\lambda)\right)$	$=$	$\langle[\Delta_{14}]+\alpha[\Delta_{23}]+\alpha[\Delta_{32}]+\alpha[\Delta_{41}],[\Delta_{21}]\rangle$

Table 3.3: Second cohomology space of four-dimensional one-generated nilpotent bicommutative algebras.

3.3.3 Central extensions of C_{01}^4

Let us use the following notations:

$$\nabla_1=[\Delta_{14}],\nabla_2=[\Delta_{31}],\nabla_3=[\Delta_{13}]+[\Delta_{41}]+[\Delta_{22}].$$

The automorphism group of C_{01}^4 consists of invertible matrices of the form

$$\phi=\begin{pmatrix} x & 0 & 0 & 0\\ y & x^2 & 0 & 0\\ z & xy & x^3 & 0\\ t & xy & 0 & x^3\end{pmatrix}.$$

Since

$$\phi^T\begin{pmatrix} 0 & 0 & \alpha_3 & \alpha_1\\ 0 & \alpha_3 & 0 & 0\\ \alpha_2 & 0 & 0 & 0\\ \alpha_3 & 0 & 0 & 0\end{pmatrix}\phi=\begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_3^* & \alpha_1^*\\ \alpha^{***} & \alpha_3^* & 0 & 0\\ \alpha_2^* & 0 & 0 & 0\\ \alpha_3^* & 0 & 0 & 0\end{pmatrix},$$

then the action of $\mathrm{Aut}\left(C_{01}^4\right)$ on the subspace $\left\langle\sum_{i=1}^3\alpha_i\nabla_i\right\rangle$ is given by $\left\langle\sum_{i=1}^3\alpha_i^*\nabla_i\right\rangle$, where

$$\alpha_1^*=x^4\alpha_1,\quad \alpha_2^*=x^4\alpha_2,\quad \alpha_3^*=x^4\alpha_3.$$

One-dimensional central extensions

We have the following cases:

1. if $\alpha_3 \neq 0$, by choosing $x = \frac{1}{\sqrt[4]{\alpha_3}}$ we have the representative $\langle \lambda\nabla_1 + \mu\nabla_2 + \nabla_3 \rangle$;

2. if $\alpha_3 = 0$, then it must hold that $\alpha_1 \neq 0$ (otherwise, the new algebras would have two-dimensional annihilator and could be constructed as two-dimensional central extensions of three-dimensional bicommutative algebras), and by choosing $x = \frac{1}{\sqrt[4]{\alpha_1}}$, we have the representatives $\langle \nabla_1 + \lambda\nabla_2 \rangle$.

Hence, we obtain the new algebras $\mathcal{C}_{03}^5(\lambda, \mu)$ and $\mathcal{C}_{04}^5(\lambda)$, whose multiplication tables can be consulted in Table 3.4 (Subsection 3.3.9).

Two-dimensional central extensions

Consider the vector space generated by the following two cocycles:

$$\begin{aligned} \theta_1 &= \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3 \\ \theta_2 &= \beta_1\nabla_1 + \beta_2\nabla_2. \end{aligned}$$

If $\alpha_3 = 0$, we get the representative $\langle \nabla_1, \nabla_2 \rangle$. If $\alpha_3 \neq 0$, we distinguish the following cases:

1. if $\beta_2 = 0$, we have the family of representatives $\langle \nabla_1, \lambda\nabla_2 + \nabla_3 \rangle$;

2. if $\beta_2 \neq 0$, we have the family of representatives $\langle \lambda\nabla_1 + \nabla_2, \mu\nabla_1 + \nabla_3 \rangle$.

Hence, we have the new algebras $\mathcal{C}_{01}^6$, $\mathcal{C}_{02}^6(\lambda)$ and $\mathcal{C}_{03}^6(\lambda, \mu)$, whose multiplication tables can be found in Table 3.6 (Subsection 3.4.13)

3.3.4 Central extensions of $\mathcal{C}_{02}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{31}], \nabla_2 = [\Delta_{13}] + [\Delta_{22}] + [\Delta_{41}].$$

The automorphism group of $\mathcal{C}_{02}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & x & 1 & 0 \\ z & x+y & x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_2 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_2^* & 0 \\ \alpha^{***} & \alpha_2^* & 0 & 0 \\ \alpha_1^* + \alpha^{**} & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

the action of Aut $\left(\mathcal{C}_{02}^4\right)$ on the subspace $\left\langle \sum\limits_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum\limits_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = \alpha_1 - x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

One-dimensional central extensions

We can suppose that $\alpha_2 \neq 0$, as otherwise we would obtain algebras with two-dimensional annihilator which would be two-dimensional central extensions of three-dimensional bicommutative algebras. Then, by choosing $x = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$, whose associated algebra is $\mathcal{C}_{05}^5$. Its multiplication table can be consulted in Table 3.4 (Subsection 3.3.9).

Two-dimensional central extensions

Since the cohomology space has dimension 2, we just obtain the new algebra $\mathcal{C}_{04}^6$, whose multiplication table can be found in Table 3.6 (Subsection 3.4.13).

3.3.5 Central extensions of $\mathcal{C}_{03}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \nabla_2 = [\Delta_{41}].$$

The automorphism group of C_{03}^4 consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & xy & x^3 & 0 \\ t & xz & x^2y & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & 0 & 0 \\ \alpha^{**} & 0 & 0 & 0 \\ \alpha^{***} & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

the action of Aut $\left(C_{03}^4\right)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^3\alpha_1, \quad \alpha_2^* = x^5\alpha_2.$$

One-dimensional central extensions

We can suppose again that $\alpha_2 \neq 0$. So, we have the following cases:

1. if $\alpha_1 \neq 0$, by choosing $x = \sqrt{\frac{\alpha_1}{\alpha_2}}$ we have the representative $\langle \nabla_1 + \nabla_2 \rangle$;

2. if $\alpha_1 = 0$, by choosing $x = \frac{1}{\sqrt[5]{\alpha_2}}$ we have the representative $\langle \nabla_2 \rangle$.

Hence, we get the new algebras C_{06}^5 and C_{07}^5. Their multiplication tables can be consulted in Table 3.4 (Subsection 3.3.9).

Two-dimensional central extensions

Since the cohomology space has dimension 2, we have just the new algebra C_{05}^6, whose multiplication table can be found in Table 3.6 (Subsection 3.4.13).

3.3.6 Central extensions of $\mathcal{C}_{04}^4$

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \nabla_2 = [\Delta_{14}] + [\Delta_{22}] + [\Delta_{31}].$$

The automorphism group of $\mathcal{C}_{04}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & x & 1 & 0 \\ z & x+y & x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & \alpha_2 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha^{***} & \alpha_2^* \\ \alpha_1^* + \alpha^{***} & \alpha_2^* & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the action of $\mathrm{Aut}\left(\mathcal{C}_{04}^4\right)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = \alpha_1 + x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

One-dimensional central extensions

We can suppose that $\alpha_2 \neq 0$. Then, by choosing $x = -\frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$, with associated algebra $\mathcal{C}_{08}^5$. Its multiplication table can be found in Table 3.4 (Subsection 3.3.9).

Two-dimensional central extensions

Since the cohomology space has dimension 2, we get the new algebra $\mathcal{C}_{06}^6$, whose multiplication table can be consulted in Table 3.6 (Subsection 3.4.13).

3.3.7 Central extensions of C_{05}^4

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \nabla_2 = [\Delta_{14}] + [\Delta_{22}] + [\Delta_{23}] + [\Delta_{31}] + [\Delta_{32}] + [\Delta_{41}].$$

The automorphism group of C_{05}^4 consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 2x & 1 & 0 \\ z & x^2+x+2y & 3x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_2 & 0 \\ \alpha_2 & \alpha_2 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha^{***} & \alpha_2^* \\ \alpha_1^* + \alpha^{**} + \alpha^{***} & \alpha_2^* + \alpha^{***} & 0 & 0 \\ \alpha_2^* + \alpha^{***} & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

the action of $\mathrm{Aut}\left(C_{05}^4\right)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = \alpha_1 - 2x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

One-dimensional central extensions

We can suppose that $\alpha_2 \neq 0$. Then, by choosing $x = \frac{\alpha_1}{2\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$. Hence, we obtain the new algebra C_{09}^5, whose multiplication table can be consulted in Table 3.4 (Subsection 3.3.9).

Two-dimensional central extensions

Since the cohomology space has dimension 2, we have the new algebra C_{07}^6. Its multiplication table can be found in Table 3.6 (Subsection 3.4.13).

3.3.8 Central extensions of $\mathcal{C}_{06}^{4}(\lambda)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \nabla_2 = [\Delta_{14}] + \alpha[\Delta_{23}] + \alpha[\Delta_{32}] + \alpha[\Delta_{41}].$$

The automorphism group of $\mathcal{C}_{06}^{4}(\lambda)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 \\ y & 0 & x^3 & 0 \\ z & (1+\lambda)\,xy & 0 & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & 0 & \lambda\alpha_2 & 0 \\ 0 & \lambda\alpha_2 & 0 & 0 \\ \lambda\alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & 0 & \alpha_2^* \\ \alpha_1^* + \lambda\alpha^{**} & 0 & \lambda\alpha_2^* & 0 \\ 0 & \lambda\alpha_2^* & 0 & 0 \\ \lambda\alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}\left(\mathcal{C}_{06}^{4}(\lambda)\right)$ on the subspace $\left\langle \sum\limits_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum\limits_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^3\alpha_1 + (1-\lambda)\,\lambda x^2 y\alpha_2, \quad \alpha_2^* = x^5\alpha_2.$$

One-dimensional central extensions

We can suppose that $\alpha_2 \neq 0$. So, we have the following cases:

1. if $\lambda \neq 0, 1$, by choosing $y = -\frac{x\alpha_1}{(1-\lambda)\lambda\alpha_2}$ and $x = \frac{1}{\sqrt[5]{\alpha_2}}$ we have the representative $\langle \nabla_2 \rangle$;
2. if $\lambda = 0$ or $\lambda = 1$, and $\alpha_1 = 0$, by choosing $x = \frac{1}{\sqrt[5]{\alpha_2}}$ we get again the representative $\langle \nabla_2 \rangle$;
3. if $\lambda = 0$ or $\lambda = 1$, and $\alpha_1 \neq 0$, by choosing $x = \sqrt{\frac{\alpha_1}{\alpha_2}}$ we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.

Hence, we obtain the family of algebras $\mathcal{C}_{10}^5(\lambda)$, associated with the representative $\langle\nabla_2\rangle$, and the algebras $\mathcal{C}_{11}^5$ and $\mathcal{C}_{12}^5$ associated with $\langle\nabla_1+\nabla_2\rangle$ for the values $\lambda=0$ and $\lambda=1$, respectively. Their multiplication tables can be consulted in Table 3.4 (Subsection 3.3.9).

Two-dimensional central extensions

Since the cohomology space has dimension 2, we get just the new algebra $\mathcal{C}_{08}^6(\lambda)$, whose multiplication table can be found in Table 3.6 (Subsection 3.4.13).

3.3.9 Classification theorem

The results of the Subsection 3.3.1 and Subsections 3.3.3 to 3.3.8 yield the following theorem.

Theorem 3.3.1. *Let* A *be a five-dimensional one-generated nilpotent bicommutative algebra. Then,* A *is isomorphic to an algebra from the following Table 3.4:*

$\mathcal{C}_{01}^5$	$e_1e_1=e_2$	$e_1e_2=e_4$	$e_2e_1=e_3$	$e_3e_1=e_5$
$\mathcal{C}_{02}^5(\lambda)$	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=e_5$	
	$e_2e_1=\lambda e_3+e_4$	$e_2e_2=\lambda e_5$	$e_3e_1=\lambda e_5$	
$\mathcal{C}_{03}^5(\lambda,\mu)$	$e_1e_1=e_2$	$e_1e_2=e_4$	$e_1e_3=e_5$	$e_1e_4=\lambda e_5$
	$e_2e_1=e_3$	$e_2e_2=e_5$	$e_3e_1=\mu e_5$	$e_4e_1=e_5$
$\mathcal{C}_{04}^5(\lambda)$	$e_1e_1=e_2$	$e_1e_2=e_4$	$e_1e_4=e_5$	
	$e_2e_1=e_3$	$e_3e_1=\lambda e_5$		
$\mathcal{C}_{05}^5$	$e_1e_1=e_2$	$e_1e_2=e_4$	$e_1e_3=e_5$	$e_2e_1=e_3$
	$e_2e_2=e_5$	$e_3e_1=e_4$	$e_4e_1=e_5$	
$\mathcal{C}_{06}^5$	$e_1e_1=e_2$	$e_1e_2=e_5$	$e_2e_1=e_3$	
	$e_3e_1=e_4$	$e_4e_1=e_5$		
$\mathcal{C}_{07}^5$	$e_1e_1=e_2$	$e_2e_1=e_3$	$e_3e_1=e_4$	$e_4e_1=e_5$
$\mathcal{C}_{08}^5$	$e_1e_1=e_2$	$e_1e_2=e_3$	$e_1e_3=e_4$	$e_1e_4=e_5$
	$e_2e_1=e_4$	$e_2e_2=e_5$	$e_3e_1=e_5$	

C_{09}^5	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_4 + e_5$	$e_2e_3 = e_5$	
	$e_3e_1 = e_4 + e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
$C_{10}^5(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_2e_1 = \lambda e_3$	$e_2e_2 = \lambda e_4$	$e_2e_3 = \lambda e_5$	
	$e_3e_1 = \lambda e_4$	$e_3e_2 = \lambda e_5$	$e_4e_1 = \lambda e_5$	
C_{11}^5	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
	$e_1e_4 = e_5$	$e_2e_1 = e_5$		
C_{12}^5	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_2e_1 = e_3 + e_5$	$e_2e_2 = e_4$	$e_2e_3 = e_5$	
	$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	

Table 3.4: Five-dimensional one-generated nilpotent bicommutative algebras.

3.4 Algebraic classification of six-dimensional one-generated nilpotent bicommutative algebras

In this section, we will base on the list of five-dimensional one-generated nilpotent bicommutative algebras of Table 3.4 (Subsection 3.3.9).

3.4.1 Second cohomology space of five-dimensional one-generated nilpotent bicommutative algebras

The necessary information about coboundaries, cocycles and second cohomology spaces of the algebras in Table 3.4 (Subsection 3.3.9) is displayed in the following Table 3.5.

$\mathrm{Z}^2\left(C_{01}^5\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \Delta_{22} + \Delta_{41}, \Delta_{14}, \Delta_{21}, \Delta_{31}, \Delta_{51} \right\rangle$
$\mathrm{B}^2\left(C_{01}^5\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{H}^2\left(C_{01}^5\right)$	$=$	$\left\langle [\Delta_{13}] + [\Delta_{22}] + [\Delta_{41}], [\Delta_{14}], [\Delta_{51}] \right\rangle$

$\mathrm{Z}^2\left(C^5_{02}(\lambda)\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} - \lambda\Delta_{14}, \Delta_{14} + \Delta_{22} + \Delta_{31}, \Delta_{15} + \lambda\Delta_{23} + \lambda\Delta_{32} + \lambda\Delta_{51}, \Delta_{21}, \Delta_{41} \right\rangle$
$\mathrm{B}^2\left(C^5_{02}(\lambda)\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \lambda\Delta_{22} + \lambda\Delta_{31}, \Delta_{21} \right\rangle$
$\mathrm{H}^2\left(C^5_{02}(\lambda)\right)$	$=$	$\left\langle [\Delta_{14}] + [\Delta_{22}] + [\Delta_{31}], [\Delta_{15}] + \alpha[\Delta_{23}] + \lambda[\Delta_{32}] + \lambda[\Delta_{51}], [\Delta_{41}] \right\rangle$
$\mathrm{Z}^2\left(C^5_{03}(\lambda,\mu)_{\lambda=0 \text{ or } \mu\neq 1/\lambda}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \Delta_{22} + \Delta_{41}, \Delta_{14}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{B}^2\left(C^5_{03}(\lambda,\mu)_{\lambda=0 \text{ or } \mu\neq 1/\lambda}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \lambda\Delta_{14} + \Delta_{22} + \mu\Delta_{31} + \Delta_{41}, \Delta_{21} \right\rangle$
$\mathrm{H}^2\left(C^5_{03}(\lambda,\mu)_{\lambda=0 \text{ or } \mu\neq 1/\lambda}\right)$	$=$	$\left\langle [\Delta_{14}], [\Delta_{31}] \right\rangle$
$\mathrm{Z}^2\left(C^5_{03}(\lambda,1/\lambda)_{\lambda\neq 0}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \Delta_{22} + \Delta_{41}, \Delta_{14}, \lambda\Delta_{15} + \Delta_{23} + \lambda\Delta_{24} + \Delta_{32} + \lambda\Delta_{42} + \Delta_{51}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{B}^2\left(C^5_{03}(\lambda,1/\lambda)_{\lambda\neq 0}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \lambda\Delta_{14} + \Delta_{22} + (1/\lambda)\,\Delta_{31} + \Delta_{41}, \Delta_{21} \right\rangle$
$\mathrm{H}^2\left(C^5_{03}(\lambda,1/\lambda)_{\lambda\neq 0}\right)$	$=$	$\left\langle [\Delta_{14}], \lambda[\Delta_{15}] + [\Delta_{23}] + \alpha[\Delta_{24}] + [\Delta_{32}] + \alpha[\Delta_{42}] + [\Delta_{51}], [\Delta_{31}] \right\rangle$
$\mathrm{Z}^2\left(C^5_{04}(\lambda)\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \Delta_{22} + \Delta_{41}, \Delta_{14}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{B}^2\left(C^5_{04}(\lambda)\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{14} + \lambda\Delta_{31}, \Delta_{21} \right\rangle$
$\mathrm{H}^2\left(C^5_{04}(\lambda)\right)$	$=$	$\left\langle [\Delta_{13}] + [\Delta_{22}] + [\Delta_{41}], [\Delta_{31}] \right\rangle$
$\mathrm{Z}^2\left(C^5_{05}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \Delta_{22} + \Delta_{41}, \Delta_{14} + \Delta_{23} + \Delta_{32} + \Delta_{51}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{B}^2\left(C^5_{05}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12} + \Delta_{31}, \Delta_{13} + \Delta_{22} + \Delta_{41}, \Delta_{21} \right\rangle$
$\mathrm{H}^2\left(C^5_{05}\right)$	$=$	$\left\langle [\Delta_{14}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{51}], [\Delta_{31}] \right\rangle$
$\mathrm{Z}^2\left(C^5_{06}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \Delta_{22} + \Delta_{51}, \Delta_{21}, \Delta_{31}, \Delta_{41} \right\rangle$
$\mathrm{B}^2\left(C^5_{06}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12} + \Delta_{41}, \Delta_{21}, \Delta_{31} \right\rangle$
$\mathrm{H}^2\left(C^5_{06}\right)$	$=$	$\left\langle [\Delta_{13}] + [\Delta_{22}] + [\Delta_{51}], [\Delta_{41}] \right\rangle$
$\mathrm{Z}^2\left(C^5_{07}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{31}, \Delta_{41}, \Delta_{51} \right\rangle$
$\mathrm{B}^2\left(C^5_{07}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{21}, \Delta_{31}, \Delta_{41} \right\rangle$
$\mathrm{H}^2\left(C^5_{07}\right)$	$=$	$\left\langle [\Delta_{12}], [\Delta_{51}] \right\rangle$
$\mathrm{Z}^2\left(C^5_{08}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14} + \Delta_{22} + \Delta_{31}, \Delta_{15} + \Delta_{23} + \Delta_{32} + \Delta_{41}, \Delta_{21} \right\rangle$
$\mathrm{B}^2\left(C^5_{08}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \Delta_{21}, \Delta_{14} + \Delta_{22} + \Delta_{31} \right\rangle$
$\mathrm{H}^2\left(C^5_{08}\right)$	$=$	$\left\langle [\Delta_{15}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{41}], [\Delta_{21}] \right\rangle$
$\mathrm{Z}^2\left(C^5_{09}\right)$	$=$	$\left\langle \Delta_{11}, \Delta_{12}, \Delta_{13} + \Delta_{22} + \Delta_{31}, \Delta_{14} + \Delta_{22} + \Delta_{23} + \Delta_{31} + \Delta_{32} + \Delta_{41}, \Delta_{15} + \Delta_{23} + \Delta_{24} + \Delta_{32} + \Delta_{33} + \Delta_{41} + \Delta_{42} + \Delta_{51}, \Delta_{21} \right\rangle$

$\mathrm{B}^2\left(\mathcal{C}^5_{09}\right)$	$=$	$\big\langle \Delta_{11}, \Delta_{12}+\Delta_{21}, \Delta_{13}+\Delta_{21}+\Delta_{22}+\Delta_{31}, \Delta_{14} +\Delta_{22}+\Delta_{23}+\Delta_{31}+\Delta_{32}+\Delta_{41}\big\rangle$
$\mathrm{H}^2\left(\mathcal{C}^5_{09}\right)$	$=$	$\big\langle [\Delta_{15}]+[\Delta_{23}]+[\Delta_{24}]+[\Delta_{32}]+[\Delta_{33}]+[\Delta_{41}] +[\Delta_{42}]+[\Delta_{51}], [\Delta_{21}]\big\rangle$
$\mathrm{Z}^2\left(\mathcal{C}^5_{10}(\lambda)\right)$	$=$	$\big\langle \Delta_{11}, \Delta_{12}, \Delta_{13}+\lambda\Delta_{22}+\lambda\Delta_{31}, \Delta_{14}+\lambda\Delta_{23} +\lambda\Delta_{32}+\lambda\Delta_{41}, \Delta_{15}+\lambda\Delta_{24}+\lambda\Delta_{33}+\lambda\Delta_{42} +\lambda\Delta_{51}, \Delta_{21}\big\rangle$
$\mathrm{B}^2\left(\mathcal{C}^5_{10}(\lambda)\right)$	$=$	$\big\langle \Delta_{11}, \Delta_{12}+\lambda\Delta_{21}, \Delta_{13}+\lambda\Delta_{22}+\lambda\Delta_{31}, \Delta_{14} +\lambda\Delta_{23}+\lambda\Delta_{32}+\lambda\Delta_{41}\big\rangle$
$\mathrm{H}^2\left(\mathcal{C}^5_{10}(\lambda)\right)$	$=$	$\big\langle [\Delta_{15}]+\alpha[\Delta_{24}]+\alpha[\Delta_{33}]+\alpha[\Delta_{42}]+\alpha[\Delta_{51}], [\Delta_{21}]\big\rangle$
$\mathrm{Z}^2\left(\mathcal{C}^5_{11}\right)$	$=$	$\big\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}+\Delta_{22}+\Delta_{31}, \Delta_{21}\big\rangle$
$\mathrm{B}^2\left(\mathcal{C}^5_{11}\right)$	$=$	$\big\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}+\Delta_{21}\big\rangle$
$\mathrm{H}^2\left(\mathcal{C}^5_{11}\right)$	$=$	$\big\langle [\Delta_{15}]+[\Delta_{22}]+[\Delta_{31}], [\Delta_{21}]\big\rangle$
$\mathrm{Z}^2\left(\mathcal{C}^5_{12}\right)$	$=$	$\big\langle \Delta_{11}, \Delta_{12}, \Delta_{13}+\Delta_{22}+\Delta_{31}, \Delta_{14}+\Delta_{23}+\Delta_{32} +\Delta_{41}, \Delta_{15}+\Delta_{22}+\Delta_{24}+\Delta_{31}+\Delta_{33}+\Delta_{42}+\Delta_{51}, \Delta_{21}\big\rangle$
$\mathrm{B}^2\left(\mathcal{C}^5_{12}\right)$	$=$	$\big\langle \Delta_{11}, \Delta_{12}+\Delta_{21}, \Delta_{13}+\Delta_{22} +\Delta_{31}, \Delta_{14}+\Delta_{21}+\Delta_{23}+\Delta_{32}+\Delta_{41}\big\rangle$
$\mathrm{H}^2\left(\mathcal{C}^5_{12}\right)$	$=$	$\big\langle [\Delta_{15}]+[\Delta_{22}]+[\Delta_{24}]+[\Delta_{31}]+[\Delta_{33}]+[\Delta_{42}] +[\Delta_{51}], [\Delta_{21}]\big\rangle$

Table 3.5: Second cohomology space of five-dimensional one-generated nilpotent bicommutative algebras.

Remark 3.4.1. The extensions of the algebras $\mathcal{C}^5_{03}(\lambda,\mu)_{\lambda=0 \text{ or } \mu\neq 1/\lambda}$ and $\mathcal{C}^5_{04}(\lambda)$ have two-dimensional annihilator, and have already been constructed as two-dimensional central extensions of four-dimensional bicommutative algebras. Then, in the following subsections we will study only the central extensions of the other algebras.

3.4.2 Central extensions of $\mathcal{C}^5_{01}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{14}], \quad \nabla_2 = [\Delta_{51}], \quad \nabla_3 = [\Delta_{13}]+[\Delta_{22}]+[\Delta_{41}].$$

The automorphism group of $\mathcal{C}_{01}^5$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & xy & x^3 & 0 & 0 \\ t & xy & 0 & x^3 & 0 \\ s & xz & x^2y & 0 & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_3 & \alpha_1 & 0 \\ 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_3^* & \alpha_1^* & 0 \\ \alpha^{***} & \alpha_3^* & 0 & 0 & 0 \\ \alpha^{****} & 0 & 0 & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

then the action of $\mathrm{Aut}\left(\mathcal{C}_{01}^5\right)$ on the subspace $\left\langle \sum_{i=1}^{3} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{3} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^4\alpha_1, \quad \alpha_2^* = x^5\alpha_2, \quad \alpha_3^* = x^4\alpha_3.$$

We can suppose that $\alpha_2 \neq 0$; otherwise, we would obtain algebras with two-dimensional annihilator which have already been constructed as two-dimensional central extensions of four-dimensional bicommutative algebras. We have the following cases:

1. if $\alpha_3 \neq 0$, by choosing $x = \frac{\alpha_3}{\alpha_2}$ we have the family of representatives $\langle \lambda\nabla_1 + \nabla_2 + \nabla_3 \rangle$;
2. if $\alpha_3 = 0, \alpha_1 \neq 0$, by choosing $x = \frac{\alpha_1}{\alpha_2}$ we have the representative $\langle \nabla_1 + \nabla_2 \rangle$;
3. if $\alpha_3 = 0, \alpha_1 = 0$, by choosing $x = \frac{1}{\sqrt[4]{\alpha_2}}$ we have the representative $\langle \nabla_2 \rangle$.

Hence, we obtain the following new algebras: $\mathcal{C}_{09}^6(\lambda)$, $\mathcal{C}_{10}^6$ and $\mathcal{C}_{11}^6$. Their multiplication tables can be found in Table 3.6 (Subsection 3.4.13).

3.4.3 Central extensions of $\mathcal{C}_{02}^{5}(\lambda)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{41}], \quad \nabla_2 = [\Delta_{14}]+[\Delta_{22}]+[\Delta_{31}], \quad \nabla_3 = [\Delta_{15}]+\alpha[\Delta_{23}]+\alpha[\Delta_{32}]+\alpha[\Delta_{51}].$$

The automorphism group of $\mathcal{C}_{01}^{5}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & (1+\lambda)\,xy & x^3 & 0 & 0 \\ t & xy & 0 & x^3 & 0 \\ s & \lambda y^2 + (1+\lambda)\,xz & (1+2\lambda)\,x^2y & \alpha\,(1-\lambda)\,x^2y & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_2 & \alpha_3 \\ 0 & \alpha_2 & \lambda\alpha_3 & 0 & 0 \\ \alpha_2 & \lambda\alpha_3 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & 0 \\ \lambda\alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha^{***} & \alpha_2^* & \alpha_3^* \\ \alpha^{****} & \alpha_2^* + \lambda\alpha^{***} & \lambda\alpha_3^* & 0 & 0 \\ \alpha_2^* + \lambda\alpha^{***} & \lambda\alpha_3^* & 0 & 0 & 0 \\ \alpha_1^* & 0 & 0 & 0 & 0 \\ \lambda\alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}\left(\mathcal{C}_{02}^{5}(\lambda)\right)$ on the subspace $\left\langle \sum_{i=1}^{3} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{3} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^4\alpha_1 + (1-\lambda)\,\alpha^2 x^3 y\alpha_3, \quad \alpha_2^* = x^4\alpha_2 + (1-\lambda)\,\lambda x^3 y\alpha_3, \quad \alpha_3^* = x^5\alpha_3.$$

We can suppose that $\alpha_3 \neq 0$. We have the following cases:

1. if $\lambda = 0$ or $\lambda = 1$, and $\alpha_1 = \alpha_2 = 0$, then we have the representative $\langle \nabla_3 \rangle$;
2. if $\lambda = 0$ or $\lambda = 1$, $\alpha_2 = 0$ and $\alpha_1 \neq 0$, by choosing $x = \frac{\alpha_1}{\alpha_3}$ we have the representative $\langle \nabla_1 + \nabla_3 \rangle$;
3. if $\lambda = 0$ or $\lambda = 1$, and $\alpha_2 \neq 0$, by choosing $x = \frac{\alpha_2}{\alpha_3}$ we have the representative $\langle \mu\nabla_1 + \nabla_2 + \nabla_3 \rangle$;

4. if $\lambda \neq 0, 1$ and $\alpha_1 - \lambda\alpha_2 = 0$, by choosing $y = -\frac{x\alpha_2}{(1-\lambda)\lambda\alpha_3}$ and $x = \frac{1}{\sqrt[5]{\alpha_3}}$ we have the representative $\langle \nabla_3 \rangle$;
5. if $\lambda \neq 0, 1$ and $\alpha_1 - \lambda\alpha_2 \neq 0$, by choosing $y = -\frac{x\alpha_2}{(1-\lambda)\lambda\alpha_3}$ and $x = \frac{\alpha_1 - \lambda\alpha_2}{\alpha_3}$ we have the representative $\langle \nabla_1 + \nabla_3 \rangle$.

Therefore, we obtain the families $\mathcal{C}^6_{12}(\lambda)$ and $\mathcal{C}^6_{13}(\lambda)$ associated with the representatives $\langle \nabla_3 \rangle$ and $\langle \nabla_1 + \nabla_3 \rangle$, respectively, and the families $\mathcal{C}^6_{14}(\mu)$ and $\mathcal{C}^6_{15}(\mu)$, associated with $\langle \mu\nabla_1 + \nabla_2 + \nabla_3 \rangle$ for the values $\lambda = 0$ and $\lambda = 1$, respectively. Their multiplication tables can be consulted in Table 3.6 (Subsection 3.4.13).

3.4.4 Central extensions of $\mathcal{C}^5_{03}(\lambda, 1/\lambda)_{\lambda\neq 0}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{14}], \quad \nabla_2 = [\Delta_{31}], \quad \nabla_3 = \lambda[\Delta_{15}] + [\Delta_{23}] + \alpha[\Delta_{24}] + [\Delta_{32}] + \alpha[\Delta_{42}] + [\Delta_{51}].$$

The automorphism group of $\mathcal{C}^5_{03}(\lambda, 1/\lambda)_{\lambda\neq 0}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & xy & x^3 & 0 & 0 \\ t & xy & 0 & x^3 & 0 \\ s & (1+1/\lambda)\,xz + (1+\lambda)\,xt + y^2 & (2+1/\lambda)\,x^2y & (2+\lambda)\,x^2y & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_1 & \lambda\alpha_3 \\ 0 & 0 & \alpha_3 & \lambda\alpha_3 & 0 \\ \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ 0 & \lambda\alpha_3 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha^{***} & \alpha_1^* + \lambda\alpha^{***} & \lambda\alpha_3^* \\ \alpha^{****} & \alpha^{***} & \alpha_3^* & \lambda\alpha_3^* & 0 \\ \alpha_2^* + (1/\lambda)\,\alpha^{***} & \alpha_3^* & 0 & 0 & 0 \\ \alpha^{***} & \lambda\alpha_3^* & 0 & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}\left(\mathcal{C}^5_{03}(\lambda, 1/\lambda)_{\lambda\neq 0}\right)$ on the subspace $\left\langle \sum_{i=1}^{3} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{3} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^4\alpha_1 + (1-\lambda)\,\lambda x^3 y\alpha_3, \quad \alpha_2^* = x^4\alpha_2 + (1-1/\lambda)\,x^3 y\alpha_3, \quad \alpha_3^* = x^5\alpha_3.$$

We can suppose that $\alpha_3 \neq 0$. We have the following cases:

1. if $\lambda = 1$, and $\alpha_1 = \alpha_2 = 0$, then we get the representative $\langle \nabla_3 \rangle$;
2. if $\lambda = 1$, $\alpha_2 = 0$ and $\alpha_1 \neq 0$, by choosing $x = \frac{\alpha_1}{\alpha_3}$ we have the representative $\langle \nabla_1 + \nabla_3 \rangle$;
3. if $\lambda = 1$, and $\alpha_2 \neq 0$, by choosing $x = \frac{\alpha_2}{\alpha_3}$ we have the representative $\langle \mu\nabla_1 + \nabla_2 + \nabla_3 \rangle$;
4. if $\lambda \neq 1$ and $\alpha_1 + \alpha^2\alpha_2 = 0$, by choosing $y = -\frac{x\alpha_2}{(1-1/\lambda)\lambda\alpha_3}$ and $x = \frac{1}{\sqrt[5]{\alpha_3}}$ we get the representative $\langle \nabla_3 \rangle$;
5. if $\lambda \neq 1$ and $\alpha_1 + \alpha^2\alpha_2 \neq 0$, by choosing $y = -\frac{x\alpha_2}{(1-1/\lambda)\lambda\alpha_3}$ and $x = \frac{\alpha_1+\alpha^2\alpha_2}{\alpha_3}$ we have the representative $\langle \nabla_1 + \nabla_3 \rangle$.

Hence, we get the new algebras $C^6_{16}(\lambda)_{\lambda\neq 0}$, $C^6_{17}(\lambda)_{\lambda\neq 0}$ and $C^6_{18}(\mu)$, associated with $\langle \nabla_3 \rangle$ and $\langle \nabla_1 + \nabla_3 \rangle$ for every value of $\lambda \neq 0$, and with $\langle \mu\nabla_1 + \nabla_2 + \nabla_3 \rangle$ for $\lambda = 1$, respectively. Their multiplication tables can be consulted in Table 3.6 (Subsection 3.4.13).

3.4.5 Central extensions of C^5_{05}

Let us use the following notations:

$$\nabla_1 = [\Delta_{31}], \quad \nabla_2 = [\Delta_{14}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{51}].$$

The automorphism group of C^5_{05} consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & x & 0 & 1 & 0 \\ z & x+y & x & 0 & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 \\ \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & 0 & \alpha_2^* & 0 \\ \alpha^{***} & 0 & \alpha_2^* & 0 & 0 \\ \alpha_1^* + \alpha^{**} & \alpha_2^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

then the action of Aut $\left(C_{05}^5\right)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = \alpha_1 - x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

We can suppose that $\alpha_2 \neq 0$. Then, by choosing $x = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$, whose associated algebra is C_{19}^6. Its multiplication table can be found in Table 3.6 (Subsection 3.4.13).

3.4.6 Central extensions of C_{06}^5

Let us use the following notations:

$$\nabla_1 = [\Delta_{41}], \quad \nabla_2 = [\Delta_{13}] + [\Delta_{22}] + [\Delta_{51}].$$

The automorphism group of C_{06}^5 consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & x & 1 & 0 & 0 \\ z & y & x & 1 & 0 \\ t & x+z & y & x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_2^* & 0 & 0 \\ \alpha^{***} & \alpha_2^* & 0 & 0 & 0 \\ \alpha^{****} & 0 & 0 & 0 & 0 \\ \alpha_1^* + \alpha^{**} & 0 & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

then the action of Aut $\left(C_{06}^5\right)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = \alpha_1 - x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

We can suppose that $\alpha_2 \neq 0$. Then, by choosing $x = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$. Hence, we obtain the new algebra C_{20}^6, whose multiplication table can be consulted in Table 3.6 (Subsection 3.4.13).

3.4.7 Central extensions of $\mathcal{C}^5_{07}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = [\Delta_{51}].$$

The automorphism group of $\mathcal{C}^5_{07}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & xy & x^3 & 0 & 0 \\ t & xz & x^2y & x^4 & 0 \\ s & xt & x^2z & x^3y & x^5 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & 0 & 0 & 0 \\ \alpha^{**} & 0 & 0 & 0 & 0 \\ \alpha^{***} & 0 & 0 & 0 & 0 \\ \alpha^{****} & 0 & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

then the action of $\mathrm{Aut}\left(\mathcal{C}^5_{07}\right)$ on the subspace $\left\langle \sum\limits_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum\limits_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^3\alpha_1, \quad \alpha_2^* = x^6\alpha_2.$$

We can suppose that $\alpha_2 \neq 0$. We have the following cases:

1. if $\alpha_1 \neq 0$, by choosing $x = \sqrt[3]{\frac{\alpha_1}{\alpha_2}}$ we have the representative $\langle \nabla_1 + \nabla_2 \rangle$;
2. if $\alpha_1 = 0$, by choosing $x = \frac{1}{\sqrt[6]{\alpha_2}}$ we have the representative $\langle \nabla_2 \rangle$.

Hence, we get the new algebras $\mathcal{C}^6_{21}$ and $\mathcal{C}^6_{22}$, whose multiplication tables can be consulted in Table 3.6 (Subsection 3.4.13).

3.4.8 Central extensions of $\mathcal{C}^5_{08}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = [\Delta_{15}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{41}].$$

The automorphism group of $\mathcal{C}_{08}^5$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & x & 0 & 1 & 0 \\ z & x+y & x & 0 & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha^{***} & 0 & \alpha_2^* \\ \alpha_1^* + \alpha^{***} & 0 & \alpha_2^* & 0 & 0 \\ 0 & \alpha_2^* & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the action of Aut $\left(\mathcal{C}_{08}^5\right)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = \alpha_1 + x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

We can suppose that $\alpha_2 \neq 0$. Then, by choosing $x = -\frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$, with associated algebra $\mathcal{C}_{23}^6$. Its multiplication table can be found in Table 3.6 (Subsection 3.4.13).

3.4.9 Central extensions of $\mathcal{C}_{09}^5$

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = [\Delta_{15}] + [\Delta_{23}] + [\Delta_{24}] + [\Delta_{32}] + [\Delta_{33}] + [\Delta_{41}] + [\Delta_{42}] + [\Delta_{51}].$$

The automorphism group of $\mathcal{C}_{09}^5$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & 2x & 0 & 1 & 0 \\ z & x+2y & 3x & 0 & 1 \end{pmatrix}.$$

Since

$$\phi^T\begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & 0 & \alpha_2 & \alpha_2 & 0 \\ 0 & \alpha_2 & \alpha_2 & 0 & 0 \\ \alpha_2 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix}\phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha^{***} & 0 & \alpha_2^* \\ \alpha_1^* + \alpha^{**} + \alpha^{***} & \alpha^{***} & \alpha_2^* & \alpha_2^* & 0 \\ \alpha^{***} & \alpha_2^* & \alpha_2^* & 0 & 0 \\ \alpha_2^* & \alpha_2^* & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

then the action of $\operatorname{Aut}\left(\mathcal{C}_{09}^5\right)$ on the subspace $\left\langle \sum\limits_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum\limits_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = \alpha_1 - 2x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

We can suppose that $\alpha_2 \neq 0$. Then, by choosing $x = \frac{\alpha_1}{2\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$. Hence, we get the new algebra $\mathcal{C}_{24}^6$, whose multiplication table can be found in Table 3.6 (Subsection 3.4.13).

3.4.10 Central extensions of $\mathcal{C}_{10}^5(\lambda)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = [\Delta_{15}] + \alpha[\Delta_{24}] + \alpha[\Delta_{33}] + \alpha[\Delta_{42}] + \alpha[\Delta_{51}].$$

The automorphism group of $\mathcal{C}_{10}^5(\lambda)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 \\ 0 & 0 & x^3 & 0 & 0 \\ y & 0 & 0 & x^4 & 0 \\ z & (1+\lambda)xy & 0 & 0 & x^5 \end{pmatrix}.$$

Since

$$\phi^T\begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & 0 & 0 & \lambda\alpha_2 & 0 \\ 0 & 0 & \lambda\alpha_2 & 0 & 0 \\ 0 & \lambda\alpha_2 & 0 & 0 & 0 \\ \lambda\alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix}\phi = \begin{pmatrix} \alpha^* & \alpha^{**} & 0 & 0 & \alpha_2^* \\ \alpha_1^* + \lambda\alpha^{**} & 0 & 0 & \lambda\alpha_2^* & 0 \\ 0 & 0 & \lambda\alpha_2^* & 0 & 0 \\ 0 & \lambda\alpha_2^* & 0 & 0 & 0 \\ \lambda\alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}\left(\mathcal{C}_{10}^{5}(\lambda)\right)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = x^3\alpha_1 + (1-\lambda)\lambda x^2 y \alpha_2, \quad \alpha_2^* = x^6\alpha_2.$$

We can suppose that $\alpha_2 \neq 0$. We have the following cases:

1. if $\lambda = 0$ or $\lambda = 1$, and $\alpha_1 \neq 0$, by choosing $x = \sqrt[3]{\frac{\alpha_1}{\alpha_2}}$ we have the representative $\langle \nabla_1 + \nabla_2 \rangle$;
2. if $\lambda = 0$ or $\lambda = 1$, and $\alpha_1 = 0$, by choosing $x = \frac{1}{\sqrt[6]{\alpha_2}}$ we have the representative $\langle \nabla_2 \rangle$;
3. if $\lambda \neq 0, 1$, by choosing $y = -\frac{x\alpha_1}{(1-\lambda)\lambda\alpha_2}$ and $x = \frac{1}{\sqrt[6]{\alpha_2}}$ we have the representative $\langle \nabla_2 \rangle$.

Hence, we obtain the new algebras $\mathcal{C}_{25}^{6}(\lambda)$, $\mathcal{C}_{26}^{6}$ and $\mathcal{C}_{27}^{6}$, associated with $\langle \nabla_2 \rangle$ for every value of λ, and with $\nabla_1 + \nabla_2$ for $\lambda = 0$ or $\lambda = 1$, respectively. Their multiplication tables can be consulted in Table 3.6 (Subsection 3.4.13).

3.4.11 Central extensions of $\mathcal{C}_{11}^{5}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = [\Delta_{15}] + [\Delta_{22}] + [\Delta_{31}].$$

The automorphism group of $\mathcal{C}_{11}^{5}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & x & 1 & 0 & 0 \\ z & y & x & 1 & 0 \\ t & x+z & y & x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha^{***} & \alpha^{****} & \alpha_2^* \\ \alpha_1^* + \alpha^{****} & \alpha_2^* & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the action of $\mathrm{Aut}\left(\mathcal{C}_{11}^5\right)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\alpha_1^* = \alpha_1 + x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

We can suppose that $\alpha_2 \neq 0$. Then, by taking $x = -\frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$, with associated algebra $\mathcal{C}_{28}^6$. Its multiplication table can be found in Table 3.6 (Subsection 3.4.13).

3.4.12 Central extensions of $\mathcal{C}_{12}^5$

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = [\Delta_{15}] + [\Delta_{22}] + [\Delta_{24}] + [\Delta_{31}] + [\Delta_{33}] + [\Delta_{42}] + [\Delta_{51}].$$

The automorphism group of $\mathcal{C}_{12}^5$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & 2x & 1 & 0 & 0 \\ z & x^2 + 2y & 3x & 1 & 0 \\ t & x(1+2y) + 2z & 3x^2 + 3y & 4x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & \alpha_2 & 0 & \alpha_2 & 0 \\ \alpha_2 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha^{***} & \alpha^{****} & \alpha_2^* \\ \alpha_1^* + \alpha^{**} + \alpha^{****} & \alpha_2^* + \alpha^{***} & \alpha^{****} & \alpha_2^* & 0 \\ \alpha_2^* + \alpha^{***} & \alpha^{****} & \alpha_2^* & 0 & 0 \\ \alpha^{****} & \alpha_2^* & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

then the action of $\operatorname{Aut}\left(C_{12}^{5}\right)$ on the subspace $\left\langle \sum\limits_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum\limits_{i=1}^{2} \alpha_i^{*} \nabla_i \right\rangle$, where

$$\alpha_1^* = \alpha_1 - 3x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

We can suppose that $\alpha_2 \neq 0$. Then, by taking $x = \frac{\alpha_1}{3\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$. Hence, we obtain the new algebra C_{29}^{6}, whose multiplication table can be consulted in Table 3.6 (Subsection 3.4.13).

3.4.13 Classification theorem

Summarising the results from Subsection 3.4.2 to Subsection 3.4.12, we obtain the following theorem.

Theorem 3.4.2. *Let* A *be a six-dimensional one-generated nilpotent bicommutative algebra. Then,* A *is isomorphic to an algebra from the following Table 3.6.*

C_{01}^{6}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_4 = e_5$	
	$e_2e_1 = e_3$	$e_3e_1 = e_6$		
$C_{02}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_6$	$e_1e_4 = e_5$
	$e_2e_1 = e_3$	$e_2e_2 = e_6$	$e_3e_1 = \lambda e_6$	$e_4e_1 = e_6$
$C_{03}^{6}(\lambda, \mu)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_6$	$e_1e_4 = \lambda e_5 + \mu e_6$
	$e_2e_1 = e_3$	$e_2e_2 = e_6$	$e_3e_1 = e_5$	$e_4e_1 = e_6$
C_{04}^{6}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_6$	$e_2e_1 = e_3$
	$e_2e_2 = e_6$	$e_3e_1 = e_4 + e_5$	$e_4e_1 = e_6$	
C_{05}^{6}	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = e_3$	$e_3e_1 = e_4$
	$e_4e_1 = e_6$			
C_{06}^{6}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_6$
	$e_2e_1 = e_4 + e_5$	$e_2e_2 = e_6$	$e_3e_1 = e_6$	
C_{07}^{6}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_6$
	$e_2e_1 = e_3 + e_4 + e_5$		$e_2e_2 = e_4 + e_6$	$e_2e_3 = e_6$
	$e_3e_1 = e_4 + e_6$	$e_3e_2 = e_6$	$e_4e_1 = e_6$	

$C_{08}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_6$
	$e_2e_1 = \lambda e_3 + e_5$	$e_2e_2 = \lambda e_4$	$e_2e_3 = \lambda e_6$	
	$e_3e_1 = \lambda e_4$	$e_3e_2 = \lambda e_6$	$e_4e_1 = \lambda e_6$	
$C_{09}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_6$	
	$e_1e_4 = \lambda e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_6$	
	$e_3e_1 = e_5$	$e_4e_1 = e_6$	$e_5e_1 = e_6$	
C_{10}^{6}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_4 = e_6$	
	$e_2e_1 = e_3$	$e_3e_1 = e_5$	$e_5e_1 = e_6$	
C_{11}^{6}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$	
	$e_3e_1 = e_5$	$e_5e_1 = e_6$		
$C_{12}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_5 = e_6$
	$e_2e_1 = \lambda e_3 + e_4$	$e_2e_2 = \lambda e_5$	$e_2e_3 = \lambda e_6$	
	$e_3e_1 = \lambda e_5$	$e_3e_2 = \lambda e_6$	$e_5e_1 = \lambda e_6$	
$C_{13}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_5 = e_6$
	$e_2e_1 = \lambda e_3 + e_4$	$e_2e_2 = \lambda e_5$	$e_2e_3 = \lambda e_6$	$e_3e_1 = \lambda e_5$
	$e_3e_2 = \lambda e_6$	$e_4e_1 = e_6$	$e_5e_1 = \lambda e_6$	
$C_{14}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
	$e_1e_4 = e_6$	$e_1e_5 = e_6$	$e_2e_1 = e_4$	
	$e_2e_2 = e_6$	$e_3e_1 = e_6$	$e_4e_1 = \lambda e_6$	
$C_{15}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = e_6$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_5 + e_6$	$e_2e_3 = e_6$
	$e_3e_1 = e_5 + e_6$	$e_3e_2 = e_6$	$e_4e_1 = \lambda e_6$	$e_5e_1 = e_6$
$C_{16}^{6}(\lambda)_{\lambda\neq 0}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = \lambda e_5$
	$e_1e_5 = \lambda e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_5$	$e_2e_3 = e_6$
	$e_2e_4 = \lambda e_6$	$e_3e_1 = (1/\lambda)\, e_5$	$e_3e_2 = e_6$	
	$e_4e_1 = e_5$	$e_4e_2 = \lambda e_6$	$e_5e_1 = e_6$	

$C^6_{17}(\lambda)_{\lambda\neq 0}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = \lambda e_5 + e_6$
	$e_1e_5 = \lambda e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_5$	$e_2e_3 = e_6$
	$e_2e_4 = \lambda e_6$	$e_3e_1 = (1/\lambda)\, e_5$	$e_3e_2 = e_6$	
	$e_4e_1 = e_5$	$e_4e_2 = \lambda e_6$	$e_5e_1 = e_6$	
$C^6_{18}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = e_5 + \lambda e_6$
	$e_1e_5 = e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_5$	$e_2e_3 = e_6$
	$e_2e_4 = e_6$	$e_3e_1 = e_5 + e_6$	$e_3e_2 = e_6$	
	$e_4e_1 = e_5$	$e_4e_2 = e_6$	$e_5e_1 = e_6$	
C^6_{19}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = e_6$
	$e_2e_1 = e_3$	$e_2e_2 = e_5$	$e_2e_3 = e_6$	$e_3e_1 = e_4$
	$e_3e_2 = e_6$	$e_4e_1 = e_5$	$e_5e_1 = e_6$	
C^6_{20}	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_1e_3 = e_6$	$e_2e_1 = e_3$
	$e_2e_2 = e_6$	$e_3e_1 = e_4$	$e_4e_1 = e_5$	$e_5e_1 = e_6$
C^6_{21}	$e_1e_1 = e_2$	$e_1e_2 = e_6$	$e_2e_1 = e_3$	
	$e_3e_1 = e_4$	$e_4e_1 = e_5$	$e_5e_1 = e_6$	
C^6_{22}	$e_1e_1 = e_2$	$e_2e_1 = e_3$	$e_3e_1 = e_4$	
	$e_4e_1 = e_5$	$e_5e_1 = e_6$		
C^6_{23}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_4$	$e_2e_2 = e_5$	$e_2e_3 = e_6$
	$e_3e_1 = e_5$	$e_3e_2 = e_6$	$e_4e_1 = e_6$	
C^6_{24}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_4 + e_5$	$e_2e_3 = e_5 + e_6$
	$e_2e_4 = e_6$	$e_3e_1 = e_4 + e_5$	$e_3e_2 = e_5 + e_6$	$e_3e_3 = e_6$
	$e_4e_1 = e_5 + e_6$	$e_4e_2 = e_6$	$e_5e_1 = e_6$	
$C^6_{25}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = \lambda e_3$	$e_2e_2 = \lambda e_4$	$e_2e_3 = \lambda e_5$
	$e_2e_4 = \lambda e_6$	$e_3e_1 = \lambda e_4$	$e_3e_2 = \lambda e_5$	$e_3e_3 = \lambda e_6$
	$e_4e_1 = \lambda e_5$	$e_4e_2 = \lambda e_6$	$e_5e_1 = \lambda e_6$	

$\mathcal{C}_{26}^{6}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
	$e_1e_4 = e_5$	$e_1e_5 = e_6$	$e_2e_1 = e_6$	
$\mathcal{C}_{27}^{6}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_6$	$e_2e_2 = e_4$	$e_2e_3 = e_5$
	$e_2e_4 = e_6$	$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_3e_3 = e_6$
	$e_4e_1 = e_5$	$e_4e_2 = e_6$	$e_5e_1 = e_6$	
$\mathcal{C}_{28}^{6}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_5$	$e_2e_2 = e_6$	$e_3e_1 = e_6$
$\mathcal{C}_{29}^{6}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_5$	$e_2e_2 = e_4 + e_6$	$e_2e_3 = e_5$
	$e_2e_4 = e_6$	$e_3e_1 = e_4 + e_6$	$e_3e_2 = e_5$	$e_3e_3 = e_6$
	$e_4e_1 = e_5$	$e_4e_2 = e_6$	$e_5e_1 = e_6$	

Table 3.6: Six-dimensional one-generated nilpotent bicommutative algebras.

3.5 Geometric classification of four-dimensional nilpotent bicommutative algebras

In the present section, we study the decomposition into irreducible components of the variety of four-dimensional nilpotent bicommutative algebras. The main result is the following theorem.

Theorem 3.5.1. *The variety of four-dimensional nilpotent bicommutative algebras has two irreducible components defined by the rigid algebra $\mathcal{B}_{10}^{4}$ and the infinite family of algebras $\mathcal{B}_{24}^{4}(\lambda)$.*

Proof. As a preliminary step, we compute the dimension of the space of derivations of all four-dimensional nilpotent bicommutative algebras, displayed in Table 3.7 below.

A	dim Der A
$\mathcal{B}_{01}^4$	6
$\mathcal{B}_{02}^4(\lambda)$	6
$\mathcal{B}_{03}^4$	4
$\mathcal{B}_{04}^4(\lambda)$	4
$\mathcal{B}_{05}^4$	4
$\mathcal{B}_{06}^4(\lambda)_{\lambda\neq 0}$	5
$\mathcal{B}_{07}^4$	4
$\mathcal{B}_{08}^4$	5
$\mathcal{B}_{09}^4$	5
$\mathcal{B}_{10}^4$	2
$\mathcal{B}_{11}^4$	3
$\mathcal{B}_{12}^4$	3
$\mathcal{B}_{13}^4$	4
$\mathcal{B}_{14}^4$	3
$\mathcal{B}_{15}^4$	4
$\mathcal{B}_{16}^4$	4
$\mathcal{B}_{17}^4$	5
$\mathcal{B}_{18}^4$	4
$\mathcal{B}_{19}^4$	5
$\mathcal{B}_{20}^4$	3
$\mathcal{B}_{21}^4$	4
$\mathcal{B}_{22}^4$	3
$\mathcal{B}_{23}^4$	3
$\mathcal{B}_{24}^4(\lambda)$	3 $(\lambda \neq 0, 1)$

Table 3.7: Dimension of the space of derivations.

After having checked that $\dim \operatorname{Der}\left(\mathcal{B}_{10}^4\right) < \dim \operatorname{Der}(\mathrm{A})$ for all four-dimensional nilpotent bicommutative algebra A, $\mathrm{A} \neq \mathcal{B}_{10}^4$, it follows that there are not nilpotent bicommutative algebras degenerating to $\mathcal{B}_{10}^4$. Also, the dimension of the square of $\mathcal{B}_{10}^4$ is 2, and $\mathcal{B}_{24}^4(\lambda)$ has three-dimensional square, so it cannot degenerate from $\mathcal{B}_{10}^4$. Therefore, if we prove that these two algebras degenerate to the rest of the nilpotent bicommutative algebras of dimension 4, the theorem is proved.

Recall that the full description of the degeneration system of four-dimensional trivial bicommutative algebras was given in [149]. Using the cited result, we have that the variety of four-dimensional trivial bicommutative algebras has two irreducible components given by the two following families of algebras:

$$\begin{array}{lllllll}
\mathfrak{N}_{02}(\lambda) & : & e_1e_1 = e_3 & e_1e_2 = e_4 & e_2e_1 = -\lambda e_3 & e_2e_2 = -e_4 & \\
\mathfrak{N}_{03}(\lambda) & : & e_1e_1 = e_4 & e_1e_2 = \lambda e_4 & e_2e_1 = -\lambda e_4 & e_2e_2 = e_4 & e_3e_3 = e_4.
\end{array}$$

The algebra $\mathcal{B}_{10}^4$ degenerates to both $\mathfrak{N}_{02}(\lambda)$ and $\mathfrak{N}_{03}(\lambda)$. We will explain in detail the degeneration $\mathcal{B}_{10}^4 \to \mathfrak{N}_{03}(\lambda)_{\lambda\neq 0,\pm i}$; as for $\mathcal{B}_{10}^4 \to \mathfrak{N}_{02}(\lambda)$, it is similar, but easier. It can be found in Table 3.8.

Let us consider the following parametric basis of $\mathcal{B}_{10}^4 : \left\{ F_i^t = \sum\limits_{j=i}^{4} a_{ij}(t)\, e_j \right\}$. The multiplication table in the new basis is given below:

$$\begin{array}{lll}
F_1^tF_1^t & = & \frac{a_{11}a_{12}}{a_{33}}F_3^t + \frac{a_{11}a_{13}+a_{11}a_{12}+a_{12}a_{13}-\frac{a_{11}a_{12}a_{34}}{a_{33}}}{a_{44}}F_4^t, \\
F_1^tF_2^t & = & \frac{a_{11}a_{22}}{a_{33}}F_3^t + \frac{a_{11}a_{23}+a_{13}a_{22}-\frac{a_{11}a_{22}a_{34}}{a_{33}}}{a_{44}}F_4^t, \\
F_1^tF_3^t & = & \frac{a_{11}a_{33}}{a_{44}}F_4^t, \quad F_2^tF_1^t \ = \ \frac{a_{11}a_{22}+a_{12}a_{23}}{a_{44}}F_4^t, \\
F_2^tF_2^t & = & \frac{a_{22}a_{23}}{a_{44}}F_4^t, \quad F_3^tF_1^t \ = \ \frac{a_{12}a_{33}}{a_{44}}F_4^t, \\
F_3^tF_2^t & = & \frac{a_{22}a_{33}}{a_{44}}F_4^t.
\end{array}$$

To make the computations easier, we will consider a new basis $\{f_1, f_2, f_3, f_4\}$ of $\mathfrak{N}_{03}(\lambda)$ such that

$$f_2 f_3 = 0, \quad f_3 f_3 = 0, \quad f_4 \mathfrak{N}_{03}(\lambda) = \mathfrak{N}_{03}(\lambda) f_4 = 0.$$

Such a basis can be defined as

$$f_1 = e_1, \quad f_2 = e_2, \quad f_3 = e_1 + \lambda e_2 + i\sqrt{\alpha^2+1}\, e_3, \quad f_4 = e_4.$$

The multiplication table of $\mathfrak{N}_{03}(\lambda)$ with this new basis is

$$\begin{array}{rclrclrcl}
f_1 f_1 &=& f_4; & f_2 f_2 &=& f_4; & f_3 f_1 &=& \left(1-\alpha^2\right) f_4; \\
f_1 f_2 &=& \lambda f_4; & f_2 f_1 &=& -\lambda f_4; & f_3 f_2 &=& 2\lambda f_4. \\
f_1 f_3 &=& \left(1+\alpha^2\right) f_4; & & & & & &
\end{array}$$

Some routine calculations show that by taking

$$\begin{array}{rclrclrclrcl}
\alpha_{11} &=& \left(1+\alpha^2\right) t, & \alpha_{12} &=& \left(1-\alpha^2\right) t, & \alpha_{13} &=& -2t, & \alpha_{14} &=& 0, \\
& & & \alpha_{22} &=& 2\lambda t, & \alpha_{23} &=& -2\lambda t, & \alpha_{24} &=& 0, \\
& & & & & & \alpha_{33} &=& -4\alpha^2 t, & \alpha_{34} &=& -4\alpha^2 t \frac{\alpha^2-3}{\alpha^2+1}, \\
& & & & & & & & & \alpha_{44} &=& -4\alpha^2 t^2,
\end{array}$$

we obtain exactly

$$\begin{array}{rclrclrcl}
F_1^0 F_1^0 &=& F_4^0; & F_2^0 F_2^0 &=& F_4^0; & F_3^0 F_1^0 &=& \left(1-\alpha^2\right) F_4^0; \\
F_1^0 F_2^0 &=& \lambda F_4^0; & F_2^0 F_1^0 &=& -\lambda F_4^0; & F_3^0 F_2^0 &=& 2\lambda F_4^0. \\
F_1^0 F_3^0 &=& \left(1+\alpha^2\right) F_4^0; & & & & & &
\end{array}$$

Then, it suffices to take

$$E_1^t = F_1^t, \quad E_2^t = F_2^t, \quad E_3^t = \frac{i}{\sqrt{\alpha^2+1}}\left(F_1^t + F_2^t - F_3^t\right), \quad E_4^t = F_4^t,$$

so that we have the desired degeneration $\mathcal{B}_{10}^4 \to \mathfrak{N}_{03}(\lambda)$ by the method described in Subsection 3.1.2. Namely,

$$
\begin{aligned}
E_1^t &= t\left(\left(1+\alpha^2\right)e_1 + \left(1-\alpha^2\right)e_2 - 2e_3\right);\\
E_2^t &= 2\lambda t\left(e_2 - e_3\right);\\
E_3^t &= \frac{it}{\sqrt{1+\alpha^2}}\left(\left(1+\alpha^2\right)e_1 + \left(1+\alpha^2\right)e_2 - 2\left(1-\alpha^2\right)e_3 + \frac{4\alpha^2\left(\alpha^2-3\right)}{1+\alpha^2}e_4\right);\\
E_4^t &= -4\alpha^2 t^2 e_4,
\end{aligned}
$$

is a parameterised basis for $\mathcal{B}_{10}^4 \to \mathfrak{N}_{03}(\lambda)$.

Regarding the pure nilpotent bicommutative algebras, similar computations show that $\mathcal{B}_{10}$, $\mathcal{B}_{12}$, $\mathcal{B}_{14}$, $\mathcal{B}_{20}$ or $\mathcal{B}_{24}^4(\lambda)$ degenerate to them. The explicit degenerations can be seen in Table 3.8 below.

$\mathcal{B}_{10}^4$	$\to$	$\mathfrak{N}_{02}^4(\lambda)$	$E_1^t = t\left(e_1+e_3\right)$ $E_2^t = -te_1 + t(1-\lambda)e_2$ $E_3^t = t^2e_4$ $E_4^t = t^2(1-\lambda)\left(e_3+e_4\right)$
$\mathcal{B}_{10}^4$	$\to$	$\mathcal{B}_{01}^4$	$E_1^t = te_1 + e_2$ $E_2^t = t\left(e_3+e_4\right)$ $E_3^t = te_4$ $E_4^t = te_2$
$\mathcal{B}_{10}^4$	$\to$	$\mathcal{B}_{02}^4(\lambda)_{\lambda\neq 0}$	$E_1^t = e_1 + \lambda e_2$ $E_3^t = \lambda e_4$ $E_2^t = \alpha\left(e_3+e_4\right)$ $E_4^t = t\left(e_2+e_3\right)$
$\mathcal{B}_{20}^4$	$\to$	$\mathcal{B}_{03}^4$	$E_1^t = te_1$ $E_2^t = t^2e_2$ $E_3^t = t^3e_3$ $E_4^t = t^3e_4$

$\mathcal{B}^4_{10}$	$\to \mathcal{B}^4_{04}(\lambda)$	$E^t_1 = -t^2e_1 - \lambda t^2e_2 + \left((\alpha+1)t^2 + t^4\right)e_3$ $E^t_2 = \lambda t^4e_3 + \left(t^4\left(\alpha - (\alpha+1)\left(\alpha+1+t^2\right)\right)\right)e_4$ $E^t_3 = t^3e_1 - \lambda t^3e_3$ $E^t_4 = -\lambda t^6e_4$
$\mathcal{B}^4_{10}$	$\to \mathcal{B}^4_{05}$	$E^t_1 = t^2\left(e_1 + e_2 + (it-2)e_3\right)$ $E^t_2 = t^4\left(e_3 + (2it-3)e_4\right)$ $E^t_3 = it^3\left(e_2 - e_3\right)$ $E^t_4 = t^6e_4$
$\mathcal{B}^4_{10}$	$\to \mathcal{B}^4_{06}(\lambda)_{\lambda\neq 0}$	$E^t_1 = e_1 + \lambda e_2 - \alpha\left(1 + \lambda t^{-1}\right)e_3$ $E^t_2 = \lambda e_3 - \alpha^2\left(1 + (1+\lambda)t^{-1}\right)e_4$ $E^t_3 = te_2$ $E^t_4 = \lambda e_4$
$\mathcal{B}^4_{10}$	$\to \mathcal{B}^4_{07}$	$E^t_1 = t^4e_1 - t^2e_2 + \left(t^2 - t^4 + t^6\right)e_3$ $E^t_2 = -t^6e_3 + \left(-t^4 + t^6 - 2t^8 + t^{10}\right)e_4$ $E^t_3 = t^5e_1 + t^3e_3$ $E^t_4 = t^8e_4$
$\mathcal{B}^4_{10}$	$\to \mathcal{B}^4_{08}$	$E^t_1 = te_1 + e_2 - \left(1 + t^{-1}\right)e_3$ $E^t_2 = te_3 - \left(2 + t^{-1}\right)e_4$ $E^t_3 = te_2 + t^2e_3$ $E^t_4 = te_4$
$\mathcal{B}^4_{20}$	$\to \mathcal{B}^4_{09}$	$E^t_1 = te_1$ $E^t_2 = t^2e_2$ $E^t_3 = t^2e_3$ $E^t_4 = t^3e_4$
$\mathcal{B}^4_{10}$	$\to \mathcal{B}^4_{11}$	$E^t_1 = t^{-1}e_1$ $E^t_2 = t^{-1}e_2$ $E^t_3 = t^{-2}e_3$ $E^t_4 = t^{-3}e_4$

$\mathcal{B}_{10}^4$	$\to$ $\mathcal{B}_{12}^4$	$E_1^t = te_1 + e_3$ $E_2^t = e_2$ $E_3^t = te_3 + e_4$ $E_4^t = te_4$
$\mathcal{B}_{10}^4$	$\to$ $\mathcal{B}_{13}^4$	$E_1^t = te_1$ $E_3^t = te_3$ $E_2^t = e_2$ $E_4^t = te_4$
$\mathcal{B}_{10}^4$	$\to$ $\mathcal{B}_{14}^4$	$E_1^t = e_1$ $E_2^t = te_2 + e_3$ $E_3^t = te_3 + e_4$ $E_4^t = te_4$
$\mathcal{B}_{10}^4$	$\to$ $\mathcal{B}_{15}^4$	$E_1^t = e_1$ $E_2^t = te_2$ $E_3^t = te_3$ $E_4^t = te_4$
$\mathcal{B}_{14}^4$	$\to$ $\mathcal{B}_{16}^4$	$E_1^t = t^{-1}e_1$ $E_3^t = t^{-3}e_3$ $E_2^t = t^{-2}e_2$ $E_4^t = t^{-4}e_4$
$\mathcal{B}_{10}^4$	$\to$ $\mathcal{B}_{17}^4$	$E_1^t = t^{-1}e_1$ $E_2^t = e_2$ $E_3^t = t^{-1}e_3$ $E_4^t = t^{-2}e_4$
$\mathcal{B}_{12}^4$	$\to$ $\mathcal{B}_{18}^4$	$E_1^t = t^{-2}e_1$ $E_2^t = t^{-1}e_2$ $E_3^t = t^{-3}e_3$ $E_4^t = t^{-4}e_4$

$\mathcal{B}_{10}^4$	$\to$	$\mathcal{B}_{19}^4$	$E_1^t = e_1$ $E_2^t = t^{-1}e_2$ $E_3^t = t^{-1}e_3$ $E_4^t = t^{-2}e_4$
$\mathcal{B}_{24}^4\left(\frac{1}{t}\right)$	$\to$	$\mathcal{B}_{20}^4$	$E_1^t = te_1 + \frac{t}{1-t}e_2$ $E_2^t = t^2e_2 + (1+t)\frac{t}{1-t}e_3 + \frac{t}{(1-t)^2}e_4$ $E_3^t = t^2e_3 + (1+2t)\frac{t}{1-t}e_4$ $E_4^t = t^2e_4$
$\mathcal{B}_{20}^4$	$\to$	$\mathcal{B}_{21}^4$	$E_1^t = t^{-1}e_1$ $E_2^t = t^{-2}e_2$ $E_3^t = t^{-3}e_3$ $E_4^t = t^{-4}e_4$
$\mathcal{B}_{24}^4(t)$	$\to$	$\mathcal{B}_{22}^4$	$E_1^t = te_1 + te_2$ $E_2^t = t^2e_2 + \left(t^2+t^3\right)e_3 + t^3e_4$ $E_3^t = t^3e_3 + \left(t^3+2t^4\right)e_4$ $E_4^t = t^4e_4$
$\mathcal{B}_{24}^4(1-t)$	$\to$	$\mathcal{B}_{23}^4$	$E_1^t = te_1 + te_2$ $E_2^t = t^2e_2 + \left(2t^2-t^3\right)e_3 + t^2(1-t)e_4$ $E_3^t = t^3e_3 + \left(3t^3-2t^4\right)e_4$ $E_4^t = t^4e_4$

Table 3.8: Degenerations of four-dimensional nilpotent bicommutative algebras.

$\square$

CHAPTER 4

Non-associative central extensions of null-filiform associative algebras

In this chapter, we give the algebraic classification of alternative, left-alternative, Jordan, bicommutative, left-commutative, assosymmetric, Novikov and left-symmetric central extensions of null-filiform associative algebras.

Introduction

Null-filiform algebras are nothing but one-generated nilpotent algebras. However, in this chapter we will employ the term null-filiform, since this is the common terminology in the literature.

The study of (split and non-split) central extensions of null-filiform algebras was initiated in [8], where all Leibniz central extensions of null-filiform Leibniz algebras were described. The associative non-split central extensions of the unique null-filiform associative algebra of dimension n, which we will denote by μ_0^n, were studied in [138] within the framework of associative algebras, and it was proved that the only non-split associative central extension of μ_0^n is μ_0^{n+1}. However, the null-filiform associative algebras can be considered as elements of more general varieties of algebras, such as alternative, left-alternative, Jordan, bicommutative, left-commutative, assosymmetric, Novikov or left-symmetric, among others (note that the right-alternative, right-commutative and right-symmetric cases are analogous to their respective left counterparts). In particular, it was proven in [39] that the null-filiform algebra μ_0^3 admits the trivial extension μ_0^4 and also another non-trivial bicommutative extension. We also recovered this result in Chapter 3: with our notations,

$\mu_0^3 = \mathcal{B}_{02}^3(1)$ admits the trivial extension $\mu_0^4 = \mathcal{B}_{24}^4(1)$, but also $\mathcal{B}_{23}^4$. Then, it is reasonable to wonder whether there will be non-trivial extensions in the aforementioned varieties of algebras.

The main result of this chapter is the classification of the isomorphism classes of central extensions of the null-filiform associative algebra over several varieties of non-associative algebras, as the ones mentioned above. This is in part summarised in Table 4.1, at the end of Section 4.5.

The structure of the present chapter is as follows. Section 1 is devoted to formalise the method of Skjelbred and Sund over any variety of non-associative algebras defined by a set of polynomial identities. Section 2 presents a quick review of null-filiform algebras. Respectively, Section 4.3, Section 4.4 and Section 4.5 deal with left-alternative and alternative, Jordan, and left-commutative and bicommutative central extensions. Finally, Section 4.6 deals with the assosymmetric, Novikov and left-symmetric cases, which happen to come out as a trivial corollary of the left-commutative and bicommutative cases.

Throughout the chapter, F will denote a field. Unless otherwise specified, this field will be arbitrary and all vector spaces, tensor products, (multi)linear maps and automorphism groups will be taken over F. Given a non-negative integer n, $[n]$ will denote the set $\{i \in \mathbb{Z} \mid 1 \le i \le n\}$. Also, we fix the following notation:

$$\begin{aligned} \mathrm{AS}&: \text{ variety of associative algebras;}\\ \mathrm{AL}&: \text{ variety of left-alternative algebras;}\\ \mathrm{J}&: \text{ variety of Jordan algebras;}\\ \mathrm{LC}&: \text{ variety of left-commutative algebras;}\\ \mathrm{RC}&: \text{ variety of right-commutative algebras;}\\ \mathrm{BC}&: \text{ variety of bicommutative algebras.}\end{aligned}$$

4.1 Non-associative central extensions

A variety M of non-associative algebras over F is defined by a set of identities $\{E_i\}_{i\in I}$ of the form

$$E_i: \quad \sum_{j=1}^{t_i} p_{i,j}\left(z_1,\ldots,z_\ell\right) = 0,$$

where $Z = \{z_1,\ldots,z_\ell\}$ is a finite alphabet, $p_{i,j} = p_{i,j}\left(z_1,\ldots,z_\ell\right)$ is a non-associative word in Z of length $n_{i,j} \ge 2$, with a coefficient either 1 or -1 and $Z_{i,j} \subseteq Z$ is the set of letters which occur in $p_{i,j}$. Note that it might be $I = \emptyset$.

As $n_{i,j} \geq 2$, each $p_{i,j}$ can be expressed uniquely as a product

$$p_{i,j} = p_{i,j}^1 p_{i,j}^2,$$

where $Z_{i,j} = Z_{i,j}^1 \cup Z_{i,j}^2$ and each $p_{i,j}^k$, with $k \in \{1, 2\}$, either has length one or is another concatenation of products. Recursively, we will obtain $n_{i,j} - 1$ factors $p_{i,j}^{\alpha}$ with $|Z_{i,j}^{\alpha}| = 1$ and α of the form $\alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{\ell}$, with $\alpha_k \in \{1, 2\}$ for all k.

Let A be an algebra in the variety M, and let V be a vector space over F. Following the Skjelbred-Sund method, we introduce the *cocycles* of A with respect to V as the bilinear maps $\theta \colon \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ satisfying the set of identities $\left\{ \widetilde{E}_i \right\}_{i \in I}$, where

$$\widetilde{E}_i \colon \quad \sum_{j=1}^{t_i} \theta \left(p_{i,j}^1, p_{i,j}^2 \right) = 0.$$

These elements form a vector space over F, which we will denote by $\mathrm{Z}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V})$.

We also define the *coboundaries* of A with respect to V as follows. Let f be a linear map from A to V, and set $\delta f \colon \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ with $(\delta f)(x, y) = f(xy)$. It is clear from

$$\sum_{j=1}^{t_i} \delta f \left(p_{i,j}^1, p_{i,j}^2 \right) = \sum_{j=1}^{t_i} f \left(p_{i,j} \right) = f(0) = 0,$$

for all $i \in I$, that $\delta f \in \mathrm{Z}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V})$. We define $\mathrm{B}^2 (\mathrm{A}, \mathrm{V}) = \{\delta f \,:\, f \in \mathrm{Hom}(\mathrm{A}, \mathrm{V})\}$; it is a linear subspace of $\mathrm{Z}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V})$. The quotient space $\mathrm{Z}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V}) / \mathrm{B}^2 (\mathrm{A}, \mathrm{V})$ is called the *second cohomology* space and is denoted by $\mathrm{H}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V})$.

Consider the group $\mathrm{Aut}(\mathrm{A})$ of automorphisms of the algebra A. If θ is a cocycle and $\phi \in \mathrm{Aut}(\mathrm{A})$, we define $\phi \cdot \theta \colon \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ by $\phi \cdot \theta (x, y) = \theta(\phi(x), \phi(y))$. The automorphism ϕ preserves the product, so

$$\phi \left(p_{i,j} \left(x_1, \ldots, x_{\ell} \right) \right) = p_{i,j} \left(\phi\left(x_1\right), \ldots, \phi\left(x_{\ell}\right) \right);$$

from this, we have that $\phi \cdot \theta \in \mathrm{Z}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V})$. This induces an action of $\mathrm{Aut}(\mathrm{A})$ on $\mathrm{Z}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V})$. Moreover, $\theta = \delta f$ if and only if $\phi \cdot \theta = \delta(f \circ \phi)$, so the action is inherited by the quotient $\mathrm{H}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V})$. Although this is a right action, we will write it on the left to follow the usual convention. The orbit of an element $\theta \in \mathrm{Z}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V})$ will be denoted by $\mathrm{Orb}(\theta)$, and the orbit of $[\theta] \in \mathrm{H}_{\mathrm{M}}^2 (\mathrm{A}, \mathrm{V})$, by $\mathrm{Orb}([\theta])$.

For every bilinear map $\theta \colon \mathrm{A} \times \mathrm{A} \to \mathrm{V}$, we can define the algebra $\mathrm{A}_{\theta} = \mathrm{A} \oplus \mathrm{V}$ with the product $[x + v, y + w]_{\theta} = xy + \theta(x, y)$.

Lemma 4.1.1. *The algebra* A_θ *belongs to the variety* M *if and only if* $\theta \in \mathrm{Z}^2_{\mathrm{M}}(\mathrm{A}, \mathrm{V})$.

Proof. By definition, A_θ belongs to M if it satisfies the identities $\left\{E_i\right\}_{i\in I}$, i.e. if it satisfies

$$\sum_{j=1}^{t_i} p_{i,j}\left(x_1+v_1,\dots,x_\ell+v_\ell\right)=0,$$

for all $i \in I$ and all $x_k \in \mathrm{A}$, $v_k \in \mathrm{V}$, with $k \in [\ell]$. For the sake of brevity, we will write

$$\sum_{j=1}^{t_i} p_{i,j}\left(X+V\right)=0;$$

also, when we make reference to these identities in the algebra A, we will write

$$\sum_{j=1}^{t_i} p_{i,j}\left(X\right)=0.$$

It holds that

$$p_{i,j}\left(X+V\right)=[p^1_{i,j}\left(X+V\right),p^2_{i,j}\left(X+V\right)]_\theta.$$

An easy induction in the cardinal $|Z_{i,j}|$ shows that

$$p_{i,j}\left(X+V\right)=p_{i,j}\left(X\right)+\theta\left(p^1_{i,j}\left(X\right),p^2_{i,j}\left(X\right)\right).$$

As A belongs to the variety M, it follows that A_θ satisfies the identities $\left\{E_i\right\}_{i\in I}$ if and only if θ satisfies $\left\{\widetilde{E}_i\right\}_{i\in I}$. □

In all that follows, we will consider A_θ just for $\theta \in \mathrm{Z}^2_{\mathrm{M}}(\mathrm{A}, \mathrm{V})$. Then, it is easy to check that the algebra A_θ of the variety M is a central extension of A by V. We define the *dimension* of the extension as the dimension of V.

The particular identities of the variety M are not involved any more in the development of the method. Thus, we refer the reader to Subsection 3.1.1, and limit to expose the final procedure to construct all non-split central extensions A_θ of a given algebra A of dimension $n-s$ in the variety M.

1. Determine $\mathrm{H}^2_{\mathrm{M}}(\mathrm{A}, F)$, $\mathrm{Ann}(\mathrm{A})$ and Aut A.
2. Determine the set of (Aut A)-orbits on $T_s(\mathrm{A})$.

3. For each orbit, construct the algebra of the variety M associated with a representative of it.

Let A be an algebra and fix a basis $\{e_1, e_2, \dots, e_n\}$ of A. We define the bilinear form $\Delta_{i,j} : \mathrm{A} \times \mathrm{A} \to F$ by $\Delta_{i,j}(e_l, e_m) = \delta_{i,l}\delta_{j,m}$. Then the set $\{\Delta_{i,j} \mid i, j \in [n]\}$ is a basis for the linear space of bilinear forms on A; in particular, every $\theta \in \mathrm{Z}^2(\mathrm{A}, \mathrm{V})$ can be uniquely written as $\theta = \sum_{i,j=1}^{n} c_{i,j}\Delta_{i,j}$, with $c_{i,j} \in F$.

4.2 Null-filiform associative algebras

For an algebra A of an arbitrary variety M, we consider the series

$$\mathrm{A}^1 = \mathrm{A}, \qquad \mathrm{A}^{i+1} = \sum_{k=1}^{i} \mathrm{A}^k \mathrm{A}^{i+1-k}, \qquad i \geq 1.$$

An n-dimensional algebra A is called *null-filiform* if $\dim \mathrm{A}^i = (n+1) - i$, for all $i \in [n+1]$. Clearly, this condition is satisfied if and only if the index of nilpotency of A is maximum. For a nilpotent algebra, being null-filiform is equivalent to being one-generated, as we commented before.

All null-filiform associative algebras were described in [68].

Theorem 4.2.1. *An arbitrary n-dimensional null-filiform associative algebra is isomorphic to the algebra:*

$$\mu_0^n : \quad e_i e_j = e_{i+j}, \quad i, j \in [n],$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra μ_0^n and we define $e_m = 0$ for all $m > n$.

Using the procedure explained in Section 4.1, we can easily find all associative central extensions of μ_0^n. Let $\nabla_j = \sum_{k=1}^{j} \Delta_{k,j+1-k}$, for $j \in [n]$. We need the following result from [138].

Proposition 4.2.2. *Let μ_0^n be the null-filiform associative algebra of dimension n. Then*

$$\mathrm{Z}^2_{\mathrm{AS}}(\mu_0^n, F) = \langle \nabla_j \mid j \in [n] \rangle, \quad \mathrm{B}^2(\mu_0^n, F) = \langle \nabla_j \mid j \in [n-1] \rangle,$$
$$\mathrm{H}^2_{\mathrm{AS}}(\mu_0^n, F) = \mathrm{Z}^2_{\mathrm{A}}(\mu_0^n, F) / \mathrm{B}^2(\mu_0^n, F) = \langle [\nabla_n] \rangle.$$

Remark 4.2.3. The above result appears in [138] for $F = \mathbb{C}$, the field of complex numbers. Nevertheless, it is immediate to check, e.g. using the methods appearing shortly in this chapter, that it does hold over an arbitrary field.

As $\dim \mathrm{H}^2_{\mathrm{AS}}\left(\mu_0^n, F\right) = 1$, the next result follows easily.

Theorem 4.2.4. *Every non-split one-dimensional associative central extension of μ_0^n is isomorphic to μ_0^{n+1}.*

In the following sections we will study non-associative central extensions of μ_0^n. It is easy to see that μ_0^{n+1} is an associative central extension of μ_0^n. All extensions of this type will be called *trivial*. The basis $\left\{e_1, \ldots, e_n\right\}$ of μ_0^n will always be assumed to satisfy the relations from Theorem 4.2.1.

4.3 Alternative and left-alternative central extensions

Throughout this section, we assume that the characteristic of the field F is different from 2. Recall that an algebra A is said to be *left-alternative* (respectively, *right-alternative*) if it satisfies the identity

$$x\,(xy) = (xx)\,y \quad \text{(respectively, } (xy)\,y = x\,(yy)\text{)},$$

for all $x, y \in$ A. Also, if A is both left-alternative and right-alternative, it is called *alternative*.

Let us consider μ_0^n as a left-alternative algebra. Note also that the linearisation of the left-alternative identity for μ_0^n leads to

$$e_i\left(e_j e_k\right) + e_j\left(e_i e_k\right) = \left(e_i e_j\right) e_k + \left(e_j e_i\right) e_k,$$

for $i, j, k \in [n]$. Then, its space of cocycles is formed by all the bilinear maps $\theta : \mu_0^n \times \mu_0^n \to F$ satisfying $\theta\left(e_i, e_j e_k\right) + \theta\left(e_j, e_i e_k\right) = \theta\left(e_i e_j, e_k\right) + \theta\left(e_j e_i, e_k\right)$. This can be expressed as

$$\theta\left(e_i, e_{j+k}\right) + \theta\left(e_j, e_{i+k}\right) = 2\theta\left(e_{i+j}, e_k\right), \tag{4.3.1}$$

for $i, j, k \in [n]$, and considering that $e_m = 0$ for $m > n$.

Theorem 4.3.1. *Assume that* $\operatorname{char}(F) \neq 2$. *Then all left-alternative and all alternative central extensions of μ_0^n are trivial.*

Proof. The first step is to compute $\mathrm{Z}^2_{\mathrm{LA}}\left(\mu_0^n, F\right)$. Let $\theta = \sum_{i,j} c_{i,j}\Delta_{i,j}$ be an arbitrary cocycle of μ_0^n considered as a left-alternative algebra. The identity (4.3.1) leads to

$$c_{i,j+k} + c_{j,i+k} = 2c_{i+j,k}$$

for $i, j, k \in [n]$, with the assumption that $c_{i,j} = 0$ if $i > n$ or $j > n$. Given integers m, s such that $m, s-m \in [n]$ and $m \geq 2$, and taking $i = m-1$, $j = 1$ and $k = s-m$ in the above equation, we get

$$2c_{m,s-m} = c_{m-1,s-(m-1)} + c_{1,s-1}. \tag{4.3.2}$$

Claim: For all $i, j \in [n]$, $c_{i,j} = c_{1,i+j-1}$. In particular, $c_{i,j} = 0$ if $i + j \geq n+2$.

The claim will follow by induction on i, the case $i = 1$ being trivial. So assume that $i \geq 2$ and that the claim holds for $i-1$. Taking $m = i$ and $s = i+j$ in (4.3.2), we get

$$2c_{i,j} = c_{i-1,j+1} + c_{1,i+j-1}.$$

If $j = n$, then $j+1, i+j-1 \geq n+1$ so $c_{i-1,j+1} = 0 = c_{1,i+j-1}$ and it follows from the identity above that $c_{i,j} = 0$, as $\mathrm{char}\,(F) \neq 2$. In particular, $c_{i,j} = c_{1,i+j-1}$ holds. Otherwise, if $j < n$, then we can use the induction hypothesis to obtain $2c_{i,j} = c_{i-1,j+1} + c_{1,i+j-1} = 2c_{1,i+j-1}$, whence the claim.

So, it is clear that $\mathrm{Z}^2_{\mathrm{LA}}\left(\mu_0^n, F\right)$ is in the linear space spanned by $\left\{\nabla_j\right\}_{j=1}^n$; it is also immediate to see that every element from $\mathrm{Z}^2_{\mathrm{AS}}\left(\mu_0^n, F\right)$ is a left-alternative cocycle. Thus, $\mathrm{H}^2_{\mathrm{LA}}\left(\mu_0^n, F\right) = \mathrm{H}^2_{\mathrm{AS}}\left(\mu_0^n, F\right)$, and every left-alternative central extension is trivial. Since any alternative extension is left-alternative, it follows that every alternative central extension is trivial as well. □

4.4 Jordan central extensions

Throughout this section, we assume that the characteristic of the field F is different from 2 and 3. Recall that a commutative algebra A is said to be *Jordan* if it satisfies

$$x^2(yx) = \left(x^2 y\right)x,$$

for all $x, y \in \mathrm{A}$. It is immediate to check that the commutative algebra μ_0^n satisfies the previous identity and therefore is a Jordan algebra.

The space of Jordan cocycles of μ_0^n is formed by all the bilinear maps $\theta\colon\ \mu_0^n \times \mu_0^n \to F$ satisfying

$$\theta(x, y) = \theta(y, x); \tag{4.4.1}$$

$$\theta\left(x^2, yx\right) = \theta\left(x^2 y, x\right). \tag{4.4.2}$$

Equivalently, since $\mathrm{char}\,(F) \neq 2, 3$, we have

$$\theta\left(e_i, e_j\right) = \theta\left(e_j, e_i\right),$$

$$\begin{aligned}&\theta\left(e_{i+\ell}, e_{j+k}\right) + \theta\left(e_{j+\ell}, e_{i+k}\right) + \theta\left(e_{k+\ell}, e_{i+j}\right)\\ &= \theta\left(e_i, e_{j+k+\ell}\right) + \theta\left(e_j, e_{i+k+\ell}\right) + \theta\left(e_k, e_{i+j+\ell}\right),\end{aligned}$$

for $i, j, k, \ell \in [n]$. In particular, taking $i = j$, we see that every cocycle must satisfy

$$2\theta\left(e_{i+\ell}, e_{i+k}\right) + \theta\left(e_{\ell+k}, e_{2i}\right) = 2\theta\left(e_i, e_{i+k+\ell}\right) + \theta\left(e_k, e_{2i+\ell}\right). \tag{4.4.3}$$

Theorem 4.4.1. *All Jordan central extensions of μ_0^n are trivial.*

Proof. We will prove that $\left\{\nabla_j\right\}_{j=1}^n$ is a basis for $\mathrm{Z}^2_{\mathrm{J}}\left(\mu_0^n, F\right)$.

Let $\theta = \sum_{i,j} c_{i,j} \Delta_{i,j}$ be an arbitrary cocycle in $\mathrm{Z}^2_{\mathrm{J}}\left(\mu_0^n, F\right)$. Note that $c_{i,j} = c_{j,i}$, for all $i, j \in [n]$. By (4.4.3), we also have

$$c_{k,2a+b} = 2c_{a+b,a+k} + c_{b+k,2a} - 2c_{a,a+b+k}, \tag{4.4.4}$$

for all $a, b, k \in [n]$, with the assumption that $c_{i,j} = 0$ in case $i > n$ or $j > n$.

Claim: For all $i, j \in [n]$, $c_{i,j} = c_{1,i+j-1}$. In particular, $c_{i,j} = 0$ if $i + j \geq n + 2$.

The proof is by induction on i. If either $i = 1$ or $j = 1$, the claim is trivial, by commutativity. So we can assume that $i, j \geq 2$. If $i = 2$, then

$$c_{2,j} = \theta\left(e_1^2, e_{j-1}e_1\right) = \theta\left(e_{j+1}, e_1\right) = \theta\left(e_1, e_{j+1}\right) = c_{1,j+1}.$$

Let us assume thus that $i \geq 3$ and $j \geq 2$. There are unique integers a, b with $b \in \{1, 2\}$ such that $i = 2a + b$. Notice that $1 \leq a < a + b < i \leq n$. Then, using (4.4.4) with $k = j$, we get

$$c_{i,j} = c_{2a+b,j} = 2c_{a+b,a+j} + c_{b+j,2a} - 2c_{a,a+b+j}.$$

There are two cases to consider.

Case1: $a+b+j \le n$. In this case, we can use the inductive hypothesis and the commutativity to get

$$\begin{aligned} 2c_{a+b,a+j} + c_{b+j,2a} - 2c_{a,a+b+j} &= 2c_{1,2a+b+j-1} + c_{1,2a+b+j-1} - 2c_{1,2a+b+j-1} \\ &= c_{1,2a+b+j-1} = c_{1,i+j-1}. \end{aligned}$$

Case2: $a+b+j \ge n+1$. Then $c_{a,a+b+j} = 0$ and we have $c_{i,j} = 2c_{a+b,a+j} + c_{2a,b+j}$. Either $a+j \ge n+1$ and hence $c_{a+b,a+j} = 0$, or $a+j \le n$ and the inductive hypothesis says that $c_{a+b,a+j} = c_{1,2a+b+j-1} = 0$, because $2a+b+j-1 \ge a+b+j \ge n+1$. In any case, $c_{a+b,a+j} = 0$. Similarly, $c_{2a,b+j} = 0$ and we conclude that $c_{i,j} = 0 = c_{1,i+j-1}$, because $i+j-1 \ge a+b+j \ge n+1$.

The claim is thus established. It remains to observe that the $\left\{\nabla_j\right\}_{j=1}^{n}$ are indeed Jordan cocycles. In [138], it is proved that $\nabla_j \in \mathrm{Z}^2_{\mathrm{AS}}\left(\mu_0^n, F\right)$, i.e. they verify $\nabla_j(xy,z) = \nabla_j(x,yz)$. Clearly, also $\nabla_j(x,y) = \nabla_j(y,x)$. These conditions are indeed stricter than conditions (4.4.1) and (4.4.2), so it is clear that $\nabla_j \in \mathrm{Z}^2_{\mathrm{J}}\left(\mu_0^n, F\right)$. This implies that $\mathrm{H}^2_{\mathrm{J}}\left(\mu_0^n, F\right) = \mathrm{H}^2_{\mathrm{AS}}\left(\mu_0^n, F\right)$ and, according to [138], the only Jordan central extension of μ_0^n is μ_0^{n+1}. □

4.5 Left-commutative and bicommutative central extensions

Recall that an algebra A is said to be *left-* (respectively, *right-*) commutative if it satisfies

$$x(yz) = y(xz) \quad \text{(respectively, } (xy)z = (xz)y\text{)},$$

for all $x, y, z \in$ A. Equivalently, A is left- (respectively, right-) commutative if and only if the left (respectively, right) multiplication operators commute. In case A is both left- and right-commutative, we say that A is *bicommutative*.

The main results in this section will require the field F to be algebraically closed and of sufficiently large characteristic (which for simplicity we will assume to be zero, when necessary), while others hold for arbitrary fields. Unless otherwise is stated, it should be assumed that the field F is arbitrary.

Proposition 4.5.1. *Let $n \ge 2$ and recall the bilinear forms $\nabla_j = \sum_{k=1}^{j} \Delta_{k,j+1-k}$, defined for $j \in [n]$. Then the following hold:*

(a) $\dim \mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right) = 2n-1$ *and* $\left\{\Delta_{i,1} \mid 2 \leq i \leq n\right\} \cup \left\{\nabla_j \mid j \in [n]\right\}$ *is a basis of* $\mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$.

(b) $\dim \mathrm{H}^2_{\mathrm{LC}}\left(\mu_0^n, F\right) = n$ *and the classes* $\left\{\left[\Delta_{i,1}\right] \mid 2 \leq i \leq n\right\} \cup \left\{\left[\nabla_n\right]\right\}$ *form a basis of* $\mathrm{H}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$.

(c) *In the bicommutative case we have that* $\dim \mathrm{H}^2_{\mathrm{BC}}\left(\mu_0^n, F\right) = 2$, *and* $\left\{\left[\Delta_{2,1}\right], \left[\nabla_n\right]\right\}$ *is a basis.*

Proof. The space $\mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$ consists of the bilinear forms $\theta = \sum_{i,j=1}^n c_{i,j}\Delta_{i,j}$, with $c_{i,j} \in F$, satisfying $\theta\left(e_i, e_j e_k\right) = \theta\left(e_j, e_i e_k\right)$, for all $i, j, k \in [n]$.

Claim: If $j \geq 2$, then $c_{i,j} = c_{1,i+j-1}$. In particular, if $i + j \geq n + 2$ then $c_{i,j} = 0$.

For $j \geq 2$ we have

$$c_{i,j} = \theta\left(e_i, e_j\right) = \theta\left(e_i, e_1 e_{j-1}\right) = \theta\left(e_1, e_i e_{j-1}\right) = c_{1,i+j-1}. \tag{4.5.1}$$

Thus, if $i+j \geq n+2$, then necessarily $j \geq 2$ and (4.5.1) gives $c_{i,j} = 0$. This establishes the claim.

Hence, $c_{i,2} = c_{i-1,3} = \cdots = c_{2,i} = c_{1,i+1}$ for all $i \in [n-1]$ and we can write

$$\theta = \sum_{i=1}^n c_{i,1}\Delta_{i,1} + \sum_{j=2}^n c_{1,j}\left(\nabla_j - \Delta_{j,1}\right) = \sum_{i=2}^n \left(c_{i,1} - c_{1,i}\right)\Delta_{i,1} + \sum_{j=1}^n c_{1,j}\nabla_j, \tag{4.5.2}$$

which shows that $\mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right) \subseteq \left\langle \left\{\Delta_{i,1} \mid 2 \leq i \leq n\right\} \cup \left\{\nabla_j \mid j \in [n]\right\}\right\rangle$.

It remains to prove the reverse inclusion, i.e. that the $\Delta_{i,1}$ and the ∇_j are left-commutative cocycles. Let $k, \ell, m \in [n]$. Then

$$\Delta_{i,1}\left(e_k, e_\ell e_m\right) = 0 = \Delta_{i,1}\left(e_\ell, e_k e_m\right),$$

so indeed $\Delta_{i,1} \in \mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$. As $\left\langle \nabla_j \mid j \in [n-1]\right\rangle = \mathrm{B}^2\left(\mu_0^n, F\right) \subseteq \mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$, it remains to show that ∇_n is a cocycle. This follows immediately from the fact that

$$\nabla_n\left(e_k, e_\ell e_m\right) = \begin{cases} 1 & \text{if } k+\ell+m = n+1; \\ 0 & \text{otherwise.} \end{cases}$$

This concludes the proof of (a). Part (b) is clear by $\mathrm{B}^2\left(\mu_0^n, F\right) = \left\langle \nabla_j \mid j \in [n-1]\right\rangle$.

We precede the proof of (c) with some general considerations. Let A be an algebra and denote by $\nu : \mathrm{A}\times\mathrm{A} \to \mathrm{A}$ its multiplication map. The opposite algebra A^{op} is the algebra with the same underlying vector space and with multiplication $\nu\circ\tau$, where $\tau : \mathrm{A}\times\mathrm{A} \to \mathrm{A}\times\mathrm{A}$, $(a,b)\mapsto(b,a)$ is the flip. Clearly, A is right-commutative if and only if A^{op} is left-commutative and $\theta \in \mathrm{Z}^2_{\mathrm{RC}}(\mathrm{A}, F)$ if and only if $\theta\circ\tau \in \mathrm{Z}^2_{\mathrm{LC}}(\mathrm{A}^{\mathrm{op}}, F)$. Moreover, $\mathrm{Z}^2_{\mathrm{BC}}(\mathrm{A}, F) = \mathrm{Z}^2_{\mathrm{LC}}(\mathrm{A}, F) \cap \mathrm{Z}^2_{\mathrm{RC}}(\mathrm{A}, F)$.

We return now to the case of the algebra μ_0^n. This algebra is commutative, so it coincides with its opposite algebra. Let $\theta \in \mathrm{Z}^2_{\mathrm{BC}}\left(\mu_0^n, F\right)$. Then $\theta \in \mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$ and by (a) we can write θ as in (4.5.2). What is more, for such a θ we have $\theta \in \mathrm{Z}^2_{\mathrm{BC}}\left(\mu_0^n, F\right)$ if and only if $\theta\circ\tau \in \mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$. Seeing that $\Delta_{i,j}\circ\tau = \Delta_{j,i}$, we obtain $\nabla_j\circ\tau = \nabla_j$, for all $j \in [n]$.

Thence, in view of the results in (a), and the observation that $\Delta_{1,2} = \nabla_2 - \Delta_{2,1}$, we have that $\theta\circ\tau \in \mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$ if and only if $c_{i,1} = c_{1,i}$ for all $3 \le i \le n$. This proves that $\left\{\nabla_j \mid j \in [n]\right\} \cup \left\{\Delta_{2,1}\right\}$ is a basis of $\mathrm{Z}^2_{\mathrm{BC}}\left(\mu_0^n, F\right)$. Finally, given our description of $\mathrm{B}^2\left(\mu_0^n, F\right)$, we immediately obtain the basis $\left\{\left[\Delta_{2,1}\right], \left[\nabla_n\right]\right\}$ of $\mathrm{H}^2_{\mathrm{BC}}\left(\mu_0^n, F\right)$. □

4.5.1 The automorphism group of μ_0^n

We denote by $\mathrm{Aut}\left(\mu_0^n\right)$ the automorphism group of μ_0^n. Let $\phi \in \mathrm{Aut}\left(\mu_0^n\right)$. Then we identify ϕ with its matrix $\left(\phi_{i,j}\right)_{i,j\in[n]}$ relative to the basis $\left\{e_1, \dots, e_n\right\}$. It is easy to see that the automorphisms of μ_0^n are precisely those linear endomorphisms $\phi = \left(\phi_{i,j}\right)_{i,j\in[n]}$ with $\phi_{1,1} \neq 0$, $\phi_{2,1}, \dots, \phi_{n,1} \in F$ arbitrary and $\phi_{i,j}$ with $2 \le j \le n$ determined by $\phi\left(e_j\right) = \phi\left(e_1\right)^j$. In other words,

$$\phi_{i,j} = \sum_{k_1+\cdots+k_j=i} \phi_{k_1,1}\cdots\phi_{k_j,1},$$

for all $i, j \in [n]$.

It follows that $\phi_{i,i} = \phi_{1,1}^i$ and $\phi_{i,j} = 0$ if $j > i$. For $j < i$ we also have

$$\phi_{i,j} = j\phi_{1,1}^{j-1}\phi_{i-j+1,1} + p_{i,j}\left(\phi_{1,1}, \phi_{2,1}, \dots, \phi_{i-j,1}\right), \tag{4.5.3}$$

for some polynomial $p_{i,j}$ with coefficients in F which depends only on i and j.

Write $\phi = \sum_{i,j=1}^{n} \phi_{i,j} E_{i,j}$, where we think of $E_{i,j}$ both as the matrix unit with all entries equal to zero except for the entry (i, j) which equals 1, and also as the linear endomorphism of μ_0^n defined by $E_{i,j}\left(e_k\right) = \delta_{j,k} e_i$. Then,

$$\begin{aligned}\phi \cdot \Delta_{s,t}\left(e_k, e_\ell\right) = \Delta_{s,t}\left(\phi\left(e_k\right), \phi\left(e_\ell\right)\right) &= \sum_{k \leq i} \sum_{\ell \leq j} \phi_{i,k} \phi_{j,\ell} \Delta_{s,t}\left(e_i, e_j\right) \\ &= \sum_{k \leq s,\ \ell \leq t} \phi_{s,k} \phi_{t,\ell}.\end{aligned}$$

Thence, the formula for the action is given as

$$\phi \cdot \Delta_{s,t} = \sum_{k \leq s,\ \ell \leq t} \phi_{s,k} \phi_{t,\ell} \Delta_{k,\ell}. \tag{4.5.4}$$

4.5.2 The orbit decomposition of $\mathrm{H}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$

To state the main result of this subsection we need an additional definition. For $0 \leq i \leq n$, let $R(i, n)$ be the multiplicative subgroup of F^* consisting of all $(i+1)$-st powers of all $(n+1)$-st roots of unity in F^*. In case F is algebraically closed of characteristic zero, there exists some primitive $(n+1)$-st root of unity ζ and $R(i, n) = \left\langle \zeta^{i+1} \right\rangle$ is the cyclic group generated by ζ^{i+1}. Denote the quotient group by

$$F_{(i,n)} = F^* / R(i, n), \tag{4.5.5}$$

and for $\mu \in F^*$, let $\overline{\mu} = \mu R(i, n)$ be the corresponding coset.

Theorem 4.5.2. *Assume that F is algebraically closed of characteristic zero and let $n \geq 2$. The following elements of $\mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$ give a complete list of distinct representatives of the orbits of the automorphism group* Aut $\left(\mu_0^n\right)$ *on* $\mathrm{H}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$:

(a) 0;

(b) $\left\{\Delta_{i,1} \mid 2 \leq i \leq n\right\}$;

(c) $\left\{\nabla_n + \mu \Delta_{n,1} \mid \mu \in F\right\}$;

(d) $\left\{\nabla_n + \overline{\mu} \Delta_{i,1} \mid 2 \leq i \leq n-1,\ \overline{\mu} \in F_{(i,n)}\right\}$.

From this result and Proposition 4.5.1, we immediately deduce the orbit space decomposition in the bicommutative case.

Corollary 4.5.3. *Assume that F is algebraically closed of characteristic zero and let $n \geq 2$. The following elements of $\mathrm{Z}^2_{\mathrm{BC}}\left(\mu_0^n, F\right)$ give a complete list of distinct representatives of the orbits of the automorphism group* $\mathrm{Aut}\left(\mu_0^n\right)$ *on* $\mathrm{H}^2_{\mathrm{BC}}\left(\mu_0^n, F\right)$:

(Case $n = 2$**)**

(a) 0;

(b) $\Delta_{2,1}$;

(c) $\left\{\nabla_2 + \mu\Delta_{2,1} \mid \mu \in F\right\}$.

(Case $n > 2$**)**

(a) 0;

(b) $\Delta_{2,1}$;

(c) ∇_n;

(d) $\left\{\nabla_n + \overline{\mu}\Delta_{2,1} \mid \overline{\mu} \in F_{(2,n)}\right\}$.

We devote the remainder of this section to the proof of Theorem 4.5.2.

Lemma 4.5.4. *Assume that F is algebraically closed of characteristic zero. Fix $i \geq 2$. Given scalars $a_{1,1}, \dots, a_{i-1,1} \in F$ with $a_{1,1} \neq 0$, we define, for $2 \leq \ell, m \leq i$:*

$$a_{\ell,m} = \sum_{j_1+\cdots+j_m=\ell} a_{j_1,1} \cdots a_{j_m,1}. \tag{4.5.6}$$

Note that $a_{\ell,m}$ depends only on $a_{1,1}, \dots, a_{\ell-m+1,1}$. Then, for any $0 \leq k \leq i-1$, the map

$$\begin{array}{rccc} \rho_k : & F^* \times F^k & \to & F^* \times F^k \\ & \left(a_{1,1}, \dots, a_{k+1,1}\right) & \mapsto & \left(a_{i,i}a_{1,1}, a_{i,i-1}a_{1,1}, \dots, a_{i,i-k}a_{1,1}\right) \end{array}$$

is onto and $(i+1)$-to-1, i.e. $\left|\rho_k^{-1}\left(\lambda_1, \dots, \lambda_{k+1}\right)\right| = i+1$, for all $\left(\lambda_1, \dots, \lambda_{k+1}\right) \in F^* \times F^k$.

Proof. The proof is by induction on $0 \leq k \leq i-1$. First, notice that (4.5.6) implies that $a_{\ell,\ell} = a_{1,1}^{\ell}$. If $k = 0$, then $\rho_0\left(a_{1,1}\right) = a_{1,1}^{i+1}$ and the relation $\left|\rho_0^{-1}(\lambda)\right| = i+1$ for any $\lambda \in F^*$ follows from both our assumptions on F.

Now assume that $k \geq 1$ and take $(\lambda_1, \ldots, \lambda_{k+1}) \in F^* \times F^k$. By the induction hypothesis, $\left|\rho_{k-1}^{-1}(\lambda_1, \ldots, \lambda_k)\right| = i + 1$, so choose $(a_{1,1}, \ldots, a_{k,1}) \in F^* \times F^{k-1}$ such that $\rho_{k-1}(a_{1,1}, \ldots, a_{k,1}) = (\lambda_1, \ldots, \lambda_k)$. As in (4.5.3), we have $a_{i,i-k}a_{1,1} = (i-k)\, a_{1,1}^{i-k} a_{k+1,1} + a_{1,1} p_{i,i-k}(a_{1,1}, \ldots, a_{k,1})$, with $p_{i,i-k}$ a polynomial in $a_{1,1}, \ldots, a_{k,1}$. Thus, we can set

$$a_{k+1,1} = \frac{\lambda_{k+1} - a_{1,1} p_{i,i-k}(a_{1,1}, \ldots, a_{k,1})}{(i-k)\, a_{1,1}^{i-k}}, \tag{4.5.7}$$

and then $\rho_k(a_{1,1}, \ldots, a_{k+1,1}) = (\rho_{k-1}(a_{1,1}, \ldots, a_{k,1}), a_{i,i-k}a_{1,1}) = (\lambda_1, \ldots, \lambda_{k+1})$.

The above shows that the $i+1$ distinct solutions of the problem for $k-1$ give rise to $i+1$ distinct solutions of the problem for k. Conversely, each solution $(a_{1,1}, \ldots, a_{k+1,1})$ of the latter determines a solution $(a_{1,1}, \ldots, a_{k,1})$ of the former and, by (4.5.7), $a_{k+1,1}$ is completely determined by $a_{1,1}, \ldots, a_{k,1}$, so there are exactly $i+1$ solutions of the problem for k. By induction, the proof is complete. □

Now we can start computing orbits of $\mathrm{Aut}(\mu_0^n)$ on $\mathrm{Z}^2_{\mathrm{LC}}(\mu_0^n, F)$.

Proposition 4.5.5. *Assume that F is algebraically closed of characteristic zero. For $i \in [n]$ we have*

$$\mathrm{Orb}(\Delta_{i,1}) = \left\{\lambda_1 \Delta_{i,1} + \lambda_2 \Delta_{i-1,1} + \cdots + \lambda_i \Delta_{1,1} \mid \lambda_1, \ldots, \lambda_i \in F, \lambda_1 \neq 0\right\}.$$

Proof. Let $\phi = (\phi_{i,j}) \in \mathrm{Aut}(\mu_0^n)$. Then, by (4.5.4) we have that $\phi \cdot \Delta_{i,1} = \phi_{1,1}^{i+1} \Delta_{i,1} + \sum_{k=1}^{i-1} \phi_{i,k}\phi_{1,1}\Delta_{k,1}$ and as $\phi_{1,1} \neq 0$, the direct inclusion in the statement is proved.

Conversely, given $(\lambda_1, \ldots, \lambda_i) \in F^* \times F^{i-1}$, Lemma 4.5.4 gives $(a_{1,1}, \ldots, a_{i,1}) \in F^* \times F^{i-1}$ such that $\lambda_{i-k+1} = a_{i,k}a_{1,1}$, for all $k \in [i]$. Since $a_{1,1} \neq 0$, there exists $\phi \in \mathrm{Aut}(\mu_0^n)$ such that $\phi_{k,1} = a_{k,1}$, for all $k \in [i]$. For any such $\phi \in \mathrm{Aut}(\mu_0^n)$ and $j \in [i]$, we have

$$a_{i,j} = \sum_{k_1+\cdots+k_j=i} a_{k_1,1} \cdots a_{k_j,1} = \sum_{k_1+\cdots+k_j=i} \phi_{k_1,1} \cdots \phi_{k_j,1} = \phi_{i,j},$$

so $\phi \cdot \Delta_{i,1} = \sum_{k=1}^{i} \phi_{i,k}\phi_{1,1}\Delta_{k,1} = \sum_{k=1}^{i} a_{i,k}a_{1,1}\Delta_{k,1} = \sum_{k=1}^{i} \lambda_{i-k+1,1}\Delta_{k,1}$. This proves the reverse inclusion. □

Recall that the ∇_j, for $j \in [n-1]$, form a basis of $\mathrm{B}^2(\mu_0^n, F)$.

Lemma 4.5.6. *Let* $\phi = \left(\phi_{i,j}\right) \in \mathrm{Aut}\left(\mu_0^n\right)$. *In* $\mathrm{H}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$ *we have* $\phi \cdot \left[\nabla_n\right] = \phi_{1,1}^{n+1}\left[\nabla_n\right]$.

Proof. If $i+j > n+1$, then $\Delta_{i,j}$ does not occur in $\phi \cdot \nabla_n$. Otherwise, for $i+j \leq n+1$, the coefficient of $\Delta_{i,j}$ in the expression for $\phi \cdot \nabla_n$ is

$$\alpha_{i,j} = \sum_{k=i}^{n+1-j} \phi_{k,i}\phi_{n+1-k,j},$$

so that $\phi \cdot \nabla_n = \sum_{i+j\leq n+1} \alpha_{i,j}\Delta_{i,j}$.

Since $\mathrm{Aut}\left(\mu_0^n\right)$ acts on $\mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$ and on $\mathrm{B}^2\left(\mu_0^n, F\right)$, we must have $\phi \cdot \nabla_n \in \mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right) \setminus \mathrm{B}^2\left(\mu_0^n, F\right)$. Thus, by the proof of Proposition 4.5.1, we have $\alpha_{1,k} = \alpha_{2,k-1} = \cdots = \alpha_{k-1,2}$ for every $2 \leq k \leq n$. Moreover, by reparameterising the summation index, we obtain $\alpha_{i,j} = \sum_{\ell=j}^{n+1-i} \phi_{n+1-\ell,i}\phi_{\ell,j} = \alpha_{j,i}$; in particular, $\alpha_{k,1} = \alpha_{1,k}$. It follows that

$$\phi \cdot \nabla_n = \sum_{k=1}^{n} \alpha_{1,k}\nabla_k.$$

Now, as $\left[\nabla_k\right] = 0$, for all $k \in [n-1]$, we get $\phi \cdot \left[\nabla_n\right] = \alpha_{1,n}\left[\nabla_n\right] = \phi_{1,1}^{n+1}\left[\nabla_n\right]$. □

Proposition 4.5.7. *Assume that* F *is algebraically closed of characteristic zero. Let* $\mu \in F^*$. *We have*

$$\mathrm{Orb}\left(\left[\nabla_n + \mu\Delta_{n,1}\right]\right) = \left\{ \lambda_1\left[\nabla_n\right] + \mu\lambda_1\left[\Delta_{n,1}\right] + \sum_{k=2}^{n-1} \lambda_{n+1-k}\left[\Delta_{k,1}\right] \mid \lambda_1, \ldots, \lambda_{n-1} \in F, \lambda_1 \neq 0 \right\}.$$

Proof. Let $\phi = \left(\phi_{i,j}\right) \in \mathrm{Aut}\left(\mu_0^n\right)$. Then, using Lemma 4.5.6 and recalling that $\left[\Delta_{1,1}\right] = \left[\nabla_1\right] = 0$,

$$\phi \cdot \left[\nabla_n + \mu\Delta_{n,1}\right] = \phi_{1,1}^{n+1}\left[\nabla_n\right] + \mu\phi_{n,n}\phi_{1,1}\left[\Delta_{n,1}\right] + \mu\sum_{k=2}^{n-1} \phi_{n,k}\phi_{1,1}\left[\Delta_{k,1}\right]. \quad (4.5.8)$$

Since $\phi_{n,n} = \phi_{1,1}^n$ and $\phi_{1,1} \neq 0$, the direct inclusion in the statement follows.

Conversely, let $\lambda_1, \dots, \lambda_{n-1} \in F$, with $\lambda_1 \neq 0$. Consider the map ρ_{n-1} from Lemma 4.5.4. By that result, for any $\mu \in F^*$, there are $\left(a_{1,1}, \dots, a_{n,1}\right) \in F^* \times F^{n-1}$ such that $\rho_{n-1}\left(a_{1,1}, \dots, a_{n,1}\right) = \left(\lambda_1, \frac{\lambda_2}{\mu}, \dots, \frac{\lambda_{n-1}}{\mu}, 0\right)$. Let $\phi \in \operatorname{Aut}\left(\mu_0^n\right)$ be determined by $\phi_{k,1} = a_{k,1}$, for all $k \in [n]$. Then $\phi_{i,j} = a_{i,j}$ for all $i, j \in [n]$. It follows that $\phi_{1,1}^{n+1} = \phi_{n,n}\phi_{1,1} = \lambda_1$ and similarly $\mu\phi_{n,k}\phi_{1,1} = \lambda_{n+1-k}$ for all $2 \leq k \leq n-1$. Thence, by (4.5.8),

$$\phi \cdot \left[\nabla_n + \mu\Delta_{n,1}\right] = \lambda_1 \left[\nabla_n\right] + \mu\lambda_1 \left[\Delta_{n,1}\right] + \sum_{k=2}^{n-1} \lambda_{n+1-k} \left[\Delta_{k,1}\right],$$

proving the reverse inclusion. □

Proposition 4.5.8. *Assume that F is algebraically closed of characteristic zero. For $2 \leq i \leq n-1$ and $\mu \in F^*$ we have*

$$\begin{aligned}&\operatorname{Orb}\left(\left[\nabla_n + \mu\Delta_{i,1}\right]\right)\\ &= \left\{\lambda^{n+1}\left[\nabla_n\right] + \lambda^{i+1}\mu\left[\Delta_{i,1}\right] + \sum_{j=2}^{i-1} \lambda_{i+1-j}\left[\Delta_{j,1}\right] \mid \lambda_2, \dots, \lambda_{i-1} \in F, \lambda \in F^*\right\}.\end{aligned}$$

Proof. Let $\phi = \left(\phi_{i,j}\right) \in \operatorname{Aut}\left(\mu_0^n\right)$. Then,

$$\phi \cdot \left[\nabla_n + \mu\Delta_{i,1}\right] = \phi_{1,1}^{n+1}\left[\nabla_n\right] + \sum_{k=2}^{i} \phi_{i,k}\phi_{1,1}\mu\left[\Delta_{k,1}\right].$$

So, as before, to prove the result it suffices to assume that $i \geq 3$ and to show that given $\phi_{1,1}, \mu \in F^*$, the elements $\phi_{i,k}\phi_{1,1}\mu$, with $2 \leq k \leq i-1$, can take on arbitrary values in F, for appropriate choices of $\phi_{2,1}, \dots, \phi_{i-1,1} \in F$.

Considering ρ_{i-2}, we know that for any $\lambda_2, \dots, \lambda_{i-1} \in F$ there are $i+1$ solutions to the equation

$$\rho_{i-2}\left(a_{1,1}, \dots, a_{i-1,1}\right) = \left(\phi_{1,1}^{i+1}, \frac{\lambda_2}{\mu}, \dots, \frac{\lambda_{i-1}}{\mu}\right).$$

As $a_{1,1}^{i+1} = \phi_{1,1}^{i+1}$, exactly one of the above solutions satisfies $a_{1,1} = \phi_{1,1}$ and the remainder of the proof goes as before. □

The final ingredient in the proof of Theorem 4.5.2 explains the relevance of the factor group $F_{(i,n)}$, defined in (4.5.5).

Lemma 4.5.9. *Fix $2 \leq i \leq n-1$. For $\mu, \mu' \in F^*$ we have*

$$\mathrm{Orb}\left(\left[\nabla_n + \mu\Delta_{i,1}\right]\right) = \mathrm{Orb}\left(\left[\nabla_n + \mu'\Delta_{i,1}\right]\right) \Leftrightarrow \overline{\mu} = \overline{\mu'},$$

where $\overline{\mu}, \overline{\mu'} \in F_{(i,n)}$.

Remark 4.5.10. In view of this result, it makes sense to write $\mathrm{Orb}\left(\left[\nabla_n + \overline{\mu}\Delta_{i,1}\right]\right)$, for $\mu \in F^*$, and we can parameterise the orbits of the form above by the elements of $F_{(i,n)}$.

Proof. For the direct implication, suppose that $\left[\nabla_n + \mu'\Delta_{i,1}\right] \in \mathrm{Orb}\left(\left[\nabla_n + \mu\Delta_{i,1}\right]\right)$. Then there is some $\lambda \in F^*$ such that $\lambda^{n+1} = 1$ and $\mu' = \lambda^{i+1}\mu$, so $\mu'/\mu \in R(i,n)$.

Conversely, if $\mu' = \lambda^{i+1}\mu$ for some $\lambda \in F^*$ with $\lambda^{n+1} = 1$, then

$$\left[\nabla_n + \mu'\Delta_{i,1}\right] = \lambda^{n+1}\left[\nabla_n\right] + \lambda^{i+1}\mu\left[\Delta_{i,1}\right] \in \mathrm{Orb}\left(\left[\nabla_n + \mu\Delta_{i,1}\right]\right),$$

which shows that the respective orbits coincide. □

We are now ready to conclude our main result of this section.

Proof of Theorem 4.5.2. Let $0 \neq \theta = \lambda_1\left[\nabla_n\right] + \sum_{j=2}^n \lambda_j\left[\Delta_{j,1}\right] \in \mathrm{H}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$. Then:

- If $\lambda_1 = 0$, then $\theta \in \mathrm{Orb}\left(\left[\Delta_{i,1}\right]\right)$, where $i = \max\left\{j \mid \lambda_j \neq 0\right\}$. (See Proposition 4.5.5.)

- If $\lambda_1 \neq 0$ and $\lambda_n \neq 0$, then $\theta \in \mathrm{Orb}\left(\left[\nabla_n + \mu\Delta_{n,1}\right]\right)$, where $\mu = \lambda_n/\lambda_1 \neq 0$. (See Proposition 4.5.7.)

- If $\lambda_1 \neq 0$ and $\lambda_j = 0$ for all $2 \leq j \leq n$, then $\theta \in \mathrm{Orb}\left(\left[\nabla_n\right]\right)$. (See Lemma 4.5.6.)

- If $\lambda_1 \neq 0$, $\lambda_n = 0$ and $\lambda_j \neq 0$ for some $2 \leq j \leq n-1$, define $\mu \in F^*$ as follows: choose some $\lambda \in F^*$ such that $\lambda^{n+1} = \lambda_1$; then take $\mu = \lambda_i/\lambda^{i+1}$. Then, $\theta \in \mathrm{Orb}\left(\left[\nabla_n + \overline{\mu}\Delta_{i,1}\right]\right)$, where $i = \max\left\{2 \leq j \leq n-1 \mid \lambda_j \neq 0\right\}$ (See Proposition 4.5.8.) By Lemma 4.5.9, $\overline{\mu}$ is uniquely determined by θ.

The fact that distinct elements given in the statement of the Theorem yield disjoint orbits follows easily from the description of the orbits in Proposition 4.5.5, Proposition 4.5.7, Proposition 4.5.8, Lemma 4.5.6 and Lemma 4.5.9. □

4.5.3 The orbit decomposition of $T_1\left(\mu_0^n\right)$ in the left-commutative and bicommutative varieties

Observe that $\mathrm{Ann}\left(\mu_0^n\right) = Fe_n$, so in order to get a non-split central extension of μ_0^n with one-dimensional annihilator associated with a cocycle θ, we must have $e_n \notin \mathrm{Ann}(\theta)$. This excludes the orbit representatives 0 and $\Delta_{i,1}$, for $i < n$. Notice also that for non-zero cocycles θ and θ', the one-dimensional spaces $\langle[\theta]\rangle$ and $\left\langle\left[\theta'\right]\right\rangle$ are in the same $\mathrm{Aut}\left(\mu_0^n\right)$-orbit in the corresponding Grassmannian if and only if there is $\lambda \in F^*$ such that $\lambda\left[\theta'\right]$ is in the orbit of $[\theta]$. In particular, if the $\mathrm{Aut}\left(\mu_0^n\right)$-orbit of $[\theta]$ is closed under the multiplicative action of F^*, then $\langle[\theta]\rangle$ and $\left\langle\left[\theta'\right]\right\rangle$ are in the same orbit if and only if $[\theta]$ and $\left[\theta'\right]$ are in the same orbit.

In our context, all orbits are closed under multiplication by non-zero scalars, except for the orbits of the form $\mathrm{Orb}\left(\left[\nabla_n + \overline{\mu}\Delta_{i,1}\right]\right)$, with $2 \le i \le n-1$ and $\overline{\mu} \in F_{(i,n)}$. Given $\epsilon \in F^*$, Proposition 4.5.8 shows that $\epsilon^{n+1}\left[\nabla_n + \overline{\mu}\Delta_{i,1}\right] = \left[\nabla_n + \overline{\epsilon^{n-i}\mu}\Delta_{i,1}\right]$. Since F is algebraically closed, for a fixed $i \le n-1$, ϵ^{n-i} can take on any non-zero value as $\epsilon \in F^*$ varies. Hence, $\left[\nabla_n + \overline{\mu}\Delta_{i,1}\right]$ and $\left[\nabla_n + \overline{\mu'}\Delta_{j,1}\right]$ define one-dimensional spaces in the same $\mathrm{Aut}\left(\mu_0^n\right)$-orbit of the Grassmannian if and only if $i = j$.

Combining all the results from Section 4.5, we obtain our main result below.

Theorem 4.5.11. *Assume that F is algebraically closed of characteristic zero and let $n \ge 2$. The following elements of $\mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$ parameterise the distinct orbits of* $\mathrm{Aut}\left(\mu_0^n\right)$ *on the subspace $T_1\left(\mu_0^n\right)$ of the Grassmannian $G_1\left(\mathrm{H}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)\right)$, i.e. they parameterise the distinct isomorphism classes of non-split left-commutative central extensions of the n-dimensional null-filiform algebra μ_0^n with one-dimensional annihilator:*

(a) $\Delta_{n,1}$;

(b) $\left\{\nabla_n + \mu\Delta_{n,1} \mid \mu \in F\right\}$ $\left(\mu = 0\right.$ *gives the trivial extension* $\left.\mu_0^{n+1}\right)$*;*

(c) $\left\{\nabla_n + \Delta_{i,1} \mid 2 \leq i \leq n-1\right\}$.

In the bicommutative case, the corresponding representatives are:

(Case** $n = 2$**)

(a) $\Delta_{2,1}$;

(b) $\left\{\nabla_2 + \mu\Delta_{2,1} \mid \mu \in F\right\}$
($\mu = 0$ *gives the trivial extension* μ_0^3*).*

(Case** $n > 2$**)

(a) ∇_n *(trivial extension* μ_0^{n+1}*);*

(b) $\nabla_n + \Delta_{2,1}$.

Remark 4.5.12. Note that Theorem 4.5.11 and the previous discussion only make reference to non-split central extensions with one-dimensional annihilator. However, it is also possible to construct non-split left-commutative central extensions of μ_0^n with two-dimensional annihilator, whose isomorphism classes are parameterised by the cocycles $\Delta_{k,1}$ for $2 \leq k \leq n-1$. In the bicommutative case for $n > 2$, also $\Delta_{2,1}$ is a representative of an isomorphism class of non-split central extensions with two-dimensional annihilator.

The explicit description of the multiplication in the central extensions referenced in Theorems 4.5.2 and 4.5.11, in Corollary 4.5.3 and in Remark 4.5.12 can be found in the following table. Only the non-zero products of the basis elements $\left\{e_1, \ldots, e_{n+1}\right\}$ are displayed.

Cocycle	Multiplication, $i, j \in [n]$	
$\Delta_{n,1}$	$e_i e_j = e_{i+j}$	if $i + j \leq n$
	$e_n e_1 = e_{n+1}$	

$\Delta_{k,1}$ $(2 \leq k \leq n-1)$	$e_i e_j = e_{i+j}$ $e_k e_1 = e_{k+1} + e_{n+1}$	if $i + j \leq n$ and $(i, j) \neq (k, 1)$
$\nabla_n + \Delta_{k,1}$ $(2 \leq k \leq n-1)$	$e_i e_j = e_{i+j}$ $e_k e_1 = e_{k+1} + e_{n+1}$	if $i + j \leq n + 1$ and $(i, j) \neq (k, 1)$
$\nabla_n + \mu\Delta_{n,1}$ $(\mu \in F)$	$e_i e_j = e_{i+j}$ $e_n e_1 = (1 + \mu)\, e_{n+1}$	if $i + j \leq n + 1$ and $i \neq n$ ($\mu = 0$ gives the trivial extension)

Table 4.1: Isomorphism classes of one-dimensional non-split left-commutative and bicommutative central extensions of the n-dimensional null-filiform algebra μ_0^n.

4.6 Assosymmetric, Novikov, and left-symmetric central extensions

4.6.1 Assosymmetric central extensions

Recall that an algebra A is said to be *assosymmetric* if it satisfies the identities

$$(xy)\,z - x\,(yz) = (yx)\,z - y\,(xz) = (xz)\,y - x\,(zy)\,,$$

for all $x, y, z \in$ A. Now, to find the assosymmetric central extensions of μ_0^n, we need cocycles $\theta :\ \mu_0^n \times \mu_0^n \to F$ satisfying

$$\theta\left(e_i e_j, e_k\right) - \theta\left(e_i, e_j e_k\right) = \theta\left(e_j e_i, e_k\right) - \theta\left(e_j, e_i e_k\right) = \theta\left(e_i e_k, e_j\right) - \theta\left(e_i, e_k e_j\right),$$

for all $i, j, k \in [n]$. Note that for the algebra μ_0^n, these two equalities reduce to

$$\theta\left(e_i, e_{j+k}\right) = \theta\left(e_j, e_{i+k}\right) \quad \text{and} \quad \theta\left(e_{i+j}, e_k\right) = \theta\left(e_{i+k}, e_j\right),$$

for all $i, j, k \in [n]$, with the usual convention that $e_m = 0$ if $m > n$. This means that every assosymmetric cocycle of μ_0^n is in $\mathrm{Z}^2_{\mathrm{BC}}\left(\mu_0^n, F\right)$. On the other hand, it is easy to see that every element from $\mathrm{Z}^2_{\mathrm{BC}}\left(\mu_0^n, F\right)$ is an assosymmetric cocycle.

4.6.2 Novikov central extensions

Recall that an algebra A is said to be *Novikov* if it satisfies the identities

$$(xy)\,z = (xz)\,y \quad \text{and} \quad (xy)\,z - x\,(yz) = (yx)\,z - y\,(xz)\,,$$

for all $x, y, z \in$ A. Now, to find all the Novikov central extensions of μ_0^n, we require cocycles $\theta : \mu_0^n \times \mu_0^n \to F$ satisfying

$$\theta\left(e_i e_j, e_k\right) = \theta\left(e_i e_k, e_j\right) \quad \text{and} \quad \theta\left(e_i e_j, e_k\right) - \theta\left(e_i, e_j e_k\right) = \theta\left(e_j e_i, e_k\right) - \theta\left(e_j, e_i e_k\right),$$

for all $i, j, k \in [n]$. For the algebra μ_0^n, these identities reduce to

$$\theta\left(e_{i+j}, e_k\right) = \theta\left(e_{i+k}, e_j\right) \quad \text{and} \quad \theta\left(e_i, e_{j+k}\right) = \theta\left(e_j, e_{i+k}\right),$$

with $e_m = 0$ if $m > n$. As these are exactly the identities for the bicommutative cocycles of $\mathrm{Z}^2_{\mathrm{BC}}\left(\mu_0^n, F\right)$, this case also reduces to the bicommutative case.

4.6.3 Left-symmetric central extensions

Recall that an algebra A is said to be *left-symmetric* if it satisfies the identity

$$(xy)\,z - x\,(yz) = (yx)\,z - y\,(xz),$$

for all $x, y, z \in$ A. To find all the left-symmetric central extensions of μ_0^n, we need cocycles $\theta : \mu_0^n \times \mu_0^n \to F$ satisfying

$$\theta\left(e_i e_j, e_k\right) - \theta\left(e_i, e_j e_k\right) = \theta\left(e_j e_i, e_k\right) - \theta\left(e_j, e_i e_k\right),$$

for all $i, j, k \in [n]$. Note that for the algebra μ_0^n, we obtain just the relation

$$\theta\left(e_i, e_{j+k}\right) = \theta\left(e_j, e_{i+k}\right), \quad \text{for all } i, j, k \in [n],$$

where $e_m = 0$ if $m > n$. Thus, θ is the same as a left-commutative cocycle from $\mathrm{Z}^2_{\mathrm{LC}}\left(\mu_0^n, F\right)$, and it follows that this case reduces to the left-commutative case.

CHAPTER 5

Central extensions of axial algebras

In this chapter, we develop a further adaptation of the method of Skjelbred-Sund explained in Chapter 3 and Chapter 4 to construct central extensions of a certain class of algebras which is not defined by polynomial identities, namely axial algebras. Although all the examples throughout the chapter are already known in the literature, it is our hope that this technique will allow us to find new examples in the near future. We also use our method to prove that all axial central extensions (with respect to a maximal set of axes) of simple finite-dimensional Jordan algebras are split.

Introduction

Axial algebras are a recent class of non-associative commutative algebras introduced by Hall, Rehren and Shpectorov [110] in 2015. They can be seen as a certain generalisation of commutative, associative algebras, and as a common frame for Majorana algebras [110, 233], Jordan algebras [111, 112] and other types of algebras appearing in mathematical physics. They are also related to code algebras [53].

The relevance of Majorana and axial algebras lies on the fact that they provide an axiomatic approach to vertex operator algebras (VOAs), complex algebraic structures rooted in theoretical physics. In mathematics, the best-known VOA is the moonshine $V^{\#}$, constructed by Frenkel, Lepowsky and Meurman in [89], and whose automorphism group is the Monster M, the largest sporadic finite simple group. This object shows a link to the theory of modular functions, and was key in the proof of Borcherds [29] of the monstrous moonshine conjecture on the connection between the Monster and modular functions. The rigorous development of the theory of VOAs, an important tool for the proof, is also due to Borcherds [28].

After the cited paper of Hall, Rehren and Shpectorov [110], it began a systematic study of axial algebras. An interesting and active direction in this study is the description of n-generated axial algebras of a certain type. So, two-generated axial algebras of Jordan type η over fields of characteristic different from 2 were classified in [111] by Hall, Rehren and Shpectorov. Rehren proved in [196, 197] that the dimension of primitive two-generated axial algebras of Monster type (α, β) does not exceed 8 if the characteristic of the ground field is not 2 and $\alpha \notin \{2\beta, 4\beta\}$. Later, Franchi, Mainardis and Shpectorov constructed an infinite dimensional two-generated primitive axial algebra of Monster type $\left(2, \frac{1}{2}\right)$, today known as *Highwater algebra* [87], and they classified all two-generated primitive axial algebras of Monster type $(2\beta, \beta)$ over a field of characteristic not 2 in [88]. Also, a classification of primitive symmetric two-generated axial algebras of Monster type was given between Yabe's [235], Franchi and Mainardis' [85] and Franchi, Mainardis and McInroy's [86]. On the other hand, Gorshkov and Staroletov showed that a three-generated primitive axial algebra of Jordan type has dimension at most 9 [104]; Khasraw, McInroy and Shpectorov enumerated all the three-generated axial algebras of Monster type (α, β) of a certain subclass, the so-called 4-algebras [153].

We cite some other directions in the research on axial algebras. Khasraw, McInroy and Shpectorov described the structure of axial algebras [152]. De Medts and Van Couwenberghe introduced axial representations of groups and modules over axial algebras as new tools to study axial algebras [66]. Axial algebras have been also studied from a computational approach in McInroy and Shpectorov's [173] (see also [187, 205]), and from a categorical point of view in De Medts, Peacock, Shpectorov and Van Couwenberghe's [65].

On the other hand, the study of algebras generated by idempotents has a proper interest. Rowen and Segev described all associative and Jordan algebras generated by two idempotents [199]; Brešar proved that a finite-dimensional (unital) algebra is zero product determined if and only if it is generated by idempotents [34]; Hu and Xiao proved that finite-dimensional algebras generated by idempotents can be characterised homologically by their irreducible modules [117], and so on.

This chapter is organised as follows. The introductory Section 5.1 provides some basic definitions about axial algebras. We also give a classification of complex two-dimensional axial algebras and describe some of the main properties of these algebras. Section 5.2 is devoted to a detailed explanation of an adaptation of the Skjelbred-Sund method [209] for the construction of central extensions of axial algebras: we describe the conditions that ensure that a given central extension of an axial algebra will also be axial (Theorem 5.2.12) and prove that an axial algebra with non-zero annihilator is a central extension of another axial algebra of smaller dimension (Theorem 5.2.13). In Section 5.3, we apply the methods developed in Section 5.2 to prove that a complex finite-dimensional simple Jordan algebra does not have non-split axial central extensions with respect to a maximal set of axes.

Unless otherwise stated, all algebras and vector spaces throughout the paper are assumed to be of arbitrary dimension and over an arbitrary field F. The generating sets for the algebras are assumed to be finite.

5.1 Preliminaries on axial algebras

Let F be a field, $\mathcal{F} \subseteq F$ a subset, and $\star : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$ a symmetric binary operation. The pair $(\mathcal{F}, \star)$ is called a *fusion law* over F, and will be denoted simply by $\mathcal{F}$ whenever there is not possibility of confusion. We say that the fusion law $(\mathcal{F}, \star)$ is contained in the fusion law $(\mathcal{G}, \odot)$ if $\mathcal{F} \subseteq \mathcal{G}$ and, for every $\lambda, \mu \in \mathcal{F}$, it holds that $\lambda \star \mu \subseteq \lambda \odot \mu$. Also, if $\mathcal{F} \subseteq \mathcal{G}$, we will denote by $(\mathcal{G}, \star)$ the fusion law resulting from setting $\lambda \star \mu = \emptyset$ for every $\lambda \in \mathcal{G} \setminus \mathcal{F}$ and every $\mu \in \mathcal{G}$.

The values of any fusion law $(\mathcal{F}, \star)$ can be displayed in a symmetric square table. This is the most common way to represent them; we employ it in Table 5.6 and Table 5.7. Following the conventions in the literature (e.g. [110], [111]), we will abuse notation in the tables by not writing the set symbols in unitary sets, or using a blank entry to mean the empty set. On other occasions, we will limit to write explicitly the relevant products of the fusion laws, as in Table 5.3 and Table 5.5.

Let A be a commutative algebra. For any element $x \in \mathrm{A}$, we denote by $\mathrm{Spec}\,(x)$ the spectrum of the endomorphism $L_x : \mathrm{A} \to \mathrm{A}$, $y \mapsto xy$, and by A^x_λ the eigenspace associated with an eigenvalue $\lambda \in \mathrm{Spec}\,(x)$. If $\mu \notin \mathrm{Spec}\,(x)$, we assume $\mathrm{A}^x_\mu = 0$. Also, given a subset $\mathcal{S} \subseteq \mathrm{Spec}\,(x)$, we denote $\mathrm{A}^x_{\mathcal{S}} = \bigoplus_{\lambda \in \mathcal{S}} \mathrm{A}^x_\lambda$ and $\mathrm{A}^x_\emptyset = \{0\}$.

Let $(\mathcal{F}, \star)$ be a fusion law over F. An element $a \in \mathrm{A}$ is called an $\mathcal{F}$-*axis* if the following conditions hold:

1. a is idempotent;
2. a is semisimple;
3. $\mathrm{Spec}\,(a) \subseteq \mathcal{F}$ and $\mathrm{A}^a_\lambda \mathrm{A}^a_\mu \subseteq \mathrm{A}^a_{\lambda \star \mu}$, for all $\lambda, \mu \in \mathrm{Spec}\,(a)$.

Recall that, if A has finite dimension, by a being semisimple we just mean that L_a is diagonalisable. In the infinite-dimensional case, a must satisfy the next two conditions:

(i) For every $x \in \mathrm{A}$, there exists a finite-dimensional subspace $W_x \subseteq \mathrm{A}$ stable by L_a such that $x \in W_x$.

(ii) For every subspace $W \subseteq \mathrm{A}$ of finite dimension stable by L_a, the restriction $L_a|_W$ is diagonalisable.

An $\mathcal{F}$*-axial* algebra over F is a pair (A, X), where A is a commutative algebra over F and X is a finite set of $\mathcal{F}$-axes that generate A. If the fusion law is clear, we will simply refer to axes and axial algebras.

We will now recall some basic definitions regarding axial algebras. For more information, see for instance [110–112, 152].

Note that, from conditions (1) and (3) above, any axis $a \in \mathrm{A}$ satisfies $\mathrm{A}_1^a \neq 0$. An axis $a \in \mathrm{A}$ is called *primitive* if $\dim \mathrm{A}_1^a = 1$. If an axial algebra (A, X) is generated by primitive axes, then it is called *primitive*. In this case, $\mathrm{A}_1^a \mathrm{A}_\lambda^a \subseteq \mathrm{A}_\lambda^a$, for all $\lambda \in \operatorname{Spec}(a)$.

A two-generated axial algebra $\left(\mathrm{A}, \left\{a_1, a_2\right\}\right)$ is called *symmetric* if it admits a flip, i.e. if there exists an automorphism switching the generating axes a_1 and a_2.

An axial algebra (A, X) is said to be *m-closed* if A is spanned by products of axes of X of length at most m.

Also, we say that an axial algebra (A, X) admits a *Frobenius form* if there exists a (non-zero) bilinear form $(\cdot,\cdot) : \mathrm{A} \times \mathrm{A} \to F$ which associates with the product of A, i.e.

$$(x, yz) = (xy, z)$$

for all $x, y, z \in \mathrm{A}$. Note that this form is necessarily symmetric [110, Proposition 3.5].

The *radical* $R(\mathrm{A}, \mathrm{X})$ of a primitive axial algebra (A, X) is the unique largest ideal of A containing no axes from X. If (A, X) admits a Frobenius form, the radical of the form and $R(\mathrm{A}, \mathrm{X})$ are closely related (see [152]).

Sometimes, the fusion law $(\mathcal{F}, \star)$ is graded by a finite abelian group T, in the sense that there exists a partition $\left\{\mathcal{F}_t \mid t \in T\right\}$ such that for all $s, t \in T$,

$$\mathcal{F}_s \star \mathcal{F}_t \subseteq \mathcal{F}_{st}.$$

In these cases, it is induced a T-grading on A for each axis a, namely $\mathrm{A} = \oplus_{t \in T} \mathrm{A}_{\mathcal{F}_t}^a$.

Let T^* be the group of linear characters of T. For each axis a, there exists a group homomorphism, $\tau_a : T^* \to \mathrm{Aut}\,(\mathrm{A})$, where $\tau_a(\chi) : \mathrm{A} \to \mathrm{A}$, with $\chi \in T^*$, is defined by the linear extension of

$$\begin{aligned} \tau_a(\chi) : \mathrm{A} &\to \mathrm{A} \\ u &\mapsto \chi(t)\,u, \end{aligned}$$

for $u \in A_t^a$.

The automorphisms of the type $\tau_a(\chi)$ are called *Miyamoto automorphisms*, and the image $T_a := \mathrm{Im}\,\tau_a$ is called the *axial subgroup* of $\mathrm{Aut}\,(\mathrm{A})$ corresponding to a. If Z is a set of axes of A, the subgroup

$$G(\mathrm{Z}) := \langle T_a \mid a \in \mathrm{Z} \rangle \subseteq \mathrm{Aut}\,(\mathrm{A})$$

is known as the *Miyamoto group* of A with respect to Z.

In this paper, we will restrict to dealing with C_2-gradings. Note that, in this setting, $T_a \subseteq \mathrm{Aut}\,(\mathrm{A})$ has order two for every axis a. Let us write $T_a = \left\{\mathrm{id}_A, \phi_a\right\}$.

A set of axes Z is called *closed* if $\phi(\mathrm{Z}) = \mathrm{Z}$ for any $\phi \in G(\mathrm{Z})$. The minimal closed set of axes containing Z exists and it is called the *closure* of Z, $\overline{\mathrm{Z}}$.

As an example, we select here the algebras of dimension 2 over $\mathbb{C}$ which are axial from the classification in [151], and provide some information about their basic features.

Example 5.1.1. Consider the set

$$\left\{ \left(\alpha, \beta, \frac{\alpha\beta - (\alpha-1)(\beta-1)}{4\alpha\beta - 1}\right) \mid \alpha, \beta \in \mathbb{C}, \alpha, \beta \neq \frac{1}{2}, \alpha\beta \neq \frac{1}{4} \right\}.$$

Note that there exists an action of $\mathcal{S}_3$ on such set, and choose a set of representatives of the orbits, Δ, such that $\beta \neq 0, 1$, $\alpha + \beta \neq 1$ and $\alpha \neq \beta \neq \frac{\alpha\beta-(\alpha-1)(\beta-1)}{4\alpha\beta-1} \neq \alpha$. Set

$$\kappa = \left\{ (\alpha, \beta) \in \mathbb{C}^2 \mid \alpha, \beta \neq \frac{1}{2}, \alpha\beta \neq \frac{1}{4}, \left(\alpha, \beta, \frac{\alpha\beta - (\alpha-1)(\beta-1)}{4\alpha\beta - 1}\right) \in \Delta \right\}.$$

Also, the cyclic group C_2 acts on $\mathbb{C} \setminus \{0, 1\}$ by taking ${}^{-1}(\alpha) = \alpha^{-1}$. We will fix a certain set of representatives of the orbits under this action and denote it by $\mathbb{C}^*_{>1}$.

Let us denote:

$$
\begin{array}{llllll}
A & : & E_1\left(0,0,0,0\right) & : & e_1e_2 = 0 \\
B & : & E_1\left(-1,-1,-1,-1\right) & : & e_1e_2 = -e_1 - e_2 \\
C\left(\alpha\right)_{\alpha\neq 0,\pm\frac{1}{2},\pm 1} & : & E_1\left(\alpha,\alpha,\alpha,\alpha\right)_{\alpha\neq 0,\pm\frac{1}{2},\pm 1} & : & e_1e_2 = \alpha\left(e_1+e_2\right) \\
D\left(\beta\right)_{(0,\beta)\in\kappa} & : & E_1\left(0,\beta,0,\beta\right)_{(0,\beta)\in\kappa} & : & e_1e_2 = \beta e_2 \\
E\left(\alpha,\beta\right)_{(\alpha,\beta)\in\kappa,\alpha\neq 0} & : & E_1\left(\alpha,\beta,\alpha,\beta\right)_{(\alpha,\beta)\in\kappa,\alpha\neq 0} & : & e_1e_2 = \alpha e_1 + \beta e_2 \\
F & : & E_2\left(\frac{1}{2},0,0\right) & : & e_1e_2 = \frac{1}{2}e_1 \\
G\left(\beta\right)_{\beta\neq 0,\frac{1}{2},1} & : & E_2\left(\frac{1}{2},\beta,\beta\right)_{\beta\neq 0,\frac{1}{2},1} & : & e_1e_2 = \frac{1}{2}e_1 + \beta e_2 \\
H\left(\gamma\right)_{\gamma\in\mathbb{C}^*_{>1},\gamma\neq 2} & : & E_3\left(\frac{1}{2},\frac{1}{2},\gamma\right)_{\gamma\in\mathbb{C}^*_{>1},\gamma\neq 2} & : & e_1e_2 = \frac{1}{2}\gamma e_1 + \frac{1}{2\gamma}e_2 \\
I & : & E_5\left(\frac{1}{2}\right) & : & e_1e_2 = \frac{1}{2}\left(e_1+e_2\right),
\end{array}
$$

where $e_1e_1 = e_1$ and $e_2e_2 = e_2$ in all instances.

All the previous algebras, alongside with certain sets of generating axes, are two-closed axial algebras. The following Table 5.2 summarises information about some of their basic features.

We limit to provide minimal set of axes X such that (A,X) is axial. The fusion laws are displayed in Table 5.3, with the following conventions: we only write the non-zero products, and we assume that $1 \star \lambda = \lambda$ for all $\lambda \in \mathcal{F}$, $\lambda \neq 0$. Write

$$
\begin{array}{lcr}
a_3 & = & e_1 + e_2; \\
a_4 & = & -\left(e_1+e_2\right); \\
a_5 & = & \frac{1}{1+2\alpha}\left(e_1+e_2\right); \\
a_6 & = & e_1 + \left(1-2\beta\right)e_2; \\
a_7 & = & \frac{1-2\alpha}{1-4\alpha\beta}e_1 + \frac{1-2\beta}{1-4\alpha\beta}e_2; \\
a_\alpha & = & \alpha e_1 + \left(1-\alpha\right)e_2,
\end{array}
$$

where α, β are elements of $\mathbb{C}$ whose requirements vary from case to case, and denote by $\mathfrak{F}_2$ the free group generated by two involutions. Note that, regarding Frobenius forms, we just offer an example for each axial algebra.

A	X	$(\mathcal{F}, \star)$	Symmetric	$(\cdot, \cdot)$
A	$\{e_1, e_2\}$	$(\mathcal{F}_A, \star_A)$	Yes	$(e_1, e_1) = 1$
	$\{e_1, a_3\}$		No	$(e_1, e_2) = 0$
	$\{e_2, a_3\}$			$(e_2, e_2) = 1$
B	$\{e_1, e_2\}$	$(\mathcal{F}_B, \star_B)$	Yes	$(e_1, e_1) = -2$
	$\{e_1, a_4\}$			$(e_1, e_2) = 1$
	$\{e_2, a_4\}$			$(e_2, e_2) = -2$
$C(\alpha)$	$\{e_1, e_2\}$	$(\mathcal{F}_{C1}, \star_{C1})$	Yes	$(e_1, e_1) = (1-\alpha)/\alpha$
	$\{e_1, a_5\}$	$(\mathcal{F}_{C2}, \star_{C2})$	No	$(e_1, e_2) = 1$
	$\{e_2, a_5\}$			$(e_2, e_2) = (1-\alpha)/\alpha$
$D(\beta)$	$\{e_1, e_2\}$	$(\mathcal{F}_{D1}, \star_{D1})$	No	$(e_1, e_1) = 1$
	$\{e_1, a_6\}$	$(\mathcal{F}_{D2}, \star_{D2})$		$(e_1, e_2) = 0$
	$\{e_2, a_6\}$	$(\mathcal{F}_{D3}, \star_{D3})$		$(e_2, e_2) = 0$
$E(\alpha, \beta)$	$\{e_1, e_2\}_{(\alpha \neq 1)}$	$(\mathcal{F}_{E1}, \star_{E1})$	No	$(e_1, e_1) = (1-\beta)/\alpha$
	$\{e_1, a_7\}$	$(\mathcal{F}_{E2}, \star_{E2})$		$(e_1, e_2) = 1$
	$\{e_2, a_7\}_{(\alpha \neq 1)}$	$(\mathcal{F}_{E3}, \star_{E3})$		$(e_2, e_2) = (1-\alpha)/\beta$
F	$\{e_1, e_2\}$	$(\mathcal{F}_F, \star_F)$	No	$(e_1, e_1) = 0$
				$(e_1, e_2) = 0$
				$(e_2, e_2) = 1$
$G(\beta)$	$\{e_1, e_2\}$	$(\mathcal{F}_G, \star_G)$	No	$(e_1, e_1) = 2(1-\beta)$
				$(e_1, e_2) = 1$
				$(e_2, e_2) = 1/2\beta$
$H(\gamma)$	$\{e_1, e_2\}$	$(\mathcal{F}_H, \star_H)$	$\gamma = -1$	$(e_1, e_1) = (2\gamma - 1)/\gamma^2$
				$(e_1, e_2) = 1$
				$(e_2, e_2) = (2-\gamma)\gamma$

I	$\{a_\alpha, a_\beta\}_{\alpha \neq \beta}$	$(\mathcal{F}_I, \star_I)$	$\alpha \neq -\beta$	$(e_1, e_1) = 1$ $(e_1, e_2) = 1$ $(e_2, e_2) = 1$

Table 5.2: Complex axial algebras of dimension 2.

Fusion law	$\mathcal{F}$	
$(\mathcal{F}_A, \star_A)$	$\{1, 0\}$	$0 \star_A 0 = 0$
$(\mathcal{F}_B, \star_B)$	$\{1, -1\}$	$(-1) \star_B (-1) = 1$
$(\mathcal{F}_{C1}, \star_{C1})$	$\{1, \alpha\}$	$\alpha \star_{C1} \alpha = \{1, \alpha\}$
$(\mathcal{F}_{C2}, \star_{C2})$	$\{1, \alpha, \lambda\}$	$\alpha \star_{C2} \alpha = \{1, \alpha\}$, $\lambda \star_{C2} \lambda = 1$
$(\mathcal{F}_{D1}, \star_{D1})$	$\{1, \beta, 0\}$	$\beta \star_{D1} \beta = \beta$, $0 \star_{D1} 0 = \{1, 0\}$
$(\mathcal{F}_{D2}, \star_{D2})$	$\{1, \beta, 1-\beta\}$	$\beta \star_{D2} \beta = \beta$, $(1-\beta) \star_{D2} (1-\beta) = 1-\beta$
$(\mathcal{F}_{D3}, \star_{D3})$	$\{1, 1-\beta, 0\}$	$(1-\beta) \star_{D3} (1-\beta) = 1-\beta$, $0 \star_{D3} 0 = \{1, 0\}$
$(\mathcal{F}_{E1}, \star_{E1})$	$\{1, \beta, \lambda\}$	$\beta \star_{E1} \beta = \{1, \beta\}$, $\lambda \star_{E1} \lambda = \{1, \lambda\}$
$(\mathcal{F}_{E2}, \star_{E2})$	$\{1, \alpha, \beta\}$	$\alpha \star_{E2} \alpha = \{1, \alpha\}$, $\beta \star_{E2} \beta = \{1, \beta\}$
$(\mathcal{F}_{E3}, \star_{E3})$	$\{1, \alpha, \lambda\}$	$\alpha \star_{E3} \alpha = \{1, \alpha\}$, $\lambda \star_{E3} \lambda = \{1, \lambda\}$
$(\mathcal{F}_F, \star_F)$	$\{1, 1/2, 0\}$	$1/2 \star_F 1/2 = 1/2$, $0 \star_F 0 = \{1, 0\}$
$(\mathcal{F}_G, \star_G)$	$\{1, \beta, 1/2\}$	$\beta \star_G \beta = \{1, \beta\}$, $1/2 \star_G 1/2 = \{1, 1/2\}$

$(\mathcal{F}_H, \star_H)$	$\left\{1, \frac{1}{2\gamma}, \gamma/2\right\}$	$\frac{1}{2\gamma} \star_H \frac{1}{2\gamma} = \left\{1, \frac{1}{2\gamma}\right\}$, $\gamma/2 \star_H \gamma/2 = \{1, \gamma/2\}$
$(\mathcal{F}_I, \star_I)$	$\{1, 1/2\}$	

Table 5.3: Fusion laws in Table 5.2.

(For $D3$, $\lambda = \frac{1}{1+2\alpha}$, and for $E1$ and $E3$, $\lambda = \frac{1-\alpha-\beta}{1-4\alpha\beta}$).

We indicate now some other properties of the above algebras not displayed in Table 5.2. The only algebras which are not primitive are $(A, \{e_1, a_3\})$ and $(A, \{e_2, a_3\})$; the only ones that have non-zero radical, $(D(\beta), \{e_1, a_6\})$ and $(I, \{a_\alpha, a_\beta\})$, with $R(D(\beta), \{e_1, a_6\}) = \langle e_2\rangle$ and $R(I, \{a_\alpha, a_\beta\}) = \langle e_1 - e_2\rangle$, respectively. For the sake of completeness, we point out that the algebras A, B, $C(\alpha)$, $E(\alpha, \beta)$, $G(\beta)$ and $H(\gamma)$ are in fact simple.

Note that, between the fusion laws of Table 5.3, only $(\mathcal{F}_B, \star_B)$, $(\mathcal{F}_{C2}, \star_{C2})$ and $(\mathcal{F}_I, \star_I)$ admit C_2-gradings. We show now in detail the explicit Miyamoto groups of B, $C(\alpha)$ and I.

Consider first B. The fusion law $(\mathcal{F}_B, \star_B)$ admits the C_2-grading $(\mathcal{F}_B)_1 = \{1\}$, $(\mathcal{F}_B)_{-1} = \{-1\}$. The Miyamoto automorphisms with respect to the axes e_1, e_2 and a_4 are

$$\begin{aligned} \phi_{e_i} : \ & B \to B \\ & e_i \mapsto e_i \\ & e_j \mapsto -e_i - e_j, \end{aligned}$$

for $i, j \in \{1, 2\}$, $i \neq j$, and

$$\begin{aligned} \phi_{a_4} : \ & B \to B \\ & e_1 \mapsto e_2 \\ & e_2 \mapsto e_1. \end{aligned}$$

As a consequence, the Miyamoto group with respect to $\mathrm{X} = \{e_1, e_2\}$ is

$$G(\mathrm{X}) = \left\langle \phi_{e_1}, \phi_{e_2} \mid \phi_{e_1}^2 = \phi_{e_2}^2 = (\phi_{e_1}\phi_{e_2})^3 = (\phi_{e_2}\phi_{e_1})^3 = 1 \right\rangle \simeq \mathcal{S}_3,$$

and also, denoting $\mathrm{X}_i = \{e_i, a_4\}$,

$$G(\mathrm{X}_i) = \left\langle \phi_{e_i}, \phi_{a_4} \mid \phi_{e_i}^2 = \phi_{a_4}^2 = \left(\phi_{e_i}\phi_{a_4}\right)^3 = \left(\phi_{a_4}\phi_{e_i}\right)^3 = 1 \right\rangle \simeq \mathcal{S}_3,$$

for $i \in \{1, 2\}$, $i \neq j$. The closure of any of the generating sets of axes considered in Table 5.2 is $\overline{X} = \{e_1, e_2, a_4\}$.

Regarding $C(\alpha)$, the fusion law $(\mathcal{F}_2, \star_2)$ admits the C_2-grading $(\mathcal{F}_{C2})_1 = \{1, \alpha\}$, $(\mathcal{F}_{C2})_{-1} = \{\lambda\}$. The Miyamoto automorphisms with respect to the axes e_1 and e_2 are $\mathrm{id}_{C(\alpha)}$, whereas

$$\begin{aligned} \phi_{a_5} : C(\alpha) &\to C(\alpha) \\ e_1 &\mapsto e_2 \\ e_2 &\mapsto e_1. \end{aligned}$$

Then, the Miyamoto groups with respect to $\mathrm{X}_i = \{e_i, a_5\}$ are

$$G(\mathrm{X}_i) = \left\langle \phi_{a_5} \mid \phi_{a_5}^2 = 1 \right\rangle \simeq C_2,$$

for $i \in \{1, 2\}$. The closure of both generating sets of axes considered in Table 5.2 is $\overline{X} = \{e_1, e_2, a_5\}$.

Write $\mathrm{X} = \{e_1, e_2\}$. Since $(\mathcal{F}_{C1}, \star_{C1}) \subseteq (\mathcal{F}_{C2}, \star_{C2})$, we could also consider $(C(\alpha), \mathrm{X})$ as an $(\mathcal{F}_{C2}, \star_{C2})$-axial algebra. In this case, the Miyamoto group reduces to $G(\mathrm{X}) = \{\mathrm{id}_{C(\alpha)}\}$, and $\overline{\mathrm{X}} = \mathrm{X}$.

Finally, consider I. Now, the C_2-grading of $(\mathcal{F}_I, \star_I)$ is $(\mathcal{F}_I)_1 = \{1\}$, $(\mathcal{F}_I)_{-1} = \left\{\frac{1}{2}\right\}$. The Miyamoto group with respect to $\mathrm{X} = \{a_\alpha, a_\beta\}_{\alpha \neq \beta}$ is

$$G(\mathrm{X}) = \left\langle \phi_{a_\alpha}, \phi_{a_\beta} \mid \phi_{a_\alpha}^2 = \phi_{a_\beta}^2 = 1 \right\rangle \simeq \mathfrak{F}_2,$$

where

$$\begin{aligned} \phi_{a_\alpha} : I &\to I \\ e_1 &\mapsto (2\alpha - 1) e_1 + 2(1 - \alpha) e_2 \\ e_2 &\mapsto 2\alpha e_1 + (1 - 2\alpha) e_2, \end{aligned}$$

for any $\alpha \in \mathbb{C}$. Also, it holds that $\overline{X} = \{a_{\alpha + \mathbb{Z}(\alpha - \beta)}\}$.

5.2 Central extensions of axial algebras

Let A be an algebra and V a vector space. Let also $\theta\colon \mathrm{A}\times\mathrm{A}\to\mathrm{V}$ be a bilinear map, and define $\mathrm{A}_\theta = \mathrm{A}\oplus\mathrm{V}$, which can be given structure of algebra with the product $[x+v, y+w]_\theta = xy+\theta(x,y)$. It is immediate to check that A_θ is a central extension of dimension $\dim\mathrm{V}$ of A with respect to V. Also, it can be seen that every commutative central extension of A of arbitrary dimension is isomorphic to A_θ for some V and θ in the above conditions.

In the following paragraphs, we recall some basic features of this approach to central extensions which will be needed later (for more details about the Skjelbred-Sund method, see Chapter 3 and Chapter 4 and references therein).

Let $f\colon \mathrm{A}\to\mathrm{V}$ be a linear map and define the bilinear map $\delta f\colon \mathrm{A}\times\mathrm{A}\to\mathrm{V}$ by $\delta f(x,y) = f(xy)$. The set $\{\delta f \mid f\colon \mathrm{A}\to\mathrm{V} \text{ is linear}\}$ is a linear subspace of the bilinear maps from A to V, so we can consider the quotient space. Note that if $\theta' = \theta+\delta f$ for some $f\colon \mathrm{A}\to\mathrm{V}$, then the map $\varphi\colon \mathrm{A}_\theta\to\mathrm{A}_{\theta'}$ defined by $\varphi(x+v) = x+v+f(x)$ is an isomorphism. Therefore, the isomorphy class of A_θ does not depend on the representatives θ of the equivalence class $[\theta]$.

Given a bilinear map $\theta\colon \mathrm{A}\times\mathrm{A}\to\mathrm{V}$ and a basis $\{e_\gamma\}_{\gamma\in\Gamma}$ of V, there exist $|\Gamma|$ unique bilinear maps $\theta_\gamma\colon \mathrm{A}\times\mathrm{A}\to F$ such that $\theta(x,y) = \sum_{\gamma\in\Gamma}\theta_\gamma(x,y)\,e_i$. A central extension A_θ is said to have an *annihilator component* if there exist an ideal I and a subspace of $\operatorname{Ann}\mathrm{A}_\theta$, J, such that $\mathrm{A}_\theta = I\oplus J$. The central extensions with annihilator component are called *split*; without annihilator component, *non-split*. If the dimensions of A and V are finite, for a non-split central extension A_θ, it holds that the set $\{[\theta_\gamma]\}_{\gamma\in\Gamma}$ is linearly independent; the converse is also true under the hypothesis $\operatorname{Ann}\mathrm{A}_\theta = \mathrm{V}$.

In the subsequent, we study when a central extension of an axial algebra is axial, in terms of the bilinear map θ. Let us fix the following notation. We will denote by L_x and L^θ_{x+v} the operators of left multiplication in A and in A_θ, respectively. The spectrum of L^θ_{x+v} will be denoted by $\operatorname{Spec}_\theta(x+v)$. Also, we write $\theta_x^\perp = \{y\in\mathrm{A} \mid \theta(x,y)=0\}$, and denote by $P\colon \mathrm{A}_\theta\to\mathrm{A}$ the natural projection onto A.

The following results are easy consequences of the definitions:

Lemma 5.2.1. *Let* A *be a commutative algebra,* V *a vector space and* $\theta\colon \mathrm{A}\times\mathrm{A}\to\mathrm{V}$ *a bilinear map. Then,* A_θ *is commutative if and only if* θ *is symmetric.*

Lemma 5.2.2. *Let* V *be a vector space and* A *an algebra with an element* $x \in \mathrm{A}$ *such that* $L_x : \mathrm{A} \to \mathrm{A}$ *is diagonalisable. Choose a symmetric bilinear map* $\theta : \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ *such that* $\ker\left(L_x\right) \subseteq \theta_x^{\perp}$. *Then if* $\left\{e_\alpha\right\}_{\alpha \in \mathcal{A}}$ *is a basis of* A *diagonalising* L_x *and* $L_x\left(e_\alpha\right) = \lambda_\alpha e_\alpha$, $\lambda_\alpha \in F$, *for any* α, *we may construct a basis of* A_θ *given by*

$$B := \left\{e_\alpha + \lambda_\alpha^{-1}\theta\left(x, e_\alpha\right) \mid \lambda_\alpha \neq 0\right\} \sqcup \left\{e_\alpha \mid \lambda_\alpha = 0\right\} \sqcup B_{\mathrm{V}}$$

where B_{V} *is any basis of* V. *The basis* B *diagonalises* L_{x+v} *for any* $v \in \mathrm{V}$.

Lemma 5.2.3. *Let* A *be a commutative algebra,* V *a vector space and* $\theta : \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ *a bilinear map. Then,* $\operatorname{Spec}_\theta(x+v) = \operatorname{Spec}(x) \cup \{0\}$ *for all semisimple* $x \in \mathrm{A}$ *and all* $v \in \mathrm{V}$, *and the eigenspaces of* L_{x+v}^{θ} *are*

$$\left(\mathrm{A}_\theta\right)_\lambda^{x+v} = \left\{y + \lambda^{-1}\theta(x, y) \mid y \in \mathrm{A}_\lambda^x\right\},$$

for $\lambda \in \operatorname{Spec}(x)$, $\lambda \neq 0$, *and*

$$\left(\mathrm{A}_\theta\right)_0^{x+v} = \left\{y + w \in \mathrm{A}_\theta \mid y \in \mathrm{A}_0^x \cap \theta_x^{\perp}\right\},$$

recalling that we mean $\mathrm{A}_0^x = 0$ *if* $0 \notin \operatorname{Spec}(x)$.

Furthermore we have:

Lemma 5.2.4. *Let* $B = \left\{e_\alpha + v_\alpha\right\}_{\alpha \in \mathcal{A}}$ *be a basis of* A_θ *diagonalising* L_{x+v}. *Then* $\ker\left(L_x\right) \subseteq \theta_x^{\perp}$ *and* L_x *is diagonalisable.*

Proof. We have $L_{x+v}\left(e_\alpha + v_\alpha\right) = \lambda_\alpha\left(e_\alpha + v_\alpha\right)$, which implies $xe_\alpha = \lambda_\alpha e_\alpha$ and $\theta\left(x, e_\alpha\right) = \lambda_\alpha v_\alpha$ for any $\alpha \in \mathcal{A}$. Note that when $\lambda_\alpha \neq 0$ we have $v_\alpha = \lambda_\alpha^{-1}\theta\left(x, e_\alpha\right)$. Define $S := \left\{\alpha \in \mathcal{A} \mid \lambda_\alpha \neq 0\right\}$ and $T := \mathcal{A} \setminus S$. So

$$B = \left\{e_\alpha + \lambda_\alpha^{-1}\theta\left(x, e_\alpha\right) \mid \alpha \in S\right\} \sqcup \left\{e_\alpha + v_\alpha \mid \alpha \in T\right\}.$$

The set $\left\{e_\alpha\right\}_{\alpha \in \mathcal{A}}$ is a system of linear generators of A and so a suitable subset $\left\{e_\alpha\right\}_{\alpha \in \mathcal{A}'}$ is a basis of A. In this basis we can distinguish those e_α's whose λ_α is non-zero and those whose $\lambda_\alpha = 0$. So we have a basis of A of the form $B' = \left\{e_\alpha \mid \lambda_\alpha \neq 0\right\}_{\alpha \in \mathcal{A}'} \sqcup$

$\{e_\alpha \mid \lambda_\alpha = 0\}_{\alpha\in\mathcal{A}'}$. Take $z \in \ker(L_x)$ and write $z = \sum_\alpha k_\alpha e_\alpha$ with $k_\alpha \in F$ relative to the basis B'. Then

$$0 = xz = \sum_\alpha k_\alpha x e_\alpha = \sum_\alpha k_\alpha \lambda_\alpha e_\alpha$$

where the λ_α's in the last sum are those which are non-zero. Consequently $k_\alpha\lambda_\alpha = 0$, that is, $k_\alpha = 0$. Thus $z = \sum_\alpha k_\alpha e_\alpha$ where the sum is extended to those indices α for which $\lambda_\alpha = 0$. So $\theta(x,z) = \sum_\alpha k_\alpha \theta(x, e_\alpha) = \sum_\alpha k_\alpha \lambda_\alpha v_\alpha = 0$ and $z \in \theta_x^\perp$. The fact that L_x is diagonalisable follows now from the fact that $xe_\alpha = \lambda_\alpha e_\alpha$ for any $\alpha \in \mathcal{A}'$. □

Fix a fusion law $(\mathcal{F}, \star)$. Unless otherwise stated, all axial algebras will be assumed to be axial with respect to $(\mathcal{F}, \star)$.

Let (A, X) an axial algebra, V a vector space and $\theta\colon \mathrm{A}\times\mathrm{A} \to \mathrm{V}$ a symmetric bilinear map such that $\{[\theta_\gamma]\}_{\gamma\in\Gamma}$ are linearly independent. Let $\{\mathrm{X}^i\}_{i\in I}$ be the family of minimal sets of axes that generate A, $\mathrm{X}^i = \left\{a_1^i, \ldots, a_{r^i}^i\right\}$. In particular, each X^i is linearly independent and can be extended to a basis $B^i = \left\{a_j^i\right\}_{j\in J}$ of A.

Set $\omega_j^i = \theta\left(a_j^i, a_j^i\right)$ and define $f_j^i\colon \mathrm{A} \to \mathrm{V}$ by $f_j^i\left(a_k^i\right) = \omega_j^i \delta_{jk}$, for $j = 1, \ldots, r^i$ and $k \in J$. Then, consider

$$\theta^i = \theta - \sum_{j=1}^{r^i} \delta f_j^i.$$

It is clear that $[\theta] = [\theta^i]$; moreover, it holds that $\theta^i\left(a_k^i, a_k^i\right) = 0$ for all $k = 1, \ldots, r^i$:

$$\theta^i\left(a_k^i, a_k^i\right) = \theta\left(a_k^i, a_k^i\right) - \sum_{j=1}^{r^i} \delta f_j\left(a_k^i, a_k^i\right) = \omega_k^i - \sum_{j=1}^{r^i} f_j\left(a_k^i\right) = \omega_k^i - \omega_k^i = 0.$$

For the sake of simplicity, we will drop the superindex i whenever X is assumed to be a minimal set of axes generating A. Also, when X is linearly independent, we can assume without loss of generality that $\theta\left(a_j, a_j\right) = 0$ for all $j = 1, \ldots, r$.

Let us establish another piece of notation. Let $a \in \mathrm{X}$ and $\lambda, \mu \in \mathrm{Spec}(a)$. For $x \in \mathrm{A}_\lambda^a$ and $y \in \mathrm{A}_\mu^a$, write

$$xy = \sum_{0\neq\nu\in\lambda\star\mu} z_\nu + z_0,$$

where $z_\nu \in \mathrm{A}_\nu^a$ and $z_0 \in \mathrm{A}_0^a$.

Proposition 5.2.5. *Let* (A, X) *be an axial algebra,* V *a vector space and* $\theta : \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ *a symmetric bilinear map. Let* $x \in \mathrm{A}$ *be semisimple and take* $v \in \mathrm{V}$*. Then,* $x + v \in \mathrm{A}_\theta$ *is semisimple if and only if*

$$\ker L_x \subseteq \theta_x^{\perp}. \tag{5.2.1}$$

Furthermore, if an axis $a \in \mathrm{X}$ *satisfies condition* (5.2.1)*, the eigenspace decomposition of* A_θ *according to* $a + v$ *follows the fusion law* $(\mathcal{F} \cup \{0\}, \star)$ *if and only if for every* $\lambda, \mu \in \operatorname{Spec}(a)$ *such that* $0 \notin \lambda \star \mu$*, it holds that*

$$\theta(x, y) = \sum_{\nu \in \lambda \star \mu} \nu^{-1} \theta\left(a, z_\nu\right) \tag{5.2.2}$$

for all $x \in \mathrm{A}^a_\lambda$*,* $y \in \mathrm{A}^a_\mu$*.*

Proof. The first part of the proposition follows trivially from Lemmas 5.2.2 and 5.2.4.

Take $a \in \mathrm{X}$ satisfying condition (5.2.1) and $v \in \mathrm{V}$. By Lemma 5.2.3, the eigenspace decomposition of A_θ according to $a+v$ follows the fusion law $(\mathcal{F} \cup \{0\}, \odot)$ for some symmetric binary operation $\odot$.

Throughout the rest of the proof, we will denote $\mathrm{B} = \mathrm{A}_\theta$ for the sake of simplicity. Lemma 5.2.3 gives the description of B^{a+v}_λ for all $\lambda \in \operatorname{Spec}_\theta(a + v)$; moreover, under the present hypotheses, we can particularise

$$\mathrm{B}^{a+v}_0 = \left\{x + v \in \mathrm{B} \mid x \in \mathrm{A}^a_0\right\}.$$

Take $\lambda, \mu \in \operatorname{Spec}(a)$, $x + u \in \mathrm{B}^{a+v}_\lambda$ and $y + w \in \mathrm{B}^{a+v}_\mu$. Then,

$$\begin{aligned}
&[x + u, y + w]_\theta = xy + \theta(x, y) = \\
&\sum_{0 \neq \nu \in \lambda \star \mu} \left(z_\nu + \nu^{-1} \theta\left(a, z_\nu\right)\right) + \left(z_0 + \left(\theta(x, y) - \sum_{0 \neq \nu \in \lambda \star \mu} \nu^{-1} \theta\left(a, z_\nu\right)\right)\right) \\
&\in \sum_{0 \neq \nu \in \lambda \star \mu} \mathrm{B}^{a+v}_\nu \oplus \mathrm{B}^{a+v}_0.
\end{aligned}$$

Then, it is clear that $\mathrm{B}^{a+v}_\lambda \mathrm{B}^{a+v}_\mu \subseteq \mathrm{B}^{a+v}_{\lambda \star \mu}$ if and only if $0 \in \lambda \star \mu$ or condition (5.2.2) holds for every $x \in \mathrm{A}^a_\lambda$ and every $y \in \mathrm{A}^a_\mu$.

Also, if $0 \notin \operatorname{Spec}(a)$, $\mathrm{B}^{a+v}_0 \mathrm{B}^{a+v}_\lambda = \mathrm{B}^{a+v}_0 \mathrm{B}^{a+v}_0 = \{0\}$ for all $\lambda \in \operatorname{Spec}(a)$. The result follows. □

Note that conditions (5.2.1) and (5.2.2) do not depend on the representative of $[\theta]$. Set $\theta' = \theta + \delta f$ for some linear map $f\colon \mathrm{A} \to \mathrm{V}$, and take $x \in \mathrm{A}$ and $a \in \mathrm{X}$. Given $y \in \ker L_x$, we have that $\delta f(x, y) = f(xy) = 0$, and therefore θ' satisfies condition (5.2.1) if and only if θ does. Also, given $\lambda, \mu \in \operatorname{Spec}(a)$ such that $0 \notin \lambda \star \mu$, we can write

$$\delta f(x, y) = f(xy) = \sum_{\nu \in \lambda \star \mu} f\left(z_\nu\right) = \sum_{\nu \in \lambda \star \mu} \nu^{-1} f\left(a z_\nu\right) = \sum_{\nu \in \lambda \star \mu} \nu^{-1} \delta f\left(a, z_\nu\right).$$

We conclude that θ' satisfies condition (5.2.2) if and only if θ does.

Corollary 5.2.6. *Let* (A, X) *be a two-dimensional axial algebra,* V *a vector space and* $\theta\colon \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ *a non-zero symmetric bilinear map. Take an axis* $a \in \mathrm{X}$*. Then, condition* (5.2.1) *is satisfied if and only if* $0 \notin \operatorname{Spec}(a)$*.*

Proof. Assume that $0 \in \operatorname{Spec}(a)$ and that $\{a, b\}$ is a minimal set of axes, with $\theta(a, a) = \theta(b, b) = 0$, and note that $\{a, b\}$ is also a basis of A. Note that $a \in \theta_a^{\perp}$ but $b \notin \theta_a^{\perp}$, as otherwise θ would be the zero map. Then, $\theta_a^{\perp} = \langle a \rangle$. By hypothesis, $\ker L_a$ is non-zero and $a \notin \ker L_a$. It follows that condition (5.2.1) is not satisfied. The converse is trivial. □

We introduce now a notion of *cocycles* for axial algebras. Note that we don't intend to relate them to any theory of cohomology for axial algebras; instead, the choice of the term "cocycle" is motivated because they will help to describe the extensions of axial algebras.

Definition 5.2.7. Let (A, X) be an $(\mathcal{F}, \star)$-axial algebra, V a vector space and $\theta\colon \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ a symmetric bilinear map. We say that θ is a cocycle relative to a subset $\mathrm{X}' \subseteq \mathrm{X}$ if condition (5.2.1) is satisfied for all $a \in \mathrm{X}'$, and if, for every $\lambda, \mu \in \mathcal{F}$ such that $0 \notin \lambda \star \mu$, condition (5.2.2) holds for all $a \in \mathrm{X}'$ such that $\lambda, \mu \in \operatorname{Spec}(a)$, all $x \in \mathrm{A}_\lambda^a$ and all $y \in \mathrm{A}_\mu^a$. The vector space formed by them will be denoted by $\mathrm{Z}\left(\mathrm{A}, \mathrm{V}; \mathrm{X}'\right)$.

The next technical lemma will be needed for the main results of this section.

Lemma 5.2.8. *Let* (A, X) *be an axial algebra,* V *a vector space and* $\theta\colon \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ *a symmetric bilinear map such that* $\left\{\left[\theta_\gamma\right]\right\}_{\gamma \in \Gamma}$ *are linearly independent. Assume that*

$\mathrm{X} = \left\{a_j\right\}_{j=1}^{r}$ *is a minimal set of axes generating* A. *Then,* X *is a minimal generating set for* A_θ.

Proof. Let us denote by $\langle \mathrm{X} \rangle$ the subalgebra of A generated by X, by $\langle \mathrm{X} \rangle_\theta$ the subalgebra of A_θ generated by X. Once we prove that $\mathrm{A} \subseteq \langle \mathrm{X} \rangle_\theta$, we will know that $\langle \mathrm{X} \rangle_\theta = \mathrm{A}_\theta$ by the linear independence of $\left\{[\theta_\gamma]\right\}_{\gamma\in\Gamma}$.

Let $B = \left\{a_j\right\}_{j\in J}$ be a basis of A extending X, and denote $J_r = J \setminus \{1, \dots, r\}$. Since $\mathrm{A} = \langle \mathrm{X} \rangle$, for $j \in J_r$ we can express each a_j as a finite sum $a_j = \sum_{l=1}^{m_j} \alpha_{j,l} \prod \mathrm{X}_{j,l}$, where $\alpha_{j,l} \in F$ and $\prod \mathrm{X}_{j,l}$ denotes a product of elements of X with a certain arrangement of brackets. Set $\prod \mathrm{X}_{j,l} = \left(\prod \mathrm{X}_{j,l}^1\right)\left(\prod \mathrm{X}_{j,l}^2\right)$, where $\prod \mathrm{X}_{j,l}^1$ and $\prod \mathrm{X}_{j,l}^2$ are products of elements of X with strictly smaller length than $\prod \mathrm{X}_{j,l}$. Set also $\omega_j = \sum_{l=1}^{m_j} \alpha_{j,l}\theta\left(\prod \mathrm{X}_{j,l}^1, \prod \mathrm{X}_{j,l}^2\right)$, and define the homomorphism $f : \mathrm{A} \to \mathrm{V}$ by $f\left(a_j\right) = 0$ for $j = 1, \dots, r$, and $f\left(a_j\right) = \omega_j$, for $j \in J_r$. Consider $\theta' = \theta - \delta f$. Since A_θ and $\mathrm{A}_{\theta'}$ are isomorphic and X is preserved by the isomorphism, it is enough to show that $\mathrm{A} \subseteq \langle \mathrm{X} \rangle_{\theta'}$. We proceed by induction in the largest length L of the products $\prod \mathrm{X}_{j,l}$ for $l = 1, \dots, m_j$. If $L = 1$, it is trivial that $a_k \in \langle \mathrm{X} \rangle_{\theta'}$. In the general case,

$$
\begin{aligned}
&\sum_{l=1}^{m_k} \alpha_{k,l}\left(\left[\prod \mathrm{X}_{k,l}^1, \prod \mathrm{X}_{k,l}^2\right]_{\theta'}\right)\\
=&\sum_{l=1}^{m_k} \alpha_{k,l}\prod \mathrm{X}_{k,l} + \sum_{l=1}^{m_k} \alpha_{k,l}\theta'\left(\prod \mathrm{X}_{k,l}^1, \prod \mathrm{X}_{k,l}^2\right) - \sum_{l=1}^{m_k} \alpha_{k,l}\left(\delta f\left(\prod \mathrm{X}_{k,l}^1, \prod \mathrm{X}_{k,l}^2\right)\right)\\
=&a_k + \omega_k - f\left(\sum_{l=1}^{m_k} \alpha_{k,l}\prod \mathrm{X}_{k,l}\right) = a_k + \omega_k - f\left(a_k\right) = a_k,
\end{aligned}
$$

and by induction $a_k \in \langle \mathrm{X} \rangle_{\theta'}$.

Finally, we prove the minimality of X. If there existed a subset $\mathrm{X}' \subsetneq \mathrm{X}$ generating A_θ, $P\left(\mathrm{X}'\right) = \mathrm{X}' \subsetneq \mathrm{X}$ would be a set of axes generating $P\left(\mathrm{A}_\theta\right) = \mathrm{A}$, a contradiction. □

We put together all the previous results in the following proposition.

Proposition 5.2.9. *Let* (A, X) *be an axial algebra,* V *a vector space and* $\theta : \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ *a symmetric bilinear map such that* $\left\{\left[\theta_\gamma\right]\right\}_{\gamma \in \Gamma}$ *are linearly independent. Assume that* $\mathrm{X} = \left\{a_j\right\}_{j=1}^{r}$ *is a minimal set of axes generating* A*. The pair* $\left(\mathrm{A}_\theta, \mathrm{X}\right)$ *is* $(\mathcal{F} \cup \{0\}, \odot)$*-axial if and only if condition* (5.2.1) *holds for all* $j = 1, \ldots, r$*, for some fusion law* $(\mathcal{F} \cup \{0\}, \odot)$ *containing* $(\mathcal{F}, \star)$*. Furthermore, we can take* $\odot = \star$ *if and only if* $\theta \in \mathrm{Z}(\mathrm{A}, \mathrm{V}; \mathrm{X})$*.*

Proof. As X is minimal, we may assume that $\theta\left(a_j, a_j\right) = 0$ for all $j = 1, \ldots, r$. Simply observe that the elements a_j are idempotents in A_θ for $j = 1, \ldots, r$, and use Lemma 5.2.3, Lemma 5.2.8 and Proposition 5.2.5. □

Note that, if $(\mathcal{F}, \star)$ is a minimal fusion law for (A, X) in the sense that (A, X) is not axial for any fusion law strictly contained in $(\mathcal{F}, \star)$, $(\mathcal{F} \cup \{0\}, \odot)$ is minimal for $\left(\mathrm{A}_\theta, \mathrm{X}\right)$, too.

Example 5.2.10. Consider the two-dimensional axial algebras over $\mathbb{C}$ described in Example 5.1.1. By Corollary 5.2.6, only $\boldsymbol{B}$, $\boldsymbol{C}(\alpha)$, $\boldsymbol{D}(\beta)$ (with respect to $\left\{e_1, a_6\right\}$), $\boldsymbol{E}(\alpha, \beta)$, $\boldsymbol{G}(\beta)$, $\boldsymbol{H}(\gamma)$ and $\boldsymbol{I}$ admit an axial extension. We select their commutative non-split central extensions of dimension 1 from the classification in [44]: all of them are given by the representative θ determined by $\theta\left(e_i, e_i\right) = 0$ for $i \in \{1, 2\}$ and $\theta\left(e_1, e_2\right) = 1$. Now we can apply Proposition 5.2.9 to find out their axial structures, shown in Table 5.4. The corresponding fusion laws are displayed in Table 5.5, with the same conventions as in Table 5.3.

A_θ	X	$(\mathcal{G}, \odot)$	$\theta \in \mathrm{Z}(\mathrm{A}, \mathrm{V}; \mathrm{X})$
$\boldsymbol{B}_\theta$	$\left\{e_1, e_2\right\}$	$\left(\mathcal{F}_B \cup \{0\}, \odot_B\right)$	No
	$\left\{e_1, a_4\right\}$		
	$\left\{e_2, a_4\right\}$		
$\boldsymbol{C}(\alpha)_\theta$	$\left\{e_1, e_2\right\}$	$\left(\mathcal{F}_{C1} \cup \{0\}, \odot_{C1}\right)$	No
	$\left\{e_1, a_5\right\}$	$\left(\mathcal{F}_{C2} \cup \{0\}, \odot_{C2}\right)$	
	$\left\{e_2, a_5\right\}$		
$\boldsymbol{D}(\beta)_\theta$	$\left\{e_1, a_6\right\}$	$\left(\mathcal{F}_{D2} \cup \{0\}, \odot_{D2}\right)$	No

$E(\alpha,\beta)_\theta$	$\{e_1,e_2\}_{(\alpha\neq 1)}$	$(\mathcal{F}_{E1}\cup\{0\},\odot_{E1})$	No
	$\{e_1,a_7\}$	$(\mathcal{F}_{E2}\cup\{0\},\odot_{E2})$	
	$\{e_2,a_7\}_{(\alpha\neq 1)}$	$(\mathcal{F}_{E3}\cup\{0\},\odot_{E3})$	
$G(\beta)_\theta$	$\{e_1,e_2\}$	$(\mathcal{F}_{G}\cup\{0\},\odot_{G})$	No
$H(\gamma)_\theta$	$\{e_1,e_2\}$	$(\mathcal{F}_{H}\cup\{0\},\odot_{H})$	No
I_θ	$\{a_\alpha,a_\beta\}_{\alpha\neq\beta}$	$(\mathcal{F}_{I}\cup\{0\},\odot_{I})$	No

Table 5.4: Axial central extensions of complex axial algebras of dimension 2.

Fusion law	$\mathcal{F}$	
$(\mathcal{F}_B\cup\{0\},\odot_B)$	$\{1,-1,0\}$	$(-1)\odot_B(-1)=\{1,0\}$
$(\mathcal{F}_{C1}\cup\{0\},\odot_{C1})$	$\{1,\alpha,0\}$	$\alpha\odot_{C1}\alpha=\{1,\alpha,0\}$
$(\mathcal{F}_{C2}\cup\{0\},\odot_{C2})$	$\{1,\alpha,\lambda,0\}$	$\alpha\odot_{C2}\alpha=\{1,\alpha,0\}$ $\lambda\odot_{C2}\lambda=\{1,0\}$
$(\mathcal{F}_D\cup\{0\},\odot_D)$	$\{1,\beta,1-\beta,0\}$	$\beta\odot_D\beta=\{\beta,0\}$ $(1-\beta)\odot_D(1-\beta)=1-\beta$
$(\mathcal{F}_{E1}\cup\{0\},\odot_{E1})$	$\{1,\beta,\lambda,0\}$	$\beta\odot_{E1}\beta=\{1,\beta,0\}$ $\lambda\odot_{E1}\lambda=\{1,\lambda,0\}$
$(\mathcal{F}_{E2}\cup\{0\},\odot_{E2})$	$\{1,\alpha,\beta,0\}$	$\alpha\odot_{E2}\alpha=\{1,\alpha,0\}$ $\beta\odot_{E2}\beta=\{1,\beta,0\}$
$(\mathcal{F}_{E3}\cup\{0\},\odot_{E3})$	$\{1,\alpha,\lambda,0\}$	$\alpha\odot_{E3}\alpha=\{1,\alpha,0\}$ $\lambda\odot_{E3}\lambda=\{1,\lambda,0\}$
$(\mathcal{F}_G\cup\{0\},\odot_G)$	$\{1,\beta,1/2,0\}$	$\beta\odot_G\beta=\{1,\beta,0\}$ $1/2\odot_G 1/2=\{1,1/2,0\}$
$(\mathcal{F}_H\cup\{0\},\odot_H)$	$\left\{1,\frac{1}{2\gamma},\gamma/2,0\right\}$	$\frac{1}{2\gamma}\odot_H\frac{1}{2\gamma}=\left\{1,\frac{1}{2\gamma},0\right\}$ $\gamma/2\odot_H\gamma/2=\{1,\gamma/2,0\}$
$(\mathcal{F}_I\cup\{0\},\odot_I)$	$\{1,1/2,0\}$	$1/2\odot_I 1/2=0$

Table 5.5: Fusion laws in Table 5.4.

We highlight now that, in dimension greater than 2, it is possible to simultaneously have $0 \in \operatorname{Spec}(a)$ and condition (5.2.1) satisfied (cf. Corollary 5.2.6). Indeed, consider the axial algebra $(\mathrm{A}, \mathrm{X}) = \left(I_\theta, \left\{e_1, e_2\right\}\right)$, the axis $a = e_1$ and the bilinear map $\theta' : \mathrm{A} \times \mathrm{A} \to \mathbb{C}$ defined by $\theta'\left(e_2, e_3\right) = 1$ as the only non-zero slot (note that $[\theta'] \neq 0$). Then, $\left(\theta'\right)_a^{\perp} = \mathrm{A}$, so condition (5.2.1) is satisfied. However, $0 \in \operatorname{Spec}(a)$, since $ae_3 = 0$.

Note that none of the bilinear maps θ of Example 5.2.10 is a cocycle in the sense of Definition 5.2.7. We present now an example, already known in the literature (see [235, Section 3.5]), to illustrate that this is not necessarily the case.

Example 5.2.11. Let F be a field of characteristic not 2, and A the 4-dimensional algebra over F with basis $\left\{a_{-1}, a_0, a_1, a_2\right\}$ and commutative product given by

$$\begin{aligned}
a_i a_{i+1} &= 2\left(a_i + a_{i+1}\right) - \frac{1}{2}\left(a_{-1} + 2a_0 + 2a_1 + a_2\right), \quad i = -1, 0, 1;\\
a_{-1}a_2 &= \frac{1}{2}\left(a_{-1} + a_2\right);\\
a_i a_{i+2} &= a_{i-1} + a_i - a_{i+1}, \quad i = -1, 0,
\end{aligned}$$

where we understand that $a_{-2} = -a_{-1} + a_1 + a_2$. It is routine to check that $\left(\mathrm{A}, \left\{a_0, a_1\right\}\right)$ is an axial algebra of Monster type $\left(2, \frac{1}{2}\right)$ (i.e. regarding the fusion law $\mathcal{M}\left(2, \frac{1}{2}\right)$ displayed in Table 5.6) with eigenspaces

$$\begin{aligned}
\mathrm{A}_1^{a_0} &= F a_0;\\
\mathrm{A}_0^{a_0} &= F\left(a_{-1} + 2a_0 - a_1 - 2a_2\right) =: Fu;\\
\mathrm{A}_2^{a_0} &= F\left(a_{-1} - a_1\right) =: Fv;\\
\mathrm{A}_{1/2}^{a_0} &= F\left(a_{-1} - a_2\right) =: Fw;\\
\mathrm{A}_1^{a_1} &= F a_1;\\
\mathrm{A}_0^{a_1} &= F\left(2a_{-1} + a_0 - 2a_1 - a_2\right) =: Fu';\\
\mathrm{A}_2^{a_1} &= F\left(a_0 - a_2\right) =: Fv';\\
\mathrm{A}_{1/2}^{a_1} &= F\left(a_{-1} - a_2\right) = Fw.
\end{aligned}$$

Take a symmetric bilinear map $\theta : \mathrm{A} \times \mathrm{A} \to F$ with $[\theta] \neq 0$. Then, θ is a cocycle relative to $\left\{a_0, a_1\right\}$ if and only if the following equations are satisfied:

$\star$	1	0	2	$1/2$
1	1		2	$1/2$
0		0	2	$1/2$
2	2	2	$\{1,0\}$	$1/2$
$1/2$	$1/2$	$1/2$	$1/2$	$\{1,0,2\}$

Table 5.6: Fusion law $\mathcal{M}(2,1/2)$.

$$\begin{aligned}
&\theta\left(a_0,u\right)=0;\\
&\theta(u,v)=\frac{1}{2}\theta\left(a_0,uv\right);\\
&\theta(u,w)=2\theta\left(a_0,uw\right);\\
&\theta(v,w)=2\theta\left(a_0,vw\right);\\
&\theta\left(a_1,u'\right)=0;\\
&\theta\left(u',v'\right)=\frac{1}{2}\theta\left(a_1,u'v'\right);\\
&\theta\left(u',w\right)=2\theta\left(a_1,u'w\right);\\
&\theta\left(v',w\right)=2\theta\left(a_1,v'w\right).
\end{aligned}\tag{5.2.3}$$

Assuming $\theta\left(a_i,a_i\right)=0$ for $i=-1,\dots,2$, the equations (5.2.3) give rise to an easy system with solution

$$\begin{aligned}
&\theta\left(a_{-1},a_1\right)=\theta\left(a_0,a_2\right)=0,\\
&\theta\left(a_{-1},a_0\right)=\theta\left(a_{-1},a_2\right)=\theta\left(a_0,a_1\right)=\theta\left(a_1,a_2\right).
\end{aligned}$$

Take θ in such conditions and set $\theta\left(a_{-1},a_0\right)=1$ so that $[\theta]\neq 0$. Then $\theta\in \mathrm{Z}\left(\mathrm{A},F;\left\{a_0,a_1\right\}\right)$ and $\left(\mathrm{A}_\theta,\left\{a_0,a_1\right\}\right)$ is an axial algebra of Monster type $\left(2,\frac{1}{2}\right)$ by Proposition 5.2.9. In particular, A_θ is the algebra $\mathrm{IV}_3\left(\frac{1}{2},2\right)$ of [235].

The next result deals with the general case in which the set X of generating axes of (A, X) is not minimal.

Theorem 5.2.12. *Let* (A, X) *be an axial algebra,* V *a vector space and* $\theta : \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ *a symmetric bilinear map such that* $\{[\theta_\gamma]\}_{\gamma \in \Gamma}$ *are linearly independent. The pair* $(\mathrm{A}_\theta, \mathrm{X}^i)$ *is* $(\mathcal{F} \cup \{0\}, \odot^i)$*-axial if and only if condition* (5.2.1) *holds for all* $a^i_j \in \mathrm{X}^i$*,* $j = 1, \ldots, r^i$*, for some fusion law* $(\mathcal{F} \cup \{0\}, \odot^i)$ *containing* $(\mathcal{F}, \star)$*. Furthermore, we can take* $\odot^i = \star$ *if and only if* $\theta \in \mathrm{Z}(\mathrm{A}, \mathrm{V}; \mathrm{X}^i)$*.*

Also, define the set

$$\mathrm{Y} = \{a + \theta(a, a) \mid a \in \mathrm{X} \text{ satisfies condition } (5.2.1)\}.$$

For every $\mathrm{Y}' \subseteq \mathrm{Y}$ *such that there exists* $i \in I$ *with* $\mathrm{X}^i \subseteq P(\mathrm{Y}')$*,* $(\mathrm{A}_\theta, \mathrm{Y}')$ *is* $(\mathcal{F} \cup \{0\}, \odot)$*-axial for some fusion law* $(\mathcal{F} \cup \{0\}, \odot)$ *containing* $(\mathcal{F}, \star)$*. We can take* $\odot = \star$ *if and only if* $\theta \in \mathrm{Z}(\mathrm{A}, \mathrm{V}; P(\mathrm{Y}'))$*.*

Proof. The first part follows directly from Proposition 5.2.9.

Regarding the second part, it is clear that Y is composed of idempotent elements. By Lemma 5.2.8, the set $\left\{a^i_j + \theta\left(a^i_j, a^i_j\right)\right\}_{j=1}^{r^i}$ generates A_θ, and consequently, so does Y'. Lemma 5.2.2 ensures that the elements of Y' are semisimple too. Finally, given $\lambda, \mu \in \mathcal{F}$, set $\lambda \odot \mu = \lambda \star \mu$ if θ is a cocycle relative to $P(\mathrm{Y}')$ or $\lambda \odot \mu = \lambda \star \mu \cup \{0\}$ otherwise, and $0 \odot 0 = 0 \odot \lambda = \emptyset$ for all $\lambda \in \mathcal{F}$ if $0 \notin \mathcal{F}$. Using Proposition 5.2.5, it is obvious that the fusion law $(\mathcal{F} \cup \{0\}, \odot)$ defined in this way satisfies the conditions of the theorem. □

Note that, contrary to the situation described after Proposition 5.2.9, if $(\mathcal{F}, \star)$ is a minimal fusion law for (A, X), $(\mathcal{F} \cup \{0\}, \odot)$ does not need to be minimal for $(\mathrm{A}_\theta, \mathrm{Y})$, with the notation of Theorem 5.2.12.

Now, we present an important result that justifies the importance of studying central extensions of axial algebras.

Theorem 5.2.13. *Let* (B, Y) *be an* $(\mathcal{F}, \star)$*-axial algebra with* $\mathrm{Ann}(B) \neq 0$*. Then, there exists another* $(\mathcal{F}, \star)$*-axial algebra* (A, X) *and a cocycle* $\theta \in \mathrm{Z}(\mathrm{A}, \mathrm{Ann}(\mathrm{B}); \mathrm{X})$ *such that* $\mathrm{B} = \mathrm{A}_\theta$*. Also, if* Y *is a minimal generating set of axes for* B*,* X *is a minimal generating set of axes for* A*.*

Proof. Take a linear complement A of $\mathrm{Ann}(\mathrm{B})$ and set $P_\mathrm{A} : \mathrm{B} \to \mathrm{A}$ defined by $P_\mathrm{A}(x + v) = x$, with $x \in \mathrm{A}$ and $v \in \mathrm{Ann}(\mathrm{B})$. We endow A with the product

$xy = P_{\mathrm{A}}([x,y])$, where $[,]$ denotes the product in B, in order to give it structure of algebra. Note that, with this structure, P is a homomorphism of algebras: indeed, for all $x+v, y+w \in \mathrm{B}$,

$$P_{\mathrm{A}}([x+v, y+w]) = P_{\mathrm{A}}\left([P_{\mathrm{A}}(x+v), P_{\mathrm{A}}(y+w)]\right) = P_{\mathrm{A}}(x+v)\, P_{\mathrm{A}}(y+w).$$

Set $\mathrm{X} = P_{\mathrm{A}}(\mathrm{Y})$. Since Y is a generating set for B, X generates $P_{\mathrm{A}}(\mathrm{B}) = \mathrm{A}$. Take $a \in \mathrm{X}$ such that $a = P_{\mathrm{A}}(b)$ for a certain $b \in \mathrm{Y}$. Then

$$aa = P_{\mathrm{A}}(b)\, P_{\mathrm{A}}(b) = P_{\mathrm{A}}([b,b]) = P_{\mathrm{A}}(b) = a,$$

and a is idempotent.

Write L_b^{B} for the left multiplication by b operator in B, and $\mathrm{Spec}_{\mathrm{B}}(b)$ for its spectrum. We reserve the notations L_a and $\mathrm{Spec}(a)$ for their correspondences in A. It is clear that $0 \in \mathrm{Spec}_{\mathrm{B}}(b)$ and $\mathrm{Ann}(\mathrm{B}) \subseteq \mathrm{B}_0^b$. Choose a basis $\left\{z_\beta^b\right\}_{\beta\in\mathcal{B}}$ of $\mathrm{Ann}(\mathrm{B})$ and complete it to a basis $\left\{z_\beta^b\right\}_{\beta\in\mathcal{B}'}$ of B formed by eigenvectors of L_b^{B}, with $[b, z_\beta^b] = \lambda_\beta z_\beta^b$. The elements $\left\{P\left(z_\alpha^b\right)\right\}_{\alpha\in\mathcal{B}'\setminus\mathcal{B}}$ form a basis for A, and are in fact eigenvectors of L_a with respect to λ_β. Note that $\mathrm{Spec}(a) = \mathrm{Spec}_{\mathrm{B}}(b)$ if and only if $\mathrm{B}_0^b \neq \mathrm{Ann}(\mathrm{B})$; otherwise, $\mathrm{Spec}(a) = \mathrm{Spec}_{\mathrm{B}}(b) \setminus \{0\}$.

The above explanations show that, for every $\lambda \in \mathrm{Spec}(a)$, $\mathrm{A}_\lambda^a = P_{\mathrm{A}}\left(\mathrm{B}_\lambda^b\right)$, and therefore

$$\mathrm{A}_\lambda^a \mathrm{A}_\mu^a = P_{\mathrm{A}}\left(\mathrm{B}_\lambda^b\right) P_{\mathrm{A}}\left(\mathrm{B}_\mu^b\right) = P_{\mathrm{A}}[\mathrm{B}_\lambda^b, \mathrm{B}_\mu^b] \subseteq P_{\mathrm{A}}\left(\mathrm{B}_{\lambda\star\mu}^b\right) = \mathrm{A}_{\lambda\star\mu}^a,$$

for all $\lambda, \mu \in \mathrm{Spec}(a)$, and assuming that $\mathrm{A}_0^a = \{0\}$ if $0 \notin \mathrm{Spec}(a)$, as always.

Summing up, we have proved that (A, X) is an $(\mathcal{F}, \star)$-axial algebra.

Now, define $\theta\colon \mathrm{A}\times\mathrm{A} \to \mathrm{Ann}(\mathrm{B})$ by $\theta(x,y) = [x,y] - xy$, and construct A_θ in the usual way. For $x+v, y+w \in \mathrm{A}_\theta$, we have that

$$[x+v, y+w]_\theta = xy + \theta(x,y) = [x,y],$$

so $\mathrm{A}_\theta = B$, and $\left(\mathrm{A}_\theta, \mathrm{Y}\right)$ is $(\mathcal{F}, \star)$-axial. Then, Proposition 5.2.5 yields that $\theta \in \mathrm{Z}(\mathrm{A}, \mathrm{Ann}(B); \mathrm{X})$.

Finally, assume that Y is a minimal generating set of axes for B. Take a minimal generating set of axes for A, $\mathrm{X}' \subseteq \mathrm{X}$. Then, X' would also generate $\mathrm{A}_\theta = \mathrm{B}$ by Lemma 5.2.8. It follows that also $\mathrm{Y}' = \left\{b \in \mathrm{Y} \mid P(b) \in \mathrm{X}'\right\} \subsetneq \mathrm{Y}$ would generate B, a contradiction. □

Remark 5.2.14. Note that, in the conditions of Theorem 5.2.13, in some cases we can find another fusion law $(\mathcal{G}, \odot) \subseteq (\mathcal{F}, \star)$ such that (A, X) is $(\mathcal{G}, \odot)$-axial. Namely, if $\mathrm{B}_0^b = \mathrm{Ann}(\mathrm{B})$ for all $b \in \mathrm{Y}$, we can set $\mathcal{G} = \mathcal{F} \setminus \{0\}$ and $\lambda \odot \mu = \lambda \star \mu \setminus \{0\}$ for all $\lambda, \mu \in \mathcal{G}$. On the contrary, assume that there exists $b \in B$ such that $\mathrm{B}_0^b \neq \mathrm{Ann}(\mathrm{B})$. Then, we must set $\mathcal{G} = \mathcal{F}$ and, given $\lambda, \mu \in \mathrm{Spec}_{\mathrm{B}}(b) = \mathrm{Spec}(a)$, we can set $\lambda \odot \mu = \lambda \star \mu \setminus \{0\}$ if and only if $\Pi_0^b\left([\mathrm{B}_\lambda^b, \mathrm{B}_\mu]\right) \subseteq \mathrm{Ann}(\mathrm{B})$, where $\Pi_0^b : \mathrm{B} \to \mathrm{B}_0^b$ is the natural projection; otherwise, we must set $\lambda \odot \mu = \lambda \star \mu$.

The following corollary is a direct consequence of Theorem 5.2.13.

Corollary 5.2.15. *Let* A *be an algebra,* V *a vector space and* $\theta : \mathrm{A}{\times}\mathrm{A} \to \mathrm{V}$ *a bilinear form. If* $\left(\mathrm{A}_\theta, \mathrm{Y}\right)$ *is axial with respect to some fusion law* $(\mathcal{F}, \star)$*, then* $(\mathrm{A}, P(\mathrm{Y}))$ *is also* $(\mathcal{F}, \star)$*-axial.*

We finish this section relating some properties of an axial algebra (A, X) with those of its central extensions. First, we provide an easy lemma whose proof is left to the reader.

Lemma 5.2.16. *Let* (A, X) *be an axial algebra admitting a Frobenius form* $(\cdot, \cdot)$*. Then,* $\mathrm{Ann}(\mathrm{A})$ *is contained in the radical of* $(\cdot, \cdot)$*.*

Proposition 5.2.17. *Let* (A, X) *be an* $(\mathcal{F}, \star)$*-axial algebra admitting an* $(\mathcal{F} \cup \{0\}, \odot)$*-axial central extension, and take* Y *as in Theorem 5.2.12. Then:*

1. (A, X) *is primitive if and only if* $\left(\mathrm{A}_\theta, \mathrm{Y}\right)$ *is primitive.*
2. (A, X) *admits a Frobenius form if and only if* $\left(\mathrm{A}_\theta, \mathrm{Y}\right)$ *admits a Frobenius form.*
3. *The radical of* $\left(\mathrm{A}_\theta, \mathrm{Y}\right)$ *is* $R\left(\mathrm{A}_\theta, \mathrm{Y}\right) = R(\mathrm{A}, \mathrm{X}) \oplus \mathrm{V}$*. Conversely, the radical of* (A, X) *is* $R(\mathrm{A}, \mathrm{X}) = P\left(R\left(\mathrm{A}_\theta, \mathrm{Y}\right)\right)$*.*

4. *If* (A, X) *is m-closed for a certain* $m \in \mathbb{N}$*, then* $\left(\mathrm{A}_\theta, \mathrm{Y}\right)$ *is at most* $(m+1)$*-closed.*

5. *If* (A, X) *is symmetric with flip* τ *and there exists an automorphism* $\varphi \in \mathrm{Aut}\,(\mathrm{V})$ *such that* $\varphi\left(\theta\left(x, y\right)\right) = \theta\left(\tau\left(x\right), \tau\left(y\right)\right)$ *for all* $x, y \in \mathrm{A}$*, then* $\left(\mathrm{A}_\theta, \mathrm{Y}\right)$ *is symmetric with flip* τ_θ *defined by* $\tau_\theta\left(x+v\right) = \tau\left(x\right) + \varphi\left(v\right)$*.*

6. *Assume that both* $(\mathcal{F}, \star)$ *and* $(\mathcal{F} \cup \{0\}, \odot)$ *admit* C_2*-gradings such that the one of* $(\mathcal{F} \cup \{0\}, \odot)$ *contains that of* $(\mathcal{F}, \star)$*. Then, the axial subgroups* $T_a = \left\{\mathrm{id}_{\mathrm{A}}, \phi_a\right\} \subseteq \mathrm{Aut}\,(\mathrm{A})$ *and* $T_{a+\theta(a,a)} = \left\{\mathrm{id}_{\mathrm{A}_\theta}, \phi_{a+\theta(a,a)}\right\} \subseteq \mathrm{Aut}\left(\mathrm{A}_\theta\right)$ *are isomorphic, for all* $a \in \mathrm{X}$*. Moreover, if the assignment* $\phi_a \mapsto \phi_{a+\theta(a,a)}$ *gives rise to a homomorphism between the Miyamoto groups* $\boldsymbol{G}\,(\mathrm{X})$ *and* $\boldsymbol{G}\,(\mathrm{Y})$*, then it is bijective and* $\boldsymbol{G}\,(\mathrm{X}) \simeq \boldsymbol{G}\,(\mathrm{Y})$*.*

Proof.

1. It follows from Lemma 5.2.3.

2. Let $(\cdot, \cdot)$ be a Frobenius form for A. Then, $(\cdot, \cdot)_\theta : \mathrm{A}_\theta \times \mathrm{A}_\theta \to F$ defined by $(x+v, y+w)_\theta = (x, y)$ is a Frobenius form for A_θ.

   Conversely, given a Frobenius form $(\cdot, \cdot)_\theta$ for A_θ, define a bilinear form in A, $(\cdot, \cdot) : \mathrm{A} \times \mathrm{A} \to F$, by $(x, y) = (x, y)_\theta$. By Lemma 5.2.16,

   $$(x, yz) = (x, yz)_\theta = \left(x, [y, z]_\theta\right)_\theta = \left([x, y]_\theta, z\right)_\theta = (xy, z)_\theta = (xy, z)$$

   for all $x, y \in \mathrm{A}$, so $(\cdot, \cdot)$ is a Frobenius form for A.

3. It follows from the definition of radical.

4. It follows from the proof of Lemma 5.2.8.

5. Routine.

6. Routine. □

Example 5.2.18. Proposition 5.2.17 and Example 5.1.1 allow us to obtain some basic properties of the algebras in Example 5.2.10. Note that the additional conditions of 5.2.17(5) only hold for $\left(B, \left\{e_1, e_2\right\}\right)$ and $\left(I, \left\{e_1, e_2\right\}\right)$; in both cases, it suffices to take $\varphi = \mathrm{id}_{\mathbb{C}}$. However, these conditions are not necessary: the map on $\left(B_\theta, \left\{e_1, a_4\right\}\right)$ defined by $\tau\left(e_1\right) = a_4$, $\tau\left(e_2\right) = e_2$ and $\tau\left(e_3\right) = -e_3$ is indeed a flip.

On the other hand, $\left(D\left(\alpha\right), \left\{e_i, a_6\right\}\right)$ for $i = 1, 2$, and (B, X) and (I, X) for all choices X satisfy the additional conditions in 5.2.17(6). Therefore, the Miyamoto groups of $\left(D\left(\alpha\right)_\theta, \left\{e_i, a_6\right\}\right)$ for $i = 1, 2$, $\left(B_\theta, \mathrm{X}\right)$ and $\left(I_\theta, \mathrm{X}\right)$ are isomorphic to C_2, $\mathcal{S}_3$ and $\mathfrak{F}_2$, respectively.

5.3 Axial central extensions of simple Jordan algebras

One of the most well-known features of the variety of Jordan algebras is the Peirce decomposition. This can be naturally expressed in the language of axial algebras: every idempotent of a Jordan algebra is an $\mathcal{J}\left(\frac{1}{2}\right)$-axis, where $\mathcal{J}\left(\frac{1}{2}\right)$ is the fusion law displayed in Table 5.7. Then, every Jordan algebra generated by its idempotent elements is $\mathcal{J}\left(\frac{1}{2}\right)$-axial.

$\star$	1	0	1/2
1	1		1/2
0		0	1/2
1/2	1/2	1/2	$\{1, 0\}$

Table 5.7: Fusion law $\mathcal{J}\left(\frac{1}{2}\right)$.

It turns out (cf. [9, 10]) that every finite-dimensional simple Jordan algebra over $\mathbb{C}$ is generated by idempotents, so we can apply the results of Section 5.2 to study which of their non-split central extensions are $\mathcal{J}\left(\frac{1}{2}\right)$-axial. The aim of this section is to prove the following theorem.

Theorem 5.3.1. *Let $\boldsymbol{J}$ be a finite-dimensional simple Jordan algebra over $\mathbb{C}$. There do not exist $\mathcal{J}\left(\frac{1}{2}\right)$-axial non-split central extensions $\boldsymbol{J}_\theta$ of $\boldsymbol{J}$ with respect to the set*

$$\mathrm{Y} = \{a + \theta(a, a) \mid a \in \mathrm{X} \text{ is semisimple}\}.$$

The significance of Theorem 5.3.1 lies on the fact that it generalizes the non-existence of non-split central extensions in the variety of Jordan algebras of such an algebra $\boldsymbol{J}$ [98].

To prove it, we will rely on the classification of the finite-dimensional simple Jordan algebras over $\mathbb{C}$ [9, 10].

- Type $\mathfrak{A}$: algebras of complex $n \times n$-matrices $\mathcal{M}_n(\mathbb{C})$, with product

$$XY = \frac{1}{2}(X \circ Y + Y \circ X), \tag{5.3.1}$$

  where $\circ$ denotes the usual product of matrices.

- Type $\mathfrak{B}$: algebras of complex symmetric $n \times n$-matrices $Sym_n(\mathbb{C})$, with product given by (5.3.1).

- Type $\mathfrak{C}$: algebras of complex $\boldsymbol{J}_n$-symmetric $n \times n$-matrices

$$Sym_n(\boldsymbol{J}, \mathbb{C}) = \left\{X \in \mathcal{M}_n(\mathbb{C}) \mid \boldsymbol{J}_n^{-1} X^T \boldsymbol{J}_n = X\right\},$$

  where

$$\boldsymbol{J}_n = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix},$$

  with product given by (5.3.1).

- Type $\mathfrak{D}$: algebras with underlying vector space $\mathbb{C}^n$ and product given by

$$xy = \left(x^T e_n\right) y + \left(y^T e_n\right) x - \left(x^T y\right) e_n,$$

  where $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{C}^n$.

- Type $\mathfrak{E}$: the algebra of 3×3-hermitian matrices over $\mathbb{O}_{\mathbb{C}}$
$$Herm_3\left(\mathbb{O}_{\mathbb{C}}\right) = \left\{X \in \mathcal{M}_3\left(\mathbb{O}_{\mathbb{C}}\right) \mid X^T = X^*\right\},$$
where X^* means the conjugate matrix of X, with product given by (5.3.1).

Proof. We will deal with each type of the classification separately. Recall that, in any case, if e is an idempotent in A, then $e + \theta(e,e)$ is an idempotent in A_θ.

- Type $\mathfrak{A}$.

  Let A be an algebra of type $\mathfrak{A}$, V a vector space over $\mathbb{C}$ and $\theta\colon \mathrm{A}\times\mathrm{A} \to \mathrm{V}$ a bilinear map. Consider the idempotents $a_i = E_{ii}$ for $i = 1,\dots,n$, with Peirce decompositions
$$\begin{aligned}
\mathrm{A}_1^{a_i} &= \mathbb{C}a_i;\\
\mathrm{A}_0^{a_i} &= \operatorname{span}\left\{E_{jk} \mid j,k = 1,\dots,n,\ j,k \neq i\right\};\\
\mathrm{A}_{1/2}^{a_i} &= \operatorname{span}\left\{E_{ij}, E_{ki} \mid j,k = 1,\dots,n,\ j,k \neq i\right\},
\end{aligned}$$
  and assume, without loss of generality, that $\theta\left(a_i,a_i\right) = 0$ for all $i = 1,\dots,n$. We apply Proposition 5.2.5 to determine what values θ must have in order that the idempotents a_i are in fact $\mathcal{J}\left(\frac{1}{2}\right)$-axes in A_θ. From condition (5.2.1), it follows that
$$\theta\left(E_{ii}, E_{jk}\right) = 0$$
  for $j,k \neq i$, and from condition (5.2.2), we obtain that
$$\begin{aligned}
&\theta\left(E_{ij}, E_{kl}\right) = 0, \quad j,l \neq i,\ j \neq k,\ (i,j) \neq (k,l);\\
&\theta\left(E_{ij}, E_{jk}\right) = \theta\left(E_{ii}, E_{ik}\right), \quad j,k \neq i.
\end{aligned}$$

  We consider also the idempotents $a_{ij} = E_{ii} + E_{ij}$ for $i,j = 1,\dots,n$, $i \neq j$, with eigenspace decomposition

$$\begin{aligned}
\mathrm{A}_1^{a_{ij}} &= \mathbb{C}a_{ij};\\
\mathrm{A}_0^{a_{ij}} &= \operatorname{span}\left\{E_{ik}-E_{jk}, E_{lk} \mid k,l=1,\dots,n,\ k,l\neq i,\ l\neq j\right\};\\
\mathrm{A}_{1/2}^{a_{ij}} &= \operatorname{span}\left\{E_{ii}-E_{jj}-E_{ji}, E_{ik}, E_{li}+E_{lj} \mid k,l=1,\dots,n,\ k,l\neq i,\ l\neq j\right\},
\end{aligned}$$

and study what values θ must take so that the idempotents $a_{ij}+\theta\left(a_{ij},a_{ij}\right)$ are $\mathcal{J}\left(\frac{1}{2}\right)$-axes in A_θ. We obtain from condition (5.2.1) that

$$\theta\left(E_{ij},E_{ij}\right)=0,$$

and from condition (5.2.2), that

$$\theta\left(E_{ij},E_{ji}\right)=0,$$

for $j\neq i$. Finally, it is easy to check that $\theta=\delta f$ for

$$\begin{aligned}
f:\ &\mathrm{A}\to\mathrm{V}\\
&E_{ij}\mapsto\begin{cases}2\theta\left(E_{ii},E_{ij}\right), & \text{if } i\neq j;\\ 0, \quad \text{if } i=j,\end{cases}
\end{aligned}$$

and therefore $[\theta]=0$.

- Type $\mathfrak{B}$.

  Let A be an algebra of type $\mathfrak{B}$, V a vector space over $\mathbb{C}$ and $\theta:\ \mathrm{A}\times\mathrm{A}\to\mathrm{V}$ a bilinear map. Consider the idempotents $a_i=E_{ii}$ for $i=1,\dots,n$, with Peirce decompositions

$$\begin{aligned}
\mathrm{A}_1^{a_i} &= \mathbb{C}a_i;\\
\mathrm{A}_0^{a_i} &= \operatorname{span}\left\{E_{jk}+E_{kj} \mid j,k\neq i\right\};\\
\mathrm{A}_{1/2}^{a_i} &= \operatorname{span}\left\{E_{ij}+E_{ji} \mid j\neq i\right\},
\end{aligned}$$

and assume, without loss of generality, that $\theta\left(a_i, a_i\right) = 0$ for all $i = 1, \ldots, n$. Applying Proposition 5.2.5, we determine what values θ must have in order that the idempotents a_i are $\mathcal{J}\left(\frac{1}{2}\right)$-axes also in A_θ. From condition (5.2.1),

$$\begin{aligned} \theta\left(E_{ii}, E_{jj}\right) &= 0; \\ \theta\left(E_{ii}, E_{jk} + E_{kj}\right) &= 0, \end{aligned}$$

for $j, k \neq i$, $j \neq k$, and from condition (5.2.2),

$$\begin{aligned} \theta\left(E_{ij} + E_{ji}, E_{kl} + E_{lk}\right) &= 0; \\ \theta\left(E_{ij} + E_{ji}, E_{jk} + E_{kj}\right) &= \theta\left(E_{ii}, E_{ik} + E_{ki}\right), \end{aligned}$$

where the indexes i, j, k, l must take different values.

Now, we take into account the idempotents $a_{ij} = \frac{1}{2}\left(E_{ii} + E_{jj} + E_{ij} + E_{ji}\right)$ for $i, j = 1, \ldots, n$, $i \neq j$, with eigenspace decomposition

$$\begin{aligned} \mathrm{A}_1^{a_{ij}} &= \mathbb{C} a_{ij}; \\ \mathrm{A}_0^{a_{ij}} &= \operatorname{span}\left\{E_{ii} + E_{jj} - E_{ij} - E_{ji}, E_{ik} + E_{ki} - E_{jk} - E_{jk},\right. \\ &\qquad \left. E_{kl} + E_{lk} \mid k, l \neq i, j\right\}; \\ \mathrm{A}_{1/2}^{a_{ij}} &= \operatorname{span}\left\{E_{ii} - E_{jj}, E_{ik} + E_{ki} + E_{jk} + E_{jk} \mid k \neq i, j\right\}. \end{aligned}$$

By condition (5.2.1), we obtain that

$$\theta\left(E_{ij} + E_{ji}, E_{ij} + E_{ji}\right) = 0$$

is a necessary condition so that $a_{ij} + \theta\left(a_{ij}, a_{ij}\right)$ is semisimple in A_θ and follows the fusion law $\mathcal{J}\left(\frac{1}{2}\right)$. Then $\theta = \delta f$ for

$$
\begin{aligned}
f:\ &\mathrm{A} \to \mathrm{V}\\
E_{ij}+E_{ji} &\mapsto \begin{cases} 2\theta\left(E_{ii}, E_{ij}+E_{ji}\right), & \text{if } i \neq j;\\ 0, \quad \text{if } i=j, \end{cases}
\end{aligned}
$$

so $[\theta]=0$.

- Type $\mathfrak{C}$.

  Let A be an algebra of type $\mathfrak{C}$, V a vector space over $\mathbb{C}$ and $\theta:\ \mathrm{A}\times\mathrm{A} \to \mathrm{V}$ a bilinear map. Note that $\left\{E_{ij}+E_{(n+j)(n+i)}, E_{i(n+j)}-E_{j(n+i)}, E_{(n+i)j}-E_{(n+j)i}\right\}_{i,j=1}^{n}$ is a basis of A. As we have done in the previous cases, we will obtain conditions on θ necessary to preserve the semisimplicity of the idempotents of A in A_θ and the fusion law $\mathcal{J}\left(\frac{1}{2}\right)$.

  Consider first the idempotents $a_i = E_{ii}+E_{(n+i)(n+i)}$, with Peirce decomposition

$$
\begin{aligned}
\mathrm{A}_1^{a_i} &= \mathbb{C}a_i;\\
\mathrm{A}_0^{a_i} &= \operatorname{span}\left\{E_{jk}+E_{(n+k)(n+j)}, E_{j(n+k)}-E_{k(n+j)}, E_{(n+j)k}-E_{(n+k)j} \mid j,k \neq i\right\};\\
\mathrm{A}_{1/2}^{a_i} &= \operatorname{span}\left\{E_{ij}+E_{(n+j)(n+i)}, E_{ji}+E_{(n+i)(n+j)}, E_{i(n+j)}-E_{j(n+i)},\right.\\
&\qquad \left. E_{(n+i)j}-E_{(n+j)i} \mid j \neq i\right\}.
\end{aligned}
$$

  Following the same steps as for algebras of type $\mathfrak{A}$, we obtain

$$
\begin{aligned}
&\theta\left(E_{ii}+E_{(n+i)(n+i)}, E_{ii}+E_{(n+i)(n+i)}\right)=0;\\
&\theta\left(E_{ii}+E_{(n+i)(n+i)}, E_{jk}+E_{(n+k)(n+j)}\right)=0, \quad j,k \neq i;\\
&\theta\left(E_{ij}+E_{(n+j)(n+i)}, E_{kl}+E_{(n+l)(n+k)}\right)=0, \quad j,l \neq i,\ j \neq k,\ (i,j)\neq(k,l);\\
&\theta\left(E_{ij}+E_{(n+j)(n+i)}, E_{jk}+E_{(n+k)(n+j)}\right)=\theta\left(E_{ii}+E_{(n+i)(n+i)}, E_{ik}+E_{(n+k)(n+i)}\right),\\
&\hspace{25em} j,k \neq i;\\
&\theta\left(E_{ij}+E_{(n+j)(n+i)}, E_{ij}+E_{(n+j)(n+i)}\right)=0, \quad j \neq i;\\
&\theta\left(E_{ij}+E_{(n+j)(n+i)}, E_{ji}+E_{(n+i)(n+j)}\right)=0, \quad j \neq i.
\end{aligned}
$$

But more information can be obtained from these idempotents. After a careful inspection and the application of conditions (5.2.1) and (5.2.2), respectively, we obtain that

$$\begin{aligned}
&\theta\left(E_{ii}+E_{(n+i)(n+i)}, E_{j(n+k)}-E_{k(n+j)}\right)=0, \quad j,k\neq i,\ j\neq k;\\
&\theta\left(E_{ii}+E_{(n+i)(n+i)}, E_{(n+j)k}-E_{(n+k)j}\right)=0, \quad j,k\neq i,\ j\neq k;
\end{aligned}$$

and

$$\begin{aligned}
&\theta\left(E_{ij}+E_{(n+j)(n+i)}, E_{k(n+l)}-E_{l(n+k)}\right)=0, \quad k,l\neq i,j,\ k\neq l;\\
&\theta\left(E_{ij}+E_{(n+j)(n+i)}, E_{(n+k)l}-E_{(n+l)k}\right)=0, \quad k,l\neq i,j,\ k\neq l;\\
&\theta\left(E_{i(n+j)}-E_{j(n+i)}, E_{k(n+l)}-E_{l(n+k)}\right)=0, \quad j,k,l\neq i,\ k\neq l;\\
&\theta\left(E_{(n+i)j}-E_{(n+j)i}, E_{(n+k)l}-E_{(n+l)k}\right)=0, \quad j,k,l\neq i,\ k\neq l;\\
&\theta\left(E_{i(n+j)}-E_{j(n+i)}, E_{(n+k)l}-E_{(n+l)k}\right)=0, \quad j,k,l\neq i,\ k,l\neq j,\ k\neq l;\\
&\theta\left(E_{i(n+j)}-E_{j(n+i)}, E_{jk}+E_{(n+k)(n+j)}\right)=0, \quad j,k\neq i,\ j\neq k;\\
&\theta\left(E_{i(n+k)}-E_{k(n+i)}, E_{jk}+E_{(n+k)(n+j)}\right)=\theta\left(E_{ii}+E_{(n+i)(n+i)}, E_{i(n+j)}-E_{j(n+i)}\right),\\
&\hspace{25em} j,k\neq i;\\
&\theta\left(E_{(n+i)k}-E_{(n+k)i}, E_{jk}+E_{(n+k)(n+j)}\right)=0 \quad j,k\neq i,\ j\neq k;\\
&\theta\left(E_{(n+i)j}-E_{(n+j)i}, E_{jk}+E_{(n+k)(n+j)}\right)=\theta\left(E_{ii}+E_{(n+i)(n+i)}, E_{(n+i)k}-E_{(n+k)i}\right),\\
&\hspace{25em} j,k\neq i;\\
&\theta\left(E_{i(n+j)}-E_{j(n+i)}, E_{(n+j)k}-E_{(n+k)j}\right)=\theta\left(E_{ii}+E_{(n+i)(n+i)}, E_{ik}+E_{(n+k)(n+i)}\right),\\
&\hspace{25em} j,k\neq i.
\end{aligned}$$

Finally, take the idempotents $a_{ij}=E_{ii}+E_{(n+i)(n+i)}+E_{ij}+E_{(n+j)(n+i)}+E_{i(n+j)}-E_{j(n+i)}$, whose Peirce decomposition is

$$\begin{aligned}
\mathrm{A}_1^{a_{ij}} &=\mathbb{C}a_{ij};\\
\mathrm{A}_0^{a_{ij}} &=\operatorname{span}\big\{E_{ij}+E_{(n+j)(n+i)}+E_{i(n+j)}-E_{j(n+i)}-E_{jj}-E_{(n+j)(n+j)},\\
&\qquad E_{kl}+E_{(n+l)(n+k)}, E_{k(n+l)}-E_{l(n+k)}, E_{(n+k)l}-E_{(n+l)k},\\
&\qquad E_{ik}+E_{(n+k)(n+i)}-E_{jk}-E_{(n+k)(n+j)}, E_{i(n+k)}-E_{k(n+i)}
\end{aligned}$$

$$
\begin{aligned}
&+ E_{kj} + E_{(n+j)(n+k)}, E_{ik} + E_{(n+k)(n+i)} - E_{(n+j)k} + E_{(n+k)j},\\
&E_{i(n+k)} - E_{k(n+i)} - E_{j(n+k)} + E_{k(n+j)} \mid k, l \neq i, j\big\};\\
\mathrm{A}_{1/2}^{a_{ij}} = \operatorname{span}\big\{&E_{ii} + E_{(n+i)(n+i)} - E_{jj} - E_{(n+j)(n+j)} - E_{ji} - E_{(n+i)(n+j)},\\
&E_{ji} + E_{(n+i)(n+j)} - E_{(n+j)i} + E_{(n+i)j}, E_{ij} + E_{(n+j)(n+i)},\\
&E_{i(n+j)} - E_{j(n+i)}, E_{ik} + E_{(n+k)(n+i)}, E_{i(n+k)} - E_{k(n+i)},\\
&E_{jk} + E_{(n+k)(n+j)} - E_{(n+j)k} + E_{(n+k)j} - E_{(n+i)k} + E_{(n+k)i}, E_{kj}\\
&+ E_{(n+j)(n+k)} + E_{ki} + E_{(n+i)(n+k)} - E_{j(n+k)} + E_{k(n+j)} \mid k \neq i, j\big\}.
\end{aligned}
$$

From condition (5.2.2), we obtain

$$
\begin{aligned}
&\theta\left(E_{ij} + E_{(n+j)(n+i)}, E_{i(n+j)} - E_{j(n+i)}\right) = 0;\\
&\theta\left(E_{ij} + E_{(n+j)(n+i)}, E_{(n+i)j} - E_{(n+j)i}\right) = 0;\\
&\theta\left(E_{i(n+j)} - E_{j(n+i)}, E_{(n+i)j} - E_{(n+j)i}\right) = 0;
\end{aligned}
$$

and after that, we can apply condition (5.2.1) to get

$$
\begin{aligned}
&\theta\left(E_{i(n+j)} - E_{j(n+i)}, E_{i(n+j)} - E_{j(n+i)}\right) = 0;\\
&\theta\left(E_{(n+i)j} - E_{(n+j)i}, E_{(n+i)j} - E_{(n+j)i}\right) = 0.
\end{aligned}
$$

In summary, we have that $\theta = \delta f$ for

$$
\begin{aligned}
f :\ & \mathrm{A} \to \mathrm{V}\\
E_{ij} + E_{(n+j)(n+i)} &\mapsto \begin{cases} 2\theta\left(E_{ii} + E_{(n+i)(n+i)}, E_{ij} + E_{(n+j)(n+i)}\right), & \text{if } i \neq j;\\ 0, \quad \text{if } i = j; \end{cases}\\
E_{i(n+j)} - E_{j(n+i)} &\mapsto 2\theta\left(E_{ii} + E_{(n+i)(n+i)}, E_{i(n+j)} - E_{j(n+i)}\right);\\
E_{(n+i)j} - E_{(n+j)i} &\mapsto 2\theta\left(E_{ii} + E_{(n+i)(n+i)}, E_{(n+i)j} - E_{(n+j)i}\right),
\end{aligned}
$$

and therefore $[\theta] = 0$.

- Type $\mathfrak{D}$.

Let A be an algebra of type $\mathfrak{D}$, V a vector space over $\mathbb{C}$ and $\theta :\, \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ a bilinear map. Consider the idempotents $a_i = \frac{1}{2}\left(Ie_i + e_n\right)$ for $i = 1, \ldots, n-1$, where I stands for the imaginary unit. The corresponding Peirce decompositions are

$$\begin{aligned} \mathrm{A}_1^{a_i} &= \mathbb{C}a_i; \\ \mathrm{A}_0^{a_i} &= \operatorname{span}\left\{Ie_i - e_n \mid i \neq n\right\}; \\ \mathrm{A}_{1/2}^{a_i} &= \operatorname{span}\left\{e_j \mid j \neq i, n\right\}. \end{aligned}$$

Assume, without loss of generality, that $\theta\left(a_i, a_i\right) = 0$ for all $i = 1, \ldots, n-1$; it follows that

$$\theta\left(e_i, e_n\right) = I\theta\left(e_n, e_n\right)$$

for $i \neq n$. In order that the idempotents a_i, are $\mathcal{J}\left(\frac{1}{2}\right)$-axes in A_θ, it must hold

$$\theta\left(e_i, e_i\right) = -\theta\left(e_n, e_n\right)$$

for $i \neq n$, by condition (5.2.1), and that

$$\theta\left(e_i, e_j\right) = 0$$

for $i, j \neq n, i \neq j$, by condition (5.2.2). Then, we have that $\theta = \delta f$ for

$$\begin{aligned} f :\, &\mathrm{A} \to \mathrm{V} \\ &e_i \mapsto \begin{cases} I\theta\left(e_n, e_n\right), & \text{if } i \neq n; \\ \theta\left(e_n, e_n\right), & \text{if } i = n, \end{cases} \end{aligned}$$

and therefore $[\theta] = 0$.

- Type $\mathfrak{E}$.

Let us establish some notation: for $x \in \mathbb{O}_{\mathbb{C}}$, write $x = x_0 + \sum_{q=1}^{7} I_q x_q$, with $x_q \in \mathbb{C}$, $X^* = x_0 - \sum_{q=1}^{7} I_q x_q$ and $xE_{ij} = \sum_{q=0}^{7} x_q E_{ij}^q$. Also, take $q, r \in$

$\{1,\ldots,7\}$, $q \neq r$. If $I_q I_r = I_s$, for some $s \in \{1,\ldots,7\}$, write $q \cdot r = s$, $s/r = q$ and $q \backslash s = r$; if $I_q I_r = -I_s$, write $q \cdot r = -s$, $s/r = -q$ and $q \backslash s = -r$. Finally, for $q = 1,\ldots,7$, set $E_{ij}^{-q} = -E_{ij}^{q}$.

Let A be the 27-dimensional algebra $Herm_3\left(\mathbb{O}_{\mathbb{C}}\right)$, and set the basis

$$\left\{E_{ii}^0, E_{ij}^0 + E_{ji}^0, E_{ij}^q - E_{ji}^q\right\}_{1\leq i<j\leq 3, t=1,\ldots,7}.$$

The idempotents $a_i = E_{ii}^0$ have Peirce decomposition

$$\begin{aligned}
\mathrm{A}_1^{a_i} &= \mathbb{C}a_i;\\
\mathrm{A}_0^{a_i} &= \operatorname{span}\left\{E_{jj}^0, E_{jk}^0 + E_{kj}^0, E_{jk}^q - E_{kj}^q \mid j,k \neq i,\ j \neq k,\ q = 1,\ldots,7\right\};\\
\mathrm{A}_{1/2}^{a_i} &= \operatorname{span}\left\{E_{ij}^0 + E_{ji}^0, E_{ij}^q - E_{ji}^q \mid j \neq i,\ q = 1,\ldots,7\right\}.
\end{aligned}$$

Consider also a complex vector space V a bilinear map $\theta\colon \mathrm{A}\times\mathrm{A} \to \mathrm{V}$. Without loss of generality, we will assume that $\theta\left(a_i, a_i\right) = 0$. Applying condition (5.2.1), we know that we must impose

$$\begin{aligned}
\theta\left(E_{ii}^0, E_{jj}^0\right) &= 0;\\
\theta\left(E_{ii}^0, E_{jk}^0 + E_{kj}^0\right) &= 0;\\
\theta\left(E_{ii}^0, E_{jk}^q - E_{kj}^q\right) &= 0,
\end{aligned}$$

for $j,k \neq i$, $j \neq k$ and $q = 1,\ldots,7$, so that the idempotents a_i are semisimple in A_θ; from condition (5.2.2), we obtain that each a_i is a $\mathcal{J}\left(\frac{1}{2}\right)$-axis if and only if

$$\begin{aligned}
\theta\left(E_{ij}^0 + E_{ji}^0, E_{jk}^0 + E_{kj}^0\right) &= \theta\left(E_{ii}^0, E_{ik}^0 + E_{ki}^0\right),\\
\theta\left(E_{ij}^0 + E_{ji}^0, E_{jk}^q - E_{kj}^q\right) &= \theta\left(E_{ii}^0, E_{ik}^q - E_{ki}^q\right),\\
\theta\left(E_{ij}^q - E_{ji}^q, E_{jk}^q - E_{kj}^q\right) &= -\theta\left(E_{ii}^0, E_{ik}^0 + E_{ki}^0\right),\\
\theta\left(E_{ij}^q - E_{ji}^q, E_{jk}^r - E_{kj}^r\right) &= \theta\left(E_{ii}^0, E_{ik}^{q\cdot r} - E_{ki}^{q\cdot r}\right),
\end{aligned}$$

for $j, k \neq i$, $j \neq k$ and $q, s \in \{1, \ldots, 7\}$, $q \neq r$.

Additionally, from the idempotents $a_{ij}^0 = \frac{1}{2}\left(E_{ii}^0 + E_{jj}^0 + E_{ij}^0 + E_{ji}^0\right)$, with Peirce decomposition

$$
\begin{aligned}
\mathrm{A}_1^{a_{ij}^0} &= \mathbb{C}a_{ij}^0; \\
\mathrm{A}_0^{a_{ij}^0} &= \operatorname{span}\big\{E_{kk}^0, E_{ii}^0 + E_{jj}^0 - E_{ij}^0 - E_{ji}^0, E_{ik}^0 + E_{ki}^0 - E_{jk}^0 - E_{kj}^0, \\
&\qquad E_{ik}^q - E_{ki}^q - E_{jk}^q + E_{kj}^q \mid j, k \neq i,\ j \neq k,\ q = 1, \ldots, 7\big\}; \\
\mathrm{A}_{1/2}^{a_{ij}^0} &= \operatorname{span}\big\{E_{ii}^0 - E_{jj}^0, E_{ij}^q - E_{ji}^q, E_{ik}^0 + E_{ki}^0 + E_{jk}^0 + E_{kj}^0, \\
&\qquad E_{ik}^q - E_{ki}^q + E_{jk}^q - E_{kj}^q \mid j \neq i,\ q = 1, \ldots, 7\big\}.
\end{aligned}
$$

and from conditions (5.2.1) and (5.2.2), respectively, we obtain the next necessary conditions in order that the idempotents $a_{ij}^0 + \theta\left(a_{ij}^0, a_{ij}^0\right)$ are $\mathcal{J}\left(\frac{1}{2}\right)$-axes in A_θ:

$$
\begin{aligned}
\theta\left(E_{ij}^0 + E_{ji}^0, E_{ij}^0 + E_{ji}^0\right) &= 0, \\
\theta\left(E_{ij}^0 + E_{ji}^0, E_{ij}^q - E_{ji}^q\right) &= 0,
\end{aligned}
$$

for $q = 1, \ldots, 7$.

Finally, from the idempotents $a_{ij}^q = \frac{1}{2}\left(E_{ii}^0 + E_{jj}^0 + E_{ij}^q - E_{ji}^q\right)$, for $q = 1, \ldots, 7$, with Peirce decomposition

$$
\begin{aligned}
\mathrm{A}_1^{a_{ij}^q} &= \mathbb{C}a_{ij}^q; \\
\mathrm{A}_0^{a_{ij}^q} &= \operatorname{span}\big\{E_{kk}^0, E_{ii}^0 + E_{jj}^0 - E_{ij}^q + E_{ji}^q, E_{ik}^0 + E_{ki}^0 + E_{jk}^q - E_{kj}^q, \\
&\qquad E_{ik}^q - E_{ki}^q - E_{jk}^0 - E_{kj}^0, \\
&\qquad E_{ik}^r - E_{ki}^r + E_{jk}^{r/q} - E_{kj}^{r/q} \mid i, j \neq k,\ r = 1, \ldots, 7,\ r \neq q\big\}; \\
\mathrm{A}_{1/2}^{a_{ij}^q} &= \operatorname{span}\big\{E_{ii}^0 - E_{jj}^0, E_{ij}^0 + E_{ji}^0, E_{ij}^r - E_{ji}^r, E_{ik}^0 + E_{ki}^0 - E_{jk}^q + E_{kj}^q, \\
&\qquad E_{ik}^q - E_{ki}^q + E_{jk}^0 + E_{kj}^0, \\
&\qquad E_{ik}^r - E_{ki}^r - E_{jk}^{r/q} + E_{kj}^{r/q} \mid i, j \neq k,\ r = 1, \ldots, 7,\ r \neq q\big\}.
\end{aligned}
$$

and, again, from conditions (5.2.1) and (5.2.2), respectively, it follows that

$$\theta\left(E^q_{ij} - E^q_{ji}, E^q_{ij} - E^q_{ji}\right) = 0$$

and

$$\theta\left(E^q_{ij} - E^q_{ji}, E^r_{ij} - E^r_{ji}\right) = 0,$$

for $r = 1, \dots, 7$, $r \neq q$, are also necessary conditions so that the idempotents $a^q_{ij} + \theta\left(a^q_{ij}, a^q_{ij}\right)$ are $\mathcal{J}\left(\frac{1}{2}\right)$-axes in A_θ.

Then, we have that $\theta = \delta f$ for

$$\begin{aligned} f : \mathrm{A} &\to \mathrm{V} \\ E^0_{ij} + E^0_{ji} &\mapsto \begin{cases} 2\theta\left(E^0_{ii}, E^0_{ij} + E^0_{ji}\right), & \text{if } i \neq j; \\ 0, \quad \text{if } i = j; \end{cases} \\ E^q_{ij} - E^q_{ji} &\mapsto 2\theta\left(E^0_{ii}, E^q_{ij} - E^q_{ji}\right), \end{aligned}$$

and therefore $[\theta] = 0$.

□

Part III

Some tapas

Introduction

This final part of the manuscript includes some works which do not lie within any of the two main lines of the thesis, developed in the previous parts. However, we can find a connection between the first Chapter 6 of this part and Chapter 2: indeed, both of them deal with the introduction of a non-abelian tensor product of different algebraic structures. In particular, in Chapter 6 we work in the category **XSLie** of crossed modules of Lie superalgebras. In such chapter we collect the contents in the article [80], a joint work with Tahereh Fakhr Taha and Manuel Ladra. We introduce also two definitions which are crucial for the development of the contents, namely, the analogues of the Whitehead's quadratic functor for supermodules and for abelian crossed modules of Lie superalgebras. The properties of these objects are studied in Chapter 7, inspired by some work in progress with Manuel Ladra. Supermodules and Lie superalgebras were already introduced in Part I; we devote the following paragraphs to explain why it is interesting to study crossed modules.

Crossed modules of groups were first introduced by Whitehead in the decade of 1940 [229–231] in the context of algebraic homotopy theory. A crossed module is a triple (H, G, ∂) of two groups H and G and a homomorphism of groups $\partial\colon H \to G$ together with an action of G on H satisfying some compatibility conditions. They can be interpreted as simultaneous generalisations of normal subgroups and of modules over a group or, from a categorical point of view, as an internal category in the category **Grp** of groups, or as an internal group object in the category **Cat** of categories. Throughout the history of crossed modules, they have been applied to different branches of mathematics as combinatorial group theory [116, 231], homological algebra [96, 97, 166, 179], algebraic topology [35], differential geometry [167] or cryptography [119].

Since their definition by Whitehead, crossed modules have been generalised in diverse directions. One of the most important was in the ambit of Lie algebras in [142], when they were used in the study of the cyclic homology of associative algebras. Crossed modules of Lie algebras were afterwards employed as suitable coefficients of a non-abelian cohomology for T-algebras (where T denotes a theory of K-Lie algebras) [161], and to study the non-abelian homology of Lie algebras [109]. A further generalisation is due to Janelidze [128], who defined internal crossed modules in any semi-abelian category. From a purely algebraic approach, crossed modules of Lie superalgebras were defined in [237] (see also [91]).

Finally, the last Chapter 8 follows a completely different direction than the rest of the manuscript. It lies within the framework of automated deduction in geometry, and is devoted to study the differences of two ways (namely Rabinowitsch's trick and ideal saturation) of introducing negative statements in the procedures for automatic proving of geometric theorems. It corresponds to the article [158], a joint work with Manuel Ladra and Tomás Recio.

We gratefully thank CESGA (Centro de Supercomputación de Galicia, Santiago de Compostela, Spain) for providing access to the FinisTerrae 2 supercomputer, employed to carry out the computations corresponding to this last chapter.

The structure of each chapter will be described in its own introduction.

CHAPTER 6

The non-abelian tensor and exterior products of crossed modules of Lie superalgebras

In this chapter, we introduce the notions of non-abelian tensor and exterior products of two ideal graded crossed submodules of a given crossed module of Lie superalgebras. We also study some of their basic properties and their connection with the second homology of crossed modules of Lie superalgebras.

Introduction

As we already commented in Chapter 2, the non-abelian tensor and exterior products of Lie algebras were first introduced by Ellis in [76]. Between their main properties, studied in that paper, we remark that for any Lie algebra L, $H_2(L) \cong \ker(L \wedge L \to L)$. These constructions have been generalised to different structures in order to obtain similar characterisations of the second homology H_2.

One of these generalisations was in the direction of crossed modules of Lie algebras. First, it was defined a tensor product for abelian crossed modules of Lie algebras in [75]. After that, Ravanbod and Salemkar [193] generalised this construction defining the non-abelian tensor product of two ideal crossed submodules of a given crossed module of Lie algebras (T, L, ∂), as well as the exterior product. They also characterised the second homology crossed module $H_2(T, L, \partial)$ as the kernel of the commutator map $(T, L, \partial) \wedge (T, L, \partial) \to (T, L, \partial)$. This homology for crossed modules of Lie algebras was introduced by Casas, Inassaridze and Ladra in [49], employing the general theory of cotriple homology of Barr and Beck [20]. Given a crossed module of Lie algebras (T, L, ∂), its homology crossed modules $H_n(T, L, \partial)$ are defined to be the simplicial derived functors of the abelianisation functor between the categories of crossed modules of Lie algebras and abelian crossed modules. This can be seen indeed as a generalisation of the Eilenberg-MacLane homology of Lie algebras. Also, the authors gave a Hopf formula for the second homology of a crossed module.

Another generalisation of the work of Ellis was given in [91], where García-Martínez, Khmaladze and Ladra introduced the non-abelian tensor and exterior products of Lie superalgebras. They also defined their homology, obtaining the Hopf formula for the second homology of Lie superalgebras and extending the five-term exact sequence one term to the left. Moreover, it was proved that, given a Lie superalgebra L, $H_2(L) \cong \ker(L \wedge L \to L)$.

In this chapter, we present a new generalisation of [76] for crossed modules of Lie superalgebras. After defining the tensor product of abelian crossed modules of Lie superalgebras, we introduce the non-abelian tensor and exterior products of two graded ideal crossed submodules of a given crossed module of Lie superalgebras. Also, generalising the work [49] of Casas, Inassaridze and Ladra, we define the second homology of a crossed module of Lie superalgebras, give a Hopf formula, and study some applications of the exterior product on the matter. In particular, we obtain an expression for the second homology $H_2(T, L, \partial)$ for any crossed module of Lie superalgebras (T, L, ∂): it is isomorphic to the kernel of the commutator map $(T, L, \partial) \wedge (T, L, \partial) \to (T, L, \partial)$.

A relevant aspect of this chapter is the introduction of the Whitehead's quadratic functor of supermodules, whose construction differs from the one for modules [208], and which turns out to be fundamental for our purposes.

This chapter is organised as follows. In Section 6.1, we recall some general aspects of Lie superalgebras and their non-abelian tensor product, as well as introduce the definition of the Whitehead's quadratic functor of supermodules. Section 6.2 consists of some generalities on crossed modules of Lie superalgebras which will be needed in the subsequent sections. Section 6.3 introduces the non-abelian tensor and exterior products of two graded ideal crossed submodules and studies some of their main properties. Finally, in Section 6.4, we define the cotriple homology of crossed modules of Lie superalgebras and explore the relation between the second homology and the non-abelian exterior products.

Throughout this chapter, R will denote a unital commutative ring in which 2 has an inverse. Unless otherwise stated, all the (super)modules and (super)algebras in this chapter will be considered over R.

6.1 Preliminaries

The definition of Lie superalgebras was already recalled in Chapter 2. However, we will need more information besides the definition, so we decided to include it again for the sake of convenience. Also, in this section we recall the construction of the non-abelian tensor product of Lie superalgebras, introduced in [91]. This reference also offers a detailed introduction to Lie superalgebras.

We define a *supermodule* as a module M endowed with a grading in $\mathbb{Z}/2\mathbb{Z}$. We write $M = M_{\bar{0}} \oplus M_{\bar{1}}$; the elements in $M_{\bar{0}}$ will be called even or of degree $\bar{0}$, and the elements in $M_{\bar{1}}$, odd or of degree $\bar{1}$. The element 0 will be assumed to have both degrees. We denote the degree of an element m with $|m|$. Non-zero elements of $M_{\bar{0}} \cup M_{\bar{1}}$ will be called *homogeneous*. The direct sum $V{+}W$ of two superspaces has the following induced grading: $(V{+}W)_{\bar{0}} = V_{\bar{0}}{+}W_{\bar{0}}$ and $(V{+}W)_{\bar{1}} = V_{\bar{1}}{+}W_{\bar{1}}$. The *homomorphisms* of supermodules are just homomorphisms of modules. They form a supermodule with the following grading: a homomorphism is even if it preserves the degree of the elements, and it is odd if it changes such degree.

A *Lie superalgebra* is a supermodule $M = M_{\bar{0}} \oplus M_{\bar{1}}$ endowed with a bilinear operation $[\,,]$ such that $|[m, m']| = |m| + |m'|$, and verifying

$$[m, m'] = -(-1)^{|m||m'|}[m', m],$$

$$[m, [m', m'']] = [[m, m'], m''] + (-1)^{|m||m'|}[m', [m, m'']], \tag{6.1.1}$$

$$[m_{\bar{1}}, [m_{\bar{1}}, m_{\bar{1}}]] = 0, \tag{6.1.2}$$

for $m, m', m'' \in M$ homogeneous elements, $m_{\bar{1}} \in M_{\bar{1}}$ and considering $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. Note that equation (6.1.2) follows from equation (6.1.1) when 3 has an inverse in R. Henceforth, we will assume that we are dealing with homogeneous elements when their degrees appear in any formula.

A *homomorphism* of Lie superalgebras f is a homomorphism of supermodules such that $f([m, m']) = [f(m), f(m')]$. Note that, to that purpose, it is necessary that f preserves the degrees of the elements. Also, we consider *graded subalgebras*, submodules N of M with the grading $N_{\bar{0}} = M_{\bar{0}} \cap N$ and $N_{\bar{1}} = M_{\bar{1}} \cap N$, verifying that $[n, n'] \in N$ for all $n, n' \in N$. Furthermore, if $[n, m] \in N$ for any $m \in M$ and $n \in N$, we say that N is a *graded ideal* of M; for example, the commutator (with respect to the subcategory **Ab** of abelian objects, i.e. the abelian Lie superalgebras) $[M, M]$ generated by the elements $[m, m']$, with $m, m' \in M$. Note that a subalgebra is graded if and only if it is generated by homogeneous elements. Also, the quotient of a Lie superalgebra by a graded ideal is another Lie superalgebra, with the induced grading.

We recall now the concept of action of Lie superalgebras, necessary to construct the non-abelian tensor product. An *action* of the Lie superalgebra M on N is a R-bilinear map $M \times N \to N$ with even degree (i.e. $|{}^m n| = |m| + |n|$) verifying

$$
\begin{aligned}
{}^{[m,m']}n &= {}^m({}^{m'}n) - (-1)^{|m||m'|}({}^{m'}({}^m n));\\
{}^m[n,n'] &= [{}^m n, n'] + (-1)^{|m||n|}[n, {}^m n'],
\end{aligned}
$$

for all $m, m' \in M$ and $n, n' \in N$.

Given two Lie superalgebras M and N acting on each other, their *non-abelian tensor product* $M \otimes N$ was defined in [91] as the quotient of the free Lie superalgebra generated by the elements $m \otimes n$ with $|m \otimes n| = |m| + |n|$, by the ideal generated by the following homogeneous elements

$$
\begin{aligned}
&\lambda(m \otimes n) - \lambda m \otimes n,\\
&\lambda(m \otimes n) - m \otimes \lambda n,\\
&(m + m') \otimes n - m \otimes n - m' \otimes n,\\
&m \otimes (n + n') - m \otimes n - m \otimes n',\\
&[m, m'] \otimes n - m \otimes {}^{m'}n + (-1)^{|m||m'|}(m' \otimes {}^m n),\\
&m \otimes [n, n'] - (-1)^{(|m|+|n|)|n'|}({}^{n'}m \otimes n) + (-1)^{|m||n|}({}^n m \otimes n'),\\
&{}^n m \otimes {}^{m'}n' + (-1)^{|m||n|}[m \otimes n, m' \otimes n'],
\end{aligned}
$$

for all $\lambda \in R$, $m, m' \in M$ and $n, n' \in N$.

Note that to define a homomorphism with domain $M \otimes N$, it suffices to define it on generators $m \otimes n$ and extend linearly to any element, provided that it respects the defining relations of the non-abelian tensor product.

Also, we recall the definition of the semidirect product in Lie superalgebras. If a superalgebra M acts over N, we can define the *semidirect product* $N \rtimes M$ as the direct sum $N \oplus M$ endowed with the Lie bracket

$$
[n + m, n' + m'] = [n, n'] + {}^m n' - (-1)^{|n||m'|}({}^{m'}n) + [m, m'].
$$

Finally, an extension of Lie superalgebras is a surjective homomorphism $f : M \to N$. It is central (relative to **Ab**) when $\ker f \subseteq Z(M) = \{m \in M \mid [m, m'] = 0 \text{ for all } m' \in M\}$.

Now, we will introduce the generalised version for supermodules of Whitehead's universal quadratic functor. The version for modules was given in [208].

Definition 6.1.1. Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a supermodule, and let $R^{M_{\bar{0}}}$ be the free supermodule, concentrated in degree zero, generated by the elements $e_{m_{\bar{0}}}$ for all $m_{\bar{0}} \in M_{\bar{0}}$. We define the supermodule $\Gamma(M)$ as the direct sum

$$R^{M_{\bar{0}}} \oplus \left(M \otimes_R M\right),$$

subject to the relations

$$\begin{aligned}
&e_{\lambda m_{\bar{0}}} = \lambda^2 e_{m_{\bar{0}}};\\
&e_{m_{\bar{0}}+m'_{\bar{0}}} - e_{m_{\bar{0}}} - e_{m'_{\bar{0}}} = m_{\bar{0}} \otimes m'_{\bar{0}};\\
&m \otimes m' = (-1)^{|m||m'|} m' \otimes m,
\end{aligned}$$

where $\lambda \in R$, $m_{\bar{0}}, m'_{\bar{0}} \in M_{\bar{0}}$ and $m, m' \in M$, and with the induced grading.

The properties of this supermodule will be studied in Chapter 7.

6.2 Crossed modules of Lie superalgebras

The crossed modules of Lie superalgebras have been scarcely studied; the authors have knowledge of only two references, namely [91] and [237], in which the definition is offered, but they are not carefully analysed. In this section, we intend to give some other definitions and results on the subject, following a categorical approach.

Definition 6.2.1. A *crossed module* of Lie superalgebras (T, L, ∂) consists of two Lie superalgebras T and L, a homomorphism $\partial\colon T \to L$ and an action of L on T such that the following conditions are fulfilled for all $t, t' \in T$ and $l \in L$:

1. $\partial\left({}^{l}t\right) = [l, \partial(t)]$,

2. ${}^{\partial(t)}t' = [t, t']$.

The simplest examples are (L, L, id) and (P, L, i), where L is a Lie superalgebra, P is a graded ideal and i is the inclusion, and taking the Lie bracket as action. Other examples somewhat more involved can be central extensions of Lie superalgebras (T, L, ∂), together with the action ${}^{l}t = [t', t]$ for any $t' \in T$ such that $\partial(t') = l$, or

$(T, \mathrm{Der}(T), \partial)$, where $\partial(t) = \mathrm{ad}\, t$ for all $t \in T$, together with the action given by ${}^{D}t = D(t)$.

In what follows, crossed module will make reference to a crossed module of Lie superalgebras, unless otherwise stated.

Definition 6.2.2. A *morphism* of crossed modules $\psi : (T, L, \partial) \to \left(T', L', \partial'\right)$ is a pair $\left(\psi_1, \psi_2\right)$ of homomorphisms of Lie superalgebras, $\psi_1 : T \to T'$ and $\psi_2 : L \to L'$, such that $\psi_1({}^{l}t) = {}^{\psi_2(l)}\psi_1(t)$ for all $t \in T$, $l \in L$, and that the following diagram is commutative:

$$\begin{tikzcd} T \arrow[r, "\partial"] \arrow[d, "\psi_1"'] & L \arrow[d, "\psi_2"] \\ T' \arrow[r, "\partial'"'] & L'. \end{tikzcd}$$

We say that ψ is injective (surjective) if both ψ_1 and ψ_2 are injective (surjective).

Note that the morphisms of crossed modules of Lie superalgebras have structure of R-supermodule.

The category formed by crossed modules and the morphisms between them will be denoted by **XSLie**. A morphism ψ in this category is an isomorphism if and only if ψ_1 and ψ_2 are isomorphisms of Lie superalgebras. **XSLie** is a semiabelian category; therefore, it has a zero object (namely, the trivial crossed module $(0, 0, 0)$), and also kernels, cokernels and images. While the kernel of a morphism ψ is the crossed module $\left(\ker \psi_1, \ker \psi_2, \partial_|\right)$, its image is $\left(\mathrm{Im}\, \psi_1, \mathrm{Im}\, \psi_2, {\partial'}_|\right)$. We also have the obvious notion of exact sequence.

The subobjects in **XSLie** of a given crossed module (T, L, ∂) can be characterised as the crossed modules (M, P, σ) such that M and P are graded Lie subalgebras of T and L, the homomorphism σ is the restriction of ∂ to M, and the action of P on M is induced by the one of L on T. From now on, we will write (M, P, ∂) instead of (M, P, σ). Furthermore, (M, P, ∂) is a normal subobject when P is a graded ideal of L and the elements ${}^{l}m$ and ${}^{p}t$ belong to M for all $l \in L$, $m \in M$, $p \in P$ and $t \in T$. These normal subobjects are called *graded ideal crossed submodules*. Other interesting objects are the regular quotient objects of (T, L, ∂), which are characterised as $(T/M, L/P, \partial)$ where (M, P, ∂) is a graded ideal crossed submodule of (T, L, ∂), and the action of $\frac{L}{P}$ on $\frac{T}{M}$ is the induced. We denote $\frac{(T,L,\partial)}{(M,P,\partial)} := (T/M, L/P, \partial)$. Note that we denote again by ∂ the homomorphism induced in the quotient. Also, the product in **XSLie** of two objects (T_1, L_1, ∂_1) and (T_2, L_2, ∂_2) is given by $(T_1 \oplus T_2, L_1 \oplus L_2, \langle \partial_1, \partial_2 \rangle)$. We will denote it by $(T_1, L_1, \partial_1) \oplus (T_2, L_2, \partial_2)$.

The abelian objects of **XSLie** are just the crossed modules (A, B, ∂) with A and B abelian Lie superalgebras. Note that the action of B on A becomes trivial, and the only condition on ∂ is to be a homomorphism of supermodules. We are going to work with respect to the Birkhoff subcategory **Ab**. The functor $(\ )_{\mathrm{ab}} :$ **XSLie** $\to$ **Ab**, left adjoint of the inclusion, gives the abelianisation $(T, L, \partial)_{\mathrm{ab}}$ of a crossed module; namely, $(T/[L,T], L/[L,L], \partial)$, where $[L,T]$ denotes the graded subalgebra generated by the elements ${}^{l}t$, with $l \in L$ and $t \in T$. Therefore, the perfect crossed modules are those satisfying $T = [L,T]$ and $L = [L,L]$, and the commutator of (T, L, ∂) is $[(T, L, \partial), (T, L, \partial)] = ([L,T], [T,T], \partial)$. Given two normal subobjects (M, P, ∂) and (N, Q, ∂) of (T, L, ∂), the Higgins commutator $[(M, P, \partial), (N, Q, \partial)]$ can be characterised as the graded ideal crossed submodule $([Q, M] + [P, N], [P, Q], \partial)$.

The same reasoning as in [51, Proposition 18] for crossed modules of Lie algebras enables us to assure that the centre $Z(T, L, \partial)$ of a crossed module in **XSLie** is

$$Z(T, L, \partial) = \left(T^L, Z(L) \cap \mathrm{st}_L(T), \partial\right),$$

where $Z(L)$ is the centre of L, $T^L = \{t \in T \mid {}^{l}t = 0 \text{ for all } l \in L\}$ and $\mathrm{st}_L(T) = \{l \in L \mid {}^{l}t = 0 \text{ for all } t \in T\}$.

We include now a definition for abelian Lie superalgebras, which will be necessary for Theorem 6.3.10 (cf. [189] for abelian groups and [75] for abelian Lie algebras).

Definition 6.2.3. The *tensor product* of two arbitrary abelian crossed modules (T, L, ∂) and (T', L', ∂') is defined as the abelian crossed module

$$(T, L, \partial) \otimes_R (T', L', \partial') = \left(\operatorname{coker} \alpha_R, L \otimes L', \delta_R\right),$$

where $\alpha_R : T \otimes T' \to (T \otimes L') \oplus (L \otimes T')$ is defined by $\alpha_R(t \otimes t') = -t \otimes \partial'(t') + \partial(t) \otimes t'$, and $\delta_R :$ coker $\alpha_R \to L \otimes L'$ is induced by $\langle \partial \otimes \mathrm{id}, \mathrm{id} \otimes \partial' \rangle$. To construct all the previous tensor products, we consider trivial actions.

Given two crossed modules (M, L, ∂) and (N, L, σ), there are actions of M on N and of N on M given, respectively, by ${}^{m}n = {}^{\partial(m)}n$ and ${}^{n}m = {}^{\sigma(n)}m$, which allow us to construct the non-abelian tensor product $M \otimes N$ (note that we can always construct $M \otimes L$ by considering (M, L, ∂) and (L, L, id)). Consider the graded submodule $M \square N$ of $M \otimes N$, generated by the elements

$$m_{\bar{0}} \otimes n_{\bar{0}}, \text{ with } \partial\left(m_{\bar{0}}\right) = \sigma\left(n_{\bar{0}}\right),$$
$$m \otimes n + (-1)^{|m'||n'|}\, m' \otimes n', \text{ with } \partial(m) = \sigma\left(n'\right) \text{ and } \partial\left(m'\right) = \sigma(n),$$

being $m_{\bar{0}} \in M_{\bar{0}}$, $n_{\bar{0}} \in N_{\bar{0}}$, $m, m' \in M$ and $n, n' \in N$. In [91], it is proved that $M \square N$ is a graded ideal in the centre of $M \otimes N$, and the non-abelian exterior product $M \wedge N$ is defined as the quotient $(M \otimes N) / (M \square N)$. The images of the elements $m \otimes n$ in the quotient are denoted by $m \wedge n$.

In the following proposition, we recall some elementary properties of the non-abelian tensor and exterior products. The proof is left to the reader (see also [91]), except the one of the last item, which we offer below.

Proposition 6.2.4. *Let (M, L, ∂) and (N, L, σ) be two crossed modules. Then:*

1. *There is a Lie superalgebra homomorphism $\xi : M \otimes N \to L$ with $\xi(m \otimes n) = {}^{m}\sigma(n) = -(-1)^{|m||n|}\left({}^{n}\partial(m)\right)$.*
2. *The triple $(M \otimes N, L, \xi)$ is a crossed module with the action of L on $M \otimes N$ defined by ${}^{l}(m \otimes n) = {}^{l}m \otimes n + (-1)^{|l||m|}\, m \otimes {}^{l}n$.*
3. *The non-abelian tensor products $M \otimes N$ and $N \otimes M$ are isomorphic, through the map $m \otimes n \mapsto -(-1)^{|m||n|}\, n \otimes m$.*
4. *If the actions of L on M and N are trivial, then $M \otimes N$ is isomorphic to $M_{\mathrm{ab}} \otimes N_{\mathrm{ab}}$, being $M_{\mathrm{ab}} = M/[M, M]$ and $N_{\mathrm{ab}} = N/[N, N]$, and where $M_{\mathrm{ab}} \otimes N_{\mathrm{ab}}$ is formed with M_{ab} and N_{ab} acting trivially on each other. Also, $M \wedge N \cong M_{\mathrm{ab}} \wedge N_{\mathrm{ab}}$.*
5. *There is an isomorphism*
$$\begin{aligned}(M \oplus N) \wedge (M \oplus N) &\to (M \wedge M) \oplus (N \wedge N) \oplus \left(M_{\mathrm{ab}} \otimes N_{\mathrm{ab}}\right)\\ (m + n) \wedge \left(m' + n'\right) &\mapsto m \wedge m' + n \wedge n' + \overline{m} \otimes \overline{n'} - (-1)^{|m||m'|}\, \overline{m'} \otimes \overline{n}.\end{aligned}$$
The inverse homomorphism is given by
$$\begin{aligned}(M \wedge M) \oplus (N \wedge N) \oplus \left(M_{\mathrm{ab}} \otimes N_{\mathrm{ab}}\right) &\to (M \oplus N) \wedge (M \oplus N)\\ m \wedge m' + n \wedge n' + \overline{m''} \otimes \overline{n''} &\mapsto m \wedge m' + n \wedge n' + m'' \wedge n''.\end{aligned}$$

6. *If (M, L, ∂) is perfect, then $L\square M$ vanishes, and $L \otimes M$ and $L \wedge M$ are isomorphic.*

7. *Given U a graded ideal of M, the following sequence of Lie superalgebras is exact:*
$$U \wedge M \to M \wedge M \to \frac{M}{U} \wedge \frac{M}{U} \to 0.$$

8. *If M is perfect, the homomorphism*
$$u\colon M \otimes M \to M$$
given by $u\left(m \otimes m'\right) = [m, m']$ is the universal central extension.

9. *The following sequence of Lie superalgebras,*
$$\Gamma\left(M_{\mathrm{ab}}\right) \xrightarrow{\eta} M \otimes M \xrightarrow{\pi} M \wedge M \longrightarrow 0$$
where $\eta\left(e_{m_{\bar{0}}} + m \otimes m'\right) = m_{\bar{0}} \otimes m_{\bar{0}} + m \otimes m' + (-1)^{|m||m'|}\, m' \otimes m$, is exact.

Proof. We just offer the proof of the last item, as it involves the Whitehead's universal quadratic functor. We must check that the morphism η is well defined. On the one hand, it is straightforward to see that it preserves the relations in Definition 6.1.1. On the other hand, take $[m, m'] \in [M, M]_{\bar{0}}$. It holds that
$$\eta\left(e_{[m,m']}\right) = [m, m'] \otimes [m, m'] = -(-1)^{|m||m'|}\left({}^{m'}m\right) \otimes {}^{m}m' = [m \otimes m', m \otimes m'] = 0.$$

Now, let $[m, m'] \in [M, M]$.
$$\begin{aligned}
\eta\left([m, m'] \otimes \mu\right) =& [m, m'] \otimes \mu + (-1)^{\left(|m|+|m'|\right)|\mu|}\, \mu \otimes [m, m'] \\
=& -(-1)^{|m||m'|}\, m' \otimes [m, \mu] + (-1)^{|m||m'|}(-1)^{\left(|m|+|m'|\right)|\mu|}\, [\mu, m'] \otimes m \\
& + (-1)^{\left(|m|+|m'|\right)|\mu|}\, \mu \otimes [m, m'] \\
=& (-1)^{|m||\mu|}(-1)^{|m||m'|}\, m' \otimes [\mu, m] - (-1)^{\left(|m|+|m'|\right)|\mu|}\, \mu \otimes [m, m'] \\
& - (-1)^{|m||m'|}(-1)^{|m||\mu|}\, m' \otimes [\mu, m] + (-1)^{\left(|m|+|m'|\right)|\mu|}\, \mu \otimes [m, m'] \\
& = 0.
\end{aligned}$$

Analogously, $\eta\left(\mu \otimes [m, m']\right) = 0$. Therefore, the morphism η is well defined. Finally, it is clear that $\operatorname{Im} \eta = M \square M = \ker \pi$, and π is surjective, so the sequence is exact. $\square$

We finish this section with some considerations about central extensions. An extension in **XSLie** is just a surjective morphism; it is central (relative to **Ab**) if and only if $\ker \psi \subseteq Z\left(T', L', \partial'\right)$.

The proofs of the following results are similar to the ones offered in [50] for crossed modules of Lie algebras; therefore, we omit them. The previous results needed to carry out these proofs can be found in [91].

Lemma 6.2.5. *Let $\psi\colon \left(T', L', \partial'\right) \to (T, L, \partial)$ be a central extension with $\left(T', L', \partial'\right)$ perfect, and let $\phi\colon \left(T'', L'', \partial''\right) \to (T, L, \partial)$ be another central extension. If there exists a morphism of central extensions $\theta\colon \left(T', L', \partial'\right) \to \left(T'', L'', \partial''\right)$, then it is unique.*

Also, it is easy to construct a counterexample to the previous result in the case in which $\left(T', L', \partial'\right)$ is not perfect (see [50]).

Lemma 6.2.6. *Let (T, L, ∂) be a crossed module. Then, $(L \otimes T, L \otimes L, \mathrm{id} \otimes \partial)$ is also a crossed module. Moreover, if (T, L, ∂) is perfect, so is $(L \otimes T, L \otimes L, \mathrm{id} \otimes \partial)$.*

The previous lemmas help to prove the following theorem.

Theorem 6.2.7. *Let (T, L, ∂) be a perfect crossed module. The morphism*

$$\upsilon\colon\ (L \otimes T, L \otimes L, \mathrm{id} \otimes \partial) \to (T, L, \partial)$$

defined by $\upsilon_1(l \otimes t) = {}^{l}t$ and $\upsilon_2\left(l \otimes l'\right) = [l, l']$ is a universal central extension of (T, L, ∂).

From this theorem and the observation below Lemma 6.2.5 is obtained the following corollary.

Corollary 6.2.8. *A crossed module (T, L, ∂) admits a universal central extension if and only if it is perfect.*

6.3 The non-abelian tensor and exterior products of crossed modules

The present section is devoted to the definition of the non-abelian tensor and exterior products of graded ideal crossed submodules, and their basic properties.

Let (M, P, ∂) and (N, Q, ∂) two graded ideal crossed submodules of a crossed module (T, L, ∂). Considering also the crossed modules (M, L, ∂), (N, L, ∂), (P, L, i) and (Q, L, i), we can construct the non-abelian tensor products $M \otimes N$, $M \otimes Q$, $P \otimes N$ and $P \otimes Q$. By Proposition 6.2.4 (1), there is a homomorphism $\xi\colon P \otimes N \to L$, and also Proposition 6.2.4 (2) ensures the existence of an action of L on $M \otimes Q$. This yields an action of $P \otimes N$ on $M \otimes Q$; similarly, of $P \otimes Q$ on $P \otimes N$ and $M \otimes Q$.

Lemma 6.3.1. *With the above assumptions, the actions of $P \otimes N$ on $M \otimes Q$, of $P \otimes Q$ on $P \otimes N$ and of $P \otimes Q$ on $M \otimes Q$, respectively, are given explicitly by:*

1. ${}^{p\otimes n}(m \otimes q) = {}^{p}n \otimes {}^{m}q.$
2. ${}^{p\otimes q}\left(p' \otimes n\right) = {}^{p}q \otimes {}^{p'}n.$
3. ${}^{p\otimes q}\left(m \otimes q'\right) = (-1)^{|m||q'|}(-1)^{(|p|+|q|)(|m|+|q'|)}\left({}^{q'}m\right) \otimes {}^{p}q.$

Proof. Routine. □

The action of $P \otimes N$ on $M \otimes Q$ allows us to construct the semidirect product $(M \otimes Q) \rtimes (P \otimes N)$, and the homomorphisms of Lie superalgebras defined by

$$\begin{aligned} \alpha\colon\ M \otimes N &\to (M \otimes Q) \rtimes (P \otimes N), \\ m \otimes n &\mapsto -m \otimes \partial(n) + \partial(m) \otimes n, \end{aligned}$$

$$\begin{aligned} \beta\colon\ (M \otimes Q) \rtimes (P \otimes N) &\to P \otimes Q. \\ m \otimes q + p \otimes n &\mapsto \partial(m) \otimes q + p \otimes \partial(n). \end{aligned}$$

Lemma 6.3.1 helps to check that the previous maps are indeed homomorphisms, and also to prove the following lemma, fundamental for our main definitions.

Lemma 6.3.2. *In the previous conditions, we have*

1. *The image of* α *is a graded ideal in* $(M \otimes Q) \rtimes (P \otimes N)$.
2. *There is an action of* $P \otimes Q$ *on* $(M \otimes Q) \rtimes (P \otimes N)$ *determined by*

$$^{p' \otimes q'}(m \otimes q + p \otimes n) = {}^{p' \otimes q'}(m \otimes q) + {}^{p' \otimes q'}(p \otimes n) .$$

3. *If* $\delta : \operatorname{coker} \alpha \to P \otimes Q$ *is the homomorphism induced by* β*, then the triple* $(\operatorname{coker} \alpha, P \otimes Q, \delta)$ *is a crossed module with the action induced by the part* (2).

Proof. Similar to the proof of [193, Lemma 3.1]. □

Let I be the graded subalgebra of $\operatorname{coker} \alpha$ generated by the elements

$$x \otimes y + (-1)^{|x||y|} y \otimes x + \partial \left(z_{\bar{0}}\right) \otimes z_{\bar{0}} + \operatorname{Im} \alpha,$$
$$x \otimes y + (-1)^{|x||y|} y \otimes x + \partial (z) \otimes z' + (-1)^{|z||z'|} \partial \left(z'\right) \otimes z + \operatorname{Im} \alpha,$$

such that $x, z, z' \in M \cap N$, $z_{\bar{0}} \in M_{\bar{0}} \cap N_{\bar{0}}$ and $y \in P \cap Q$. We easily check that $\delta (I) \subseteq P \square Q$.

Lemma 6.3.3. *The crossed module* $(I, P \square Q, \delta)$ *is a graded ideal crossed submodule of* $(\operatorname{coker} \alpha, P \otimes Q, \delta)$.

Proof. As $P \square Q$ is a graded ideal of $P \otimes Q$, we just have to prove that the action of $P \otimes Q$ on I remains in I and that the action of $P \square Q$ on $\operatorname{coker} \alpha$ also lies in I (in fact, we will prove that it is trivial). Applying Lemma 6.3.1, and with the usual notations, we have that

$$\begin{aligned}
&^{p \otimes q}\left(x \otimes y + (-1)^{|x||y|} y \otimes x + \partial \left(z_{\bar{0}}\right) \otimes z_{\bar{0}}\right) + \operatorname{Im} \alpha \\
&= (-1)^{|x||y|} (-1)^{(|p|+|q|)(|x|+|y|)} ({}^{y}x) \otimes {}^{p}q + (-1)^{|x||y|} ({}^{p}q) \otimes {}^{y}x + {}^{p}q \otimes [z_{\bar{0}}, z_{\bar{0}}] + \operatorname{Im} \alpha \\
&= (-1)^{|x||y|} (-1)^{(|p|+|q|)(|x|+|y|)} \left({}^{y}x \otimes {}^{p}q + (-1)^{(|p|+|q|)(|x|+|y|)} ({}^{p}q) \otimes {}^{y}x\right) + \operatorname{Im} \alpha \in I;
\end{aligned}$$

$$
\begin{aligned}
&{}^{p\otimes q}\left(x\otimes y+(-1)^{|x||y|}\,y\otimes x+\partial(z)\otimes z'+(-1)^{|z||z'|}\,\partial\left(z'\right)\otimes z\right)+\operatorname{Im}\alpha\\
&=(-1)^{|x||y|}\,(-1)^{(|p|+|q|)(|x|+|y|)}\,({}^{y}x)\otimes{}^{p}q\\
&\qquad+(-1)^{|x||y|}\,({}^{p}q)\otimes{}^{y}x+{}^{p}q\otimes[z,z']+(-1)^{|z||z'|}\,({}^{p}q)\otimes[z',z]+\operatorname{Im}\alpha\\
&=(-1)^{|x||y|}\,(-1)^{(|p|+|q|)(|x|+|y|)}\left({}^{y}x\otimes{}^{p}q+(-1)^{(|p|+|q|)(|x|+|y|)}\,({}^{p}q)\otimes{}^{y}x\right)+\operatorname{Im}\alpha\in I;
\end{aligned}
$$

therefore, the first condition holds. As for the second one, $P\square Q$ is generated by the elements $p_{\bar{0}}\otimes q_{\bar{0}}$ and $p\otimes q+(-1)^{|p'||q'|}\,p'\otimes q'$, with $i\left(p_{\bar{0}}\right)=i\left(q_{\bar{0}}\right)$, $i\left(p\right)=i\left(q'\right)$ and $i\left(p'\right)=i\left(q\right)$. Motivated by these restrictions, we will work with the elements $l_{\bar{0}}\otimes l_{\bar{0}}$ and $l\otimes l'+(-1)^{|l||l'|}\,l'\otimes l$. Applying again Lemma 6.3.1, we see that

$$
{}^{l_{\bar{0}}\otimes l_{\bar{0}}}\left(m\otimes q+p\otimes n+\operatorname{Im}\alpha\right)=(-1)^{|m||q|}\,({}^{q}m)\otimes[l_{\bar{0}},l_{\bar{0}}]+[l_{\bar{0}},l_{\bar{0}}]\otimes{}^{p}n+\operatorname{Im}\alpha=0;
$$

$$
\begin{aligned}
&{}^{l\otimes l'+(-1)^{|l||l'|}l'\otimes l}\left((m\otimes q,p\otimes n)+\operatorname{Im}\alpha\right)\\
&=(-1)^{|m||q|}\,(-1)^{\left(|l|+|l'|\right)(|m|+|q|)}\left({}^{q}m\otimes[l,l']+(-1)^{|l||l'|}\,({}^{q}m)\otimes[l',l]\right)\\
&\qquad+[l,l']\otimes{}^{p}n+(-1)^{|l||l'|}\,[l',l]\otimes{}^{p}n+\operatorname{Im}\alpha=0.
\end{aligned}
$$

Therefore, the action of $P\square Q$ on $\operatorname{coker}\alpha$ is trivial, and the lemma is proved. □

Now, we present the main definitions of this chapter.

Definition 6.3.4. Let (M,P,∂) and (N,Q,∂) two graded ideal crossed submodules of a crossed module (T,L,∂). With the previous notations, we define the *non-abelian tensor product* of (M,P,∂) and (N,Q,∂) as

$$
(M,P,\partial)\otimes(N,Q,\partial)=(\operatorname{coker}\alpha,P\otimes Q,\delta),
$$

and the *exterior product* as

$$
(M,P,\partial)\wedge(N,Q,\partial)=\frac{(\operatorname{coker}\alpha,P\otimes Q,\delta)}{(I,P\square Q,\delta)}=\left(\frac{\operatorname{coker}\alpha}{I},P\wedge Q,\delta\right).
$$

For coherence with the theory of Lie algebras and superalgebras, we will denote $(I,P\square Q,\delta)$ as $(M,P,\partial)\square(N,Q,\partial)$.

The reader should note that the only role played by the crossed module (T, L, ∂) in the definition above is to provide a common environment for (M, P, ∂) and (N, Q, ∂), which is used to determine the actions specified in Lemma 6.3.1.

As in the case of crossed modules of Lie algebras, we find the following particular cases:

Proposition 6.3.5.

1. *Let P and Q be two graded ideals of a Lie superalgebra L. Then, we have:*

   (a) $(P, P, \mathrm{id}) \otimes (Q, Q, \mathrm{id}) \cong (P \otimes Q, P \otimes Q, \mathrm{id})$*;*

   (b) $(P, P, \mathrm{id}) \wedge (Q, Q, \mathrm{id}) \cong (P \wedge Q, P \wedge Q, \mathrm{id})$*;*

   (c) $(0, P, i) \otimes (0, Q, i) \cong (0, P \otimes Q, i)$*;*

   (d) $(0, P, i) \wedge (0, Q, i) \cong (0, P \wedge Q, i)$*;*

2. *Let (T, L, ∂) be a crossed module, and consider the exterior product, $(T, L, \partial) \wedge (T, L, \partial)$. There is a isomorphism of Lie superalgebras*

$$\nu : \frac{\operatorname{coker} \alpha}{I} \to L \wedge T$$
$$\overline{t \otimes l + l' \otimes t'} + I \mapsto l' \wedge t' - (-1)^{|l||t|}\, l \wedge t,$$

   such that $(\nu, \mathrm{id}) : (T, L, \partial) \wedge (T, L, \partial) \to (L \wedge T, L \wedge L, \mathrm{id} \wedge \partial)$ is an isomorphism of crossed modules.

Proof.

1. Routine.

2. To check that ν is well defined, it suffices to define $\tilde{\nu} : (T \otimes L) \rtimes (L \otimes T) \to L \otimes T$ by $\tilde{\nu}\left(t \otimes l + l' \otimes t'\right) = l' \otimes t' - (-1)^{|l||t|}\, l \otimes t$. Noting that $\tilde{\nu}\,(\mathrm{Im}\,\alpha)$ is contained in $L \square T$, we see that $\tilde{\nu}$ induces a homomorphism $\bar{\nu} : \operatorname{coker}\alpha \to L \wedge T$ with kernel $\ker \bar{\nu} = I$. Then, ν is well defined. The rest of the proof is routine.

$\square$

The rest of this section is devoted to the study of some basic properties of the non-abelian tensor and exterior products of crossed modules of Lie superalgebras, which extend the ones offered in Proposition 6.2.4 (4)–(9) for Lie superalgebras. To that purpose, we need first this useful lemma.

Lemma 6.3.6. *Let (T, L, ∂) be a crossed module such that ∂ is surjective or the action of L on T is trivial. Then:*

1. *For all $t \in T$ and $l, l' \in L$, it holds that $\partial\left({}^{l}t\right) \otimes t' = -(-1)^{|t'|(|l|+|t|)}\,\partial\left(t'\right) \otimes {}^{l}t$.*
2. *The graded ideal $I \subseteq \operatorname{coker}\alpha$ is abelian.*

Proof. We will give the proof for the case in which ∂ is surjective, and the other case is trivial.

1. Routine.
2. Let $x, y \in I$. Clearly, $[x, y] = {}^{\delta(x)}y$, with $\delta(x) \in L\square L \subseteq L \otimes L$; so, it suffices to prove that the action of $L \otimes L$ on I is trivial. It is enough to consider $\lambda \otimes \lambda' \in L \otimes L$ and the generators of I. Using Lemma 6.3.1, part (1) and the surjectivity of ∂, we see that

$$\begin{aligned}
&{}^{\lambda\otimes\lambda'}\left(t \otimes l + (-1)^{|t||l|}\, l \otimes t + \partial\left(z_{\bar 0}\right) \otimes z_{\bar 0}\right)\\
&= -(-1)^{|t||l|}\,(-1)^{(|t|+|l|)(|\lambda|+|\lambda'|)}\,(-1)^{|\lambda||\lambda'|}\left({}^{l}t\right) \otimes [\lambda', \lambda]\\
&\qquad\qquad - (-1)^{|t||l|}\,(-1)^{|\lambda||\lambda'|}\,[\lambda', \lambda] \otimes {}^{l}t - (-1)^{|\lambda||\lambda'|}\,[\lambda', \lambda] \otimes [z_{\bar 0}, z_{\bar 0}]\\
&= -(-1)^{|t||l|}\,(-1)^{(|t|+|l|)(|\lambda|+|\lambda'|)}\,(-1)^{|\lambda||\lambda'|}\left({}^{l}t\right) \otimes [\lambda', \partial(\tau)]\\
&\qquad\qquad - (-1)^{|t||l|}\,(-1)^{|\lambda||\lambda'|}\,[\lambda', \partial(\tau)] \otimes {}^{l}t\\
&= -(-1)^{|t||l|}\,(-1)^{(|t|+|l|)(|\lambda|+|\lambda'|)}\,(-1)^{|\lambda||\lambda'|}\left({}^{l}t\right) \otimes \partial\left({}^{\lambda'}\tau\right)\\
&\qquad\qquad - (-1)^{|t||l|}\,(-1)^{|\lambda||\lambda'|}\,\partial\left({}^{\lambda'}\tau\right) \otimes {}^{l}t\\
&= (-1)^{|t||l|}\,(-1)^{|\lambda||\lambda'|}\,(-1)^{(|t|+|l|)(|\lambda|+|\lambda'|)}\left(-{}^{l}t \otimes \partial\left({}^{\lambda'}\tau\right) + \partial\left({}^{l}t\right) \otimes {}^{\lambda'}\tau\right)\\
&\in \operatorname{Im}\alpha,
\end{aligned}$$

for $z_0 \in T_{\bar 0}$, $t, \tau \in T$ and $l \in L$. Also, for $t, z, z', \tau \in T$ and $l \in L$,

$$
\begin{aligned}
&{}^{\lambda\otimes\lambda'}\left(t\otimes l+(-1)^{|t||l|}\, l\otimes t+\partial\left(z\right)\otimes z'+(-1)^{\left(|z||z'|\right)}\,\partial\left(z'\right)\otimes z\right)\\
&=-\,(-1)^{|t||l|}\,(-1)^{|\lambda||\lambda'|}\,(-1)^{(|t|+|l|)(|\lambda|+|\lambda'|)}\left({}^{l}t\right)\otimes\partial\left({}^{\lambda'}\tau\right)\\
&\qquad+(-1)^{|t||l|}\,(-1)^{|\lambda||\lambda'|}\,(-1)^{(|t|+|l|)(|\lambda|+|\lambda'|)}\,\partial\left({}^{l}t\right)\otimes{}^{\lambda'}\tau\\
&\qquad\quad-(-1)^{|\lambda||\lambda'|}\,[\lambda',\lambda]\otimes[z,z']-(-1)^{\left(|z||z'|\right)}\,(-1)^{|\lambda||\lambda'|}\,[\lambda',\lambda]\otimes[z',z]\\
&=(-1)^{|t||l|}\,(-1)^{|\lambda||\lambda'|}\,(-1)^{(|t|+|l|)(|\lambda|+|\lambda'|)}\left(-{}^{l}t\otimes\partial\left({}^{\lambda'}\tau\right)+\partial\left({}^{l}t\right)\otimes{}^{\lambda'}\tau\right)\\
&\in \operatorname{Im}\alpha.
\end{aligned}
$$

The lemma is proved.

□

Proposition 6.3.7. *Let (T, L, ∂) be a perfect crossed module, such that ∂ is surjective or the action of L on T is trivial. Then,*

$$(T, L, \partial)\otimes(T, L, \partial)\cong(T, L, \partial)\wedge(T, L, \partial)\,.$$

Proof. We will show that $(T, L, \partial)\,\square\,(T, L, \partial)=(I, L\square L, \delta)$ is zero. As (T, L, ∂) is perfect, so is L, and Proposition 6.2.4 (6) implies that $L\square L$ is zero. We just have to check that I is also zero. It suffices to show that the generators

$$ {}^{\lambda}\tau\otimes y+(-1)^{(|\lambda|+|\tau|)|y|}\,y\otimes{}^{\lambda}\tau+\partial\left({}^{l}t\right)\otimes{}^{l}t+\operatorname{Im}\alpha, \tag{6.3.1}$$

with $l, \lambda, y\in L$, $t, \tau\in T$ and $|t|=|l|$, and

$$ {}^{\lambda}\tau\otimes y+(-1)^{(|\lambda|+|\tau|)|y|}\,y\otimes{}^{\lambda}\tau+\partial\left({}^{l}t\right)\otimes{}^{l'}t'+(-1)^{(|l|+|t|)(|l'|+|t'|)}\,\partial\left({}^{l'}t'\right)\otimes{}^{l}t+\operatorname{Im}\alpha, \tag{6.3.2}$$

with $l, l', \lambda, y\in L$ and $t, t', \tau\in T$, are zero.

Clearly, if the action of L on T is zero, the assertion is trivial. Therefore, let us assume that ∂ is surjective and $y=\partial(x)$. Applying Lemma 6.3.6 (1), we see that ${}^{\lambda}\tau\otimes y+(-1)^{(|\lambda|+|\tau|)|y|}\,y\otimes{}^{\lambda}\tau={}^{\lambda}\tau\otimes\partial(x)-\partial\left({}^{\lambda}\tau\right)\otimes x\in\operatorname{Im}\alpha$. Also, for $|t|=|l|$, $\partial\left({}^{l}t\right)\otimes{}^{l}t=-\partial\left({}^{l}t\right)\otimes{}^{l}t$, hence it is zero, and the generator (6.3.1) is zero. Regarding the generator (6.3.2), $\partial\left({}^{l}t\right)\otimes{}^{l'}t'+(-1)^{(|l|+|t|)(|l'|+|t'|)}\,\partial\left({}^{l'}t'\right)\otimes{}^{l}t=\partial\left({}^{l}t\right)\otimes{}^{l'}t'-\partial\left({}^{l}t\right)\otimes{}^{l'}t'=0$, so it is also zero. This completes the proof. □

Theorem 6.3.8. *Let (T, L, ∂) be a perfect crossed module such that ∂ is surjective or the action of L on T is trivial. The morphism*

$$\kappa = \left(\kappa_1, \kappa_2\right) : (T, L, \partial) \otimes (T, L, \partial) \to (T, L, \partial)$$

given by $\kappa_1 \left(t \otimes l + l' \otimes t' + \operatorname{Im} \alpha\right) = {}^{l'}t' - (-1)^{|t||l|} \left({}^{l}t\right)$ *and* $\kappa_2 \left(l \otimes l'\right) = [l, l']$ *is a universal central extension of* (T, L, ∂)*.*

Proof. In view of Theorem 6.2.7, we just have to prove that there exists an isomorphism $\omega : (T, L, \partial) \otimes (T, L, \partial) \to (L \otimes T, L \otimes L, \mathrm{id} \otimes \partial)$ such that $\upsilon \omega = \kappa$. Indeed, the combination of Proposition 6.3.7, Proposition 6.3.5 (2) and Proposition 6.2.4 (6) gives the desired isomorphism. □

Proposition 6.3.9. *Let (M, P, ∂) and (N, Q, ∂) be two graded ideal crossed submodules of (T, L, ∂).*

1. *If the commutator $[(M, P, \partial), (N, Q, \partial)]$ is trivial, then*

$$(M, P, \partial) \otimes (N, Q, \partial) \cong (M, P, \partial)_{\mathrm{ab}} \otimes (N, Q, \partial)_{\mathrm{ab}} .$$

2. *The following natural sequence of crossed modules is exact:*

$$(M, P, \partial) \wedge (T, L, \partial) \to (T, L, \partial) \wedge (T, L, \partial) \to \frac{(T, L, \partial)}{(M, P, \partial)} \wedge \frac{(T, L, \partial)}{(M, P, \partial)} \to 0.$$

Proof.

1. It follows from the definitions, Proposition 6.2.4 (4) and the fact that, in our conditions, $\frac{M}{[P,M]} \otimes Q_{\mathrm{ab}}$ and $M \otimes Q$ are isomorphic (similarly, $P_{\mathrm{ab}} \otimes \frac{N}{[Q,N]}$ and $P \otimes N$).

2. We give a sketch of the proof.

   Let us fix the following terminology:

$$(M, P, \partial) \wedge (T, L, \partial) = \left(\operatorname{coker} \tilde{\alpha}/\tilde{I}, P \wedge L, \tilde{\delta}\right);$$
$$(T, L, \partial) \wedge (T, L, \partial) = (\operatorname{coker} \alpha/I, L \wedge L, \delta);$$

$$\frac{(T,L,\partial)}{(M,P,\partial)} \wedge \frac{(T,L,\partial)}{(M,P,\partial)} = \left(\operatorname{coker} \bar{\alpha}/\bar{I}, \frac{L}{P} \wedge \frac{L}{P}, \bar{\delta}\right).$$

Let $\varpi_1 :$ coker $\alpha/I \to$ coker $\bar{\alpha}/\bar{I}$ and $\varpi_2 :\ L \wedge L \to L/P \wedge L/P$ be the surjective homomorphisms induced by the projections $T \to T/M$ and $L \to L/P$. Then, $\varpi = \left(\varpi_1, \varpi_2\right) :\ (T,L,\partial) \wedge (T,L,\partial) \to \dfrac{(T,L,\partial)}{(M,P,\partial)} \wedge \dfrac{(T,L,\partial)}{(M,P,\partial)}$ is a surjective morphism of crossed modules. Also, consider the following homomorphism induced by the natural inclusions, $M \otimes L \rtimes P \otimes T \to T \otimes L \rtimes L \otimes T$, which in turn induces $\rho_1 :$ coker $\tilde{\alpha}/\tilde{I} \to$ coker α/I, and $\rho_2 :\ P \wedge L \to L \wedge L$. They provide another morphism of crossed modules $\rho = \left(\rho_1, \rho_2\right) :\ (M,P,\partial) \wedge (T,L,\partial) \to (T,L,\partial) \wedge (T,L,\partial)$. To see that $\ker \varpi = \operatorname{Im} \rho$, we define the homomorphism $\vartheta :\ (T/M \otimes L/P) \rtimes (L/P \otimes T/M) \to$ coker ρ_1 given by $\vartheta\left((t+M) \otimes (l+P) + \left(l'+P\right) \otimes \left(t'+M\right)\right) = \overline{t \otimes l + l' \otimes t'} + \operatorname{Im} \rho_1$. It induces an isomorphism $\tilde{\vartheta} :$ coker $\bar{\alpha}/\bar{I} \to$ coker ρ_1 with inverse induced by ϖ_1; this makes clear that $\ker \varpi_1 = \operatorname{Im} \rho_1$. Also, $\ker \varpi_2 = \operatorname{Im} \rho_2$ by Proposition 6.2.4 (7). This concludes the proof.

□

Theorem 6.3.10. *Let $\left(T_1, L_1, \partial_1\right)$ and $\left(T_2, L_2, \partial_2\right)$ be two arbitrary crossed modules. Then*

$$\begin{aligned}
&\left(\left(T_1, L_1, \partial_1\right) \oplus \left(T_2, L_2, \partial_2\right)\right) \wedge \left(\left(T_1, L_1, \partial_1\right) \oplus \left(T_2, L_2, \partial_2\right)\right) \\
\cong &\left(\left(T_1, L_1, \partial_1\right) \wedge \left(T_1, L_1, \partial_1\right)\right) \\
&\qquad \oplus \left(\left(T_2, L_2, \partial_2\right) \wedge \left(T_2, L_2, \partial_2\right)\right) \\
&\qquad\qquad \oplus \left(\left(T_1, L_1, \partial_1\right)_{\mathrm{ab}} \otimes_R \left(T_2, L_2, \partial_2\right)_{\mathrm{ab}}\right),
\end{aligned}$$

where the last crossed module is the tensor product of abelian objects of Definition 6.2.3.

Proof. Denoting $\overline{T_i} = T_i/[L_i, T_i]$ and $\overline{L_i} = \left(L_i\right)_{\mathrm{ab}}$, for $i \in \{1,2\}$, we will construct an isomorphism $\psi = (\chi, \zeta)$ between both crossed modules

$$\begin{tikzcd}
\left(L_1 \oplus L_2\right) \wedge \left(T_1 \oplus T_2\right) \arrow[r, "\chi"] \arrow[d, "\mathrm{id} \wedge (\partial_1 \oplus \partial_2)"'] & \left(L_1 \wedge T_1\right) \oplus \left(L_2 \wedge T_2\right) \oplus \operatorname{coker} \alpha_R \arrow[d, "(\mathrm{id} \wedge \partial_1) \oplus (\mathrm{id} \wedge \partial_2) \oplus \delta_R"] \\
\left(L_1 \oplus L_2\right) \wedge \left(L_1 \oplus L_2\right) \arrow[r, "\zeta"] & \left(L_1 \wedge L_1\right) \oplus \left(L_2 \wedge L_2\right) \oplus \left(\overline{L_1} \otimes \overline{L_2}\right).
\end{tikzcd}$$

The homomorphism ζ will be the isomorphism defined in Proposition 6.2.4 (5); as for χ, we define it on generators as

$$\chi\left(\left(l_1 + l_2\right) \wedge \left(t_1 + t_2\right)\right) = l_1 \wedge t_1 + l_2 \wedge t_2 - (-1)^{|t_1||l_2|}\, \overline{t_1} \otimes \overline{l_2} + \overline{l_1} \otimes \overline{t_2} + \operatorname{Im} \alpha_R.$$

Routine calculations show that χ is well defined and it extends to an homomorphism of Lie superalgebras. We construct the inverse homomorphism as follows: consider the maps

$$\begin{aligned}
\varphi^1 &: L_1 \wedge T_1 \to \left(L_1 \oplus L_2\right) \wedge \left(T_1 \oplus T_2\right), & l_1 \wedge t_1 &\mapsto l_1 \wedge t_1;\\
\varphi^2 &: L_2 \wedge T_2 \to \left(L_1 \oplus L_2\right) \wedge \left(T_1 \oplus T_2\right), & l_2 \wedge t_2 &\mapsto l_2 \wedge t_2;\\
\varphi^3 &: \overline{T_1} \otimes \overline{L_2} \to \left(L_1 \oplus L_2\right) \wedge \left(T_1 \oplus T_2\right), & \overline{t_1} \otimes \overline{l_2} &\mapsto -(-1)^{|t_1||l_2|}\, l_2 \wedge t_1;\\
\varphi^4 &: \overline{L_1} \otimes \overline{T_2} \to \left(L_1 \oplus L_2\right) \wedge \left(T_1 \oplus T_2\right), & \overline{l_1} \otimes \overline{t_2} &\mapsto l_1 \wedge t_2;
\end{aligned}$$

all of them are well defined and extend to Lie superalgebra homomorphisms. Also, φ^3 and φ^4 define another homomorphism, $\tilde{\varphi} \colon \operatorname{coker} \alpha_R \to \left(L_1 \oplus L_2\right) \wedge \left(T_1 \oplus T_2\right)$ with $\tilde{\varphi}\left(\overline{t_1} \otimes \overline{l_2} + \overline{l_1} \otimes \overline{t_2} + \operatorname{Im} \alpha\right) = \varphi^3\left(\overline{t_1} \otimes \overline{l_2}\right) + \varphi^4\left(\overline{l_1} \otimes \overline{t_2}\right)$.

Finally, we construct $\varphi = \langle \varphi^1, \varphi^2, \tilde{\varphi} \rangle$

$$\varphi \colon \left(L_1 \wedge T_1\right) \oplus \left(L_2 \wedge T_2\right) \oplus \operatorname{coker} \alpha_R \to \left(L_1 \oplus L_2\right) \wedge \left(T_1 \oplus T_2\right),$$

which is an inverse to χ. The pair of isomorphisms (χ, ζ) satisfies the conditions to be an isomorphism of crossed modules, and the theorem is proved.

□

We will finish this section by studying the kernel of the projection

$$\pi = \left(\pi_1, \pi_2\right) \,:\, (T, L, \partial) \otimes (T, L, \partial) \to (T, L, \partial) \wedge (T, L, \partial)\,.$$

First, we present a definition which will be fundamental for the last theorem of the section (cf. [189] for the case of abelian crossed modules of groups, and [193] for abelian crossed modules of Lie algebras).

Definition 6.3.11. Let (A, B, ∂) an abelian crossed module, and denote by $B\underline{\otimes}A$ the tensor product $B \otimes A$ subject to the homogeneous relation

$$\partial\,(a) \otimes a' = (-1)^{|a||a'|}\,\partial\left(a'\right) \otimes a,$$

for all $a, a' \in A$. Consider also the Lie homomorphism

$$\begin{aligned} f\,:\, A \otimes A &\to \left(B\underline{\otimes}A\right) \oplus \Gamma\,(A) \\ a \otimes a' &\mapsto \partial\,(a) \otimes a' - a \otimes a', \end{aligned}$$

and denote $\widetilde{\Gamma}\,(A, B, \partial) \;:=\;$ coker f. Then we define $\Gamma\,(A, B, \partial)$ to be the abelian crossed module $\left(\widetilde{\Gamma}\,(A, B, \partial)\,, \Gamma\,(B)\,, \partial_\Gamma\right)$, where ∂_Γ is determined by

$$\partial_\Gamma\left(b \otimes a + e_{a_{\bar{0}}} + \alpha \otimes \alpha'\right) = e_{\partial(a_{\bar{0}})} + b \otimes \partial\,(a) + \partial\,(\alpha) \otimes \partial\left(\alpha'\right).$$

Theorem 6.3.12. *Let (T, L, ∂) be a crossed module such that ∂ is surjective or the action of L on T is trivial. Then, there is an exact sequence*

$$\Gamma\left((T, L, \partial)_{\mathrm{ab}}\right) \xrightarrow{(\eta_1,\eta_2)} (T, L, \partial) \otimes (T, L, \partial) \xrightarrow{(\pi_1,\pi_2)} (T, L, \partial) \wedge (T, L, \partial) \longrightarrow 0.$$

Proof. Let $\overline{T} \;=\; T/[L, T]$ and $\overline{L} \;=\; L_{\mathrm{ab}}$. We want to find a morphism of crossed modules $\eta :\, \Gamma\left((T, L, \partial)_{\mathrm{ab}}\right) \to (T, L, \partial) \otimes (T, L, \partial)$ such that Im $\eta = (I, L\Box L, \delta)$; i.e. two surjective homomorphisms of Lie superalgebras η_1, η_2 making commutative

$$\begin{tikzcd} \widetilde{\Gamma}\left((T, L, \partial)_{\mathrm{ab}}\right) \arrow[r, "\partial_\Gamma"] \arrow[d, "\eta_1"'] & \Gamma\left(\overline{L}\right) \arrow[d, "\eta_2"] \\ I \arrow[r, "\delta"'] & L\Box L. \end{tikzcd}$$

Note that such homomorphisms preserve the actions of the crossed modules, as $L\square L$ acts trivially on $I \subseteq \operatorname{coker} \alpha$, as it was pointed out in Lemma 6.3.3.

The component η_2 will be the homomorphism η given in Proposition 6.2.4 (9). As for η_1, it will be induced by a homomorphism $g = \langle g_1, g_2 \rangle \colon \left(\overline{L}\underline{\otimes}\overline{T}\right) \oplus \Gamma\left(\overline{T}\right) \to I$ which vanishes on $\operatorname{Im} f$.

We define $\widetilde{g_1} \colon \overline{L}\otimes\overline{T} \to I$ on generators as $\widetilde{g_1}\left(\bar{l} \otimes \bar{t}\right) = (-1)^{|l||t|}\, t\otimes l + l\otimes t + \operatorname{Im}\alpha$. Easy calculations lead us to check that $\widetilde{g_1}$ is well defined, and also that it induces another Lie homomorphism $g_1 \colon \overline{L}\underline{\otimes}\overline{T} \to I$.

We will explain the construction of g_2 more carefully, due to the complexity of the object $\Gamma\left(\overline{T}\right)$ comparatively to the classical case of modules. We define

$$\begin{aligned} &\widetilde{h_1} \colon \overline{T_{\bar{0}}} \to I, && \overline{t_{\bar{0}}} \mapsto \partial\left(t_{\bar{0}}\right) \otimes t_{\bar{0}} + \operatorname{Im}\alpha; \\ &h_2 \colon \overline{T} \otimes \overline{T} \to I, && \bar{t} \otimes \overline{t'} \mapsto \partial(t) \otimes t' + (-1)^{|t||t'|}\, \partial\left(t'\right) \otimes t + \operatorname{Im}\alpha. \end{aligned}$$

The map $\widetilde{h_1}$ is well defined: if $t_{\bar{0}} = t'_{\bar{0}} + x$, with $x \in [L, T]$, we have that

$$\begin{aligned} \widetilde{h_1}\left(\overline{t_{\bar{0}}}\right) &= \partial\left(t_{\bar{0}}\right) \otimes t_{\bar{0}} + \operatorname{Im}\alpha \\ &= \partial\left(t'_{\bar{0}}\right) \otimes t'_{\bar{0}} + \partial\left(t'_{\bar{0}}\right) \otimes x + \partial(x) \otimes t'_{\bar{0}} + \partial(x) \otimes x + \operatorname{Im}\alpha = \widetilde{h_1}\left(\overline{t'_{\bar{0}}}\right), \end{aligned}$$

as Lemma 6.3.6 (1) assures that $\partial\left(t'_{\bar{0}}\right)\otimes x = -\partial(x)\otimes t'_{\bar{0}}$ (and, in particular, $\partial(x)\otimes x = -\partial(x) \otimes x = 0$). Furthermore, the map is a homomorphism of supermodules, and it induces another homomorphism

$$\begin{aligned} h_1 \colon R^{\overline{T_{\bar{0}}}} &\to I \\ e_{\overline{t_{\bar{0}}}} &\mapsto \widetilde{h_1}\left(\overline{t_{\bar{0}}}\right). \end{aligned}$$

Regarding h_2, Lemma 6.3.6 (1) again allows to check that it is well defined, and it leads to another homomorphism of supermodules. We define now

$$\begin{aligned} \widetilde{g_2} = \langle h_1, h_2 \rangle \colon R^{\overline{T_{\bar{0}}}} \oplus \left(\overline{T} \otimes \overline{T}\right) &\to I \\ e_{\overline{t_{\bar{0}}}} + \bar{t} \otimes \overline{t'} &\mapsto h_1\left(e_{\overline{t_{\bar{0}}}}\right) + h_2\left(\bar{t} \otimes \overline{t'}\right). \end{aligned}$$

It is easy to see that $\widetilde{g_2}$ vanishes on the relations of Definition 6.1.1; then, it induces a homomorphism of supermodules $g_2 \colon \Gamma\left(\overline{T}\right) \to I$. Giving to $\Gamma\left(\overline{T}\right)$ the trivial structure of abelian Lie superalgebra, g_2 will be also homomorphism of Lie superalgebras due to Lemma 6.3.6 (2).

Finally, $g = \langle g_1, g_2 \rangle \colon \left(\overline{L}\,\underline{\otimes}\,\overline{T}\right) \oplus \Gamma\left(\overline{T}\right) \to I$ is a homomorphism of Lie superalgebras which vanishes on $\operatorname{Im} f$, as $g\left(f\left(\bar{t} \otimes \overline{t'}\right)\right) \in \operatorname{Im} \alpha$. Then, g induces $\eta_1 \colon \widetilde{\Gamma}\left((T, L, \partial)_{\mathrm{ab}}\right) \to I$, which is surjective and satisfies $\delta\eta_1 = \eta_2\partial_\Gamma$. The theorem is proved. □

Note that Proposition 6.3.7 can also be obtained as a trivial corollary of Theorem 6.3.12.

6.4 Applications to the second homology of crossed modules

This section is devoted to deal with the connection between the exterior products with the second homologies of crossed modules of Lie superalgebras, and generalise some known results of the second homology of Lie superalgebras to the second homology of crossed modules.

It will be shown that the category **XSLie** of crossed modules of Lie superalgebras is an algebraic category, that is, there exists a tripleable forgetful functor from **XSLie** to the category of $\mathbb{Z}_2$-graded sets, **2−Set**.

Firstly we construct an adjoint pair of functors $\mathbf{2{-}Set} \underset{\mathcal{U}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathbf{XSLie}$. In fact, the usual forgetful functor $\mathcal{U}_1 \colon$ **SLie** $\to$ **2−Set** has a left adjoint $\mathcal{F}_1 \colon$ **2−Set** $\to$ **SLie**, where $\mathcal{F}_1(X)$ is the free Lie superalgebra on X. On the other hand, there is a faithful functor $\mathcal{U}_2 \colon$ **XSLie** $\to$ **SLie**, which assigns to a crossed module (T, L, ∂) the direct product of Lie superalgebras $T \times L$. Now, we define the functor $\mathcal{F}_2 \colon$ **SLie** $\to$ **XSLie** as follows: for any Lie superalgebra M, let $\mathcal{F}_2(M)$ denote the inclusion crossed module $\left(\overline{M}, M * M, \mathrm{inc}\right)$, where $M * M$ is the coproduct of M with itself, with the natural inclusions $i_1, i_2 \colon M \to M * M$, and $\overline{M}$ is the kernel of the retraction $p_2 \colon M * M \to M$ determined by the conditions $p_2 i_1 = 0$ and $p_2 i_2 = \mathrm{id}_M$.

Proposition 6.4.1. *The functor $\mathcal{F}_2$ is left adjoint to the functor $\mathcal{U}_2$.*

Proof. Let M be a Lie superalgebra. Our assertion is that the homomorphism

$$\left(i_1, i_2\right) : M \to \overline{M} \times (M * M)$$

is a universal arrow from M to the functor $\mathcal{U}_2$.

Given a crossed module (T, L, ∂), let $f_T : M \to T, f_L : M \to L$ be two homomorphisms of Lie superalgebras and define the homomorphism $\left(f_T, f_L\right) : M \to T \times L = \mathcal{U}_2 (T, L, \partial)$. Denoting by $i : T \to T \rtimes L$, $j : L \to T \rtimes L$ the natural inclusions, we construct the following commutative diagram with split exact sequences of Lie superalgebras:

$$\begin{tikzcd}
\overline{M} \arrow[r, tail, "\mathrm{inc}"] \arrow[d, dashed, "g_T"'] & M * M \arrow[r, shift left, "p_2"] \arrow[d, dashed, "h"'] & M \arrow[l, shift left, "i_2"] \arrow[d, "f_L"] \\
T \arrow[r, tail, "i"] & T \rtimes L \arrow[r, shift left, "p"] & L, \arrow[l, shift left, "j"]
\end{tikzcd}$$

where h is the Lie superalgebra homomorphism satisfying $hi_1 = if_T$ and $hi_2 = jf_L$ defined by the universal property of the coproduct, and g_T is the restriction of h to $\overline{M}$. It also holds that $ph = f_L p_2$, as can be easily checked. Again, the universal property of the coproduct defines a unique Lie superalgebra homomorphism $g_L : M * M \to L$ satisfying $g_L i_1 = \partial f_T$ and $g_L i_2 = f_L$.

Next, we will show that $\left(g_T, g_L\right) : \left(\overline{M}, M * M, \mathrm{inc}\right) \to (T, L, \partial)$ is the unique morphism of crossed modules satisfying $\left(g_T \times g_L\right)\left(i_1, i_2\right) = \left(f_T, f_L\right)$. Let $m \in M$ and $\bar{m} \in \overline{M}$ be arbitrary. We have

$$g_T[i_1(m), \bar{m}] = [f_T(m), g_T(\bar{m})] = {}^{\partial f_T(m)} g_T(\bar{m}) = {}^{g_L i_1(m)} g_T(\bar{m}),$$
$$g_T[i_2(m), \bar{m}] = [j f_L(m), i g_T(\bar{m})] = {}^{f_L(m)} g_T(\bar{m}) = {}^{g_L i_2(m)} g_T(\bar{m});$$

it follows that $g_T[z, \bar{m}] = {}^{g_L(z)} g_T(\bar{m})$ for any $z \in M * M$ and $\bar{m} \in \overline{M}$.

Furthermore, since $\overline{M}$ is the graded ideal of $M * M$ generated by $\mathrm{Im}\, i_1$, it holds that

$$\partial g_T i_1(m) = \partial f_T(m) = g_L i_1(m),$$

$$\begin{aligned}\partial g_T[z, i_1(m)] &= \partial\left({}^{g_L(z)}g_T i_1(m)\right) = \partial\left({}^{g_L(z)}f_T(m)\right) = [g_L(z), \partial f_T(m)]\\ &= [g_L(z), g_L i_1(m)] = g_L[z, i_1(m)],\end{aligned}$$

and hence $\partial g_T = g_L$ inc.

We conclude that (g_T, g_L) is a morphism of crossed modules such that

$$(g_T \times g_L)(i_1, i_2) = (f_T, f_L),$$

and it is clearly the unique satisfying this condition.

□

Composing the functors $\mathcal{U}_2$ and $\mathcal{U}_1$, we have the underlying set functor

$$\mathcal{U} = \mathcal{U}_1\mathcal{U}_2 : \ \mathbf{XSLie} \to \mathbf{2{-}Set}, \quad (T, L, \partial) \mapsto T \times L,$$

which assigns to any crossed module (T, L, ∂) the cartesian product of the underlying $\mathbb{Z}_2$-graded sets of the Lie superalgebras T and L. Therefore, the functor $\mathcal{F} = \mathcal{F}_2\mathcal{F}_1 : \ \mathbf{2{-}Set} \to \mathbf{XSLie}$ is left adjoint to $\mathcal{U}$. Let $\mathbb{G} = (G, \varepsilon, \delta)$ be the cotriple on the category **XSLie** generated by the adjoint pair of functors $\mathcal{U}$ and $\mathcal{F}$, $G = \mathcal{F}\mathcal{U}$.

Following [79], which extends the Barr and Beck's definition of cotriple homology [20] to the semiabelian context, we define the cotriple homology of crossed modules of Lie superalgebras.

Definition 6.4.2. The n-th cotriple homology of the crossed module (T, L, ∂) is defined by the formula

$$H_n(T, L, \partial) = H_{n-1}N\mathbb{G}(T, L, \partial)_{\mathrm{ab}}, \qquad n \geq 1.$$

This defines a functor $H_n : \ \mathbf{XSLie} \to \mathbf{Ab}$, for any $n \geq 1$.

Analogously to the theory developed in [76] for Lie algebras, we can develop a relative homology theory for Lie superalgebras. In particular, we obtain the following results.

Let $0 \to P \to L \xrightarrow{\psi} Q \to 0$ be a short exact sequence of Lie superalgebras, and $\mathbb{S} = (S, \varepsilon_1, \delta_1)$ the cotriple on the category **SLie** generated by the adjoint pair of functors $\mathcal{U}_1$ and $\mathcal{F}_1$, $S = \mathcal{F}_1\mathcal{U}_1$, $\varepsilon_1 : S \to 1_{\mathbf{SLie}}$ and $\delta_1 : S \to S^2$. Consider the surjective simplicial Lie superalgebras homomorphism

$$\psi_{*\mathrm{ab}} : \ (S^*L)_{\mathrm{ab}} \to (S^*Q)_{\mathrm{ab}}.$$

The n-th relative derived functor $H_n(L; P)$ is defined as the n-th homotopy group $\pi_n\left(\ker\left(\psi_{*\mathrm{ab}}\right)\right)$.

Proposition 6.4.3 (see [76, Theorem 34])**.** *For any graded ideal P of a Lie superalgebra L, there is the following natural long exact sequence:*

$$\cdots \to H_{n+1}(L/P) \to H_n(L;P) \to H_n(L) \to H_n(L/P) \to \cdots \to H_1(L/P) \to 0.$$

Proposition 6.4.4 (see [76, Theorem 35])**.** *For any graded ideal P of a Lie superalgebra P, we have the isomorphism*

$$H_2(L;P) \cong \ker(L \wedge P \to P).$$

Applying Proposition 6.4.3 and Proposition 6.4.4, as well as [91, Proposition 6.3], we obtain the following fundamental result.

Proposition 6.4.5. *If (Y, F, μ) is a projective crossed module, then*

$$(Y, F, \mu) \wedge (Y, F, \mu) \cong ([F, Y], [F, F], \mu).$$

Given a projective presentation $0 \to (V, R, \mu) \to (Y, F, \mu) \to (T, L, \partial) \to 0$ of the crossed module (T, L, ∂), the Hopf formula provided in [79] assures that

$$H_2(T, L, \partial) \cong \left(\frac{V \cap [F, Y]}{[R, Y] + [F, V]}, \frac{R \cap [F, F]}{[F, R]}, \overline{\mu}\right).$$

The above proposition and the Hopf formula allow us to prove the following theorem.

Theorem 6.4.6. *With the above notation, there is an isomorphism*

$$(T, L, \partial) \wedge (T, L, \partial) \cong \frac{[(Y, F, \mu), (Y, F, \mu)]}{[(V, R, \mu), (Y, F, \mu)]}.$$

In particular, $H_2(T, L, \partial) \cong \ker((T, L, \partial) \wedge (T, L, \partial) \to (T, L, \partial))$.

Proof. Consider the exact sequence

$$(V, R, \mu) \wedge (Y, F, \mu) \to (Y, F, \mu) \wedge (Y, F, \mu) \to (T, L, \partial) \wedge (T, L, \partial) \to 0,$$

where the first morphism is ρ from Proposition 6.3.9 (2), and the second one is given by the projective presentation of (T, L, ∂). Then,

$$(T, L, \partial) \wedge (T, L, \partial) \cong \frac{(Y, F, \mu) \wedge (Y, F, \mu)}{\operatorname{Im} \rho}.$$

Identifying $(Y, F, \mu) \wedge (Y, F, \mu)$ with $(F \wedge Y, F \wedge F, \mathrm{id} \wedge \mu)$ by Proposition 6.3.5 (1b), it is easy to see that $\operatorname{Im} \rho_1$ is isomorphic to the ideal of $F \wedge Y$ generated by all $f \wedge v$ and $r \wedge y$ with $f \in F$, $v \in V$, $r \in R$, $y \in Y$, and that $\operatorname{Im} \rho_2$ is isomorphic to the ideal of $F \wedge F$ generated by all $r \wedge f$ with $r \in R$ and $f \in F$. By the isomorphism in Proposition 6.4.5, it holds that

$$(T, L, \partial) \wedge (T, L, \partial) \cong \frac{([F, Y], [F, F], \mu)}{([F, V] + [R, Y], [R, F], \mu)} = \frac{[(Y, F, \mu), (Y, F, \mu)]}{[(V, R, \mu), (Y, F, \mu)]}. \qquad \square$$

CHAPTER 7

On Whitehead's quadratic functor for supermodules

We devote this chapter to study the properties of Whitehead's quadratic functor for supermodules and for abelian crossed modules of Lie superalgebras, introduced in Chapter 6.

Introduction

Whitehead first introduced his quadratic functor Γ for abelian groups in [232]. He used it to construct a long exact sequence in the context of homotopy theory, yielding an invariant for four-dimensional CW-complexes. This construction was later generalised by Simson and Tyc [208] for arbitrary modules over a commutative ring R in connection with the study of stable derived functors. They explored some of its basic properties, proving in particular that this object satisfies a universal property regarding quadratic maps between R-modules: namely, every quadratic map $b\colon M \to N$ factorises through $\Gamma(M)$. After this, Ellis also related this version of Whitehead's quadratic functor to his just introduced non-abelian tensor and exterior products of Lie algebras in [76].

Further generalisations of Whitehead's quadratic functor were given in the context of abelian crossed modules, both of groups [189] and of Lie algebras [193]. The definition of these objects led to advances in the homology theory of crossed modules [189] and in the study of non-abelian tensor and exterior products of crossed modules of groups [200] and Lie algebras [193].

In Chapter 6, we offered two new generalisations of Whitehead's quadratic functor, namely for supermodules and for abelian crossed modules of Lie superalgebras. We devote the present chapter to study some of their properties. In particular, we prove that the version for supermodules satisfies a universal property regarding quadratic maps between supermodules in the sense of [180]. However, these maps do not seem to have been introduced in the framework of crossed modules, so we offer a definition of quadratic maps between abelian crossed modules of Lie superalgebras (with its straightforward particular case for abelian crossed modules of Lie algebras) to be those maps factorising through $\Gamma(A, B, \partial)$.

The structure of this chapter is as follows. The first preliminary Section 7.1 recalls some definitions and properties which will be needed later. In Section 7.2, we study some properties of Whitehead's quadratic functor for supermodules introduced in Definition 6.1.1 (see Chapter 6, page 185), such as its relation with quadratic maps of supermodules, with the symmetric algebra $S^2(M)$ of a supermodule M and with the non-abelian tensor and exterior products of Lie superalgebras. Finally, Section 7.3 addresses similar problems for Whitehead's quadratic functor for abelian crossed modules of Lie superalgebras, according to Definition 6.3.11 (see Chapter 6, page 200). In particular, we introduce a definition for quadratic maps between abelian crossed modules of Lie superalgebras.

Throughout this chapter, R will denote a unital commutative ring. Unless otherwise stated, all the (super)modules and (super)algebras in this chapter will be considered over R.

7.1 Preliminaries

We devote this section to recall some basic definitions and properties regarding Whitehead's quadratic functor of modules and also regarding supermodules.

Let M be a module. Recall from [208] that the *universal quadratic functor* $\Gamma(M)$ is defined to be the module generated by the elements e_m, for all $m \in M$, subject to the relations

$$
\begin{aligned}
0 &= e_{\lambda m} - \lambda^2 e_m;\\
0 &= e_{\lambda m+m'} + \lambda e_m + \lambda e_{m'} - \lambda e_{m+m'} - e_{\lambda m} - e_{m'};\\
0 &= e_{m+m'+m''} + e_m + e_{m'} + e_{m''} - e_{m+m'} - e_{m+m''} - e_{m'+m''},
\end{aligned}
$$

for all $m, m', m'' \in M$ and $\lambda \in R$.

This object is strongly related to the *quadratic maps* between modules, namely those maps $\varphi\colon M \to N$ satisfying:

1. $\varphi(\lambda m) = \lambda^2 \varphi(m)$.
2. The associated symmetric function $b_\varphi\colon M \times M \to N$ defined by $b_\varphi(m, m') = \varphi(m + m') - \varphi(m) - \varphi(m')$ is bilinear.

Indeed, the map $\gamma\colon M \to \Gamma(M)$ defined by $\gamma(m) = e_m$ is quadratic. Also, given any quadratic map φ from M to another module N, there exists a unique homomorphism of modules $h\colon \Gamma(M) \to N$ making the following diagram commutative:

$$\begin{tikzcd}
M \arrow[rr, "\gamma"] \arrow[dr, "\varphi"'] & & \Gamma(M) \arrow[dl, dashed, "h"] \\
& N. &
\end{tikzcd}$$

This homomorphism h is determined by $h(e_m) = \varphi(m)$.

We recall now a less common construction of the same object $\Gamma(M)$, which can be found in [114] and is more suitable for our purposes. To do so, it is convenient to handle the notation R^M, the free module generated by the elements e_m for all $m \in M$.

Indeed, construct the module $\Gamma'(M)$ as the direct sum $R^M \oplus (M \otimes_R M)$ subject to the following relations:

$$\begin{aligned}
&e_{\lambda m} = \lambda^2 e_m;\\
&e_{m+m'} - e_m - e_{m'} = m \otimes m',
\end{aligned}$$

for all $m, m' \in M$ and all $\lambda \in R$, and note that the map $\gamma'\colon M \to \Gamma'(M)$ given by $\gamma'(m) = e_m$ is quadratic. Also, $\Gamma'(M)$ satisfies the same universal property than $\Gamma(M)$: let $\varphi\colon M \to N$ be a quadratic map, and define $h'\colon \Gamma'(M) \to N$ as $h'(e_x + y \otimes z) = \varphi(x) + b_\varphi(y, z)$. This map h' is well defined and is the unique homomorphism from $\Gamma'(M)$ to N which satisfies $h'\gamma' = \varphi$. It follows that $\Gamma(M)$ and $\Gamma'(M)$ are isomorphic. As a consequence, we will henceforth identify these two objects, employing the notation $\Gamma(M)$ for both of them.

The basic properties of the module $\Gamma(M)$ were studied in [208]. Also, Ellis explored its relation with the non-abelian tensor product of Lie algebras in [76].

For a basic background in Lie superalgebras and crossed modules of Lie superalgebras, we refer the reader to Chapter 6. Additionally, we include now two definitions which will be necessary for the next section.

Recall from Chapter 2 that the tensor superalgebra $T(M)$ of a supermodule M is the sum $\bigoplus_{n\geq 0} M^{\otimes_n}$ with juxtaposition as product, where $M^{\otimes_0} := R$, $M^{\otimes_1} := M$ and $M^{\otimes_n} := M \otimes_R \overset{(n)}{\cdots} \otimes_R M$ for $n \geq 2$. To obtain the symmetric superalgebra $S(M)$, we take the quotient of $T(M)$ by the graded ideal generated by the elements $m \otimes m' - (-1)^{|m||m'|} m' \otimes m$, with $m, m' \in M$; the image of $M^{\otimes_n}$ through this quotient is denoted $S^n(M)$. We denote the class of the elements $x_1 \otimes \cdots \otimes x_n$ by $x_1 \wedge \cdots \wedge x_n$.

In particular, $S^2(M)$ has the following universal property: given another supermodule N and a symmetric bilinear map $b\colon M \times M \to N$, there exists a unique homomorphism of supermodules $\theta\colon S^2(M) \to N$ such that $\theta(x \wedge y) = b(x, y)$.

Finally, we recall from [181] the definition of quadratic maps between supermodules. Given M and N two supermodules, a homogeneous bilinear map $b\colon M \times M \to N$ is called *symmetric-alternating* if

$$
\begin{aligned}
&b(m, m') = (-1)^{|m||m'|} b(m', m);\\
&b(m_{\bar 1}, m_{\bar 1}) = 0,
\end{aligned}
$$

for all $m, m' \in M$, $m_{\bar 1} \in M_{\bar 1}$. Also, a *quadratic map* from M to N, denoted by $\varphi\colon M \to N$, is defined to be a pair $\varphi = (\varphi_{\bar 0}, b_\varphi)$ such that $\varphi_{\bar 0}\colon M_{\bar 0} \to N_{\bar 0}$ is a quadratic map between modules, and $b_\varphi\colon M \times M \to N$ is a symmetric-alternating bilinear map of degree $\bar 0$, compatible with $\varphi_{\bar 0}$ in the sense that

$$
b_\varphi(m_{\bar 0}, m'_{\bar 0}) = \varphi_{\bar 0}(m_{\bar 0} + m'_{\bar 0}) - \varphi_{\bar 0}(m_{\bar 0}) - \varphi_{\bar 0}(m'_{\bar 0}),
$$

for all $m_{\bar 0}, m'_{\bar 0} \in M_{\bar 0}$.

7.2 Whitehead's quadratic functor for supermodules

In this section, we study the basic properties of Whitehead's quadratic functor for supermodules introduced in Definition 6.1.1 (see Chapter 6, page 185). We present here a definition that slightly generalises Definition 6.1.1, as it is also valid for supermodules over rings in which 2 does not have an inverse. Note that the free supermodule $R^{M_{\bar 0}}$ is assumed to be concentrated in degree zero.

Definition 7.2.1 (cf. Definition 6.1.1)**.** Let M a supermodule. We define the supermodule $\Gamma(M)$ as the direct sum

$$
R^{M_{\bar 0}} \oplus (M \otimes_R M),
$$

subject to the homogeneous relations

$$e_{\lambda m_{\bar{0}}} = \lambda^2 e_{m_{\bar{0}}}; \tag{7.2.1}$$

$$e_{m_{\bar{0}}+m'_{\bar{0}}} - e_{m_{\bar{0}}} - e_{m'_{\bar{0}}} = m_{\bar{0}} \otimes m'_{\bar{0}}; \tag{7.2.2}$$

$$m \otimes m' = (-1)^{|m||m'|} m' \otimes m; \tag{7.2.3}$$

$$m_{\bar{1}} \otimes m_{\bar{1}} = 0, \tag{7.2.4}$$

where $\lambda \in R$, $m_{\bar{0}}, m'_{\bar{0}} \in M_{\bar{0}}$, $m_{\bar{1}} \in M_{\bar{1}}$ and $m, m' \in M$, with the induced grading.

Note that the difference with respect to Definition 6.1.1 is precisely the relation (7.2.4), which follows from relation (7.2.3) whenever 2 has an inverse in R.

Proposition 7.2.2. *The pair $\gamma = \gamma_M = (\gamma_{\bar{0}}, b_\gamma)$, with $\gamma_{\bar{0}} \colon M_{\bar{0}} \to \Gamma(M)_{\bar{0}}$ defined by $\gamma_{\bar{0}}(x_{\bar{0}}) = e_{x_{\bar{0}}}$ and $b_\gamma \colon M \times M \to \Gamma(M)$ defined by $b_\gamma(x, y) = x \otimes y$ is a quadratic map.*

Proof. Routine. □

The next proposition shows that the pair $(\Gamma(M), \gamma)$ is universal with respect to the quadratic maps with domain M. We introduce the following piece of notation: if $h \colon \Gamma(M) \to N$ is a homomorphism of supermodules, with restriction $h_{\bar{0}}$ to $\Gamma(M)_{\bar{0}}$, such that $h_{\bar{0}}\gamma_{\bar{0}} = \varphi_{\bar{0}}$ and $hb_\gamma = b_\varphi$, we will limit to write $h\gamma = \varphi$.

Proposition 7.2.3. *Let M and N be two supermodules, and let $\varphi \colon M \to N$, $\varphi = (\varphi_{\bar{0}}, b_\varphi)$, be a quadratic map. Then, there exists a unique homomorphism of supermodules $h \colon \Gamma(M) \to N$ such that $h\gamma = \varphi$.*

$$\begin{tikzcd} M_{\bar{0}} \arrow[rr, "\gamma_{\bar{0}}"] \arrow[dr, "\varphi_{\bar{0}}"'] & & \Gamma(M)_{\bar{0}} \arrow[dl, dashed, "h_{\bar{0}}"] \\ & N_{\bar{0}} & \end{tikzcd}$$

$$\begin{tikzcd} M \times M \arrow[rr, "b_\gamma"] \arrow[dr, "b_\varphi"'] & & \Gamma(M) \arrow[dl, dashed, "h"] \\ & N & \end{tikzcd}$$

Proof. It suffices to define $h\colon \Gamma(M) \to N$ by

$$h\left(e_{m_{\bar{0}}} + m' \otimes m''\right) = \varphi_{\bar{0}}\left(m_{\bar{0}}\right) + b_{\varphi}\left(m', m''\right).$$

□

We would like to stress the fact that it does not seem possible to construct $\Gamma(M)$ in a similar way to the original definition for modules [208]. The reason is that, to construct the free superalgebra over a set S, this set must admit a $\mathbb{Z}_2$-grading, and this is not the case of the underlying set of M. Thus, the construction offered in [114] appears to be much more suitable to be adapted to supermodules.

Note that Whitehead's quadratic functor for supermodules Γ is indeed an endofunctor in the category **SMod** of supermodules. It carries each object M to $\Gamma(M)$, and acts on morphisms in the following way: given $f\colon M \to N$ a morphism in **SMod**, the composition $\gamma_N f$ is a quadratic map from M to $\Gamma(N)$. Then, Proposition 7.2.3 yields the desired morphism $\Gamma(f) = h$.

The rest of this section will be devoted to prove some elementary properties of the functor Γ.

Proposition 7.2.4. *Let M and N be two supermodules. There is an isomorphism*

$$\Gamma(M \oplus N) \simeq \Gamma(M) \oplus \Gamma(N) \oplus \left(M \otimes_R N\right).$$

Proof. We will begin by constructing a quadratic map from $M \oplus N$ to $\Gamma(M) \oplus \Gamma(N) \oplus \left(M \otimes_R N\right)$. Define

$$\begin{aligned}
&\varphi^1_{\bar{0}}\colon (M \oplus N)_{\bar{0}} \to \Gamma(M)_{\bar{0}}, && m_{\bar{0}} + n_{\bar{0}} \mapsto e_{m_{\bar{0}}};\\
&\varphi^2_{\bar{0}}\colon (M \oplus N)_{\bar{0}} \to \Gamma(N)_{\bar{0}}, && m_{\bar{0}} + n_{\bar{0}} \mapsto e_{n_{\bar{0}}};\\
&\varphi^3_{\bar{0}}\colon (M \oplus N)_{\bar{0}} \to (M \otimes_R N)_{\bar{0}}, && m_{\bar{0}} + n_{\bar{0}} \mapsto m_{\bar{0}} \otimes n_{\bar{0}};\\
&b^1_{\varphi}\colon (M \oplus N) \times (M \oplus N) \to \Gamma(M), && (m+n, m'+n') \mapsto m \otimes m';\\
&b^2_{\varphi}\colon (M \oplus N) \times (M \oplus N) \to \Gamma(N), && (m+n, m'+n') \mapsto n \otimes n';\\
&b^3_{\varphi}\colon (M \oplus N) \times (M \oplus N) \to \left(M \otimes_R N\right), && (m+n, m'+n') \mapsto m \otimes n'\\
& && \qquad +(-1)^{|m||m'|} n \otimes m';
\end{aligned}$$

The pairs $\varphi^1 = (\varphi^1_{\bar{0}}, b^1_{\varphi}), \varphi^2 = (\varphi^2_{\bar{0}}, b^2_{\varphi}), \varphi^3 = (\varphi^3_{\bar{0}}, b^3_{\varphi})$ are quadratic maps, and define a natural quadratic map φ from $M \oplus N$ to $\Gamma(M) \oplus \Gamma(N) \oplus (M \otimes_R N)$. The universal property of the functor Γ exposed in Proposition 7.2.3 defines a homomorphism $h\colon \Gamma(M \oplus N) \to \Gamma(M) \oplus \Gamma(N) \oplus (M \otimes_R N)$.

On the other hand, let us consider the maps

$$\begin{aligned}
\phi^1_{\bar 0}\colon\ & M_{\bar 0} \to \Gamma(M \oplus N)_{\bar 0}, && m_{\bar 0} \mapsto e_{m_{\bar 0}};\\
b^1_{\phi}\colon\ & M \times M \to \Gamma(M \oplus N)_{\bar 0}, && m \otimes m' \mapsto m \otimes m';\\
\phi^2_{\bar 0}\colon\ & N_{\bar 0} \to \Gamma(M \oplus N)_{\bar 0}, && n_{\bar 0} \mapsto e_{n_{\bar 0}};\\
b^2_{\phi}\colon\ & N \times N \to \Gamma(M \oplus N)_{\bar 0}, && n \otimes n' \mapsto n \otimes n';\\
\phi^3\colon\ & M \times N \to \Gamma(M \oplus N), && (m,n) \mapsto m \otimes n.
\end{aligned}$$

The pair $\phi^1 = (\phi^1_{\bar 0}, b^1_{\phi})$ is a quadratic map, and it induces a homomorphism $\theta^1 : \Gamma(M) \to \Gamma(M \oplus N)$ thanks to the universal property of the functor Γ. Analogously, we obtain $\theta^2 : \Gamma(M) \to \Gamma(M \oplus N)$. Also, ϕ^3 is bilinear, and it induces the natural homomorphism $\theta^3 : M \otimes N \to \Gamma(M \oplus N)$. Finally, we get the homomorphism $\theta = \langle \theta^1, \theta^2, \theta^3 \rangle : \Gamma(M) \oplus \Gamma(N) \oplus \big(M \otimes_R N\big) \to \Gamma(M \oplus N)$.

It is easily checked that h and θ are inverse homomorphisms. $\square$

The following lemma will be useful to prove that the functor Γ preserves free supermodules.

Lemma 7.2.5. *Let M be a free supermodule and N an arbitrary supermodule. Let us consider an ordered basis of M, $\{x_i\}_{i\in I}$, composed by homogeneous elements and such that the elements of $M_{\bar 0}$ are less or equal than those of $M_{\bar 1}$. Set $|i| := |x_i|$. Let also $\{y_{ij}\}_{i,j\in I}$ be a family of elements of N such that $|y_{ij}| = |i| + |j|$ and $y_{ij} = (-1)^{|i||j|} y_{ji}$. Then, there exists a unique quadratic map $\varphi\colon M \to N$, $\varphi = (\varphi_{\bar 0}, b_{\varphi})$, such that $\varphi_{\bar 0}(x_i) = y_{ii}$ for $|i| = \bar 0$, and $b_{\varphi}(x_i, x_j) = y_{ij}$ for $i < j$.*

Proof. To construct φ, we will employ the procedure explained in the first example in [181, 1.10], i.e. constructing a bilinear map $b\colon M \times M \to N$ preserving the degree and defining $\varphi_{\bar 0}(m_{\bar 0}) = b(m_{\bar 0}, m_{\bar 0})$ and $b_{\varphi}(m, m') = b(m, m') + (-1)^{|m||m'|} b(m', m)$. The pair $(\varphi_{\bar 0}, b_{\varphi})$ will be a quadratic map.

Define

$$b(x_i, x_j) = \begin{cases} y_{ij} & \text{if } i \leq j, \\ & \\ 0 & \text{otherwise,} \end{cases}$$

and extend b by bilinearity. Indeed, it preserves the degree, and defines a quadratic map $\varphi = (\varphi_{\bar{0}}, b_\varphi)$ in the required conditions. The uniqueness is immediate. □

Proposition 7.2.6. *Let M be a free supermodule, and let $\{\overline{x_i}\}_{i \in I_{\bar{0}}} \cup \{\overline{x_i}\}_{i \in I_{\bar{1}}}$ be an ordered basis of M composed by homogeneous elements and such that the elements $\{x_i\}_{i \in I_{\bar{0}}}$ of $M_{\bar{0}}$ are less or equal than those of $M_{\bar{1}}$, $\{x_i\}_{i \in I_{\bar{1}}}$. Then, $\Gamma(M)$ is free with basis $\left\{\gamma_{\bar{0}}\left(x_i\right)\right\}_{i \in I_{\bar{0}}} \cup \left\{b_\gamma\left(x_i, x_j\right)\right\}_{i,j \in I_{\bar{0}} \cup I_{\bar{1}},\, i<j}$.*

Proof. Let N be a supermodule. Every quadratic map $\varphi = \left(\varphi_{\bar{0}}, b_\varphi\right) : M \to N$ determines a homomorphism h from $\Gamma(M)$ to N; conversely, every homomorphism $h' : \Gamma(M) \to N$ determines a quadratic map $\varphi' : M \to N$ with $\varphi'_{\bar{0}}\left(x_i\right) = h'\left(e_{x_i}\right)$ for $i \in I_{\bar{0}}$, and $b'_\varphi(x_i, x_j) = h'\left(x_i \otimes x_j\right)$ for $i < j$. This correspondence is bijective, and therefore, applying Lemma 7.2.5, each homomorphism of supermodules $h : \Gamma(M) \to N$ is unequivocally determined by the images $h\left(e_{x_i}\right)$ for $i \in I_{\bar{0}}$ and $h\left(x_i \otimes x_j\right)$ for $i < j$. This means that $\left\{\gamma_{\bar{0}}\left(x_i\right)\right\}_{i \in I_{\bar{0}}} \cup \left\{b_\gamma\left(x_i, x_j\right)\right\}_{i,j \in I_{\bar{0}} \cup I_{\bar{1}},\, i<j}$ is a basis for $\Gamma(M)$. □

Next, we relate the universal quadratic and symmetric functors Γ and S^2.

Proposition 7.2.7. *Assume that* 2 *has an inverse in R, and let M be a supermodule. Then,*

$$S^2(M) \cong \Gamma(M).$$

Proof. The canonical map $b_\gamma : M \times M \to \Gamma(M)$ is bilinear and therefore induces a homomorphism of supermodules $g : S^2(M) \to \Gamma(M)$. On the other hand, the pair $\varphi = (\varphi_{\bar{0}}, b_\varphi) : M \to S^2(M)$ defined as $\varphi_{\bar{0}}(m_{\bar{0}}) = m_{\bar{0}} \wedge m_{\bar{0}}$ and $b_\varphi(m, m') = m \wedge m' + (-1)^{|m||m'|} m' \wedge m = 2m \wedge m'$ is a quadratic map and induces a homomorphism $h : \Gamma(M) \to S^2(M)$. It is easy to check that $hg = 2\,\mathrm{id}$ and $gh = 2\,\mathrm{id}$; therefore, h and $\frac{g}{2}$ are inverse. □

The next properties will relate Whitehead's quadratic functor for supermodules to the non-abelian tensor product of Lie superalgebras. We will need the following notation.

Let (M, P, ∂) and (N, P, σ) be two crossed modules of Lie superalgebras, and consider the following graded Lie subalgebra of $M \oplus N$:

$$M \times_P N = \{m + n \in M \oplus N \mid \partial(m) = \sigma(n)\}\,.$$

Set also

$$\langle M, N\rangle = \left\{-(-1)^{|m||n|}\,({}^{n}m) + {}^{m}n\right\}.$$

It is easy to check that $\langle M, N\rangle$ is a graded ideal of $M \times_P N$, and also that the quotient $Q := \frac{M\times_P N}{\langle M,N\rangle}$ is abelian. We will denote its elements by $\overline{m+n}$.

Proposition 7.2.8. *Let (M, P, ∂) and (N, P, σ) be two crossed modules of Lie superalgebras, and define $\psi : \Gamma(Q) \to M \otimes N$ by*

$$\psi\left(e_{\overline{\mu_{\bar 0}+\nu_{\bar 0}}} + \overline{m+n} \otimes \overline{m'+n'}\right) = m_{\bar 0} \otimes n_{\bar 0} + m \otimes n' + (-1)^{|m||n|} m' \otimes n.$$

Then, the following sequence of Lie superalgebras

$$\Gamma(Q) \xrightarrow{\ \psi\ } M \otimes N \xrightarrow{\ \pi\ } M \wedge N \longrightarrow 0$$

is exact.

Proof. It is clear that, if ψ is well defined, the sequence is exact, since $\operatorname{Im}\psi = M\square N = \ker\pi$. To see that ψ is well defined, let us consider $\widetilde{\psi} : R^{(M\times_P N)_{\bar 0}} \oplus \left(\left(M \times_P N\right) \otimes_R \left(M \times_P N\right)\right) \to M \otimes N$ and check that it induces ψ. Let us prove firstly the following equality:

$$\widetilde{\psi}\left((m+n) \otimes \left(-(-1)^{|m'||n'|}\left({}^{n'}m'\right) + {}^{m'}n'\right)\right) = 0.$$

We will prove it using the properties of crossed modules and the relations of the non-abelian tensor product. If $\partial(m) = \sigma(n) \neq 0$, it holds that $|m| = |n|$ and

$$\begin{aligned}&\widetilde{\psi}\left((m+n) \otimes \left(-(-1)^{|m'||n'|}\left({}^{n'}m'\right) + {}^{m'}n'\right)\right)\\ &= m \otimes {}^{m'}n' - (-1)^{|n|(|m'|+|n'|)}(-1)^{|m'||n'|}\, n \otimes {}^{n'}m'\end{aligned}$$

$$= [m,m'] \otimes n + (-1)^{|m||m'|} m' \otimes {}^{\partial(m)}n' - (-1)^{|n|(|m'|+|n'|)} (-1)^{|m'||n'|} \left({}^{n'}m'\right) \otimes n$$
$$= [m,m'] \otimes n + (-1)^{|m||m'|} m' \otimes {}^{\sigma(n)}n' - (-1)^{|n|(|m'|+|n'|)} (-1)^{|m'||n'|} \left({}^{n'}m'\right) \otimes n$$
$$= [m,m'] \otimes n + (-1)^{|m||m'|} m' \otimes [n,n'] - (-1)^{|n|(|m'|+|n'|)} (-1)^{|m'||n'|} \left({}^{n'}m'\right) \otimes n$$
$$= [m,m'] \otimes n + (-1)^{|n'|(|m'|+|n|)} (-1)^{|m||m'|} \left({}^{n'}m'\right) \otimes n$$
$$- (-1)^{|n||m'|} (-1)^{|m||m'|} \left({}^{\sigma(n)}m'\right) \otimes n' - (-1)^{|m||m'|} (-1)^{|n||n'|} (-1)^{|m'||n'|} \left({}^{n'}m'\right) \otimes n$$
$$= [m,m'] \otimes n - (-1)^{|m||m'|} (-1)^{|m||m'|} \left({}^{\partial(m)}m'\right) \otimes n' = [m,m'] \otimes n - [m,m'] \otimes n$$
$$= 0.$$

If $\partial(m) = \sigma(n) = 0$, the computations are similar but less involved, so we omit them. Analogously, $\widetilde{\psi}\left(\left(-(-1)^{|m'||n'|}\left({}^{n'}m'\right) + {}^{m'}n'\right) \otimes (m+n)\right) = 0$.

Now we check that $\widetilde{\psi}\left(e_{\mu_{\bar{0}}+\nu_{\bar{0}}+\langle M,N\rangle_{\bar{0}}}\right) = \widetilde{\psi}\left(e_{\mu_{\bar{0}}+\nu_{\bar{0}}}\right) = \mu_{\bar{0}} \otimes \nu_{\bar{0}}$ for all $\mu_{\bar{0}} + \nu_{\bar{0}} \in \left(M \times_P N\right)_{\bar{0}}$. Indeed, if $\partial(m) = \sigma(n) = 0$ the assertion is trivial; otherwise,

$$\widetilde{\psi}\left(e_{\mu_{\bar{0}}+\nu_{\bar{0}}-(-1)^{|m||n|}({}^{n}m)+{}^{m}n}\right)$$
$$= \widetilde{\psi}\left(e_{\mu_{\bar{0}}+\nu_{\bar{0}}}\right) + \widetilde{\psi}\left(e_{-(-1)^{|m||n|}({}^{n}m)+{}^{m}n}\right) + \widetilde{\psi}\left(\left(\mu_{\bar{0}}+\nu_{\bar{0}}\right) \otimes \left(-(-1)^{|m||n|}\left({}^{n}m\right) + {}^{m}n\right)\right)$$
$$= \mu_{\bar{0}} \otimes \nu_{\bar{0}} - (-1)^{|m||n|} \left({}^{n}m\right) \otimes {}^{m}n + \mu_{\bar{0}} \otimes {}^{m}n - (-1)^{|m||n|} \nu_{\bar{0}} \otimes {}^{n}m$$
$$= \mu_{\bar{0}} \otimes \nu_{\bar{0}} + [m \otimes n, m \otimes n] = \mu_{\bar{0}} \otimes \nu_{\bar{0}};$$

therefore, we can define $\overline{\psi}\colon\ R^{Q_{\bar{0}}} \oplus (Q \otimes Q) \to M \otimes N$.

We can easily check that $\overline{\psi}$ respects the relations (7.2.1)–(7.2.4); therefore, it induces ψ, which is a well-defined homomorphism. □

As a particular case, we recover Proposition 6.2.4(9) (see Chapter 6, page 189): for any Lie superalgebra M, there exists an exact sequence

$$\Gamma(M_{\mathrm{ab}}) \xrightarrow{\psi} M \otimes M \xrightarrow{\pi} M \wedge M \longrightarrow 0; \tag{7.2.5}$$

also, given I, J two graded ideals of M, the following sequence is exact:

$$\Gamma\left(\frac{I \cap J}{[I,J]}\right) \xrightarrow{\psi} I \otimes J \xrightarrow{\pi} I \wedge J \longrightarrow 0. \tag{7.2.6}$$

We finish this section with a sufficient condition for ψ to be injective.

Proposition 7.2.9. *Let M be a Lie superalgebra such that M^{ab} is a free supermodule. Then, the homomorphism ψ in the sequence* (7.2.5) *is injective.*

Proof. Let $\rho\colon M \otimes M \to M_{\mathrm{ab}} \otimes M_{\mathrm{ab}}$ be the natural projection, and consider $\left\{\overline{x_i}\right\}_{i\in I_{\bar 0}} \cup \left\{\overline{x_i}\right\}_{i\in I_{\bar 1}}$ an ordered basis of M_{ab}, composed by homogeneous elements and such that the elements $\{x_i\}_{i\in I_{\bar 0}}$ of $M_{\bar 0}$ are less or equal than those of $M_{\bar 1}$. Then, $\left\{\overline{x_i} \otimes \overline{x_j}\right\}_{i,j\in I_{\bar 0}\cup I_{\bar 1}}$ is a basis of $M_{\mathrm{ab}} \otimes M_{\mathrm{ab}}$, and $\left\{\left(e_{\overline{x_i}}\right)\right\}_{i\in I_{\bar 0}} \cup \left\{\overline{x_i} \otimes \overline{x_j}\right\}_{i,j\in I_{\bar 0}\cup I_{\bar 1},\ i<j}$ is a basis of $\Gamma(M_{\mathrm{ab}})$ by Proposition 7.2.6. The composition $\rho\psi$ carries the elements of a basis of $\Gamma(M_{\mathrm{ab}})$ to elements of a basis of $M_{\mathrm{ab}} \otimes M_{\mathrm{ab}}$, and therefore it is injective. It follows that ψ is injective. □

7.3 Whitehead's quadratic functor for abelian crossed modules of Lie superalgebras

A Whitehead's quadratic functor for abelian crossed modules was first defined by Pirashvili in [189] for abelian groups. Later, an analogue for abelian crossed modules of Lie algebras was given in [193]. Definition 6.3.11 (see Chapter 6, page 200) generalises this concept for abelian crossed modules of Lie superalgebras; as in Section 7.2, we include here a slightly more general definition which admits supermodules over rings without inverse of 2.

Definition 7.3.1 (cf. Definition 6.3.11)**.** Let (A, B, ∂) an abelian crossed module of Lie superalgebras, and denote by $B \underline{\otimes} A$ the tensor product $B \otimes A$ subject to the homogeneous relation

$$\partial(a) \otimes a' = (-1)^{|a||a'|}\partial(a') \otimes a,$$

for all $a, a' \in A$. Consider also the Lie homomorphism

$$\begin{aligned} f\colon A \otimes A &\to \left(B \underline{\otimes} A\right) \oplus \Gamma(A) \\ a \otimes a' &\mapsto \partial(a) \otimes a' - a \otimes a', \end{aligned}$$

and denote $\widetilde{\Gamma}(A, B, \partial) := \operatorname{coker} f$. Then we define $\Gamma(A, B, \partial)$ to be the abelian crossed module $\left(\widetilde{\Gamma}(A, B, \partial), \Gamma(B), \partial_\Gamma\right)$, where ∂_Γ is determined by

$$\partial_\Gamma\left(b \otimes a + \left(e_{a_{\bar{0}}} + \alpha \otimes \alpha'\right)\right) = e_{\partial(a_{\bar{0}})} + b \otimes \partial(a) + \partial(\alpha) \otimes \partial\left(\alpha'\right).$$

We define now the quadratic maps of abelian crossed modules of Lie superalgebras to be precisely the maps which factorise through $\Gamma(A, B, \partial)$. We highlight that we do not have knowledge of any reference in the literature dealing with this concept, and also that our definition admits the quadratic maps of abelian crossed modules of Lie algebras as a particular case.

Definition 7.3.2. Let (A, B, ∂) and (C, D, σ) be two abelian crossed modules of Lie superalgebras. We define a *quadratic map* between them as two pairs $\xi = \left(\xi_{\bar{0}}, b_\xi\right)$ and $\varphi = \left(\varphi_{\bar{0}}, b_\varphi\right)$ such that:

1. $\varphi : B \to D$ is a quadratic map between supermodules.
2. $\xi = \left(\xi_{\bar{0}}, b_\xi\right) : A \to C$ is such that:
   - $\xi_{\bar{0}} : A_{\bar{0}} \to C_{\bar{0}}$ is a quadratic map between modules;
   - $b_\xi : B \times A \to C$ is bilinear and satisfies:
     $$\begin{aligned} &b_\xi\left(\partial\left(a_{\bar{0}}\right), a'_{\bar{0}}\right) = \xi_{\bar{0}}\left(a_{\bar{0}} + a'_{\bar{0}}\right) - \xi_{\bar{0}}\left(a_{\bar{0}}\right) - \xi_{\bar{0}}\left(a'_{\bar{0}}\right),\\ &b_\xi\left(\partial(a), a'\right) = (-1)^{|a||a'|} b_\xi\left(\partial\left(a'\right), a\right),\\ &b_\xi\left(a_{\bar{1}}, a_{\bar{1}}\right) = 0,\end{aligned}$$
     for all $a_{\bar{0}}, a'_{\bar{0}} \in A_{\bar{0}}$, $a_{\bar{1}} \in A_{\bar{1}}$ and $a, a' \in A$;
3. $\sigma_{\bar{0}}\xi_{\bar{0}} = \varphi_{\bar{0}}\partial_{\bar{0}}$, where $\partial_{\bar{0}}$ and $\sigma_{\bar{0}}$ denote the restrictions of ∂ and σ to $A_{\bar{0}}$ and $C_{\bar{0}}$, respectively, and $\sigma b_\xi = b_\varphi(\mathrm{id} \otimes \partial)$.

We denote these pairs by $\Upsilon = (\xi, \varphi) : (A, B, \partial) \to (C, D, \sigma)$.

Trivially, the composition of a homomorphism of abelian crossed modules and a quadratic map is again a quadratic map.

The following easy results explore the relation between $\Gamma(A, B, \partial)$ and quadratic maps of abelian crossed modules of Lie superalgebras.

Proposition 7.3.3. *The canonical map*

$$\Omega = (\eta, \gamma) = \left(\left(\eta_{\bar{0}}, b_\eta \right), \left(\gamma_{\bar{0}}, b_\gamma \right) \right) : (A, B, \partial) \to \Gamma(A, B, \partial),$$

with $\eta_{\bar{0}} : A_{\bar{0}} \to \widetilde{\Gamma}(A, B, \partial)_{\bar{0}}$ *defined by* $\eta_{\bar{0}}\left(a_{\bar{0}}\right) = e_{a_{\bar{0}}}$, $b_\eta : B \times A \to \widetilde{\Gamma}(A, B, \partial)$ *defined by* $b_\eta(b, a) = b \otimes a$, $\gamma_{\bar{0}} : B_{\bar{0}} \to \Gamma(B)_{\bar{0}}$ *defined by* $\gamma_{\bar{0}}\left(b_{\bar{0}}\right) = e_{b_{\bar{0}}}$ *and* $b_\gamma : B \times B \to \Gamma(B)$ *defined by* $b_\gamma\left(b, b'\right) = b \otimes b'$, *is a quadratic map.*

Proof. It is routine. □

Proposition 7.3.4. *Given any quadratic map* $\Upsilon : (A, B, \partial) \to (C, D, \sigma)$, *with* $\Upsilon = (\xi, \varphi) = \left(\left(\xi_{\bar{0}}, b_\xi \right), \left(\varphi_{\bar{0}}, b_\varphi \right) \right)$, *there exists a unique morphism of crossed modules* $H = \left(h_1, h_2 \right)$ *such that* $h_1 \eta = \xi$ *and* $h_2 \gamma = \varphi$. *To summarise, we will write* $\Omega H = \Upsilon$.

$$\begin{tikzcd}
 & B_{\bar{0}} \arrow[rrr, "\gamma_{\bar{0}}"] \arrow[ddr, "\varphi_{\bar{0}}"] & & & \Gamma(B)_{\bar{0}} \arrow[ddll, dashed, "h_2"] \\
A_{\bar{0}} \arrow[ur, "\partial_{\bar{0}}"] \arrow[rrr, "\eta_{\bar{0}}"] \arrow[ddr, "\xi_{\bar{0}}"'] & & & \widetilde{\Gamma}(A, B, \partial)_{\bar{0}} \arrow[ur, "\partial_{\Gamma_{\bar{0}}}"'] \arrow[ddll, dashed, "h_1"] & \\
 & & D_{\bar{0}} & & \\
 & C_{\bar{0}} \arrow[ur, "\sigma_{\bar{0}}"] & & &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & B \times B \arrow[rrr, "b_\gamma"] \arrow[ddr, "b_\varphi"] & & & \Gamma(B) \arrow[ddll, dashed, "h_2"'] \\
B \times A \arrow[ur, "\mathrm{id} \times \partial"] \arrow[rrr, "b_\eta"] \arrow[dddr, "b_\xi"'] & & & \widetilde{\Gamma}(A, B, \partial) \arrow[ur, "\partial_\Gamma"'] \arrow[ddll, dashed, "h_1"] & \\
 & & D & & \\
 & C \arrow[ur, "\sigma"] & & &
\end{tikzcd}
$$

Proof. The morphism $H = \left(h_1, h_2\right)$ given by

$$
\begin{aligned}
&h_1\left(\beta \otimes \alpha + \left(\gamma\left(\alpha'_{\bar{0}}\right) + a \otimes a'\right)\right) = \xi_{\bar{0}}\left(\alpha'_{\bar{0}}\right) + b_\xi\left(\beta, \alpha\right) + b_\xi\left(\partial\left(a\right), a'\right), \\
&h_2\left(\gamma\left(\beta_{\bar{0}}\right) + b \otimes b'\right) = \varphi_{\bar{0}}\left(\beta_{\bar{0}}\right) + b_\varphi\left(b, b'\right)
\end{aligned}
$$

satisfies the required conditions. □

Proposition 7.3.4 makes it clear that Whitehead's quadratic functor for abelian crossed modules is an actual functor. Indeed, let $f = \left(f_1, f_2\right) : (A, B, \partial) \to (C, D, \sigma)$ be a homomorphism. The composition $\Omega_{(C,D,\sigma)} f$ is a quadratic map, and therefore induces a morphism $h = \left(h_1, h_2\right) : \Gamma(A, B, \partial) \to \Gamma(C, D, \sigma)$, which is functorial and will be called $\Gamma(f)$.

We devote the rest of the section to investigate the properties of Whitehead's quadratic functor of abelian crossed modules, following a similar line as in Section 7.2.

To do so, for an abelian crossed module (A, B, ∂), we introduce the abelian crossed module $S^2(A, B, \partial)$ as a crossed module analogue of $S^2(A)$ for an abelian Lie superalgebra A (cf. [189] for abelian crossed modules of groups).

Definition 7.3.5. Let (A, B, ∂) be an abelian crossed module. We define the abelian crossed module $S^2(A, B, \partial)$ as $(B\underline{\otimes}A, S^2(B), \partial_S)$, with $\partial_S(\overline{b \otimes a}) = b \wedge \partial(a)$.

As well as other second symmetric powers, $S^2(A, B, \partial)$ satisfies the following interesting universal property.

Proposition 7.3.6. *Let (A, B, ∂) and (C, D, σ) be two abelian crossed modules, and consider two bilinear maps $f_1 : B \times A \to C$ and $f_2 : B \times B \to D$ satisfying*

(i) $f_1(\partial(a), a') = (-1)^{|a||a'|} f_1(\partial(a'), a)$, *for all* $a, a' \in A$;

(ii) f_2 *is symmetric;*

(iii) $\sigma f_1 = f_2(\mathrm{id} \times \partial)$.

Then, there exists a unique morphism of abelian crossed modules $g : S^2(A, B, \partial) \to (C, D, \sigma)$, $g = (g_1, g_2)$, *making the following diagram commutative:*

$$\begin{tikzcd}
 & B \times B \arrow[rr, "s_2"] \arrow[ddr, "f_2"] & & S^2(B) \arrow[ddl, dashed, "g_2"] \\
B \times A \arrow[ur, "\mathrm{id} \times \partial"] \arrow[rrr, "s_1"] \arrow[ddr, "f_1"'] & & & B\underline{\otimes}A \arrow[u, "\partial_S"'] \arrow[ddll, dashed, "g_1"] \\
 & & D & \\
 & C \arrow[ur, "\sigma"] & &
\end{tikzcd}$$

where $s_1(b, a) = b \otimes a$ *and* $s_2(b, b') = b \wedge b'$ *for all* $a \in A$ *and* $b, b' \in B$.

Proof. Define g by $g_1(b \otimes a) = f_1(b, a)$ and $g_2(b \wedge b') = f_2(b, b')$. The rest of the proof is routine. □

We need also a result about the adjointness of the tensor product of abelian crossed modules of Lie superalgebras, introduced in Chapter 6 (Definition 6.2.3) following [75] (consult [189] for abelian groups and [75] for abelian Lie algebras).

Given (C, D, σ) and (E, F, χ) two abelian crossed modules of Lie superalgebras, there is a homomorphism of abelian Lie superalgebras

$$\epsilon : \ \mathrm{Hom}\,(D, E) \to \mathrm{Hom}\,((C, D, \sigma), (E, F, \chi))$$

given by $\epsilon(f) = (f\sigma, \chi f)$. We define the abelian crossed module

$$\mathbf{Hom}\,((C, D, \sigma), (E, F, \chi)) := (\mathrm{Hom}\,(D, E), \mathrm{Hom}\,((C, D, \sigma), (E, F, \chi)), \epsilon)\,.$$

Proposition 7.3.7. *Let (A, B, ∂), (C, D, σ) and (E, F, χ) be three abelian crossed modules of Lie superalgebras. There is a natural isomorphism of R-supermodules*

$$\begin{aligned}&\mathrm{Hom}\,\big((A, B, \partial) \otimes_R (C, D, \sigma), (E, F, \chi)\big)\\ &\simeq \mathrm{Hom}\,((A, B, \partial), \mathbf{Hom}\,((C, D, \sigma), (E, F, \chi)))\,.\end{aligned}$$

Proof. First, note that a morphism Δ from (A, B, ∂) to $\mathbf{Hom}\,((C, D, \sigma), (E, F, \chi))$ is characterised by a pair of homomorphisms of abelian Lie superalgebras $\Delta_1 : A \to \mathrm{Hom}\,(D, E)$ and $\Delta_2 : B \to \mathrm{Hom}\,((C, D, \sigma), (E, F, \chi))$ satisfying

$$\big(\Delta_2(\partial(a))\big)_1(c) = \Delta_1(a)(\sigma(c))\,, \tag{7.3.1}$$

$$\big(\Delta_2(\partial(a))\big)_2(d) = \chi\big(\Delta_1(a)(d)\big)\,, \tag{7.3.2}$$

$$\chi\Big(\big(\Delta_2(b)\big)_1(c)\Big) = \big(\Delta_2(b)\big)_2(\sigma(c))\,, \tag{7.3.3}$$

for all $a \in A$, $b \in B$, $c \in C$ and $d \in D$. Now, given a morphism $f = \big(f_1, f_2\big) \in \mathrm{Hom}\,\big((A, B, \partial) \otimes_R (C, D, \sigma), (E, F, \chi)\big)$, if we define

$$\begin{aligned}\Delta_1(a)(d) &= f_1(a \otimes d)\,;\\ \big(\Delta_2(b)\big)_1(c) &= f_1(c \otimes b)\,;\\ \big(\Delta_2(b)\big)_2(d) &= f_2(b \otimes d)\,,\end{aligned}$$

it is easy to check that Δ_1 and Δ_2 are homomorphisms of abelian Lie superalgebras satisfying the conditions (7.3.1)–(7.3.3).

Conversely, given a morphism

$$\Delta = \left(\Delta_1, \Delta_2\right) \in \mathrm{Hom}\left((A, B, \partial), \mathbf{Hom}\left((C, D, \sigma), (E, F, \chi)\right)\right),$$

we define $f = \left(f_1, f_2\right) : (A, B, \partial) \otimes_R (C, D, \sigma) \to (E, F, \chi)$ by

$$f_1\left(a \otimes d + b \otimes c\right) = \Delta_1(a)(d) + \left(\Delta_2(b)\right)_1(c);$$
$$f_2(b \otimes d) = \left(\Delta_2(c)\right)_2(d).$$

Condition (7.3.1) assures that f_1 is well defined, and conditions (7.3.2) and (7.3.3) tell us that $\chi f_1 = f_2\mu$. Also, f_1 and f_2 preserve the degrees, and the pair $f = \left(f_1, f_2\right)$ is a homomorphism of abelian crossed modules of Lie superalgebras.

This correspondence is lineal and bijective, and we obtain the desired natural isomorphism. □

Proposition 7.3.8. *Let (A, B, ∂) and (C, D, σ) be two abelian crossed modules. Then, there is an isomorphism*

$$\Gamma((A, B, \partial) \oplus (C, D, \sigma)) \simeq \Gamma(A, B, \partial) \oplus \Gamma(C, D, \sigma) \oplus \left((A, B, \partial) \otimes_R (C, D, \sigma)\right).$$

Proof. First, we define quadratic maps Υ^1, Υ^2 and Υ^3 from $(A, B, \partial) \oplus (C, D, \sigma) = (A \oplus C, B \oplus D, \langle \partial, \sigma \rangle)$ to, respectively, $\Gamma(A, B, \partial)$, (C, D, σ) and $(A, B, \partial) \otimes (C, D, \sigma) = (\mathrm{coker}\alpha_R, B \otimes D, \mu)$. Set $\varphi^1 : B \oplus D \to \Gamma(B)$, $\varphi^2 : B \oplus D \to \Gamma(D)$ and $\varphi^3 : B \oplus D \to B \otimes D$ in the same way as in Proposition 7.2.4. Also, set

$$\begin{aligned}
&\xi^1_{\bar 0} : A_{\bar 0} \oplus C_{\bar 0} \to \widetilde{\Gamma}(A, B, \partial)_{\bar 0}, && a_{\bar 0} + b_{\bar 0} \mapsto e_{a_{\bar 0}};\\
&\xi^2_{\bar 0} : A_{\bar 0} \oplus C_{\bar 0} \to \widetilde{\Gamma}(C, D, \sigma)_{\bar 0}, && a_{\bar 0} + b_{\bar 0} \mapsto e_{b_{\bar 0}};\\
&\xi^2_{\bar 0} : A_{\bar 0} \oplus C_{\bar 0} \to (\mathrm{coker}\ \alpha)_{\bar 0}, && a_{\bar 0} + b_{\bar 0} \mapsto \partial(a_{\bar 0}) \otimes c;\\
&b^1_\xi : (B \oplus D) \times (A \oplus C) \to \widetilde{\Gamma}(A, B, \partial), && (b + d, a + c) \mapsto b \otimes a;\\
&b^2_\xi : (B \oplus D) \times (A \oplus C) \to \widetilde{\Gamma}(C, D, \sigma), && (b + d, a + c) \mapsto d \otimes c;\\
&b^3_\xi : (B \oplus D) \times (A \oplus C) \to \mathrm{coker}\ \alpha, && (b + d, a + c) \mapsto (-1)^{|a||d|} a \otimes d + b \otimes c;
\end{aligned}$$

We define $\xi^i = (\xi^i_{\bar 0}, b^i_\xi)$ and $\Upsilon^i = (\xi^i, \varphi^i)$, $i \in \{1, 2, 3\}$. It is routine to prove that the Υ^i are quadratic maps, and so is their sum Υ from $(A, B, \partial) \oplus (C, D, \sigma)$ to

$\Gamma(A,B,\partial)\oplus(C,D,\sigma)\oplus((A,B,\partial)\otimes(C,D,\sigma))$. By Proposition 7.3.4, Υ induces a morphism of crossed modules

$$\begin{aligned} H=(h_1,h_2)\colon\ \Gamma((A,B,\partial)\oplus(C,D,\sigma)) \to \Gamma(A,B,\partial)\oplus(C,D,\sigma) \\ \oplus\big((A,B,\partial)\otimes_R(C,D,\sigma)\big)\,. \end{aligned}$$

Now, we construct a morphism from

$$\Gamma(A,B,\partial)\oplus(C,D,\sigma)\oplus\big((A,B,\partial)\otimes_R(C,D,\sigma)\big)$$

to $\Gamma\left((A,B,\partial)\oplus(C,D,\sigma)\right)$. On the one hand, the maps

$$\begin{aligned} &\varrho^1_{\bar 0}\colon\ A_{\bar 0}\to\widetilde{\Gamma}\,(A\oplus C,B\oplus D,\langle\partial,\sigma\rangle)_{\bar 0}, && a_{\bar 0}\mapsto e_{a_{\bar 0}};\\ &b^1_{\varrho}\colon\ B\times A\to\widetilde{\Gamma}\,(A\oplus C,B\oplus D,\langle\partial,\sigma\rangle), && (b,a)\mapsto b\otimes a, \end{aligned}$$

together with the analogue to ϕ^1 from Proposition 7.2.4, conform a quadratic map $\Phi^1=((\varrho^1_{\bar 0},b^1_{\varrho}),\phi^1)$, which induces a morphism $\Theta^1\colon\ \Gamma(A,B,\partial)\to\Gamma((A,B,\partial)\oplus(C,D,\sigma))$. Analogously, we get another morphism $\Theta^2\colon\ \Gamma(C,D,\sigma)\to\Gamma((A,B,\partial)\oplus(C,D,\sigma))$. On the other hand, we define

$$\begin{aligned} &\Delta_1(a)\colon\ D\to\widetilde{\Gamma}\,(A\oplus C,B\oplus D,\langle\partial,\sigma\rangle), && d\mapsto(-1)^{|a||d|}d\otimes a;\\ &(\Delta_2(b))_1\colon\ C\to\widetilde{\Gamma}\,(A\oplus C,B\oplus D,\langle\partial,\sigma\rangle), && c\mapsto b\otimes c;\\ &(\Delta_2(b))_2\colon\ D\to\Gamma(B\oplus D), && d\mapsto b\otimes d, \end{aligned}$$

for all $a\in A$, $b\in B$, $c\in C$ and $d\in D$. These maps induce two homomorphisms of Lie superalgebras, $\Delta_1\colon\ A\to\mathrm{Hom}(D,E)$ and

$$\Delta_2\colon\ B\to\mathrm{Hom}\left((C,D,\sigma),\Gamma\left((A,B,\partial)\oplus(C,D,\sigma)\right)\right),$$

satisfying the conditions (7.3.1)–(7.3.3); therefore, they define a morphism Δ from (A,B,∂) to $\mathbf{Hom}((C,D,\sigma),(E,F,\chi))$, which by Proposition 7.3.7 determines another morphism $\Theta^3\colon\ (A,B,\partial)\otimes(C,D,\sigma)\to\Gamma\left((A,B,\partial)\oplus(C,D,\sigma)\right)$. The morphisms Θ^1, Θ^2 and Θ^3 determine the morphism

$$\Theta\colon\ \Gamma(A,B,\partial)\oplus(C,D,\sigma)\oplus((A,B,\partial)\otimes(C,D,\sigma))\to\Gamma((A,B,\partial)\oplus(C,D,\sigma)),$$

which is the inverse of H. $\square$

Proposition 7.3.9. *Assume that* 2 *has an inverse in* R*, and let* (A, B, ∂) *be an abelian crossed module. Then, the abelian crossed modules* $\Gamma(A, B, \partial)$ *and* $S^2(A, B, \partial)$ *are isomorphic.*

Proof. On the one hand, the pair of canonical maps $b_\eta : B \times A \to \widetilde{\Gamma}(A, B, \partial)$ and $b_\gamma : B \times B \to \Gamma(B)$ from Proposition 7.3.3 satisfy conditions (i)–(iii) of Proposition 7.3.6. Therefore, they induce a morphism $G = (g_1, g_2)$ between $S^2(A, B, \partial)$ and $\Gamma(A, B, \partial)$. On the other hand, we will define a quadratic map between (A, B, ∂) and $\Gamma(A, B, \partial)$. Consider the pair $\varphi = \left(\varphi_{\bar{0}}, b_\varphi\right)$ from Proposition 7.2.7, and also $x_{\bar{0}} : A_{\bar{0}} \to \left(B \underline{\otimes} A\right)_{\bar{0}}$ and $b_\xi : B \times A \to B \underline{\otimes} A$, defined by $x_{\bar{0}}(a_{\bar{0}}) = \partial(a_{\bar{0}}) \otimes a_{\bar{0}}$ and $b_\xi(b, a) = 2b \otimes a$, respectively. It holds that $\Upsilon = \left(\left(x_{\bar{0}}, b_\xi\right), \varphi\right)$ is a quadratic map; then, Proposition 7.3.4 ensures that there exists a morphism $H = (h_1, h_2)$ from $\Gamma(A, B, \partial)$ to $S^2(A, B, \partial)$. Some quick calculations show that $HG = 2\,\mathrm{id}$ and $GH = 2\,\mathrm{id}$; then, $S^2(A, B, \partial)$ and $\Gamma(A, B, \partial)$ are isomorphic. □

To obtain an exact sequence similar to that of Proposition 7.2.8 in the context of crossed modules, it would be necessary to handle non-abelian tensor and exterior products of crossed modules acting on each other in the sense of [51, 182]. As far as we are concerned, these concepts have not been defined yet, so we limit to deal with a particular case which slightly generalises Theorem 6.3.12 (see Chapter 6, page 200).

Proposition 7.3.10. *Let* (T, L, ∂) *be a crossed module such that* ∂ *is surjective or the action of* L *on* T *is trivial, and let* (M, P, ∂) *and* (N, Q, ∂) *be two graded ideal crossed submodules. Then, there is an exact sequence*

$$\Gamma\left(\frac{(M, P, \partial) \cap (N, Q, \partial)}{[(M, P, \partial), (N, Q, \partial)]}\right) \xrightarrow{(\phi,\psi)} (M, P, \partial) \otimes (N, Q, \partial) \xrightarrow{(\pi_1,\pi_2)} (M, P, \partial) \wedge (N, Q, \partial) \longrightarrow 0.$$

Proof. This proof is step-by-step analogous to the one of Theorem 6.3.12 (see Chapter 6, page 200). □

We finish this chapter by highlighting a significant difference with respect to the cases of modules and supermodules. Namely, Whitehead's quadratic functor for abelian crossed modules of Lie superalgebras does not preserve free objects, defined in the following way. Consider the forgetful functor $\mathcal{U}$ from the subcategory **Ab** of **XSLie** to the category $\mathbf{Set_2}$ of $\mathbb{Z}_2$-graded sets, which carries each (A, B, ∂) to the disjoint union $\left(A_{\bar{0}} \times B_{\bar{0}}\right) \sqcup \left(A_{\bar{1}} \times B_{\bar{1}}\right)$. It can be checked that this functor has a left adjoint, which we will denote by $\mathcal{ABF}$ and which carries each set $X = X_{\bar{0}} \cup X_{\bar{1}}$ to the abelian crossed module $\left(\bigoplus_X R, \left(\bigoplus_X R\right) \oplus \left(\bigoplus_X R\right), \iota_1\right)$, where ι denotes the inclusion into the first component, and being $\bigoplus_X R = \left(\bigoplus_{X_{\bar{0}}} R\right) \oplus \left(\bigoplus_{X_{\bar{1}}} R\right)$. It will be called the *free functor*. One can check that this is the same approach to free crossed modules as the one given in [48] in the context of crossed modules of groups.

Set $X = \{*\}$, concentrated in degree $\bar{0}$, and $R = \mathbb{Z}$. Then, the free abelian crossed module over X of is $\mathcal{ABF}(X) = \left(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \iota_1\right)$. Now, consider the second component of $\Gamma\left(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \iota_1\right)$,

$$\Gamma(\mathbb{Z} \oplus \mathbb{Z}) = \Gamma(\mathbb{Z}) \oplus \Gamma(\mathbb{Z}) \oplus \mathbb{Z} \otimes \mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

where $\Gamma(\mathbb{Z}) \simeq \mathbb{Z}$ follows from Proposition 7.2.6. It is clear that $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ cannot be expressed as $\left(\bigoplus_Y \mathbb{Z}\right) \oplus \left(\bigoplus_Y \mathbb{Z}\right)$ for any set Y, and therefore $\Gamma(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}, \iota_1)$ is not free.

CHAPTER 8

Dealing with negative conditions in automated proving: tools and challenges. The unexpected consequences of Rabinowitsch's trick

In the algebraic-geometry-based theory of automated proving and discovery, it is often required that the user includes, as complementary hypotheses, some intuitively obvious non-degeneracy conditions. Traditionally there are two main procedures to introduce such conditions into the hypotheses set. The aim of this chapter is to present these two approaches, namely Rabinowitsch's trick and the ideal saturation computation, and to discuss in detail the close relationships and subtle differences that exist between them, highlighting the advantages and drawbacks of each one. We also present a carefully developed example which illustrates the previous discussion. Moreover, the chapter will analyse the impact of each of these two methods in the formulation of statements with negative theses, yielding rather unexpected results if Rabinowitsch's trick is applied.

Introduction

The framework of this chapter is the automated theorem proving and discovery theory initiated, forty years ago, by Wu on his seminal paper [234, "On the decision problem and the mechanisation of theorem-proving in elementary geometry"], based in computational (complex) algebraic geometry. This theory has evolved, along the years, yielding a variety of methods that have been recognised to be a quite successful approach to automated reasoning in elementary geometry, as already shown, long ago, by the quantity and quality of the performing examples in [56]. In this chapter, we will follow the protocol and notation described in [58, Chapter 6, Section 4], quite similar to that of [59], [194] or [238]. Its recent implementation (see [1]) in a free mathematics software, with millions of users worldwide, brings again to the frontline some pending issues.

The point we will address in this chapter deals with the most convenient way(s) of handling hypotheses and theses that describe negative assertions, such as "consider two different points" (i.e. two points that are not equal), or "let A, B, C be the vertices of a non-degenerate triangle" (i.e. three points that A, B, C neither coincide nor lie on a line), etc. The relevance of clarifying this issue is not just restricted to extending the mechanical proving method to handle a larger kind of statements. In fact, as we will show in the next Section 8.1 of this chapter, non-degeneracy conditions arise in a natural way along with the traditional protocol for theorem proving of purely affirmative statements.

It happens that, in order to introduce, as input for the standard algebraic geometry algorithms, the requirement to avoid such degeneracies, the given polynomial inequalities $p_1(x_1, \dots, x_n) \neq 0$ must be expressed by means of equations. In the tradition of automated theorem proving this conversion has been achieved through two possible approaches, that we will describe in detail in Section 8.2: Rabinowitsch's trick and ideal saturation. We plan to deal in the future with some recent methods for introducing negative hypotheses, such as the ones based on comprehensive Gröbner systems (see [177]), or those in the `ZariskiFrames` package `https://github.com/homalg-project/ZariskiFrames` (see [16]), using ideal membership and syzygies, as they are not yet implemented in most popular dynamic geometry programs, such as `GeoGebra` [2].

Rabinowitsch's trick is an old companion to automated proving in geometry [135], where it has been used to formulate negations of equalities, the so-called "disequality" relations. Despite its antiquity, the current validity and interest of this approach can be confirmed, for example, by considering the recent research of Kapur, Sun, Wang and Zhou [136] on a generalisation of the "trick". See also [16, Example 6.1] for a sound, abstract, description of this trick, i.e. the replacement of a locally closed set $A \setminus B$, where A, B are algebraic subsets of an affine space F^n, by an algebraic set in F^{n+1}, the so-called "Rabinowitsch cover", such that the set-theoretic projection of the cover is exactly the locally closed set and, thus, the closure in the Zariski topology of the projection of the cover can be computed through elimination.

On the other hand, ideal saturation is a direct algebraic way to compute $\overline{A \setminus B}$ without requiring to replace the locally closed set $A \setminus B$ by an algebraic set on a higher dimensional affine space and then projecting it down to F^n. Its relation to Rabinowitsch's trick is well known in commutative algebra, and the potential impact of using saturation as an alternative to Rabinowitsch's trick for theorem proving has been already highlighted in [59, Section 5], but it seems a detailed and general analysis of the pros and cons of both approaches, regarding their faithfulness as translations of negative statements, has never been thoroughly addressed.

Thus, the main contribution of this chapter is to study, in detail, the different implications of adopting each of these formulations for describing negative theses and hypotheses. Section 8.3 focuses on the consequences of both methods for stating negative theses, suggesting that saturation could be considered as more reliable in this context (see Proposition 8.3.8), while Section 8.4 deals with the introduction of negative hypotheses, clarifying the different, albeit close, results in each of the two alternate proposals. Section 8.5 discusses in detail an example, showing the computational pros and cons of both approaches. Finally, Section 8.6 establishes some conclusions for the future consideration of automatic theorem proving software developers.

Throughout this chapter, F will denote an algebraically closed field of characteristic 0, and all the vector spaces, affine spaces and polynomial rings will be considered over F. Also, for simplicity of notation, we will denote by H (respectively T) the collection of polynomials involved in the hypotheses (respectively theses) and the ideal generated by them.

8.1 A short digest on automatic proving and discovery by algebraic geometry methods

Roughly speaking, the computational algebraic geometry theorem proving method proceeds by assigning coordinates and equations to the elements (points, lines, circles, etc.) and conditions (perpendicularity, incidence, etc.) of the involved geometric hypotheses H and theses T. In this way the geometric statement, which we will symbolise as $H \Rightarrow T$, is translated as an inclusion $V(H) \subseteq V(T)$ between the solution set, $V(H)$, of the system of equations $\{H = 0\}$ and that, $V(T)$, of the set of equations $\{T = 0\}$. Finally, this inclusion needs to be tested by some computational algebraic geometry methods.

One practical protocol to perform this test is the refutational approach introduced by Kapur [135], in which testing the inclusion $V(H) \subseteq V(T)$ turns to deciding if $V(H) \setminus V(T)$ is empty or not. This is, obviously, equivalent to showing that its Zariski-closure $\overline{V(H) \setminus V(T)}$ is empty or not. But this fact can be checked by testing the emptiness of the corresponding Rabinowitsch cover (i.e. by determining the membership of 1 in the defining ideal of the cover) since the projection of an algebraic set is empty if and only if the set is empty. In fact, when T is a single thesis, verifying that $H \wedge \neg T$ is empty is equivalent, by the Weak Nullstellensatz, to simply checking if $1 \in H^e + (Tt - 1)$, where H^e is the extension of H to the polynomial ring with an extra variable t. In what follows, we will assume indeed that T consists of a single thesis, as it is straightforward to adapt all the obtained results to the general case

As it is well known, the reduction from the Strong to the Weak Nullstellensatz, by introducing $Tt - 1 = 0$ as the algebraic, affirmative formulation for $\neg\{T = 0\}$ is, precisely, the role of the so-called Rabinowitsch's trick, see [191]. In summary, there is a special, and since long, relation between automatic proving in geometry and this particular trick.

Going a little bit further, let us remark that, in the algebraic geometry approach to automated theorem proving, it happens that in most cases we are not actually dealing with proving, but with discovering! Indeed, many statements that seem obviously true to human intuition, turn out to be false in the algebraic formulation, in the sense of not fully satisfying $V(H) \subseteq V(T)$. Thus, the main task for the automatic reasoning theory turns out to be devising algorithms to find out extra hypotheses that will constrain the set $V(H)$ in order to fit inside $V(T)$: i.e. to discover how to modify a given statement so that it becomes true! See [195] for a thorough reflection and bibliographic references on this involved issue.

A key ingredient in this framework is the concept of set of independent variables modulo the hypotheses ideal, i.e. a set of variables such that no polynomial, in these variables alone, belongs to the ideal H. Examples of sets of free variables are the collection of three times two coordinates describing the vertices of an arbitrary triangle, or only one of the coordinates of a point constrained to be in a circle, etc.

Among the many different sets of independent variables for a particular hypotheses ideal, we will consider sets of maximum cardinality: in this way, the remaining variables will satisfy some algebraic dependence over the independent ones and thus, except at some special cases, they are finitely determined for each setting of the independent variables. Therefore, it is exclusively in terms of the independent variables that we will consider reasonable to formulate the extra hypotheses needed to modify some given geometric statement, to turn it strictly true.

Obviously, it is crucial to automatically find such conditions: this can be done, roughly speaking, by elimination of the extra variable t and the non-independent variables (say, $x_{s+1}, \ldots, x_n$) in the ideal $H^e+(Tt-1)$. Indeed, the zero set of $H^e+(Tt-1)$ exactly corresponds to the "failure cases" where H and $\neg T$ simultaneously hold.

Thus, by adding to the given hypotheses H the negation of any of the polynomials in the elimination ideal of $H^e+(Tt-1)$, we will get a true statement. These additional, negative hypotheses (such as these two given points must be truly different, the given triangle should not collapse to a line, etc.) expressed in terms of the independent variables, are known as non-degeneracy conditions.

Of course, it can happen that the elimination of t and the dependent variables in the ideal $H^e+(Tt-1)$ is just the zero ideal and, in this case, the only non-degeneracy condition turns out to be $0 \neq 0$, so that it does not hold over any instance of the geometric hypotheses. Notice that this zero-ideal case is the only possibility for getting an empty hypotheses statement when adding the negation of an equation $h' = 0$, where h' arises from the elimination of $H^e + (Tt - 1)$ in terms of independent variables. Thus, the name *generally true* is reserved to statements where this elimination ideal $(H^e + (Tt - 1)) \cap F\left[x_1, \ldots, x_s\right]$ is not zero; logically, the name *non generally true* is applied to those statements in which the former ideal is zero (that is, generally true is false). Note that we are employing the terminology of [56] and some recent papers as [1, 59, 194], but recall that in some classic references as [58] or [176] these statements are called *generically true* and *non generically true*, respectively.

When the elimination ideal is zero, it is advisable to consider, instead, the elimination of the same dependent variables, but now in the ideal of hypotheses and thesis $H+T$. If this elimination is not zero, the given statement is labeled as *generally false* (and *non generally false* if it is zero).

Obviously, when the elimination of the dependent variables in $H+T$ is not zero, adding as complementary, affirmative hypotheses the equations of a basis of this elimination ideal we are led to a new statement, and we should restart again our protocol.

Notice that the new hypotheses variety could be empty if and only if $H+T = (1)$, i.e. if the elimination ideal turns out to be (1), and from there we can conclude the truth of whatever statement. It is the *extremely false* case, in which the hypothesis variety has nothing in common with the thesis variety. Our approach does not disregard this option; but routinely checking that the analysed statements have a non empty hypotheses set should be included in any theorem proving algorithm, in order to detect trivialities.

On the other hand, if the elimination in $H + T$ yields zero, we are in the non generally false and non generally true case. See [58, Chapter 6, Section 4] or [59], [194] for further details on the whole algorithm.

Thus, the general automatic proving procedure ends, either with a generally true statement (after discovering some new, affirmative and/or negative conditions), or arriving at a neither generally false nor generally true situation, a quite challenging context, yielding as well to the discovery of new statements, but in a more involved way. See [30] and [238] for some recent advances concerning this last issue.

8.2 Introducing negative conditions: different ways…

Summarising, negative conditions appear naturally in classical automated theorem proving in two different circumstances:

- to refute the thesis T, in order to establish if the given statement is generally true or not;

- if generally true, to add, as complementary, negative hypotheses, some newly discovered non-degeneracy conditions.

On the other hand, the high complexity of the polynomial Gröbner basis algorithms involved in the method explained before [58] compels the user to manually introduce, before starting to run the proving algorithm, some intuitive, easy-to-guess non-degeneracy conditions, to attempt to simplify the computation.

Bearing this in mind, we think that the second item above should be extended and reformulated as follows:

- to add, at different stages of the proving protocol, as complementary hypotheses, human or automatically guessed non-degeneracy conditions.

Thus, an important task is to find ways to introduce both the refutation of a thesis and non-degeneracy conditions, so that it reflects (as closely as possible) the geometric meaning of the added condition (i.e. to avoid some degenerate cases, to negate some theses) and expresses it by means of equations.

As mentioned above, traditionally (at least since [135]) the negation of a given geometric property described by the equation $f = 0$, is handled as an equation by adding some auxiliary variable t and considering the equation $ft - 1 = 0$ as representing $\neg\{f = 0\}$, emulating Rabinowitsch's trick. It is easy to generalise this approach to refutation for the case of having to negate the conjunction or the disjunction of several conditions, see [59, Appendix].

However, the avoidance of some condition $f = 0$ can be expressed considering the Zariski closure of the difference $V(H) \setminus V(f)$, i.e. by considering as new hypotheses the polynomial equations expressing the smallest set that satisfies the given hypotheses and not the condition $f = 0$. In general, if I, J are ideals of a polynomial ring $F\left[x_1, \dots, x_n\right]$, the *saturation of I by J* is defined as $\mathtt{Sat}(I, J) = I : J^\infty = \cup_n (I : J^n)$ (see [58, 59]), where $I : J = \{g \in F\left[x_1, \dots, x_n\right] \mid gJ \subseteq I\}$, and it satisfies $V(\mathtt{Sat}(I, J)) = \overline{V(I) \setminus V(J)}$. When the ideal J is principal, $J = (j)$, we merely denote $\mathtt{Sat}(I, j)$.

In summary, the other option we are considering here to include non-degeneracy conditions $\neg\{f = 0\}$ is to saturate the ideal of hypotheses by the ideal $J = (f)$. Again, it is straightforward the generalisation of this idea of saturation to the case of several conditions (see [59, Appendix]). As mentioned in the previous section, we could say that saturation is a direct way to compute $V(H) \setminus V(f)$ without requiring adding one extra variable and then eliminating it.

This second option could seem, at first glance, more sophisticated than the implementation of Rabinowitsch's trick. But there is not a big difference. In fact, [59, Proposition 6 and Corollary 2 of Appendix] shows that the saturation of the ideal I by another ideal $J = \prod_{i=1}^{r} J_i$, where $J_i = (f_{i1}, \dots, f_{il_i})$, satisfies:

$$\mathtt{Sat}(I, J) = \Big(I^e + \big((f_{11}t_1 - 1) \cdots (f_{1l_1}t_1 - 1), \dots, (f_{r1}t_r - 1) \cdots (f_{rl_r}t_r - 1)\big)\Big) \cap F\left[x_1, \dots, x_n\right];$$

in particular, it holds that

$$\mathtt{Sat}(I, f) = (I^e + (ft - 1)) \cap F\left[x_1, \dots, x_n\right], \qquad (8.2.1)$$

as it is stated in [58, Theorem 14 of Chapter 4, Section 4].

Remark 8.2.1. Although formula (8.2.1) relates saturation and elimination theoretically, it does not implies that, computationally speaking, saturation requires elimination, see, for instance, [17].

Thus, the actual dilemma is: do we want to add non-degeneracy conditions as in Rabinowitsch's trick, by carrying around within H an extra, alien variable, which should be eliminated at the end of the theorem proving or discovery process, i.e. when considering the ideal $H^e+(Tt-1)$; or should we deal, from the beginning, with the non degeneracy conditions expressed in terms of the "given" variables of our statement, by saturation? And, actually, does it imply any difference concerning theorem proving?

This dilemma does not seem to appear concerning the use of Rabinowitsch's trick or saturation in order to determine if a theorem is generally true by refuting the thesis. It is easy to prove that both approaches yield the same result: it follows from (8.2.1) and the fact that

$$(H^e + (Tt-1)) \cap F\left[x_1, \dots, x_s\right] = (H^e + (Tt-1)) \cap F\left[x_1, \dots, x_n\right] \cap F\left[x_1, \dots, x_s\right].$$

8.3 …different consequences: introducing negative theses

In this section, we will analyse the different behaviour of both approaches (Rabinowitsch, saturation) regarding the introduction of negative theses. Note that this task is not as important as the introduction of negative hypotheses (which will be tackled in the following section), because the former just enlarges the realm of classical automated theorem proving, while the latter appears naturally on it.

Example 8.3.1. Let us consider the following quite artificial statement: given a general triangle, with free vertices $A(0,0), B(1,0), C(c_1,c_2)$, we assert that $c_2 \neq 0$, i.e. that C does not lie in the AB line. Intuitively this statement seems *generally true*. We consider the ideal $H = (0)$ as the zero ideal, since there are no hypotheses; moreover, both variables, c_1, c_2 are free.

Then, we apply the protocol and start computing $H+(Tt-1)$, with $T := \{c_2 \neq 0\}$. Using Rabinowitsch's trick we should consider $T := \{c_2 z - 1 = 0\}$, with an auxiliary variable z, and then proceed to compute the elimination of the variables z and t in the ideal $H + \big((c_2 z - 1)t - 1\big) = (0) + \big((c_2 z - 1)t - 1\big) = \big((c_2 z - 1)t - 1\big)$. The obvious result is (0), so the statement is non generally true and we should proceed by considering the ideal $H + T$, i.e. $(0) + (c_2 z - 1)$ and eliminating the variable z here. The result is, again, zero. So we are stuck in the non generally true and non generally false case!

On the other hand, if we model the thesis T as the saturation of $H = (0)$ by c_2 we get $\mathtt{Sat}((0),(c_2)) = (0)$, so the thesis should be considered to be $T = 0$ and, then, we start checking if this thesis is generally true, but computing the ideal $H + (Tt - 1)$, i.e. ideal $(0) + (1) = (1)$. We get that the statement is not only generally but always true, since the only non-degeneracy condition is $1 \neq 0$!

As we can see in the above example, we can obtain quite different results following both methods. But this is not just a particular behaviour in some cases. In general, we can state the following unexpected facts:

Proposition 8.3.2. *The introduction of a negative thesis $T_1 := \{p \neq 0\}$ by using Rabinowitsch's trick, always yields a non generally true statement $H \Rightarrow T_1$.*

Proof. Let us assume that $\{x_1, \dots, x_s\}$ are the independent variables for H. Then let us prove that the closure of the projection over this affine space, of the variety $V(H) \cap V((pz-1)t-1)$ lying in the space of the variables $\{x_1, \dots, x_s, x_{s+1}, \dots, x_n, z, t\}$, is the whole $\{x_1, \dots, x_s\}$-space. In fact, take any point $(x_1, \dots, x_s)$ in the projection of $V(H)$, which is, by definition of independent variables, dense in the affine space, and we will prove that it is also in the projection of $V(H) \cap V((pz-1)t-1)$.

First we notice that, because $(x_1, \dots, x_s)$ lies in the projection of $V(H)$, there are values of $x_{s+1}, \dots, x_n$ such that $(x_1, \dots, x_s, x_{s+1}, \dots, x_n)$ is in $V(H)$. If, for one of these points in $V(H)$, it happens that $p(x_1, \dots, x_n) = 0$, then by taking $t = -1$ and an arbitrary value of z, we will have that the point $(x_1, \dots, x_n, z, -1)$ is in $V(H) \cap V((pz-1)t-1)$. On the other hand, if $p(x_1, \dots, x_n) \neq 0$, we consider a value of $z \neq 1/p(x_1, \dots, x_n)$, so that $pz-1 \neq 0$. Finally, by taking $t = 1/(pz-1)$, we will have, again, that the point $\left(x_1, \dots, x_n, z, 1/(pz-1)\right)$ is in $V(H) \cap V((pz-1)t-1)$. □

This surprising result could suggest the idea that this method fails to model geometric problems with negative thesis. Indeed, an intuitive approximation might consider that the two successive negations involved here (one, for the negative thesis; two, for the refutational approach required for checking generally true statements), would be equivalent to simply verifying the thesis in an affirmative way. That is, common sense is prone to conclude that verifying if a negative thesis (introduced through Rabinowitsch's trick) is generally true would be equivalent to verifying if the corresponding affirmative thesis is generally false, but see some of the examples and propositions below. In fact, postponing the elimination of z until the end of the process, which is the essence of Rabinowitsch's trick, forces us to consider the formulation of the negative thesis $pz - 1$ just as a simple statement in $F\left[x_1, \dots, x_n, z\right]$, rather than the negation of something, yet with subtle relations with the corresponding affirmative statement.

Notice that, in general, we have [30, Proposition 2.3]:

Proposition 8.3.3. *A statement $H \Rightarrow T$ cannot be simultaneously generally true and generally false, that is, it cannot happen that both $H + T$ and $H^e + (Tt - 1)$ have, at the same time, some non-zero polynomials in the independent variables alone.*

On the other hand, it is easy to notice that, by applying directly the definitions, we have:

Proposition 8.3.4. *If a statement with an affirmative thesis $T_2 := \{p = 0\}$ is generally true, then the same statement but with the negative thesis (formulated through Rabinowitsch's trick) $T_1 := \{p \neq 0\}$ will be generally false, and conversely.*

Putting together the two precedent propositions, it follows immediately the following result:

Proposition 8.3.5. *If a statement with an affirmative thesis $T_2 := \{p = 0\}$ is generally false, then the same statement, but with the corresponding negative thesis (formulated through Rabinowitsch's trick) $T_1 := \{p \neq 0\}$ will be non generally false.*

Proof. If a statement with an affirmative thesis $T_2 := \{p = 0\}$ is generally false, then Proposition 8.3.3 says that it cannot be generally true and thus, by Proposition 8.3.4, the corresponding negative thesis (formulated through Rabinowitsch's trick) $T_1 := \{p \neq 0\}$ will be non generally false. □

Corollary 8.3.6. *In summary: if $H \Rightarrow T_2$ is generally false, then necessarily*

- *$H \Rightarrow T_2$ is non generally true;*
- *$H \Rightarrow T_1$ is non generally false and non generally true.*

On the other hand, if $H \Rightarrow T_2$ is non generally false, then, there are two options: if $H \Rightarrow T_2$ is also generally true, we will have that $H \Rightarrow T_1$ is generally false as well, and non generally true; and if $H \Rightarrow T_2$ is non generally false and non generally true, then $H \Rightarrow T_1$ is also non generally false and non generally true.

Example 8.3.7. If we consider $H := \{(y-1)(y-2) = 0\}$ in the variables $\{x, y\}$, with x the only independent variable, and $T_2 := \{y - 1 = 0\}$, $T_1 := \{(y-1)z - 1 = 0\}$ it is easy to check that $H \Rightarrow T_2$ is both non generally true and non generally false, and the same happens for $H \Rightarrow T_1$. On the other hand, if we take $T_2 := \{y - 3 = 0\}$, $T_1 := \{(y-3)z - 1 = 0\}$, we obtain that $H \Rightarrow T_2$ is non generally true, but generally false, while $H \Rightarrow T_1$ is both non generally true and non generally false. Finally, if we take $H := \{(y - 1) = 0\}$ in the variables $\{x, y\}$, and $T_2 := \{y - 1 = 0\}$, $T_1 := \{(y - 1)z - 1 = 0\}$ it is easy to check that $H \Rightarrow T_2$ is generally true, but non generally false, while $H \Rightarrow T_1$ will be no generally true, but generally false. This covers all possibilities.

Similarly to Proposition 8.3.4, for the formulation of negative thesis using saturation, we have the following:

Proposition 8.3.8. *If the statement $H \Rightarrow T_2$ is generally true then the statement $H \Rightarrow T_1$, formulated by introducing the negative thesis $T_1 := \{p \neq 0\}$ by using saturation, is generally false. Analogously, if $H \Rightarrow T_2$ is generally false then the statement $H \Rightarrow T_1$, formulated by introducing the negative thesis $T_1 := \{p \neq 0\}$ by using saturation, is a generally true statement.*

Proof. If $H \Rightarrow T_2$ is generally true, it means that the elimination of dependent variables modulo H in $H^e + (pt - 1)$ is not zero. Let J be this elimination ideal. Now, $H \Rightarrow T_1 := \{\mathtt{Sat}(H, p)\}$, expressed by saturation, is generally false because if we eliminate the dependent variables in $H+T_1$ we get $(H + \mathtt{Sat}(H, p)) \cap F\left[x_1, \ldots, x_s\right] = \left(H + \left((H^e + (pt - 1)) \cap F\left[x_1, \ldots, x_n\right]\right)\right) \cap F\left[x_1, \ldots, x_s\right]$, where the inner intersection just means the elimination of t. Obviously, this intersection contains J, which is already an ideal in the independent variables, so this inclusion is not affected by adding H or by the outer intersection, hence $(H + \mathtt{Sat}(H, p)) \cap F\left[x_1, \ldots, x_s\right] = \left(H + \left((H^e + (pt - 1)) \cap F\left[x_1, \ldots, x_n\right]\right)\right) \cap F\left[x_1, \ldots, x_s\right] \supseteq J \neq (0)$.

Concerning the second statement in this proposition, let us assume that $\mathtt{Sat}(H, p)$ is principal, for simplicity; say, generated by q. Then there is a power of p, such as p^r, satisfying that $qp^r \in H$. Next, notice that if $H + (p)$ has some polynomial in the independent variables (i.e. if $H \Rightarrow T_2$ is generally false), then the same happens for

$H + (p^n)$, for whatever power of p; thus, the elimination of the independent variables in $H + (p^r)$ will also be not zero. On the other hand, we observe that p^r is in $H^e + (\mathtt{Sat}(H, p)t - 1)$, since equality (8.2.1) tells us that the elimination of t in this ideal is $\mathtt{Sat}\,(H, \mathtt{Sat}(H, p))$, and $p^r q \in H$. Thus, $H^e + (\mathtt{Sat}(H, p)t - 1) \supseteq H + (p^r)$ and it follows the corresponding elimination is not zero. □

Example 8.3.9. If we consider $H := \{(y-1)(y-2) = 0\}$ in the variables $\{x, y\}$, with x the only independent variable, and $T_2 := \{y - 1 = 0\}$, $T_1 := \mathtt{Sat}(((y-1)(y-2)), (y-1))$=$\{y - 2 = 0\}$ it is easy to check that $H \Rightarrow T_2$ is both non generally true and non generally false, and the same happens for $H \Rightarrow T_1$. On the other hand, if we take $T_2 := \{y - 3 = 0\}$, $T_1 := \mathtt{Sat}(((y-1)(y-2)),$
$(y-3))$=$\{(y-1)(y-2) = 0\}$, we obtain that $H \Rightarrow T_2$ is non generally true, but generally false, while $H \Rightarrow T_1$ is generally true and non generally false. Finally, if we take $H := \{(y-1) = 0\}$ in the variables $\{x, y\}$, and $T_2 := \{y - 1 = 0\}$, $T_1 := \mathtt{Sat}((y-1), (y-1)) = (1)$ it is easy to check that $H \Rightarrow T_2$ is generally true, but non generally false, while $H \Rightarrow T_1$ will be non generally true, but generally false. We see, comparing with Example 8.3.7, that when $H \Rightarrow T_2$ is generally false, the behaviour with saturation is diverse from the one with Rabinowitsch.

8.4 ...different consequences: introducing negative hypotheses

In [59, Example 6 of Section 5] it is presented one specific example showing how both methods (Rabinowitsch, saturation) differ in a common context, yielding, if a non-degeneracy hypothesis is introduced using Rabinowitsch's trick, an interesting theorem discovering the conditions for the orthic triangle of a given triangle with non-collinear vertices to be equilateral. On the other hand, if the non-collinearity of the vertices is introduced by saturation, there is no discovery at all. Although the concept of theorem discovery in [59] and the one from [194] we have just recalled here in the introduction (which has been recently implemented in some popular mathematical software [1,2]) are practically the same, the framework is a little bit different: in [59] the approach is slightly more sophisticated.

In order to analyse the behaviour of both approaches to non-degeneracy hypotheses, let us denote:

$$\begin{aligned} H_1 &:= H^e + (ft - 1), \\ H_2 &:= \mathtt{Sat}(H, f). \end{aligned}$$

Thus, H_1, H_2 are the enlarged ideals of hypotheses, corresponding to the two possibilities of introducing non-degeneracy conditions such as $f \neq 0$ over an ideal of hypotheses H of a given statement.

Remark 8.4.1. With the above notation, it holds that $H_2^e \subseteq H_1$, since the saturation is equal to the elimination of the variable t in the ideal on the right side (i.e. the contraction into the ring $F\left[x_1, \ldots, x_n\right]$), and the contraction and, then, extension of an ideal, is always contained in the given ideal.

Example 8.4.2. The following trivial example $H = (0)$, $f = x$, shows that, in general, the inclusion $H_2^e \subset H_1$ is strict. We find that $H_2 = \mathtt{Sat}(H, f) = (0)$, its extension is, again, (0) and $H_1 = H^e + (ft - 1) = (xt - 1)$. In this case, the saturation does not assimilate the information contained in $x \neq 0$: the zero set of $\mathtt{Sat}(H, f)$ includes the points where $x = 0$, although we wanted to avoid such instances. This situation is due to the early consideration of the closure in the saturation method.

Taking Remark 8.4.1 into account, we are ready to present the following basic result.

Proposition 8.4.3. *Notation as above. The set of hypotheses H_1 provides, in general, more additional conditions for discovery that the set H_2; that is:*

$$(H_2 + T) \cap F\left[x_1, \ldots, x_s\right] \subseteq (H_1 + T) \cap F\left[x_1, \ldots, x_s\right]. \tag{8.4.1}$$

Proof. Using the properties of extension and contraction of ideals, it is clear that $H_2 + T \subseteq (H_2 + T)^{ec} = (H_2^e + T)^c$, where $(-)^c$ symbolises the contraction of the ideals to the ring $F\left[x_1, \ldots, x_n\right]$. By Remark 8.4.1, we can state that $(H_2^e + T)^c \subseteq (H_1 + T)^c$. Thus, intersecting both $H_2 + T$ and $(H_1 + T)^c$ with $F\left[x_1, \ldots, x_s\right]$, the inclusion (8.4.1) holds. □

Corollary 8.4.4. *It follows that if the statement $H_2 \Rightarrow T$ is generally false, $H_1 \Rightarrow T$ will also be generally false. Replacing in the proposition above the polynomial T with $Tt' - 1$, we obtain the same inclusion and, thus $H_2 \Rightarrow T$ generally true implies that $H_1 \Rightarrow T$ is, as well, generally true.*

Example 8.4.5. Again, we present a simple example to illustrate that the inclusion in (8.4.1) is, in general, strict. Let us retake the previous Example 8.4.2, with thesis $T = (x)$ (essentially, the same example which appears at the beginning of [59, Section 5]). Clearly, the variable x is independent. Recall that $H_1 = (xt - 1)$ and $H_2 = (0)$. If we add T to our ideals and eliminate all variables except x (i.e. the variable t), we obtain, respectively, the sets (1) and (x). It is clear that adding the condition $1 = 0$ to the hypotheses set makes the hypotheses variety empty, from which we could conclude whatever we wanted. So, this is not an interesting option. But neither is the set obtained from saturation, because it leads to a contradiction with $x \neq 0$.

So, the previous example ends up in a discovery, but not in a proper one.

In order to gain a better understanding of discovery from each one of the approaches, we will now precise the difference between the two sets of derived additional conditions (a result which is similar to [59, Proposition 3]). Remark that we just consider the case in which the non-degeneracy condition introduced by the user is formulated in terms of the independent variables; a reasonable restriction, since these variables are the only ones we can freely manipulate.

Lemma 8.4.6. *Notation as above. Assume that $f \in F\left[x_1, \ldots, x_s\right]$. Then it holds that:*

$$\begin{aligned}&(H_2 + T) \cap F\left[x_1, \ldots, x_s\right] = (H_1 + T) \cap F\left[x_1, \ldots, x_s\right] \\ &\Leftrightarrow (H_2 + T) \cap F\left[x_1, \ldots, x_s\right] = \mathtt{Sat}\left((H_2 + T) \cap F\left[x_1, \ldots, x_s\right], f\right).\end{aligned}$$

Proof. The statement immediately follows if we prove the equality

$$(H^e + (ft - 1) + T) \cap F\left[x_1, \ldots, x_s\right] = \mathtt{Sat}\left((\mathtt{Sat}(H, f) + T) \cap F\left[x_1, \ldots, x_s\right], f\right), \tag{8.4.2}$$

stating that the non-degeneracy conditions found employing Rabinowitsch's trick are the saturation of those provided by the saturation method.

First of all, we recall that, by equality (8.2.1), it holds that $(H^e + (ft-1) + T) \cap F\left[x_1, \dots, x_n\right] = \mathtt{Sat}(H+T, f)$. Let us continue proving that the right side of this equality can be regarded as follows

$$\mathtt{Sat}(H+T, f) = \mathtt{Sat}\left(\mathtt{Sat}(H, f) + T, f\right). \tag{8.4.3}$$

It is clear that $\mathtt{Sat}(H+T, f) \subseteq \mathtt{Sat}\left(\mathtt{Sat}(H, f) + T, f\right)$, since $H \subseteq \mathtt{Sat}(H, f)$. Conversely, if $g \in \mathtt{Sat}(\mathtt{Sat}(H, f)+T, f)$, there exists $n \in \mathbb{N}_{>0}$ such that $gf^n = a+b$, with $a \in \mathtt{Sat}(H, f)$ and $b \in T$. Again, there exists $m \in \mathbb{N}_{>0}$ such that $af^m \in H$; so, $gf^{n+m} = af^m + bf^m \in H+T$. Thus, it follows by definition that $g \in \mathtt{Sat}(H+T, f)$.

Now, in order to end proving the initial equality (8.4.2) it remains to exhibit that

$$\begin{aligned}&\mathtt{Sat}\left(\mathtt{Sat}(H, f) + T, f\right) \cap F\left[x_1, \dots, x_s\right] \\ &= \mathtt{Sat}\left((\mathtt{Sat}(H, f) + T) \cap F\left[x_1, \dots, x_s\right], f\right).\end{aligned}$$

Let $g \in \mathtt{Sat}\left(\mathtt{Sat}(H, f) + T, f\right) \cap F\left[x_1, \dots, x_s\right]$. It is clear that $gf^n \in \mathtt{Sat}(H, f) + T$ for some $n \in \mathbb{N}_{>0}$. Both g and f^n belong to $F\left[x_1, \dots, x_s\right]$, whence gf^n belongs to the same ring, too. So, $g \in \mathtt{Sat}\left((\mathtt{Sat}(H, f) + T) \cap F\left[x_1, \dots, x_s\right], f\right)$. The converse follows trivially from the fact that both $(\mathtt{Sat}(H, f) + T) \cap F\left[x_1, \dots, x_s\right]$ and (f) lie on $F\left[x_1, \dots, x_s\right]$. □

The previous lemma allows us to conclude the following result.

Theorem 8.4.7. *The statement $H_1 \Rightarrow T$ is generally false if and only if the statement $H_2 \Rightarrow T$ is also generally false; analogously, $H_1 \Rightarrow T$ generally true is equivalent to $H_2 \Rightarrow T$ generally true.*

Proof. The first assertion follows from Lemma 8.4.6 and the fact that an ideal is zero if and only if its saturation by another, non-zero, arbitrary ideal, is zero. For the second one, it suffices to replace the ideal T in Lemma 8.4.6 with $Tt' - 1$ and to reason as above. □

Theorem 8.4.7 says that the method employed for introducing the non-degeneracy conditions does not affect whether the theorem is generally true or generally false, but it can provide different sets of additional hypotheses for discovery (see Example 8.4.5). Henceforth, we will denote them by:

$$\begin{aligned}\mathcal{H}_1 &:= (H_1 + T) \cap F\left[x_1, \dots, x_s\right], \\ \mathcal{H}_2 &:= (H_2 + T) \cap F\left[x_1, \dots, x_s\right].\end{aligned}$$

Recall (see equation (8.4.2)) that $\mathcal{H}_1 = \mathtt{Sat}(\mathcal{H}_2, f)$. Following the traditional protocol for discovery, the next step would be to consider the statements:

$$\begin{aligned}St_A :&\quad H_1 + \mathcal{H}_1^e \Rightarrow T, \\ St_B :&\quad H_2 + \mathcal{H}_2^e \Rightarrow T,\end{aligned}$$

and continue to check out whether the new theorems are generally true or not. But we have already seen in Example 8.4.5 that $\mathcal{H}_2$ may be generated by elements contradicting the negation $\neg\{f = 0\}$. Therefore, it might be interesting to saturate again $H_2 + \mathcal{H}_2^e$ by f and to change St_B by

$$St_{B'} : \quad \mathtt{Sat}(H_2 + \mathcal{H}_2^e, f) \Rightarrow T,$$

where $\mathtt{Sat}(H_2 + \mathcal{H}_2^e, f) = \mathtt{Sat}(H + \mathcal{H}_2^e, f)$ (it follows from repeating the proof of equality (8.4.3)). Note that, also, $H_1 + \mathcal{H}_1^e = H^e + \mathcal{H}_1^e + (ft - 1)$. Finally, and with the goal of generalising as well as making easier the proof of the following results, we will consider

$$\begin{aligned}St_1 :&\quad H^e + \mathcal{H}_1^e + (ft-1) \Rightarrow T, \\ St_2 :&\quad H^e + \mathcal{H}_2^e + (ft-1) \Rightarrow T, \\ St_3 :&\quad \mathtt{Sat}(H + \mathcal{H}_1^e, f) \Rightarrow T, \\ St_4 :&\quad \mathtt{Sat}(H + \mathcal{H}_2^e, f) \Rightarrow T,\end{aligned}$$

being $St_1 = St_A$ and $St_4 = St_{B'}$.

Recall that, by enlarging the set of hypotheses H with $\mathcal{H}_1^e$ or $\mathcal{H}_2^e$, some of the independent variables $\{x_1, \dots, x_s\}$ ruling the initial theorem $H \Rightarrow T$, could become dependent. In that case we would have to deal with two new sets of independent variables: $\Lambda_1 := \{x_1, \dots, x_{s_1}\}$, the independent variables in $H + \mathcal{H}_1^e$, as well as $\Lambda_2 := \{x_1, \dots, x_{s_2}\}$ for $H + \mathcal{H}_2^e$. Since the inclusion $H_2^e \subseteq H_1$ holds, it follows that $\Lambda_1 \subseteq \Lambda_2$.

Nevertheless, we will avoid such subtleties by restricting, in what follows, to the case in which f is formulated just in terms of the variables in Λ_1: f "as independent as possible".

Aiming to establish the relationships among the statements St_1, St_2, St_3 and St_4, we present the following useful lemma.

Lemma 8.4.8. *With the above notation, it holds that*

$$\mathtt{Sat}(H + \mathcal{H}_1^e, f) = \mathtt{Sat}(H + \mathcal{H}_2^e, f). \tag{8.4.4}$$

In addition,

$$\begin{aligned} &\left(H^e + \mathcal{H}_1^e + (ft-1) + (Tt'-1)\right) \cap F\left[x_1, \dots, x_n, t'\right] \\ &= \left(H^e + \mathcal{H}_2^e + (ft-1) + (Tt'-1)\right) \cap F\left[x_1, \dots, x_n, t'\right]. \end{aligned} \tag{8.4.5}$$

Proof. We infer from Lemma 8.4.6 that

$$\mathtt{Sat}(H + \mathcal{H}_1^e, f) = \mathtt{Sat}(H + \mathtt{Sat}(\mathcal{H}_2, f)^e, f),$$

being the term in the right side equal to $\mathtt{Sat}\left(H + \mathtt{Sat}(\mathcal{H}_2^e, f), f\right)$ (see [59, Appendix]). From here, it suffices to essentially repeat the proof of equality (8.4.3) to have equation (8.4.4) proved: the two sets of additional hypotheses $\mathcal{H}_1$ and $\mathcal{H}_2$ yield the same ideal when added to H and saturated by f.

As for (8.4.5), employing equality (8.2.1) we see that both

$$\left(H^e + \mathcal{H}_1^e + (ft-1) + (Tt'-1)\right) \cap F\left[x_1, \dots, x_n, t'\right] = \mathtt{Sat}\left(H^e + \mathcal{H}_1^e + (Tt'-1), f\right)$$

and

$$\left(H^e + \mathcal{H}_2^e + (ft-1) + (Tt'-1)\right) \cap F\left[x_1, \dots, x_n, t'\right] = \mathtt{Sat}\left(H^e + \mathcal{H}_2^e + (Tt'-1), f\right)$$

hold. Again, Lemma 8.4.6 and the idea lying under equality (8.4.3) enable us to state:

$$\begin{aligned} &\mathtt{Sat}\left(H^e + \mathcal{H}_1^e + (Tt'-1), f\right) \\ &= \mathtt{Sat}\left(H^e + \mathtt{Sat}(\mathcal{H}_2, f)^e + (Tt'-1), f\right) \\ &= \mathtt{Sat}\left(H^e + \mathtt{Sat}(\mathcal{H}_2^e, f) + (Tt'-1), f\right) \\ &= \mathtt{Sat}\left(H^e + \mathcal{H}_2^e + (Tt'-1), f\right), \end{aligned}$$

and we are done. $\square$

Remark 8.4.9. Note that the only requirement for Lemma 8.4.8 to hold is that $f \in F\left[x_1, \dots, x_s\right]$, since this is an essential hypotheses in Lemma 8.4.6.

Now, we are ready to state our main results.

Theorem 8.4.10. *Notation as above. The statements St_1, St_2, St_3 and St_4 are generally true if and only if one of them is generally true. Furthermore, St_3 and St_4 have exactly the same hypotheses, and the non-degeneracy conditions of St_1 are equal to those of St_2 provided that $\Lambda_1 = \Lambda_2$.*

Proof. Firstly, we observe that, with the assumption that $f \in F\left[x_1, \dots, x_{s_1}\right]$, by Theorem 8.4.7, St_1 generally true is equivalent to St_3 generally true; the same happens with St_2 and St_4. If we prove that St_3 is generally true if and only if so is St_4, we are done. Indeed, this follows trivially from Lemma 8.4.8, since $St_3 = St_4$.

In addition, equality (8.4.5) of Lemma 8.4.8 and the inclusion $\Lambda_1 \subseteq \Lambda_2$ tell us that

$$\begin{aligned} &\left(H^e + \mathcal{H}_1^e + (ft-1) + (Tt'-1)\right) \cap F\left[x_1, \dots, x_{s_1}\right] \\ &\subseteq \left(H^e + \mathcal{H}_2^e + (ft-1) + (Tt'-1)\right) \cap F\left[x_1, \dots, x_{s_2}\right]. \end{aligned}$$

The previous inclusion trivially switches to an equality if $\Lambda_1 = \Lambda_2$. □

Corollary 8.4.11. *With the above notation, St_A is generally true if and only if St_B is generally true as well.*

Proof. Since St_A and St_1 are the same statement, Theorem 8.4.10 enables us to state that St_A generally true is equivalent to St_2 generally true; i.e.

$$\left(H^e + \mathcal{H}_2^e + (ft-1)^e + (Tt'-1)\right) \cap F\left[x_1, \dots, x_{s_2}\right] \neq (0).$$

Let us prove that

$$\begin{aligned} &\left(H^e + \mathcal{H}_2^e + (ft-1) + (Tt'-1)\right) \cap F\left[x_1, \dots, x_{s_2}\right] \\ &= \mathtt{Sat}\left(\left(H_2^e + \mathcal{H}_2^e + (Tt'-1)\right) \cap F\left[x_1, \dots, x_{s_2}\right], f\right). \end{aligned} \tag{8.4.6}$$

Indeed, by equality (8.2.1), the reasoning in (8.4.3) and $f \in F\left[x_1, \dots, x_{s_1}\right]$, we have that

$$\begin{aligned}
&\mathtt{Sat}\left(\left(H_2^e + \mathcal{H}_2^e + (Tt' - 1)\right) \cap F\left[x_1, \dots, x_{s_2}\right], f\right) \\
&= \mathtt{Sat}\left(\mathtt{Sat}(H^e, f) + \mathcal{H}_2^e + (Tt' - 1), f\right) \cap F\left[x_1, \dots, x_{s_2}\right] \\
&= \mathtt{Sat}\left(H^e + \mathcal{H}_2^e + (Tt' - 1), f\right) \cap F\left[x_1, \dots, x_{s_2}\right] \\
&= \left(H^e + \mathcal{H}_2^e + (Tt' - 1) + (ft - 1)\right) \cap F\left[x_1, \dots, x_n, t'\right] \cap F\left[x_1, \dots, x_{s_2}\right] \\
&= \left(H^e + \mathcal{H}_2^e + (ft - 1) + (Tt' - 1)\right) \cap F\left[x_1, \dots, x_{s_2}\right].
\end{aligned}$$

Therefore, (8.4.6) is proved and then

$$\left(H^e + \mathcal{H}_2^e + (ft - 1) + (Tt' - 1)\right) \cap F\left[x_1, \dots, x_{s_2}\right] \neq (0)$$

if and only if $\left(H_2^e + \mathcal{H}_2^e + (Tt' - 1)\right) \cap F\left[x_1, \dots, x_{s_2}\right] \neq (0)$; i.e. St_2 is generally true if and only if St_B is generally true.

In conclusion, St_A generally true is equivalent to St_B generally true. □

8.5 Our experiences

In this section, we would like to present a particular example in order to show in some detail the described situation, having the calculations been carried out using the software `Singular` [67] in the FinisTerrae 2 supercomputer. Our example is based on the already cited theorem about the orthic triangle taken from [59, Example 6 of Section 5]. We wish to show that the orthic triangle associated with an equilateral triangle is also equilateral (see Fig. 8.1). Since we want to address the theorem from the point of view of discovery, we decide to ignore the hypothesis about the original triangle being equilateral and take a completely arbitrary one, with the purpose of obtaining, automatically, a necessary condition on this general triangle for the corresponding orthic triangle to be equilateral.

So, we take $A(0,0)$, $B(x_1, 0)$ and $C(x_2, x_3)$ as the vertices of the main triangle, and set $D(x_2, 0)$, $E(x_4, x_5)$ and $F(x_6, x_7)$ the vertices of the orthic one. The independent variables are $\{x_1, x_2, x_3\}$. We force the segments $\overline{AE}$ and $\overline{BC}$ to be perpendicular, as well as E to be collinear with B and C; analogously, $\overline{BF}$ and $\overline{AC}$ must be perpendicular, and F must be aligned with A and C. By construction, it is obvious that the point D is collinear with A and B, and that $\overline{CD}$ is perpendicular to $\overline{AC}$. As for our desired conclusion, we state it using two polynomials, each one forcing two sides of the orthic triangle to have the same length; i.e. one for $AB = AC$, and another for $AB = BC$. We deal with them separately, distinguishing between theses T and T', respectively, and theorems $H \Rightarrow T$ and $H \Rightarrow T'$.

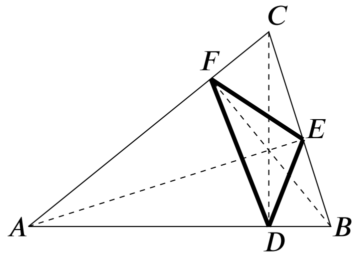

Figure 8.1: Orthic triangle

Besides the main hypotheses, we choose to add (based on human intuition and hoping to help to simplify the computations) as non-degeneracy conditions those which force the triangle ABC not to collapse to a line, i.e. $x_1 \neq 0$ and $x_3 \neq 0$. These two conditions can be summarised in just one: $f = x_1x_3 \neq 0$. We introduce this new hypothesis $f \neq 0$, both employing Rabinowitsch's trick and by saturation, in each of the two theorems $H \Rightarrow T$ and $H \Rightarrow T'$. It is easy to check that any of the statements $H_1 \Rightarrow T$, $H_2 \Rightarrow T$, $H_1 \Rightarrow T'$ and $H_2 \Rightarrow T'$ is generally false. However, different formulations of the introduced non-degeneracy hypothesis can lead to different sets of additional affirmative hypotheses for the discovery of a true statement. More precisely, for the theorem $H \Rightarrow T$ with the notation of 8.4, we get that

$$\mathcal{H}_2^T = x_1\mathcal{H}_1^T = \left(x_1\,(x_1 - 2x_2)\,(x_1x_2 - x_2^2 + x_3^2)\,(-x_1x_2 + x_2^2 + x_3^2)\right).$$

Nevertheless, for the theorem $H \Rightarrow T'$, we obtain

$$\mathcal{H}_1^{T'} = \mathcal{H}_2^{T'} = \left(x_2\,(-x_1^2 + x_2^2 + x_3^2)\,(-x_1^2x_2 + 2x_1x_2^2 + 2x_1x_3^2 - x_2^3 - x_2x_3^2)\right).$$

We appreciate that the factor x_1, which was forced to be different from zero by introducing the non-degeneracy hypothesis, appears as a zero condition in $\mathcal{H}_2^T$, due to the early closure of the saturation, similarly to what happens in Example 8.4.5. Informally speaking, we can say that adding the negation $x_1x_3 \neq 0$ is not so conclusive when we deal with saturation as it is when employing Rabinowitsch's trick.

It is also trivial to check that, if the triangle ABC is equilateral, the additional conditions in $\mathcal{H}_1^T$, $\mathcal{H}_2^T$, $\mathcal{H}_1^{T'}$ or $\mathcal{H}_2^{T'}$ all vanish. That is, equilateral triangles yield equilateral orthic triangles. Nevertheless, we also see that there are other possible configurations for the given triangle which make these additional, necessary conditions to vanish: configurations that should be carefully analysed, since, if also sufficient, they would bring some unexpected statements regarding the regularity of the orthic triangle.

Now, a direct computation shows that the statement St_B (according to the previous section notation) is generally true for both T and T', but by brute force we are not able to decide if St_A is, as well, generally true. Yet, applying our precedent results, we can conclude that not only is $St_A = St_1$ generally true, but the same applies to St_2 and $St_3 = St_4$, too. Moreover, the calculations for St_4 reveal that, in addition to being generally true, we do not need to consider non-degeneracy conditions both for T and for T'; again, and since $\Lambda_1^T = \Lambda_2^T = \Lambda_1^{T'} = \Lambda_2^{T'} = \{x_1, x_3\}$, Theorem 8.4.10 allows us to state that $St_A = St_1$ and St_2 do not need additional non-degeneracy conditions as well.

What is interesting to emphasise here is that the computer is not able to arrive at these conclusions by directly following the Rabinowitsch approach in St_A and confirming that the theses T and T' are generally true, while it encounters no difficulties with the saturation approach of St_B or St_4. So, we think that this example clearly illustrates the practical advantages, in some cases, of using saturation instead of Rabinowitsch's trick. Also, the example highlights the applicability of the results in the previous section.

Yet, we should make clear here that we do not know the reason for the failure of the FinisTerrae 2 supercomputer concerning the computation, via `Singular`, of the Rabinowitsch approach for this particular example. We would like to note that the saturation-based computation uses `Singular`'s implementation of saturation which does not involve elimination (see `https://github.com/Singular/Sources/blob/Release-4-1-2/Singular/LIB/elim.lib#L739`). We want to thank the referees for pointing out this, as well as for bringing up our attention to reference [17], for a detailed account of the complex, reciprocal relation between saturation and elimination (saturation via elimination, elimination via saturation).

8.6 Conclusions

At this point, the reader could wonder: which one of the presented methods is better? The answer is not totally objective. We encourage to implement the saturation method in the automated proving and discovery software, due to the scarce effectiveness of Rabinowitsch's trick when dealing with negative theses and to the practical objections exposed in Section 8.5. But there is a counterpart: by considering the early closure of the saturation $\texttt{Sat}(H, f)$, we can lose essential information about the negation $\neg\{f = 0\}$, and f may appear further as a zero equation in the additional hypotheses yielded by saturation, transgressing, in a certain sense, the restrictions imposed by the introduced non-degeneracy conditions. This fact is precisely what differentiates both methods, and what could persuade us to employ sometimes Rabinowitsch's trick, if we want to preserve the negation of f until the end of the procedure, in order to remain faithful to some a priori stated non-degeneracy condition. In our opinion, in this case, it should be decided by the user, through the corresponding dialogue with the involved automatic theorem proving software.

Finally, we must recall here the existence of some new algorithmic tools to deal with constructible sets, as commented in the Introduction, that deserve future consideration and implementation in some dynamic geometry program provided with automated reasoning modules.

Results

The results presented in this thesis appear in the following research articles:

- **“Restricted Lie algebras having a distributive lattice of restricted subalgebras”**, *Linear and Multilinear Algebra* (2019).

  DOI: 10.1080/03081087.2019.1708238.

- **“The algebraic and geometric classification of nilpotent bicommutative algebras”**, *Algebras and Representation Theory*, Vol. 23, No. 6 (2020), 2331–2347.

  DOI: 10.1007/s10468-019-09944-x.

- **“One-generated nilpotent bicommutative algebras”**, accepted for publication in *Algebra Colloquium*.

- **“Non-associative central extensions of null-filiform associative algebras”**, *Journal of Algebra*, Vol. 560 (2020), 1190–1210.

  DOI: 10.1016/j.jalgebra.2020.06.013.

- **“The non-abelian tensor and exterior products of crossed modules of Lie superalgebras”**, *Journal of Algebra and its Applications* (2021).

  DOI: 10.1142/S0219498822501699.

- **“Dealing with negative conditions in automated proving: tools and challenges. The unexpected consequences of Rabinowitsch’s trick”**, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, Vol. 114, No. 4 (2020), Paper 162, 16 pp.

  DOI: 10.1007/s13398-020-00874-8.

Also, we have included a considerable amount of work in progress, which be plan to submit to some international journals in the near future.

Conclusions

Throughout this dissertation, we have fully achieved the objectives stated. Namely:

In Chapter 1, we characterise restricted Lie algebras with distributive and Boolean lattices of restricted subalgebras, and we study the structural properties of finite-dimensional restricted Lie algebras over algebraically closed fields whose lattice of restricted subalgebras is dually atomistic, atomistic, lower or upper semimodular, or in which every restricted subalgebra is a quasi-ideal. Most of these properties turn out to be weaker than their counterparts in the ordinary Lie algebra case, hence there are more examples of algebras satisfying them.

In Chapter 2, we introduce the definition of the non-abelian tensor product of restricted Lie superalgebras and study some of its structural and categorical properties. In particular, we obtain that a restricted Lie superalgebra admits a universal central extension if and only if it is perfect, and offer a specific construction of this extension in terms of the non-abelian tensor product.

In Chapter 3, we find the twenty-four isomorphism classes of nilpotent pure bicommutative algebras over $\mathbb{C}$ of dimension 4, the twelve isomorphism classes of one-generated nilpotent bicommutative algebras over $\mathbb{C}$ of dimension 5 and the twenty-nine isomorphism classes of one-generated nilpotent bicommutative algebras over $\mathbb{C}$ of dimension 6. Also, we find that the variety of four-dimensional complex nilpotent bicommutative algebras has two irreducible components determined by a rigid algebra and an infinite family of algebras.

In Chapter 4, we prove that the null-filiform associative algebra μ_0^n over fields of suitable characteristic does not admit non-split non-trivial central extensions within the varieties of alternative, left-alternative (and right-alternative) or Jordan algebras. If the ground field is algebraically closed and has characteristic 0, we give the algebraic classification of all non-split central extensions with one-dimensional annihilator of μ_0^n within the varieties of left-commutative (or right-commutative) and bicommutative algebras. We also reduce the cases of assosymmetric, Novikov and left-symmetric (or right-symmetric) algebras to the situation in the bicommutative and left-commutative (or right-commutative) varieties.

In Chapter 5, we develop a further adaptation of the method of Skjelbred-Sund explained in Chapters 3 and 4 to construct central extensions of axial algebras. We prove that every axial algebra with non-zero annihilator is isomorphic to a certain central extension of an axial algebra of smaller dimension, and also use our results to prove that all axial central extensions (with respect to a maximal set of axes) of simple finite-dimensional Jordan algebras are split.

In Chapter 6, we introduce the notions of non-abelian tensor and exterior products of two ideal graded crossed submodules of a given crossed module of Lie superalgebras. We study some of their basic structural and homological properties; in particular, we characterise the second homology $H_2(T, L, \partial)$ in terms of the non-abelian exterior product. We also define new versions of Whitehead's quadratic functor for supermodules and for abelian crossed modules of Lie superalgebras.

In Chapter 7, we study the properties of Whitehead's quadratic functor for supermodules and for abelian crossed modules of Lie superalgebras, introduced in Chapter 6, and generalised in Chapter 7 to rings in which 2 does not necessarily have an inverse. In particular, we explore their relation with quadratic maps of supermodules and introduce a notion of quadratic maps of abelian crossed modules, which we relate with the Whitehead's quadratic functor for abelian crossed modules of Lie superalgebras.

In Chapter 8, we compare the methods of Rabinowitsch's trick and saturation for introducing negative theses and negative hypotheses in the standard procedures for the algebraic-geometry-based theory of automated proving and discovery of geometric theorems. We conclude that Rabinowitsch's trick is not suitable for the introduction of negative theses, whereas for negative hypotheses it presents theoretical advantages and practical disadvantages compared to saturation.

Resumo

Resumo abreviado

O marco xeral no que se encadra esta tese é a teoría das álxebras non asociativas. Nela tratamos diversos problemas que concernentes ás álxebras de Lie restrinxidas, ás extensións centrais de diferentes clases de álxebras e aos módulos cruzados de superálxebras de Lie. A saber, estudamos as relacións entre as propiedades estruturais dunha álxebra de Lie restrinxida e as do seu retículo de subálxebras restrinxidas; definimos un produto tensor non abeliano para superálxebras de Lie restrinxidas e para submódulos cruzados graduados ideais dun módulo cruzado de superálxebras de Lie, e exploramos as súas propiedades dende puntos de vista estrutural, categórico e homolóxico; empregamos extensións centrais para clasificar álxebras biconmutativas nilpotentes; e calculamos extensións centrais das álxebras nulfiliformes asociativas e de álxebras axiais. Ademais, incluímos un capítulo final adicado a comparar os dous métodos principais (o truco de Rabinowitsch e a saturación) para introducir condicións negativas nos procedementos usuais da teoría de probas e descubrimentos automáticos.

As relacións entre a estrutura dun grupo e a do seu retículo de subgrupos está altamente desenvolvida e atraeu a atención de moitos destacados alxebristas (véxase por exemplo a monografía [202] ou a revisión [185]). Segundo Schmidt [202], a orixe deste tema remóntase a Dedekind, quen estudou o retículo de ideais nun anel de enteiros alxébricos; descubriu e empregou a identidade modular, tamén chamada a lei de Dedekind, ao calcular os ideais. Porén, os comezos reais do estudo de retículos de subgrupos datan dos arredores de 1930. Un dos primeiros logros destacados neste contexto foi a caracterización de Ore dos grupos con retículo de subgrupos distributivos: son exactamente os grupos localmente cíclicos [183]. Dende entón, a modularidade, distributividade e condicións reticulares relacionadas con elas foron estudadas en moitos contextos. O retículo de submódulos dun módulo sobre un anel é modular, e polo tanto tamén o é o retículo de subgrupos dun grupo abeliano. O retículo de subgrupos normais dun grupo tamén é modular, pero o retículo de todos os subgrupos non o é, en xeral [122, 123]. O retículo de ideais dun anel tamén é modular. A distributividade do retículo de submódulos dun módulo foi investigado en [46, 212, 218], e a do retículo de ideais (pola dereita) dun anel ou dunha álxebra asociativa, en [37, 131, 218].

O estudo do retículo de subálxebras e de ideais dunha álxebra de Lie de dimensión finita foi popular na segunda metade do século pasado, especialmente nas décadas de 1980 e 1990 (véxase, por exemplo, [13, 18, 24, 33, 92–95, 100, 154, 160, 216, 217, 220–223]), pero logo o interese desvaneceuse. Unha posible razón disto é que as condicións estudadas eran demasiado fortes e poucas álxebras as satisfacían; un exemplo paradigmático disto é a caracterización das álxebras de Lie con retículo de subálxebras distributivo [154, Theorem 2.1]. Non obstante, o retículo de subálxebras restrinxidas dunha álxebra de Lie restrinxida é radicalmente diferente: por exemplo, non todo elemento xera unha subálxebra restrinxida de dimensión 1. Entón, podemos esperar que se cumpran resultados máis interesantes que no caso das álxebras de Lie ordinarias e, como veremos, este é o caso.

No primeiro Capítulo 1, caracterizamos as álxebras de Lie restrinxidas con retículo de subálxebras restrinxidas distributivo e Booleano nos seguintes resultados.

Teorema 1.2.5. *Unha álxebra de Lie restrinxida L sobre un corpo F de característica $p > 0$ é distributiva se e só se L é abeliana e, para toda subálxebra restrinxida H de L, L/H non contén subálxebras restrinxidas minimais isomorfas diferentes.*

Corolario 1.2.11. *Sexa L unha álxebra de Lie restrinxida L sobre un corpo F de característica $p > 0$. Entón L é Booleana se e só se $L \simeq \bigoplus_{I\in\mathcal{F}} I$, onde $\mathcal{F}$ é unha familia de álxebras de Lie restrinxidas non isomorfas dúas a dúas, cada unha das cales non contén subálxebras restrinxidas propias e distintas de cero.*

A continuación, estúdanse álxebras de Lie restrinxidas de dimensión finita e con corpos bases case sempre alxebricamente pechados cuxos retículos de subálxebras restrinxidas son dualmente atomísticos, atomísticos, semimodulares por abaixo e semimodulares por enriba. Tamén se estudan álxebras de Lie restrinxidas nas que toda subálxebra restrinxida é un case-ideal restrinxido. Enumeramos a continuación os resultados máis relevantes neste respecto.

Proposición 1.3.2. *Sexa L unha álxebra de Lie restrinxida sobre un corpo alxebricamente pechado F. Entón, L é soluble ou semisimple.*

Proposición 1.3.3. *Sexa L unha álxebra de Lie restrinxida soluble sobre calquera corpo F. Se L é dualmente atomística, entón*

$$L \simeq \left(\mathcal{L}/\langle \bar{f}_1 \rangle_p \oplus \cdots \oplus \mathcal{L}/\langle \bar{f}_r \rangle_p \oplus F x_{r+1} \oplus \cdots \oplus F x_n\right) \dotplus F b,$$

onde $r \geq 0$, pero $r \neq n$, b é toral, $\mathcal{L} = \langle x \rangle_p$ é unha álxebra de Lie restrinxida cíclica libre e $\bar{f}_i = \sum_{k=0}^{s_i} \lambda_k x^{[p]k}$ é un elemento de $\mathcal{L}$ tal que $f_i = \sum_{k=0}^{s_i} \lambda_k t^k$ é un elemento irreducible do anel de polinomios nesgados $F[t, \sigma]$.

Proposición 1.3.6. *Sexa F un corpo alxebricamente pechado de característica $p > 0$. Unha álxebra de Lie restrinxida sobre F é atomística se e só se cada subálxebra restrinxida cíclica p-nilpotente ten dimensión 1.*

Teorema 1.6.1. *Sexa L unha álxebra de Lie restrinxida sobre un corpo alxebricamente pechado F. As seguintes condicións son equivalentes:*

(i) L é semimodular por enriba;

(ii) L é modular;

(iii) toda subálxebra restrinxida de L é un case-ideal restrinxido.

Ademais, se se cumpre algunha das afirmacións previas, entón L é case abeliano ou nilpotente de clase como moito 2.

As superálxebras de Lie apareceron orixinalmente asociadas a certos grupos xeneralizados, hoxe coñecidos como supergrupos formais de Lie, na década de 1930. Porén, non foi ata corenta anos despois cando estes obxectos acadaron importancia real nas comunidades física e matemática, debido á súa conexión coa teoría da supersimetría (véxase [21, 192], por exemplo). Esta teoría pretendía proporcionar un tratamento unificado para bosóns e fermións, as dúas clases de partículas elementais que compoñen o universo, así como modelar as transicións entre elas. As superálxebras de Lie son un obxecto clave neste contexto, o que motivou un profundo estudo non só dende a perspectiva da física matemática, senón tamén dende unha aproximación puramente alxébrica. Exemplos disto poden ser a aclamada clasificación das superálxebras de Lie simples de dimensión finita sobre un corpo alxebricamente pechado de característica 0 feita por Kac [133], a clasificación homóloga real [206] ou os resultados parciais cara á clasificación das álxebras de Lie simples de dimensión infinita (por exemplo [134]).

Ao igual que ocorre no caso non graduado, para traballar con superálxebras de Lie modulares convén coñecer as superálxebras de Lie restrinxidas. Dende a súa introdución no 1988 da man de Mikhalëv [175], probaron sobradamente ser útiles na obtención de novos resultados na teoría de representacións ou na clasificación de superálxebras de Lie modulares (véxase [15, 31, 32, 174, 186, 225, 226], entre outros). Estas superálxebras tamén foron estudadas en moitas outras referencias coma [25, 81, 165, 219, 224], por exemplo.

Por outra banda, os produtos tensores non abelianos teñen unha longa historia na literatura especializada. A súa primeira aparición foi no contexto dos grupos: en efecto, Loday e Brown [36] definiron un produto tensor entre dous grupos non necesariamente abelianos actuando un sobre o outro, ao cal chamaron non abeliano para evitar confusións co coñecido produto tensor de $\mathbb{Z}$-módulos. Tamén introduciron un cociente deste obxecto, chamado o produto exterior non abeliano. Despois disto, Ellis construíu os seus produtos tensor e exterior non abelianos de álxebras de Lie en [76]. Estes produtos foron obxecto de diversas xeneralizacións en distintas direccións, como pode ser para álxebras de Lie restrinxidas [157], para módulos cruzados de álxebras de Lie [75, 193], ou para superálxebras de Lie [91]. En todos os casos, ditos produtos foron empregados para obter resultados acerca da (co)homoloxía en baixas dimensións das estruturas alxébricas respectivas, así como para atopar unha expresión explícita das extensións centrais universais de obxectos perfectos, afondando deste xeito nos resultados que afirman que un obxecto é perfecto se e só se admite unha extensión central universal [142, 156, 180].

No Capítulo 2, estendemos distintos resultados de [76, 91, 142, 156, 157, 180] ao introducir un produto tensor non abeliano de superálxebras de Lie restrinxidas, estudar as súas propiedades básicas e relacionalo con extensións centrais relativas á subcategoría de Birkhoff dos obxectos abelianos **Ab**. Queremos salientar que a nosa construción xeneraliza a da corta nota [157], ofrecendo polo tanto algúns resultados novos no ámbito das álxebras de Lie restrinxidas. Porén, non definimos un produto exterior non abeliano de superálxebras de Lie restrinxidas, nin tampouco tratamos con ningunha aplicación (co)homolóxica das nosas conclusións.

Tamén é de destacar que, aínda que nós nos centramos principalmente na subcategoría de Birkhoff **Ab**, existen outras subcategorías de Birkhoff que sería interesante estudar, a saber a subcategoría **0pSLie** de superálxebras de Lie restrinxidas nas que a p-aplicación é identicamente cero, ou a intersección **sAb** de **Ab** e **0pSLie**, é dicir, a subcategoría formada polas superálxebras de Lie restrinxidas abelianas con p-aplicación cero.

$$\begin{tikzcd}
& \mathbf{pSLie} & \\
\mathbf{Ab} \arrow[ur] & & \mathbf{0pSLie} \arrow[ul] \\
& \mathbf{sAb} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Polo tanto, este capítulo 2 debe ser entendido como un primeiro paso cara a un estudo exhaustivo das relacións entre os diferentes tipos de extensións centrais de superálxebras de Lie restrinxidas e as súas correspondentes teorías de (co)homoloxía.

A principal definición introducida é a seguinte.

Definición 2.2.1. *Sexan L e M dúas superálxebras de Lie restrinxidas actuando unha sobre a outra, e sexa $X_{L,M}$ o conxunto de símbolos $x \otimes m$, para x e m elementos homoxéneos de L e M, respectivamente. Dótese a $X_{L,M}$ coa $\mathbb{Z}_2$-graduación $|x \otimes m| = |x| + |m|$. O produto tensor non abeliano $L \otimes M$ defínese como a superálxebra de Lie restrinxida xerada por $X_{L,M}$ e suxeita ás seguintes relacións:*

$$\lambda(x \otimes m) = \lambda x \otimes m = x \otimes \lambda m;$$

$$(x + y) \otimes m = x \otimes m + y \otimes m;$$

$$x \otimes (m + n) = x \otimes m + x \otimes n;$$

$$[x, y] \otimes m = x \otimes {}^{y}m - (-1)^{|x||y|} y \otimes {}^{x}m,$$

$$x \otimes [m, n] = (-1)^{|x||n|}(-1)^{|m||n|}({}^{n}x \otimes m) - (-1)^{|x||m|}({}^{m}x \otimes n);$$

$${}^{m}x \otimes {}^{y}n = -(-1)^{|x||m|}[x \otimes m, y \otimes n];$$

$$x_{\bar{0}}^{[p]} \otimes m = x_{\bar{0}} \otimes \left({}^{x_{\bar{0}}^{p-1}} m\right);$$

$$x \otimes m_{\bar{0}}^{[p]} = \left({}^{m_{\bar{0}}^{p-1}} x\right) \otimes m_{\bar{0}},$$

para todo $\lambda \in F$, $x, y \in L$, $m, n \in M$, $x_{\bar{0}} \in L_{\bar{0}}$ e $m_{\bar{0}} \in M_{\bar{0}}$.

O principal resultado obtido en relación ao produto tensor non abeliano de superálxebras de Lie restrinxidas é o seguinte.

Teorema 2.3.4. *Unha superálxebra de Lie restrinxida L admite unha extensión central universal se e só se é perfecta.*

Dita extensión do Teorema 2.3.4 non é outra que

$$\begin{aligned} \mu\colon\ L \otimes L &\to L \\ (x, y) &\mapsto [x, y]. \end{aligned}$$

A variedade das álxebras biconmutativas, tamén coñecidas como LR-álxebras, está definida polas identidades polinomiais de conmutatividade pola esquerda e pola dereita. Explicitamente, unha álxebra $(\mathrm{A}, \cdot)$ dise biconmutativa se

$$x\,(yz) \;=\; y\,(xz)\,, \qquad (xy)\,z \;=\; (xz)\,y,$$

para todo $x, y, z \in \mathrm{A}$.

O primeiro exemplo coñecido de álxebra conmutativa por un lado é a álxebra de Witt nunha variable simétrica pola dereita, que se remonta ao ano 1857 [54]. Esta álxebra satisfai a a identidade da conmutatividade pola esquerda, pero non pola dereita, polo que non é biconmutativa. Os exemplos máis sinxelos de álxebras biconmutativas son as álxebras conmutativas e asociativas. Nótese que as álxebras biconmutativas son Lie admisibles: o conmutador $[x, y] = xy - yx$ define unha estrutura de Lie asociada en A.

As álxebras biconmutativas sobre $\mathbb{R}$ aparecen naturalmente en conexión coa xeometría, en particular coas accións afíns simplemente transitivas de grupos de Lie nilpotentes [38]. Estas álxebras biconmutativas son completas (é dicir, os operadores de multiplicación pola esquerda L_x son nilpotentes para todo $x \in \mathrm{A}$) e as súas estruturas de álxebras de Lie asociadas son nilpotentes. Este feito motiva a clasificación de [39] para dimensións $n \leq 4$, na que os autores limítanse a considerar álxebras biconmutativas reais e completas con estrutura de álxebra de Lie asociada nilpotente. As álxebras biconmutativas tamén foron estudadas en [40, 70–74]; salientamos as clasificacións alxébrica e xeométrica das álxebras biconmutativas de dimensión 2 sobre un corpo alxebricamente pechado de [151].

No Capítulo 3, tamén ofrecemos unha clasificación parcial das álxebras biconmutativas de dimensión $n \leq 4$, pero dende un enfoque diferente ao de [39]. Por unha parte, traballamos sobre o corpo base $\mathbb{C}$, non sobre $\mathbb{R}$. Por outra, clasificamos álxebras biconmutativas nilpotentes, as cales conforman unha clase máis ampla que a das álxebras biconmutativas completas con estrutura de Lie asociada nilpotente (véxase [40, Proposition 2.2]). Ademais, a clasificación de [39] depende fortemente da clasificación das álxebras de Lie nilpotentes, mentres que a nosa desenvólvese completamente dentro da variedade das álxebras biconmutativas, unha vez feita unha selección preliminar das álxebras nilpotentes sobre $\mathbb{C}$ de dimensións 2 e 3 satisfacendo as identidades de conmutatividade pola esquerda e pola dereita.

Amosamos a continuación as clasificacións obtidas.

Teorema 3.2.2. *Sexa* A *unha álxebra biconmutativa pura nilpotente de dimensión* 4 *sobre* $\mathbb{C}$*. Entón,* A *é isomorfa a unha das álxebras na seguinte Táboa 3.2:*

$\mathcal{B}^4_{01}$	$e_1e_1 = e_2$	$e_2e_1 = e_3$			
$\mathcal{B}^4_{02}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_2e_1 = \lambda e_3$		
$\mathcal{B}^4_{03}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$		
$\mathcal{B}^4_{04}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = \lambda e_4$	$e_3e_3 = e_4$	
$\mathcal{B}^4_{05}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_3e_3 = e_4$
$\mathcal{B}^4_{06}(\lambda)\ \lambda \neq 0$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_4$	$e_2e_1 = \lambda e_4$	
$\mathcal{B}^4_{07}$	$e_1e_1 = e_2$	$e_2e_1 = e_4$	$e_3e_3 = e_4$		
$\mathcal{B}^4_{08}$	$e_1e_1 = e_2$	$e_1e_3 = e_4$	$e_2e_1 = e_4$		
$\mathcal{B}^4_{09}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_3e_1 = e_4$		
$\mathcal{B}^4_{10}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_3e_2 = e_4$	
$\mathcal{B}^4_{11}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_3e_2 = e_4$		
$\mathcal{B}^4_{12}$	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_3e_2 = e_4$	
$\mathcal{B}^4_{13}$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_3e_2 = e_4$		
$\mathcal{B}^4_{14}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_2e_2 = e_4$	
$\mathcal{B}^4_{15}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$		
$\mathcal{B}^4_{16}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_4$		
$\mathcal{B}^4_{17}$	$e_1e_2 = e_3$	$e_1e_3 = e_4$			
$\mathcal{B}^4_{18}$	$e_1e_1 = e_4$	$e_1e_2 = e_3$	$e_3e_2 = e_4$		
$\mathcal{B}^4_{19}$	$e_1e_2 = e_3$	$e_3e_2 = e_4$			
$\mathcal{B}^4_{20}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$	$e_3e_1 = e_4$	
$\mathcal{B}^4_{21}$	$e_1e_1 = e_2$	$e_2e_1 = e_3$	$e_3e_1 = e_4$		
$\mathcal{B}^4_{22}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	
$\mathcal{B}^4_{23}$	$e_1e_1 = e_2$ $e_2e_1 = e_3 + e_4$	$e_1e_2 = e_3$ $e_2e_2 = e_4$	$e_1e_3 = e_4$ $e_3e_1 = e_4$		

$\mathcal{B}^4_{24}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$
	$e_2e_1 = \lambda e_3$	$e_2e_2 = \lambda e_4$	$e_3e_1 = \lambda e_4$

Táboa 3.2: Álxebras biconmutativas nilpotentes puras de dimensión 4.

Teorema 3.3.1. *Sexa* A *unha álxebra biconmutativa nilpotente de dimensión* 5 *sobre* $\mathbb{C}$*, xerada por un único elemento. Entón,* A *é isomorfa a unha das álxebras na seguinte Táboa 3.4:*

C^5_{01}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$	$e_3e_1 = e_5$
$C^5_{02}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
	$e_2e_1 = \lambda e_3 + e_4$	$e_2e_2 = \lambda e_5$	$e_3e_1 = \lambda e_5$	
$C^5_{03}(\lambda,\mu)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = \lambda e_5$
	$e_2e_1 = e_3$	$e_2e_2 = e_5$	$e_3e_1 = \mu e_5$	$e_4e_1 = e_5$
$C^5_{04}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_4 = e_5$	
	$e_2e_1 = e_3$	$e_3e_1 = \lambda e_5$		
C^5_{05}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = e_3$
	$e_2e_2 = e_5$	$e_3e_1 = e_4$	$e_4e_1 = e_5$	
C^5_{06}	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = e_3$	
	$e_3e_1 = e_4$	$e_4e_1 = e_5$		
C^5_{07}	$e_1e_1 = e_2$	$e_2e_1 = e_3$	$e_3e_1 = e_4$	$e_4e_1 = e_5$
C^5_{08}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_2e_1 = e_4$	$e_2e_2 = e_5$	$e_3e_1 = e_5$	
C^5_{09}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_4 + e_5$	$e_2e_3 = e_5$	
	$e_3e_1 = e_4 + e_5$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	
$C^5_{10}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_2e_1 = \lambda e_3$	$e_2e_2 = \lambda e_4$	$e_2e_3 = \lambda e_5$	
	$e_3e_1 = \lambda e_4$	$e_3e_2 = \lambda e_5$	$e_4e_1 = \lambda e_5$	

C^5_{11}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
	$e_1e_4 = e_5$	$e_2e_1 = e_5$		
C^5_{12}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_2e_1 = e_3 + e_5$	$e_2e_2 = e_4$	$e_2e_3 = e_5$	
	$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_4e_1 = e_5$	

Táboa 3.4: Álxebras biconmutativas nilpotentes de dimensión 5 xeradas por un elemento.

Teorema 3.4.2. *Sexa* A *unha álxebra biconmutativa nilpotente de dimensión* 6 *sobre* $\mathbb{C}$*, xerada por un único elemento. Entón,* A *é isomorfa a unha das álxebras na seguinte Táboa 3.6:*

C^6_{01}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_4 = e_5$	
	$e_2e_1 = e_3$	$e_3e_1 = e_6$		
$C^6_{02}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_6$	$e_1e_4 = e_5$
	$e_2e_1 = e_3$	$e_2e_2 = e_6$	$e_3e_1 = \lambda e_6$	$e_4e_1 = e_6$
$C^6_{03}(\lambda, \mu)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_6$	$e_1e_4 = \lambda e_5 + \mu e_6$
	$e_2e_1 = e_3$	$e_2e_2 = e_6$	$e_3e_1 = e_5$	$e_4e_1 = e_6$
C^6_{04}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_6$	$e_2e_1 = e_3$
	$e_2e_2 = e_6$	$e_3e_1 = e_4 + e_5$	$e_4e_1 = e_6$	
C^6_{05}	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_2e_1 = e_3$	$e_3e_1 = e_4$
	$e_4e_1 = e_6$			
C^6_{06}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_6$
	$e_2e_1 = e_4 + e_5$	$e_2e_2 = e_6$	$e_3e_1 = e_6$	
C^6_{07}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_6$
	$e_2e_1 = e_3 + e_4 + e_5$		$e_2e_2 = e_4 + e_6$	$e_2e_3 = e_6$
	$e_3e_1 = e_4 + e_6$	$e_3e_2 = e_6$	$e_4e_1 = e_6$	

$C_{08}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_6$
	$e_2e_1 = \lambda e_3 + e_5$	$e_2e_2 = \lambda e_4$	$e_2e_3 = \lambda e_6$	
	$e_3e_1 = \lambda e_4$	$e_3e_2 = \lambda e_6$	$e_4e_1 = \lambda e_6$	
$C_{09}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_6$	
	$e_1e_4 = \lambda e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_6$	
	$e_3e_1 = e_5$	$e_4e_1 = e_6$	$e_5e_1 = e_6$	
C_{10}^{6}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_4 = e_6$	
	$e_2e_1 = e_3$	$e_3e_1 = e_5$	$e_5e_1 = e_6$	
C_{11}^{6}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$	
	$e_3e_1 = e_5$	$e_5e_1 = e_6$		
$C_{12}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_5 = e_6$
	$e_2e_1 = \lambda e_3 + e_4$	$e_2e_2 = \lambda e_5$	$e_2e_3 = \lambda e_6$	
	$e_3e_1 = \lambda e_5$	$e_3e_2 = \lambda e_6$	$e_5e_1 = \lambda e_6$	
$C_{13}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_5 = e_6$
	$e_2e_1 = \lambda e_3 + e_4$	$e_2e_2 = \lambda e_5$	$e_2e_3 = \lambda e_6$	$e_3e_1 = \lambda e_5$
	$e_3e_2 = \lambda e_6$	$e_4e_1 = e_6$	$e_5e_1 = \lambda e_6$	
$C_{14}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	
	$e_1e_4 = e_6$	$e_1e_5 = e_6$	$e_2e_1 = e_4$	
	$e_2e_2 = e_6$	$e_3e_1 = e_6$	$e_4e_1 = \lambda e_6$	
$C_{15}^{6}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = e_6$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_5 + e_6$	$e_2e_3 = e_6$
	$e_3e_1 = e_5 + e_6$	$e_3e_2 = e_6$	$e_4e_1 = \lambda e_6$	$e_5e_1 = e_6$
$C_{16}^{6}(\lambda)_{\lambda\neq 0}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = \lambda e_5$
	$e_1e_5 = \lambda e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_5$	$e_2e_3 = e_6$
	$e_2e_4 = \lambda e_6$	$e_3e_1 = (1/\lambda)\, e_5$	$e_3e_2 = e_6$	
	$e_4e_1 = e_5$	$e_4e_2 = \lambda e_6$	$e_5e_1 = e_6$	

$C^6_{17}(\lambda)_{\lambda\neq0}$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = \lambda e_5 + e_6$
	$e_1e_5 = \lambda e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_5$	$e_2e_3 = e_6$
	$e_2e_4 = \lambda e_6$	$e_3e_1 = (1/\lambda)\, e_5$	$e_3e_2 = e_6$	
	$e_4e_1 = e_5$	$e_4e_2 = \lambda e_6$	$e_5e_1 = e_6$	
$C^6_{18}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = e_5 + \lambda e_6$
	$e_1e_5 = e_6$	$e_2e_1 = e_3$	$e_2e_2 = e_5$	$e_2e_3 = e_6$
	$e_2e_4 = e_6$	$e_3e_1 = e_5 + e_6$	$e_3e_2 = e_6$	
	$e_4e_1 = e_5$	$e_4e_2 = e_6$	$e_5e_1 = e_6$	
C^6_{19}	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_1e_4 = e_6$
	$e_2e_1 = e_3$	$e_2e_2 = e_5$	$e_2e_3 = e_6$	$e_3e_1 = e_4$
	$e_3e_2 = e_6$	$e_4e_1 = e_5$	$e_5e_1 = e_6$	
C^6_{20}	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_1e_3 = e_6$	$e_2e_1 = e_3$
	$e_2e_2 = e_6$	$e_3e_1 = e_4$	$e_4e_1 = e_5$	$e_5e_1 = e_6$
C^6_{21}	$e_1e_1 = e_2$	$e_1e_2 = e_6$	$e_2e_1 = e_3$	
	$e_3e_1 = e_4$	$e_4e_1 = e_5$	$e_5e_1 = e_6$	
C^6_{22}	$e_1e_1 = e_2$	$e_2e_1 = e_3$	$e_3e_1 = e_4$	
	$e_4e_1 = e_5$	$e_5e_1 = e_6$		
C^6_{23}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_4$	$e_2e_2 = e_5$	$e_2e_3 = e_6$
	$e_3e_1 = e_5$	$e_3e_2 = e_6$	$e_4e_1 = e_6$	
C^6_{24}	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_4 + e_5$	$e_2e_3 = e_5 + e_6$
	$e_2e_4 = e_6$	$e_3e_1 = e_4 + e_5$	$e_3e_2 = e_5 + e_6$	$e_3e_3 = e_6$
	$e_4e_1 = e_5 + e_6$	$e_4e_2 = e_6$	$e_5e_1 = e_6$	
$C^6_{25}(\lambda)$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = \lambda e_3$	$e_2e_2 = \lambda e_4$	$e_2e_3 = \lambda e_5$
	$e_2e_4 = \lambda e_6$	$e_3e_1 = \lambda e_4$	$e_3e_2 = \lambda e_5$	$e_3e_3 = \lambda e_6$
	$e_4e_1 = \lambda e_5$	$e_4e_2 = \lambda e_6$	$e_5e_1 = \lambda e_6$	

$\mathcal{C}_{26}^{6}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	
	$e_1e_4 = e_5$	$e_1e_5 = e_6$	$e_2e_1 = e_6$	
$\mathcal{C}_{27}^{6}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_6$	$e_2e_2 = e_4$	$e_2e_3 = e_5$
	$e_2e_4 = e_6$	$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_3e_3 = e_6$
	$e_4e_1 = e_5$	$e_4e_2 = e_6$	$e_5e_1 = e_6$	
$\mathcal{C}_{28}^{6}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_5$	$e_2e_2 = e_6$	$e_3e_1 = e_6$
$\mathcal{C}_{29}^{6}$	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$
	$e_1e_5 = e_6$	$e_2e_1 = e_3 + e_5$	$e_2e_2 = e_4 + e_6$	$e_2e_3 = e_5$
	$e_2e_4 = e_6$	$e_3e_1 = e_4 + e_6$	$e_3e_2 = e_5$	$e_3e_3 = e_6$
	$e_4e_1 = e_5$	$e_4e_2 = e_6$	$e_5e_1 = e_6$	

Táboa 3.6: Álxebras biconmutativas nilpotentes de dimensión 6 xeradas por un elemento.

O Capítulo 3 complétase coa clasificación xeométrica das álxebras biconmutativas nilpotentes de dimensión 4 sobre $\mathbb{C}$.

Teorema 3.5.1. *A variedade das álxebras biconmutativas nilpotentes de dimensión* 4 *sobre* $\mathbb{C}$ *ten dúas compoñentes irreducibles definidas pola álxebra ríxida* $\mathcal{B}_{10}^{4}$ *e pola familia infinita de álxebras* $\mathcal{B}_{24}^{4}(\lambda)$.

As álxebras nulfiliformes non son máis que álxebras nilpotentes xeradas por un único elemento. Non obstante, no Capítulo 4 empregaremos o termo nulfiliforme, xa que esta é a terminoloxía máis común na literatura especializada.

O estudo das extensións centrais (separables e non separables) das álxebras nulfiliformes iniciouse en [8], onde se describen todas as extensións centrais na variedade das álxebras de Leibniz das álxebras de Leibniz nulfiliformes. As extensións centrais non separables da única álxebra asociativa nulfiliforme de dimensión n, que denotaremos por μ_0^n, foron estudadas en [138] no marco das álxebras asociativas. Probouse que a única extensión central asociativa non separable de μ_0^n é μ_0^{n+1}. Porén, as álxebras asociativas nulfiliformes poden ser consideradas dentro de variedades de álxebras máis xerais, como as álxebras alternativas, alternativas pola esquerda, de Jordan, biconmutativas, biconmutativas pola esquerda, asosimétricas, simétricas pola esquerda ou de Novikov, entre outras (nótese que os casos alternativa pola dereita, conmutativa pola dereita e simétrica pola dereita son análogos ás súas contrapartidas pola esquerda). En particular, en [39] probouse que a álxebra nulfiliforme μ_0^3 admite a extensión trivial μ_0^4 pero tamén outra extensión biconmutativa non trivial. Este resultado recupérase no Capítulo 3: coas nosas notacións, $\mu_0^3 = \mathcal{B}_{02}^3(1)$ admite a extensión trivial $\mu_0^4 = \mathcal{B}_{24}^4(1)$ e tamén $\mathcal{B}_{23}^4$. Polo tanto, semella razoable preguntarse se existirán extensións non triviais nas variedades de álxebras previamente mencionadas.

O resultado principal do Capítulo 4 é a clasificación das clases de isomorfismo das extensións centrais da álxebra nulfiliforme asociativa μ_0^n nas distintas variedades de álxebras non asociativas mencionadas no parágrafo anterior. Mentres que nas variedades das álxebras alternativas, alternativas pola esquerda (e pola dereita) e de Jordan non existen extensións centrais non triviais de μ_0^n, para as álxebras biconmutativas e conmutativas pola esquerda (ou pola dereita) obtemos o seguinte resultado.

Teorema 4.5.11. *Sexa F un corpo alxebricamente pechado de característica 0, e sexa $n \geq 2$. Os seguintes elementos de $\mathrm{Z}_{\mathrm{LC}}^2(\mu_0^n, F)$ parametrizan as distintas órbitas de $\mathrm{Aut}(\mu_0^n)$ no subespazo $T_1(\mu_0^n)$ da Grassmanniana $G_1(\mathrm{H}_{\mathrm{LC}}^2(\mu_0^n, F))$, é dicir, parametrizan as distintas clases de isomorfismo das extensións centrais non separables conmutativas pola esquerda con anulador de dimensión 1 da álxebra nulfiliforme de dimensión n, μ_0^n.*

(a) $\Delta_{n,1}$;

(b) $\left\{\nabla_n + \mu\Delta_{n,1} : \mu \in F\right\}$ ($\mu = 0$ *dá a extensión trivial* μ_0^{n+1});

(c) $\left\{\nabla_n + \Delta_{i,1} : 2 \leq i \leq n-1\right\}$.

No caso biconmutativo, os representantes correspondentes son:

***(Caso** $n = 2$)*

(a) $\Delta_{2,1}$;

(b) $\left\{\nabla_2 + \mu\Delta_{2,1} : \mu \in F\right\}$ ($\mu = 0$ *dá a extensión trivial* μ_0^3).

***(Caso** $n > 2$)*

(a) ∇_n *(extensión trivial* μ_0^{n+1}*);*

(b) $\nabla_n + \Delta_{2,1}$.

Nótese que o Teorema 4.5.11 só fai referencia ás extensións centrais non separables con anulador de dimensión 1. Porén, tamén é posible construír extensións centrais de μ_0^n conmutativas pola esquerda, non separables e con anulador de dimensión 2. As súas clases de isomorfismo están parametrizadas polos cociclos $\Delta_{k,1}$ para $2 \leq k \leq n-1$. No caso biconmutativo para $n > 2$, $\Delta_{2,1}$ tamén é un representante dunha clase de isomorfismo das extensións centrais non separables con anulador de dimensión 2.

A descrición explícita da multiplicación nas extensións centrais atopadas amósase na seguinte táboa, na que só están escritos os produtos dos elementos da base $\{e_1, \ldots, e_{n+1}\}$ que non son nulos.

Cociclo	Multiplicación, $i, j \in [n]$	
$\Delta_{n,1}$	$e_i e_j = e_{i+j}$	se $i + j \leq n$
	$e_n e_1 = e_{n+1}$	
$\Delta_{k,1}$	$e_i e_j = e_{i+j}$	se $i + j \leq n$ e $(i, j) \neq (k, 1)$
$(2 \leq k \leq n-1)$	$e_k e_1 = e_{k+1} + e_{n+1}$	
$\nabla_n + \Delta_{k,1}$	$e_i e_j = e_{i+j}$	se $i + j \leq n+1$ e $(i, j) \neq (k, 1)$
$(2 \leq k \leq n-1)$	$e_k e_1 = e_{k+1} + e_{n+1}$	
$\nabla_n + \mu\Delta_{n,1}$	$e_i e_j = e_{i+j}$	se $i + j \leq n+1$ e $i \neq n$
$(\mu \in F)$	$e_n e_1 = (1+\mu) e_{n+1}$	($\mu = 0$ dá a extensión trivial)

Táboa 4.1: Clases de isomorfismo de extensións centrais non separables conmutativas pola esquerda e biconmutativas da álxebra nulfiliforme de dimensión n, μ_0^n.

As extensións centrais nas variedades das álxebras asosimétricas, simétricas pola esquerda (e pola dereita) e de Novikov resultan ser unha consecuencia trivial das extensións centrais biconmutativas e conmutativas pola esquerda (e pola dereita).

As álxebras axiais son unha clase recente de álxebras non asociativas conmutativas, que foi introducida no ano 2015 por Hall, Rehren e Shpectorov [110]. Podemos velas como unha certa xeneralización das álxebras conmutativas e asociativas, e tamén como un marco común para as álxebras de Majorana [110, 233], de Jordan [111, 112] e outros tipos de álxebras típicas da física matemática. Ademais, tamén están relacionadas coas álxebras de códigos [53].

A relevancia das álxebras de Majorana e as álxebras axiais xace no feito de que estas proporcionan unha aproximación axiomática ás álxebras de operadores de vértices (VOAs), estruturas alxébricas complicadas que xurdiron na física teórica. En matemáticas, o exemplo máis coñecido de VOA é o *moonshine* $V^{\#}$, construído por Frenkel, Lepowsky e Meurman en [89], e cuxo grupo de automorfismos é o Monstro $\mathbb{M}$, o maior grupo finito simple esporádico. Este obxecto fai patente unha conexión coa teoría das funcións modulares, e foi clave na proba de Borcherds [29] da conxectura *monstrous moonshine* sobre a relación entre o Monstro e as funcións modulares. O desenvolvemento rigoroso da teoría das VOAs, unha ferramenta moi importante para dita proba, débese tamén a Borcherds [28].

Despois do xa nomeado artigo de Hall, Rehren e Shpectorov [110], comezou un estudo sistemático das álxebras axiais. Unha dirección interesante e activa neste estudo é a descrición das álxebras dun certo tipo xeradas por n elementos. Neste sentido, todas al álxebras axiais xeradas por dous elementos dee tipo Jordan η sobre un corpo de característica distinta de 2 foron clasificadas por Hall, Rehren e Shpectorov en [111]. Rehren probou en [196, 197] que a dimensión das álxebras axiais primitivas de tipo Monstro (α, β) xeradas por dous elementos non excede 8 se a característica do corpo base é distinta de 2 e $\alpha \notin \{2\beta, 4\beta\}$. Despois disto, Franchi, Mainardis e Shpectorov construíron unha álxebra axial primitiva de tipo Monstro $\left(2, \frac{1}{2}\right)$, xerada por dous elementos e de dimensión infinita, hoxe coñecida como a álxebra *Highwater* [87], e tamén clasificaron todas as álxebras axiais primitivas de tipo Monstro $(2\beta, \beta)$ sobre un corpo de característica distinta de 2 xeradas por dous elementos en [88]. Finalmente, entre Yabe [235], Franchi e Mainardis [85] e Franchi, Mainardis e McInroy [86] clasificaron todas as álxebras axiais primitivas e simétricas de tipo Monstro xeradas por dous elementos. Respecto das álxebras xeradas por tres elementos, Gorshkov e Staroletov amosaron que as álxebras axiais primitivas de tipo Jordan teñen dimensión como moito 9; pola súa banda, Khasraw, McInroy e Shpectorov enumeraron en [153] todas as álxebras axiais xeradas por tres elementos dunha certa subclase (as chamadas 4-álxebras) das álxebras de tipo Monstro (α, β).

Citamos tamén algunhas outras direccións na investigación en álxebras axiais. Khasraw, McInroy e Shpectorov describiron a estrutura das álxebras axiais [152]. De Medts e Van Couwenberghe introduciron as representacións axiais de grupos e módulos sobre álxebras axiais como novas ferramentas para estudar as álxebras axiais [66]. Estas álxebras tamén foron estudadas dende un punto de vista computacional no artigo [173] de McInroy e Shpectorov (véxase tamén [187, 205]), e dende un punto de vista categórico por De Medts, Peacock, Shpectorov e Van Couwenberghe [65].

Por outra banda, o estudo das álxebras xeradas por idempotentes ten interese en si mesmo. Rowen e Segev describiron todas as álxebras asociativas e de Jordan xeradas por dous idempotentes en [199]; Brešar probou en [34] que unha álxebra unitaria de dimensión finita é determinada por produto cero se e só se está xerada por idempotentes; Hu e Xiao probaron en [117] que as álxebras de dimensión finita xeradas por idempotentes poden ser caracterizadas homoloxicamente polos seus módulos irreducibles, etc.

No Capítulo 5, describimos un método para construír álxebras axiais como extensións centrais A_θ doutras álxebras axiais dadas A. Para enunciar os principais resultados obtidos, precisamos introducir antes as seguintes notación e definición.

Sexa $a \in \mathrm{X}$ e $\lambda, \mu \in \operatorname{Spec}(a)$. Para $x \in \mathrm{A}_\lambda^a$ e $y \in \mathrm{A}_\mu^a$, escribimos

$$xy = \sum_{0 \neq \nu \in \lambda \star \mu} z_\nu + z_0,$$

onde $z_\nu \in \mathrm{A}_\nu^a$ e $z_0 \in \mathrm{A}_0^a$.

Definición 5.2.7. *Sexa* (A, X) *unha álxebra* $(\mathcal{F}, \star)$*-axial,* V *un espazo vectorial e* $\theta \colon \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ *unha aplicación bilinear simétrica. Dicimos que* θ *é un cociclo relativo a un subconxunto* $\mathrm{X}' \subseteq \mathrm{X}$ *se*

$$\ker L_a \subseteq \theta_a^\perp \tag{5.2.1}$$

satisfaise para todo $a \in \mathrm{X}'$*, e se para todo* $\lambda, \mu \in \mathcal{F}$ *tales que* $0 \notin \lambda \star \mu$*, entón*

$$\theta(x, y) = \sum_{\nu \in \lambda \star \mu} \nu^{-1} \theta\left(a, z_\nu\right) \tag{5.2.2}$$

satisfaise para todo $a \in \mathrm{X}'$ *tal que* $\lambda, \mu \in \operatorname{Spec}(a)$*, todo* $x \in \mathrm{A}_\lambda^a$ *e todo* $y \in \mathrm{A}_\mu^a$*. Denotaremos o espazo vectorial formado por tales aplicacións bilineares por* $\mathrm{Z}\left(\mathrm{A}, \mathrm{V}; \mathrm{X}'\right)$.

A continuación presentamos os resultados máis importantes do Capítulo 5.

Teorema 5.2.12. *Sexa* (A, X) *unha álxebra axial,* V *un espacio vectorial e* $\theta \colon \mathrm{A} \times \mathrm{A} \to \mathrm{V}$ *unha aplicación bilinear simétrica tal que* $\{[\theta_\gamma]\}_{\gamma \in \Gamma}$ *son linearmente independentes. O par* $\left(\mathrm{A}_\theta, \mathrm{X}^i\right)$ *é* $\left(\mathcal{F} \cup \{0\}, \odot^i\right)$*-axial se e só se a condición* (5.2.1) *satisfaise para todo* $a_j^i \in \mathrm{X}^i$*,* $j = 1, \ldots, r^i$*, para algunha regra de fusión* $\left(\mathcal{F} \cup \{0\}, \odot^i\right)$ *contendo a* $(\mathcal{F}, \star)$*. Ademais, podemos tomar* $\odot^i = \star$ *se e só se* $\theta \in \mathrm{Z}\left(\mathrm{A}, \mathrm{V}; \mathrm{X}^i\right)$.

Defínase tamén o conxunto

$$\mathrm{Y} = \{a + \theta\,(a,a) \,:\, a \in \mathrm{X} \textit{ satisfai a condición } (5.2.1)\}.$$

Para todo $\mathrm{Y}' \subseteq \mathrm{Y}$ *tal que exista* $i \in I$ *con* $\mathrm{X}^i \subseteq P\left(\mathrm{Y}'\right)$, $\left(\mathrm{A}_\theta, \mathrm{Y}'\right)$ *é* $(\mathcal{F} \cup \{0\}, \odot)$*-axial para algunha regra de fusión* $(\mathcal{F} \cup \{0\}, \odot)$ *contendo a* $(\mathcal{F}, \star)$*. Podemos tomar* $\odot = \star$ *se e só se* $\theta \in \mathrm{Z}\left(\mathrm{A}, \mathrm{V}; P\left(\mathrm{Y}'\right)\right)$*.*

Teorema 5.2.13. *Sexa* (B, Y) *unha álxebra* $(\mathcal{F}, \star)$*-axial con* $\mathrm{Ann}(B) \neq 0$*. Entón, existe outra álxebra* $(\mathcal{F}, \star)$*-axial* (A, X) *e un cociclo* $\theta \in \mathrm{Z}\,(\mathrm{A}, \mathrm{Ann}\,(\mathrm{B})\,; \mathrm{X})$ *tales que* $\mathrm{B} = \mathrm{A}_\theta$*. Ademais, se* Y *é un conxunto minimal de eixos xeradores de* B*, entón* X *é un conxunto minimal de eixos xeradores de* A*.*

Unha consecuencia interesante do Teorema 5.2.12 é o seguinte resultado.

Teorema 5.3.1. *Sexa* J *unha álxebra de Jordan simple e de dimensión finita sobre* $\mathbb{C}$*. Non existen extensións centrais* $\mathcal{J}\left(\frac{1}{2}\right)$*-axiais non separables* J_θ *con respecto ao conxunto*

$$\mathrm{Y} = \{a + \theta\,(a,a) \,:\, a \in \mathrm{X} \text{ é semisimple}\}.$$

Como xa comentamos uns parágrafos atrás, os produtos tensor e exterior non abelianos de álxebras de Lie foron introducidos por Ellis en [76]. Entre as súas propiedades, estudadas nese artigo, destacamos que para toda álxebra de Lie L, $H_2(L) \cong \ker(L \wedge L \to L)$. Estas construcións foron xeneralizadas a diferentes estruturas para obter caracterizacións similares da segunda homoloxía H_2.

Unha destas xeneralizacións foi na dirección dos módulos cruzados de álxebras de Lie. En primeiro lugar, definiuse un produto tensor para módulos cruzados abelianos de álxebras de Lie en [75]. Despois, Ravanbod e Salemkar [193] xeneralizaron esta construción definindo o produto tensor non abeliano de dous submódulos cruzados ideais dun módulo cruzado de álxebras de Lie (T, L, ∂) dado, así como o seu produto exterior. Tamén caracterizaron o segundo módulo cruzado de homoloxía $H_2(T, L, \partial)$ como o núcleo da aplicación conmutador $(T, L, \partial) \wedge (T, L, \partial) \to (T, L, \partial)$. Esta homoloxía para módulos cruzados de álxebras de Lie foi introducida por Casas, Inassaridze e Ladra en [49], empregando a teoría xeral da homoloxía do cotriple de Barr e Beck [20]. Dado un módulo cruzado de álxebras de Lie (T, L, ∂), os seus módulos cruzados de homoloxía $H_n(T, L, \partial)$ defínense como os funtores derivados simpliciais do funtor abelianización entre as categorías de módulos cruzados de álxebras de Lie e os módulos cruzados abelianos de álxebras de Lie. En efecto, esta teoría pode verse como unha xeneralización da homoloxía de Eilenberg-MacLane para álxebras de Lie. Ademais, os autores ofrecen unha fórmula de Hopf para a segunda homoloxía dun módulo cruzado.

Outra xeneralización do traballo de Ellis foi dada en [91], onde García-Martínez, Khmaladze e Ladra introduciron os produtos tensor e exterior non abelianos de superálxebras de Lie. Tamén definiron a súa homoloxía, obtendo unha fórmula de Hopf para a segunda homoloxía de superálxebras de Lie e estendendo a sucesión exacta de cinco termos un termo cara á esquerda. Ademais, probaron que, dada unha superálxebra de Lie L, $H_2(L) \cong \ker(L \wedge L \to L)$.

No Capítulo 6, presentamos unha nova xeneralización de [76] para módulos cruzados de superálxebras de Lie. A definición principal do capítulo é a seguinte.

Definición 6.3.4. *Sexan (M, P, ∂) e (N, Q, ∂) dous submódulos cruzados ideais graduados dun módulo cruzado de superálxebras de Lie (T, L, ∂), e consideremos os homomorfismos*

$$\begin{aligned} \alpha\colon\ M \otimes N &\to (M \otimes Q) \rtimes (P \otimes N), \\ m \otimes n &\mapsto -m \otimes \partial(n) + \partial(m) \otimes n, \end{aligned}$$

$$\begin{aligned} \beta\colon\ (M \otimes Q) \rtimes (P \otimes N) &\to P \otimes Q. \\ m \otimes q + p \otimes n &\mapsto \partial(m) \otimes q + p \otimes \partial(n). \end{aligned}$$

Definimos o produto tensor non abeliano de (M, P, ∂) e (N, Q, ∂) como

$$(M, P, \partial) \otimes (N, Q, \partial) = (\operatorname{coker} \alpha, P \otimes Q, \delta),$$

onde δ é o homomorfismo inducido por β, e o produto exterior como

$$(M, P, \partial) \wedge (N, Q, \partial) = \frac{(\operatorname{coker} \alpha, P \otimes Q, \delta)}{(I, P \square Q, \delta)} = \left(\frac{\operatorname{coker} \alpha}{I}, P \wedge Q, \delta \right),$$

onde I é a subálxebra graduada de coker α *xerada polos elementos*

$$x \otimes y + (-1)^{|x||y|} y \otimes x + \partial\left(z_{\bar{0}}\right) \otimes z_{\bar{0}} + \operatorname{Im} \alpha,$$
$$x \otimes y + (-1)^{|x||y|} y \otimes x + \partial(z) \otimes z' + (-1)^{|z||z'|} \partial\left(z'\right) \otimes z + \operatorname{Im} \alpha,$$

tales que $x, z, z' \in M \cap N$, $z_{\bar{0}} \in M_{\bar{0}} \cap N_{\bar{0}}$ *e* $y \in P \cap Q$. *Por coherencia coa teoría de álxebras e superálxebras de Lie, denotamos* $(I, P\square Q, \delta)$ *como* $(M, P, \partial) \square (N, Q, \partial)$.

A continuación, estúdanse diversas propiedades estruturais e homolóxicas dos produtos tensor e exterior non abelianos da Definición 6.3.4. Destacamos as dúas seguintes.

Teorema 6.2.7. *Sexa* (T, L, ∂) *un módulo cruzado perfecto. O morfismo*

$$\upsilon : \ (L \otimes T, L \otimes L, \mathrm{id} \otimes \partial) \to (T, L, \partial)$$

definido por $\upsilon_1 (l \otimes t) = {}^{l}t$ *e* $\upsilon_2 \left(l \otimes l'\right) = [l, l']$ *é unha extensión central universal de* (T, L, ∂).

Teorema 6.4.6. *Dada unha presentación proxectiva* $0 \to (V, R, \mu) \to (Y, F, \mu) \to (T, L, \partial) \to 0$ *do módulo cruzado* (T, L, ∂)*, existe un isomorfismo*

$$(T, L, \partial) \wedge (T, L, \partial) \cong \frac{[(Y, F, \mu), (Y, F, \mu)]}{[(V, R, \mu), (Y, F, \mu)]}.$$

En particular, $H_2 (T, L, \partial) \cong \ker ((T, L, \partial) \wedge (T, L, \partial) \to (T, L, \partial))$.

Whitehead introduciu o seu funtor cadrático Γ para grupos abelianos en [232], e empregouno para construír unha sucesión exacta longa no contexto da teoría de homotopía, proporcionando un invariante para CW-complexos de dimensión 4. Esta construción foi xeneralizada máis adiante por Simson e Tyc [208] para módulos arbitrarios sobre un anel conmutativo R en relación co estudo de funtores derivados estables. Simson e Tyc exploraron algunhas das súas propiedades básicas, e en particular probaron que este obxecto satisfai unha propiedade universal en relación ás aplicacións cadráticas entre R-módulos: a saber, que toda aplicación cadrática $b : M \to N$ factoriza a través de $\Gamma(M)$. Despois disto, Ellis relacionou esta versión do funtor cadrático de Whitehead cos seus produtos tensor e exterior non abelianos de álxebras de Lie en [76].

Outras xeneralizacións do funtor cadrático de Whitehead apareceron no contexto de módulos cruzados abelianos, tanto de grupos [189] como de álxebras de Lie [193]. A definición destes obxectos trouxo consigo avances nas teorías de homoloxía de módulos cruzados [189] e no estudo dos produtos tensor e exterior non abelianos de módulos cruzados de grupos [200] e de álxebras de Lie [193]. Outro aspecto relevante deste capítulo é a introdución do funtor cadrático de Whitehead para supermódulos, cuxa construción difire da versión para módulos de [208] e que resulta ser fundamental para os nosos propósitos.

Definición 6.1.1. *Sexa* $M = M_{\bar{0}} \oplus M_{\bar{1}}$ *un supermódulo sobre* R*, e sexa* $R^{M_{\bar{0}}}$ *o supermódulo libre, concentrado en grao* $\bar{0}$*, xerado polos elementos* $e_{m_{\bar{0}}}$ *para todo* $m_{\bar{0}} \in M_{\bar{0}}$*. Definimos o supermódulo* $\Gamma(M)$ *como a suma directa*

$$R^{M_{\bar{0}}} \oplus (M \otimes_R M),$$

suxeita ás relacións

$$\begin{aligned} &e_{\lambda m_{\bar{0}}} = \lambda^2 e_{m_{\bar{0}}}, \\ &e_{m_{\bar{0}} + m'_{\bar{0}}} - e_{m_{\bar{0}}} - e_{m'_{\bar{0}}} = m_{\bar{0}} \otimes m'_{\bar{0}}, \\ &m \otimes m' = (-1)^{|m||m'|} m' \otimes m, \end{aligned}$$

onde $\lambda \in R$*,* $m_{\bar{0}}, m'_{\bar{0}} \in M_{\bar{0}}$ *e* $m, m' \in M$*, e coa graduación inducida.*

Tamén ofrecemos a seguinte versión para módulos cruzados abelianos de superálxebras de Lie, que nos permitirá construír unha sucesión exacta de módulos cruzados de tres termos involucrando aos produtos tensor e exterior non abelianos.

Definición 6.3.11. *Sexa* (A, B, ∂) *un módulo cruzado abeliano de superálxebras de Lie restrinxidas, e denotemos por* $B \underline{\otimes} A$ *o produto tensor* $B \otimes A$ *suxeito á relación homoxénea*

$$\partial(a) \otimes a' = (-1)^{|a||a'|} \partial\left(a'\right) \otimes a,$$

para todo $a, a' \in A$*. Consideremos tamén o homomorfismo de álxebras de Lie*

$$\begin{aligned} f :\ A \otimes A &\to \left(B \underline{\otimes} A\right) \oplus \Gamma(A) \\ a \otimes a' &\mapsto \partial(a) \otimes a' - a \otimes a', \end{aligned}$$

e denotemos $\widetilde{\Gamma}(A, B, \partial) := \operatorname{coker} f$*. Entón, defínese* $\Gamma(A, B, \partial)$ *como o módulo cruzado abeliano de superálxebras de Lie* $\left(\widetilde{\Gamma}(A, B, \partial), \Gamma(B), \partial_\Gamma\right)$*, onde* ∂_Γ *está determinado por*

$$\partial_\Gamma\left(b \otimes a + e_{a_{\bar{0}}} + \alpha \otimes \alpha'\right) = e_{\partial(a_{\bar{0}})} + b \otimes \partial(a) + \partial(\alpha) \otimes \partial\left(\alpha'\right).$$

Teorema 6.3.12. *Sexa (T, L, ∂) un módulo cruzado de superálxebras de Lie tal que ∂ é sobrexectiva ou a acción de L sobre T é trivial. Entón, existe unha sucesión exacta*

$$\Gamma\left((T, L, \partial)_{\mathrm{ab}}\right) \xrightarrow{(\eta_1,\eta_2)} (T, L, \partial) \otimes (T, L, \partial) \xrightarrow{(\pi_1,\pi_2)} (T, L, \partial) \wedge (T, L, \partial) \longrightarrow 0.$$

O Capítulo 7 adícase por completo a estudar as propiedades destas dúas novas versións do funtor cadrático de Whitehead, ademais de estendelas a supermódulos e superálxebras sobre aneis nos que 2 non ten por que ter inverso. No caso de supermódulos, destacamos o seu papel como obxecto universal respecto das aplicacións cadráticas; no caso de módulos cruzados abelianos de superálxebras de Lie, definimos como aplicacións cadráticas entre módulos cruzados abelianos aquelas aplicacións respecto das cales $\Gamma(A, B, \partial)$ xoga un papel de obxecto universal.

Proposición 7.2.3. *Sexan M e N dous supermódulos, e sexa $\varphi\colon M \to N$, $\varphi = (\varphi_{\bar{0}}, b_\varphi)$, unha aplicación cadrática. Entón, existe un único homomorfismo de supermódulos tal que $h\gamma = \varphi$.*

$$\begin{tikzcd}
M_{\bar{0}} \arrow[rr, "\gamma_{\bar{0}}"] \arrow[dr, "\varphi_{\bar{0}}"'] & & \Gamma(M)_{\bar{0}} \arrow[dl, dashed, "h_{\bar{0}}"] \\
& N_{\bar{0}} &
\end{tikzcd}$$

$$\begin{tikzcd}
M \times M \arrow[rr, "b_\gamma"] \arrow[dr, "b_\varphi"'] & & \Gamma(M) \arrow[dl, dashed, "h"] \\
& N &
\end{tikzcd}$$

Definición 7.3.2. *Sexan (A, B, ∂) e (C, D, σ) dous módulos cruzados abelianos de superálxebras de Lie. Definimos unha aplicación cadrática entre eles como dous pares $\xi = \left(\xi_{\bar{0}}, b_\xi\right)$ e $\varphi = \left(\varphi_{\bar{0}}, b_\varphi\right)$ tales que:*

1. *$\varphi\colon B \to D$ é unha aplicación cadrática entre supermódulos.*

2. $\xi = \left(\xi_{\bar{0}}, b_{\xi}\right) : A \to C$ *é tal que:*

- $\xi_{\bar{0}} : A_{\bar{0}} \to C_{\bar{0}}$ *é unha aplicación cadrática entre módulos;*
- $b_{\xi} : B \times A \to C$ *é bilinear e satisfai:*

$$b_{\xi}\left(\partial\left(a_{\bar{0}}\right), a'_{\bar{0}}\right) = \xi_{\bar{0}}\left(a_{\bar{0}} + a'_{\bar{0}}\right) - \xi_{\bar{0}}\left(a_{\bar{0}}\right) - \xi_{\bar{0}}\left(a'_{\bar{0}}\right),$$
$$b_{\xi}\left(\partial(a), a'\right) = (-1)^{|a||a'|} b_{\xi}\left(\partial\left(a'\right), a\right),$$
$$b_{\xi}\left(a_{\bar{1}}, a_{\bar{1}}\right) = 0,$$

para todo $a_{\bar{0}}, a'_{\bar{0}} \in A_{\bar{0}}$, $a_{\bar{1}} \in A_{\bar{1}}$ *e* $a, a' \in A$*;*

3. $\sigma_{\bar{0}}\xi_{\bar{0}} = \varphi_{\bar{0}}\partial_{\bar{0}}$, *onde* $\partial_{\bar{0}}$ *e* $\sigma_{\bar{0}}$ *denotan as restricións de* ∂ *e* σ *a* $A_{\bar{0}}$ *e* $C_{\bar{0}}$*, respectivamente, e* $\sigma b_{\xi} = b_{\varphi}(\mathrm{id} \otimes \partial)$.

Denotamos estes pares como $\Upsilon = (\xi, \varphi) : (A, B, \partial) \to (C, D, \sigma)$.

Proposición 7.3.4. *Dada unha aplicación cadrática* $\Upsilon : (A, B, \partial) \to (C, D, \sigma)$ *entre dous módulos cruzados abelianos de superálxebras de Lie, con* $\Upsilon = (\xi, \varphi) = \left(\left(\xi_{\bar{0}}, b_{\xi}\right), \left(\varphi_{\bar{0}}, b_{\varphi}\right)\right)$*, entón existe un único morfismo de módulos cruzados* $H = \left(h_1, h_2\right)$ *tal que* $h_1\eta = \xi$ *e* $h_2\gamma = \varphi$*. Para resumir, escribiremos* $\Omega H = \Upsilon$.

$$\begin{tikzcd}
 & B_{\bar{0}} \arrow[rrr, "\gamma_{\bar{0}}"] \arrow[ddr, "\varphi_{\bar{0}}"] & & & \Gamma(B)_{\bar{0}} \arrow[ddll, dashed, "h_2"] \\
A_{\bar{0}} \arrow[ur, "\partial_{\bar{0}}"] \arrow[rrr, "\eta_{\bar{0}}"] \arrow[ddr, "\xi_{\bar{0}}"'] & & & \widetilde{\Gamma}(A, B, \partial)_{\bar{0}} \arrow[ur, "\partial_{\Gamma_{\bar{0}}}"'] \arrow[ddll, dashed, "h_1"] & \\
 & & D_{\bar{0}} & & \\
 & C_{\bar{0}} \arrow[ur, "\sigma_{\bar{0}}"] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & B \times B \arrow[rrr, "b_\gamma"] \arrow[ddr, "b_\varphi"] & & & \Gamma(B) \arrow[ddll, dashed, "h_2"] \\
B \times A \arrow[ur, "\mathrm{id} \times \partial"] \arrow[rrr, "b_\eta"] \arrow[dddr, "b_\xi"'] & & & \widetilde{\Gamma}(A, B, \partial) \arrow[ur, "\partial_\Gamma"'] \arrow[ddll, dashed, "h_1"] & \\
 & & D & & \\
 & C \arrow[ur, "\sigma"] & & &
\end{tikzcd}$$

Mentres que a nosa versión do funtor cadrático de Whitehead para supermódulos satisfai todas as propiedades típicas da versión para módulos [76, 208] que estudamos, na versión para módulos cruzados abelianos de superálxebras de Lie atopamos unha diferenza destacable, xa que, ao contrario que nos outros casos, este funtor non preserva obxectos libres.

O Capítulo 8 encádrase nunha teoría completamente distinta á do resto da tese: a das probas e descubrimentos automáticos de teoremas, baseada na xeometría alxébrica (complexa) computacional, que foi iniciada fai corenta anos por Wu no artigo fundacional [234, “On the decision problem and the mechanization of theorem-proving in elementary geometry”]. Dita teoría evolucionou ao longo dos anos desenvolvendo unha gran variedade de métodos para os razoamentos automáticos en xeometría elemental, cuxa eficacia quedou sobradamente probada pola cantidade e a calidade dos exemplos presentados en referencias como [56]. Neste capítulo traballamos co protocolo e coa notación descritos en [58, Sección 4 do Capítulo 6], os cales son bastante similares aos de [59], [194] ou [238]. A súa recente implementación en software matemático libre e amplamente empregado (véxase [1]) amosa que este tema segue tendo actualidade hoxe en día, e animámonos a fixar a nosa atención nalgúns aspectos que non estaban completamente estudados.

O obxectivo deste capítulo é esclarecer cal é o modo máis adecuado de tratar con hipóteses e teses que describan condicións negativas, tales como “consideremos dous puntos diferentes” (é dicir, dous puntos que non son iguais), ou “sexan A, B, C os vértices dun triángulo non dexenerado” (é dicir, tres puntos A, B, C que non sexan coincidentes nin colineares), etc. Nótese que a relevancia de clarificar este aspecto non se restrinxe unicamente a estender métodos xa coñecidos no ámbito das probas automáticas a un tipo máis amplo de postulados. De feito, as chamadas *condicións de non dexeneración* xorden de modo natural no protocolo tradicional para as probas de teoremas con postulados puramente afirmativos.

Ocorre que, ao introducir o requirimento de evitar situacións dexeneradas nos algoritmos estándares de xeometría alxébrica, as desigualdades polinómicas do tipo $p_1(x_1, \dots, x_n) \neq 0$ téñense que expresar por medio de ecuacións. Na tradición das probas automáticas de teoremas, esta conversión levouse a cabo a través de dous procedementos: o truco de Rabinowitsch e o ideal saturación.

O truco de Rabinowitsch é un vello coñecido das probas automáticas en xeometría [135], pois sempre foi amplamente empregado para formular negacións de igualdades, tamén coñecidas como “relacións de desigualdade”. A pesar da súa antigüidade, segue mantendo a súa validez e o seu interese; non hai máis que considerar, por exemplo, a recente investigación de Kapur, Sun, Wang e Zhou [136] sobre unha xeneralización do “truco”, cuxa descrición abstracta completa pode atoparse en [16, Exemplo 6.1]. En liñas xerais, o truco de Rabinowitsch consiste en substituír un conxunto localmente pechado $A \setminus B$ contido no espazo afín K^n por un conxunto alxébrico de K^{n+1} chamado “cuberta de Rabinowitsch”, de tal xeito que a proxección conxuntista da cuberta sexa exactamente $A \setminus B$ e polo tanto a súa clausura (na topoloxía de Zariski) poida ser calculada mediante unha eliminación.

Doutra banda, o ideal saturación permítenos calcular $\overline{A \setminus B}$ dun xeito alxébrico directo, sen necesidade de substituír $A \setminus B$ por un conxunto alxébrico nun espazo afín de maior dimensión para logo proxectalo de volta a K^n. A súa relación co truco de Rabinowitsch é ben coñecida no campo da álxebra conmutativa, e o potencial impacto do seu uso nas probas de teoremas no canto de dito truco xa foi estudado en [59, Sección 5]. Porén, non atopamos na literatura existente ningunha análise sólida e detallada das vantaxes e inconvenientes de cada unha destas aproximacións no que respecta á súa fidelidade como tradución de condicións negativas.

Así, a principal contribución deste capítulo final é estudar en detalle as diferentes implicacións de adoptar cada unha destas formulacións para describir hipóteses e teses negativas. A seguinte proposición deixa claro que o truco de Rabinowitsch non é adecuado para modelar teses negativas.

Proposición 8.3.2. *A introdución dunha tese negativa* $T_1 := \{p \neq 0\}$ *empregando o truco de Rabinowitsch nunca dá lugar a unha afirmación* $H \Rightarrow T_1$ *xeralmente certa.*

Porén, a introdución de teses negativas por medio da saturación ofrece unha casuística completa de situacións, como amosa o seguinte exemplo.

Exemplo 8.3.9. *Se consideramos a hipótese* $H := \{(y-1)\cdot(y-2) = 0\}$ *nas variables* $\{x, y\}$*, sendo x a única variable independente, e as teses* $T_2 := \{y-1=0\}$ *e* $T_1 := \mathtt{Sat}(((y-1)\cdot(y-2)),(y-1))=\{y-2=0\}$*, é sinxelo comprobar que tanto* $H \Rightarrow T_2$ *como* $H \Rightarrow T_1$ *non son nin xeralmente certos nin xeralmente falsos. Se no canto das teses anteriores tomamos* $T_2 := \{y-3=0\}$ *e a correspondente* $T_1 := \mathtt{Sat}(((y-1)\cdot(y-2)),(y-3))=\{(y-1)\cdot(y-2)=0\}$*, obtemos que* $H \Rightarrow T_2$ *non é xeralmente certo pero si xeralmente falso, mentres que* $H \Rightarrow T_1$ *é xeralmente certo e non xeralmente falso. Por último, consideremos* $H := \{(y-1)=0\}$ *nas variables* $\{x, y\}$*,* $T_2 := \{y-1=0\}$ *e* $T_1 := \mathtt{Sat}((y-1),(y-1)) = (1)$*. Neste caso, tense que* $H \Rightarrow T_2$ *é xeralmente certo e non xeralmente falso, e* $H \Rightarrow T_1$ *é xeralmente falso pero non xeralmente certo.*

En cambio, na introdución de hipóteses negativas non semella haber diferenzas teóricas entre empregar o truco de Rabinowitsch ou a saturación. Denotemos por

$$H_1 := H^e + (f\cdot t - 1),$$
$$H_2 := \mathtt{Sat}(H, f).$$

os ideais de hipóteses agrandados correspondentes ás dúas posibilidades de introducir condicións de non dexeneración do tipo $f \neq 0$ sobre o ideal orixinal de hipóteses H dun postulado dado.

Teorema 8.4.7. *O teorema* $H_1 \Rightarrow T$ *é xeralmente falso se e só se o teorema* $H_2 \Rightarrow T$ *é xeralmente falso; analogamente, o postulado* $H_1 \Rightarrow T$ *é xeralmente certo se e só se tamén o é* $H_2 \Rightarrow T$.

Porén, as nosas experiencias traballando con exemplos concretos amosan que hai ocasións nas que o ordenador non experimenta problemas para determinar se un enunciado é xeralmente certo ou falso empregando a saturación, pero é incapaz de determinalo se as condicións negativas foron introducidas a través do truco de Rabinowitsch.

Chegados a este punto, poderíamos preguntarnos: cal dos dous métodos presentados é mellor? A resposta non é totalmente obxectiva. A nosa suxestión é que se implemente o método da saturación no software existente na materia, debido á escasa efectividade do truco de Rabinowitsch ao tratar con teses negativas e ás obxeccións prácticas que acabamos de comentar. Pero a saturación tamén ten desvantaxes: ao considerar a clausura do ideal $\mathtt{Sat}(H, f)$, arriscámonos a perder información esencial acerca da negación $\neg\{f = 0\}$ e incluso a atopar a ecuación $f = 0$ entre as hipóteses adicionais para o descubrimento, transgredindo en certo sentido as restricións impostas polas condicións de non dexeneración introducidas ao comezo da proba. Este feito é precisamente o que diferencia a ambos métodos, e o que podería persuadirnos para empregar o truco de Rabinowitsch nas ocasións en que desexemos permanecer fieis a algunha condición de non dexeneración establecida de antemán. De ser o caso, na nosa opinión, a decisión de que método utilizar debería ser tomada polo propio usuario a través do correspondente diálogo co software de probas automáticas de teoremas que estea empregando.

Publications

The contents of this thesis appear in the following publications. We would like to acknowledge the corresponding publishers for allowing the reproduction of the results obtained.

- **Title:** Restricted Lie algebras having a distributive lattice of restricted subalgebras.

  Year: 2019.

  Author 1: Nicola Maletesta, Università del Salento.

  Author 2: Pilar Páez-Guillán, Universidade de Santiago de Compostela.

  Author 3: Salvatore Siciliano, Università del Salento.

  Reference: Linear and Multilinear Algebra.

  DOI: 10.1080/03081087.2019.1708238.

  Editorial: Taylor and Francis Ltd. ISSN: 03081087.

  Quality indexes: Impact factor 1,736. Quartile Q1.

  Journal authorisation: see Annexe A.

  Corresponding chapter: Chapter 1.

  Contributions of the Ph.D. candidate: essential. The candidate contributed to the design of the research and proofs, to the analysis of the results and to the writing of the manuscript. In particular, she contributed to the obtention of the results on the relationship between the properties of a restricted Lie algebra and the distributivity and Booleanity of its lattice of restricted subalgebras. She also reviewed the final manuscript.

- **Title:** The algebraic and geometric classification of nilpotent bicommutative algebras.

  Year: 2020.

  Author 1: Ivan Kaygorodov, Universidade da Beira Interior.

  Author 2: Pilar Páez-Guillán, Universidade de Santiago de Compostela.

  Author 3: Vasily Voronin, Novosibirsk State University.

  Reference: Algebras and Representation Theory, Vol. 23, No. 6, 2331–2347.

  DOI: 10.1007/s10468-019-09944-x.

  Editorial: Springer. Electronic ISSN: 1572-9079. Print ISSN: 1386-923X.

  Quality indexes: Impact factor: 0,689. Quartile Q3.

  Journal authorisation: see Annexe B.

  Corresponding chapter: Chapter 3.

  Contributions of the Ph.D. candidate: essential. The candidate contributed to the design of the research and proofs, to the analysis of the results and to the writing of the manuscript. In particular, she contributed to the obtention of the results on the algebraic and geometric classifications of nilpotent bicommutative algebras. She also reviewed the final manuscript.

- **Title:** One-generated nilpotent bicommutative algebras.

  Year: 2021.

  Author 1: Ivan Kaygorodov, Universidade da Beira Interior.

  Author 2: Pilar Páez-Guillán, Universidade de Santiago de Compostela.

  Author 3: Vasily Voronin, Novosibirsk State University.

  Reference: accepted for publication in Algebra Colloquium. Not DOI yet.

  Editorial: World Scientific. Electronic ISSN: 0219-1733. Print ISSN: 1005-3867.

  Quality indexes: Impact factor: 0,429. Quartile Q4.

  Journal authorisation: see Annexe C.

Corresponding chapter: Chapter 3.

Contributions of the Ph.D. candidate: essential. The candidate contributed to the design of the research and proofs, to the analysis of the results and to the writing of the manuscript. In particular, she contributed to the obtention of the results on the algebraic classifications of one-generated nilpotent bicommutative algebras. She also reviewed the final manuscript.

- **Title:** Non-associative central extensions of null-filiform associative algebras.

  Year: 2020.

  Author 1: Ivan Kaygorodov, Universidade da Beira Interior.

  Author 2: Samuel Lopes, Universidade do Porto.

  Author 3: Pilar Páez-Guillán, Universidade de Santiago de Compostela.

  Reference: Journal of Algebra, Vol. 560, 1190–1210.

  DOI: 10.1016/j.jalgebra.2020.06.013.

  Editorial: Elsevier. ISSN: 0021-8693.

  Quality indexes: Impact factor: 0,89. Quartile Q3.

  Journal authorisation: see Annexe D.

  Corresponding chapter: Chapter 4.

  Contributions of the Ph.D. candidate: essential. The candidate contributed to the design of the research and proofs, to the analysis of the results and to the writing of the manuscript. In particular, she contributed to the obtention of the results on the classifications of central extensions of null-filiform associative algebras. She also reviewed the final manuscript.

- **Title:** The non-abelian tensor and exterior products of crossed modules of Lie superalgebras.

  Year: 2021.

  Author 1: Tahereh Fakhr Taha, Shahid Beheshti University.

  Author 2: Manuel Ladra, Universidade de Santiago de Compostela.

Author 3: Pilar Páez-Guillán, Universidade de Santiago de Compostela.

Reference: Journal of Algebra and its Applications.

DOI: 10.1142/S0219498822501699.

Editorial: World Scientific. Electronic ISSN: 1793-6829. Print ISSN: 0219-4988.

Quality indexes: Impact factor: 0,736. Quartile Q3.

Journal authorisation: see Annexe C.

Corresponding chapter: Chapter 6.

Contributions of the Ph.D. candidate: essential. The candidate contributed to the design of the research and proofs, to the analysis of the results and to the writing of the manuscript. In particular, she contributed to the obtention of the results on the algebraic and homological properties of the non-abelian tensor and exterior products of graded ideal crossed submodules of crossed modules of Lie superalgebras. She also reviewed the final manuscript.

- **Title:** Dealing with negative conditions in automated proving: tools and challenges. The unexpected consequences of Rabinowitsch's trick.

  Year: 2020.

  Author 1: Manuel Ladra, Universidade de Santiago de Compostela.

  Author 2: Pilar Páez-Guillán, Universidade de Santiago de Compostela.

  Author 3: Tomás Recio, Universidad de Cantabria.

  Reference: Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, Vol. 114, No. 4, Paper 162, 16 pp.

  DOI: 10.1007/s13398-020-00874-8.

  Editorial: Springer. Electronic ISSN: 1579-1505. Print ISSN: 1578-7303.

  Quality indexes: Impact factor: 2,169. Quartile Q1.

  Journal authorisation: see Annexe B.

  Corresponding chapter: Chapter 8.

  Contributions of the Ph.D. candidate: essential. The candidate contributed to the design of the research and proofs, to the analysis of the results and to the writing of the manuscript. In particular, she contributed to the obtention of the results on the differences and similarities of Rabinowitsch's trick and saturation for including negative conditions in automated theorem proving. She also reviewed the final manuscript.

Also, we include information about the authorship of two articles in preparation.

- **Title:** On the subalgebra lattice of a restricted Lie algebra.

  Author 1: Pilar Páez-Guillán, Universidade de Santiago de Compostela.

  Author 2: Salvatore Siciliano, Università del Salento.

  Author 3: David A. Towers, University of Lancaster.

  Corresponding chapter: Chapter 1.

  Contributions of the Ph.D. candidate: essential. The candidate contributed to the design of the research and proofs, to the analysis of the results and to the writing of the manuscript. In particular, she contributed to the obtention of the results on the relationship between the properties of a restricted Lie algebra and those of its lattice of restricted subalgebras (other than distributivity and Booleanity). She also reviewed the current manuscript.

- **Title:** On central extensions of axial algebras.

  Author 1: Ivan Kaygorodov, Universidade da Beira Interior.

  Author 2: Cándido Martín González, Universidad de Málaga.

  Author 3: Pilar Páez-Guillán, Universidade de Santiago de Compostela.

  Corresponding chapter: Chapter 5.

  Contributions of the Ph.D. candidate: essential. The candidate contributed to the design of the research and proofs, to the analysis of the results and to the writing of the manuscript. In particular, she contributed to the obtention of the results on the study of central extensions of axial algebras. She also reviewed the current manuscript.

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