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Solution of a fractional logistic ordinary differential equation

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ABSTRACT

We solve the logistic differential equation of fractional order and non-singular kernel. The analytical solution is obtained.

1. Introduction

The logistic ordinary differential equation

$$x'(t) = x(t) \cdot [1 - x(t)] \tag{1}$$

appears in many contexts. The solution, for a given initial condition $x(0) = x_0$, is

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)e^{-t}}. \tag{2}$$

For $x_0 = 1/2$, one obtains the logistic function $x(t) = \frac{1}{1+e^{-t}}$.

The logistic equation and the logistic function have many applications in different fields. In biology [1,2], medicine [3], economy [4], data security in optical networks [5] and even to study the evolution of the COVID-19 epidemic [6–8].

Different versions and generalizations of the logistic equation (1) have been considered and, in particular, the fractional versions of the logistic differential equation [9–13].

The following fractional version of the logistic differential equation has been considered

$$D^\alpha x(t) = x(t) \cdot [1 - x(t)] \tag{3}$$

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with $\alpha \in (0,1)$ and D^α the Caputo fractional derivative:

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s)\, ds.$$

Although an analytical expression for the solutions of (3) is not known, it has been solved using different techniques such Euler's numbers [10,14] or power series [9].

In this note we study another fractional version of the logistic differential equation:

$$\mathcal{D}^\alpha x(t) = x(t) \cdot [1 - x(t)] \tag{4}$$

where $\mathcal{D}$ is the fractional derivative in the Caputo–Fabrizio sense.

This paper is organized as follows. In Section 2, we recall some concepts and properties of the Caputo–Fabrizio fractional derivative and its corresponding integral. Then, we solve the Caputo–Fabrizio fractional logistic differential equation of order $\alpha \in (0,1)$ by giving an implicit representation which coincides with the expression (1) when $\alpha \to 1^-$. We use Mathematica [15], version 12.3. We also compare the solution with the corresponding solution of the Caputo fractional logistic differential equation.

2. Fractional calculus with non-singular kernel

Let $\alpha \in (0,1)$. Recall that the classical Riemann–Liouville fractional integral is given by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds$$

we have [16]

$$D^\alpha I^\alpha x(t) = x(t)$$

and

$$I^\alpha D^\alpha x(t) = x(t) + c$$

where c is an arbitrary constant. This is the same situation as for the first order derivative and the usual primitive (the integral of order one).

For a real smooth function f, the Caputo–Fabrizio fractional derivative [17] is given by

$$\mathcal{D}^\alpha f(t) = \frac{1}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s)\, ds. \tag{5}$$

It is analogous to the Caputo fractional derivative in the fact that one replaces the constant $\frac{1}{\Gamma(1-\alpha)}$ by $\frac{1}{1-\alpha}$ and the singular kernel

$$k(t,s) = (t-s)^{-\alpha}$$

by another kernel

$$\mathcal{K}(t,s) = e^{-\frac{\alpha}{1-\alpha}(t-s)}.$$

For some real applications of this operator we refer the reader to the recent articles [18,19]. Thus, the corresponding fractional integral [20] of a function g is

$$\mathcal{I}^\alpha g(t) = (1-\alpha)[g(t) - g(0)] + \alpha \int_0^t g(s)\, ds. \tag{6}$$

We have that

$$\mathcal{I}^\alpha \mathcal{D}^\alpha f(t) = f(t) + c, \tag{7}$$

c an arbitrary constant. However, as remarked in [19],

$$\mathcal{D}^\alpha \mathcal{I}^\alpha f(t) = f(t) - f(0) e^{-\frac{\alpha}{1-\alpha} t}.$$

This is quite different to the first order integral and derivative, the Caputo fractional derivative and the Riemann–Liouville fractional derivative and integral.

3. Solution of the fractional logistic differential equation

If x is a solution of (4), integrating we get

$$\mathcal{I}^\alpha \mathcal{D}^\alpha x(t) = \mathcal{I}^\alpha X(t)$$

where

$$X(t) = x(t) \cdot [1 - x(t)].$$

Therefore, using (7)

$$x(t) - x_0 = (1-\alpha)[x(t)(1-x(t)) - x_0(1-x_0)] + \alpha \int_0^t x(s)[1-x(s)]\, ds.$$

where $x_0 = x(0)$ is the initial condition. Taking the first derivative,

$$x'(t) = (1-\alpha)[x'(t)(1-x(t) - x(t)x'(t)] + \alpha x(t)[1-x(t)]$$

or

$$\alpha x'(t) + 2(1-\alpha)x(t)x'(t) = \alpha x(t)[1-x(t)]. \tag{8}$$

Observe that for $\alpha = 1$ we recover the classical logistic ordinary differential equation (1). As in that case, the constants 0 and 1 are solutions.

For $\alpha \in (0,1)$, we write the previous equation (8) as follows

$$\alpha x' - 2\alpha x x' + 2xx' = \alpha x(1-x).$$

For $x \neq 0, 1$,

$$\frac{x' - 2xx'}{x - x^2} + \frac{2}{\alpha} \cdot \frac{x'}{1-x} = 1.$$

Then,

$$\frac{d}{dt} \ln|x - x^2| - \frac{2}{\alpha} \ln|1-x| = 1,$$

and integrating

$$\ln|x - x^2| - \ln\left(|1-x|^{2/\alpha}\right) = t + c.$$

Therefore,

$$\frac{x - x^2}{(1-x)^{2/\alpha}} = e^c \cdot e^t, e^c = \frac{x_0 - x_0^2}{(1-x_0)^{2/\alpha}}.$$

Hence, we have obtained the solution in an implicit form

$$\frac{x(t) - x^2(t)}{(1-x(t))^{2/\alpha}} = \frac{x_0 - x_0^2}{(1-x_0)^{2/\alpha}} \cdot e^t. \tag{9}$$

We now plot the solutions of (4) for different values of α. We compare, for the initial value condition $x_0 = 1/2$, the classical ($\alpha = 1$) logistic differential equation (1), the Caputo–Fabrizio fractional differential equation (4) for $\alpha = 1/2$ and $\alpha = 0.1$, and the Caputo fractional logistic equation (3) with $\alpha = 1/2$ using the power series approximation developed in [9]. See Fig. 1. The solution for the ordinary logistic differential equation is above the solution of the Caputo–Fabrizio fractional logistic differential equation, and, above all, the solution of the Caputo fractional logistic differential equation.

4. Conclusions

We have solved the fractional logistic differential equation with a non-singular kernel and give an implicit solution. A figure to illustrate the result is given. Comparison with the classical logistic equation and the Caputo fractional logistic differential equations is plotted.

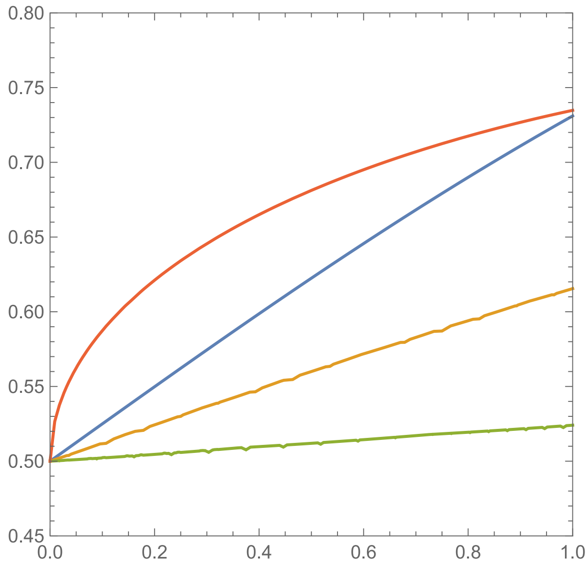

Fig. 1. Solution of the logistic differential equations for the initial condition $x_0 = 1/2$. Classical logistic equation (blue), fractional Caputo–Fabrizio logistic equation with $\alpha = 1/2$ (orange) and $\alpha = 0.1$ (green) and Caputo fractional ($\alpha = 1/2$) logistic differential equation (in red) and greater than the other solutions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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