J. Math. Anal. Appl. 511 (2022) 126010

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On first and second order linear Stieltjes differential equations

Francisco J. Fernández *, Ignacio Marquéz Albés, F. Adrián F. Tojo

CITMAga, 15782, Santiago de Compostela, Spain
Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, 15782, Facultade de Matemáticas, Campus Vida, Santiago, Spain

ARTICLE INFO

Article history:
Received 21 September 2021
Available online 13 January 2022
Submitted by S. Hencl

Keywords:
Stieltjes derivative
Second order
Uniqueness
Existence
Green's function

ABSTRACT

This work deals with the obtaining of solutions of first and second order Stieltjes differential equations. We define the notion of Stieltjes derivative on the whole domain of the functions involved, provide a notion of n-times continuously Stieltjes-differentiable functions and prove existence and uniqueness results of Stieltjes differential equations in the space of such functions. We also present the Green's functions associated to the different problems and an application to the Stieltjes harmonic oscillator.

1. Introduction

There has been a recent surge in the study of Stieltjes differential equations focused on obtaining applicable results comparable to those available for classical derivatives [2–14,16,17,19]. These works center their attention in the procuring of solutions of first order differential equations and systems. The theory developed starts with the obtaining of simple solutions, like the solution of the first order linear problem [3,4], which is identified with the exponential, in order to, later, prove existence and uniqueness results in more general settings [7,13,16]. Some of these works also provide interesting practical applications [5,9] and others generalize the framework in several ways, such as allowing for sign changing derivators [4], considering several different derivators [16] or generalizing the concept of Stieltjes derivative [15].

In any case, all of the aforementioned works restrict themselves to the first order case. The reason behind this is that, in order to study higher order problems, the notion of higher order Stieltjes derivative has to be correctly defined, which is not obvious. In fact, the first difficulty lies on the mere definition of the Stieltjes derivative, which, to the best of our knowledge, is nowhere defined in the literature on the whole domain of definition of the function, something which impedes taking a second derivative.

* Corresponding author.
E-mail addresses: fjavier.fernandez@usc.es (F.J. Fernández), ignacio.marquez@usc.es (I. Marquéz Albés), fernandoadrian.fernandez@usc.es (F.A.F. Tojo).

https://doi.org/10.1016/j.jmaa.2022.126010

In this work we provide this definition, which enables us to study second order problems. First, we consider the Stieltjes derivative in the whole of the domain of the function, which allows us to talk about the space of continuously Stieltjes-differentiable functions in the same way we speak of the space of continuously differentiable functions with the usual derivative. We can then explore the first order problem in this space, obtaining existence and uniqueness results that mirror those of the previous works. In fact, we profit from the opportunity of revisiting the solution of the first order linear problem to provide a constructive way of obtaining its solution. All of these steps are also taken with a further generalization: our functions are allowed to take real or complex values. Furthermore, we obtain the explicit expression of the Green's function of the first order linear problem with initial conditions and we construct the Stieltjes versions of the sine and cosine functions using the complex version of the Stieltjes exponential.

Once we have studied the first order problem with various degrees of regularity (something the subsequent spaces of n-times continuously Stieltjes-differentiable functions allow), we move on to study second order problems. First, we present existence and uniqueness results for the homogeneous second order problem with constant coefficients and then we study the non homogeneous case with varying degrees of regularity. Here we also obtain the explicit expression of the Green's function of the second order linear problem with initial conditions. All this work is then illustrated with an application to the Stieltjes harmonic oscillator for which we also analyze the resonance effect. Finally, in order to validate the explicit solutions obtained, we compare them with the numerical approximation of the corresponding first order linear system using the numerical scheme introduced in [2].

The structure of this work is as follows: In Section 2 we present some preliminary concepts and we prove several results related to Lebesgue-Stieltjes integral. In Section 3 we introduce the space of bounded Stieltjes differentiable functions and analyze some of its properties. We study the first order linear Stieltjes differential equation in Section 4, including in the complex case. In this section we also define the complex Stieltjes exponential and the Stieltjes version of the sine and cosine functions. In Section 5 we study the homogeneous Stieltjes second order problem with constant coefficients, the non homogeneous case and we also obtain an explicit solution for both situations. Finally, in Section 6 we present an application to the Stieltjes harmonic oscillator. We obtain the explicit solution of the overdamped, critically damped and underdamped cases, and provide an example in which the resonance effect appears. In order to validate the explicit solution obtained, we compare it with the numerical solution of the corresponding first order linear system.

2. Preliminaries

Let $[a, b] \subset \mathbb{R}$ be an interval, $\mathbb{F}$ the field $\mathbb{R}$ or $\mathbb{C}$ and $g : \mathbb{R} \to \mathbb{R}$ a left-continuous non-decreasing function. We will refer to such functions as *derivators*. For these functions, we define the set $D_g = \{d_n\}_{n\in\Lambda}$ (where $\Lambda \subset \mathbb{N}$) as the set of all discontinuity points of g, namely, $D_g = \{t \in \mathbb{R} : \Delta^+ g(t) > 0\}$ where $\Delta^+ g(t) := g(t^+) - g(t)$, $t \in \mathbb{R}$, and $g(t^+)$ denotes the right hand side limit of g at t. We also define

$$C_g := \{t \in \mathbb{R} : g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\}.$$

Observe that C_g is open in the usual topology of $\mathbb{R}$, so we can write

$$C_g = \bigcup_{n\in\widetilde{\Lambda}} \{(a_n, b_n)\} \tag{2.1}$$

where $\widetilde{\Lambda} \subset \mathbb{N}$ and $(a_k, b_k) \cap (a_j, b_j) = \emptyset$ for $k \neq j$. With this notation, we denote $N_g^- := \{a_n\}_{n\in\widetilde{\Lambda}} \backslash D_g$, $N_g^+ := \{b_n\}_{n\in\widetilde{\Lambda}} \backslash D_g$ and $N_g := N_g^- \cup N_g^+$.

Remark 2.1. For the aims of this paper, we will assume without loss of generality that $g(a) = 0$. Furthermore, we will also assume that g is continuous at $x = a$. As pointed out in [3, p. 21] and [12, Proposition 4.28], the continuity assumption has no impact in the study of differential equations, which is our final goal. Finally, in order to properly define the Stieltjes derivative in the whole $[a, b]$, we will also ask that $[a, b] \setminus C_g \neq \emptyset$.

We define $g^B : \mathbb{R} \to \mathbb{R}$ as:

$$g^B(t) = \begin{cases} \displaystyle\sum_{s \in [a,t) \cap D_g} \Delta^+ g(s), & t > a, \\ \displaystyle -\sum_{s \in [t,a) \cap D_g} \Delta^+ g(s), & t \leq a. \end{cases}$$

It is clear that g^B is a left-continuous and non-decreasing function. Moreover, the map $g^C : \mathbb{R} \to \mathbb{R}$ given by

$$g^C(t) := g(t) - g^B(t),$$

is also non-decreasing and continuous. We say g^C that is the *continuous part* of g and g^B is the *jump part* of g. Observe that both g^B and g^C are continuous at $x = a$ and $g^C(a) = g^B(a) = 0$.

Throughout this work we consider the Lebesgue–Stieltjes measure space $(\mathbb{R}, \mathcal{M}_g, \mu_g)$, where $\mathcal{M}_g$ and μ_g are the σ-algebra and measure constructed in an analogous fashion to the classical Lebesgue measure, where the length of $[c, d)$ is given by $\mu_g([c, d)) = g(d) - g(c)$. The interested reader may refer to [11] for details concerning this measure space. We must emphasize that, in the case of considering $g(t) = t$, we recover the classic Lebesgue measure space that we will denote by $(\mathbb{R}, \mathcal{L}, \mu) \equiv (\mathbb{R}, \mathcal{M}_{\mathrm{Id}}, \mu_{\mathrm{Id}})$ where Id is the identity function. Furthermore, we can define the measure space associated with the continuous part, $(\mathbb{R}, \mathcal{M}_{g^C}, \mu_{g^C})$, the jump part, $(\mathbb{R}, \mathcal{M}_{g^B}, \mu_{g^B})$, and the one associated with the derivator itself, $(\mathbb{R}, \mathcal{M}_g, \mu_g)$. If we denote by $\mathcal{B}(\tau_u)$ the Borel σ-algebra associated to τ_u, the usual topology of $\mathbb{R}$, we have that $\mathcal{B}(\tau_u) \subset \mathcal{M}_g$ and also $\mathcal{B}(\tau_u) \subset \mathcal{M}_{g^M}$, with $M = C, B$. We must mention that if $E \subset \mathbb{R}$, $\mu^*_{g^M}(E) \leq \mu^*_g(E)$, for $M = C, B$, being $\mu^*_{g^M}$ and μ^*_g the outer measures associated to g^M and g respectively, $M = C, B$. We also have that if $E \subset \mathbb{R}$ is a bounded set, then $\mu^*_g(E) < \infty$.

We have the following lemma that, in particular, provides us with a relationship between the σ-algebras $\mathcal{M}_g$, $\mathcal{M}_{g^C}$ and $\mathcal{M}_{g^B}$.

Lemma 2.2. *The following properties hold for the maps g, g^C and g^B:*

1. *Given an element $E \in \mathcal{M}_g$ there exists $H \in G_\delta$ (that is, H is a countable intersection of open sets) and $N \in \mathcal{M}_g$ such that $E \subset H$, $N \subset H$, $\mu_g(N) = 0$ and $E = H \setminus N$.*
2. *Given an element $E \in \mathcal{M}_g$ there exists $F \in F_\sigma$ (that is, F is a countable union of closed sets) and $N \in \mathcal{M}_g$ such that $\mu_g(N) = 0$, $F \cap N = \emptyset$ and $E = F \cup N$.*
3. *$\mathcal{M}_g \subset \mathcal{M}_{g^C}$.*
4. *$\mathcal{M}_{g^B} = \mathcal{P}(\mathbb{R})$.*

Proof. Since $\mathcal{B}(\tau_u) \subset \mathcal{M}_g$ and g is left-continuous, we have that

$$\mu_g(E) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu_g([a_n, b_n)) : \; E \subset \bigcup_{n \in \mathbb{N}} [a_n, b_n) \right\} = \inf \left\{ \sum_{n \in \mathbb{N}} \mu_g((a_n, b_n)) : \; E \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n) \right\}.$$

Indeed on the one hand given $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ such that $E \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n)$, we have that

$$\mu_g^*(E) \le \mu_g^*\left(\bigcup_{n\in\mathbb{N}}(a_n,b_n)\right) \le \sum_{n\in\mathbb{N}} \mu_g^*((a_n,b_n)),$$

therefore, $\mu_g(E) \le \inf\left\{\sum_{n\in\mathbb{N}} \mu_g((a_n,b_n)) : E \subset \bigcup_{n\in\mathbb{N}}(a_n,b_n)\right\}$. On the other hand, given $\varepsilon > 0$ and $\{[a_n,b_n)\}_{n\in\mathbb{N}}$ such that $E \subset \bigcup_{n\in\mathbb{N}}[a_n,b_n)$, we have, thanks to the left-continuity of g, that there exists $\{(\tilde{a}_n,b_n)\}_{n\in\mathbb{N}}$ such that $[a_n,b_n) \subset (\tilde{a}_n,b_n)$ and $\mu_g^*((\tilde{a}_n,b_n)) \le \mu_g^*([a_n,b_n)) + \varepsilon/2^n$, $n \in \mathbb{N}$. Thus, $\sum_{n\in\mathbb{N}} \mu_g^*(\tilde{a}_n,b_n) \le \sum_{n\in\mathbb{N}} \mu_g^*([a_n,b_n)) + \varepsilon$ and we conclude, taking the infimum in both sides of inequality, that $\inf\left\{\sum_{n\in\mathbb{N}} \mu_g((a_n,b_n)) : E \subset \bigcup_{n\in\mathbb{N}}(a_n,b_n)\right\} \le \mu_g(E)$. Now, we can proceed as in [1, Corollaries 15.5 and 15.8] to obtain 1 and 2, respectively.

Now, for 3, given an element $E \in \mathcal{M}_g$, there exists $F \in F_\sigma$ and $N \in \mathcal{M}_g$ such that $\mu_g(N) = 0$, $F \cap N = \emptyset$ and $E = F \cup N$. Now, $F \in \mathcal{B}(\tau_u) \subset \mathcal{M}_c$ and $\mu_{g^C}^*(N) \le \mu_g^*(N) = 0$ so we have that $N \in \mathcal{M}_{g^C}$ since $(\mathbb{R}, \mathcal{M}_{g^C}, \mu_{g^C})$ is a complete measure space. Therefore, $E \subset \mathcal{M}_{g^C}$.

Finally, for $E \in \mathcal{P}(\mathbb{R})$, we have that $E = (E \setminus D_{g^B}) \cup (E \cap D_{g^B})$. Now, $(E \setminus D_{g^B}) \subset C_{g^B}$ and then $\mu_{g^B}^*(E \setminus D_{g^B}) = 0$, so $E \setminus D_{g^B} \in \mathcal{M}_{g^B}$. Finally $E \cap D_{g^B} \in \mathcal{B}(\tau_u) \subset \mathcal{M}_{g^B}$. Therefore $E \in \mathcal{M}_{g^B}$, which finishes the proof of 4. □

We denote by $\mathcal{L}_g^1([a,b);\mathbb{F})$ the set of functions $f : [a,b) \to \mathbb{F}$ such that their real and imaginary parts, that is, $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ respectively, are measurable and $\int_{[a,b)} |f| \, \mathrm{d}\mu_g < \infty$. For this class of functions we define

$$\int_{[a,b)} f \, \mathrm{d}\mu_g = \int_{[a,b)} \operatorname{Re}(f) \, \mathrm{d}\mu_g + i \int_{[a,b)} \operatorname{Im}(f) \, \mathrm{d}\mu_g.$$

Lemma 2.3. *Given a function $f \in \mathcal{L}_g^1([a,b);\mathbb{F})$,*

$$\int_{[a,t)} f \, \mathrm{d}\mu_g = \int_{[a,t)} f \, \mathrm{d}\mu_{g^C} + \sum_{s\in[a,t)\cap D_g} f(s)\Delta^+ g(s), \ \forall t \in [a,b].$$

Proof. Given a function $f \in \mathcal{L}_g^1([a,b);\mathbb{F})$, thanks to Lemma 2.2 and the fact that $\mu_{g^C}(E) \le \mu_g(E)$ for all $E \in \mathcal{M}_g$, we have that $f \in \mathcal{L}_{g^C}^1([a,b);\mathbb{F})$. Now thanks to [18, Theorems 6.3.13, 6.12.3 and 6.12.7], and separating the real and imaginary part if necessary, we have the desired result. □

Corollary 2.4. *Given $E \in \mathcal{M}_g$ and taking $f = \chi_E$ (the characteristic function associated to E) in Lemma 2.3 we have that*

$$\mu_g(E) = \mu_{g^C}(E) + \sum_{s\in E\cap D_g} \Delta^+ g(s).$$

We now introduce a tool that will allow us to transform Lebesgue-Stieltjes integrals with respect to g^C into the usual Lebesgue ones. In particular, in light of Lemma 2.3, this means that we will have a way of transforming any Lebesgue-Stieltjes integral into a Lebesgue one.

Definition 2.5 *(Pseudo-inverse of g^C).* Given an interval $[a,b]$ and a derivator $g : \mathbb{R} \to \mathbb{R}$, we define the *pseudo-inverse* of the continuous part g^C in the interval $[0, g^C(b)]$ by:

$$\gamma : x \in [0, g^C(b)] \to \gamma(x) = \min\left\{t \in [a,b] : g^C(t) = x\right\} \in [a,b]. \tag{2.2}$$

In [7, Proposition 5.1] we can find some of the properties of the pseudo-inverse of a continuous derivator mapping the real line onto the real line. For our context, by extending linearly the map g outside of the interval $[a,b]$ we can obtain the required property, which leads to the following result.

Proposition 2.6. *We have the following properties for the pseudo-inverse of the continuous part g^C in the interval $[0, g^C(b)]$:*

- *For all $x \in [0, g^C(b)]$, $g^C(\gamma(x)) = x$.*
- *For all $t \in [a, b]$, $\gamma(g^C(t)) \leq t$.*
- *For all $t \in [a, b]$, $t \notin C_{g^C} \cup N^+_{g^C}$, $\gamma(g^C(t)) = t$.*
- *The map γ is strictly increasing.*
- *The map γ is left-continuous everywhere and continuous at every $x \in [0, g^C(b)]$, $x \notin g^C(C_g)$.*

Now we are ready to prove the following result.

Proposition 2.7. *Given an interval $[a, b]$ and a derivator $g : \mathbb{R} \to \mathbb{R}$:*

1. *The continuous part*

$$g^C : ([a, b], \mathcal{M}_{g^C}) \to ([0, g^C(b)], \mathcal{L})$$

is a measurable morphism.[1]

2. *The pseudo-inverse of the continuous part g^C*

$$\gamma : ([0, g^C(b)], \mathcal{L}) \to ([a, b], \mathcal{M}_{g^C})$$

is a measurable morphism.

Proof. Let us prove the two statements separately.

1. Let us consider a subset $E \subset [0, g^C(b)]$ such that $E \in \mathcal{L}$. We have that there exists $F \in F_\sigma$ and $N \in \mathcal{L}$ with $\mu(N) = 0$ such that $F \cap N = \emptyset$ and $E = F \cup N$. It is clear that $(g^C)^{-1}(F) \in \mathcal{B}(\tau_u)$ so, if we prove that $\mu^*_{g^C}((g^C)^{-1}(N)) = 0$, where $\mu^*_{g^C}$ is the outer Lebesgue-Stieltjes measure, we will have finished.

Now, since $\mu(N) = 0$, given $\varepsilon > 0$, there exists a countable disjoint family $\{[\widetilde{c}_n, \widetilde{d}_n)\}_{n\in\mathbb{N}}$ such that $N \subset \bigcup_{n\in\mathbb{N}}[\widetilde{c}_n, \widetilde{d}_n)$ and $\sum_{n\in\mathbb{N}}(\widetilde{d}_n - \widetilde{c}_n) < \varepsilon$. We have that $(g^C)^{-1}([\widetilde{c}_k, \widetilde{d}_k)) = [\gamma(\widetilde{c}_k), \gamma(\widetilde{d}_k))$, for all $k \in \mathbb{N}$, thus $(g^C)^{-1}(N) \subset \bigcup_{n\in\mathbb{N}}[\gamma(\widetilde{c}_n), \gamma(\widetilde{d}_n))$. Finally,

$$\mu^*_{g^C}((g^C)^{-1}N) \leq \sum_{n\in\mathbb{N}} \mu^*_{g^C}[\gamma(\widetilde{c}_n), \gamma(\widetilde{d}_n)) = \sum_{n\in\mathbb{N}} \left(g^C(\gamma(\widetilde{d}_n)) - g^C(\gamma(\widetilde{c}_n))\right) = \sum_{n\in\mathbb{N}} (\widetilde{d}_n - \widetilde{c}_n) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily chosen, we have that $\mu^*_{g^C}((g^C)^{-1}N) = 0$, which finishes the proof of 1.

2. Let us consider a subset $E \subset [a, b]$ such that $E \in \mathcal{M}_{g^C}$. We have that there exists $F \in F_\sigma$ and $N \in \mathcal{M}_{g^C}$ such that $\mu_g(N) = 0$, $F \cap N = \emptyset$ and $E = F \cup N$. Thus, we conclude that $\gamma^{-1}(E) = \gamma^{-1}(F) \cup \gamma^{-1}(N)$. Now, since γ is strictly increasing, it is a Borel map, so we have that $\gamma^{-1}(F) \in \mathcal{B}(\tau_u) \subset \mathcal{L}$. Hence, if we prove that $\mu^*(\gamma^{-1}(N)) = 0$, where μ^* is the outer Lebesgue measure, we are done. The proof in this case is analogous to the previous one, the only difference lies in that, given an interval $[c, d)$, we have that $\gamma^{-1}([c, d)) \subset [g^C(c), g^C(d)]$, thus $\mu^*(\gamma^{-1}([c, d))) \leq g^C(d) - g^C(c) = \mu^*_{g^C}([c, d))$. □

The following Corollary is in the line of [2, Lemma 1].

[1] Given two measurable spaces (X, Σ_X) and (Y, Σ_Y), we say that a function $f : X \to Y$ is a measurable morphism if $f^{-1}(F) \in \Sigma_X$, for all $F \in \Sigma_Y$.

Corollary 2.8. *Given a function $f \in \mathcal{L}^1_g([a,b);\mathbb{F})$, for every $t \in [a,b]$,*

$$\int_{[a,t)} f \, \mathrm{d}\,\mu_g = \int_{[a,t)} f \, \mathrm{d}\,\mu_{g^C} + \sum_{s\in[a,t)\cap D_g} f(s)\Delta^+ g(s) = \int_{[0,g^C(t))} \widehat{f}\,\mathrm{d}\,\mu + \sum_{s\in[a,t)\cap D_g} f(s)\Delta^+ g(s),$$

where $\widehat{f} = f\circ\gamma$, $\gamma : t \in [0, g^C(b)] \to \gamma(t)$ is given by (2.2) and μ denotes the Lebesgue measure.

Proof. We write $(X,\Sigma_X) = ([a,t),\mathcal{M}_{g^C})$ and $(Y,\Sigma_Y) = ([0,g^C(t)),\mathcal{L})$. We have, thanks to Proposition 2.7, that $\widehat{f} : Y \to \mathbb{F}$ is a measurable function and $g^C : (X,\Sigma_X) \to (Y,\Sigma_Y)$ is a measurable morphism, which (cf. [20, Exercise 1.4.38]) ensures that

$$\int_Y \widehat{f}\,\mathrm{d}\,g^C_*\mu_{g^C} = \int_X (\widehat{f}\circ g^C)\,\mathrm{d}\,\mu_{g^C},$$

where

$$g^C_*\mu_{g^C} : E \in \Sigma_Y \to g^C_*\mu_{g^C}(E) = \mu_{g^C}((g^C)^{-1}(E))$$

is the *pushforward measure* in (Y,Σ_Y). However, given an element $(c,d) \subset [0, g^C(t))$, it is clear that $g^C_*\mu_{g^C}(c,d) = \mu_{g^C}((g^C)^{-1}(c,d)) = d - c$. In particular, (cf. [1, Theorem 13.8]) $g^C_*\mu_{g^C} = \mu$. Therefore,

$$\int_Y \widehat{f}\,\mathrm{d}\,g^C_*\mu_{g^C} = \int_{[0,g^C(t))} \widehat{f}\,\mathrm{d}\,\mu.$$

Finally, since $\mu_{g^C}(C_{g^C}\cup N^+_{g^C}) = 0$ and $\gamma(g^C(s)) = s$ for all $s \in [a,b]\backslash(C_{g^C}\cup N^+_{g^C})$, we have that

$$\int_{[a,t)} (\widehat{f}\circ g^C)\,\mathrm{d}\,\mu_{g^C} = \int_{[a,t)} f\,\mathrm{d}\,\mu_{g^C}. \quad \square$$

Finally, we recall a concept of continuity introduced in [3] as well as some of its properties. To that end we define the *g-topology*, τ_g, as the family of those sets $U \subset \mathbb{R}$ such that for every $x \in U$ there exists $\delta > 0$ such that if $y \in \mathbb{R}$ satisfies $|g(y) - g(x)| < \delta$ then $y \in U$. Then, the following definition can be understood as the continuity of a function $f : (I,\tau_g) \to (\mathbb{F},\tau_u)$, see [15, Lemma 6].

Definition 2.9 *(g-continuous function).* A function $f : [a,b] \to \mathbb{F}$ is *g-continuous* at a point $t \in [a,b]$, or *continuous with respect to g* at t, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$, for every $s \in [a,b]$ with $|g(t) - g(s)| < \delta$. If f is g-continuous at every point $t \in [a,b]$, we say that f is g-continuous on $[a,b]$.

Proposition 2.10 *([3, Proposition 3.2]). If $f : [a,b] \to \mathbb{R}$ is g-continuous on $[a,b]$, then*

1. *f is continuous from the left at every $t_0 \in (a,b]$;*
2. *if g is continuous at $t_0 \in [a,b)$, then so is f;*
3. *if g is constant on some $[\alpha,\beta] \subset [a,b]$, then so is f.*

In particular, g-continuous functions on $[a,b]$ are continuous on $[a,b]$ when g is continuous on $[a,b)$.

3. The space of bounded g-differentiable functions

In the literature –see, for instance, [3,4,11,15]– authors use the following definition of Stieltjes derivative.

Definition 3.1. We define the *Stieltjes derivative*, or *g–derivative*, of function $f : [a, b] \to \mathbb{R}$ at a point $t \in [a, b]\backslash C_g$ as

$$f'_g(t) = \begin{cases} \lim\limits_{s \to t} \dfrac{f(s) - f(t)}{g(s) - g(t)}, & t \notin D_g, \\ \lim\limits_{s \to t^+} \dfrac{f(s) - f(t)}{g(s) - g(t)}, & t \in D_g, \end{cases}$$

provided the corresponding limits exist and, in that case, we say that f is *g–differentiable at* t. In particular, for $t \in N_g^+ \cup N_g^-$, the g-derivative at t must be understood in the following sense:

$$f'_g(t) = \begin{cases} \lim\limits_{s \to t^+} \dfrac{f(s) - f(t)}{g(s) - g(t)}, & t \in N_g^+, \\ \lim\limits_{s \to t^-} \dfrac{f(s) - f(t)}{g(s) - g(t)}, & t \in N_g^-. \end{cases} \tag{3.1}$$

Remark 3.2. Observe that the points of C_g are excluded from the definition of g–derivative. This is because the corresponding limit cannot be considered at those points since they are in a neighborhood where the corresponding function is not defined. Observe also that the previous definition is also valid for functions with values in $\mathbb{C}$.

Remark 3.3. Taking into account Definition 3.1 and given a function $f : [a, b] \to \mathbb{R}$, the following conditions will be necessary for the existence of the g-derivative in all of the points of $[a, b]\backslash C_g$:

- If $a \in [a, b] \setminus C_g$, then $a \notin N_g^-$. Indeed if $a \in N_g^-$, to calculate the g-derivative at a we need to know the values of f to the left of a, which are not defined. Observe that $C_g \cap N_g = \emptyset$ therefore the previous condition is equivalent to $a \notin N_g^-$.
- If $b \in [a, b] \setminus C_g$, then $b \notin N_g^+ \cup D_g$. Indeed if $b \in N_g^+ \cup D_g$, to calculate the g-derivative at b we need to know the values of f to the right of b, which are not defined. Observe that $C_g \cap N_g = C_g \cap D_g = \emptyset$ therefore the previous condition is equivalent to $b \notin N_g^+ \cup D_g$.
- There exists $f(t^+)$ for every $t \in (a, b) \cap D_g$ (which is also a sufficient condition for the existence of the g-derivative at that point).
- Given $t \in (a, b) \cap N_g^-$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, if $s < t$ with $g(t) - g(s) < \delta$ then, $|f(s) - f(t)| < \varepsilon$. We say, in that case, that f is *g-continuous from the left at* t. To check this fact it is enough to observe that g is left continuous (in the usual sense) at t. The function f might not be g-continuous at t. Indeed, take for instance

$$g : t \in \mathbb{R} \to g(t) = \begin{cases} t, & t \le 1, \\ 1, & 1 \le t \le 2, \\ t - 1, & t \ge 2. \end{cases} \tag{3.2}$$

  Then,

$$f : t \in [0, 3] \to f(t) = \begin{cases} t, & 0 \le t \le 1, \\ t + 1, & 1 < t \le 3, \end{cases}$$

is g-differentiable at $t = 1$ since

$$\lim_{s\to 1^-}\frac{f(s)-f(1)}{g(s)-g(1)} = \lim_{s\to 1^-}\frac{s-1}{s-1} = 1.$$

Observe that g is continuous at $t = 1$, but f is not, so f cannot be g-continuous at that point.

- Given $t \in (a,b)\cap N_g^+$, and $\varepsilon > 0$, there exists $\delta > 0$ such that, if $s > t$ with $g(s) - g(t) < \delta$ then, $|f(s) - f(t)| < \varepsilon$. We say, in that case, that f is *g-continuous from the right at t*. To check this fact it is enough to observe that g is right continuous (in the usual sense) at t. Observe that, once again, the f might not be g-continuous at such points. Indeed, take for instance g as in (3.2) and

$$f : t \in [0,3] \to f(t) = \begin{cases} t, & 0 \le t < 2, \\ t+1, & 2 \le t \le 3. \end{cases}$$

In this case, f is g-differentiable at $t = 2$ but f is not g-continuous at such point.

- Given $t \in (a,b)\backslash(C_g \cup D_g \cup N_g)$, f is g-continuous at t. In particular, f is continuous at t since g is continuous at those points.

We conclude that, interestingly enough, the g-differentiability of a function at a point of N_g does not imply the g-continuity of the function at the point. The g-differentiability of a function only guarantees the g-continuity at the points of $(a,b)\backslash(C_g \cup D_g \cup N_g)$.

Definition 3.4 *($\mathcal{C}_g^1([a,b];\mathbb{F})$ space).* Let $g : \mathbb{R} \to \mathbb{R}$ be such that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g$. We say that $f : [a,b] \to \mathbb{F}$ belongs to $\mathcal{C}_g^1([a,b];\mathbb{F})$ if the following conditions are met:

1. $f \in \mathcal{C}_g([a,b];\mathbb{F})$,
2. $\exists f_g'(x)$, for every $x \in [a,b]\backslash C_g$,
3. $\exists h \in \mathcal{C}_g([a,b];\mathbb{F})$ such that $h(x) = f_g'(x)$, for every $x \in [a,b]\backslash C_g$.

Unless necessary, we will write $\mathcal{C}_g^1([a,b])$ instead of $\mathcal{C}_g^1([a,b];\mathbb{F})$ for brevity.

Let us show now that if we assume that $b \notin C_g$ (observe that, in that case, the hypothesis $[a,b] \setminus C_g \neq \emptyset$ is trivially satisfied) the previous definition is consistent insofar as the function given by 3, if it exists, it is unique.

Proposition 3.5. *Let $[a,b] \subset \mathbb{R}$ be a closed interval, $g : \mathbb{R} \to \mathbb{R}$ a derivator such that $a \notin N_g^-$ and $b \notin C_g \cup N_g^+ \cup D_g$ and $f \in \mathcal{C}_g([a,b];\mathbb{F})$ be g-differentiable at every $x \in [a,b]\backslash C_g$. If h_1, $h_2 \in \mathcal{C}_g([a,b])$ are such that $h_1(x) = h_2(x) = f_g'(x)$, for every $x \in [a,b]\backslash C_g$, then $h_1 = h_2$.*

Proof. Let us show that $h_1(x) = h_2(x)$ for every $x \in C_g$. Given $\widetilde{x} \in C_g$, there exists a unique connected component of C_g, (a_n, b_n), such that $\widetilde{x} \in (a_n, b_n)$. Let us see that $h_1(\widetilde{x}) = h_2(\widetilde{x}) = f_g'(b_n)$. Indeed, since h_1 is g-continuous, we have, by Proposition 2.10, that h_1 is constant on (a_n, b_n) and left-continuous, therefore, $h_1(\widetilde{x}) = h_1(b_n) = f_g'(b_n)$ for every $x \in (a_n, b_n)$. The case of h_2 is proven analogously. □

Remark 3.6. Observe that if $b \in C_g$, given a function $f \in \mathcal{C}_g^1([a,b])$ the function $h \in \mathcal{C}_g([a,b])$ such that $f_g'(x) = h(x)$ for every $x \in [a,b] \setminus C_g$ is not uniquely defined in a neighborhood of point b since we can not compute the g-derivative at $x = b_n$, with $b \in (a_n, b_n) \subset C_g$.

A possible way of defining the g-derivative at the points of C_g, which is coherent with the definition of the space $\mathcal{C}_g^1$, follows from the previous proof. Indeed, we can generalize Definition 3.1 in the following terms.

Definition 3.7. Let $[a,b] \subset \mathbb{R}$ be a closed interval and $g : \mathbb{R} \to \mathbb{R}$ a derivator such that $a \notin N_g^-$ and $b \notin C_g \cup N_g^+ \cup D_g$. We define the *Stieltjes derivative*, or *g–derivative*, of a function $f : [a,b] \to \mathbb{F}$ at a point $t \in [a,b]$ as

$$f'_g(t) = \begin{cases} \lim\limits_{s\to t} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g \cup C_g, \\ \lim\limits_{s\to t^+} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \in D_g, \\ \lim\limits_{s\to b_n^+} \dfrac{f(s)-f(b_n)}{g(s)-g(b_n)}, & t \in (a_n, b_n) \subset C_g, \end{cases} \tag{3.3}$$

with a_n, b_n as in (2.1); provided the corresponding limits exist. In that case, we say that f is *g–differentiable at t*. The g-derivative in the points N_g must be understood as in (3.1).

Remark 3.8. It follows from the Definition 3.7 that, for $t \in D_g$, $f'_g(t)$ exists if and only if $f(t^+)$ exists and, in that case,

$$f'_g(t) = \frac{f(t^+) - f(t)}{\Delta^+ g(t)}.$$

Similarly, for any $t \in (a_n, b_n) \subset C_g$, we have that $f'_g(t)$ exists if and only if $f'_g(b_n)$ exists and, in that case, $f'_g(t) = f'_g(b_n)$.

The following result which includes some basic properties of the Stieltjes derivative is a generalization of [12, Proposition 3.13].

Proposition 3.9. *Let $[a,b] \subset \mathbb{R}$ be a closed interval and $g : \mathbb{R} \to \mathbb{R}$ a derivator such that $a \notin N_g^-$ and $b \notin C_g \cup N_g^+ \cup D_g$. Given an element $t \in [a,b]$ we denote by:*

$$t^* = \begin{cases} t, & t \notin C_g, \\ b_n, & t \in (a_n, b_n) \subset C_g, \end{cases}$$

with a_n, b_n as in (2.1). If f_1, f_2 are two g-differentiable functions at t, then:

- *The function $\lambda_1 f_1 + \lambda_2 f_2$ is g-differentiable at t for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and*

$$(\lambda_1 f_1 + \lambda_2 f_2)'_g(t) = \lambda_1 (f_1)'_g(t) + \lambda_2 (f_2)'_g(t).$$

- *The product $f_1 f_2$ is g-differentiable at t and*

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t^*) + (f_2)'_g(t) f_1(t^*) + (f_1)'_g(t)\,(f_2)'_g(t) \Delta^+ g(t^*). \tag{3.4}$$

- *If $f_2(t^*)\left(f_2(t^*) + (f_2)'_g(t)\,\Delta^+ g(t^*)\right) \neq 0$, the quotient f_1/f_2 is g-differentiable at t and*

$$\left(\frac{f_1}{f_2}\right)'_g(t) = \frac{(f_1)'_g(t) f_2(t^*) - (f_2)'_g(t) f_1(t^*)}{f_2(t^*)\left(f_2(t^*) + (f_2)'_g(t)\,\Delta^+ g(t^*)\right)} \tag{3.5}$$

Proof. We only need to show that the result holds for $t \in C_g$ as any other case is covered by [12, Proposition 3.13].

Let $t \in (a_n, b_n) \subset C_g$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Then, it follows from Remark 3.8 that f_1, f_2 are g-differentiable at b_n. Now, [12, Proposition 3.13] ensures that $\lambda_1 f_1 + \lambda_2 f_2$ and $f_1 f_2$ are g-differentiable at b_n and, provided that $f_2(b_n)\,(f_2(b_n) + (f_2)'_g(b_n)\,\Delta^+ g(b_n)) \neq 0$, so is f_1/f_2. Furthermore, we also have that

$$\begin{aligned}
(\lambda_1 f_1 + \lambda_2 f_2)'_g(b_n) &= \lambda_1 (f_1)'_g(b_n) + \lambda_2 (f_2)'_g(b_n),\\
(f_1 f_2)'_g(b_n) &= (f_1)'_g(b_n) f_2(b_n) + (f_2)'_g(b_n) f_1(b_n) + (f_1)'_g(b_n)\,(f_2)'_g(b_n)\Delta^+ g(b_n),\\
\left(\frac{f_1}{f_2}\right)'_g(b_n) &= \frac{(f_1)'_g(b_n) f_2(b_n) - (f_2)'_g(b_n) f_1(b_n)}{f_2(b_n)\,(f_2(b_n) + (f_2)'_g(t)\,\Delta^+ g(b_n))}.
\end{aligned}$$

Now the result follows from Remark 3.8 and the fact that $t^* = b_n$ in this case. $\square$

Remark 3.10. Other expressions for (3.4) and (3.5) can be obtained when the functions are also g-continuous on $[a, b]$. Under this condition, we have that

$$\begin{aligned}
(f_1 f_2)'_g(t) &= (f_1)'_g(t) f_2(t) + (f_2)'_g(t) f_1(t) + (f_1)'_g(t)\,(f_2)'_g(t)\Delta^+ g(t^*),\\
\left(\frac{f_1}{f_2}\right)'_g(t) &= \frac{(f_1)'_g(t) f_2(t) - (f_2)'_g(t) f_1(t)}{f_2(t)\,(f_2(t) + (f_2)'_g(t)\,\Delta^+ g(t^*))}.
\end{aligned}$$

Indeed, the formulas are clear for $t \notin C_g$, so we shall focus on the case $t \in (a_n, b_n) \subset C_g$ for some a_n, b_n in (2.1). In that case, and since g is left-continuous, we have that g is constant on $[t, b_n]$, which forces the same character onto f_1 and f_2. Therefore, it follows that $f_1(t^*) = f_1(t)$ and $f_2(t^*) = f_2(t)$, from which the formulas follow.

Definition 3.11. Let $g : \mathbb{R} \to \mathbb{R}$ be such that $a \notin N_g^-$ and $b \notin C_g \cup N_g^+ \cup D_g$. Given $k \in \mathbb{N}$, we define $\mathcal{C}_g^0([a, b]; \mathbb{F}) = \mathcal{C}_g([a, b]; \mathbb{F})$ and $\mathcal{C}_g^k([a, b]; \mathbb{F})$ recursively as

$$\mathcal{C}_g^k([a, b]) = \{f \in \mathcal{C}^{k-1}([a, b]; \mathbb{F}) :\ (f_g^{(k-1)})'_g \in \mathcal{C}_g([a, b]; \mathbb{F})\},$$

where $f_g^{(0)} = f$ and $f_g^{(k)} = (f_g^{(k-1)})'_g$, $k \in \mathbb{N}$. We also define $\mathcal{C}_g^\infty([a, b]; \mathbb{F}) := \bigcap_{n \in \mathbb{N}} \mathcal{C}_g^k([a, b]; \mathbb{F})$. Unless necessary, we will write $\mathcal{C}_g^k([a, b])$ instead of $\mathcal{C}_g^k([a, b]; \mathbb{F})$ for brevity.

Now we endow $\mathcal{C}_g^k([a, b])$ with a normed space structure. First, observe that g-continuous functions on $[a, b]$ are not necessarily bounded [3, Example 3.3], so we will restrict ourselves to the space $\mathcal{BC}_g([a, b])$ of bounded g-continuous functions. This is a Banach space [3, Theorem 3.4] with the supremum norm

$$\|f\|_0 = \sup\{|f(x)| :\ x \in [a, b]\}.$$

Definition 3.12. Let $g : \mathbb{R} \to \mathbb{R}$ be such that $a \notin N_g^-$ and $b \notin C_g \cup N_g^+ \cup D_g$. We define:

$$\mathcal{BC}_g^1([a, b]; \mathbb{F}) := \{f \in \mathcal{C}_g^1([a, b]; \mathbb{F}) :\ f,\ f'_g \in \mathcal{BC}_g([a, b]; \mathbb{F})\}.$$

Analogously, given $k \in \mathbb{N}$,

$$\mathcal{BC}_g^k([a, b]; \mathbb{F}) = \{f \in \mathcal{C}_g^k([a, b]; \mathbb{F}) :\ f_g^{(n)} \in \mathcal{BC}_g([a, b]; \mathbb{F}),\ \forall n = 0, \dots, k\}$$

and we will denote by $\mathcal{BC}_g^0([a, b]; \mathbb{F}) = \mathcal{BC}_g([a, b]; \mathbb{F})$.

In the following results we will assume that $[a,b] \subset \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a derivator such that $a \notin N_g^-$ and $b \notin C_g \cup N_g^+ \cup D_g$. We have that $\mathcal{BC}_g^k([a,b]) \equiv \mathcal{BC}_g^k([a,b];\mathbb{F})$ is a normed vector space with the norm

$$\begin{aligned} \|\cdot\|_k : \mathcal{BC}_g^k([a,b]) &\mapsto \mathbb{R} \\ f &\mapsto \|f\|_k = \sum_{0\le i\le k} \|f_g^{(i)}\|_0 \end{aligned}$$

Before proving it is also a Banach space, we will present the following lemma.

Lemma 3.13. *We have the continuous embedding $\mathcal{BC}_g^1([a,b]) \hookrightarrow \mathcal{AC}_g([a,b])$. Furthermore, for every $f \in \mathcal{BC}_g^1([a,b])$,*

$$f(x) = f(a) + \int_{[a,x)} f_g'(s)\, \mathrm{d}\,\mu_g,\ \forall x \in [a,b].$$

Proof. This is an immediate consequence of [11, Theorem 6.2] and [11, Corollary 6.3]. Indeed, given $f \in \mathcal{BC}_g^1([a,b])$, it is clear that $f \in \mathcal{BC}_g([a,b])$. In particular, f is continuous from the left at the points in $(a,b] \cap D_g$ and constant at the intervals where g is. On the other hand, $f_g' \in \mathcal{BC}_g([a,b]) \subset \mathcal{L}_g^1([a,b])$. Hence, by the aforementioned results, $f \in \mathcal{AC}_g([a,b])$ and, furthermore,

$$f(x) = f(a) + \int_{[a,x)} f_g'(s)\, \mathrm{d}\,\mu_g,\ \forall x \in [a,b]. \quad \square$$

We derive the following result from the previous lemma.

Lemma 3.14. *Let $h \in \mathcal{BC}_g([a,b])$ and consider the function*

$$H : x \in [a,b] \to H(x) = \int_{[a,x)} h(s)\, \mathrm{d}\,\mu_g.$$

We have that $H_g'(x) = h(x)$, for every $x \in [a,b]$ and, therefore, $H \in \mathcal{BC}_g^1([a,b])$.

Proof. Indeed, on the one hand, given that $h \in \mathcal{BC}_g([a,b]) \subset \mathcal{L}_g^1([a,b))$, it holds that $H \in \mathcal{AC}_g([a,b])$, so it is enough to prove that $H_g'(x) = h(x)$ for every $x \in [a,b]$ to get the result. We study three different cases:

- For $x \in D_g$, it is clear that

$$\begin{aligned} H_g'(x) &= \lim_{s\to x^+} \frac{H(s)-H(x)}{g(s)-g(x)} = \lim_{s\to x^+} \frac{1}{g(s)-g(x)} \int_{[x,s)} h(s)\, \mathrm{d}\,\mu_g \\ &= \lim_{s\to x^+} \frac{1}{g(s)-g(x)} \left(\int_{\{x\}} h(s)\, \mathrm{d}\,\mu_g + \int_{(x,s)} h(s)\, \mathrm{d}\,\mu_g \right) = \lim_{s\to x^+} \frac{h(x)\Delta^+ g(x)}{g(s)-g(x)} = h(x). \end{aligned}$$

- For $x \in [a,b]\backslash(C_g \cup D_g)$, let us compute the limit

$$\lim_{s\to x} \frac{H(s)-H(x)}{g(s)-g(x)},$$

on the domain of the function, namely, $D_x = \{s \in [a,b] : g(s) \neq g(t)\}$. Fix $\varepsilon > 0$. Since h is g-continuous and g is continuous at x, there exists $\delta > 0$ such that $|h(u) - h(x)| < \varepsilon$ if $|u - x| < \delta$. Define $[\![x,s)\!) := [\min\{x,s\}, \max\{x,s\})$. Now, for $s \in [a,b] \cap D_x$, $|u - s| < \delta$, we have that

$$\begin{aligned}\left|\frac{H(s)-H(x)}{g(s)-g(x)} - h(x)\right| &= \left|\frac{\operatorname{sgn}(s-x)}{g(s)-g(x)} \int_{[\![x,s)\!)} h(u) \, \mathrm{d}\,\mu_g(u) - h(x)\right| \\ &= \frac{1}{|g(s)-g(x)|}\left|\int_{[\![x,s)\!)} (h(u)-h(x)) \, \mathrm{d}\,\mu_g(u)\right| \\ &\leq \frac{1}{|g(s)-g(x)|}\int_{[\![x,s)\!)} |h(u)-h(x)| \, \mathrm{d}\,\mu_g(u) \leq \frac{1}{|g(s)-g(x)|}\int_{[\![x,s)\!)} \varepsilon \, \mathrm{d}\,\mu_g(u) = \varepsilon.\end{aligned}$$

Thus,

$$\lim_{s\to x} \frac{H(s)-H(x)}{g(s)-g(x)} = h(x).$$

- Finally, for $x \in (a_n, b_n) \subset C_g$, it holds that

$$H_g'(x) = H_g'(b_n) = h(b_n) = h(x),$$

where the first equality comes from the definition of the g-derivative at the points of C_g and the last is a consequence of the g-continuity of h. □

Theorem 3.15. *$(\mathcal{BC}_g^k([a,b]), \|\cdot\|_k)$ is a Banach space.*

Proof. Let us check the case $k = 1$ (the case $k \geq 2$ is analogous). Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{BC}_g^1([a,b])$ be a Cauchy sequence. Then, $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{BC}_g([a,b])$ and $\{(f_n)_g'\}_{n\in\mathbb{N}} \subset \mathcal{BC}_g([a,b])$ are Cauchy sequences in the Banach space $\mathcal{BC}_g([a,b])$ so there exist f, $h \in \mathcal{BC}_g([a,b])$ such that $f_n \to f$ and $(f_n)_g' \to h$ in $\mathcal{BC}_g([a,b])$. Let us check that $f_g'(x)$ exists for every $x \in [a,b]$ and that, furthermore, $f_g' = h$. Indeed, let $\varepsilon > 0$. Since $(f_n)_g' \to h$, there exists $N \in \mathbb{N}$ such that $\|(f_n)_g' - h\|_0 \leq \varepsilon/(g(a) - g(b))$. Now, using Lemma 3.13, we have that

$$f_n(x) - f_n(a) = \int_{[a,x)} (f_n)_g'(s) \, \mathrm{d}\,\mu_g, \ \forall x \in [a,b],$$

whence, for $n \geq N$,

$$\left|\int_{[a,x)} (f_n)_g'(s) \, \mathrm{d}\,\mu_g - \int_{[a,x)} h(s) \, \mathrm{d}\,\mu_g\right| \leq \int_{[a,x)} \left|(f_n)_g'(s) - h(s)\right| \mathrm{d}\,\mu_g \leq \varepsilon, \quad \forall x \in [a,b].$$

This means that

$$\lim_{n\to\infty} \int_{[a,x)} (f_n)_g'(s) \, \mathrm{d}\,\mu_g = \int_{[a,x)} h(s) \, \mathrm{d}\,\mu_g$$

uniformly on $[a,b]$. Thus,

$$\lim_{n\to\infty}(f_n(x)-f_n(a))=\lim_{n\to\infty}\int\limits_{[a,x)}(f_n)'_g(s)\,\mathrm{d}\,\mu_g=\int\limits_{[a,x)}h(s)\,\mathrm{d}\,\mu_g$$

uniformly on $[a,b]$. Hence,

$$f(x)=f(a)+\int\limits_{[a,x)}h(s)\,\mathrm{d}\,\mu_g.$$

Since $h\in\mathcal{BC}_g([a,b])$, by Lemma 3.14, we get that $f'_g(x)=h(x)$ for all $x\in[a,b]$ as we wanted to show. □

Let us study now the properties of the functions in $\mathcal{BC}^n_g([a,b];\mathbb{F})$.

Remark 3.16. Observe that, given $f_1,\,f_2\in\mathcal{BC}^1_g([a,b])$, the product $f_1\,f_2\notin\mathcal{BC}^1_g([a,b])$ in general. This happens because the product $(f_1)'_g(t)\,(f_2)'_g(t)\Delta^+g(t^*)$ might not be g-continuous. Indeed, take the following derivator

$$g:t\in\mathbb{R}\to g(t)=\begin{cases}t, & t\le 0,\\ t+2, & t>1,\end{cases}$$

and the function

$$f:t\in[-1,1]\to f(t)=\begin{cases}t, & -1\le t\le 0,\\ 2, & 0<t\le 1.\end{cases}$$

Observe that $C_g=\emptyset$, therefore $t^*=t$ for all $t\in[-1,1]$. It is easy to check that $f\in\mathcal{BC}_g([a,b])$ and its derivative,

$$f'_g:t\in[-1,1]\to f'_g(t)=\begin{cases}1, & -1\le t<0,\\ \dfrac{f(0^+)-f(0)}{\Delta^+g(0)}=1, & t=0,\\ 0, & 0<t\le 1,\end{cases}$$

is also g-continuous. On the other hand,

$$f^2:t\in[-1,1]\to f^2(t)=\begin{cases}t^2, & -1\le t\le 0,\\ 4, & 0<t\le 1,\end{cases}$$

is g-continuous, but

$$(f^2)'_g:t\in[-1,1]\to (f^2)'_g(t)=\begin{cases}2t, & -1\le t<0,\\ \dfrac{f^2(0^+)-f^2(0)}{\Delta^+g(0)}=2, & t=0,\\ 0, & 0<t\le 1,\end{cases}$$

is not g-continuous since $\lim_{t\to 0^-}(f^2)'_g(t)=0\neq 2=(f^2)'_g(0)$. The problem, as mentioned before, relies on the g-continuity (or lack thereof) of the term $(f)'_g(t)\,(f)'_g(t)\Delta^+g(t^*)$. Indeed, given an element $t\in[-1,1]$, we have that

$$(f^2)'_g(t)=2f'_g(t)\,f(t)+(f'_g(t))^2\Delta^+g(t^*).$$

Now, even though

$$\lim_{t\to 0^-} 2f'_g(t)\, f(t) = 2f'_g(0)\, f(0),$$

the same does not happen for

$$\lim_{t\to 0^-} (f'_g(t))^2 \Delta^+ g(t^*) = 0 \neq (f'_g(0))^2 \Delta^+ g(0^*) = 2.$$

The problem in the lack of continuity of the previous term can be solved if one of the functions involved in the product is also continuous at the points of the discontinuity of the derivator, that is, if it is also continuous in the usual sense, as the following proposition shows.

Proposition 3.17. *Given $f_1 \in \mathcal{BC}^1_g([a,b]) \cap \mathcal{C}([a,b])$ and $f_2 \in \mathcal{BC}^1_g([a,b])$, it holds that $f_1 f_2 \in \mathcal{BC}^1_g([a,b])$ and*

$$(f_1 f_2)'_g(x) = (f_1)'_g(x) f_2(x) + (f_2)'_g(x) f_1(x), \ \forall x \in [a,b]. \tag{3.6}$$

In particular, if $f_1 \in \mathcal{BC}^1_g([a,b]) \cap \mathcal{BC}_{g^C}([a,b])$ and $f_2 \in \mathcal{BC}^1_g([a,b])$, we have that (3.6) holds.

Proof. By the continuity of f_1, given $x \in [a,b] \cap D_g$, we have that $(f_1)'_g(x) = 0$. Hence, Proposition 3.9 and the definition of the g-derivative at the points of C_g imply that

$$(f_1 f_2)'_g(x) = (f_1)'_g(x) f_2(x) + (f_2)'_g(x) f_1(x), \ \forall x \in [a,b].$$

Therefore, $(f_1 f_2)'_g \in \mathcal{BC}_g([a,b])$ and, thus, $f_1 f_2 \in \mathcal{BC}^1_g([a,b])$. □

In the following corollary, which can be obtained from Proposition 3.17 using induction, we provide a generalization of Proposition 3.17.

Corollary 3.18. *Given $f_1 \in \mathcal{BC}^n_g([a,b]) \cap \mathcal{BC}^{n-1}_{g^C}([a,b])$ and $f_2 \in \mathcal{BC}^n_g([a,b])$, we have that $f_1 f_2 \in \mathcal{BC}^n_g([a,b])$.*

4. First order linear Stieltjes differential equations

To simplify the notation we will work on the interval $[a,b] = [0,T]$. In this section we will analyze a first order linear Stieltjes differential equation where the coefficients and data are complex valued functions. Additionally, we will prove further properties of the solution and we will show that, under some regularity conditions for coefficients and data, it is possible to obtain solutions in the space $\mathcal{BC}^1_g([0,T];\mathbb{F})$. In order to correctly define the regular solutions in the space $\mathcal{BC}^1_g([0,T];\mathbb{F})$ we will assume that $0 \notin N^-_g$ and $T \notin N^+_g \cup D_g \cup C_g$. This consideration is not necessary when looking for solutions in the space of the absolutely continuous functions.

4.1. The homogeneous case

Let us consider the first order homogeneous linear problem

$$\begin{cases} v'_g(t) - \beta(t)\, v(t) = 0, \ g - a.e.\, t \in [0,T), \\ v(0) = v_0, \end{cases} \tag{4.1}$$

where $\beta \in \mathcal{L}^1_g([0,T];\mathbb{F})$ and $v_0 \in \mathbb{F}$. The solution of problem (4.1) was given, for the first time, in [3] for the real case. In this section we will analyze the existence of solution in the complex case and see how to recover the particular cases studied in [3]. Apart from the generalization proposed here for the complex case, we present a constructive proof of the expression of the solution which brings light to the nature of the structure of the solutions of problem (4.1).

For the work ahead, we will need to use the chain rule for the Stieltjes derivative. In [12, Proposition 3.15] we can find a version of the result for the derivative of real valued functions at a continuity point of the derivator. Here, we introduce the following more general version.

Proposition 4.1. *Let $t \in [0,T]$, $f : [0,T] \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{F}$. Then, the following hold:*

1. *If $t^* \in [0,T] \setminus (D_g \cup C_g)$ (where $t*$ is as in Proposition 3.9) and there exist $h'(f(t^*))$ and $f'_g(t)$, then $h \circ f$ is g-differentiable at t and*

$$(h \circ f)'_g(t) = h'(f(t^*)) f'_g(t). \tag{4.2}$$

2. *If $t \in D_g$ and*

$$f(s) = f(t), \quad s \in (t, t+\delta) \text{ for some } \delta > 0, \tag{4.3}$$

then $f'_g(t) = (h \circ f)'_g(t) = 0$. In particular, (4.2) holds provided $h'(f(t))$ exists.

3. *Suppose that $t \in D_g$ and condition (4.3) does not hold. If $f(t^+)$ exists, h is continuous at $f(t^+)$ and the limit*

$$\lim_{s \to t^+} \frac{h(f(s)) - h(f(t))}{f(s) - f(t)} \tag{4.4}$$

exists, then there exist $f'_g(t)$ and $(h \circ f)'_g(t)$ and

$$(h \circ f)'_g(t) = \frac{h(f(t^+)) - h(f(t))}{f(t^+) - f(t)} f'_g(t).$$

Proof. First, observe that 1 follows directly from [12, Proposition 3.15] in the case $t \in [0,T] \setminus (D_g \cup C_g)$ and from the definition of the g-derivative at points of C_g for the case $t \in (a_n, b_n) \subset C_g$ and $b_n \notin D_g$. Noting that (4.3) guarantees that $f(t^+) = f(t)$ and $(h \circ f)(t^+) = (h \circ f)(t)$ is enough to obtain 2. Finally, for 3, the hypotheses ensure that $f'_g(t)$ exists and

$$\lim_{s \to t^+} h \circ f(s) = h(f(t^+)),$$

so $(h \circ f)'_g(t)$ also exists. On the other hand, given that (4.3) does not hold, we can find $\{t_n\}_{n \in \mathbb{N}} \subset [0,T]$ such that $t_n \to t$, $t_n > t$, and $f(t_n) \neq f(t)$ for all $n \in \mathbb{N}$. Hence, given (4.4), we have that

$$\begin{aligned}(h \circ f)'_g(t) &= \lim_{n \to \infty} \frac{h(f(t_n)) - h(f(t))}{g(t_n) - g(t)} \\ &= \lim_{n \to \infty} \frac{h(f(t_n)) - h(f(t))}{f(t_n) - f(t)} \frac{f(t_n) - f(t)}{g(t_n) - g(t)} = \frac{h(f(t^+)) - h(f(t))}{f(t^+) - f(t)} f'_g(t). \quad \square\end{aligned}$$

For the following theorem, we will denote by $\ln(z) := \ln|z| + i \operatorname{Arg}(z)$ for $z \in \mathbb{C}$ the principal branch of the complex logarithm where Arg is the principal argument.

Theorem 4.2. *Assume $\mu_g(\overline{D_g} \setminus D_g) = 0$. Let $\beta \in \mathcal{L}^1_g([0,T), \mathbb{F})$ be such that $1 + \beta(t)\Delta^+ g(t) \neq 0$ for every $t \in [0,T) \cap D_g$. Then there exists a unique solution $v \in \mathcal{AC}_g([0,T]; \mathbb{F})$ of problem (4.1) which, furthermore, is of the form*

$$v(t) = v^B(t)\, v^C(t),$$

where $v^B \in \mathcal{AC}_{g^B}([0,T];\mathbb{F})$ *is the unique solution of the problem*

$$\begin{cases} v'_{g^B}(t) - \beta(t)\, v(t) = 0, \; g^B - a.e.\, t \in [0,T), \\ v(0) = 1, \end{cases} \tag{4.5}$$

given by

$$v^B(t) = \prod_{s\in[0,t)\cap D_g} \big(1 + \beta(s)\Delta^+ g(s)\big)\,; \tag{4.6}$$

and $v^C \in \mathcal{AC}_{g^C}([0,T];\mathbb{F})$ *is the unique solution of*

$$\begin{cases} v'_{g^C}(t) - \beta(t)\, v(t) = 0, \; g^C - a.e.\, t \in [0,T), \\ v(0) = v_0, \end{cases} \tag{4.7}$$

given by

$$v^C(t) = u(g^C(t)), \tag{4.8}$$

where $u \in \mathcal{AC}([0,T];\mathbb{F})$ *is the unique solution of*

$$\begin{cases} u'(t) = \widehat{\beta}(t)\, u(t), \; a.e.\, t \in [0, g^C(T)), \\ u(0) = v_0, \end{cases} \tag{4.9}$$

where $\widehat{\beta} = \beta \circ \gamma$ *and* γ *is provided by Definition 2.5.*

Furthermore, v *can be written as*

$$v(t) = v_0 \exp\left(\int_{[0,t)} \widetilde{\beta}(s) \; \mathrm{d}\,\mu_g \right), \tag{4.10}$$

with

$$\widetilde{\beta}(t) = \begin{cases} \beta(t), & t \in [0,T)\backslash D_g, \\ \dfrac{\ln\left(1 + \beta(t)\Delta^+ g(t)\right)}{\Delta^+ g(t)}, & t \in [0,T) \cap D_g. \end{cases}$$

Proof. *Existence and uniqueness:* If v solves (4.1), then (x, y) where $x := \operatorname{Re} v$ and $y := \operatorname{Im} v$ solves the real system

$$\begin{cases} x'_g(t) - \operatorname{Re}\beta(t)\, x(t) + \operatorname{Im}\beta(t)\, y(t) = 0, \; g - a.e.\, t \in [0,T), \\ y'_g(t) - \operatorname{Im}\beta(t)\, x(t) - \operatorname{Re}\beta(t)\, y(t) = 0, \; g - a.e.\, t \in [0,T), \\ x(0) = \operatorname{Re} v_0, \; y(0) = \operatorname{Im} v_0, \end{cases} \tag{4.11}$$

and vice-versa, that is, the function $x + iy$, where (x, y) is a solution of (4.11), solves (4.1). Now, it is easy to see that (4.11) satisfies the conditions of [6, Theorem 4.3] with $L = |\operatorname{Re}(\beta)| + |\operatorname{Im}(\beta)|$, so it has a unique solution on $[0, T]$. Hence, (4.1) has a unique solution there as well.

Expression of the solution: Given the nature of problem (4.1) where the g-derivative has to be a multiple of itself it is only natural to use an *ansatz* of the form

$$v(t) = v_0 \exp\left(\int_{[0,t)} \widetilde{\beta}(s) \, \mathrm{d}\, \mu_g \right),$$

with $\widetilde{\beta} \in \mathcal{L}^1_g([0,T);\mathbb{F})$. Given that $\mathcal{M}_g \subset \mathcal{M}_{g^C}$ and $\mathcal{M}_g \subset \mathcal{M}_{g^B}$, it is clear that if $\widetilde{\beta} \in \mathcal{L}^1_g([0,T);\mathbb{F})$, then $\widetilde{\beta} \in \mathcal{L}^1_{g^C}([0,T);\mathbb{F})$ and $\widetilde{\beta} \in \mathcal{L}^1_{g^B}([0,T);\mathbb{F})$. Furthermore,

$$\begin{aligned} v(t) &= v_0 \exp\left(\int_{[0,t)} \widetilde{\beta}(s) \, \mathrm{d}\, \mu_{g^B} + \int_{[0,t)} \widetilde{\beta}(s) \, \mathrm{d}\, \mu_{g^C} \right) \\ &= v_0 \exp\left(\int_{[0,t)} \widetilde{\beta}(s) \, \mathrm{d}\, \mu_{g^B} \right) \exp\left(\int_{[0,t)} \widetilde{\beta}(s) \, \mathrm{d}\, \mu_{g^C} \right) = v^B(t)\, v^C(t), \end{aligned}$$

where $v^B(t) := \exp\left(\int_{[0,t)} \widetilde{\beta}(s) \, \mathrm{d}\, \mu_g^B\right)$ and $v^C(t) := v_0 \exp\left(\int_{[0,t)} \widetilde{\beta}(s) \, \mathrm{d}\, \mu_g^C\right)$. From the definition we deduce that $v^B \in \mathcal{AC}_{g^B}([0,T];\mathbb{F})$ and $v^C \in \mathcal{AC}_{g^C}([0,T];\mathbb{F})$. Hence, given the g^B-continuity of v^B, we have that $(v^B)'_g(t) = 0$ for every $t \in [0,T)\backslash(\overline{D_g} \cup C_g)$ and, thanks to the g^C-continuity of v^C, it holds that $(v^C)'_g(t) = 0$ for every $t \in [0,T) \cap D_g$. Thus, by Proposition 3.9,

$$v'_g(t) = \begin{cases} (v^B)'_g(t)\, v^C(t), & t \in [0,T) \cap D_g, \\ v^B(t)\, (v^C)'_g(t), & g - a.e.\, t \in [0,T)\backslash(\overline{D_g} \cup C_g). \end{cases}$$

This implies that we will have a different equation for each of the components of the solution:

$$(v^B)'_g(t) = \beta(t)\, v^B(t),\ t \in [0,T) \cap D_g, \tag{4.12}$$

$$(v^C)'_g(t) = \beta(t)\, v^C(t),\ g - a.e.\, t \in [0,T)\backslash(\overline{D_g} \cup C_g). \tag{4.13}$$

We will start studying equation (4.12). For $t \in [0,T) \cap D_g$ we have that

$$(v^B)'_g(t) = \frac{v^B(t^+) - v^B(t)}{\Delta^+ g(t)} = (v^B)'_{g^B}(t).$$

Now, if we develop equation (4.12):

$$\frac{v^B(t^+) - v^B(t)}{\Delta^+ g(t)} = \beta(t)\, v^B(t)$$

and we get that

$$v^B(t^+) = v^B(t)(1 + \beta(t)\, \Delta^+ g(t)),\ t \in [0,T) \cap D_g. \tag{4.14}$$

In order to get a solution candidate for equation (4.12), define $h(t) = \ln(1 + \beta(t)\Delta^+ g(t))/\Delta^+ g(t)$ if $t \in D_g$, $h(t) = 0$ if $t \notin D_g$. Then, taking into account that $\mu_{g^B}(t) = 0$ for every $t \notin D_g$, we define

$$\begin{aligned} H(t) &:= \exp\left(\int_{[0,t)} h(t)\, \mathrm{d}\,\mu_{g^B}\right) = \exp\left(\int_{[0,t)} \frac{\ln(1+\beta(s)\Delta^+ g(s))}{\Delta^+ g(s)}\, \mathrm{d}\,\mu_{g^B}\right) \\ &= \exp\left(\sum_{s\in[0,t)\cap D_g} \ln(1+\beta(s)\Delta^+ g(s))\right) \\ &= \exp\left(\sum_{s\in[0,t)\cap D_g} \left(\ln\left|1+\beta(s)\Delta^+ g(s)\right| + i\ \mathrm{Arg}(1+\beta(s)\Delta^+ g(s))\right)\right). \end{aligned}$$

To show that H is well defined, let us check that the series

$$\sum_{s\in[0,t)\cap D_g} \ln\left|1+\beta(s)\Delta^+ g(s)\right| \quad \text{and} \quad \sum_{s\in[0,t)\cap D_g} \mathrm{Arg}(1+\beta(s)\Delta^+ g(s))$$

are absolutely convergent. We have that

$$\sum_{s\in[0,T)\cap D_g} \left|\ln\left|1+\beta(s)\Delta^+ g(s)\right|\right| = \sum_{s\in A} \left|\ln\left|1+\beta(s)\Delta^+ g(s)\right|\right| + \sum_{s\in B} \left|\ln\left|1+\beta(s)\Delta^+ g(s)\right|\right|,$$

where

$$\begin{aligned} A &= \left\{s\in[0,T)\cap D_g:\ \left|1+\beta(s)\Delta^+ g(s)\right| \geq 1\right\} = \left\{s\in[0,T)\cap D_g:\ \ln\left|1+\beta(s)\Delta^+ g(s)\right| \geq 0\right\}, \\ B &= \left\{s\in[0,T)\cap D_g:\ \left|1+\beta(s)\Delta^+ g(s)\right| < 1\right\} = \left\{s\in[0,T)\cap D_g:\ \ln\left|1+\beta(s)\Delta^+ g(s)\right| < 0\right\}. \end{aligned}$$

In order to bound the sum on A it is enough to take into account that $0 \leq \ln(1+x) \leq x$ for every $x \in [0,\infty)$:

$$\sum_{s\in A} \left|\ln\left|1+\beta(s)\Delta^+ g(s)\right|\right| = \sum_{s\in A} \ln\left|1+\beta(s)\Delta^+ g(s)\right| \leq \sum_{s\in A} \ln\left(1+|\beta(s)|\Delta^+ g(s)|\right) \leq \sum_{s\in A} |\beta(s)|\Delta^+ g(s) < \infty,$$

because $\beta \in \mathcal{L}^1_{g^B}([0,T),\mathbb{F})$.

Now, let us focus on the sum on B. For any $s \in B$, taking into account that $1+\beta(s)\Delta^+ g(s) \neq 0$, we have that

$$\begin{aligned} 0 < \left|1+\beta(s)\,\Delta^+ g(s)\right|^2 &= [1+\mathrm{Re}(\beta(s))\,\Delta^+ g(s)]^2 + [\mathrm{Im}(\beta(s))\,\Delta^+ g(s)]^2 \\ &= 1+2\ \mathrm{Re}(\beta(s))\,\Delta^+ g(s) + |\beta(s)\,\Delta^+ g(s)|^2 < 1. \end{aligned}$$

In particular, $2\ \mathrm{Re}(\beta(s))\,\Delta^+ g(s) + |\beta(s)\,\Delta^+ g(s)|^2 < 0$ which yields $\mathrm{Re}(\beta(s)) < 0$. Now, we can consider the following sets:

$$\begin{aligned} B_1 &= \left\{s\in B:\ 0 < 1+2\ \mathrm{Re}(\beta(s))\Delta^+ g(s) + |\beta(s)\Delta^+ g(s)|^2 < \frac{1}{2}\right\}, \\ B_2 &= \left\{s\in B:\ \frac{1}{2} \leq 1+2\ \mathrm{Re}(\beta(s))\Delta^+ g(s) + |\beta(s)\Delta^+ g(s)|^2 < 1\right\}. \end{aligned}$$

Observe that $B = B_1 \cup B_2$. The definition of B_1 implies that

$$1 > 2\,|\,\mathrm{Re}(\beta(s))|\Delta^+ g(s) - |\beta(s)\Delta^+ g(s)|^2 > \frac{1}{2},\ \forall s \in B_1.$$

Therefore,

$$|\operatorname{Re}(\beta(s))|\Delta^+ g(s) > \frac{1}{4},\ \forall s \in B_1.$$

Hence, we have that B_1 is finite since, otherwise, we would have that $\beta \notin \mathcal{L}^1_{g^B}([0,T),\mathbb{F})$, which is a contradiction. For the elements in the set B_2 we have that:

$$\frac{1}{2} \geq 2\,|\operatorname{Re}(\beta(s))|\Delta^+ g(s) - |\beta(s)\Delta^+ g(s)|^2 > 0,\ \forall s \in B_2.$$

Thus, if we take into account that $\ln(1/(1-x)) \leq 2x$, for every $x \in [0,1/2]$,

$$\begin{aligned}\left|\ln\left|1+\beta(s)\Delta^+ g(s)\right|\right| &= \frac{1}{2}\left|\ln\left(1+2\operatorname{Re}(\beta(s))\Delta^+ g(s) + |\beta(s)\Delta^+ g(s)|^2\right)\right| \\ &= \frac{1}{2}\ln\left(1/\left(1+2\operatorname{Re}(\beta(s))\Delta^+ g(s) + |\beta(s)\Delta^+ g(s)|^2\right)\right) \\ &= \frac{1}{2}\ln\left(1/\left(1-\left(2\,|\operatorname{Re}(\beta(s))|\Delta^+ g(s) - |\beta(s)\Delta^+ g(s)|^2\right)\right)\right) \\ &\leq 2\left(2\,|\operatorname{Re}(\beta(s))|\Delta^+ g(s) - |\beta(s)\Delta^+ g(s)|^2\right) \leq 4|\operatorname{Re}(\beta(s))|\Delta^+ g(s).\end{aligned}$$

Hence,

$$\sum_{s\in B}\left|\ln\left|1+\beta(s)\Delta^+ g(s)\right|\right| < \infty.$$

Let us now bound the term associated with the argument. Taking into account that $|\operatorname{atan}(x)| \leq |x|$ for every $x \in \mathbb{R}$, we have that

$$\sum_{s\in[0,T)\cap D_g}\left|\operatorname{Arg}(1+\beta(s)\Delta^+ g(s))\right| \leq \sum_{s\in[0,T)\cap D_g}\frac{|\operatorname{Im}(\beta(s))\Delta^+ g(s)|}{|1+\operatorname{Re}(\beta(s))\Delta^+ g(s)|}.$$

Let us divide the set $[0,T)\cap D_g$ into the subsets

$$\begin{aligned}\widetilde{B}_1 &= \left\{s \in [0,T)\cap D_g \ :\ |\operatorname{Re}(\beta(s))|\Delta^+ g(s) > 1/2\right\}, \\ \widetilde{B}_2 &= ([0,T)\cap D_g)\backslash\widetilde{B}_1.\end{aligned}$$

Observe that $\widetilde{B}_1$ must be of finite cardinality. On the other hand, given $t \in \widetilde{B}_2$,

$$\left|1+\operatorname{Re}(\beta(s))\Delta^+ g(s)\right| \geq \frac{1}{2}.$$

Thus,

$$\sum_{s\in B_2}\frac{|\operatorname{Im}(\beta(s))\Delta^+ g(s)|}{|1+\operatorname{Re}(\beta(s))\Delta^+ g(s)|} \leq 2\sum_{s\in B_2}\left|\operatorname{Im}(\beta(s))\Delta^+ g(s)\right| < \infty.$$

Hence, we conclude that H is well defined. In order to prove that H is a solution of (4.12), we observe that, given $t \in [0,T)\cap D_g$,

$$H(t^+) = \lim_{s\to t^+}\exp\left(\int_{[0,s)}\frac{\ln(1+\beta(s)\Delta^+ g(s))}{\Delta^+ g(s)}\,\mathrm{d}\,\mu_{g^B}\right)$$

$$=\lim_{s\to t^+}\exp\left(\int_{[0,t)}\frac{\ln(1+\beta(s)\Delta^+g(s))}{\Delta^+g(s)}\,\mathrm{d}\,\mu_{g^B}+\ln(1+\beta(t)\Delta^+g(t))+\int_{(t,s)}\frac{\ln(1+\beta(s)\Delta^+g(s))}{\Delta^+g(s)}\,\mathrm{d}\,\mu_{g^B}\right)$$

$$=(1+\beta(t)\Delta^+g(t))\exp\left(\int_{[0,t)}\frac{\ln(1+\beta(s)\Delta^+g(s))}{\Delta^+g(s)}\,\mathrm{d}\,\mu_{g^B}\right)=(1+\beta(t)\Delta^+g(t))H(t),$$

so equation (4.14) holds and $v^B:=H$ is a solution of (4.12). Observe that, given any set $A\subset[0,T)\backslash D_g$, we have $A=(A\backslash\overline{D_g})\cup(A\cap(\overline{D_g}\backslash D_g))$ thus $\mu^*_{g^B}(A)\le\mu^*_{g^B}(A\backslash\overline{D_g})+\mu^*_{g^B}(A\cap(\overline{D_g}\backslash D_g))\le\mu^*_{g^B}(A\backslash\overline{D_g})+\mu^*_g(\overline{D_g}\backslash D_g)=0$. Therefore v^B satisfies (4.5) and, moreover,

$$v^B(t)=\exp\left(\sum_{s\in[0,t)\cap D_g}\ln(1+\beta(s)\Delta^+g(s))\right)=\prod_{s\in[0,t)\cap D_g}\left(1+\beta(s)\Delta^+g(s)\right).$$

Let us now study equation (4.13). First, observe that, given an element $t\in[0,T)\backslash(\overline{D_g}\cup C_g)$, there exists $\delta>0$ such that g is continuous on $(t-\delta,t+\delta)$. In the case $t\in N_g^-$ we further know that g is strictly increasing on the interval $(t-\delta,t]$, and constant on $(t,t+\delta)$. In the case $t\in N_g^+$, g would be constant on $(t-\delta,t)$ and strictly increasing on $[t,t+\delta)$. In any case (observe that, if $t\in N_g^-$ we have to take the limit from the left and in the case $t\in N_g^+$ the limit from the right, respectively):

$$(v^C)'_g(t)=\lim_{s\to t}\frac{v^C(s)-v^C(t)}{g(s)-g(t)}=\lim_{s\to t}\frac{v^C(s)-v^C(t)}{g^C(s)-g^C(t)}=(v^C)'_{g^C}(t).$$

Hence, taking into account that $\mu^*_g(A)=0\Leftrightarrow\mu^*_{g^C}(A)=0$ for any $A\subset[0,T)\backslash\overline{D_g}$, together with the fact that $C_g=C_{g^C}$, we see that equation (4.13) is equivalent to

$$(v^C)'_{g^C}(t)=\beta(t)\,v^C(t),\ g^C-a.e.\,t\in[0,T)\backslash(\overline{D_g}\cup C_{g^C}). \tag{4.15}$$

Let us observe that $\mu_{g^C}(\overline{D_g}\cup C_{g^C})\le\mu_{g^C}(\overline{D_g}\backslash D_g)+\mu_{g^C}(D_g)+\mu_{g^C}(C_{g^C})=0$, since $\mu_{g^C}(\overline{D_g}\backslash D_g)\le\mu_g(\overline{D_g}\backslash D_g)=0$ by hypothesis. Therefore, (4.15) is equivalent to:

$$(v^C)'_{g^C}(t)=\beta(t)\,v^C(t),\ g^C-a.e.\,t\in[0,T). \tag{4.16}$$

Now we will see that $v^C(t):=u(g^C(t))$, with $u\in\mathcal{AC}([0,g^C(T)];\mathbb{F})$ the solution of (4.9) satisfies equation (4.16). On the one hand, we have that $\widehat{\beta}=\beta\circ\gamma\in\mathcal{L}^1([0,g^C(T)];\mathbb{F})$. Indeed, the measurability is a consequence of Proposition 2.7. Now, using a similar argument as the one in the proof of Corollary 2.8:

$$\int_{[0,g^C(T))}|\widehat{\beta}|\,\mathrm{d}\,\mu=\int_{[0,T)}|\beta|\,\mathrm{d}\,\mu_{g^C}\le\int_{[0,T)}|\beta|\,\mathrm{d}\,\mu_g<\infty.$$

Thus, (4.9) admits a unique solution

$$u(t)=v_0\exp\left(\int_{[0,t)}\widehat{\beta}(s)\,\mathrm{d}\,\mu\right)\in\mathcal{AC}([0,g^C(T)];\mathbb{F}).$$

In particular, $v^C(t)=u(g^C(t))$ is such that $(v^C)'_g(t)=0$ for every $t\in D_g$. Indeed,

$$(v^C)'_g(t) = \lim_{s\to t^+} \frac{v^C(s) - v^C(t)}{g(s) - g(t)} = \lim_{s\to t^+} \frac{u(g^C(s)) - u(g^C(t))}{g(s) - g(t)} = 0,$$

thanks to the continuity of the composition $u \circ g^C$. On the other hand, since $u \in \mathcal{AC}([0, g^C(T)]; \mathbb{F})$ is the solution of (4.9), there exists a Lebesgue-null set $N \subset [0, g^C(T)]$ such that

$$u'(t) = \widehat{\beta}(t)\, u(t),\ \forall t \in [0, g^C(T)]\backslash N.$$

In particular,

$$u'(g^C(t)) = \widehat{\beta}(g^C(t))\, u(g^C(t)),\ \forall t \in [0,T]\backslash (g^C)^{-1}(N),$$

whence, by Proposition 4.1,

$$(v^C)'_{g^C}(t) = u'(g^C(t)) = \widehat{\beta}(g^C(t))\, u(g^C(t)),\ \forall t \in [0,T]\backslash (g^C)^{-1}(N).$$

Taking into account that $\gamma(g^C(t)) = t$ for every $t \in [0,T]\backslash(C_{g^C} \cup N^+_{g^C})$, that $\mu_{g^C}(C_{g^C} \cup N^+_{g^C}) = 0$ and that $\mu_{g^C}((g^C)^{-1}(N)) = 0$ (see the proof of Proposition 2.7), we deduce that

$$(v^C)'_{g^C}(t) = \beta(t)\, v^C(t),\ g^C\text{-a.e. } t \in [0,T).$$

Last, in regard to v^C, using a reasoning similar to the one used in the proof of the Corollary 2.8, we have that

$$v^C(t) = u(g^C(t)) = v_0 \exp\left(\int_{[0,g^C(t))} \widehat{\beta}(s)\ \mathrm{d}\,\mu \right) = v_0 \exp\left(\int_{[0,t)} \beta(s)\ \mathrm{d}\,\mu_{g^C} \right).$$

Finally, let us check that $v := v^C\, v^B$ is in the space $\mathcal{AC}_g([0,T]; \mathbb{F})$. To show this, let us define

$$\widetilde{\beta}(t) = \begin{cases} \beta(t), & t \in [0,T)\backslash D_g, \\ \dfrac{\ln(1 + \beta(t)\Delta^+ g(t))}{\Delta^+ g(t)}, & t \in [0,T) \cap D_g, \end{cases}$$

and check that

$$v(t) = v_0 \exp\left(\int_{[0,t)} \widetilde{\beta}(s)\ \mathrm{d}\,\mu_g \right) = v_0 \exp\left(\int_{[0,t)\backslash D_g} \widetilde{\beta}(s)\ \mathrm{d}\,\mu_g + \sum_{s\in[0,t)\cap D_g} \widetilde{\beta}(s)\Delta^+ g(s) \right).$$

Indeed, on the one hand,

$$\begin{aligned} v(t) &= v_0 \left[\prod_{s\in[0,t)\cap D_g} (1 + \beta(s)\Delta^+ g(s)) \right] \exp\left(\int_{[0,g^C(t))} \widehat{\beta}(s)\ \mathrm{d}\,\mu \right) \\ &= v_0 \exp\left(\int_{[0,g^C(t))} \widehat{\beta}(s)\ \mathrm{d}\,\mu + \sum_{s\in[0,t)\cap D_g} \frac{\ln(1 + \beta(s)\Delta^+ g(s))}{\Delta^+ g(s)} \Delta^+ g(s) \right). \end{aligned}$$

Now, thanks to the fact that $\mu(D_g) = 0$ as it is a countable set, we see that

$$\int_{[0,g^C(t))} (\beta\circ\gamma)(s)\, \mathrm{d}\,\mu = \int_{[0,g^C(t))} (\widetilde{\beta}\circ\gamma)(s)\, \mathrm{d}\,\mu.$$

Thus, by Corollary 2.8,

$$\int_{[0,t)} \widetilde{\beta}(s)\, \mathrm{d}\,\mu_g = \int_{[0,g^C(t))} \widehat{\beta}(s)\, \mathrm{d}\,\mu + \sum_{s\in[0,t)\cap D_g} \frac{\ln(1+\beta(s)\Delta^+ g(s))}{\Delta^+ g(s)}\Delta^+ g(s).$$

Finally, it is clear that $\widetilde{\beta}\in\mathcal{L}^1_g([0,T);\mathbb{F})$, therefore $v\in\mathcal{AC}_g([0,T);\mathbb{F})$. □

Remark 4.3. We must take into account the following remarks:

1. If g^C is constant, then the solution of (4.1) is reduced to $v_0\, v^B$ and the hypothesis $\mu_g(\overline{D_g}\backslash D_g)=0$ is not necessary.

2. The hypothesis $\mu_g(\overline{D_g}\backslash D_g)=0$ that appears in the statement of Theorem 4.2 has been used to express the solution of (4.1) as the product of the solutions of the problems (4.5) and (4.7). This hypothesis is not essential to guarantee the existence of a solution of problem (4.1). Even in the case $\mu_g(\overline{D_g}\backslash D_g)\neq 0$, we will have (4.10) is well defined and a valid solution of problem (4.1). Indeed, since $\widetilde{\beta}\in\mathcal{L}^1_g([0,T);\mathbb{F})$, we have that

$$\left(\int_{[0,t)} \widetilde{\beta}(s)\, \mathrm{d}\,\mu_g\right)'_g (t) = \widetilde{\beta}(t),\ g\text{-a.e. } t\in[0,T).$$

Therefore, (4.2) ensures that

$$\left(\exp\left(\int_{[0,t)} \widetilde{\beta}(s)\, \mathrm{d}\,\mu_g\right)\right)'_g (t) = \beta(t)\, \exp\left(\int_{[0,t)} \beta(s)\, \mathrm{d}\,\mu_g\right),\ g\text{-a.e. } t\in[0,T)\setminus D_g. \tag{4.17}$$

Now, given $t\in[0,T)\cap D_g$,

$$\begin{aligned}\lim_{s\to t^+} \exp\left(\int_{[0,s)} \widetilde{\beta}(s)\, \mathrm{d}\,\mu_g\right) &= \lim_{s\to t^+} \exp\left(\int_{[0,t)} \widetilde{\beta}(s)\, \mathrm{d}\,\mu_g + \ln(1+\beta(t)\Delta^+ g(t)) + \int_{(t,s)} \widetilde{\beta}(s)\, \mathrm{d}\,\mu_g\right)\\ &= (1+\beta(t)\Delta^+ g(t))\, \exp\left(\int_{[0,t)} \widetilde{\beta}(s)\, \mathrm{d}\,\mu_g\right),\end{aligned}$$

so equation (4.17) is also satisfied for the points of D_g.

Remark 4.4. The previous result is a generalization of the results in [3, Section 6] for several reasons.

1. The solution obtained is valid in the complex case, whereas in [3] it is only applied to the real case. The generalization to the complex case is immediate considering the complex exponential and the principal branch of the complex logarithm.

2. We have proven that the hypothesis

$$\sum_{s\in[0,T)\cap D_g} \big|\ln|1+\beta(s)\Delta^+ g(s)|\big| < \infty$$

occurring in [3, Definition 6.1 and Lemma 6.5] is not necessary, it being a direct consequence of $\beta \in \mathcal{L}^1_g([0,T);\mathbb{F})$ and $g(T) < \infty$. This was also proven in [13, Lemma 3.1] for the real case.

3. The solution obtained generalizes that in [3, Lemma 6.5]. Indeed, in the particular case $\beta \in \mathcal{L}^1_g([0,T);\mathbb{R})$ and given that $1+\beta(t)\Delta^+ g(t) \neq 0$ for every $t \in [0,T)\cap D_g$, we have that, for every $t \in [0,T)\cap D_g$,

$$\operatorname{Arg}(1+\beta(t)\Delta^+ g(t)) = \begin{cases} \pi, & 1+\beta(t)\Delta^+ g(t) < 0, \\ 0, & 1+\beta(t)\Delta^+ g(t) > 0. \end{cases}$$

Hence, if we write $T^-_\beta := \{t \in [0,T)\cap D_g :\ 1+\beta(t)\Delta^+ g(t) < 0\}$ and $T^+_\beta = \{t \in [0,T)\cap D_g :\ 1+\beta(t)\Delta^+ g(t) > 0\}$ (observe that T^-_β is of finite cardinality), we have that

$$\ln\left(1+\beta(t)\Delta^+ g(t)\right) = \begin{cases} \ln\left|1+\beta(t)\Delta^+ g(t)\right|, & t \in T^+_\beta, \\ \ln\left|1+\beta(t)\Delta^+ g(t)\right| + i\pi, & t \in T^-_\beta. \end{cases}$$

Taking into account the previous observations,

$$\begin{aligned} v(t) &= \exp\left(\int_{[0,t)\setminus D_g} \beta(s)\, \mathrm{d}\,\mu_g + \sum_{t\in[0,t)\cap D_g} \ln\left|1+\beta(t)\Delta^+ g(t)\right| + i \sum_{s\in[0,t)\cap T^-_\beta} \pi \right) \\ &= \cos\left(\sum_{s\in[0,t)\cap T^-_\beta} \pi \right) \exp\left(\int_{[0,t)\setminus D_g} \beta(s)\, \mathrm{d}\,\mu_g + \sum_{t\in[0,t)\cap D_g} \ln\left|1+\beta(t)\Delta^+ g(t)\right| \right). \end{aligned}$$

Hence, if $T^-_\beta = \{t_1,\dots,t_k\}$ and $t_{k+1} := T$, we get

$$v(t) = \begin{cases} \exp\left(\displaystyle\int_{[0,t)\setminus D_g} \beta(s)\, \mathrm{d}\,\mu_g + \sum_{t\in[0,t)\cap D_g} \ln\left|1+\beta(t)\Delta^+ g(t)\right| \right), & t \in [0,t_1], \\ \cos(j\,\pi)\exp\left(\displaystyle\int_{[0,t)\setminus D_g} \beta(s)\, \mathrm{d}\,\mu_g + \sum_{t\in[0,t)\cap D_g} \ln\left|1+\beta(t)\Delta^+ g(t)\right| \right), & \begin{array}{l} t \in (t_j, t_{j+1}], \\ j = 1,\dots,k, \end{array} \end{cases}$$

which is precisely the solution in [3, Lemma 6.5].

4. In the case that there exists some element $t \in [0,T)\cap D_g$ such that $1+\beta(t)\Delta^+ g(t) = 0$, the set

$$T^0_\beta := \{t \in [0,T)\cap D_g :\ 1+\beta(t)\Delta^+ g(t) = 0\}$$

is of finite cardinality and, therefore, if we denote by $t^0_\beta := \min T^0_\beta$ if $T^0_\beta \neq \emptyset$, $t^0_\beta := T$ otherwise, we have that

$$v(t) = \begin{cases} u(g^C(t)) \displaystyle\prod_{s\in[0,t)\cap D_g} \left(1+\beta(s)\Delta^+ g(s)\right), & t \in [0, t^0_\beta], \\ 0, & t \in (t^0_\beta, T]. \end{cases}$$

Taking into account that we are assuming that g is continuous at $t = 0$, we have that $t^0_\beta = \min T^0_\beta > 0$. Thus, $v(t) \neq 0$ for every $t \in [0, t^0_\beta]$.

Definition 4.5. Given an element $\beta \in \mathcal{L}^1_g([0,T);\mathbb{F})$ and $v_0 = 1$, we denote the solution of problem (4.1) constructed in Theorem 4.2 by $\exp_g(\beta;0,t) \in \mathcal{AC}_g([0,T];\mathbb{F})$ and call it the *complex g-exponential* map or just *g-exponential* map.

In the following result we present some important properties of the complex g-exponential function.

Proposition 4.6. *Let $\beta, \beta_1, \beta_2 \in \mathcal{L}^1_g([0,T);\mathbb{F})$. The following properties hold:*

1. *If $a = \operatorname{Re}\beta$ and $b = \operatorname{Im}\beta$ then*

$$\begin{aligned}\exp_g(\beta;0,t) = &\prod_{u\in[0,t)\cap D_g} \big(1 + a(u)\Delta^+ g(u) + ib(u)\Delta^+ g(u)\big) \exp\left(\int_{[0,g^C(t))} (a\circ\gamma)\,\mathrm{d}\,\mu\right) \\ &\cdot \left[\cos\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right) + i\sin\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right)\right].\end{aligned} \tag{4.18}$$

2. $\overline{\exp_g(\beta;0,t)} = \exp_g(\overline{\beta};0,t)$, *for every* $t \in [0,T]$.
3. *Given $n \in \mathbb{N}$, $\exp_g(\beta;0,t)^n = \exp_g(p_n(\beta);0,t) \in \mathcal{AC}_g([0,T];\mathbb{F})$, where*

$$p_n(\beta)(t) = n\,\beta(t) + \sum_{k=2}^{n} \binom{n}{k} \beta(t)^k \Delta^+ g(t)^{k-1}, \quad n \in \mathbb{N}.$$

4. *Given $n \in \mathbb{N}$, $\exp_g(\beta;0,t)^{-n} = \exp_g(q_n(\beta);0,t) \in \mathcal{AC}_g([0,t^0_\beta];\mathbb{F})$, where*

$$q_n(\beta)(t) = -\frac{p_n(\beta)(t)}{1 + p_n(\beta)(t)\,\Delta^+ g(t)}, \quad n \in \mathbb{N}.$$

   Observe that $\exp_g(\beta;0,t)^{-n}$ is not well defined in $(t^0_\beta, T]$ since $\exp_g(\beta;0,\cdot) = 0$ in that set.
5. *For all $t \in [0,T)$,*

$$\exp_g(\beta_1;0,t)\exp_g(\beta_2;0,t) = \exp_g(\beta_1 + \beta_2 + \beta_1\beta_2\Delta^+ g;0,t). \tag{4.19}$$

Proof. 1. Indeed,

$$\begin{aligned}&\exp_g(a+bi;0,t) \\ &= \exp\left(\int_{[0,t)\setminus D_g} a(s)\,\mathrm{d}\,\mu_g + i\int_{[0,t)\setminus D_g} b(s)\,\mathrm{d}\,\mu_g\right)\exp\left(\sum_{u\in[0,t)\cap D_g} \ln(1 + (a(u)+ib(u))\Delta^+ g(u))\right) \\ &= \exp\left(\int_{[0,t)\setminus D_g} a(s)\,\mathrm{d}\,\mu_g\right)\prod_{u\in[0,t)\cap D_g}\big(1 + a(u)\Delta^+ g(u) + ib(u)\Delta^+ g(u)\big) \\ &\quad\cdot\left[\cos\left(\int_{[0,t)\setminus D_g} b(s)\,\mathrm{d}\,\mu_g\right) + i\sin\left(\int_{[0,t)\setminus D_g} b(s)\,\mathrm{d}\,\mu_g\right)\right].\end{aligned}$$

Now the formula is obtained reasoning as in Corollary 2.8.

2. This property is clear from the definition of the complex conjugate.
3. Observe that $p_n(\beta) \in \mathcal{L}^1_g([0,T);\mathbb{F})$ since

$$\|p_n(\beta)\|_{\mathcal{L}^1_g([0,T);\mathbb{F})} \leq n\,\|\beta\|_{\mathcal{L}^1_g([0,T)} + \sum_{k=2}^{n}\binom{n}{k}\sum_{t\in[0,T)\cap D_g}\big(|\beta(t)|\Delta^+g(t)\big)^k < \infty.$$

Thus the solution of problem (4.1) where we consider $p_n(\beta)$ instead of β is given by $v = v^B v^C$ where

$$\begin{aligned} v^B(t) &= \prod_{s\in[0,t)\cap D_g}\left(1+\left(n\,\beta(s)+\sum_{k=2}^{n}\binom{n}{k}\beta(s)^k\Delta^+g(s)^{k-1}\right)\Delta^+g(s)\right),\\ &= \prod_{s\in[0,t)\cap D_g}\left(1+n\,\beta(s)\Delta^+g(s)+\sum_{k=2}^{n}\binom{n}{k}\beta(s)^k\Delta^+g(s)^k\right),\\ &= \prod_{s\in[0,t)\cap D_g}\left(\sum_{k=0}^{n}\binom{n}{k}\beta(s)^k\Delta^+g(s)^k\right) = \prod_{s\in[0,t)\cap D_g}\big(1+\beta(s)\Delta^+g(s)\big)^n\\ &= \left(\prod_{s\in[0,t)\cap D_g}\big(1+\beta(s)\Delta^+g(s)\big)\right)^n,\\ v^C(t) &= \exp\left(\int_{[0,t)}\left(n\,\beta(s)+\sum_{k=2}^{n}\binom{n}{k}\beta(s)^k\Delta^+g(s)^{k-1}\right)\mathrm{d}\,\mu_{g^C}\right) = \exp\left(n\int_{[0,t)}\beta(s)\,\mathrm{d}\,\mu_{g^C}\right)\\ &= \left[\exp\left(\int_{[0,t)}\beta(s)\,\mathrm{d}\,\mu_{g^C}\right)\right]^n. \end{aligned}$$

Hence, $\exp_g(\beta;0,t)^n = \exp_g(p_n(\beta);0,t)$.

4. Observe that $q_n(\beta) \in \mathcal{L}^1_g([0,T);\mathbb{F})$ since

$$\|q_n\|_{\mathcal{L}^1_g([0,t^0_\beta T);\mathbb{F})} \leq n\,\|\beta\|_{\mathcal{L}^1_g([0,T)} + \sum_{t\in[0,T)\cap D_g}\frac{|p_n(\beta)(t)\,\Delta^+g(t)|}{|1+p_n(\beta)(t)\,\Delta^+g(t)|} < \infty,$$

because $p_n(\beta) \in \mathcal{L}^1_g([0,T);\mathbb{F})$. Therefore, the solution to problem (4.1), where we consider $q_n(\beta)$ instead of β, is given by $v = v^B v^C$ where

$$\begin{aligned} v^B(t) &= \prod_{s\in[0,t)\cap D_g}\left(1-\frac{n\,\beta(s)+\sum_{k=2}^{n}\binom{n}{k}\beta(s)^k\Delta^+g(s)^{k-1}}{1+\left(n\,\beta(s)+\sum_{k=2}^{n}\binom{n}{k}\beta(s)^k\Delta^+g(s)^{k-1}\right)\Delta^+g(s)}\Delta^+g(s)\right),\\ &= \prod_{s\in[0,t)\cap D_g}\left(\frac{1}{1+\left(n\,\beta(s)+\sum_{k=2}^{n}\binom{n}{k}\beta(s)^k\Delta^+g(s)^{k-1}\right)\Delta^+g(s)}\right)\\ &= \left(\prod_{s\in[0,t)\cap D_g}\big(1+\beta(s)\Delta^+g(s)\big)\right)^{-n},\\ v^C(t) &= \exp\left(\int_{[0,t)}-\frac{n\,\beta(s)+\sum_{k=2}^{n}\binom{n}{k}\beta(s)^k\Delta^+g(s)^{k-1}}{1+\left(n\,\beta(s)+\sum_{k=2}^{n}\binom{n}{k}\beta(s)^k\Delta^+g(s)^{k-1}\right)\Delta^+g(s)}\,\mathrm{d}\,\mu_{g^C}\right) \end{aligned}$$

$$= \exp\left(\int_{[0,t)} -n\,\beta(s)\,\mathrm{d}\,\mu_{g^C}\right) = \left(\exp\left(\int_{[0,t)} \beta(s)\,\mathrm{d}\,\mu_{g^C}\right)\right)^{-n}.$$

Hence, $\exp_g(\beta;0,t)^{-n} = \exp_g(q_n(\beta);0,t)$.

5. Observe that $\tilde{\beta} = \beta_1 + \beta_2 + \beta_1\beta_2\Delta^+ g \in \mathcal{L}^1_g([0,T);\mathbb{F})$. Indeed,

$$\begin{aligned}\|\tilde{\beta}\|_{\mathcal{L}^1_g([0,T);\mathbb{F})} \le& \|\beta_1\|_{\mathcal{L}^1_g([0,T);\mathbb{F})} + \|\beta_2\|_{\mathcal{L}^1_g([0,T);\mathbb{F})} + \|\beta_1\beta_2\Delta^+ g\|_{\mathcal{L}^1_g([0,T);\mathbb{F})}\\ \le& \|\beta_1\|_{\mathcal{L}^1_g([0,T);\mathbb{F})} + \|\beta_2\|_{\mathcal{L}^1_g([0,T);\mathbb{F})} + \frac{1}{2}\sum_{s\in[0,T)\cap D_g}\left(|\beta_1(s)|^2\Delta^+ g(s)^2 + |\beta_2(s)|^2\Delta^+ g(s)^2\right).\end{aligned}$$

Now, since $\beta_k \in \mathcal{L}^1_g([0,T);\mathbb{F})$, we have that $\sum_{s\in[0,T)\cap D_g}|\beta_k(s)|\Delta^+ g(s) < \infty$, $k=1,2$. Thus it is clear that $\sum_{s\in[0,T)\cap D_g}|\beta_k(s)|^2\Delta^+ g(s)^2 < \infty$, $k=1,2$, and we obtain that $\|\tilde{\beta}\|_{\mathcal{L}^1_g([0,T);\mathbb{F})} < \infty$. Therefore, the solution to problem (4.1) associated to $\tilde{\beta}$ is given by:

$$\begin{aligned}\exp_g(\tilde{\beta};0,t) =& \prod_{s\in[0,t)\cap D_g}\left(1+\tilde{\beta}(s)\Delta^+ g(s)\right)\exp\left(\int_{[0,t)}\tilde{\beta}(s)\,\mathrm{d}\,\mu_{g^C}\right)\\ =& \prod_{s\in[0,t)\cap D_g}\left(1+\beta_1(s)\Delta^+ g(s)\right)\left(1+\beta_2(s)\Delta^+ g(s)\right)\exp\left(\int_{[0,t)}(\beta_1(s)+\beta_2(s))\,\mathrm{d}\,\mu_{g^C}\right)\\ =& \prod_{s\in[0,t)\cap D_g}\left(1+\beta_1(s)\Delta^+ g(s)\right)\exp\left(\int_{[0,t)}\beta_1(s)\,\mathrm{d}\,\mu_{g^C}\right)\\ &\times\prod_{s\in[0,t)\cap D_g}\left(1+\beta_2(s)\Delta^+ g(s)\right)\exp\left(\int_{[0,t)}\beta_2(s)\,\mathrm{d}\,\mu_{g^C}\right)\\ =& \exp_g(\beta_1;0,t)\,\exp_g(\beta_2;0,t). \quad \square\end{aligned}$$

Remark 4.7. We must mention that both the expression for the g-exponential (4.10), and the properties 3 (for $n=2$) and 4 (for $n=1$) of the Proposition 4.6 can be obtained as particular cases of Proposition 8.5.4 and Theorems 8.5.6 and 8.5.8 in [18], respectively. In our case we have developed the theory within the framework of Stieltjes differential equations to keep it self-contained.

4.2. g-Sine and g-cosine

Let us see now how to use the complex g-exponential map in order to define the *g-sine* and *g-cosine* functions. We observe that the presence of jumps in the derivator prevents us from expressing the exponential as the product of its real and imaginary parts. Indeed, thanks to (4.19)

$$\exp_g(a+bi;0,t) = \exp_g(a;0,t)\,\exp_g\left(\frac{i\,b}{1+a\,\Delta^+ g};0,t\right) \ne \exp_g(a;0,t)\,\exp_g(i\,b;0,t). \tag{4.20}$$

This fact will have its repercussion when we study the case of second order linear equations. In view of expression (4.18), it might be interesting to consider the case $a=0$, in order to define the g-sine and g-cosine.

Definition 4.8 *(g-sine and g-cosine).* Let $b \in \mathcal{L}^1_g([0,T];\mathbb{F})$. We define $\sin_g(b;0,t)$ and $\cos_g(b;0,t)$, as the first and second components, respectively, of the unique solution in $\mathcal{AC}_g([0,T];\mathbb{F}^2)$ of the following linear system:

$$\begin{cases} \begin{pmatrix} \sin_g(b;0,t) \\ \cos_g(b;0,t) \end{pmatrix}'_g (t) = \begin{pmatrix} 0 & b(t) \\ -b(t) & 0 \end{pmatrix} \begin{pmatrix} \sin_g(b;0,t) \\ \cos_g(b;0,t) \end{pmatrix}, \ g-a.e.\, t \in [0,T), \\ \sin_g(b;0,0) = 0, \ \cos_g(b;0,0) = 1. \end{cases} \tag{4.21}$$

Remark 4.9. Observe that (4.21) has, indeed, a unique solution in $\mathcal{AC}_g([0,T];\mathbb{F}^2)$ as it satisfies the conditions of [3, Theorem 7.3] with $L = |b|$.

Proposition 4.10. *Given $b \in \mathcal{L}^1_g([0,T];\mathbb{R})$, we have that*

$$\begin{aligned} \sin_g(b;0,t) &= \frac{\exp_g(bi;0,t) - \exp_g(-bi;0,t)}{2i}, \\ \cos_g(b;0,t) &= \frac{\exp_g(bi;0,t) + \exp_g(-bi;0,t)}{2}. \end{aligned} \tag{4.22}$$

Furthermore, developing the previous expressions,

$$\begin{aligned} &\sin_g(b;0,t) \\ &= \prod_{u\in[0,t)\cap D_g} \left|1+b(u)\,i\,\Delta^+g(u)\right| \sin\left(\sum_{u\in[0,t)\cap D_g} \operatorname{atan}(b(u)\,\Delta^+g(u))\right) \cos\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right) \\ &\quad + \prod_{u\in[0,t)\cap D_g} \left|1+b(u)\,i\,\Delta^+g(u)\right| \cos\left(\sum_{u\in[0,t)\cap D_g} \operatorname{atan}(b(u)\,\Delta^+g(u))\right) \sin\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right), \\ &\cos_g(b;0,t) \\ &= \prod_{u\in[0,t)\cap D_g} \left|1+b(u)\,i\,\Delta^+g(u)\right| \cos\left(\sum_{u\in[0,t)\cap D_g} \operatorname{atan}(b(u)\,\Delta^+g(u))\right) \cos\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right) \\ &\quad - \prod_{u\in[0,t)\cap D_g} \left|1+b(u)\,i\,\Delta^+g(u)\right| \sin\left(\sum_{u\in[0,t)\cap D_g} \operatorname{atan}(b(u)\,\Delta^+g(u))\right) \sin\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right). \end{aligned}$$

Proof. Indeed, differentiating the equations in (4.22),

$$\begin{aligned} \left(\sin_g(b;0,t)\right)'_g(t) &= \frac{1}{2i}\left(ib(t)\exp_g(bi;0,t) + ib(t)\exp_g(-bi;0,t)\right) \\ &= b(t)\frac{1}{2}\left(\exp_g(bi;0,t) + \exp_g(-bi;0,t)\right) \\ &= b(t)\cos_g(b;0,t), \ g-a.e.\, t\in[0,T). \\ \left(\cos(b;0,t)\right)'_g(t) &= \frac{1}{2}\left(ib(t)\exp_g(bi;0,t) - ib(t)\exp_g(-bi;0,t)\right) \\ &= -b(t)\frac{1}{2i}\left(\exp_g(bi;0,t) - \exp_g(-bi;0,t)\right) \\ &= -b(t)\sin_g(b;0,t), \ g-a.e.\, t\in[0,T). \end{aligned}$$

Observe also that $\overline{\exp_g(bi;0,t)} = \exp_g(-bi;0,t)$. Hence,

$$\begin{aligned}\cos_g(b;0,t) &= \mathrm{Re}(\exp_g(bi;0,t)),\\ \sin_g(b;0,t) &= \mathrm{Im}(\exp_g(bi;0,t)).\end{aligned}$$

In particular, we can obtain the explicit expression of the g-sine and the g-cosine separating the real and imaginary parts of $\exp_g(bi;0,t)$. We have that

$$\begin{aligned}\sin_g(b;0,t) = &\quad \cos\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right)\mathrm{Im}\left(\prod_{u\in[0,t)\cap D_g}\big(1+b(u)\,i\,\Delta^+g(u)\big)\right)\\ &+\sin\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right)\mathrm{Re}\left(\prod_{u\in[0,t)\cap D_g}\big(1+b(u)\,i\,\Delta^+g(u)\big)\right),\\ \cos_g(b;0,t) = &\quad \cos\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right)\mathrm{Re}\left(\prod_{u\in[0,t)\cap D_g}\big(1+b(u)\,i\,\Delta^+g(u)\big)\right)\\ &-\sin\left(\int_{[0,g^C(t))} (b\circ\gamma)\,\mathrm{d}\,\mu\right)\mathrm{Im}\left(\prod_{u\in[0,t)\cap D_g}\big(1+b(u)\,i\,\Delta^+g(u)\big)\right).\end{aligned}$$

Hence, in order to obtain the result it is enough to observe that

$$\begin{aligned}&\prod_{u\in[0,t)\cap D_g}\big(1+b(u)\,i\,\Delta^+g(u)\big) = \exp\left(\sum_{u\in[0,t)\cap D_g}\log\big(1+b(u)\,i\,\Delta^+g(u)\big)\right)\\ &=\exp\left(\sum_{u\in[0,t)\cap D_g}\log\big|1+b(u)\,i\,\Delta^+g(u)\big| + i\sum_{u\in[0,t)\cap D_g}\mathrm{Arg}\big(1+b(u)\,i\,\Delta^+g(u)\big)\right)\\ &=\exp\left(\sum_{u\in[0,t)\cap D_g}\log\big|1+b(u)\,i\,\Delta^+g(u)\big|\right)\exp\left(i\sum_{t\in[0,t)\cap D_g}\mathrm{atan}\big(b(u)\,\Delta^+g(u)\big)\right)\\ &=\prod_{u\in[0,t)\cap D_g}\big|1+b(u)\,i\,\Delta^+g(u)\big|\\ &\quad\cdot\left[\cos\left(\sum_{u\in[0,t)\cap D_g}\mathrm{atan}(b(u)\,\Delta^+g(u))\right)+i\,\sin\left(\sum_{u\in[0,t)\cap D_g}\mathrm{atan}(b(u)\,\Delta^+g(u))\right]\right). \quad\square\end{aligned}$$

Remark 4.11. Observe that, in a similar way to Remark 4.7, Proposition 4.10 provides the expressions for $\cos_{dg}(t,0)$ and $\sin_{dg}(t,0)$ given in [18, Definition 8.5.13].

4.3. The non homogeneous case

In this section we will study the linear non homogeneous problem:

$$\begin{cases} v_g'(t) = \beta(t)\,v(t) + f(t), \ g-a.e.\,t\in[0,T),\\ v(0)=v_0,\end{cases} \tag{4.23}$$

where β, $f\in\mathcal{L}^1_g([0,T);\mathbb{F})$ and $v_0\in\mathbb{F}$. We have the following result whose proof can be achieved using the techniques employed in [13, Theorems 3.5 and 4.6].

Proposition 4.12. *Let β, $f \in \mathcal{L}^1_g([0,T);\mathbb{F})$. Then the map $v : [0, t^0_\beta] \to \mathbb{F}$ defined as:*

$$v(t) = v_0 \exp_g(\beta; 0, t) + \exp_g(\beta; 0, t) \int\limits_{[0,t)} \exp_g(\beta; 0, s)^{-1} \frac{f(s)}{1 + \beta(s)\Delta^+ g(s)} \,\mathrm{d}\,\mu_g, \tag{4.24}$$

is the unique solution in $\mathcal{AC}_g([0, t^0_\beta]; \mathbb{F})$ of problem (4.23) in the interval $[0, t^0_\beta)$.

Remark 4.13. Considering (4.23), observe that the set $A = \{s \in [0,T) \cap D_g :\ |1 + \beta(s)\Delta^+ g(s)| < 1/2\}$ has at most finite cardinality since $\beta \in \mathcal{L}^1_g([0,T];\mathbb{F})$. Indeed, given $s \in [0,T) \cap D_g$ such that $1 + \beta(s)\Delta^+ g(s) \neq 0$,

$$\begin{aligned} 0 < |1 + \beta(s)\Delta^+ g(s)| < \frac{1}{2} &\Leftrightarrow 0 < 1 + 2\operatorname{Re}(\beta(s))\Delta^+ g(s) + |\beta(s)\Delta^+ g(s)|^2 < \frac{1}{4} \\ &\Leftrightarrow \frac{3}{4} < 2|\operatorname{Re}(\beta(s))|\Delta^+ g(s) - |\beta(s)\Delta^+ g(s)|^2 < 1, \end{aligned}$$

so $|\operatorname{Re}(\beta(s))|\Delta^+ g(s) > 3/8$ and then A has finite cardinality. Therefore:

$$\sum_{s \in [0,T) \cap D_g} \frac{|f(s)|\Delta^+ g(s)}{|1 + \beta(s)\Delta^+ g(s)|} \le \sum_{s \in A} \frac{|f(s)|\Delta^+ g(s)}{|1 + \beta(s)\Delta^+ g(s)|} + 2 \sum_{s \in ([0,T) \cap D_g) \setminus A} |f(s)|\Delta^+ g(s) < \infty.$$

Remark 4.14. Observe that, by Proposition 4.12, if we define

$$G(t,s) = \exp_g(\beta; 0, t) \frac{\exp_g(\beta; 0, s)^{-1}}{1 + \beta(s)\Delta^+ g(s)} \chi_{[0,t)}(s), \quad t, s \in [0,T),$$

G is the *Green's function* associated to problem (4.23), that is, its solution can be expressed as

$$v(t) = v_0 \exp_g(\beta; 0, t) + \int\limits_{[0,T]} G(t,s) f(s) \,\mathrm{d}\,\mu_g(s).$$

Now, following the same idea as in the previous section, we will see that, under certain hypotheses on the set D_g, it is possible to decompose the solution of (4.23) in terms of the solution of two problems associated with the continuous and discrete parts of the derivator.

Corollary 4.15. *Assume $\mu_g(\overline{D_g} \setminus D_g) = 0$ and let β, $f \in \mathcal{L}^1_g([0,T);\mathbb{F})$. Then the unique g-absolutely continuous solution of (4.23) in the interval $[0, t^0_\beta)$ given by (4.24) can be expressed in the following terms:*

$$v(t) = v^C(t)\, \widetilde{v}^B(t) + v^B(t)\, \widetilde{v}^C(t), \ \forall t \in [0, t^0_\beta],$$

where

- $v^C \in \mathcal{AC}_{g^C}([0, t^0_\beta]; \mathbb{F})$ *is the unique solution of (4.7) in the interval $[0, t^0_\beta)$ given by (4.8),*
- $v^B \in \mathcal{AC}_{g^B}([0, t^0_\beta]; \mathbb{F})$ *is the unique solution of (4.5) in the interval $[0, t^0_\beta)$ given by (4.6),*
- $\widetilde{v}^B \in \mathcal{AC}_{g^B}([0, t^0_\beta]; \mathbb{F})$ *is the unique solution of*

$$\begin{cases} (\widetilde{v}^B)'_{g^B}(t) = \beta(t)\, \widetilde{v}^B(t) + \dfrac{f(t)}{v^C(t)(1 + \beta(t)\Delta^+ g(t))}, \ g^B - a.e.\, t \in [0, t^0_\beta), \\ \widetilde{v}^B(0) = \dfrac{1}{2}, \end{cases} \tag{4.25}$$

given by

$$\widetilde{v}^B(t)=v^B(t)\left[\frac{1}{2}+\sum_{s\in[0,t)\cap D_g} v^B(s)^{-1}\frac{f(s)\Delta^+g(s)}{v^C(s)(1+\beta(s)\Delta^+g(s))}\right],$$

- $\widetilde{v}^C\in\mathcal{AC}_{g^C}([0,t_\beta^0];\mathbb{F})$ *is the unique solution of*

$$\begin{cases}(\widetilde{v}^C)'_{g^C}(t)=\beta(t)\,\widetilde{v}^C(t)+\dfrac{f(t)}{v^B(t)},\ g^C-a.e.\,t\in[0,t_\beta^0),\\ \widetilde{v}^C(0)=\dfrac{v_0}{2},\end{cases}\tag{4.26}$$

given by

$$\widetilde{v}^C(t)=v^C(t)\left[\frac{v_0}{2}+\int_{[0,t)} v^C(s)^{-1}\frac{f(s)}{v^B(s)}\,\mathrm{d}\,\mu_{g^C}\right].$$

Proof. Let us consider the solution of (4.23) given by (4.24) and the decomposition $\exp_g(\beta;0,t)=v^B(t)v^C(t)$ given by Proposition 4.2. We have that

$$\begin{aligned}v(t)&=v^B(t)\,v^C(t)\left[v_0+\int_{[0,t)} v^C(s)^{-1}\,\frac{f(s)}{v^B(s)}\,\mathrm{d}\,\mu_{g^C}+\sum_{s\in[0,t)\cap D_g} v^B(s)^{-1}\,\frac{f(s)\Delta^+g(s)}{v^B(s)(1+\beta(s)\Delta^+g(s))}\right]\\
&=v^B(t)\,v^C(t)\left[\frac{v_0}{2}+\int_{[0,t)} v^C(s)^{-1}\,\frac{f(s)}{v^B(s)}\,\mathrm{d}\,\mu_{g^C}(s)\right]\\
&\quad+v^C(t)\,v^B(t)\left[\frac{v_0}{2}+\sum_{s\in[0,t)\cap D_g} v^B(s)^{-1}\,\frac{f(s)\Delta^+g(s)}{v^B(s)(1+\beta(s)\Delta^+g(s))}\right]\\
&=v^B(t)\,\widetilde{v}^C(t)+v^C(t)\,\widetilde{v}^B(t).\end{aligned}$$

Finally, by Proposition 4.12 we have that $\widetilde{v}^C\in\mathcal{AC}_{g^C}([0,t_\beta^0];\mathbb{F})$ is the unique solution of (4.26) and $\widetilde{v}^B\in\mathcal{AC}_{g^B}([0,t_\beta^0];\mathbb{F})$ is the unique solution of (4.25). □

4.4. Additional regularity

In order to correctly define regular solutions, throughout this section we will assume that $g:\mathbb{R}\to\mathbb{R}$ is a derivator such that $0\notin N_g^-$ and $T\notin N_g^+\cup D_g\cup C_g$. We also assume that $t_\beta^0=T$, otherwise, we redefine T by taking $\min\{T,t_\beta^0\}$.

Let us check now that we can obtain solutions of (4.1) with greater regularity in the case $\beta\in\mathcal{BC}_g([0,T];\mathbb{F})$. We need the following result, which we state for scalar equations.

Proposition 4.16 *([3, Proposition 7.6]). Let $x\in\mathcal{AC}_g([0,T];\mathbb{R})$ be a solution of*

$$x'_g(t)=f(t,x(t)),\quad g\text{-}a.a.\ t\in[0,T).$$

If $f(\cdot,x(\cdot))$ is g-continuous on $[0,T]$, then

$$x'_g(t)=f(t,x(t))\quad \textit{for all } t\in[0,T)\backslash C_g.$$

We have the following corollary.

Corollary 4.17. *Let* $\beta \in \mathcal{BC}_g([0,T];\mathbb{F})$*, then the problem*

$$\begin{cases} v_g'(t) - \beta(t)\, v(t) = 0, \ \forall\, t \in [0,T], \\ v(0) = v_0, \end{cases} \tag{4.27}$$

admits a unique solution in the space $\mathcal{BC}_g^1([0,T];\mathbb{F})$*.*

Proof. Indeed, on the one hand, we have that $\mathcal{BC}_g([0,T];\mathbb{F}) \subset \mathcal{L}_g^1([0,T);\mathbb{F})$, so there exists a unique solution $v \in \mathcal{AC}_g([0,T];\mathbb{F})$ given by (4.10). Let us see that $v \in \mathcal{BC}_g^1([0,T];\mathbb{F})$ and that v satisfies equation (4.27) on all of the interval $[0,T]$. First observe that $\beta\, v \in \mathcal{BC}_g([0,T];\mathbb{F})$, so, thanks to Proposition 4.16 we have that $v_g'(t) = \beta(t)\, v(t)$ for all $t \in [0,T) \setminus C_g$. Observe that we can extend the result to $t = T$ thanks to the fact that $T \notin N_g^+ \cup D_g$. Finally, thanks to Definition 3.7, we have the desired result since $v_g' = \beta\, v \in \mathcal{BC}_g([0,T];\mathbb{F})$. □

Remark 4.18. In the case where $\beta \in \mathcal{BC}_g^1([0,T])$ we cannot ensure that $v \in \mathcal{BC}_g^2([0,T];\mathbb{F})$ since the product of two $\mathcal{BC}_g^1([0,T];\mathbb{F})$ functions is not, in general, a $\mathcal{BC}_g^1([0,T];\mathbb{F})$ function. However, if β and its g-derivatives are also continuous, we can recover the desired regularity as a consequence of Corollary 3.18.

Corollary 4.19. *Let* $\beta \in \mathcal{BC}_g^n([0,T];\mathbb{F}) \cap \mathcal{BC}_{g^C}^{n-1}([0,T];\mathbb{F})$*, with* $n \in \mathbb{N}$*, then the problem* (4.27) *admits a unique solution in the space* $\mathcal{BC}_g^{n+1}([0,T];\mathbb{F})$*.*

Let us now consider the non homogeneous case.

Corollary 4.20. *Let* $\beta,\ f \in \mathcal{BC}_g([0,T];\mathbb{F})$*, then the problem*

$$\begin{cases} v_g'(t) = \beta(t)\, v(t) + f(t), \ \forall t \in [0,T], \\ v(0) = v_0, \end{cases} \tag{4.28}$$

admits a unique solution in the space $\mathcal{BC}_g^1([0,T];\mathbb{F})$*.*

Corollary 4.21. *Let* $\beta \in \mathcal{BC}_g^n([0,T];\mathbb{F}) \cap \mathcal{BC}_{g^C}^{n-1}([0,T];\mathbb{F})$ *and* $f \in \mathcal{BC}_g^n([0,T];\mathbb{F})$*, with* $n \in \mathbb{N}$*, then problem* (4.28) *admits a unique solution in the space* $\mathcal{BC}_g^{n+1}([0,T];\mathbb{F})$*.*

Example 4.22. Consider any derivator g and the equation

$$\begin{cases} v_g'(t) = x\, v(t) + \exp_g(z;0,t), \ \forall t \in [0,T], \\ v(0) = 1, \end{cases} \tag{4.29}$$

where $x, z \in \mathbb{F}$ are constants. Defining $\beta(t) := x,\ f(t) := \exp_g(z;0,t)$ we have that $\beta,\ f \in \mathcal{BC}_g^\infty([0,T];\mathbb{F})$. By Corollary 4.19, problem (4.29) has a unique solution $v \in \mathcal{C}_g^\infty([0,T];\mathbb{F})$, which, by Proposition 4.12, is provided by expression (4.24) as

$$v(t) = \exp_g(x;0,t) + \exp_g(x;0,t) \int\limits_{[0,t)} \exp_g(x;0,s)^{-1} \, \frac{\exp_g(z;0,s)}{1 + x\,\Delta^+ g(s)} \, \mathrm{d}\,\mu_g(s).$$

Now, by Proposition 4.6,

$$\begin{aligned}\exp_g(x;0,s)^{-1}\exp_g(z;0,s) &= \exp_g\left(-\frac{x}{1+x\,\Delta^+ g};0,s\right)\exp_g(z;0,s)\\ &= \exp_g\left(-\frac{x}{1+x\,\Delta^+ g}+z-\frac{x\,z\,\Delta^+ g}{1+x\,\Delta^+ g};0,s\right)=\exp_g\left(\frac{z-x}{1+x\,\Delta^+ g};0,s\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\int\limits_{[0,t)}\exp_g(x;0,s)^{-1}\frac{\exp_g(z;0,s)}{1+x\,\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s) &= \int\limits_{[0,t)}\frac{1}{1+x\,\Delta^+ g(s)}\exp_g\left(\frac{z-x}{1+x\,\Delta^+ g};0,s\right)\mathrm{d}\,\mu_g(s)\\ &=(z-x)^{-1}\int\limits_{[0,t)}\left(\exp_g\left(\frac{z-x}{1+x\,\Delta^+ g};0,\cdot\right)\right)_g'(s)\,\mathrm{d}\,\mu_g(s)\\ &=(z-x)^{-1}\left[\exp_g\left(\frac{z-x}{1+x\,\Delta^+ g};0,t\right)-\exp_g\left(\frac{z-x}{1+x\,\Delta^+ g};0,0\right)\right]\\ &=(z-x)^{-1}\left[\exp_g(x;0,t)^{-1}\exp_g(z;0,t)-1\right].\end{aligned}$$

Finally,

$$\begin{aligned}v(t) &= \exp_g(x;0,t)+\exp_g(x;0,t)\,(z-x)^{-1}\left[\exp_g(x;0,t)^{-1}\exp_g(z;0,t)-1\right]\\ &= \exp_g(x;0,t)+(z-x)^{-1}\left[\exp_g(z;0,t)-\exp_g(x;0,t)\right].\end{aligned}$$

Observe that, differentiating v_g' again, we obtain that $v_g''-(x+z)\,v_g'+x\,z\,v=0$, so, for any values P, $Q\in\mathbb{C}$, taking $x=(-P+\sqrt{P^2-4Q})/2$, $z=P-x$, v solves the equation $v_g''+P\,v_g'+Q\,v=0$.

This fact illustrates how we can obtain a solution of a second order problem from a first order problem. In the next section we study this type of problems.

5. Linear g-differential problems of second order with constant coefficients

In this section we consider g-differential problems of second order with constant coefficients. Since we will assume that the coefficients are constant, we will look for solutions in the space $\mathcal{BC}_g^2([0,T];\mathbb{F})$. Once again, we assume that $0\notin N_g^-$ and $T\notin N_g^+\cup D_g\cup C_g$.

5.1. The homogeneous case

Let us consider the second order homogeneous linear Cauchy problem

$$\begin{cases} v_g''(t)+P\,v_g'(t)+Q\,v(t)=0,\ \forall t\in[0,T],\\ v(0)=x_0,\\ v_g'(0)=v_0,\end{cases}\tag{5.1}$$

where P, Q, x_0, $v_0\in\mathbb{F}$. We start by defining what we understand as a solution of problem (5.1).

Definition 5.1. We say $v\in\mathcal{BC}_g^2([0,T];\mathbb{F})$ is a solution of (5.1) if it satisfies the equation

$$v_g''(t)+P\,v_g'(t)+Q\,v(t)=0,\ \forall t\in[0,T]$$

and the initial conditions $v(0)=x_0$ and $v_g'(0)=v_0$.

We have the following lemma, whose proof is straightforward from the linearity of the g-derivative.

Lemma 5.2. *Let* $x_0,\ v_0 \in \mathbb{F}$ *and* $v_1,\ v_2 \in \mathcal{BC}_g^2([0,T];\mathbb{F})$ *be such that*

$$(v_k)_g''(t) + P\,(v_k)_g'(t) + Q\,v_k(t) = 0,\ \forall t \in [0,T],\ k = 1,2. \tag{5.2}$$

If $v_1(0)\,(v_2)_g'(0) - v_2(0)\,(v_1)_g'(0) \neq 0$*, then* $v = c_1\,v_1 + c_2\,v_2$ *is a solution of* (5.1)*, where*

$$\begin{aligned} c_1 =& \frac{(v_2)_g'(0)\,x_0 - v_0\,v_2(0)}{v_1(0)\,(v_2)_g'(0) - v_2(0)\,(v_1)_g'(0)}, \\ c_2 =& \frac{v_0\,v_1(0) - (v_1)_g'(0)\,x_0}{v_1(0)\,(v_2)_g'(0) - v_2(0)\,(v_1)_g'(0)}. \end{aligned}$$

Theorem 5.3. *For* (5.1)*, the following hold:*

- *If* $P^2 - 4\,Q \neq 0$*, then, defining* $\lambda_1 = (-P + \sqrt{P^2 - 4\,Q})/2$ *and* $\lambda_2 = (-P - \sqrt{P^2 - 4\,Q})/2$*, we have that*

$$v(t) = \left(\frac{v_0 - \lambda_2\,x_0}{\lambda_1 - \lambda_2}\right) \exp_g(\lambda_1;0,t) - \left(\frac{v_0 - \lambda_1\,x_0}{\lambda_1 - \lambda_2}\right) \exp_g(\lambda_2;0,t)$$

  is a solution of (5.1)*. Furthermore,* $v \in \mathcal{BC}_g^\infty([0,T];\mathbb{F})$ *and it is the unique solution in that space.*
- *If* $P^2 - 4\,Q = 0$*, then, taking* $\lambda = -P/2$*,*

$$v(t) = x_0\,\exp_g(\lambda;0,t) + (v_0 - \lambda\,x_0)\,\exp_g(\lambda;0,t) \int\limits_{[0,t)} \frac{1}{1 + \lambda\,\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s)$$

  is a solution of (5.1)*. Furthermore,* $v \in \mathcal{BC}_g^\infty([0,T];\mathbb{F})$ *and it is the unique solution in that space.*

Proof. We consider the *characteristic equation* of problem (5.1),

$$\lambda^2 + P\,\lambda + Q = 0.$$

If $P^2 - 4\,Q \neq 0$, let $v_1 = \exp_g(\lambda_1;0,t)$ and $v_2(t) = \exp_g(\lambda_2;0,t)$. By Corollary 4.19 we have that $v_1,\ v_2 \in \mathcal{BC}_g^\infty([0,T];\mathbb{F}) \subset \mathcal{BC}_g^2([0,T];\mathbb{F})$. Furthermore, it can be checked that both functions satisfy (5.2). On the other hand,

$$v_1(0)\,(v_2)_g'(0) - v_2(0)\,(v_1)_g'(0) = \lambda_2 - \lambda_1 \neq 0.$$

Hence, by Lemma 5.2, there exists a solution of problem (5.1) given by

$$v(t) = \left(\frac{v_0 - \lambda_2\,x_0}{\lambda_1 - \lambda_2}\right) \exp_g(\lambda_1;0,t) - \left(\frac{v_0 - \lambda_1\,x_0}{\lambda_1 - \lambda_2}\right) \exp_g(\lambda_2;0,t).$$

If $P^2 - 4\,Q = 0$ we get the double root $\lambda = -P/2$ of the characteristic equation. Observe that the left hand side of the equation occurring in (5.1) can be written as $(\partial_g + P/2)^2 v$ where ∂_g denotes the g-derivative operator. Hence, we define $v_1(t) := \exp_g(\lambda;0,t)$, which is a solution of $(\partial_g + P/2)v = 0$ and consider the unique solution of

$$\begin{cases} (v_2)_g'(t) = \lambda\,v_2(t) + v_1(t),\ g - a.e.\,t \in [0,T), \\ v_2(0) = 0. \end{cases}$$

Since $(\partial_g + P/2)v_1 = 0$, it is clear that $(\partial_g + P/2)^2 v_2 = 0$. Furthermore, $v_2(0) = 0$ and $(v_2)'_g(0) = 1$, so v_2 is the solution we are looking for. By Corollary 4.19, $v_1 \in \mathcal{BC}_g^\infty([0,T];\mathbb{F})$ and, applying Corollary 4.21, $v_2 \in \mathcal{BC}_g^\infty([0,T];\mathbb{F})$ as well. Now, thanks to Proposition 4.12, we have that:

$$v_2(t) = \exp_g(\lambda;0,t) \int_{[0,t)} \frac{1}{1+\lambda\,\Delta^+ g(s)}\,\mathrm{d}\,\mu_g.$$

Since $v_2(0) = 0$, $(v_2)'_g(0) = 1$ we have that:

$$v_1(0)\,(v_2)'_g(0) - v_2(0)\,(v_1)'_g(0) = 1 \neq 0.$$

Thus, by Lemma 5.2, there exists a solution of problem (5.1) given by

$$v(t) = x_0\,\exp_g(\lambda;0,t) + (v_0 - \lambda\,x_0)\,\exp_g(\lambda;0,t) \int_{[0,t)} \frac{1}{1+\lambda\,\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s).$$

Finally, if we define $u(t) = v'_g(t) \in \mathcal{BC}_g^\infty([0,T])$ we have that the pair of functions $(u,v) \in [\mathcal{AC}_g([0,T];\mathbb{F})]^2$ satisfies the following system of differential equations:

$$\begin{cases} \begin{pmatrix} v \\ u \end{pmatrix}'_g (t) = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}, \\ \qquad v(0) = x_0,\ u(0) = v_0. \end{cases}$$

Thanks to [3, Theorem 7.3] we have that the previous system has a unique solution in $[\mathcal{AC}_g([0,T];\mathbb{F})]^2$, therefore v is the unique solution of (5.2) in the space $\mathcal{BC}_g^\infty([0,T];\mathbb{F})$. □

5.2. The non homogeneous case

In this section we focus on the non homogeneous version of the second order linear problem, namely,

$$\begin{cases} v''_g(t) + P\,v'_g(t) + Q\,v(t) = f(t),\ \forall t \in [0,T], \\ v(0) = x_0, \\ v'_g(0) = v_0, \end{cases} \tag{5.3}$$

where $P, Q, x_0, v_0 \in \mathbb{F}$ are constant values and $f \in \mathcal{AC}_g([0,T];\mathbb{F})$. Since the coefficients are constant, it will be the regularity of the term f that determines the additional regularity of the solution. As in the previous sections, we will see that it is possible to prove the uniqueness of solution when we consider the solution in the space $\mathcal{BC}_g^2([0,T];\mathbb{F})$.

Theorem 5.4. *Let $f \in \mathcal{BC}_g^n([0,T];\mathbb{F})$ and assume $1 + \lambda\,\Delta^+ g(t) \neq 0$, for all $t \in [0,T) \cap D_g$ and $\lambda \in \mathbb{F}$ such that $\lambda^2 + P\,\lambda + Q = 0$. Then, problem (5.3) has a unique solution $v \in \mathcal{BC}_g^{n+2}([0,T],\mathbb{F})$ given by*

$$\begin{aligned} v(t) =& x_0\,\exp_g(\lambda_2;0,t) + (v_0 - \lambda_2 x_0)\,\exp_g(\lambda_2;0,t) \cdot \int_{[0,t)} \frac{\exp_g(\lambda_2;0,s)^{-1}}{1+\lambda_2\Delta^+ g(s)}\,\exp_g(\lambda_1;0,s)\,\mathrm{d}\,\mu_g(s) \\ &+ \exp_g(\lambda_2;0,t) \int_{[0,t)} \frac{\exp_g(\lambda_2;0,s)^{-1}}{1+\lambda_2\Delta^+ g(s)}\,\exp_g(\lambda_1;0,s) \cdot \left(\int_{[0,s)} \frac{\exp_g(\lambda_1;0,r)^{-1}}{1+\lambda_1\Delta^+ g(r)}\,f(r)\,\mathrm{d}\,\mu_g(r) \right) \mathrm{d}\,\mu_g(s), \end{aligned} \tag{5.4}$$

where $(x-\lambda_1)(x-\lambda_2)=x^2+Px+Q$.

Proof. Let be λ_1,λ_2 the two complex eigenvalues of the characteristic polynomial $x^2+Px+Q=0$. Assume $v_2\in\mathcal{C}_g^{n+2}([0,T],\mathbb{F})$ is a solution of problem (5.3). Observe that, if we define $v_1=(v_2)'_g-\lambda_2 v_2$, it is clear that $v_1\in\mathcal{C}_g^{n+1}([0,T],\mathbb{F})$ and $(v_1)'_g-\lambda_1 v_1=f$, $v_1(0)=(v_2)'_g(0)-\lambda_2 v_2(0)=v_0-\lambda_2 x_0$, so v_1 has to solve the problem

$$\begin{cases}(v_1)'_g(t)=\lambda_1 v_1(t)+f(t),\ g-a.e.\,t\in[0,T)\\ v(0)=v_0-\lambda_2 x_0.\end{cases}\tag{5.5}$$

By Corollary 4.21, problem (5.5) has a unique solution in $\mathcal{C}_g^{n+1}([0,T],\mathbb{F})$, so v_1 is that unique solution. Furthermore, by definition, $v_1=(v_2)'_g-\lambda_2 v_2$ and $v_2(0)=x_0$, so v_2 solves the problem

$$\begin{cases}(v_2)'_g(t)=\lambda_2 v_2(t)+v_1(t),\ g-a.e.\,t\in[0,T)\\ v(0)=x_0.\end{cases}\tag{5.6}$$

By Corollary 4.21, problem (5.6) has a unique solution in $\mathcal{C}_g^{n+2}([0,T],\mathbb{F})$, so v_2 is that unique solution. This implies that, if a solution in $\mathcal{C}_g^{n+2}([0,T],\mathbb{F})$ of problem (5.3) exists, it has to be unique.

In order to obtain that unique solution it is enough to retrace the steps we have taken to prove the uniqueness. Let v_1 be the unique solution of problem (5.5) in $\mathcal{C}_g^{n+1}([0,T],\mathbb{F})$ and let v_2 be the unique solution of problem (5.6) in $\mathcal{C}_g^{n+2}([0,T],\mathbb{F})$. Clearly v_2 is a solution of problem (5.3).

In order to obtain the explicit expression of the solution, observe that, by Proposition 4.12, v_1 is of the form

$$v_1(t)=(v_0-\lambda_2 x_0)\exp_g(\lambda_1;0,t)+\exp_g(\lambda_1;0,t)\int_{[0,t)}\exp_g(\lambda_1;0,s)^{-1}\frac{f(s)}{1+\lambda_1\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s),$$

and v_2 of the form

$$\begin{aligned}v_2(t)&=x_0\exp_g(\lambda_2;0,t)+\exp_g(\lambda_2;0,t)\int_{[0,t)}\exp_g(\lambda_2;0,s)^{-1}\frac{v_1(s)}{1+\lambda_2\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s)\\
&=x_0\exp_g(\lambda_2;0,t)+(v_0-\lambda_2 x_0)\exp_g(\lambda_2;0,t)\int_{[0,t)}\exp_g(\lambda_2;0,s)^{-1}\frac{\exp_g(\lambda_1;0,s)}{1+\lambda_2\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s)\\
&\quad+\exp_g(\lambda_2;0,t)\int_{[0,t)}\exp_g(\lambda_2;0,s)^{-1}\frac{\exp_g(\lambda_1;0,s)}{1+\lambda_2\Delta^+g(s)}\\
&\quad\cdot\left(\int_{[0,s)}\exp_g(\lambda_1;0,r)^{-1}\frac{f(r)}{1+\lambda_1\Delta^+g(r)}\,\mathrm{d}\,\mu_g(r)\right)\mathrm{d}\,\mu_g(s).\end{aligned}$$

Now, thanks to Proposition 4.6,

$$\begin{aligned}\exp_g(\lambda_2;0,t)^{-1}\exp_g(\lambda_1;0,t)&=\exp_g(-\lambda_2/(1+\lambda_2\,\Delta^+g(t));0,t)\exp_g(\lambda_1;0,t)\\&=\exp_g((\lambda_1-\lambda_2)/(1+\lambda_2\,\Delta^+g(t));0,t).\end{aligned}$$

Thus,

$$v_2(t) = x_0 \exp_g(\lambda_2;0,t) + (v_0 - \lambda_2 x_0)\, \exp_g(\lambda_2;0,t) \int\limits_{[0,t)} \frac{1}{1+\lambda_2\Delta^+ g(s)}\, \exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+ g};0,s\right)\, \mathrm{d}\,\mu_g(s)$$

$$+ \exp_g(\lambda_2;0,t) \int\limits_{[0,t)} \frac{1}{1+\lambda_2\Delta^+ g(s)}\, \exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+ g};0,s\right)$$

$$\cdot \left(\int\limits_{[0,s)} \exp_g(\lambda_1;0,r)^{-1}\, \frac{f(r)}{1+\lambda_1\Delta^+ g(r)}\, \mathrm{d}\,\mu_g(r)\right)\, \mathrm{d}\,\mu_g(s). \qquad \square$$

Remark 5.5. Note that, for $\lambda \in \mathbb{F}$ such that $\lambda^2 + P\,\lambda + Q = 0$, the condition $1 + \lambda\,\Delta^+ g(t) = 0$ can only happen for a finite number of $t \in [0,T) \cap D_g$.

Remark 5.6. From the previous expression we can derive the expression of Green's function of problem (5.3) just by equating

$$\int\limits_{\mathbb{R}} G(t,r) f(r)\, \mathrm{d}\,\mu_g(r) = \exp_g(\lambda_2;0,t) \int\limits_{[0,t)} \frac{1}{1+\lambda_2\Delta^+ g(s)}\, \exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+ g};0,s\right)$$
$$\cdot \left(\int\limits_{[0,s)} \exp_g(\lambda_1;0,r)^{-1}\, \frac{f(r)}{1+\lambda_1\Delta^+ g(r)}\, \mathrm{d}\,\mu_g(r)\right)\, \mathrm{d}\,\mu_g(s). \tag{5.7}$$

Now, if we consider the product measure space $([0,T], \mathcal{M}_g \cdot \mathcal{M}_g, \mu_g \cdot \mu_g)$, we have, by Fubini's Theorem [1, Theorem 10.10],

$$\begin{aligned}
\int\limits_{\mathbb{R}} G(t,r) f(r)\, \mathrm{d}\,\mu_g(r) =& \exp_g(\lambda_2;0,t) \int\limits_{[0,T]\cdot[0,T]} \frac{1}{1+\lambda_2\Delta^+ g(s)}\, \exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+ g};0,s\right) \\
& \cdot \exp_g(\lambda_1;0,r)^{-1}\, \frac{f(r)}{1+\lambda_1\Delta^+ g(r)}\, \chi_{[0,s)}(r)\, \chi_{[0,t)}(s)\, \mathrm{d}\,\mu_g \cdot \mathrm{d}\,\mu_g \\
=& \exp_g(\lambda_2;0,t) \int\limits_{[0,T]} \int\limits_{[0,T]} \exp_g(\lambda_1;0,r)^{-1}\, \exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+ g};0,s\right) \\
& \cdot (1+\lambda_1\Delta^+ g(r))^{-1}\, (1+\lambda_2\Delta^+ g(s))^{-1}\, f(r)\, \chi_{(r,t)}(s)\, \chi_{[0,t)}(r)\, \mathrm{d}\,\mu_g(s)\, \mathrm{d}\,\mu_g(r) \\
=& \exp_g(\lambda_2;0,t) \int\limits_{[0,t)} \exp_g(\lambda_1;0,r)^{-1}\, \frac{f(r)}{1+\lambda_1\,\Delta^+ g(r)} \\
& \cdot \left(\int\limits_{(r,t)} \frac{1}{1+\lambda_2\,\Delta^+ g(s)}\, \exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+ g};0,s\right)\, \mathrm{d}\,\mu_g(s)\right)\, \mathrm{d}\,\mu_g(r).
\end{aligned}$$

Therefore, for $t, r \in [0,T]$,

$$
\begin{aligned}
G(t,r) =& \exp_g(\lambda_2;0,t)\ \exp_g(\lambda_1;0,r)^{-1}\,(1+\lambda_1\Delta^+g(r))^{-1}\,\chi_{[0,t)}(r)\\
&\cdot\int\limits_{(r,t)} \frac{1}{1+\lambda_2\,\Delta^+g(s)}\ \exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+g};0,s\right)\,\mathrm{d}\,\mu_g(s)\\
=& \exp_g(\lambda_2;0,t)\ \exp_g(\lambda_1;0,r)^{-1}\,(1+\lambda_1\Delta^+g(r))^{-1}\,\chi_{[0,t)}(r)\\
&\cdot\int\limits_{[r,t)} \frac{1}{1+\lambda_2\,\Delta^+g(s)}\ \exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+g};0,s\right)\,\mathrm{d}\,\mu_g(s)\\
&-\exp_g(\lambda_2;0,t)\ \exp_g(\lambda_2;0,r)^{-1}\,(1+\lambda_1\Delta^+g(r))^{-1}\,(1+\lambda_2\Delta^+g(r))^{-1}\,\Delta^+g(r)\,\chi_{[0,t)}(r).
\end{aligned}
$$

Observe that:

- If $\lambda_1\neq\lambda_2$,

$$
v(s)=\exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+g};0,s\right)\in\mathcal{AC}_g([r,T];\mathbb{F})
$$

  is the solution of

$$
\begin{cases}
v_g'(s)=\dfrac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+g(s)}\,v(s),\ g-a.e.\, s\in[r,T),\\
v(r)=\exp_g\left(\dfrac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+g};0,r\right).
\end{cases}
$$

  Therefore,

$$
\begin{aligned}
&\int\limits_{[r,t)} \frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+g(s)}\,\exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+g};0,s\right)\,\mathrm{d}\,\mu_g(s)\\
&=\exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+g};0,t\right)-\exp_g\left(\frac{\lambda_1-\lambda_2}{1+\lambda_2\,\Delta^+g};0,r\right)\\
&=\exp_g(\lambda_1;0,t)\ \exp_g(\lambda_2;0,t)^{-1}-\exp_g(\lambda_1;0,r)\ \exp_g(\lambda_2;0,r)^{-1}.
\end{aligned}
$$

  Thus, the Green's function in the case $\lambda_1\neq\lambda_2$ has the following expression:

$$
\begin{aligned}
G(t,r) =& \exp_g(\lambda_2;0,t)\ \exp_g(\lambda_1;0,r)^{-1}\,(1+\lambda_1\Delta^+g(r))^{-1}\,\chi_{[0,t)}(r)\\
&\cdot(\lambda_1-\lambda_2)^{-1}\left(\exp_g(\lambda_1;0,t)\ \exp_g(\lambda_2;0,t)^{-1}-\exp_g(\lambda_1;0,r)\ \exp_g(\lambda_2;0,r)^{-1}\right)\\
&-\exp_g(\lambda_2;0,t)\ \exp_g(\lambda_2;0,r)^{-1}\,(1+\lambda_1\Delta^+g(r))^{-1}\,(1+\lambda_2\Delta^+g(r))^{-1}\,\Delta^+g(r)\,\chi_{[0,t)}(r)\\
=&+(\lambda_1-\lambda_2)^{-1}\ \exp_g(\lambda_1;0,t)\ \exp_g(\lambda_1;0,r)^{-1}\,(1+\lambda_1\Delta^+g(r))^{-1}\,\chi_{[0,t)}(r)\\
&-(\lambda_1-\lambda_2)^{-1}\ \exp_g(\lambda_2;0,t)\ \exp_g(\lambda_2;0,r)^{-1}\,(1+\lambda_1\Delta^+g(r))^{-1}\,\chi_{[0,t)}(r)\\
&-\exp_g(\lambda_2;0,t)\ \exp_g(\lambda_2;0,r)^{-1}\,(1+\lambda_1\Delta^+g(r))^{-1}\,(1+\lambda_2\Delta^+g(r))^{-1}\,\Delta^+g(r)\,\chi_{[0,t)}(r)\\
=&+(\lambda_1-\lambda_2)^{-1}\ \exp_g(\lambda_1;0,t)\ \exp_g(\lambda_1;0,r)^{-1}\,(1+\lambda_1\Delta^+g(r))^{-1}\,\chi_{[0,t)}(r)\\
&-(\lambda_1-\lambda_2)^{-1}\ \exp_g(\lambda_2;0,t)\ \exp_g(\lambda_2;0,r)^{-1}\,(1+\lambda_2\Delta^+g(r))^{-1}\,\chi_{[0,t)}(r).
\end{aligned}
\tag{5.8}
$$

- If $\lambda_1=\lambda_2$, we have the following expression for the Green's function:

$$
G(r,t)=\exp_g(\lambda;0,t)\ \exp_g(\lambda;0,r)^{-1}\,(1+\lambda\,\Delta^+g(r))^{-1}\,\chi_{[0,t)}(r)\cdot\int\limits_{(r,t)}\frac{1}{1+\lambda\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g(s)
$$

$$\begin{aligned}
&= \exp_g(\lambda;0,t)\, \exp_g(\lambda;0,r)^{-1}\, (1+\lambda\,\Delta^+ g(r))^{-1}\, \chi_{[0,t)}(r) \\
&\quad \cdot \left(\int\limits_{[0,t)} \frac{1}{1+\lambda\,\Delta^+ g(s)}\, \mathrm{d}\,\mu_g(s) - \int\limits_{[0,r)} \frac{1}{1+\lambda\,\Delta^+ g(s)}\, \mathrm{d}\,\mu_g(s) - \frac{\Delta^+ g(r)}{1+\lambda\,\Delta^+ g(r)} \right) \qquad (5.9)\\
&= + \exp_g(\lambda;0,t) \left(\int\limits_{[0,t)} \frac{1}{1+\lambda\,\Delta^+ g(s)}\, \mathrm{d}\,\mu_g(s) \right) \exp_g(\lambda;0,r)^{-1}\, (1+\lambda\,\Delta^+ g(r))^{-1}\, \chi_{[0,t)}(r) \\
&\quad - \exp_g(\lambda;0,t) \left(\int\limits_{[0,r)} \frac{1}{1+\lambda\,\Delta^+ g(s)}\, \mathrm{d}\,\mu_g(s) \right) \exp_g(\lambda;0,r)^{-1}\, (1+\lambda\,\Delta^+ g(r))^{-1}\, \chi_{[0,t)}(r) \\
&\quad - \exp_g(\lambda;0,t)\, \exp_g(\lambda;0,r)^{-1}\, (1+\lambda\,\Delta^+ g(r))^{-2}\, \Delta^+ g(r)\, \chi_{[0,t)}(r).
\end{aligned}$$

Remark 5.7. Observe that we can arrive to expressions (5.8) and (5.9) using an integration by parts argument in formula (5.7). Indeed, given two elements $h_1, h_2 \in \mathcal{AC}_g([0,T];\mathbb{F})$ we have that $h_1\, h_2 \in \mathcal{AC}_g([0,T];\mathbb{F})$ and

$$(h_1\, h_2)'_g(t) = (h_1)'_g(t)\, h_2(t) + h_1(t)\, (h_2)'_g(t) + (h_1)'_g(t)\, (h_2)'(t)\, \Delta^+ g(t),\ g - a.e.\, t \in [0,T].$$

Observe that we are explicitly excluding the points of C_g in the above formula. In particular, for $t \in [0,T]$,

$$\begin{aligned}
h_1(t)\, h_2(t) - h_1(0)\, h_2(0) &= \int\limits_{[0,t)} (h_1)'_g(s)\, h_2(s)\, \mathrm{d}\,\mu_g(s) + \int\limits_{[0,t)} h_1(s)\, (h_2)'_g(s)\, \mathrm{d}\,\mu_g(s) \\
&\quad + \int\limits_{[0,t)} (h_1)'_g(s)\, (h_2)'(s)\, \Delta^+ g(s)\, \mathrm{d}\,\mu_g(s).
\end{aligned} \qquad (5.10)$$

Now we study two cases:

- Case $\lambda_1 \neq \lambda_2$. Let us consider

$$\begin{aligned}
h_1(t) &= (\lambda_1 - \lambda_2)^{-1} \exp_g(\lambda_1;0,t)\, \exp_g(\lambda_2;0,t)^{-1}, \\
h_2(t) &= \int\limits_{[0,t)} \frac{\exp_g(\lambda_1;0,r)^{-1}}{1+\lambda_1 \Delta^+ g(r)}\, f(r)\, \mathrm{d}\,\mu_g(r).
\end{aligned}$$

We have that $h_1, h_2 \in \mathcal{AC}_g([0,T];\mathbb{F})$, so

$$\begin{aligned}
&\int\limits_{[0,t)} \frac{\exp_g(\lambda_2;0,s)^{-1}}{1+\lambda_2 \Delta^+ g(s)}\, \exp_g(\lambda_1;0,s) \left(\int\limits_{[0,s)} \frac{\exp_g(\lambda_1;0,r)^{-1}}{1+\lambda_1 \Delta^+ g(r)}\, f(r)\, \mathrm{d}\,\mu_g(r) \right) \mathrm{d}\,\mu_g(s) \\
&= (\lambda_1 - \lambda_2)^{-1} \exp_g(\lambda_1;0,t)\, \exp_g(\lambda_2;0,t)^{-1} \int\limits_{[0,t)} \frac{\exp_g(\lambda_1;0,r)^{-1}}{1+\lambda_1 \Delta^+ g(r)}\, f(r)\, \mathrm{d}\,\mu_g(r) \\
&\quad - \int\limits_{[0,t)} (\lambda_1 - \lambda_2)^{-1} \exp_g(\lambda_1;0,s)\, \exp_g(\lambda_2;0,s)^{-1} \frac{\exp_g(\lambda_1;0,s)^{-1}}{1+\lambda_1 \Delta^+ g(s)}\, f(s)\, \mathrm{d}\,\mu_g(s) \\
&\quad - \int\limits_{[0,t)} \frac{\exp_g(\lambda_2;0,s)^{-1}}{1+\lambda_2 \Delta^+ g(s)}\, \exp_g(\lambda_1;0,s)\, \frac{\exp_g(\lambda_1;0,s)^{-1}}{1+\lambda_1 \Delta^+ g(s)}\, f(s)\, \Delta^+ g(s)\, \mathrm{d}\,\mu_g(s),
\end{aligned} \qquad (5.11)$$

and we recover the expression of Green's function in (5.8). Observe that by substituting (5.11) in (5.4) we obtain

$$
\begin{aligned}
v_2(t) = & \left(\frac{v_0 - \lambda_2 x_0}{\lambda_1 - \lambda_2}\right) \exp_g(\lambda_1; 0, t) - \left(\frac{v_0 - \lambda_1 x_0}{\lambda_1 - \lambda_2}\right) \exp_g(\lambda_2; 0, t) \\
& + (\lambda_1 - \lambda_2)^{-1} \exp_g(\lambda_1; 0, t) \int_{[0,t)} \frac{\exp_g(\lambda_1; 0, s)^{-1}}{1 + \lambda_1 \Delta^+ g(s)} f(s) \, \mathrm{d}\mu_g \\
& - (\lambda_1 - \lambda_2)^{-1} \exp_g(\lambda_2; 0, t) \int_{[0,t)} \frac{\exp_g(\lambda_2; 0, s)^{-1}}{1 + \lambda_2 \Delta^+ g(s)} f(s) \, \mathrm{d}\mu_g.
\end{aligned}
\tag{5.12}
$$

We have that

$$
v_h(t) = \left(\frac{v_0 - \lambda_2 x_0}{\lambda_1 - \lambda_2}\right) \exp_g(\lambda_1; 0, t) - \left(\frac{v_0 - \lambda_1 x_0}{\lambda_1 - \lambda_2}\right) \exp_g(\lambda_2; 0, t) \in \mathcal{BC}_g^\infty([0,T]; \mathbb{F})
$$

is the solution of the homogeneous equation (5.1) and

$$
\begin{aligned}
v_p(t) = & + (\lambda_1 - \lambda_2)^{-1} \exp_g(\lambda_1; 0, t) \int_{[0,t)} \frac{\exp_g(\lambda_1; 0, s)^{-1}}{1 + \lambda_1 \Delta^+ g(s)} f(s) \, \mathrm{d}\mu_g \\
& - (\lambda_1 - \lambda_2)^{-1} \exp_g(\lambda_2; 0, t) \int_{[0,t)} \frac{\exp_g(\lambda_2; 0, s)^{-1}}{1 + \lambda_2 \Delta^+ g(s)} f(s) \, \mathrm{d}\mu_g
\end{aligned}
$$

is a particular solution in the space $\mathcal{BC}_g^{n+2}([0,T], \mathbb{F})$ of the non homogeneous equation (5.3) that satisfies $v_p(0) = (v_p)_g'(0) = 0$.

- Case $\lambda_1 = \lambda_2$. We have that expression (5.4) reduces to

$$
\begin{aligned}
v_2(t) = & \, x_0 \exp_g(\lambda; 0, t) + (v_0 - \lambda x_0) \exp_g(\lambda; 0, t) \int_{[0,t)} \frac{1}{1 + \lambda \Delta^+ g(s)} \, \mathrm{d}\mu_g(s) \\
& + \exp_g(\lambda; 0, t) \int_{[0,t)} \frac{1}{1 + \lambda \Delta^+ g(s)} \left(\int_{[0,s)} \frac{\exp_g(\lambda; 0, r)^{-1}}{1 + \lambda \Delta^+ g(r)} f(r) \, \mathrm{d}\mu_g(r) \right) \mathrm{d}\mu_g(s).
\end{aligned}
\tag{5.13}
$$

Define

$$
\begin{aligned}
h_1(t) &= \int_{[0,t)} \frac{1}{1 + \lambda \Delta^+ g(s)} \, \mathrm{d}\mu_g(s), \\
h_2(t) &= \int_{[0,t)} \frac{\exp_g(\lambda; 0, s)^{-1}}{1 + \lambda \Delta^+ g(s)} f(s) \, \mathrm{d}\mu_g(s).
\end{aligned}
$$

We have that $h_1, h_2 \in \mathcal{AC}_g([0,T]; \mathbb{F})$. Hence, by formula (5.10),

$$
\int_{[0,t)} \frac{1}{1 + \lambda \Delta^+ g(s)} \left(\int_{[0,s)} \frac{\exp_g(\lambda; 0, r)^{-1}}{1 + \lambda \Delta^+ g(r)} f(r) \, \mathrm{d}\mu_g(r) \right) \mathrm{d}\mu_g(s)
$$

$$
\begin{aligned}
&= \int_{[0,t)} \frac{1}{1+\lambda\,\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s) \int_{[0,t)} \frac{\exp_g(\lambda;0,s)^{-1}}{1+\lambda\Delta^+ g(s)}\, f(s)\,\mathrm{d}\,\mu_g(s) \\
&\quad - \int_{[0,t)} \left(\int_{[0,s)} \frac{1}{1+\lambda\,\Delta^+ g(r)}\,\mathrm{d}\,\mu_g(r) \right) \frac{\exp_g(\lambda;0,s)^{-1}}{1+\lambda\Delta^+ g(s)}\, f(s)\,\mathrm{d}\,\mu_g(s) \\
&\quad - \int_{[0,t)} \frac{\exp_g(\lambda;0,s)^{-1}}{(1+\lambda\Delta^+ g(s))^2}\, f(s)\,\Delta^+ g(s)\,\mathrm{d}\,\mu_g(s).
\end{aligned} \tag{5.14}
$$

Substituting expression (5.14) in (5.13) we obtain

$$
\begin{aligned}
v_2(t) =& x_0 \exp_g(\lambda;0,t) + (v_0-\lambda x_0)\exp_g(\lambda;0,t) \int_{[0,t)} \frac{1}{1+\lambda\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s) \\
&+ \exp_g(\lambda;0,t) \int_{[0,t)} \frac{1}{1+\lambda\,\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s) \int_{[0,t)} \frac{\exp_g(\lambda;0,s)^{-1}}{1+\lambda\Delta^+ g(s)}\, f(s)\,\mathrm{d}\,\mu_g(s) \\
&- \exp_g(\lambda;0,t) \int_{[0,t)} \left(\int_{[0,s)} \frac{1}{1+\lambda\,\Delta^+ g(r)}\,\mathrm{d}\,\mu_g(r) \right) \frac{\exp_g(\lambda;0,s)^{-1}}{1+\lambda\Delta^+ g(s)}\, f(s)\,\mathrm{d}\,\mu_g(s) \\
&- \exp_g(\lambda;0,t) \int_{[0,t)} \frac{\exp_g(\lambda;0,s)^{-1}}{(1+\lambda\Delta^+ g(s))^2}\, f(s)\,\Delta^+ g(s)\,\mathrm{d}\,\mu_g(s).
\end{aligned}
$$

Observe that

$$
v_h(t) = x_0 \exp_g(\lambda;0,t) + (v_0-\lambda x_0)\exp_g(\lambda;0,t) \int_{[0,t)} \frac{1}{1+\lambda\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s) \in \mathcal{BC}_g^{\infty}([0,T];\mathbb{F})
$$

is the solution of the homogeneous equation (5.3) and

$$
\begin{aligned}
v_p(t) =& \exp_g(\lambda;0,t) \int_{[0,t)} \frac{1}{1+\lambda\,\Delta^+ g(s)}\,\mathrm{d}\,\mu_g(s) \int_{[0,t)} \frac{\exp_g(\lambda;0,s)^{-1}}{1+\lambda\Delta^+ g(s)}\, f(s)\,\mathrm{d}\,\mu_g(s) \\
&- \exp_g(\lambda;0,t) \int_{[0,t)} \left(\int_{[0,s)} \frac{1}{1+\lambda\,\Delta^+ g(r)}\,\mathrm{d}\,\mu_g(r) \right) \frac{\exp_g(\lambda;0,s)^{-1}}{1+\lambda\Delta^+ g(s)}\, f(s)\,\mathrm{d}\,\mu_g(s) \\
&- \exp_g(\lambda;0,t) \int_{[0,t)} \frac{\exp_g(\lambda;0,s)^{-1}}{(1+\lambda\Delta^+ g(s))^2}\, f(s)\,\Delta^+ g(s)\,\mathrm{d}\,\mu_g(s)
\end{aligned}
$$

is a particular solution of the non homogeneous equation (5.3) in the space $\mathcal{BC}_g^{n+2}([0,T],\mathbb{F})$ that satisfies $v_p(0) = (v_p)'_g(0) = 0$.

6. The Stieltjes harmonic oscillator

In this section we present an application related to the real solution of the Stieltjes harmonic oscillator (g-harmonic oscillator). Let $g:\mathbb{R}\to\mathbb{R}$ be a derivator such that $0 \notin N_g^-$ and $T \notin N_g^+ \cup D_g \cup C_g$ and denote by g^C its continuous part. We consider the following equation:

$$\begin{cases} v''_g(t) + 2\,\zeta\,\omega_0\,v'_g(t) + \omega_0^2\,v(t) = 0, \;\; g - a.e.\, t \in [0,T), \\ v(0) = x_0, \\ v'_g(0) = v_0, \end{cases} \tag{6.1}$$

where x_0 and v_0 are real numbers and:

- ω_0 is the undamped angular frequency of the oscillator,

$$\omega_0 = \sqrt{\frac{k}{m}},$$

  where $m > 0$ is the mass of the oscillator and $k > 0$ is a measure of the stiffness of the spring;
- ζ is the damping ratio,

$$\zeta = \frac{c}{2\sqrt{mk}},$$

  with $c > 0$, the viscous damping coefficient (resistance of the medium). If $\zeta > 1$ we have an overdamped oscillator, if $\zeta = 1$ the oscillator is critically damped and, if $\zeta < 1$, the oscillator is underdamped. Observe that the solutions of the characteristic equation are given by:

$$\lambda = \frac{1}{2}\left(-2\,\zeta\,\omega_0 \pm \sqrt{4\,\zeta^2\,\omega_0^2 - 4\,\omega_0^2}\right) = -\zeta\,\omega_0 \pm \omega_0\,\sqrt{\zeta^2 - 1}.$$

  Assume that $1 + \lambda\,\Delta^+ g(t) \neq 0$ for all $t \in [0,T) \cap D_g$ and for all λ solution of the characteristic equation.

We have the following real solution of the g-harmonic oscillator in terms of the damping ratio:

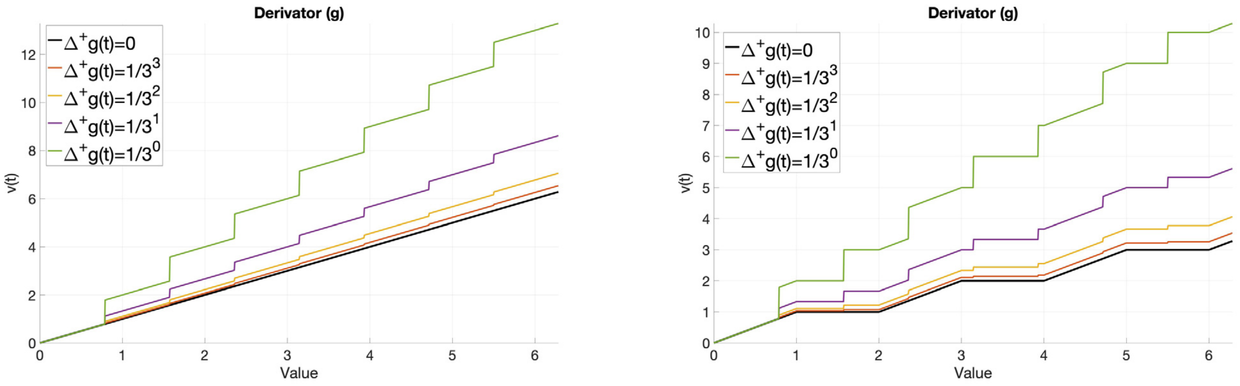
- If $\zeta > 1$, we have two real solutions of the characteristic equation, $\lambda_1 = -\zeta\,\omega_0 - \omega_0\,\sqrt{\zeta^2 - 1}$ and $\lambda_2 = -\zeta\,\omega_0 + \omega_0\,\sqrt{\zeta^2 - 1}$. Thus, the solution of (6.1) is given by

$$\begin{aligned} v(t) &= \left(\frac{v_0 - \lambda_2\,x_0}{\lambda_1 - \lambda_2}\right)\,\exp_g(\lambda_1;0,t) - \left(\frac{v_0 - \lambda_1\,x_0}{\lambda_1 - \lambda_2}\right)\,\exp_g(\lambda_2;0,t), \\ &= \left(\frac{v_0 - \lambda_2\,x_0}{\lambda_1 - \lambda_2}\right)\,\exp(\lambda_1\,g_C(t)) \prod_{s\in[0,t)\cap D_g} \left(1 + \lambda_1\,\Delta^+ g(s)\right) \\ &\quad - \left(\frac{v_0 - \lambda_1\,x_0}{\lambda_1 - \lambda_2}\right)\,\exp_g(\lambda_2\,g_C(t)) \prod_{s\in[0,t)\cap D_g} \left(1 + \lambda_2\,\Delta^+ g(s)\right). \end{aligned}$$

- If $\zeta = 1$, we have one real solution of the characteristic equation, $\lambda = -\zeta\,\omega_0$. Thus, the solution of (6.1) is given by

$$\begin{aligned} v(t) =& x_0\,\exp_g(\lambda;0,t) + (v_0 - \lambda\,x_0)\,\exp_g(\lambda;0,t) \int_{[0,t)} \frac{1}{1 + \lambda\,\Delta^+ g(s)}\,\mathrm{d}\,\mu_g, \\ =& \exp(\lambda\,g_C(t)) \prod_{s\in[0,t)\cap D_g} \left(1 + \lambda\,\Delta^+ g(s)\right) \cdot \left[x_0 + (v_0 - \lambda\,x_0)\left(g_C(t) + \sum_{s\in[0,t)\cap D_g} \frac{\Delta^+ g(s)}{1 + \lambda\,\Delta^+ g(s)}\right)\right]. \end{aligned}$$

- If $\zeta < 1$, we have a pair of conjugate complex solutions, $\lambda_1 = -\zeta\,\omega_0 + i\,\omega_0\,\sqrt{1 - \zeta^2}$ and $\lambda_2 = -\zeta\,\omega_0 - i\,\omega_0\,\sqrt{1 - \zeta^2}$. If we denote by $a = -\zeta\,\omega_0$ and $b = \omega_0\,\sqrt{1 - \zeta^2}$, we have that the solution is given by



(a) Derivator g_1.

(b) Derivator g_2.

Fig. 6.1. Derivators g_1 and g_2 associated to $l \in \{0, 1/3^3, 1/3^2, 1/3^1, 1/3^0\}$ (vertical lines have to be understood as jumps and not as a multivalued function). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$v(t) = \left(\frac{v_0 - a\,x_0 + b\,i\,x_0}{2\,b\,i}\right) \exp_g(a + b\,i; 0, t) - \left(\frac{v_0 - a\,x_0 - b\,i\,x_0}{2\,b\,i}\right) \exp_g(a - b\,i; 0, t).$$

If we take into account expression (4.20),

$$\begin{aligned} v(t) =& \exp_g(a;0,t)\left[\left(\frac{v_0 - a\,x_0}{b}\right)\frac{1}{2\,i}\left(\exp_g\left(\frac{i\,b}{1+a\,\Delta^+ g};0,t\right) - \exp_g\left(\frac{-i\,b}{1+a\,\Delta^+ g};0,t\right)\right)\right. \\ &\left. + x_0\,\frac{1}{2}\left(\exp_g\left(\frac{i\,b}{1+a\,\Delta^+ g};0,t\right) + \exp_g\left(\frac{-i\,b}{1+a\,\Delta^+ g};0,t\right)\right)\right] \\ =& \exp_g(a;0,t)\left[\left(\frac{v_0 - a\,x_0}{b}\right)\sin_g\left(\frac{b}{1+a\,\Delta^+ g};0,t\right) + x_0\,\cos_g\left(\frac{b}{1+a\,\Delta^+ g};0,t\right)\right]. \end{aligned}$$

Example 6.1. Let us study the behavior of the g-harmonic oscillator in the particular case $\omega_0 = 2$, $x_0 = v_0 = 1$. We consider the following derivators:

$$\begin{aligned} g_1(t) =& g_1^C(t) + g_1^B(t), \\ g_2(t) =& g_2^C(t) + g_2^B(t), \end{aligned}$$

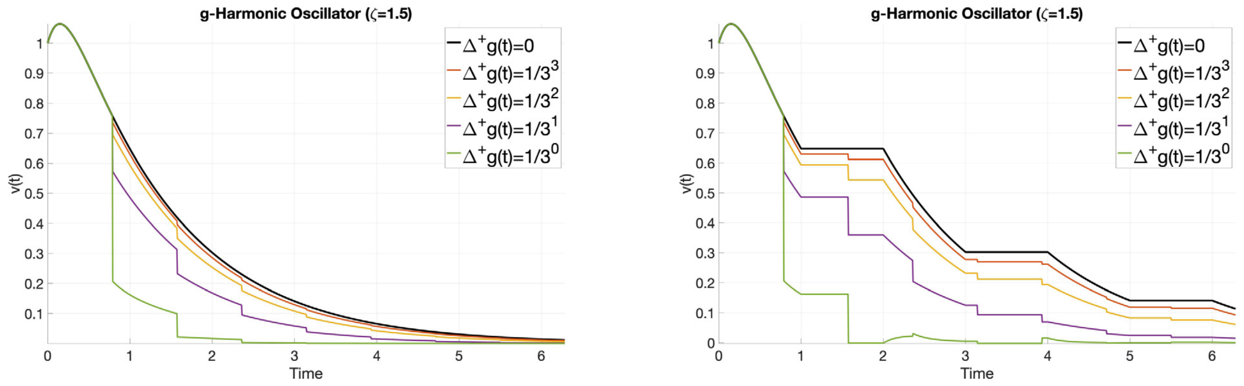
where $g_1^C(t) = t$,

$$g_2(t) = \begin{cases} \dfrac{1}{2} + \dfrac{[t]}{2}, & [t] + 1 = 2\,k,\ k \in \mathbb{N}, \\ t - \dfrac{[t]}{2}, & [t] + 1 = 2\,k - 1,\ k \in \mathbb{N}, \end{cases}$$

$[\cdot]$ denotes the floor function,

$$g_1^B(t) = g_2^B(t) = \sum_{s \in [0,t) \cap D} l,$$

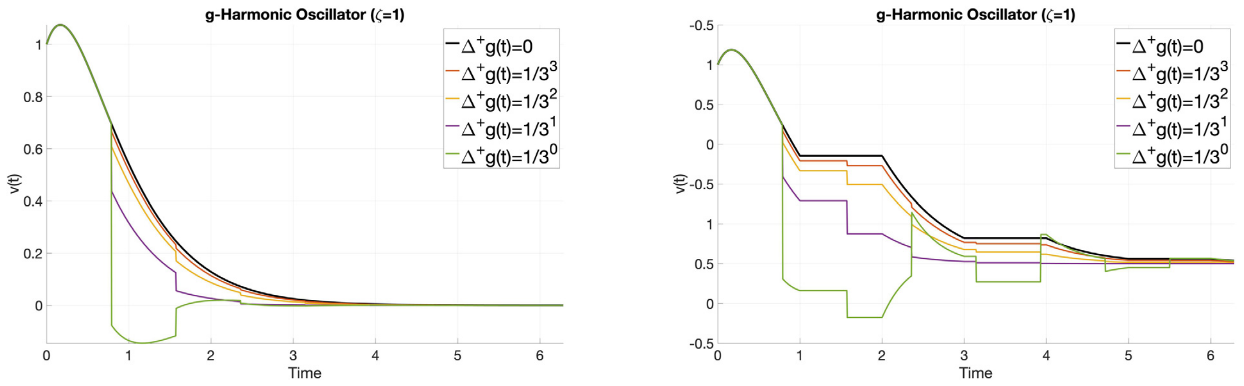
with $D = \{s \in [0, +\infty) : s = k\pi/4,\ k \in \mathbb{N}\}$ and $l \geq 0$ (observe that $g_1^B(t) = g_2^B(t) < \infty$, for all $t \in [0, \infty)$). In order to compare the effect that discontinuities in the derivator have on the solution, we have considered the cases where $l \in \{0, 1/3^3, 1/3^2, 1/3^1, 1/3^0\}$. Observe that the solution associated to the derivator g_1 and $l = 0$ corresponds to the classical solution of the harmonic oscillator. In Fig. 6.1 we can see a graphical



(a) $\zeta = 1.5$, overdamped oscillator (g_1).

(b) $\zeta = 1.5$, overdamped oscillator (g_2).

Fig. 6.2. Solution of the g-harmonic oscillator for $\zeta = 1.5$ (overdamped oscillator) associated to g_1 and g_2 (vertical lines have to be understood as jumps and not as a multivalued function).



(a) $\zeta = 1$, critically damped oscillator (g_1).

(b) $\zeta = 1$, critically damped oscillator (g_2).

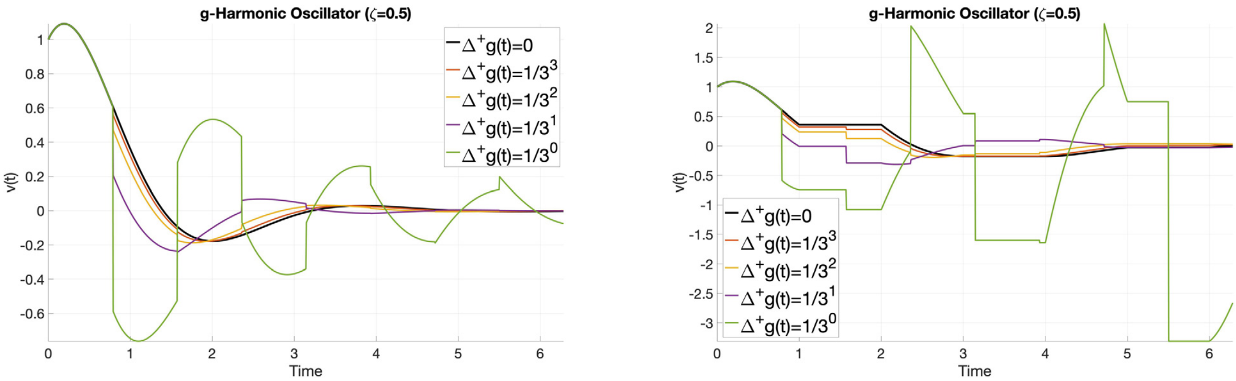
Fig. 6.3. Solution of the g-harmonic oscillator for $\zeta = 1$ (critically damped oscillator) associated to g_1 and g_2 (vertical lines have to be understood as jumps and not as a multivalued function).

representation of the derivators considered. Now, we present a graphical representation of the solution associated to $\zeta = 1.5$ (overdamped oscillator, Fig. 6.2), $\zeta = 1$ (critically damped oscillator, Fig. 6.3) and $z = 0.5$ (underdamped oscillator, Fig. 6.4).

Now we will study the effect of a g-periodic source term with the same frequency as the natural frequency of the oscillator. We will consider the non-homogeneous g-harmonic oscillator with $c = 0$ and we will consider as a source term $f(t) = \cos_g(\omega_0; 0, t)$; that is,

$$\begin{cases} v_g''(t) + \omega_0^2\, v(t) = \cos_g(\omega_0; 0, t), \ g - a.e.\, t \in [0, T), \\ v(0) = x_0, \\ v_g'(0) = v_0. \end{cases} \tag{6.2}$$

Observe that in the case $g(t) = t$ we recover the classical resonance effect. It is reasonable to expect that in the case of having a generic derivator the amplitude of the oscillations increases with t. Indeed we have that the solution v is given by



(a) $\zeta = 0.5$, underdamped oscillator (g_1). (b) $\zeta = 0.5$, underdamped oscillator (g_2).

Fig. 6.4. Solution of the g-harmonic oscillator for $\zeta = 0.5$ (underdamped oscillator) associated to g_1 and g_2 (vertical lines have to be understood as jumps and not as a multivalued function).

$$v(t) = v_h(t) + v_p(t),$$

where v_h is the solution of the homogeneous equation, namely,

$$\begin{aligned} v_h(t) =& \left(\frac{v_0 + i\,\omega_0\,x_0}{2\,i\,\omega_0}\right) \exp_g(i\,\omega_0; 0, t) - \left(\frac{v_0 - i\,\omega_0\,x_0}{2\,i\,\omega_0}\right) \exp_g(-i\,\omega_0; 0, t) \\ =& x_0 \cos_g(\omega_0; 0, t) + \frac{v_0}{\omega_0} \sin_g(\omega_0; 0, t); \end{aligned}$$

and v_p is a particular solution that satisfies $v_p(0) = (v_p)'_g(t) = 0$ given by

$$\begin{aligned} v_p(t) =& + (2\,i\,\omega_0)^{-1} \exp_g(+i\,\omega_0; 0, t) \int_{[0,t)} \frac{\exp_g(+i\,\omega_0; 0, s)^{-1}}{1 + i\,\omega_0\,\Delta^+ g(s)} \cos_g(\omega_0; 0, s)\, \mathrm{d}\,\mu_g \\ & - (2\,i\,\omega_0)^{-1} \exp_g(-i\,\omega_0; 0, t) \int_{[0,t)} \frac{\exp_g(-i\,\omega_0; 0, s)^{-1}}{1 - i\,\omega_0\,\Delta^+ g(s)} \cos_g(\omega_0; 0, s)\, \mathrm{d}\,\mu_g. \end{aligned}$$

Now,

$$\begin{aligned} & \int_{[0,t)} \frac{\exp_g(+i\,\omega_0; 0, s)^{-1}}{1 + i\,\omega_0\,\Delta^+ g(s)} \cos_g(\omega_0; 0, s)\, \mathrm{d}\,\mu_g \\ =& \frac{1}{2} \int_{[0,t)} \frac{\exp_g(+i\,\omega_0; 0, s)^{-1}}{1 + i\,\omega_0\,\Delta^+ g(s)} \left(\exp_g(i\,\omega_0; 0, s) + \exp_g(-i\,\omega_0; 0, s)\right) \mathrm{d}\,\mu_g \\ =& \frac{1}{2} \int_{[0,t)} \frac{1}{1 + i\,\omega_0\,\Delta^+ g(s)}\, \mathrm{d}\,\mu_g + \frac{1}{2} \int_{[0,t)} \exp_g(-i\,\omega_0; 0, s) \frac{\exp_g(+i\,\omega_0; 0, s)^{-1}}{1 + i\,\omega_0\,\Delta^+ g(s)}\, \mathrm{d}\,\mu_g \\ =& \frac{1}{2} \int_{[0,t)} \frac{1}{1 + i\,\omega_0\,\Delta^+ g(s)}\, \mathrm{d}\,\mu_g + \frac{1}{2} \int_{[0,t)} \frac{1}{1 + i\,\omega_0\,\Delta^+ g(s)} \exp_g\left(\frac{-2\,i\,\omega_0}{1 + i\,\omega_0\,\Delta^+ g}; 0, s\right) \mathrm{d}\,\mu_g \\ =& \frac{1}{2} \int_{[0,t)} \frac{1}{1 + i\,\omega_0\,\Delta^+ g(s)}\, \mathrm{d}\,\mu_g - \frac{1}{4\,i\,\omega_0} \left[\exp_g\left(\frac{-2\,i\,\omega_0}{1 + i\,\omega_0\,\Delta^+ g}; 0, t\right) - 1\right]. \end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\int_{[0,t)} \frac{\exp_g(-i\,\omega_0;0,s)^{-1}}{1-i\,\omega_0\,\Delta^+g(s)}\,\cos_g(\omega_0;0,s)\,\mathrm{d}\,\mu_g\\
=&\frac{1}{2}\int_{[0,t)} \frac{\exp_g(-i\,\omega_0;0,s)^{-1}}{1-i\,\omega_0\,\Delta^+g(s)}\,\big(\exp_g(+i\,\omega_0;0,s)+\exp_g(-i\,\omega_0;0,s)\big)\,\mathrm{d}\,\mu_g\\
=&\frac{1}{2}\int_{[0,t)} \exp_g(+i\,\omega_0;0,s)\,\frac{\exp_g(-i\,\omega_0;0,s)^{-1}}{1-i\,\omega_0\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g+\frac{1}{2}\int_{[0,t)}\frac{1}{1-i\,\omega_0\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g\\
=&\frac{1}{2}\int_{[0,t)} \frac{1}{1-i\,\omega_0\,\Delta^+g(s)}\,\exp_g\left(\frac{2\,i\,\omega_0}{1-i\,\omega_0\,\Delta^+g};0,s\right)\,\mathrm{d}\,\mu_g+\frac{1}{2}\int_{[0,t)}\frac{1}{1-i\,\omega_0\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g\\
=&\frac{1}{4\,i\,\omega_0}\left[\exp_g\left(\frac{2\,i\,\omega_0}{1-i\,\omega_0\,\Delta^+g};0,t\right)-1\right]+\frac{1}{2}\int_{[0,t)}\frac{1}{1-i\,\omega_0\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
v_p(t)=&+\frac{1}{4\,i\,\omega_0}\,\exp_g(+i\,\omega_0;0,t)\int_{[0,t)}\frac{1}{1+i\,\omega_0\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g\\
&+\frac{1}{8\,\omega_0^2}\,\exp_g(+i\,\omega_0;0,t)\left[\exp_g\left(\frac{-2\,i\,\omega_0}{1+i\,\omega_0\,\Delta^+g};0,t\right)-1\right]\\
&-\frac{1}{4\,i\,\omega_0}\,\exp_g(-i\,\omega_0;0,t)\int_{[0,t)}\frac{1}{1-i\,\omega_0\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g\\
&+\frac{1}{8\,\omega_0^2}\,\exp_g(-i\,\omega_0;0,t)\left[\exp_g\left(\frac{2\,i\,\omega_0}{1-i\,\omega_0\,\Delta^+g};0,t\right)-1\right].
\end{aligned}
$$

Now, taking into account that

$$
\begin{aligned}
\exp_g\left(\frac{-2\,i\,\omega_0}{1+i\,\omega_0\,\Delta^+g};0,t\right)&=\exp_g(-i\,\omega_0;0,t)\,\exp_g(+i\,\omega_0;0,t)^{-1},\\
\exp_g\left(\frac{2\,i\,\omega_0}{1-i\,\omega_0\,\Delta^+g};0,t\right)&=\exp_g(+i\,\omega_0;0,s)\,\exp_g(-i\,\omega_0;0,s)^{-1},
\end{aligned}
$$

we obtain the following expression for the particular solution:

$$
\begin{aligned}
v_p(t)=&+\frac{1}{4\,i\,\omega_0}\,\exp_g(+i\,\omega_0;0,t)\int_{[0,t)}\frac{1}{1+i\,\omega_0\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g\\
&-\frac{1}{4\,i\,\omega_0}\,\exp_g(-i\,\omega_0;0,t)\int_{[0,t)}\frac{1}{1-i\,\omega_0\,\Delta^+g(s)}\,\mathrm{d}\,\mu_g\\
&+\frac{1}{8\,\omega_0^2}\,\exp_g(-i\,\omega_0;0,t)-\frac{1}{8\,\omega_0^2}\,\exp_g(+i\,\omega_0;0,t)\\
&+\frac{1}{8\,\omega_0^2}\,\exp_g(+i\,\omega_0;0,s)-\frac{1}{8\,\omega_0^2}\,\exp_g(-i\,\omega_0;0,t).
\end{aligned}
$$

Thus,

$$v_p(t) = + \frac{1}{4\,i\,\omega_0}\ \exp_g(+i\,\omega_0;0,t) \int\limits_{[0,t)} \frac{1}{1+i\,\omega_0\,\Delta^+ g(s)}\ \mathrm{d}\,\mu_g$$
$$- \frac{1}{4\,i\,\omega_0}\ \exp_g(-i\,\omega_0;0,t) \int\limits_{[0,t)} \frac{1}{1-i\,\omega_0\,\Delta^+ g(s)}\ \mathrm{d}\,\mu_g.$$

In order to simplify the previous expression let us consider the following computations:

$$\int\limits_{[0,t)} \frac{1}{1+i\,\omega_0\,\Delta^+ g(s)}\ \mathrm{d}\,\mu_g = \int\limits_{[0,t)} \frac{1-i\,\omega_0\,\Delta^+ g(s)}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g$$
$$= \int\limits_{[0,t)} \frac{1}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g - i\,\omega_0 \int\limits_{[0,t)} \frac{\Delta^+ g(s)}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g,$$
$$\int\limits_{[0,t)} \frac{1}{1-i\,\omega_0\,\Delta^+ g(s)}\ \mathrm{d}\,\mu_g = \int\limits_{[0,t)} \frac{1+i\,\omega_0\,\Delta^+ g(s)}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g$$
$$= \int\limits_{[0,t)} \frac{1}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g + i\,\omega_0 \int\limits_{[0,t)} \frac{\Delta^+ g(s)}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g.$$

Therefore,

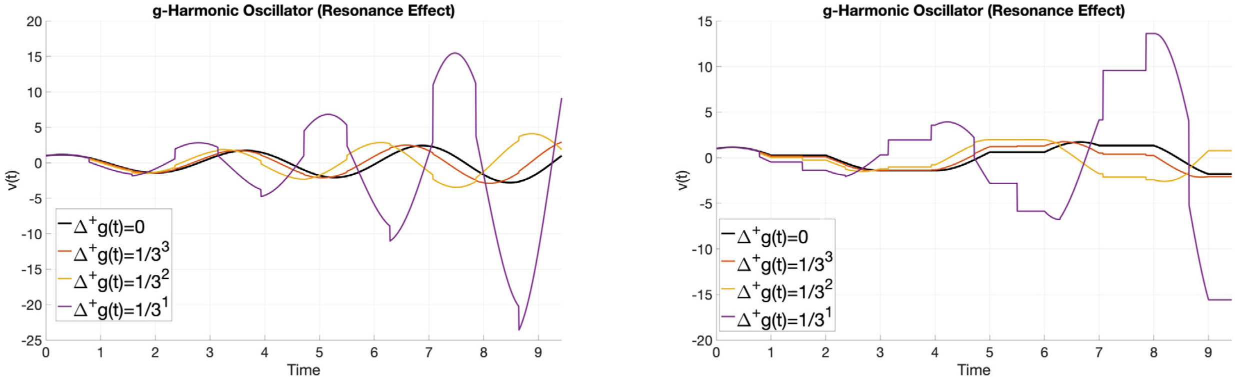
$$v_p(t) = + \frac{1}{2\,\omega_0} \int\limits_{[0,t)} \frac{1}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g \left[\frac{1}{2\,i}\left(\exp_g(+i\,\omega_0;0,t) - \exp_g(-i\,\omega_0;0,t)\right)\right]$$
$$- \frac{1}{2} \int\limits_{[0,t)} \frac{\Delta^+ g(s)}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g \left[\frac{1}{2}\left(\exp_g(+i\,\omega_0;0,t) + \exp_g(-i\,\omega_0;0,t)\right)\right]$$
$$= \frac{1}{2\,\omega_0}\ \sin_g(\omega_0;0,t) \int\limits_{[0,t)} \frac{1}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g$$
$$- \frac{1}{2}\ \cos_g(\omega_0;0,t) \int\limits_{[0,t)} \frac{\Delta^+ g(s)}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g.$$

Finally, the solution of (6.2) is given by

$$v(t) = x_0\ \cos_g(\omega_0;0,t) + \frac{v_0}{\omega_0}\ \sin_g(\omega_0;0,t) + \frac{1}{2\,\omega_0}\ \sin_g(\omega_0;0,t) \int\limits_{[0,t)} \frac{1}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g$$
$$- \frac{1}{2}\ \cos_g(\omega_0;0,t) \int\limits_{[0,t)} \frac{\Delta^+ g(s)}{1+\omega_0^2\,\Delta^+ g(s)^2}\ \mathrm{d}\,\mu_g. \tag{6.3}$$

Observe that if $g(t) = t$, we recover the classical solution and the amplitude of oscillations grows with t.

Example 6.2. Let us take the same data and derivators considered in Example 6.1. In Fig. 6.5 we can see the solution of (6.2) for some of the derivators considered in Example 6.1.



(a) Resonance effect (g_1). (b) Resonance effect (g_2).

Fig. 6.5. Solution of the g-harmonic oscillator associated to g_1 and g_2 (resonance effect) (vertical lines have to be understood as jumps and not as a multivalued function).

Table 1
Numerical errors $e_h = \max\{|v(t_j) - y_{2,j}| : \ j = 0, \dots, N+1\}$ for different values of h.

h	$1.e-1$	$1.e-2$	$1.e-3$	$1.e-4$	$1.e-5$	$1.e-6$
e_h	$4.5260e-01$	$3.8906e-03$	$3.8335e-05$	$3.8274e-07$	$3.8273e-09$	$3.6102e-11$

In order to validate the exact solution (6.3), let us compare the solution (6.3) with the numerical solution of the system:

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix}_g'(t) = \begin{pmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} \cos_g(\omega_0; 0, t) \\ 0 \end{pmatrix}, \\ u(0) = v_0, \ v(0) = x_0. \end{cases}$$

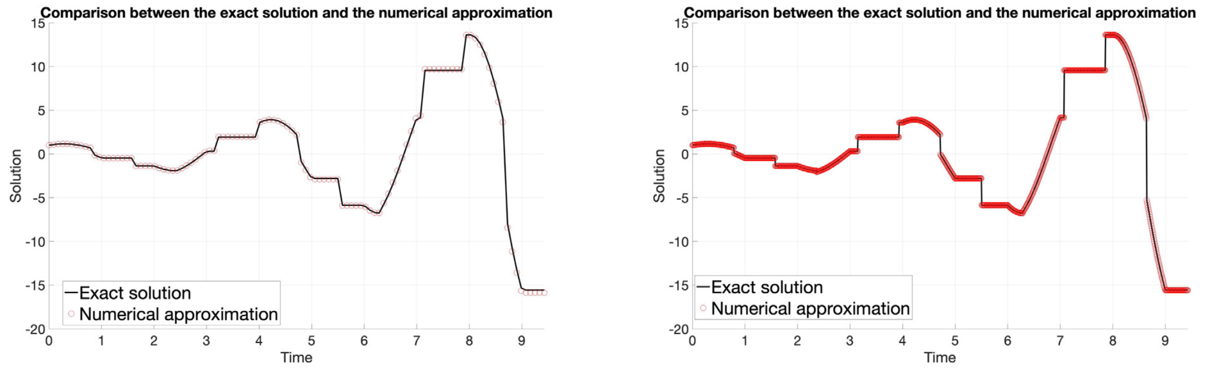
The numerical approximation of the solution of a system of Stieltjes differential equations was introduced in [2], where the authors presented a predictor-corrector numerical scheme to approximate the solution of a Stieltjes differential equation (also for systems) from a quadrature formula for the Lebesgue Stieltjes integral. For this, a finite set of times $\{t_j\}_{j=0}^{N+1} \subset [0, T]$ is considered such that $D_g \subset \{t_j\}_{j=0}^{N+1}$, $t_0 = 0$, $t_{N+1} = T$ and $t_{k+1} - t_k = h > 0$, for every $k = 1, \dots, N$. The application of the numerical method to our case is as follows. Given an element $\mathbf{y}_0 = (v_0, x_0)$, we compute $\{(\mathbf{y}_{j-1}^+, \mathbf{y}_j^*, \mathbf{y}_j)\}_{j=1}^{N+1}$ as

$$\begin{cases} y_{i,j}^+ = y_{i,k} + F_i(t_j, \mathbf{y}_j)\, \Delta^+ g(t_j), \\ y_{i,j+1}^* = y_{i,k}^+ + F_i(t_j^+, \mathbf{y}_j^+)\, (g(t_{j+1}) - g(t_j^+)), \\ y_{i,j+1} = y_{i,j}^+ + \dfrac{1}{2} \left(F_i(t_j^+, \mathbf{y}_j^+) + F_i(t_{j+1}^-, \mathbf{y}_{j+1}^*) \right) (g(t_{j+1}) - g_s(t_j^+)), \end{cases}$$

for every $j = 0, \dots, N$ and $i = 1, 2$, being $\mathbf{y}_j = (x_{1,j}, x_{2,j})$ and

$$\mathbf{F}(t, \mathbf{y}) = \begin{pmatrix} \cos_g(\omega_0; 0, t) - \omega_0^2\, y_2 \\ y_1 \end{pmatrix}.$$

In Table 1 we can see the numerical errors $e_h = \max\{|v(t_j) - y_{2,j}| : \ j = 0, \dots, N+1\}$ for different values of h, taking $g(t) = g_2^C(t) + g^B(t)$, with $\Delta^+ g = 1/3$ (see Example 6.1 for the definitions of g_2^C and g^B).



(a) Exact solution vs. numerical approximation ($h = 1.e - 1$). (b) Exact solution vs. numerical approximation ($h = 1.e - 2$).

Fig. 6.6. Comparison between the exact solution and the numerical approximation (vertical lines have to be understood as jumps and not as a multivalued function).

Finally, in Fig. 6.6, we can see the comparison between the exact solution and the numerical approximation for $h = 1.e - 1$ and $h = 1.e - 2$.

Acknowledgments

The authors would like to thank the anonymous referee for their comments, suggestions and corrections, as they have greatly contributed to improve the quality of the manuscript.

The authors were partially supported by Xunta de Galicia, project ED431C 2019/02, and by the Agencia Estatal de Investigación (AEI) of Spain under grant MTM2016-75140-P, co-financed by the European Community fund FEDER. Ignacio Márquez Albés was partially supported by Xunta de Galicia under grant ED481B-2021-074.

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