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SISTEMAS ELECTORALES: IMPLICACIONES DEMOCRÁTICAS Y ECONÓMICAS

ELECTORAL SYSTEMS: DEMOCRATIC AND ECONOMIC IMPLICATIONS

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A mi hermano Carlos,

por el gran cariño que nos tuvimos.

A mi madre y a mi padre,

quienes han sido mi inspiración y mi fuerza para seguir.

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Índice

Resumen

En esta tesis doctoral se analizan los elementos constitutivos de los sistemas electorales (circunscripciones, formas de candidatura y de votación, métodos de reparto y barreras legales) con el objeto de rediseñarlos buscando que la representación política sea más ecuánime que la obtenida mediante los mecanismos de traducción votos/escaños vigentes en la práctica. Así mismo, el análisis realizado se utiliza para introducir instrumentos más precisos de cuantificación de la calidad democrática. Por otro lado, nos inspiramos en uno de estos elementos, las barreras legales, modificándolo adecuadamente, para mejorar el desempeño del sistema de partidos en cuanto a su financiación estatal. Cabe señalar que, aunque trabajamos desde una amplia perspectiva de sistemas electorales y países, hemos focalizado nuestra atención en México, especialmente tras la última reforma electoral.

En este contexto, nuestra primera propuesta analiza el sistema electoral actual de México y detecta la existencia de paradojas y asimetrías en el tratamiento de los partidos políticos. Para subsanar tales deficiencias se sugiere una modificación, basada en el método de reparto biproporcional, de dicho sistema, que está libre de contradicciones y es más equitativa que la actual.

A continuación, nuestro interés se centra en el estudio de las desviaciones votos/escaños utilizando como herramienta los índices de desproporcionalidad. Hasta ahora la literatura sólo contemplaba índices que utilizaban como referencia la proporcionalidad exacta, entendiendo por tal la que se daría cuando el porcentaje de votos obtenidos por cada partido igualase el porcentaje de escaños asignados al mismo.

Puesto que la proporcionalidad exacta es, en la práctica, inalcanzable, ya que el número de escaños asignado a cada partido debe ser entero, nuestra primera propuesta es más realista que las existentes al considerar que cada partido no debería desviarse de su cuota (representación parlamentaria que debería recibir en proporcionalidad exacta) en más de una unidad. En esta línea, el índice de la cuota que nosotros proponemos no computa lo que denominamos desproporcionalidad forzada, es decir, la inherente al hecho de que las asignaciones de escaños deben ser números enteros, y mide en cambio la desproporcionalidad evitable o no forzada de un reparto electoral. Nuestro

índice, en comparación con los clásicos, tiene una fácil interpretación en términos de transferencia votos/escaños y goza de muchas propiedades consideradas deseables.

Por otro lado, propugnamos alternativamente toda una familia de índices de desproporcionalidad, asociados en cada caso a métodos proporcionales de asignación de escaños. Más concretamente, en virtud de sus buenas propiedades, se aboga por los índices asociados a los métodos de Sainte-Laguë y de D'Hondt. Los nuevos índices introducidos también tienen una clara interpretación al cuantificar las discrepancias entre la asignación real de asientos y la obtenida por la pura aplicación del método considerado, o sea, sin ningún tipo de interferencias tales como la existencia de circunscripciones, de umbrales de exclusión y/o de primas para el ganador.

Todos los índices de desproporcionalidad propuestos se han testado en diversos procesos electorales celebrados en las últimas décadas en países con sistemas electorales muy diferentes.

Habida cuenta de que el sistema de partidos es esencial en las democracias modernas, otra línea de investigación plantea un mecanismo, más eficiente que el actual, para estimar y distribuir el presupuesto público destinado a la financiación de los partidos políticos en México. Para determinar las asignaciones a recibir, nuestra propuesta tiene en cuenta la votación total emitida, estimada por el método de Promedios Móviles Ponderados, y no el número de electores inscritos en el padrón electoral, como actualmente se contempla. Además, para evitar tratamientos no equitativos a partidos políticos con similar número de votos, se emplea una función continua para determinar la barrera a la entrada de reparto de los recursos disponibles.

Finalmente, utilizando técnicas de optimización, se presenta una nueva alternativa de redistritación electoral, que además de conjugar algunos de los criterios más habituales en el diseño territorial (equilibrio poblacional, conformidad administrativa, contigüidad y compacidad) presta especial atención a las garantías de representación de grupos minoritarios. Dado que en México existe una población indígena significativa y que su propia ley electoral contempla el respeto a este tipo de minorías, el modelo propuesto es especialmente indicado para dicho país, habiendo sido aplicado por nosotros en el estado de Chiapas. No obstante, la evidencia de nuestros resultados sugiere que siempre que se pueda establecer la característica de grupo minoritario sobre la unidad básica territorial, el método puede ser usado exitosamente para fines prácticos de distritación político-electoral, independientemente de a qué tipo de minoría se preste atención.

Abstract

In this Ph.D. thesis, the constitutive elements of electoral systems (constituencies, forms of candidacy and voting, apportionment methods and legal barriers) are analyzed in order to redesign them, seeking to achieve more equanimous political representation than that obtained through the mechanisms of votes/seats translation usually in practice. Likewise, the analysis carried out is used to introduce more precise instruments for quantifying democratic quality. On the other hand, we are inspired by one of such electoral elements, legal barriers, modifying it appropriately in order to improve the performance of the party system in terms of its state funding. It is worth to be noted that, although we work from a broad perspective of electoral systems and countries, we have focused our attention on Mexico, especially after the last electoral reform.

In this context, the first proposal analyzes the present electoral system in Mexico and points out the existence of paradoxes and asymmetries in the treatment of political parties. To correct such deficiencies, a modification based on the biproportional apportionment method is suggested, which is free of contradictions and more equitable than the current one.

Next, our interest is focused on the study of vote/seat deviations using disproportionality indices as a tool. Up to now, literature only contemplate indexes that used exact proportionality as a reference, understood as the one that would occur when the percentage of votes obtained by each party equals that of seats assigned to it.

Since the exact proportionality is, in practice, unattainable, due to the number of seats assigned to each party must be whole, our first proposal is more realistic than the existing ones, considering that each party should not deviate from its quota (parliamentary representation it should receive in exact proportionality) more than one unit. In this sense, the proposed quota index does not compute what is called forced disproportionality, that is, that inherent to the fact that seat allocations must be whole numbers, and instead the new index measures the avoidable or unforced disproportionality of an electoral result. Our index, compared to the classic ones, is easy to understand in terms of vote/seat transfers and has many considered as desirable properties.

On the other hand, we alternatively advocate for a whole family of disproportionality indexes, associated in each case with proportional methods of seats allocation. More specifically, because of their good properties, the indices associated with the Sainte-Laguë and D'Hondt methods are considered. The new introduced indices also have a clear interpretation in quantifying the discrepancies between the actual allocation of seats and that obtained by the pure application of the considered method, that is, without any type of interferences such as the existence of districts, exclusion thresholds and/or premiums for the winner.

All the proposed disproportionality indices have been tested in various electoral processes held in recent decades in several countries with very different electoral systems.

Given that the party system is essential in modern democracies, another line of research proposes a mechanism, more efficient than the current one, to estimate and distribute the public budget destined to the financing of political parties in Mexico. To determine the shares to receive, our proposal takes into account the total vote cast, estimated by the Weighted Moving Average method, and not the number of voters registered in the electoral roll, as currently contemplated. In addition, to avoid unfair treatments of political parties with similar amounts of votes, a continuous function is used to determine the barrier to enter the distribution of available resources.

Finally, using optimization techniques, it is presented a new alternative of electoral redistricting which combines some of the most common criteria in territorial design (population balance, administrative compliance, contiguity and compactness) and also pays special attention to representation guarantees of minority groups. As in Mexico there is a significant indigenous population and given that its own electoral law contemplates respect to this type of minority, the proposed model is especially suitable for the mentioned country, having been applied by us in the state of Chiapas. However, the evidence from our results suggests that, as long as the minority group characteristic can be established over the basic territorial unit, the method can be successfully applied for practical political-electoral districting purposes, regardless of what type of minority in which attention is directed.

Estructura de la tesis doctoral

Esta tesis doctoral se presenta bajo la modalidad de tesis por compendio de artículos publicados, de acuerdo con la normativa vigente para la presentación y defensa de la tesis doctoral de la Universidad de Valladolid, aprobada por el Consejo de Gobierno el 3 de junio de 2016 (BOCyL nº 114 del 15 de junio de 2016). La tesis está constituida por un total de cinco artículos publicados en un medio de impacto, según los criterios de la ANECA para el área de conocimiento en la que se presenta la tesis. Además, opta a mención internacional, conforme a los requisitos recogidos en el artículo 15.1 de R.D. 99/2011.

El primer artículo está publicado en una revista online OpenAIRE, los artículos segundo, tercero y cuarto están publicados en revistas indexadas en SJR y el quinto artículo está publicado en una revista indexada en WOS-JCR, por lo que cumple lo establecido por la Comisión Académica del Programa de Doctorado en Economía de la Universidad de Valladolid. A continuación, se indican los artículos que se compilan en esta tesis y sus referencias de publicación:

1. Arredondo, V., Martínez-Panero, M., Peña, T., Ramírez, V.: "Electoral mathematics and asymmetrical treatment to political parties: The Mexican case". *International Journal of Law and Poltical Sciences* 11, pp. 214-222, 2017.
   DOI: 10.5281/zenodo.1340054
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2. Martínez-Panero, M., Arredondo, V., Peña, T., Ramírez, V.: "A new quota approach to electoral disproportionality". *Economies* 7 (1), pp. 1-17, 2019.
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3. Arredondo, V., Martínez-Panero, M., Palomares, A., Peña, T., Ramírez, V.: "New indexes for measuring electoral disproportionality". *Rect@* 21 (2), pp. 45-62, 2020. DOI: 10.24309/recta.2020.21.1.04

   Indexada en: Scopus, Fuente Académica Plus, ABI/INFORM, DOAJ, DIALNET

   Journal Rank: SJR – Q4 (Economics, Econometrics and Finance).

4. Arredondo, V., Vega-Esparza, R. M., Villegas-Santillán, M. T., Álvarez-Diez, R. C.: "Nueva propuesta de financiamiento público a partidos políticos en México". *Revista de Ciencias Sociales 27 (1), pp. 84-98, 2021.*

   DOI: 10.31876/rcs.v27i1

   Indexada en: Scopus, Scimago, REDALyC, DOAJ.

   Journal Rank: SJR– Q3 (Social Science).

5. Arredondo, V., Martínez-Panero, A., Peña, T. Ricca, F.: "Mathematical political districting taking care of minority groups". *Annals of Operations Research*. 305, pp. 375-402, 2021.

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   Journal Rank: WOS-JCR Q1 (Management Science and Operations Research).

A continuación, se detalla la filiación de los coautores de los citados artículos:

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Introducción

El estudio de las decisiones colectivas en el campo electoral, en lo que atañe a las sociedades y a su desarrollo democrático, es hoy en día de gran interés e importancia en los contextos internacional, nacional y local. La disciplina que analiza la toma de decisiones colectivas a partir de las preferencias individuales es la Teoría de la Elección Social (Massó [1]). Esta teoría utiliza una serie de modelos matemáticos con los que se procesan los elementos individuales (votos, preferencias, intereses, juicios, etc.) para convertirse en decisiones que representarán las preferencias colectivas en aras del bienestar social.

Según Dryzek y List [2], la Teoría de la Elección Social es:

> […] una teoría matemática de la toma de decisiones grupales. Su preocupación no es tanto la cuestión empírica de cómo los grupos realmente toman decisiones, sino las preguntas normativas y lógicas de cómo deberían, y podrían, agregar información sobre las opiniones, intereses o preferencias de los individuos en las decisiones de grupos. El aspecto normativo es la especificación de condiciones mínimas que debe cumplir un mecanismo de agregación aceptable.

Bajo esta teoría, diversos trabajos han expresado formalmente procedimientos para que, a partir de las decisiones individuales, sea posible manifestar y construir las decisiones colectivas. Arrow [3] presentó estudios en los que se analizan las decisiones colectivas ejercidas en el ámbito del capitalismo democrático, y señaló que una de las principales aplicaciones de la Elección Social está en el ejercicio del voto, finalmente considerado para la toma de decisiones en la política.

En este sentido, autores como Black [4] y Arrow [3] ponen de manifiesto que el problema de favorecer por elección a un candidato de entre un número de estos es similar a elegir una alternativa dentro de un conjunto de estas para implementar una política social. Así, asumiendo comparaciones interpersonales de utilidad, se determina un orden de preferencias sociales de acuerdo con la suma de utilidades individuales, que son representadas en una función de bienestar social. De este modo se han propuesto diversos sistemas de votación para determinar la elección de una opción dentro de un grupo de estas.

El ejercicio del voto, la forma de elegir, se articula mediante los sistemas de votación. Un sistema de votación es el conjunto las reglas que rigen la forma en la que los votantes pueden expresar sus deseos y cómo se agregan estas preferencias para llegar a un resultado final. Sin embargo, por el Teorema de Imposibilidad de Arrow se sabe que no existe ningún sistema de votación que agregue las preferencias individuales en una preferencia colectiva y que además satisfaga criterios deseables de racionalidad, a menos que las preferencias sean las de un solo individuo considerado como dictador (véase Arrow [3, pp. 59 y ss.]).

En este contexto, el análisis de las revisiones que Sen [5] aplica al Teorema de Imposibilidad de Arrow, conduce a formular un concepto más general que el de función de bienestar social: el de función de decisión social.

Sen [5], además, propone contemplar dentro de las preferencias individuales, condiciones que permitan la adición de valores individuales en las decisiones colectivas, y que posibiliten la existencia de normas para la adopción de resoluciones sociales que resulten consistentes con una clase de derechos para los individuos, entre ellos la libertad de participar en la economía, pero también la libertad de expresión y participación política.

En los regímenes democráticos, la participación política se encauza a través de los sistemas electorales. De acuerdo con Nohlen [6],

> los sistemas electorales determinan las reglas a través de las cuales los electores pueden expresar sus preferencias políticas y mediante las cuales es posible convertir los votos en escaños parlamentarios (en caso de elecciones parlamentarias) o en cargos de gobierno (en caso de elecciones para presidente, gobernador, alcalde, etc.).

La historia de la democracia representativa y de los comicios electorales se vincula con el surgimiento del régimen burgués-liberal. Los estados occidentales pasaron de administraciones feudales a estados-nación. En este tránsito, los sistemas electorales fueron evolucionando y madurando. Este hecho se dio paralelamente a la emergencia de los individuos-ciudadanos y de la democracia participativa (véase Bobbio y Matteucci [7]).

En la actualidad, cualquier sistema democrático se asienta en el sufragio libre, directo y secreto de la ciudadanía; por ello los sistemas electorales son esenciales en nuestra actual cultura política (véase Loewenstein [8]). Mediante el voto popular se define quiénes serán los actores que ocupen los espacios de toma de decisión en un estado. Las personas candidatas a cargos de elección popular presentan al electorado una plataforma política en sus "actos de captación de voto". La organización y desarrollo de los procesos electorales (en lo nacional y en lo local) tiende hoy día a lograr una mayor interés y participación ciudadana.

Concretamente, para Nohlen [6, p. 3] son cuatro los elementos de los sistemas electorales que determinan las reglas para que los ciudadanos expresen sus preferencias políticas y los votos se conviertan en escaños parlamentarios (o en cargos de gobierno):

1. *La distribución de las circunscripciones electorales*.
2. *Las formas de candidatura y de votación*.
3. *La conversión de votos en escaños*.
4. *La barrera legal*.

La aplicación de estos elementos, que pueden ser administrados a través de esquemas concretos que los integren en distintas formas, tiene claras repercusiones políticas. Así mismo, como pondremos de manifiesto más adelante, también individualmente o en combinación, han sido factores cruciales en distintos aspectos de gestación y elaboración de nuestros trabajos.

En primer lugar, el diseño de las **circunscripciones electorales** determina su número y su tamaño. Una distritación (o demarcación política del territorio) puede crearse con fines electorales; o bien, esta puede concordar con la división político-administrativa del país. El tamaño no considera la extensión territorial, sino cantidad de escaños correspondientes a cada circunscripción, que pueden clasificarse como uninominales y plurinominales. Las dimensiones de las circunscripciones son de gran importancia a efectos del sistema electoral, tanto para la relación entre votos y escaños como para las oportunidades electorales de los partidos políticos.

En cuanto a las **formas de candidatura y votación**, cabe distinguir entre candidatura individual o de lista. Ambas se vinculan con las prácticas de votación; la diferencia radica en la opción del ejercicio del voto. Consecuentemente, la distinción básica es entre el voto por un candidato individual o el voto por una lista de partido, con una gran casuística para la segunda modalidad.

Las formas de candidatura y de votación tienen una gran importancia, especialmente, en tres sentidos:

- Para la relación entre elector y candidato-parlamentario.
- Para la relación entre los candidatos-parlamentarios y su respectivo partido.
- Para la posibilidad de los partidos de estructurar la composición de los grupos políticos en el parlamento.

Por otro lado, la **conversión de votos en escaños** considera la fórmula o regla de decisión utilizada para, a partir de la voluntad popular, decidir los ganadores y perdedores en una elección. En este contexto se puede distinguir entre la regla mayoritaria (absoluta y relativa) y la de representación proporcional. De acuerdo con la primera, solo se toma en cuenta en la adjudicación de escaños a los vencedores en las circunscripciones respectivas; en el caso de la fórmula proporcional, la adjudicación de escaños resulta del porcentaje de votos que obtienen los distintos candidatos o partidos (una visión de conjunto sobre estos aspectos puede encontrarse en Penadés [9]).

Los métodos matemáticos más importantes para el reparto proporcional son los de cocientes y restos, y los de divisores.

- Entre los principales métodos de cocientes y restos se encuentran la regla de Hamilton o de los Restos Mayores y la regla de RM-Droop. Así mismo, algunos de los métodos de divisores son la regla de D'Hondt, la de Sainte-Laguë, y la regla de Adams. Estos métodos proporcionan mecanismos concretos bajo los que distribuir los escaños. Por supuesto, cada uno de ellos tiene ventajas y desventajas.

  Por ejemplo, el método de Restos Mayores, tiene la ventaja de verificar la condición de la cuota, es decir, produce repartos que difieren en menos de un escaño de la asignación ideal (y no factible, al no ser entera) que se obtendría en proporcionalidad pura. Sin embargo, la aplicación de este método podría presentar dos paradojas: la paradoja de Alabama y la Paradoja de los Votos. La

Paradoja de Alabama representa el incumplimiento de la monotonía respecto al número de escaños. Ocurre cuando al aumentar el número de escaños a repartir (manteniéndose constante el resto de las variables), algún partido puede recibir menos escaños. Por su parte, la Paradoja de los Votos (incumplimiento de la monotonía respecto al número de votos) surge cuando al cambiar el número de votantes (manteniéndose constante todo lo demás), un partido puede perder un escaño a favor de otro, incluso, si el número de votantes del primer partido crece más que el del segundo.

- Con respecto a los métodos de divisores, éstos son invulnerables a la paradoja de Alabama, y son monótonos respecto del número de votos (invulnerables a la paradoja de los votos). Por otro lado, tienen la desventaja de no verificar la condición de la cuota.

El hecho de que, en todos los casos mencionados, los métodos de reparto presenten algún tipo de paradoja es, en cierto modo, paradigmático. Balinski y Young [10] demostraron un teorema de la imposibilidad que, en este contexto electoral, es análogo al de Arrow en Elección Social. Dicho teorema establece que no existe ningún método de asignación de escaños que cumpla simultáneamente las cuatro siguientes propiedades:

- *Verificación de la cuota*: Ninguna de las diferencias entre escaños y cuotas debe ser superior a la unidad.
- *Monotonía respecto al número de escaños*: Al aumentar el número de escaños, ningún partido debería recibir menos escaños, para una asignación fija de votos.
- *Monotonía respecto de los votos*: Al comparar los resultados de dos elecciones, si el número de votos de un partido aumenta y el de otro disminuye, no debería ocurrir que el primero tuviera menos escaños, y el segundo más, de los que hubieran logrado anteriormente.
- *Homogeneidad*: La solución no debe alterarse si los números de votos se multiplican por un factor $\lambda > 0$.

A partir de este teorema sabemos que no hay una regla de asignación de escaños perfecta. A pesar de ello y con esta realidad como horizonte teórico, se trata de encontrar modelos adecuados que, en la medida de lo posible, mejoren la representación electoral.

Por último, con respecto a las **barreras legales** de entrada (o, alternativamente, umbrales de exclusión) al reparto de escaños por representación proporcional, cabe señalar que, a diferencia de las barreras naturales que el tamaño de las circunscripciones impone en el proceso de convertir votos en escaños, existen restricciones legales que se promulgan con el objetivo de excluir a los partidos pequeños de la representación parlamentaria, y de ejercer un efecto concentrador sobre el sistema de partidos.

Así, usualmente, los partidos tienen la obligación de obtener cierto número de votos o un porcentaje de votos, definidos en las constituciones y en leyes secundarias, para tener derecho a participar en distribución de escaños o curules a nivel de circunscripciones plurinominales o nacionales.

En la literatura es habitual la clasificación de sistemas electorales atendiendo a la regla utilizada para la conversión de votos en escaños o, alternativamente, al tipo de representación que se trata de conseguir.

Así, el principio de representación de mayoría permite a un partido alcanzar una mayoría absoluta en el parlamento. Esto implica la disminución de los partidos más pequeños. En cambio, el objetivo de la representación proporcional es distribuir equitativamente los escaños con base en la votación obtenida. La representación por mayoría no considera el grado de desproporción; por el contrario, la proporcional enfatiza el equilibrio entre votos y escaños obtenidos. Con la primera fórmula, una mayoría de votos a favor de los candidatos o partidos determina al ganador. El resto de la votación se pierde. En el caso de la fórmula de representación proporcional, la proporción de votos a favor determina la cantidad de escaños y todos los votos cuentan, pero su valor puede variar en función del tamaño de la circunscripción, o de la introducción de barreras legales.

Cabe señalar que, junto a los clásicos sistemas electorales de mayoría y representación proporcional, se encuentran los denominados sistemas mixtos. Desde hace tiempo se discute sobre si estos constituyen por sí mismos un tercer tipo de sistema electoral. Los sistemas mixtos combinan elementos técnicos de la representación por mayoría y de la representación proporcional. Desde esta perspectiva, Bobbio y Matteucci [7], señalan que: "los sistemas electorales clásicos son los de mayoría y el proporcional; los derivados y mixtos, constituyen modificaciones o perfeccionamiento de estos".

Sin embargo, basándose en los dos principios de representación elementales, a efectos de análisis y diseño electoral se reconoce la existencia de los tres sistemas cuando se combinan los elementos -circunscripciones, candidatura-votación, conversión de votos en escaños y barreras a la entrada- con el fin modelar múltiples tipos de sistemas electorales con fundamento en los dos principios básicos de representación.

Desde un punto de vista metodológico, los investigadores analizan los sistemas electorales a partir de enfoques teóricos o empíricos. Los primeros se denominan enfoques normativos (véase Nohlen [6]), y estudian los sistemas electorales con base en la teoría y la filosofía políticas, en oposición a la experiencia y el contraste con datos reales. Los empíricos, a su vez, se pueden dividir en comparativo-cuantitativos y comparativo-cualitativos.

Los comparativo-cuantitativos son de tipo estadístico y estudian los elementos de varios sistemas electorales más efectivamente que los normativos. Los comparativo-cualitativos investigan sistemas electorales concretos, en su contexto sociopolítico, para describir, explicar y predecir su funcionamiento y sus efectos.

Estos enfoques se han utilizado para estudiar si los efectos del sistema electoral influyen directamente sobre el sistema de partidos, o sí, además, otros factores ejercen influencia sobre aquellos. También se ha preguntado si el sistema de partidos determina el sistema electoral: cuándo los partidos lo seleccionan, y también cuándo el sistema de partidos influye sobre los efectos que puede tener aquel (Duverger [11]).

Nuestro enfoque metodológico ha sido fundamentalmente empírico, con tratamiento de datos electorales reales de distintas legislaturas y países para distintos fines. Pero, como se verá en lo que sigue, no hemos renunciado a una componente teórico-normativa en nuestros trabajos, imponiendo propiedades o criterios deseables o estudiando la idoneidad y verificación de ciertas propiedades.

En el contexto expuesto, el objetivo de esta tesis se concentra en analizar y proponer nuevas formulaciones de algunos elementos de los sistemas electorales, evaluar su representatividad y exponer nuevos planteamientos para determinar el financiamiento de partidos políticos. Para tal fin, se ha analizado la presencia de paradojas y reformulado del método de reparto de escaños en el sistema electoral mexicano. Así mismo, se han introducido nuevas medidas más realistas de desproporcionalidad electoral que las existentes. Además, se ha propuesto una nueva metodología para asignar recursos públicos a partidos políticos para el caso de México. Finalmente, se ha diseñado y aplicado un nuevo modelo de distritación electoral, prestando atención a grupos minoritarios.

Esta tesis se articula en su modalidad de compendio y se organiza como sigue.

En el **CAPÍTULO 1** se realiza un análisis crítico y se sugiere una propuesta de reforma del Sistema Electoral Mexicano.

La constitucionalización de los partidos políticos en México tuvo lugar con la reforma electoral de 1977. Desde entonces, el sistema electoral mexicano que se utiliza en la elección de la Cámara de Diputados (también llamada la Cámara Baja del Congreso) ha sido mixto. Inicialmente, la Cámara estaba compuesta por 300 diputados elegidos por mayoría relativa (MR) y 100 por representación proporcional (RP). Los diputados RP eran solo asignados a partidos con menos de 60 asientos MR que alcanzaran al menos el 1,5% del voto nacional. La asignación de escaños de RP se modificó en la reforma de 1986-1987, y el número de diputados RP aumentó a 200. La reforma también declaró que ningún partido político podría tener más de 350 diputados (70% de la Cámara) incluso si hubieran obtenido un mayor porcentaje de votos. Además, se incorporó una cláusula de gobernabilidad para garantizar la mayoría absoluta de la Cámara de Diputados para el partido con el mayor número de asientos.

Esta doble estructura se ha mantenido en las sucesivas reformas electorales llevadas a cabo a lo largo de las últimas décadas (ocho de 1986 a 2014). En todas ellas, la distribución de escaños RP no es independiente de la elección por MR, ya que aquella intenta corregir los desequilibrios de representación producidos en los distritos uninominales. El proceso de elección de los 200 escaños RP siempre ha sido bastante complejo y las fórmulas legales introducidas, difíciles de entender e interpretar. Más aún, algunos estudios han expuesto fallos del sistema electoral mexicano y han demostrado su no aplicabilidad en algunas situaciones. Además, han señalado cómo el partido en el poder, el PRI, ha sido injustamente favorecido (Balinski y Ramírez [12, 13, 14]).

Los trabajos anteriores se realizaron antes de la última reforma electoral, que tuvo lugar en 2014 y que se denominó Ley General de Instituciones y Procedimientos Electorales (LGIPE). Nosotros hemos analizado el sistema electoral mexicano después de dicha reforma, aplicada por primera vez en 2015. En ese proceso electoral se necesitaron negociaciones internas entre algunos partidos para alcanzar una asignación final de escaños RP. En nuestro estudio hemos mostrado que la aplicación de la ley actual provocó efectos no deseados, como son inconsistencias y tratamiento asimétrico entre partidos políticos favoreciendo a los partidos grandes frente a los candidatos independientes. Además, hemos detectado que con la actual ley se pueden presentar tres paradojas indeseables en un sistema electoral:

1. *Paradoja de la abstención*: Algunos votantes podrían resultar perjudicados por el hecho de votar honestamente en lugar de abstenerse.
2. *Paradoja de los votos* (incumplimiento de la monotonía respecto al número de votos): al cambiar el número de votantes (manteniéndose constante todo lo demás), un partido podría perder un escaño a favor de otro, incluso si el número de votantes del primer partido creciera más que el del segundo.
3. *Paradoja de Alabama* (incumplimiento de la monotonía respecto al número de escaños): podría suceder que al aumentar el número de escaños a repartir (manteniéndose constante todo lo demás) algún partido recibiera menos escaños.

Para evitar esos problemas, presentamos una propuesta de reforma electoral basada en el método biproporcional. Este método fue propuesto por Balinski y Demange [15] y permite corregir desajustes e incompatibilidades de representación electoral, toda vez que los escaños totales han de asignarse en una doble vía: a partidos y a circunscripciones. En tal sentido, la técnica de biproporcionalidad permite transformar la matriz que contiene los votos de todos los partidos en todas las circunscripciones en otra matriz, que contiene los escaños que corresponden a cada partido en cada circunscripción electoral, de tal forma que partidos y circunscripciones reciban escaños de acuerdo con sus votos totales y con sus poblaciones respectivamente (véase Balinski y Pukelsheim [16] para más detalles). Debido a las dificultades que conllevan los cálculos, se requiere asistencia computacional y, a estos efectos, se ha desarrollado desde la Universidad de Augsburgo el programa informático BAZI [17] para implementar el procedimiento. En 2006, a sugerencia de Pukelsheim, este sistema fue adoptado oficialmente por el cantón suizo de Zürich, que hasta entonces usaba uno muy similar al que se emplea actualmente en España. Se cambió de sistema porque algunas de las circunscripciones eran relativamente pequeñas (4 escaños), mientras que el voto estaba bastante repartido entre muchos partidos (9 partidos), condiciones en las que es imposible conseguir ningún tipo de proporcionalidad. Posteriormente, este método ha sido adoptado por seis cantones más.

Utilizando los datos reales de las últimas elecciones celebradas en México, hemos mostrado que nuestra propuesta de reforma electoral basada en la mencionada técnica biproporcional produce resultados electorales libres de contradicciones y paradojas, permitiendo un tratamiento más justo de los partidos políticos. Además, el sistema planteado es más simple y claro que el actual, por lo que previsiblemente gozaría de una mayor permanencia y estabilidad que los previos.

Los análisis realizados y el detalle de nuestra propuesta de reforma aparecen recogidos en la publicación "Electoral mathematics and asymmetrical treatment to political parties: The Mexican case".

Nuestros dos siguientes trabajos tienen como objeto el análisis y la cuantificación de la desproporcionalidad electoral en distintas formas.

En los sistemas electorales proporcionales, los partidos reciben un porcentaje de escaños de acuerdo al porcentaje de votos obtenido. Como los escaños no se pueden dividir es imposible asignar escaños de manera que su porcentaje iguale el correspondiente de votos. Este problema de reparto genera lo que se conoce como desproporcionalidad electoral, definida en Casal [18] como "el grado de diferencia producida entre la proporción de votos y la proporción de escaños obtenidos por cada partido en competencia". Como consecuencia, algunos partidos están sobrerrepresentados mientras que otros están infrarrepresentados. Además, esa desproporcionalidad puede verse incrementada por la existencia de circunscripciones, de barreras electorales y de primas al vencedor. Como señala Karpov [19] la desproporcionalidad no es meramente un problema matemático sino también político, ya que se trata de una distorsión en las preferencias de la ciudadanía.

Como se observa, la definición anterior es laxa y solo da cuenta de la existencia de desproporcionalidad electoral pero no de su medida. A este respecto, cabe señalar que no se ha llegado a un consenso sobre qué instrumento usar para medir tales distorsiones.

Un mecanismo utilizado recurrentemente en la literatura es la estimación de índices con objeto de conocer si el sistema electoral usado produce resultados más o menos próximos a la proporcionalidad exacta, entendiendo por tal la situación aquella en la que el porcentaje de votos obtenidos por cada partido igualase el porcentaje de escaños asignados al mismo[1].

Así, tanto Taagepera y Grofman [21] como Karpov [19] presentan una extensa variedad de índices (se han propuesto en la literatura alrededor de veinte) y muestran un análisis de propiedades que estos cumplen (véase también Urdánoz [22]). También, Borisyuk et al. [23] hacen una contribución notable, destacando las interrelaciones entre algunos de ellos. Por otro lado, Ocaña y Oñate [24] diseñan un programa informático mediante el que es posible calcular nueve índices de desproporcionalidad. Aunque su análisis no explica las ventajas y desventajas de los índices considerados, sí destaca el hecho de contar con una herramienta que los calcula rápidamente, que ha sido testada para que no exista posibilidad de errores.

Puesto que la proporcionalidad exacta es, en la práctica, inalcanzable, nuestra primera propuesta, expuesta en el **CAPÍTULO 2** se basa en un enfoque más realista, que no pretende que el porcentaje de escaños asignados a cada partido sea igual al porcentaje de votos obtenidos por el mismo, sino que se mantenga dentro de unos límites razonables. De acuerdo con Balinski y Young [10], el número de escaños que recibe un partido no debería desviarse de su cuota (número de escaños que debería recibir en proporcionalidad exacta) en más de una unidad. En otras palabras, ningún partido debería recibir menos escaños que su cuota redondeada hacia abajo ni más de su cuota redondeada hacia arriba. Esta propiedad, de la que ya hablamos al tratar los métodos de reparto proporcional, se conoce como verificación de la cuota o también estar dentro de la cuota.

[1] O sea, el desiderátum de "un hombre, un voto". Es significativo, a este respecto, el título de Balinski y Young [10]: Fair Representation: Meeting the Ideal of One Man, One Vote, sobre todo, sabiendo que su resultado principal es un teorema de imposibilidad, tal como se ha señalado. Aspectos históricos sobre este ideal democrático pueden encontrarse en Urdánoz [20].

En este contexto, se sitúa el índice de la cuota que nosotros proponemos. Este índice utiliza un nuevo criterio de referencia, pues a diferencia de los índices clásicos, que solo se anulan en el caso ideal de que exista proporcionalidad exacta, para que el índice de la cuota sea cero es necesario y suficiente que se satisfaga la condición de la cuota. Así, el nuevo índice no computa lo que denominamos desproporcionalidad forzada, es decir, la inherente al hecho de que las asignaciones de escaños deben ser números enteros, y mide en cambio la desproporcionalidad evitable o no forzada de un reparto electoral.

Una ventaja de nuestro índice es que, a partir del mismo, es posible calcular la cantidad mínima de escaños que es necesario transferir entre los partidos sobre e infrarrepresentados para que se satisfaga la cuota a nivel global. Otra es que, al contrario de lo que ocurre con los más importantes índices de desproporcionalidad usados en la práctica, la existencia de muchos partidos con una cuota inferior a la unidad y sin representación parlamentaria no incrementa el valor del índice de la cuota.

Asimismo, hemos demostrado que nuestro índice cumple muchas propiedades que se consideran deseables. En particular, merece la pena destacar que el índice de la cuota aventaja a los utilizados habitualmente en cuanto al cumplimiento del principio de transferencia. Por otro lado, hemos comprobado que nuestro índice no es homogéneo en relación al número de escaños, pero también hemos justificado por qué ese hecho tiene sentido en nuestro contexto.

Lo anteriormente apuntado da buena cuenta de la idoneidad del índice de la cuota introducido a efectos del cálculo de la desproporcionalidad electoral.

También hemos establecido relaciones cuantitativas entre el nuevo índice y los más relevantes que aparecen en la literatura (máxima desviación, Gallagher y Loosemore-Hanby). Finalmente, para testar su implementación, hemos calculado todos ellos en las elecciones celebradas en las últimas décadas en Suecia, Alemania y España, tres países con sistemas electorales muy distintos. Los resultados obtenidos muestran que existe una elevada correlación entre el índice de Loosemore-Hanby y el índice de la cuota, pero un importante argumento a favor del último es su interpretabilidad en términos de transferencia.

Los resultados del estudio de este índice se recogen en el artículo "A new quota approach to electoral disproportionality".

Por otro lado, en el **CAPÍTULO 3**, como paso ulterior, propugnamos toda una familia de índices de desproporcionalidad, asociados en cada caso a métodos proporcionales de asignación de escaños que, tal y como se ha indicado, se suelen dividir en dos clases: de cocientes y de divisores. De entre estos últimos, en virtud de las propiedades observadas, se aboga por los índices de desproporcionalidad asociados a los métodos de Sainte-Lague y de D'Hondt (denominados de Webster y de Jefferson, respectivamente, en el ámbito anglosajón).

En este nuevo esquema para medir la desproporcionalidad electoral, contrariamente tanto al enfoque clásico que toma como referencia la inalcanzable proporcionalidad exacta, como al utilizado en el capítulo anterior que considera como meta la verificación de la cuota, tenemos en cuenta las desviaciones respecto a otros valores factibles: en este caso, las distribuciones de asientos obtenidas cuando se utiliza un método de reparto proporcional prefijado, sin ningún tipo de interferencias, como son la existencia de circunscripciones, de umbrales de exclusión y/o de primas para el ganador. De esta manera, cuantificamos las discrepancias entre la asignación real de asientos y la obtenida por la pura aplicación del método considerado.

De nuevo, en este escenario, los índices asociados con los métodos que nosotros proponemos son fáciles de calcular y su valor tiene un significado muy claro. Cuando se multiplican por el tamaño del parlamento, muestran el número de escaños necesarios que han de ser transferidos, de partidos con sobrerrepresentación respecto al método elegido a partidos infrarepresentados, para obtener una asignación consistente con dicho método. Por lo tanto, fijado un método de reparto, estos índices cuantifican las distorsiones producidas por los umbrales electorales, las circunscripciones y la bonificación directa al partido ganador.

Es importante tener en cuenta que es posible diseñar un sistema electoral con desproporcionalidad cero, en el sentido presentado en nuestra propuesta, incluso manteniendo varias circunscripciones. Esto se puede lograr si los escaños se asignan a los partidos en proporción al total de votos, y luego, esos escaños se distribuyen a las circunscripciones utilizando un método biproporcional (ya tratado en el Capítulo 1), de modo que cada circunscripción electoral obtenga los escaños previamente establecidos.

Por otro lado, mientras que otros índices clásicos sólo toman el valor máximo 1 en situaciones irreales en las que se asignan escaños a partidos con cero votos, nuestro índice puede tomar ese valor en situaciones más realistas: por ejemplo, en circunscripciones uninominales en los que partidos locales ganasen dichos escaños, pero quedasen excluidos por una barrera a nivel nacional.

En nuestro trabajo hemos contrastado empíricamente los índices propuestos con los clásicos: Gallagher, Loosemore-Hanby y Sainte-Laguë, en 55 procesos electorales celebrados en las últimas décadas en países con sistemas electorales muy diferentes: Alemania, Suecia, España, Portugal y Reino Unido.

A partir de los argumentos expuestos, es posible concluir que los índices asociados con los métodos de Jefferson y Webster son alternativas válidas a los índices considerados hasta ahora para medir la desproporcionalidad de los resultados electorales.

El análisis relacionado con el índice del método se exhibe en el artículo "New indexes for measuring electoral disproportionality".

Por otra parte, el **CAPÍTULO 4** tiene como objetivo central proponer una nueva metodología para estimar y distribuir el presupuesto público destinado a financiar a los partidos políticos en México.

El financiamiento político-legal de los partidos en dicho país se inicia con la constitucionalización de éstos y viene estipulado en la reforma de 1977 de la Constitución Política de los Estados Unidos Mexicanos (CPEUM). Asimismo, sucesivas reformas del Código Federal Electoral desde 1987 han ido perfilando en qué tipo de actividades se invertirían los recursos asignados.

El modelo de financiación vigente toma en cuenta como variables el padrón electoral, la barrera a la entrada fija del 3% de la Votación Total Emitida (VTE) en la elección de diputados inmediata interior y el salario mínimo diario vigente para el Distrito Federal, -cuyos cambios se publican en el Diario Oficial de la Federación en México-, reemplazado el 29 de febrero de 2016 por la Unidad de Medida y Actualización (UMA) que publica diariamente el Banco de México.

Cabe señalar, no obstante, que ciudadanos y diversas organizaciones sociales se han manifestado en contra de tal modelo de financiamiento; o al menos han hecho visible su interés en conocer cómo se designa y se define instrumentalmente el presupuesto que se otorga a estas instituciones. Uno de los principales argumentos para prestar atención al análisis de la estimación del monto de recursos públicos asignados a partidos es que la cantidad asignada es considerablemente mayor a la que se otorga a dependencias relevantes para el desarrollo social y económico de este país.

En la investigación realizada se propone una nueva metodología para distribuir los recursos destinados a respaldar el quehacer de los partidos políticos. Se enfatiza la votación total emitida, estimada por el método de Promedios Móviles Ponderados, como un aspecto relevante a considerar en la estimación global de los recursos, y no el número de electores inscritos en el padrón electoral, como actualmente se contempla.

Por otro lado, hemos observado que un problema de las barreras utilizadas a efectos de reparto de fondos es su discontinuidad. Actualmente los partidos políticos que no reciben el 3% o más de votos respecto de la VTE, no tienen derecho a reparto de recursos públicos. De esta manera, si el partido A obtiene un 2,99% de la VTE, no recibe financiamiento. Por el contrario, si un partido político B cuenta con pocos votos más de los que obtiene el partido A, de modo que alcance el 3% de la VTE, este partido B sí entrará en el reparto de fondos.

Para evitar este tratamiento no equitativo a partidos políticos con similar número de votos, nuestra propuesta contempla una función continua para determinar la barrera a la entrada de reparto de recursos disponibles, inspirada en la sugerida por Ramírez et al. [25] para el reparto de escaños bajo representación proporcional.

Utilizando métodos documentales y descriptivos para el análisis de datos y obtención de resultados en los años del periodo 2008-2020, la implementación de la nueva propuesta de financiación a partidos políticos en México permite concluir que los ahorros que se generarían al aplicarla hubieran sido de un poco más del 50% de lo que actualmente reciben, con lo que, evidentemente, podrían verse beneficiadas instancias de desarrollo social y económico desprotegidas en México.

Los resultados obtenidos se recogen en el artículo “Nueva propuesta de financiamiento público a partidos políticos en México”.

El **CAPÍTULO 5** tiene como fin utilizar técnicas de optimización para proponer una nueva alternativa de redistritación electoral prestando especial atención a grupos minoritarios.

Como se ha señalado con anterioridad, un elemento central de los sistemas electorales es la división del territorio mediante distritos electorales. Un distrito electoral es la demarcación política de un territorio donde se expresan los intereses de sus electores para formar órganos de representación política. El proceso de la delimitación de esta demarcación territorial se conoce como distritación y ha sido ampliamente estudiado (véanse Grilli di Cortona et al. [26] así como Ricca y Scozzari [27]). Ello ha motivado una diversidad de modelos y métodos aplicados a la distritación política (DP), que siempre intenta ser imparcial ante los intereses de los partidos políticos, formando regiones geográficas que no favorezcan a algunos de ellos, estrategia que se conoce como *gerrymandering*.

El tratamiento de este problema requiere un enfoque multicriterio y programación matemática. Concretamente, es usual que los modelos de DP recurran a técnicas de programación matemática entera o mixta. Las diferencias entre los modelos propuestos dependen básicamente de la función objetivo que es minimizada (o maximizada), y del número y tipo de restricciones impuestas en cada formulación. Estas difieren de acuerdo con las características particulares que establece la ley electoral en cada país. La complejidad del proceso de solución depende también de cuáles de los elementos anteriores están considerados en el problema de DP y cómo estos se formulan en el correspondiente modelo matemático. Los criterios clásicos de la DP son: el equilibrio poblacional; respeto de las fronteras administrativas; contigüidad; y compacidad. En Ricca et al. [28] se definen como sigue:

- *Equilibrio poblacional*: todos los distritos deben tener la misma proporción de representación (de acuerdo con el principio “one person, one vote”). Una forma de respetar este criterio consiste en fijar una desviación máxima absoluta respecto del promedio de la población por distrito.
- *Respeto de las fronteras administrativas*: se debe evitar tanto como sea posible la división de zonas administrativas, tales como estados o municipios.

- *Contigüidad*: un distrito es contiguo si es posible conectar cualquier par de puntos en el distrito sin salir de él.
- *Compacidad*: un distrito es considerado más compacto cuanto más redonda es su forma, es decir, cuanto más regular es su frontera y menor dispersión presenta desde el centro.

En general, para la resolución de los problemas propuestos se emplean distintas técnicas de optimización, exacta y heurística. La solución exacta de modelos de optimización hace uso de métodos de programación entera o mixta, mediante los que se encuentra un valor concreto para la función objetivo tomando en cuenta las restricciones a las que esta se encuentra sujeta. Por otro lado, una solución heurística contempla un procedimiento sistemático, intuitivamente eficiente para dar solución aproximada al modelo propuesto. Este tratamiento es especialmente adecuado cuando, por ejemplo, el tamaño y complejidad del problema hace que no se puedan encontrar soluciones óptimas exactas. A este respecto puede consultarse Ríos-Mercado [29].

Además de los criterios clásicos, en la literatura se han introducido otros más debatibles como el respeto a las fronteras naturales (los distritos no deben atravesar ríos, montañas, etc.) y la homogeneidad socio-económica. También, se han discutido ejemplos donde la falta de representación de las comunidades produce soluciones inequitativas en términos de representación de los distritos electorales (Williams Jr. [30]), y adicionalmente se ha señalado la importancia de considerar la representación de las minorías y evitar la división de comunidades con intereses comunes (véase Morril [31]).

Dado que en México existe una población indígena significativa y que su propia ley electoral contempla el respeto a este tipo de minorías, nuestra aportación presenta un nuevo modelo que toma en cuenta, además de los objetivos usuales, un criterio adicional que presta atención a tales grupos minoritarios. En este sentido se ha elaborado la nueva propuesta que contempla un modelo de Programación Lineal Entera, para el que obtuvimos una solución exacta.

El modelo propuesto contempla dos etapas: en la primera centramos nuestra atención en concentrar la población perteneciente al grupo minoritario tanto como fuese posible en los distritos electorales que les corresponden; en la segunda etapa, construimos el resto de los distritos electorales.

El modelo se aplicó en un territorio real, Chiapas (México), que presenta una considerable proporción de población indígena que se encuentra en regiones con niveles considerables de pobreza y marginación. Así mismo, testamos nuestra propuesta en territorios ficticios que se representaron por medio de redes cuadriculadas comprobándose que para territorios con un mayor número de unidades aparecen problemas computacionales que hicieron necesario un enfoque heurístico.

La aplicación de nuestra propuesta permite concluir que, para el caso de Chiapas, la distritación obtenida satisface los requerimientos establecidos en la ley. Adicionalmente, se mejora el diseño distrital vigente, toda vez que éste último permite que varios municipios sean divididos y asignados a distintos distritos.

Por otro lado, los resultados con territorios ficticios permiten confirmar las ventajas de nuestro modelo. Para regiones de hasta 120 unidades (municipios) fue posible aplicar el modelo sin problemas computacionales; paralelamente, en territorios con un mayor número de unidades, el método heurístico puede dar como resultado mapas distritales óptimos en tiempos razonables.

Aunque nuestra propuesta se ha implementado al caso de Chiapas y territorios ficticios, es importante señalar que la evidencia de nuestros resultados sugiere que siempre que se pueda establecer la característica de grupo minoritario sobre la unidad básica territorial, el método puede ser usado exitosamente para fines prácticos de distritación político-electoral, independientemente de qué tipo de minoría que se representa.

Los detalles del análisis y resultados de esta propuesta de distritación se exhiben en el artículo “Mathematical and political districting taking care of minority groups”.

Previamente a su publicación, los estudios y resultados de investigación conducentes a esta Tesis Doctoral se han presentado en diversas convenciones científicas que se reseñan a continuación:

1. *A new proposal for measuring electoral disproportionality: The quota index*, 7th Summer Workshop of the Centre of Mathematics in Social Science, Auckland (Nueva Zelanda), febrero 2016.

2. *A way to determine non-forced disproportionality: the cases of Mexican and Spanish elections*, 8º Congreso Internacional del Consejo Europeo de Investigaciones Sociales de América Latina, Salamanca (España), julio 2016.

3. *Measuring non-forced electoral disproportionality*, Summer School: Computational Social Choice, European Cooperation in Science and Technology, San Sebastian (España), julio 2016.

4. *Apportionment paradoxes and the last electoral reform in Mexico*, Encuentro de la Red Española de Elección Social, Granada (España), noviembre 2016.

5. *Electoral mathematics and asymmetrical treatment to political parties: The Mexican Case*, 19th International Conference on Political Science and Political Economy, Londres (Inglaterra), enero 2017.

6. *Quota conditions in proportional apportionment*, V Jornadas sobre Sistemas de Votación, Valladolid (España), mayo 2017.

7. *A new proposal for measuring electoral disproportionality: the quota index*, 7th International Conference on Approximation Methods and Numerical Modelling in Environment and Natural Resources, Oujda (Marruecos), mayo 2017.

8. *Matemáticas y política. Fórmulas electorales inaplicables: El sistema electoral de México*, VIII Congreso Iberoamericano de Educación Matemática, Madrid (España), julio 2017.

9. *Political Districting and Attention to Minority Groups*. 87th Meeting of the European Working Group on Multicriteria Decision Aiding (EWG/MCDA), Delft (Holanda), abril 2018.

10. *Fair distributions of electoral seats under proporcional representation*, XXVI Jornadas ASEPUMA – XIV Encuentro Internacional de Profesores Universitarios para la Economía y la Empresa, Sevilla (España), junio 2018.

11. *Communities of interest and the districting problem*. European Conference on Operational Research (EURO2018), Valencia (España), julio 2018.

12. *Mathematical political districting and minorities*. XV Encuentro de la REES. Elche (España), noviembre 2018.

13. *Bounding the ideal of "One Man, One Vote"*. XXVII Jornadas ASEPUMA – XV Encuentro Internacional de Profesores Universitarios para la Economía y la Empresa, Santander (España), junio 2019.

Por otro lado, la movilidad para culminar la investigación realizada tuvo los siguientes destinos:

1. Estancia de académica en la universidad de Granada (España), bajo la tutela del Profesor-Investigador Victoriano Ramírez, noviembre 2016.

2. Estancia académica en la universidad de Roma La Sapienza (Italia), bajo la tutela de la Profesora-Investigadora Federica Ricca, septiembre- diciembre 2017.

3. Estancia académica en la universidad de Roma La Sapienza (Italia), bajo la tutela de la Profesora-Investigadora Federica Ricca, septiembre 2018.

4. Estancia académica en la universidad de Roma La Sapienza (Italia), bajo la tutela de la Profesora-Investigadora Federica Ricca, febrero-julio 2019.

Por último, cabe señalar que los artículos que se compilan en esta tesis ya han sido referenciados en el apartado que detalla la estructura de la misma,

CAPÍTULO 1: *Electoral mathematics and asymmetrical treatment to political parties: The Mexican case*

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Electoral Mathematics and Asymmetrical Treatment to Political Parties: The Mexican Case

Verónica Arredondo, Miguel Martínez-Panero, Teresa Peña, Victoriano Ramírez

***Abstract*—**The Mexican Chamber of Deputies is composed of 500 representatives: 300 of them elected by relative majority and another 200 ones elected through proportional representation in five electoral clusters (constituencies) with 40 representatives each. In this mixed-member electoral system, the seats distribution of proportional representation is not independent of the election by relative majority, as it attempts to correct representation imbalances produced in single-member districts. This two-fold structure has been maintained in the successive electoral reforms carried out along the last three decades (eight from 1986 to 2014). In all of them, the election process of 200 seats becomes complex: Formulas in the Law are difficult to understand and to be interpreted. This paper analyzes the Mexican electoral system after the electoral reform of 2014, which was applied for the first time in 2015. The research focuses on contradictions and issues of applicability, in particular situations where seats allocation is affected by ambiguity in the law and where asymmetrical treatment of political parties arises. Due to these facts, a proposal of electoral reform will be presented. It is intended to be simpler, clearer, and more enduring than the current system. Furthermore, this model is more suitable for producing electoral outcomes free of contradictions and paradoxes. This approach would allow a fair treatment of political parties and as a result an improved opportunity to exercise democracy.

***Keywords*—**Apportionment paradoxes, biproportional representation, electoral mathematics, electoral reform, Mexican electoral system, proportional representation, political asymmetry.

I. Introduction

THE constitutionalization of political parties in Mexico took place with the 1977 electoral reform. Since then, the Mexican electoral system used in the election of the Chamber of Deputies (also called the Lower House of the Congress) has been mixed. Initially, the Chamber was composed for 300 deputies elected by relative majority (RM) and 100 by proportional representation (PR). The Deputies of PR were only assigned to parties with less than 60 RM seats reaching at least 1.5% of the national vote [1].

The allocation of PR seats was modified in the electoral reform of 1986-1987. The number of deputies of PR increased to 200. The reform also stated that no political party could have more than 350 deputies (70% of the Chamber) even if they had had a higher percentage of votes. Even more, a governability clause was incorporated to guarantee an absolute majority of the Chamber of Deputies for the party with the most seats [2].

The 1990 electoral reform modified the previous governability clause. In order to reach an absolute majority, a sufficient number of seats was assigned to the party that having obtained the most seats by RM also achieved at least 35% of them. On the other hand, all the parties with at least 35% of the national vote will be given two more seats for each 1% of the votes obtained above 35% and up to 75% [3].

In the next electoral reform (1993-1994), the maximum representation limit for each political party in the Chamber of Deputies was reduced from 350 to 315 seats [4]. It was the 1996 reform which established that no political party could obtain more than 300 deputies in total and that the percentage of deputies for any political party could not exceed 8% of its percentage of the national vote, except for parties that reached this difference by winning districts by RM. Furthermore, the legal threshold to have access to PR seats was increased to 2% [5].

Electoral reforms in 2007-2008 did not consider changes in the method of seat allocations for political parties [6]. The last electoral political reform in 2014, named *Ley General de Institutions y Procedimientos Electorales (LGIPE),* increased from 2% to 3% the minimum percentage required to have right to the allocation of PR seats [7]. It is important to emphasize that, along these processes, the highest remainders method has been used in the distribution of seats. However, the implementation description of the electoral system has always been tedious and ambiguous.

Regarding the electoral system in Mexico, some studies have exposed failures in its electoral law and shown non-applicability in some situations. Furthermore, they have proven how the party in power, the PRI, was unfairly favored [8]-[10]. The hegemonic role of PRI party, which had normally achieved the control in the Chamber, is also analyzed in [11] in terms of number of seats that the parties could reach and those that actually they have reached.

There also exist qualitative studies on the effects of electoral reforms in Mexico, highlighting the political, economic and social circumstances that gave rise to the different electoral reforms. In this way, [12] explains the results of these electoral reforms in terms of the development of democracy in Mexico and illustrates the difference in treatment among political parties.

The paper is structured as follows: Section II introduces the

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basic concepts. Section III presents the implementation of the Mexican Electoral Law for the 2015 election, showing how the obtained results yield inequities concerning the treatment to political parties. Section IV deals with inconsistencies and paradoxes of the Mexican electoral system, considering specific examples that clearly make evident such failures. Section V shows how the implementation of the Electoral Law causes asymmetrical treatment of political parties: Specifically, big parties take advantage of the small ones. In Section VI a proposal for electoral reform in Mexico with a fairer treatment to political parties is suggested. Finally, in Section VII, some conclusions are presented.

II. Basic Concepts

A. Notation

Let V be the number of voters, n the number of parties and S the number of seats to be distributed; $(V_1, V_2, \ldots, V_n)$ is the vector of votes obtained by each party, so that $V = \sum_{i=1}^{n} V_i$, and $(S_1, S_2, \ldots, S_n)$ is the vector of seats assigned to each party, where $S = \sum_{i=1}^{n} S_i$; finally, v_i and s_i are the proportion of votes and seats that party i receives, respectively. Thus, $v_i = V_i/V$ and $s_i = S_i/S$.

The quota is the number of seats that the party i should receive in exact proportionality after obtaining V_i votes. That is, the quota for party i is $q_i = \frac{V_i}{V} S$, $i = 1,2, \ldots, n$.

The lower quota is the closest integer number that does not exceed q_i; it will be denoted by $\lfloor q_i \rfloor$. Likewise, the upper quota is the smallest integer number bigger than or equal to q_i; it will be denoted by $\lceil q_i \rceil$. Those apportionment methods that will be used in this paper (Highest Remainder, Sainte-Laguë and Biproportional apportionment method) will be introduced in what follows (see [13] for more details).

Under the Highest Reminder method (HR), also called Hamilton Rule, each party first receives as many seats as the integer part of its quota, $\lfloor q_i \rfloor$. Then, the remainders $(q_i - \lfloor q_i \rfloor)$ are ordered from the largest to the smallest one. Finally, the remaining seats are assigned to parties with highest reminders up to S seats are completed.

Saint-Laguë method (also called Webster rule) is one of the apportionment divisor methods which considers for each party successive quotients of its number of obtained votes V_i, calculated as

$$\frac{V_i}{s+{}^{1}/_{2}}$$

where $s = 0,1,2 \ldots S-1$.

The S highest quotients determine both the number of seats for each party and the order in which they are allocated. For practical purposes the previous quotients are equivalent to those obtained dividing V_i by $s = 1,3,5 \ldots$

Biproportional apportionment methods has been described by Balinski and Pukelsheim [14] as:

> […] a novel approach of translating electoral votes into parliamentary seats. A two-way proportionality is achieved, to districts relative to their populations, and to parties relative to their total votes. The methods apply when the electoral region is subdivided into several electoral districts, each with a prespecified "district magnitude", that is, the number of seats per district. The input data thus consists of a matrix with rows and columns corresponding to districts and parties, and entries to party votes in districts. A biproportional apportionment method converts the party votes into an apportionment matrix of corresponding seat-numbers such that, within a district, the sum of the seat-numbers matches the prespecified district magnitude, while within a party, the seat-numbers sum to the overall party seats that are proportional to the vote totals across the whole electoral region.

Due to the calculus difficulty (pointed out in [15]), computer programming is necessary to perform biproportional method. Currently there exists free software, called BAZI, which is simple to use [16].

B. General Description of LGIPE

The current Mexican Chamber of Deputies has 500 members. 300 of them are elected by RM in 300 uninominal districts. The remaining 200 seats are allocated under PR through the system of regional lists considering 5 constituencies made up of 40 seats each.

In order to obtain its regional lists registration, a political party must present RM candidates in at least 200 districts. Once this requirement satisfied, any political party that reaches at least 3% of the Valid Cast Votes (sum of the votes casted at the polls minus null votes and votes for unregistered candidates) will be allowed to participate in the distribution of PR deputies.

Concerning the number of seats that a party can reach in total (RM and PR modalities), there are two restrictions. On one hand, no political party can obtain more than 300 seats in total. And, on the other hand, the percentage of deputies obtained by a political party cannot exceed 8% of Effective National Votes (VCV minus the number of votes obtained by independent candidates and by political parties that did not reach 3% of VCV). The second limit does not apply to political parties obtaining under RM a percentage of total Chamber seats higher than the sum of its percentage of ENV plus 8%. For example, if a political party obtains 35% of ENV, it cannot receive more than 43% of all seats in the Chamber; that is, no more than 215 of the 500 deputies.

The allocation of the 200 PR seats is codified in articles 14-21 of LGIPE. The implementation of these articles is developed in the next section for 2015 elections results.

III. Implementation of LGIPE on 2015 Elections

In this section, the 2015 PR seats allocation results will be derived step-by-step. Data of votes in this section are obtained in [17].

First of all, Table I shows the basic data for applying the allocation method: VCV and ENV.

TABLE I
VCV AND ENV FOR 2015 ELECTIONS [17]

Total vote emitted	39,864,082
Nulls votes	1,900,449
Not registered	52,371
VCV	37,911,262
Independent candidates	225,029
Parties under 3% VCV	1,990,817
ENV	35,695,416

The votes and VCV percentages for each political party, the null votes and the votes cast for Independent (IND) and Non-Registered (NR) candidates in 2015 elections are shown in Table II. Likewise, results for RM seats appear in last column of Table II. It can be observed that PT and Humanista parties did not get 3% of VCV, so they had no right to participate in the distribution of PR seats. The remaining parties were allowed to participate in the allocation of PR seats.

TABLE II
DEPUTY ELECTIONS 2015 [18]

Party	Votes	VCV (%)	RM seats
PRI	11,636,957	30.70	155
PAN	8,377,535	22.10	56
PRD	4,335,321	11.44	28
MORENA	3,345,712	8.83	14
PVEM	2,757,170	7.27	29
MC	2,431,063	6.41	10
NA	1,486,626	3.92	1
PES	1,325,032	3.50	0
PT	1,134,101	2.99	6
Humanista	856,716	2.26	0
Nulls	1,900,449	-	-
IND	225,029	0.59	1
NR	52,371	-	-
Total	39,864,082	100	300

TABLE III
MAXIMUM LIMIT OF SEATS BY PARTY

Party	Votes	ENV (%)	ENV(%)+8	Limit of seats by party
PRI	11,636,957	32.60	40.60	203
PAN	8,377,535	23.47	31.47	157
PRD	4,335,321	12.15	20.15	100
MORENA	3,345,712	9.37	17.37	86
PVEM	2,757,170	7.72	15.72	78
MC	2,431,063	6.81	14.81	74
NA	1,486,626	4.17	12.16	60
PES	1,325,032	3.71	11.71	58
Total	35,695,416	100		

Once calculated the VCV and known what parties do not reach the 3% threshold, the next step consists in determining the ENV percentage for each party and checking if there is some party out of the established limits: No party may seat more than 300 deputies and no party's seat share may exceed more than eight percent its ENV share, as mentioned. These requirements can be checked in Table III. Now, articles 16, 17.1 and 17.3 of the LGIPE [7] establish how to assign the PR seats (English version of LGIPE deals with "districts" instead of):

Article 16.

1. For the allocation of representatives by proportional representation according to the provisions of section III of article 54 of the Constitution, a pure proportionality formula integrated by the following elements will be applied:
a) Natural Quotient, and
b) Largest Remainder.
2. Natural Quotient: is the result of dividing the national cast votes by the 200 representatives by proportional representation.
3. Largest Remainder: is the highest remainder between the remaining votes cast for each political party, once the allocation of seats is done by the natural quotient. The largest remainder is used when there are still seats left to distribute.

Article 17.

1. Once the formula mentioned in the previous article has been developed, the following procedure will be observed:
a) It will be decided which representatives will be allocated to each political party, according to the number of times that its votes contain the natural quotient, and
b) The representatives that will be distributed by largest remainder, if there are remaining seats to assign after applying the natural quotient, follow the descending order of the votes not used for each of the political parties in the distribution of seats.

[…]

3. Once the excess number of representatives by proportional representation has been determined, the political party affected by the terms of the previous paragraph will be allocated its corresponding seats for each district, according to the following terms:
a) The distribution ratio will be obtained by dividing the total number of votes of the political party which finds itself in this situation by the seats to be allocated to the same political party;
b) The votes obtained by the political party in each electoral district will be divided by the distribution ratio, allocating in whole numbers the seats for each of the districts, and
c) If there are still representatives left to be allocated, the largest remainder method will be used, as stated in the previous article.

Table IV shows the implementation of the procedure described for 2015 elections. To this aim, the Natural Quotient (NQ) appearing in the Mexican Law is:

$$\text{NQ} = \frac{35{,}695{,}416}{200} = 178{,}477.08$$

Notice that the method appearing in Articles 16 and 17.1 exactly corresponds with the Highest Remainder method described in Section II, where for each party, Votes/NQ coincides with its quota.

TABLE IV
ESTIMATION OF PR SEATS BY PARTY

Party	Votes/NQ	Assigned seats	Unused votes	Adjustment seats	HR seats
PRI	65.2	65	35,947		65
PAN	46.94	46	167,589	1	47
PRD	24.29	24	51,871		24
MORENA	18.75	18	133,125	1	19
PVEM	15.45	15	80,014	1	16
MC	13.62	13	110,861	1	14
NA	8.33	8	58,809		8
PES	7.42	7	75,692		7
Total	200	196	-	4	200

Table V summarizes the number of RM and PR seats obtained by each party at this moment.

TABLE V
RM AND PR ESTIMATED SEATS BY PARTY

Party	RM seats	HR seats	Total assignment	Limit of seats by party
PRI	155	65	220	203
PAN	56	47	103	157
PRD	28	24	52	100
MORENA	14	19	33	86
PVEM	29	16	45	78
MC	10	14	24	74
NA	1	8	9	60
PES	0	7	7	58
PT	6	0	0	
Total	300	200	-	-

There is not party over 300 seats. On the other hand, it can be observed that PRI party is the only one that exceeds the limit given by its percentage of ENV plus eight points (with 17 seats). For this reason, PRI gets just 48 HR seats instead of those 65 initially assigned.

In order to allocate such 48 seats per constituency according to the obtained votes in each one, the distribution ratio (DR) considered in article 17.3 is:

$$\mathrm{DR} = \frac{11{,}636{,}957}{48} = 242{,}436.60$$

Dividing the PRI votes in each constituency by this coefficient the corresponding number of seats for this party is obtained (Table VI).

TABLE VI
PRI SEATS BY CONSTITUENCY

Cons	Votes PRI	Votes PRI /DR	Deputies number	Unused votes	Reman. seats	Total seats
1st	2,336,569	9.6378	9	154,640	1	10
2nd	2,689,712	11.0945	11	22,909		11
3rd	2,334,043	9.6274	9	152,114	1	10
4th	1,585,747	6.54087	6	131,127		6
5th	2,690,886	11.0993	11	24,083		11
Total	11,636,957		46			48

Once the distribution of seats per constituency of the party that exceeds the limit has been done, the remaining seats are object of a new distribution among the other parties, taking into account ENV, is presented in Table VI.

Article 18.1a

[…]

II. The effective national votes will be divided by the number of seats of parliament to be allocated, to obtain a new natural quotient;

III. The effective national votes obtained by each political party will be divided by the new natural quotient. The result in whole numbers will be the total number of representatives to be allocated to each political party, and

IV. If there would still be seats left to be distributed, these would be allocated according to the largest remainders of the political parties.

The new natural quotient for the 2015 elections is:

$$\text{New NQ} = \frac{35{,}695{,}416 - 11{,}636{,}957}{200 - 48} = 152{,}303.59$$

The total number of seats for the remaining parties is obtained with this new natural quotient, as appears in Table VII.

TABLE VII
SEATS FOR THE REMAINING PARTIES IN 2015

Party	Votes	Seats	Not used Votes	Seats adjustment	Total seats / party
PAN	8,377,535	52	147,010	1	53
PRD	4,335,321	27	61,779		27
MORENA	3,345,712	21	21,846		21
PVEM	2,757,170	17	66,421	1	18
MC	2,431,063	15	56,873		15
NA	1,486,626	9	62,112	1	10
PES	1,325,032	8	58,797		8

Now, the procedure to distribute the seats of each party among the five constituencies is described in Article 18.2 of the LGIPE [7], it says:

Article 18.2

[…]

b) The effective vote by electoral districts will be divided by the number of seats to be allocated in each multi-member regional electoral district in order to obtain the distribution ratio in each district;

c) The effective vote for each political party in each of the multi-member regional electoral district will be divided by the distribution ratio, and the result in whole numbers is the total number of representatives to be allocated in each multi-member regional electoral district, and

d) If some seats are left to be distributed to the political parties after applying the distribution ratio, the largest remainder of votes that each political party received in the electoral districts will be used, in descending order, until they are exhausted, so that each multi-member regional electoral district has forty representatives.

According to the described process, the quotients of distribution in each district appear in Table VIII. With these results and the ENV for each party, seats for the remaining parties per constituency are obtained (Table IX).

TABLE VIII
DISTRIBUTION COEFFICIENT FOR EACH CONSTITUENCY

Cons	Votes	Votes PRI	Effective votes	PRI seats	Remaining seats	DC
1st	6,629,435	2,336,569	4,292,866	10	30	143,095.53
2nd	7,876,851	2,689,712	5,187,139	11	29	178,866.86
3rd	7,086,446	2,334,043	4,752,403	10	30	158,413.43
4th	6,467,060	1,585,747	4,881,313	6	34	143,568.03
5th	7,635,624	2,690,886	4,944,738	11	29	170,508.21

TABLE IX
PARTY SEATS PER CONSTITUENCY

Party	Cons	Effective votes	DR /Cons.	Quota	Seats	Remaining Votes	Add Seats
PAN	1st	1,790,937	143,095.53	12.52	12	73,791	1
PAN	2nd	2,707,710	178,866.86	15.14	15	24,707	
PAN	3rd	1,280,757	158,413.43	8.08	8	13,450	
PAN	4th	1,147,713	143,568.03	7.99	7	142,737	1
PAN	5th	1,450,418	170,508.21	8.51	8	86,352	1
PRD	1st	316,598	143,095.53	2.21	2	30,407	
PRD	2nd	479,996	178,866.86	2.68	2	122,262	1
PRD	3rd	922,941	158,413.43	5.83	5	130,874	1
PRD	4th	1,259,498	143,568.03	8.77	8	110,954	
PRD	5th	1,356,288	170,508.21	7.95	7	162,731	1
MORENA	1st	365,306	143,095.53	2.55	2	79,115	
MORENA	2nd	342,972	178,866.86	1.92	1	164,105	1
MORENA	3rd	806,798	158,413.43	5.09	5	14,731	
MORENA	4th	1,096,758	143,568.03	7.64	7	91,782	1
MORENA	5th	733,878	170,508.21	4.30	4	51,845	
PVEM	1st	299,898	143,095.53	2.10	2	13,707	
PVEM	2nd	569,775	178,866.86	3.19	3	33,174	1
PVEM	3rd	1,141,491	158,413.43	7.21	7	32,597	
PVEM	4th	401,659	143,568.03	2.80	2	114,523	1
PVEM	5th	344,347	170,508.21	2.02	2	3,331	
MC	1st	1,026,591	143,095.53	7.17	7	24,922	
MC	2nd	465,741	178,866.86	2.60	2	108,007	
MC	3rd	225,516	158,413.43	1.42	1	67,103	
MC	4th	366,648	143,568.03	2.55	2	79,512	1
MC	5th	346,567	170,508.21	2.03	2	5,551	
NA	1st	286,959	143,095.53	2.01	2	768	
NA	2nd	364,309	178,866.86	2.04	2	6,575	
NA	3rd	208,688	158,413.43	1.32	1	50,275	1
NA	4th	299,482	143,568.03	2.09	2	12,346	
NA	5th	327,188	170,508.21	1.92	1	156,680	1
PES	1st	206,577	143,095.53	1.44	1	63,481	1
PES	2nd	256,636	178,866.86	1.43	1	77,769	
PES	3rd	166,212	158,413.43	1.05	1	7,799	
PES	4th	309,555	143,568.03	2.16	2	22,419	
PES	5th	386,052	170,508.21	2.26	2	45,036	

Summing up, the application of the law produces the PR seats allocation appearing in Table X.

Notice that the mere implementation of the law does not always guaranty an exact allocation of 40 seats per constituency. In order to avoid this problem, one seat that should correspond to MC party in district 2 was transferred to the same party in district 4. The reason of this adjustment was that 108,007 votes of MC did not obtain representation in district 4, while with 79,512 votes in district 2 one seat were assigned. A similar fact occurred with PES (one seat from this party was transferred from district 2 to 1). With these seats movements, exactly 40 seats per constituency were achieved. Consequently, the law favors the biggest parties to assure their seats in the corresponding constituencies, while the smallest ones sometimes have to transfer seats among constituencies. Therefore, an asymmetrical treatment to some parties with respect to others may arise.

TABLE X
ALLOCATION OF TOTAL SEATS PER POLITICAL PARTY

	Constituency					
Party	1st	2nd	3rd	4th	5th	Total
PRI	10	11	10	6	11	48
PAN	13	15	8	8	9	53
PRD	2	3	6	8	8	27
MORENA	2	2	5	8	4	21
PVEM	2	4	7	3	2	18
MC	7	3	1	2	2	15
NA	2	2	2	2	2	10
PES	1	2	1	2	2	8
Independent	0	0	0	0	0	0
Total	39	42	40	39	40	200

IV. INCONSISTENCIES AND PARADOXES AFFLICTING THE MEXICAN ELECTORAL SYSTEM

The previous section showed an asymmetrical treatment among parties, where two small ones had to transfer a seat among constituencies. Nonetheless, they maintained their amount of seats in a global way.

Now a hypothetical example will demonstrate that this kind of agreements is not always possible and some party might be forced to lose seats. As a consequence, the formulas described in the Mexican electoral law may yield inconsistencies that directly demonstrate its inapplicability.

First of all, suppose that there exists eight parties whose RM seats appear in Table XI and also assume that the votes' distribution per constituency is reflected in Table XII. Notice that, although this is an unreal example, the number of votes for each party, if multiplied by 1000, would be similar to those obtained in 2015 by the participant parties.

First, all 200 PR seats are distributed using the highest remainder method (Table XIII). Notice that P1 exceeds 17 seats the limit of those it can obtain. Thus, P1 only receives 48 PR seats.

TABLE XI
NUMBER OF SEATS PER PARTY

Party	RM seats
P1	155
P2	56
P3	29
P4	28
P5	14
P6	10
P7	1
P8	0
Independent	7

TABLE XII
NUMBER OF VOTES PER PARTY PER CONSTITUENCY

	Constituency					
Party	1st	2nd	3rd	4th	5th	Total
P1	2440	2700	2400	1600	2670	11810
P2	3600	3600	360	320	600	8480
P3	800	800	720	1280	720	4320
P4	400	200	480	1920	360	3360
P5	400	200	600	960	720	2880
P6	400	400	600	640	360	2400
P7	200	400	480	160	360	1600
P8	200	200	360	160	360	1280
Total	8440	8500	6000	7040	6150	36130

TABLE XIII
MAXIMUM NUMBER OF SEATS AND REMAINDERS PER PARTY PER CONSTITUENCY

Party	Votes	%Votes	PR seats	%Max	Seats. Max.	Exceeding
P1	11810	32.69	65	40.69	203	17
P2	8480	23.47	47	31.47	157	0
P3	4320	11.96	24	19.96	99	0
P4	3360	9.30	19	17.30	86	0
P5	***2880***	7.97	16	15.97	79	0
P6	2400	6.64	13	14.64	73	0
P7	1600	4.43	9	12.43	62	0
P8	1280	3.54	7	11.54	57	0
Total		100	200			

The distribution of seats per constituency of P1 according to Article 17 of the LGIPE is presented in Table XIV. In this case, the corresponding DR is

$$\text{DR of P1} = \frac{11{,}810}{48} = 246.04$$

Then, the remaining 152 seats are distributed among the other parties. The new NQ is 24320/152 = 160. The result is shown in Table XV.

TABLE XIV
TOTAL NUMBER OF SEATS OBTAINED BY P1 PER CONSTITUENCY

Const	Votes	Seats number	Remaining voting	Adjustment seats	Total seats
1st	2,440	9	226	1	10
2nd	2,700	10	240	1	11
3rd	2,400	9	186	1	10
4th	1,600	6	124		6
5th	2,670	10	210	1	11
Total	11,810	44			48

TABLE XV
TOTAL NUMBER OF SEATS OBTAINED PER REMAINING PARTIES

Party	Votes	Seats	Total PR seats
P2	8480	53	53
P3	4320	27	27
P4	3360	21	21
P5	2880	18	18
P6	2400	15	15
P7	1600	10	10
P8	1280	8	8
Total	24320	152	152

DRs for multi-member constituencies are displayed in Table XVI. Using these DRs, the allocation of PR seats per constituency is displayed in Table XVII.

TABLE XVI
DISTRIBUTION RATIO FOR CONSTITUENCY

Constituency	Votes	Available seats	DR
1a	6,000	30	200
2a	5,800	29	200
3a	3,600	30	120
4a	5,440	34	160
5a	3,480	29	120
Total	24,320	152	

TABLE XVII
FINAL ASSIGNMENT OF SEATS PER PARTY PER CONSTITUENCY

Party/Const.	1st	2nd	3rd	4th	5th	Seats
P1	10	11	10	6	11	48
P2	18	18	3	2	5	46
P3	4	4	6	8	6	28
P4	2	1	4	12	3	22
P5	2	1	5	6	6	20
P6	2	2	5	4	3	16
P7	1	2	4	1	3	11
P8	1	1	3	1	3	9
Total	40	40	40	40	40	200

It can be observed that, for parties P2 to P8, the numbers of obtained seats in Table XVII differ from those appearing in Table XV. This is an example where the Mexican electoral system is inconsistent.

Now it will be shown how the so-called No-show paradox (The No-show paradox occurs when part of the electorate may be better off by not voting than voting (see [19] and [20]) may arise in the Mexican electoral system. Consider data of the 2015 elections in Mexico, and suppose that 900 abstaining electors decided to vote for NA party. Also assume that this do not affect the RM results (Table XVIII). In such situation, the maximum number of seats for PRI party decreases one unit (from 203 to 202) as shown in Table XIX.

TABLE XVIII
FINAL ASSIGNMENT OF RM SEATS PER PARTY

Party	Votes	RM seats
PRI	11,636,957	155
PAN	8,377,535	56
PRD	4,335,321	28
MORENA	3,345,712	14
PVEM	2,757,170	29
MC	2,431,063	10
NA	1,486,626+900	1
PES	1,325,032	0
PT	1,134,101	6
IND	225,029	1

Now, PRI party receives 65-18=47 seats and the remaining 153 seats must be distributed among the other parties. The result appears in Table XX.

TABLE XIX
MAXIMUM AND EXCEEDING SEATS PER PARTY

Party	Votes	Votes %	PR seats	% Max	Seats. Max.	Seats excess
PRI	11,636,957	32.59	65	30.59	202	18
PAN	8,377,535	23.47	47	31.47	157	0
PRD	4,335,321	12.15	24	20.15	100	0
MORENA	3,345,712	9.37	19	17.37	86	0
PVEM	2,757,170	7.72	16	15.72	78	0
MC	2,431,063	6.81	14	14.81	74	0
NA	1,486,626+900	4.17	8	12.17	60	0
PES	1,325,032	3.72	7	11.72	58	0

TABLE XX
PR SEATS PER PARTY

Party	Votes	Quota	PR Seats
PAN	8,377,535	53.27	53
PRD	4,335,321	27.58	28
MORENA	3,345,712	21.28	21
PVEM	2,757,170	17.53	18
MC	2,431,063	15.46	16
NA	1,486,626	9.45	9
PES	1,325,932	8.43	8
Total	24,059,359	153	153

Comparing with the 2015 results (see Table X), it can be observed that NA, with 900 more votes, would obtain one seat less. In other words, these voters participation would have been against their interests. Moreover, notice that two well-known apportionment paradoxes of PR (see [12]) are involved in this fact. On one hand, when allocating all 200 PR seats, the Population paradox arises: NA loses one seat in spite of obtaining 900 more votes. On the other hand, when allotting the 153 remaining seats (once the PRI excluded), a stronger version of the Alabama paradox also appears: With one more seat in the PR apportionment (153 instead of 152), and 900 more votes, NA would lose one seat.

V. EFFECTS OF THE THRESHOLD CHANGE ON THE TREATMENT TO POLITICAL PARTIES

The 2014 reform established a more restrictive requirement than previous electoral laws for parties to participate in PR allocation, increasing from 2% to 3% the exclusion threshold.

In this way, when applied the new electoral law in 2015, such change excluded two parties (concretely, the PT and the Humanist parties) that would participate with the previous codification. Even more, it should be noted that PT party obtained a little more than 2.99% of the VCV, so that with 0.001% more votes (3,337), this party would have received at least 6 seats. Thus, the threshold outlined in the LGIPE means that 1,134,101 citizens did not obtain representation in the Chamber. On the other hand, with 190,931 votes more than those obtained by PT, the PES obtained 8 PR seats, being this amount of votes just a little more than that required to obtain a seat (see last column in Table X).

The next hypothetical situation compares the real voting data from 2015 elections with those obtained if PT party had got 3,337 additional votes (needed for getting 3% of the VCV).

TABLE XXI
SEATS PER PARTY CONSIDERING SUPPOSED DATA AND SEATS ASSIGNED IN 2015 ELECTION

Party	Votes 2015	PR seats 2015	New votes	New PR Seats
PRI	11,636,957	48	11,636,957	42
PAN	8,377,535	53	8,377,535	53
PRD	4,335,321	27	4,335,321	27
MORENA	3,345,712	21	3,345,712	21
PVEM	2,757,170	18	2,757,170	17
MC	2,431,063	15	2,431,063	15
NA	1,486,626	10	1,486,626	10
PES	1,325,032	8	1,325,032	8
PT	1,134,101	0	1,137,438	7

It can be observed how the PRI, jointly with its partner in 2015 elections, PVEM, are the only benefitted parties from increasing the threshold barrier from 2% to 3%. This example shows again an asymmetrical treatment to political parties: notice how without those 3,337 additional votes of PT (needed to reach the threshold), the PRI obtains six more seats. Moreover, big parties take advantage of the votes that are obtained by independent candidates, because their votes are valid but they do not take part on the PR allocation. Again, a differentiated treatment is given to big political parties *versus* independent candidates.

VI. PROPOSAL OF APPORTIONMENT METHOD FOR MEXICAN ELECTORAL SYSTEM

As shown along this paper, the current Mexican electoral system presents inconsistencies, paradoxes and asymmetrical treatment to political parties. Due to its serious problems, a new apportionment method for the Mexican electoral system is proposed, intended be clearer, applicable, free of the inconsistencies and paradoxes, and fairer with all political parties.

The proposal is: First, it is proposed to define the VCV as all votes cast, minus the sum of null votes, and those of unregistered candidates and independent candidates. The reason for the exclusion of independent candidates' votes relies on they do not participate in the allocation of PR seats.

Second, the seats allocation to the parties is obtained according to its percentage of the ENV, with the current threshold and limits, but using the Sainte-Laguë apportionment method (Simulations with biproportional apportionment methods have confirmed Sainte-Lagüe as one of the best procedures in order to avoid seats biases [21]).

Finally, with the implementation of biproportional apportionment method, the allocation of party seats per constituency is obtained, again using the Sainte Laguë apportionment method.

Applying the first two steps to 2015 Mexican elections the results would be those appearing in Table XXII. Notice that, with this proposal, PT party has participation on PR seats, which does not happen with the current system.

Finally, the obtained seats by each party are distributed among the five constituencies, using the biproportional method (Table XXIII). Hence, with this proposal all 200 PR seats have been distributed in a comprehensive way avoiding inconsistences and paradoxes that may arise with the current system.

TABLE XXII
PR ESTIMATED SEATS PER PARTY (SAINTE-LAGUË METHOD)

Party	Votes	Percentage of VCV	RM seats	PR seats
PRI	11,636,957	30.88	155	42
PAN	8,377,535	22.23	56	53
PRD	4,335,321	11.50	28	28
MORENA	3,345,712	8.88	14	21
PVEM	2,757,170	7.32	29	17
MC	2,431,063	6.45	10	15
NA	1,486,626	3.94	1	9
PES	1,325,032	3.52	0	8
PT	1,134,101	3.01	6	7
Humanista	856,716	2.27	0	-
Nulls	1,900,449	-	-	-
IND	225,029	-	1	-
NR	52,371	-	-	-
Total	39,872,757	100	300	200

TABLE XXIII
ALLOCATION OF TOTAL SEATS PER POLITICAL PARTY BY BIPROPORTIONAL METHOD

	Constituency					
Party	1st	2nd	3rd	4th	5th	Total
PRI	9	9	9	6	9	42
PAN	13	15	8	8	9	53
PRD	2	3	6	9	8	28
MORENA	3	2	5	7	4	21
PVEM	2	3	7	3	2	17
MC	7	3	1	2	2	15
NA	2	2	2	1	2	9
PES	1	2	1	2	2	8
PT	1	1	1	2	2	7
Total	40	40	40	40	40	200

VII. CONCLUSIONS

Along this paper the Mexican Electoral System has been analyzed. To this aim, the 2015 electoral results (where internal negotiations for some parties were needed in order to reach a final allocation of PR seats) and other hypothetical (but plausible) data have been used. As shown, the application of the current law may lead to undesirable effects: Inconsistencies, paradoxes and asymmetrical treatment to political parties. It is important to point out such failures of the LGIPE because this might help to improve the Mexican electoral system and achieve a greater democratic quality in this country. Although it is known after the Balinski and Young theorem [13] that is not possible a perfect apportionment method for allocating seats, it is necessary to have a clear, fair and free of paradoxes electoral system, as much as possible.

Contrary to the current system, with the biproportional method proposed in this paper a distribution of seats without inconsistencies in any electoral situation is possible. The allocations based on this method will be free of drawbacks as those presented in this paper; even more, the differences in treatment to political parties will become reduced and agreements for interchanging their seats among the constituencies will not be needed.

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CAPÍTULO 2: *A new quota approach to electoral disproportionality*

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A New Quota Approach to Electoral Disproportionality

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Abstract: In this paper electoral disproportionality is split into two types: (1) Forced or unavoidable, due to the very nature of the apportionment problem; and (2) non-forced. While disproportionality indexes proposed in the literature do not distinguish between such components, we design an index, called "quota index", just measuring avoidable disproportionality. Unlike the previous indexes, the new one can be zero in real situations. Furthermore, this index presents an interesting interpretation concerning transfers of seats. Properties of the quota index and relationships with some usual disproportionality indexes are analyzed. Finally, an empirical approach is undertaken for different countries and elections.

Keywords: electoral systems; proportionality; electoral quota; disproportionality indexes; measurement; Spain; Sweden; Germany

1. Introduction

Electoral systems are mechanisms by which votes become seats in a parliament. In order to reflect the overall distribution of voters' preferences, some of these systems advocate for proportional representation, so that political parties will receive percentages of seats corresponding to their respective percentages of votes. Since a seat cannot be divided, it is impossible to assign exactly the obtained vote shares in seat terms. This apportionment problem generates something known as electoral disproportionality. Consequently, some parties are overrepresented while others become underrepresented. Even more, disproportionality may increase, due to the existence of many districts and electoral thresholds.

There is not an agreement about an instrument to determine such distortions generated during the process of translating votes into seats and many efforts have been made to measure them. Disproportionality indexes are usually employed to this aim and there exists a wide literature on this approach.

A survey compilation of indexes resulting from the application of different techniques is presented by Taagepera and Grofman (2003) (see also Taagepera 2007; Karpov 2008; Chessa and Fragnelli 2012; Goldenberg and Fisher 2017). These authors also develop an interesting analysis of the properties that they fulfill. From a computational point of view, Ocaña and Oñate (2011) presents a software for calculating nine disproportionality indexes. On the other hand, Koppel and Diskin (2009) and Boyssou et al. (2016) propose axiomatizations for some indexes measuring disproportionality. Finally, relationships among some disproportionality indexes appear in Borisyuk et al. (2004) and Bolun (2012).

All the proposed indexes measure deviations (in some way) from exact proportionality, which is not affordable in practice. Balinski and Young (2001, pp. 79–83) deals with a more realistic requirement concerning the apportionment problem: No party's representation should deviate from its quota (number of seats that should be received by the parties in exact proportionality) by more than one unit. In other words, no party should get less than its quota rounded down, nor more than its quota rounded up. This property is called "staying within the quota" or "verification of the quota rule". Taking into account that usually the quota is not an integer number, votes-seats disproportionality could be non-forced (if the quota rule is not satisfied), or forced, otherwise.

This paper presents a new index that just measures non-forced disproportionality, avoiding that inherent to the fact that exact proportionality is unfeasible, as pointed out. Remarkably, this index is zero if and only if the quota condition is satisfied. Even more, from this index, it is possible to obtain the minimum number of seats that would be necessary to transfer from some parties to others, so that the distribution of parliament seats will satisfy the quota condition.

The paper has the following structure: Section 2 introduces the notation and basic concepts, paying particular attention to the quota. Section 3 presents some of the most used disproportionality indexes, namely: The maximum deviation, Loosemore-Hanby and Gallagher indexes. Section 4 introduces a new index, which will be called "quota index". Section 5 shows the properties that the last index verifies and some relationships with the previous disproportionality indexes. In Section 6 the aforementioned indexes are computed for different elections in Spain, Sweden and Germany, and comparisons among them are established. In Section 7, some conclusions are presented. Finally, technical proofs, electoral results and data resources are left in Appendices A–C, respectively.

2. Notation and Basic Concepts

Let V be the number of voters, n the number of parties and S the number of seats to be distributed; $(V_1, V_2, \ldots, V_n)$ is the vector whose components are the votes obtained by each party, so that $V = \sum_{i=1}^{n} V_i$; on the other hand, $(S_1, S_2, \ldots, S_n)$ is the vector whose components are the seats assigned to each party, where $S = \sum_{i=1}^{n} S_i$; finally, we denote by v_i and s_i the proportion of votes and seats that party i receives. Thus, $v_i = V_i/V$ and $s_i = S_i/S$ are the vote and seat shares, respectively, for each party i.

A party i is overrepresented when $s_i > v_i$, and underrepresented when $s_i < v_i$. Any of these inequalities represents a distortion with respect to the voters' real preferences.

The quota (or "fair share") is the number of seats that the party i should receive in exact proportionality after obtaining V_i votes. That is, the quota for party i results $q_i = \frac{V_i}{V} S$.

In terms of the quota, a party is underrepresented if $S_i < q_i$ and overrepresented if $S_i > q_i$. Since $\sum_{i=1}^{n} S_i = \sum_{i=1}^{n} q_i$, if a party is underrepresented, at least another one will be overrepresented.

The lower quota is the closest integer number that does not exceed q_i; it will be denoted by $\lfloor q_i \rfloor$. Likewise, the upper quota is the smallest integer number bigger than or equal to q_i; it will be denoted by $\lceil q_i \rceil$. In other terms, the lower quota is obtained by rounding down q_i, and the upper quota by rounding up q_i.

Usually, for each party, quotas are fractional numbers and hence $\lceil q_i \rceil = \lfloor q_i \rfloor + 1$. Otherwise, if q_i is an integer number, then $\lfloor q_i \rfloor = \lceil q_i \rceil$. The interval whose extremes are lower and upper quotas will be called quota interval.

An apportionment satisfies the quota rule if the number of seats S_i assigned to each party differs from its quota less than one, this is: $|q_i - S_i| < 1$, or equivalently, $\lfloor q_i \rfloor \leq S_i \leq \lceil q_i \rceil$ for each $i = 1, 2, \ldots, n$.

On the other hand we will say that a party is overrepresented with respect to the upper quota if $S_i > \lceil q_i \rceil$; and is underrepresented with respect to the lower quota if $S_i < \lfloor q_i \rfloor$. Obviously, these are more restrictive requirements for parties than being merely overrepresented or underrepresented.

3. Some Indexes of Electoral Disproportionality and Their Relationship

The literature on indexes is devoted to measuring the quality of an electoral system, in some way. One of the most important issues in this context is electoral disproportionality, which could be defined as the deviation level of vote and seat shares of the participating parties in an election.

In order to determine electoral disproportionality various indexes have been proposed. As aforementioned, compilations of these indexes have been made by different authors (Taagepera and Grofman 2003; Karpov 2008). Among them, the maximum deviation index, the Loosemore-Hanby index proposed by Loosemore and Hanby (1971) and the least squares index, presented by Gallagher (1991), are some of the most frequently used ones.

3.1. Maximum Deviation Index

This index measures the maximum difference between vote and seat shares in absolute terms. The mathematical expression for this index is:

$$I_{MD} = \max_{i=1,\ldots,n} |s_i - v_i|.$$

As it can be observed, the maximum deviation index only provides information of one party that can be either the most underrepresented or overrepresented one, regardless of the deviation sizes of the other parties.

3.2. Loosemore-Hanby Index

This index adds all the deviations generated during the allocation, meaning the sum of absolute values of the differences between the vote and seat shares. Mathematically the index is defined as

$$I_{LH} = \frac{1}{2}\sum_{i=1}^{n}|s_i - v_i|.$$

The sum of absolute values of the differences between the vote and seat shares for overrepresented parties coincides with the same sum for underrepresented ones. Hence, the total sum appearing in I_{LH} is divided by two in order to obtain the seat share that has not been distributed in a completely proportional way.

3.3. Gallagher Index

The least squares index is also known as Gallagher index, and it is defined as the square root of the sum of the squared differences between vote and seat shares of every party divided by two. Formally:

$$I_G = \sqrt{\frac{1}{2}\sum_{i=1}^{n}(s_i - v_i)^2}.$$

This index takes into account both big and small deviations in the proportion of assigned seats and obtained votes. However, small differences have less influence than big differences. Consequently, this index is less sensitive than the previous one to the appearance of small parties.

3.4. Relationship among Disproportionality Indexes

Obviously, $I_{LH} = I_{MD} = I_G = 0$ if and only if there exists exact proportionality, i.e., the percentage of votes equals that of seats for each party. Some further relations among these indexes can be established. It is straightforward that $I_{LH} = I_{MD}$ if and only if there exists either just one overrepresented or just one underrepresented party. On the other hand, if there are at least two overrepresented parties jointly with another two underrepresented ones, it is straightforward that $I_{LH} > I_{MD}$.

On the other hand, Borisyuk et al. (2004) proved that $I_G \leq I_{LH}$. Besides, it is easy to check that $I_G = I_{LH}$ if and only if there is exactly one overrepresented party jointly with just one underrepresented party. In both cases, also I_{MD} reaches the same value.

Finally, taking into account the aforementioned relationships concerning the considered indexes we can assert that, if there are at least two overrepresented parties jointly with another two underrepresented ones, then $I_{LH} > \max\{I_{MD},\ I_G\}$.

4. The Quota Index

All the aforementioned indexes measure deviations between vote and seat shares, and hence, in an implicit way, they take into account the quota as a point of reference. For example, the Loosemore-Hanby index can be expressed in quota terms as

$$I_{LH} = \frac{1}{2}\sum_{i=1}^{n}|s_i - v_i| = \frac{1}{2S}\sum_{i=1}^{n}\left|\frac{S_i}{S}S - \frac{V_i}{V}S\right| = \frac{1}{2S}\sum_{i=1}^{n}|S_i - q_i|.$$

Note that $I_{LH} = 0$ if and only if $S_i = q_i$ for all the parties (this is also true for the previously considered indexes). However, as the seats are indivisible, this situation requires all q_i to be integer numbers, and this is extremely unlikely. Therefore, exact proportionality becomes almost impossible in real elections.

On the other hand, in terms of seat transference I_{LH} can be understood as the proportion of seats that we need to transfer from overrepresented parties to underrepresented ones in order to achieve exact proportionality. However, this is merely a theoretical value because, again, such exact proportionality would require the seats to be divided.

This is the reason why we have focused our attention not in exact apportionments, but in those staying within the quota, which is a more plausible condition. These considerations do not mean that we advocate for apportionment methods verifying the quota rule, as the largest remainders (a.k.a. Hamilton) rule. That is, regardless of the used method, our aim is measuring *post hoc* deviations from the quota interval.

If the quota q_i is not an integer number for some party, depending on the value of S_i, two kinds of disproportionality can be considered. We will say that in an allocation of seats, there exists non-forced disproportionality if some party does not verify the quota condition (i.e., it is overrepresented with respect to the upper quota or underrepresented with respect to the lower quota). Otherwise, the quota rule is satisfied for all the parties and we will talk about forced disproportionality, unavoidable due to the nature of the apportionment problem. Such considerations are illustrated in Figure 1.

These ideas have been taken into account in our proposal, in which we only measure non-forced disproportionality (i.e., beyond de quota interval): That is, only distances of overrepresented parties from their upper quotas or underrepresented parties from their lower quotas are considered. In this way, we have defined an index, called quota index, as

$$I_q = \frac{1}{S}max\left\{ \sum_{\substack{i=1 \\ S_i > q_i}}^{n} (S_i - \lceil q_i \rceil), \quad \sum_{\substack{i=1 \\ S_i < q_i}}^{n} (\lfloor q_i \rfloor - S_i) \right\}.$$

The value of I_q is between zero and one. The zero value corresponds to any distribution that verifies the quota rule, while the maximum disproportionality will be reached when all the seats are assigned to parties with no votes. It is worth noting that, while Loosemore-Hanby and the other aforementioned indexes are zero if and only if the apportionment is exact, I_q can be zero without this requirement. But, obviously, I_q is also zero if there exists exact proportionality.

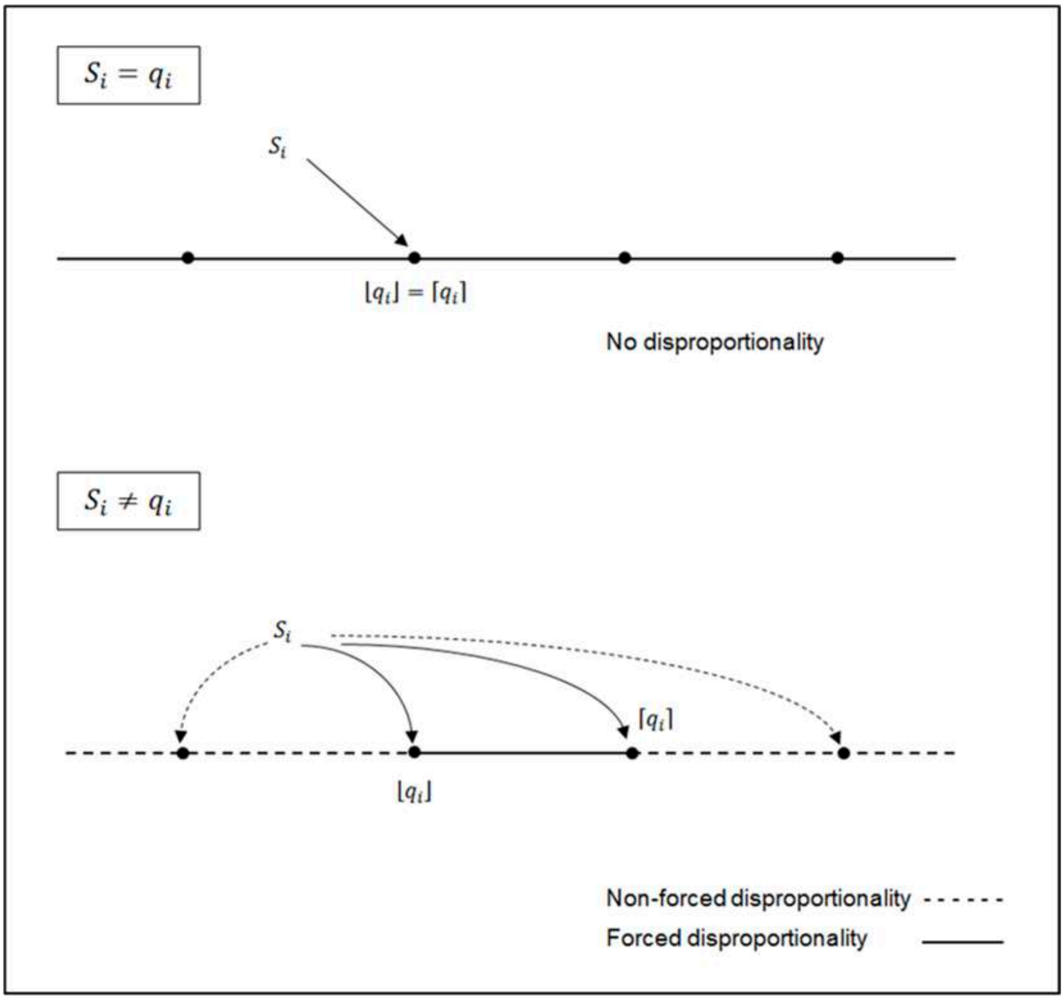

Figure 1. Types of disproportionality taking into account the quota rule.

Moreover, our index has an interesting interpretation in transference terms: The quota index I_q is the minimum proportion of seats in a parliament of S seats that we would need to transfer from overrepresented parties with respect to the upper quota to others underrepresented (or from overrepresented parties to others underrepresented with respect to the lower quota), for the quota rule to be verified. And $S \cdot I_q$ will be exactly the minimum number of seats that would have to be transferred from some parties to others for the distribution to verify the quota condition at a global level. This fact will be illustrated in Section 6 concerning 2016 Spanish elections.

5. Quota Index Analysis and Relationships with Other Indexes

Karpov (2008), Taagepera and Grofman (2003) and Taagepera (2007) propose some reasonable properties and analyze their fulfillment for several disproportionality indexes, the maximum deviation, the Loosemore-Hanby index and the Gallagher indexes among them. In this section, after showing the difference of perspectives between the above-mentioned indexes and the new one, we will test for I_q the most relevant properties appearing in the literature.

In what follows, we will formulate the above-mentioned disproportionality indexes in terms of the quota. In Section 4 we have shown that:

$$I_{LH} = \frac{1}{2S} \sum_{i=1}^{n} |S_i - q_i|.$$

In a similar way, it is easy to check that:

$$I_{MD} = \frac{1}{S} \max_{i=1,\ldots,n} |S_i - q_i|,$$

and

$$I_G = \frac{1}{S}\sqrt{\frac{1}{2}\sum_{i=1}^{n}(S_i - q_i)^2}.$$

These expressions are intended to establish in an easy way their relationships with the quota index. It will be also used in further computations.

5.1. Quota Index and Disproportionality Indexes: Difference of Scopes

In Section 4, disproportionality has been split into two types. As shown in the previous expressions, traditional approaches to this issue measure the distances from quotas, and hence they take into account both forced and non-forced disproportionality. On the other hand, the quota index just measures distances to the quota interval, and therefore just consider non-forced disproportionality. In other words, usual disproportionality indexes contemplate underrepresented on overrepresented parties, while the quota index just considers those over the upper quota or below the lower quota.

The following example illustrates these aspects.

Example 1. *Consider parties A and B, and let the number of seats to allocate* $S = 10$. *Suppose that* $S_A = 8$ *and* $q_A = 7.6$ *are the number of seats and the respective quota of the party A. Also consider that* $S_B = 2$ *and* $q_B = 2.4$ *are the number of seats and the quota of the party B, respectively. Then, we obtain:*

$$I_{MD} = I_{LH} = I_G = 0.04.$$

However, as the quota rule is verified, $I_q = 0$. *Notice that with these data all the appearing disproportionality is forced (unavoidable).*

5.2. Disproportionality Indexes Properties and Quota Index

Following Karpov (2008), some compelling properties are taken into account:

1. *Anonymity*: Any permutation of party labels does not change the value of the index.
2. *Principle of transfers*: If we transfer a seat from an overrepresented party to an underrepresented one, then the value of the index should not increase.
3. *Independence from split*: Suppose there are many parties with equal vote and seat shares, and these parties are grouped into one. Then, the value of the index calculated for all the parties in the group should be equal to the value of the index for the group considered as a whole.
4. *Scale invariance (homogeneity)*: The index should not depend on any proportional change in the number of votes or seats
5. *Zero normalization*: This property is satisfied if, when $v_i = s_i$ for all $i = 1, \ldots, n$, then the value of the index is 0.

Next, we will check the fulfillment of the previous properties by I_q.

Proposition 1. *The index* I_q *satisfies anonymity, principle of transfers and zero normalization.*

(The proof can be found in Appendix A).

Now, Example 2 shows that I_q does not satisfy the property of independence from split.

Example 2. *Suppose nine parties whose quotas and assigned seats appear in Table 1, where* $S = 6$.

Table 1. Electoral data for testing independence from split (before grouping).

	Parties								
Results	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**
q_i	1.4	1.4	0.4	0.4	0.4	0.5	0.5	0.5	0.5
S_i	3	3	0	0	0	0	0	0	0

Calculating separately I_q for all the appearing parties, we obtain:

$$I_q = max\left\{ \frac{3 - \lceil 1.4 \rceil + 3 - \lceil 1.4 \rceil}{6}, 0 \right\} = \frac{2}{6} = 0.33.$$

Now, in Table 2, I_q is calculated for parties with equal percentage of seats and quotas as unique coalitions:

Table 2. Electoral data for testing independence from split (after grouping).

	Parties Coalitions		
Results	**A + B**	**C + D + E**	**F + G + H + I**
q_i	2.8	1.2	2
S_i	6	0	0

And hence

$$I_q = max\left\{ \frac{6 - \lceil 2.8 \rceil}{6}, \frac{\lfloor 1.2 \rfloor - 0 + (2 - 0)}{6} \right\} = \frac{3}{6} = 0.5.$$

Table 3 shows the properties that I_{MD}, I_{LH}, I_G and I_q satisfy or do not (Karpov (2008) and Taagepera and Grofman (2003) for the three first indexes). A "+" sign means that the index satisfies the property and "−" means that it does not. Occasionally, these signs may appear enclosed into parentheses to point out that the corresponding property is or not satisfied under specific circumstances.

Table 3. Summary of indexes and properties.

Index	Anonymity	Transfer Principle	Independence from Split	Scale Invariance	Zero Normalizing
I_{MD}	+	(+)	−	+	+
I_{LH}	+	(+)	+	+	+
I_G	+	(+)	−	+	+
I_q	+	+	−	(−)	+

Parentheses appearing in the column relative to the Principle of Transfers in Table 3 mean that I_{MD}, I_{LH} and I_G may violate the principle of transfers in some situations, as shown in Example 3.

Example 3. *Consider parties A and B, and $S = 10$. Suppose that $q_A = 7.6$, $S_A = 8$, $q_B = 2.4$ and $S_B = 2$. That is, A is overrepresented and B is underrepresented. In this situation:*

$$I_{MD} = I_{LH} = I_G = 0.04.$$

Now, if we transfer a seat from A to B :

$$I_{MD} = I_{LH} = I_G = 0.06.$$

Note that, in this example, after the seat transference the overrepresented party becomes underrepresented, and vice versa.

However, it is easy to check that I_{MD}, I_{LH} and I_G satisfy a weaker Principle of Transfers establishing that, if a seat is transferred from an overrepresented party verifying $S_i - q_i > 0.5$ to an underrepresented one that satisfies $q_i - S_i > 0.5$, then the value of these indexes should not increase. In particular, this situation happens when a seat is transferred from an overrepresented party with respect to the upper quota to an underrepresented one with respect to the lower quota. Concerning different versions of the Principle of Transfers in electoral disproportionality and their connection with the original Dalton's Principle in more general inequality contexts, see Taagepera and Grofman (2003), Van Puyenbroeck (2008) and Goldenberg and Fisher (2017).

On the other hand, the parentheses appearing in the column relative to Scale Invariance in Table 3 means that I_q violates this property just with proportional changes in the number of seats, but not in the number of votes, as shown in Example 4.

Example 4. *Consider again parties A and B, and* $S = 10$. *Suppose that* $q_A = 7.6$, $S_A = 8$, $q_B = 2.4$ *and* $S_B = 2$. *Note that in this situation the quota rule is satisfied and hence* $I_q = 0$. *If we multiply by 10 the number of seats, that is,* $S = 100$, *we obtain* $q_A = 76$, $S_A = 80$, $q_B = 24$ *and* $S_B = 20$. *Now, the quota rule is not verified and* $I_q = 0.04$.

However, this fact should not be considered as a drawback of the index because the first situation cannot be improved by transferring seats in any way, while in the second situation if we transfer $S * I_q = 4$ *seats from party A to B, the quota rule is verified. Even more, in this case the apportionment becomes exact.*

Concerning this issue, Boyssou et al. (2016) assert that although the homogeneity with respect the number of seats "seems rather reasonable for large parliaments, a good disproportionality index should perhaps be sensitive to the size of the parliament, at least for small parliaments".

Obviously, proportional changes in the number of votes (maintaining the number of seats to allocate) do not affect the quota and consequently neither the value of I_q.

Some other properties can be considered for a disproportionality index (Taagepera and Grofman 2003; Taagepera 2007), among them:

- Informationally complete (makes use of all s_i and v_i)
- Uses data for all parties uniformly
- Does not depend on the number of parties
- Varies between 0 and 1 (or 100%)

As shown by the previous authors, these properties are satisfied by the disproportionality indexes considered along this paper, except the first one by I_{MD}. On the other hand, it is straightforward that I_q also verifies all of them.

5.3. Relationships among I_q and Disproportionality Indexes

The relationships existing among different disproportionality indexes have been shown in various ways (Borisyuk et al. 2004; Bolun 2012). In the present paper some relationships that the quota index has with the disproportionality indexes appearing above will be analyzed.

Proposition 2. *The value of the quota index is always minor than or equal to the Loosemore-Hanby index:*

$$I_q \leq I_{LH}.$$

(The proof can be found in Appendix A).

Proposition 3. *The values of the quota and the maximum deviation indexes verify the following inequality:*

$$I_q \geq I_{MD} - \frac{1}{S}.$$

(The proof can be found in Appendix A).

Obviously, $I_q = I_{LH} = I_{MD} = I_G = 0$ if there exists exact proportionality. If not, other relations can be established. It is obvious that $I_q = I_{LH}$ if and only if q_i are integer numbers for all the overrepresented parties and the maximum of the expression of I_q is reached for these ones, or q_i are integer numbers for all the underrepresented parties and the maximum is reached for them. As these situations are extremely unlikely, in general $I_q < I_{LH}$.

On the other hand, it is straightforward that $I_q = I_{MD}$ if and only if there exists either just one overrepresented party with respect to the upper quota and, in addition, its quota is an integer number or just one underrepresented party with respect to the lower quota and, in addition, its quota is an integer.

Finally, it is easy to check that $I_q = I_G$ if there is just one overrepresented party whose quota is an integer number and, in addition, there is exactly one underrepresented party. In both cases, also I_{MD} reaches the same value. Otherwise both inequalities might appear between I_q and I_G. For example, in any allocation verifying the quota with no exact proportionality, $I_q = 0 < I_G$. But if the quota condition is not satisfied, the inequality might be reversed and, in fact, $I_q < I_G$ is unlikely (see results in Section 6).

5.4. Discussion about Indexes

It can be observed that, as appearing in Table 3, none of the indexes considered along the paper is optimal. This situation is somehow analogous (in another context) to those in Social Choice theory, where is well known that there do not exist perfect voting systems nor apportionment methods, as proven by Arrow and Balinski-Young theorems, respectively.

In fact, it is possible to find examples where all the considered indexes present some weaknesses, as will be shown in what follows.

In an electoral situation where there exist non integer quotas (in fact, this is the most usual case), it is impossible to achieve exact proportionality, but it is always possible to find an apportionment verifying $I_q = 0$. It is a simple question of adjustment of each S_i in its quota interval, so that, at the end of the process $\sum_{i=1}^{n} S_i = S$. In such a situation, there are several possibilities of seat distribution staying within the quota, and this fact might be considered as a criticism, as shown in Example 5.

Example 5. *Consider parties A and B, and let the number of seats to allocate* $S = 5$*. Suppose that* $q_A = 2.4$ *and* $q_B = 2.6$*. In this situation there are two possibilities of seat distribution staying within the quota:* $S_A = 2$, $S_B = 3$ *and* $S'_A = 3$, $S'_B = 2$*. Hence,* $I_q = I'_q = 0$*.*

However, the second allocation is less compelling than the first one, because the most voted party obtains the least representation. In other terms, there exists a lack of vote/seat monotonicity in the last apportionment.

Now, notice that for the first allocation, we have $I_{MD} = I_{LH} = I_G = 0.08$*, while for the second allocation,* $I'_{MD} = I'_{LH} = I'_G = 0.12$*. Consequently, these indexes point out the first allotment as better than the second one.*

Nonetheless, Example 6 illustrates that the lack of monotonicity might not be captured (even more, it can be inversely reflected) when the usual disproportionality indexes are used.

Example 6. *Suppose eight parties whose quotas and assigned seats (in two different apportionments) appear in Table 4, where* $S = 10$*.*

The first seat distribution is intentionally arbitrary (in fact, it cannot be obtained by any divisor or quotient method). However, the second distribution is obtained by any divisor method in the parametric family (Balinski and Ramírez 1999) between Webster (Sainte-Laguë) and Jefferson (D'Hondt).

After some computations, the obtained values for quota and maximum deviation indexes in both allotments are $I_q = I'_q = 0$ *and* $I_{MD} = I'_{MD} = 0.09$*. The first apportionment presents two pair of parties,* (A, B) *and* (C, D)*, where, in each of them, the most voted is the least represented. However, in this example, unlike the* *previous one, Loosemore-Hanby and Gallagher do no detect this lack of monotonicity. Even worse, they work in the opposite way to that expected:* $I_{LH} = 0.16$, $I'_{LH} = 0.17$ *and* $I_G = 0.094$, $I'_G = 0.099$.

Table 4. Electoral data for testing indexes suitability.

Results	A	B	C	D	E	F	G	H
q_i	4.2	4.1	0.4	0.3	0.3	0.3	0.2	0.2
S_i	4	5	0	1	0	0	0	0
$S_i^{'}$	5	5	0	0	0	0	0	0

6. Implementation of the Quota Index in Some Countries and Elections

We first study in depth the case of Spain along the last 25 years, paying special attention to the transfer analysis in the 2016 elections, as well as to the overall correlation of the disproportionality indexes previously calculated.

Afterwards, we merely calculate the considered indexes for another two countries, Sweden and Germany, jointly with some comments on the results.

As a caveat, we note that slight (and hence negligible) differences might appear in the results, due to the treatment (grouped or not) of small parties without representation.

All electoral data resources appear in Appendix C.

6.1. Spain (1993–2016)

In Spain there exists a party list proportional representation system. The Spanish Chamber of Deputies has 350 members chosen in 52 districts. An exclusion threshold of 3% of the valid votes in each district is applied. All these elements have not been modified from 1977. Disproportionality partially occurs in Spain because population/representation shares are not balanced among districts. Furthermore, due to the small size of districts, some parties whose votes are scattered all over the country may obtain significantly fewer seats than other parties with a similar number of votes, if geographically concentrated. Other causes of disproportionality in Spain are the use of the D'Hondt rule, intended to favor larger parties.

Table 5 shows the values of maximum deviation, Loosemore-Hanby, Gallagher and the quota indexes for the eight most recent elections in Spain.

Table 5. Disproportionality indexes for recent Spanish elections (in percentage).

Index	1993	1996	2000	2004	2008	2011	2015	2016
I_{LH}	12.01	8.07	8.58	7.95	8.08	11.29	10.51	7.80
I_q	11.42	7.10	8.00	7.42	7.42	10.57	9.71	7.14
I_G	6.81	5.32	5.60	4.63	4.50	6.91	5.92	5.23
I_{MD}	6.33	5.39	7.04	3.97	3.92	7.89	6.21	5.86
S = 350								

In all analyzed elections, I_{LH} is bigger than I_q (as theoretically proven), while I_G is the smallest (hence, in particular, $I_q > I_G$ in all the cases). Remarkably, the quota rule never is globally satisfied, given that always $I_q \neq 0$.

Focusing our attention in 2016, it can be observed that all the obtained values have decreased from those corresponding to the previous elections. An important fact that can partially explain this issue is that IU, a left-wing party, traditionally penalized by vote dispersion, formed a coalition with the emergent party Podemos.

Following with the last Spanish elections, it is worth noting that $I_{LH} = 7.8\%$ and $S \cdot I_{LH} = 27.3$. This value corresponds to the number of seats to be transferred from some parties to others in order to achieve exact proportionality. This is a theoretical value because seats cannot be divided. On the other hand, $I_q = 7.14\%$, and hence $S \cdot I_q = 25$, which means that this is the minimum number of seats to be transferred for the apportionment to verify the quota. This is a feasible goal. Concretely, in order to achieve this aim, 20 seats belonging to PP and 5 more seats from PSOE should be transferred to C's (14), PACMA (4) and Podemos-IU-EQUO (2). The remaining 5 seats should be moved, one by one, to any other unrepresented party (see Appendix B for details).

Despite the fact that I_q solely measures non-forced disproportionality and the remaining indexes are disproportionality ones, in what follows a correlation analysis among them is undertaken in order to explore their relationships. Table 6 shows the results coming from data appearing in Table 5.

Table 6. Correlation among indexes for elections in Spain.

Indexes	I_{LH}	I_q	I_G	I_{MD}
I_{LH}	1			
I_q	0.9990	1		
I_G	0.8991	0.9033	1	
I_{MD}	0.7051	0.7110	0.9311	1

The highest correlation value appears between I_{LH} and I_q. This fact relies on the formal expression of I_q which is somehow inspired by that of I_{LH}, although the first one only measures avoidable (beyond the quota interval) disproportionality. Hence, both indexes have different interpretations, as aforementioned. On the other hand, I_{MD} and I_{LH} have the smallest correlation.

6.2. Other Countries

As aforementioned, we next calculate the considered disproportionality indexes for another two countries, Sweden (proportional system) and Germany (mixed-member system).

6.2.1. Sweden (1998–2014)

Sweden is ranked in the third position worldwide according to the 2017 democratic index developed by *The Economist Intelligence Unit* (www.eiu.com). The Swedish Parliament (Riksdag) is composed of 349 seats elected under a proportional system. 310 of them belong to fixed constituency seats and the remaining 39 are adjustment seats. Any particular party must receive at least 4% of the national votes to be assigned a seat. Fixed seats are allocated among the parties using a method known as the adjusted odd numbers method (or modified Sainte-Laguë).

The purpose of the 39 adjustment seats is to make sure that the distribution of seats among the parties over the whole country should be as proportional in relation to the number of votes as possible. The whole country is viewed as it was a single constituency and is then compared with the distribution of votes in the 29 constituencies. The adjustment seats are allocated first according to party and then according to the constituency (see www.riksdagen.se for details).

Table 7 shows the values of the I_{LH}, I_q, I_G and I_{MD} indices in all the Swedish Riksdag elections held from 1998 to 2014.

Table 7. Disproportionality indexes for recent Swedish elections (in percentage).

Index	1998	2002	2006	2010	2014
I_{LH}	2.52	2.92	6.67	2.07	4.04
I_q	1.72	2.01	5.44	1.72	3.15
I_G	1.27	1.58	3.17	1.25	2.64
I_{MD}	1.12	1.44	2.67	1.42	3.13
S = 349					

It can be observed, as in the previously studied case, that I_q is always between I_G and I_{LH}. Notice also that now the obtained results are lower than those for Spain. In fact, the Swedish electoral system produces a high proportionality unless one or several political parties have a percentage of votes just a little below the electoral threshold (4%). Concretely, this situation happened in 2006 and 2014 because the Sweden Democrats and the Feminist Initiative parties obtained 2.9% and 3.1% of the national votes, respectively.

6.2.2. Germany (1976–2017)

Germany's electoral system is a combination of "first-past-the-post" election of constituency candidates (first votes) and proportional representation on the basis of votes for the parties' States (Länder) lists (second votes). Hence, it is a mixed-member electoral system.

Concretely, half of the Members of the German Parliament (Bundestag) are elected directly from Germany's 299 constituencies, the other half via party lists in Germany's sixteen Länder. Accordingly, each voter casts two votes in the elections to the German Bundestag. The first vote, allowing voters to elect their local representatives to the Bundestag, decides which candidates are sent to Parliament from the constituencies. The second vote is cast for a party list. The 598 seats are distributed among the parties that have gained more than 5% of the second votes or at least three constituency seats. Each party receives the minimum between the number of seats obtained on the basis of the first votes and those corresponding to the second votes. The Sainte-Laguë/Schepers method is used to convert the votes into seats

In some circumstances, Parliament's size may increase during the process of allocating the seats, due to what are known as "overhang seats" and additional "balance seats" in order to maintain proportionality (see www.bundestag.de for further details).

Table 8 shows the values of the I_{LH} , I_q, I_G and I_{MD} indices in the German Bundestag elections held from 1976 to the most recent in 2017.

Table 8. Disproportionality indexes for recent German elections (in percentage).

Size	Index	1990	1994	1998	2002	2005	2009	2013	2017
	I_{LH}	8.05	3.61	4.72	8.45	4.33	6.01	15.69	5.00
	I_q	7.40	3.27	4.19	8.33	3.75	5.63	15.21	4.51
	I_G	4.62	2.22	2.75	6.15	2.28	3.14	7.83	1.95
	I_{MD}	3.85	2.15	3.11	7.98	2.05	3.92	6.29	1.45
S		662	672	669	672	614	622	631	709

Notice that, again, the values of I_q are always between those of I_G and I_{LH}. Paying attention to the historical sequence of data, the magnitude of the results obtained in 2013 is shocking. This high disproportionality arose because, in this year, parties that did not overcome the electoral threshold represented approximately 16% of the votes.

7. Conclusions

In this paper, the quota index (I_q) has been introduced and analyzed. It is worth mentioning that this index is zero if and only if the quota rule is satisfied by all the parties, i.e., when only forced (i.e., unavoidable) disproportionality arises.

Remarkably, in our approach, $I_q = 0$ can occur even if the apportionment is not exact (in fact, exact proportionality is almost impossible, due to the very nature of the apportionment problem). Moreover, I_q corresponds to the minimum percentage of seats that is necessary to transfer among parties for the quota rule to be verified. From this value it is possible to obtain the minimum number of seats (being an integer number) that it would be necessary to transfer from some parties to others, so that the seat distribution of the parliament will satisfy the quota condition.

After an electoral process, it is usual for the main party to be overrepresented. If the underrepresentation is distributed among all the other parties and none of them stays below its lower quota, then I_q will represent the surplus of the winning party calculated from its upper quota.

Notice also that, contrary to other well-known disproportionality indexes, the existence of many small parties with a quota less than the unit and without seat representation do not increase the value of I_q.

We have proven that the quota index verifies some compelling properties appearing in the literature. In particular, it is worth noting that I_q gain an advantage over some of the most relevant disproportionality indexes (maximum deviation, Loosemore-Hanby and Gallagher) when the principle of transfers is considered. On the other hand, we have checked that the quota index is not homogenous with respect to the number of seats, although it has been justified that this fact can make sense in our context.

Finally, quantitative relationships have been established among the quota and the aforementioned disproportionality indexes and all of them have been calculated for several elections in Spain, Sweden and Germany. The obtained results show that there exists a high correlation among the Loosemore-Hanby and quota indexes, but a major argument to use the last one is its interpretability in terms of seat transfer.

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Appendix A. Proofs of the Propositions

Proposition 1. *The index I_q satisfies anonymity, principle of transfers and zero normalization.*

Proof:

- *Anonymity*

Given that the value of I_q is the maximum of two arguments:

$$\sum_{\substack{i=1 \\ S_i > q_i}}^{n} (S_i - \lceil q_i \rceil) \text{ and } \sum_{\substack{i=1 \\ S_i < q_i}}^{n} (\lfloor q_i \rfloor - S_i),$$

this property is immediate since it relies on commutative property for real numbers.

- *Principle of transfers*

Consider S_h and q_h, number of seats assigned and quota for an overrepresented party h, respectively. Let S_j and q_j be the number of seats assigned and the quota for an underrepresented party j, respectively. When a seat is transferred from h to j, the following cases can happen:

(1) If party h continues being overrepresented, then

$$S_h - \lceil q_h \rceil > (S_h - 1) - \lceil q_h \rceil \geq 0.$$

(2) If party h becomes underrepresented, then

$$\lfloor q_h \rfloor - (S_h - 1) = 0.$$

In such scenarios we can find any of these situations:

(a) If party j continues being underrepresented, then

$$\lfloor q_j \rfloor - S_j > \lfloor q_j \rfloor - (S_j + 1) \geq 0.$$

(b) If party j becomes overrepresented, then

$$(S_j + 1) - \lceil q_j \rceil = 0.$$

Taking this into account, if a seat is transferred from an overrepresented party to an underrepresented one I_q does not increase its value in any of the arguments of the maximum appearing in its expression.

- *Zero normalization*

If $v_i = s_i$ for all $i = 1, \ldots, n$, then $\frac{V_i}{V} = \frac{S_i}{S}$ and $\frac{V_i}{V} S = S_i$. Furthermore, we know that $q_i = \frac{V_i}{V} S$, because $q_i = S_i$. Given that the number of assigned seats S_i is an integer number and $q_i = S_i$, q_i is also an integer number, and then $\lceil q_j \rceil = \lfloor q_i \rfloor = q_i$. Therefore:

$$\begin{aligned} I_q &= \frac{1}{S}\max\left\{ \sum_{\substack{i=1 \\ S_i > q_i}}^{n} (S_i - \lceil q_i \rceil), \sum_{\substack{i=1 \\ S_i < q_i}}^{n} (\lfloor q_i \rfloor - S_i) \right\} = \\ &= \frac{1}{S}\max\left\{ \sum_{\substack{i=1 \\ S_i > q_i}}^{n} (S_i - S_i), \sum_{\substack{i=1 \\ S_i < q_i}}^{n} (S_i - S_i) \right\} = 0. \quad \square \end{aligned}$$

Proposition 2. *The value of the quota index is always minor than or equal to the Loosemore-Hanby index:*

$$I_q \leq I_{LH}.$$

Proof: As aforementioned in Section 4, the Loosemore-Hanby index can be expressed in terms of the quota as

$$I_{LH} = \frac{1}{2S} \sum_{i=1}^{n} |S_i - q_i|.$$

On one hand, given that the sum of the terms corresponding to overrepresented parties is equal to that corresponding to underrepresented ones, we obtain

$$I_{LH} = \frac{1}{S} \sum_{\substack{i=1 \\ S_i > q_i}}^{n} (S_i - q_i) = \frac{1}{S} \sum_{\substack{i=1 \\ S_i < q_i}}^{n} (q_i - S_i).$$

On the other hand,

$$\sum_{\substack{i=1 \\ S_i > q_i}}^{n} (S_i - q_i) \geq \sum_{\substack{i=1 \\ S_i > q_i}}^{n} (S_i - \lceil q_i \rceil), \quad \sum_{\substack{i=1 \\ S_i < q_i}}^{n} (q_i - S_i) \geq \sum_{\substack{i=1 \\ S_i < q_i}}^{n} (\lfloor q_i \rfloor - S_i).$$

Thus, the maximum of the two terms appearing in the definition of I_q is less than or equal to I_{LH}. □

Proposition 3. *The value of the quota indexes and the maximum deviation verify the following inequality:*

$$I_q \geq I_{MD} - \frac{1}{S}\,.$$

Proof: Taking into account that

$$I_{MD} = \frac{1}{S} \max_{i=1,\ldots,\, n} |S_i - q_i|,$$

we consider two cases:

(a) If $\max\limits_{i=1,\ldots,\, n} |S_i - q_i|$ is reached for underrepresented party u, then $\max\limits_{i=1,\ldots,n} |S_i - q_i| = (q_u - S_u)$. Therefore:

$$\sum_{\substack{i=1 \\ S_i < q_i}}^{n} (\lfloor q_i \rfloor - S_i) \geq \lfloor q_u \rfloor - S_u \geq (q_u - S_u) - 1,$$

where the last inequality takes into account that $\lfloor q_u \rfloor \geq q_u - 1$. Dividing both extreme members by S, we have

$$\frac{1}{S} \sum_{\substack{i=1 \\ S_i < q_i}}^{n} (\lfloor q_i \rfloor - S_i) \geq \frac{(q_u - S_u)}{S} - \frac{1}{S} = \frac{\max\limits_{i=1,\ldots,n} |S_i - q_i|}{S} - \frac{1}{S}\,, I_q \geq I_{MD} - \frac{1}{S}$$

(b) If $\max\limits_{i=1,\ldots,n} |S_i - q_i|$, is reached for a party o that is overrepresented, then $\max\limits_{i=1,\ldots,n} |S_i - q_i| = (S_o - q_o)$. Therefore:

$$\sum_{\substack{i=1 \\ S_i > q_i}}^{n} (S_i - \lceil q_i \rceil) \geq S_o - \lceil q_o \rceil \geq (S_o - q_o) - 1,$$

where the last inequality takes into account that $q_o + 1 \geq \lceil q_o \rceil$. Dividing both extreme members by S, we have

$$\frac{1}{S} \sum_{\substack{i=1 \\ S_i > q_i}}^{n} (S_i - \lceil q_i \rceil) \geq \frac{(q_o - S_o)}{S} - \frac{1}{S} = \frac{\max\limits_{i=1,\ldots,n} |S_i - q_i|}{S} - \frac{1}{S}\,.$$

In consequence, $I_q \geq I_{MD} - \frac{1}{S}$. □

Appendix B.

Table A1. Spanish Electoral Results (2016).

Parties	Votes	Quotas	Seats
PP	7906.185	116.48	137
PSOE	5424.709	79.92	85
PODEMOS-IU-EQUO	3201.170	47.16	45
C's	3123.769	46.02	32
ECP	848.526	12.50	12
PODEMOS-COMPROMÍS-EUPV	655.895	9.66	9
ERC-CATSÍ	629.294	9.27	9
CDC	481.839	7.10	8
PODEMOS-EN MAREA-ANOVA-EU	344.143	5.07	5
EAJ-PNV	286.215	4.22	5
EH Bildu	184.092	2.71	2
CCa-PNC	78.080	1.15	1
PACMA	284.848	4.20	
RECORTES CERO-GRUPO VERDE	51.742	0.76	
UPyD	50.282	0.74	
VOX	46.781	0.69	
BNG-NÓS	44.902	0.66	
PCPE	26.553	0.39	
GBAI	14.289	0.21	
EB	12.024	0.18	
FE de las JONS	9.862	0.15	
SI	7.413	0.11	
SOMVAL	6.612	0.10	
CCD	6.264	0.09	
PH	3.288	0.05	
SAIn	3.221	0.05	
P-LIB	3.103	0.05	
CENTRO MODERADO	2.986	0.04	
CCD-CI	2.668	0.04	
UPL	2.307	0.03	
PCOE	1.812	0.03	
AND	1.695	0.02	
JXC	1.184	0.02	
IZAR	854	0.01	
CILUS	847	0.01	
PFyV	838	0.01	
PxC	722	0.01	
MAS	718	0.01	
UNIDAD DEL PUEBLO	684	0.01	
PREPAL	640	0.01	
Ln	617	0.01	
REPO	569	0.01	
INDEPENDIENTES-FIA	556	0.01	
IMC	351	0.01	
FME	338	0.00	
PUEDE	330	0.00	
ENTABAN	257	0.00	
FE	254	0.00	
ALCD	210	0.00	
HRTS-Ln	82	0.00	
UDT	54	0.00	
Total	23,756.674	350	350

Source: Ministry of Interior (Spain).

Appendix C. Electoral Data Resources

Spain: www.infoelectoral.mir.es/infoelectoral/min/.
Sweden: www.electionresources.org/se/.
Germany: www.bundeswahlleiter.de/bundestagswahlen/2017/publikationen.html.

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CAPÍTULO 3: *New indexes for measuring electoral disproportionality*

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NEW INDEXES FOR MEASURING ELECTORAL DISPROPORTIONALITY

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ABSTRACT: The number of representatives obtained by each political party in an electoral process must be a whole number. So, the percentage of votes for each party usually differs from the corresponding percentage of seats, forcing a certain unavoidable disproportionality. On the other hand, different elements of the electoral system (constituencies, thresholds, etc.) may produce some avoidable disproportionality. Those indexes traditionally used to analyse disproportionality take into account an unreachable exact proportionality as a reference. Instead, our more realistic approach quantifies distortions from a specific allotment, namely the seat distribution obtained when applying a proportional method to the total votes (that is, as if it were a unique constituency, without electoral thresholds or incentives to the winning party). Hence, we measure the avoidable disproportionality associated with such method. Unlike traditional indexes, we propose indexes associated with proportional allotment methods that can be zero in real situations. They are simple to calculate and allow us to decipher the number of seats assigned beyond the inevitable disproportionality which arises from the constraint of whole numbers. We are particularly interested in the indexes associated with Jefferson and Webster methods, which are compared to Gallagher, Loosemore-Hanby and Sainte-Laguë indexes for the results of 55 elections held in several countries.

Keywords: Proportional representation, disproportionality indexes, Loosemore-Hanby index, Jefferson method, Webster method.

1. Introduction

The electoral systems are mechanisms which translate votes into parliamentary seats. None of the procedures put into practice manages to faithfully reflect the general distribution of the voters' preferences. Even with the so called proportional systems, political parties do not receive seat percentages equivalent to their vote percentages. This fact generates what is known as an electoral disproportionality. Consequently some parties become overrepresented while others are underrepresented.

A component of such disproportionality is due to the fact that a seat cannot be divided, and hence it is impossible to assign the exact proportion of votes in seats terms to each party. In real practice, different methods of proportional apportionment have been proposed: D'Hondt and Sainte-Laguë (a.k.a. Jefferson and Webster, respectively, especially in the Anglo-Saxon culture), Largest Remainder (also called Hamilton), etc., which, taking into account plausible criteria, assign a whole number of seats to each party (Balinski and Young [1]; Pukelsheim [2]). Nonetheless, these methods cannot prevent the slight but not negligible[1] difference between the percentages of votes and seats that can be considered as an unavoidable structural disproportionality.

In addition to the apportionment method there are other elements of the electoral systems that may cause discordances between vote and seat percentages in elections. Mainly:

- The existence of several electoral constituencies, especially when there are many small and medium sized ones.
- The exclusion of some parties by means of electoral thresholds.
- The addition of a number of seats (bonus) assigned to the winning party.

Additional factors beyond the allocation formula influencing disproportionality are considered in Gallagher [3] and Suojanen [4], among others.

There is no unanimity on how to measure the distortions caused by translating votes into seats, and many efforts have been made to quantify such deviation. Disproportionality indexes are generally used to this end, and there is a wide literature on this subject. So, Karpov [5] compiled nineteen indexes resulting from the application of different techniques. This author also includes an interesting analysis of the properties that the indexes fulfil. Furthermore, Taagapera and Grofman [6] have made an overview of several indexes that estimate electoral disproportionality, and analyse their suitability (see also Taagapera [7]). Finally, Koppel and Diskin [8] and Boyssou et al. [9] propose characterizations for some indexes that measure disproportionality.

As far as we know, the indexes proposed in the literature use as reference the exact (non integer) number of seats that every party should receive in pure proportionality. Instead of this, in the present paper, we measure electoral results regarding feasible values, that is: The seats allotment obtained using a pre-established method of proportional representation applied to the parties' total votes without distortion elements afflicting the results (that is, as if it were a unique constituency, no electoral thresholds nor incentives to the winning party). In this way, we quantify the discrepancies between the actual allotment and that obtained using the considered apportionment method.

It is worth to note that Gallagher [3] foresaw this approach pointing out that "it would be fairer to measure disproportionality not as the difference between the actual outcome and perfect proportionality but as the difference between the actual outcome and the highest degree of proportionality that was attainable under the circumstances".

This paper is structured as follows. In the second section we briefly describe the proportional representation methods and, for them, we detail desirable properties. As a consequence, we point out as suitable procedures those of Jefferson and Webster. In the third section we survey some classic indexes of disproportionality, as well as the reasonable properties that any index should verify. In section four we

[1] By "not negligible" we mean that these differences between votes and seats shares due to rounding to whole numbers are small, but not zero. As we will show, classical disproportionality indexes take into account these differences, but our approach does not.

introduce the disproportionality index associated with a proportional representation method. We highlight in particular the indexes associated with Jefferson and Webster methods because of its properties[2]. Section five includes comparisons of the proposed indexes with those of Gallagher, Loosemore-Hanby and Sainte-Laguë (defined in Section 3) for several elections to German Bundestag, Swedish Riksdag, Spanish Congreso de los Diputados, Portugal Assembleia da República and British House of Commons. All the electoral databases used appear in Appendix A. Finally, we present some conclusions.

2. Proportional representation

The aim in this paper is to provide a new perspective for analysing electoral disproportionality by means of a new family of indexes associated with Proportional Representation (PR) methods. This is the reason why, in what follows, we present the main PR methods and their properties. A complete description of them can be found in Balinski and Young [1] and Pukelsheim [2].

2.1 *Notation*

Let $\bar{V} = \left(V_{1,}, V_{2,}, \ldots, V_n\right)$ be the total number of votes cast for the n political parties running an election and $\bar{S} = (S_1, S_2, \ldots, S_n)$ be the seats that the electoral system has assigned to the parties. $V = \sum_{i=1}^{n} V_i$ is the total number of valid votes obtained for the different candidacies and $S = \sum_{i=1}^{n} S_i$ is the size of the parliament.

We denote by $q_i = \frac{V_i}{V} S$ the quota of party i, that is, the number of seats this party should receive in pure proportionality. Generally, this number is not integer; its downward rounding is called the lower quota and its upward rounding, the upper quota.

Finally, the fraction of the total votes received by the party i is given by $v_i = V_i/V$ and the fraction of seats of party i is $s_i = S_i/S$.

2.2 *Proportional Representation (PR) methods*

A PR problem consists in a pair $(\bar{V}, S)$ where the components of $\bar{V}$ are the votes obtained by the different parties and S is the number of seats to be allotted. One solution of the PR problem is given by the integer numbers of seats for every party, $S_1, S_2, \ldots, S_n$, so that every S_i is near q_i and also $S_1 + S_2 + \cdots + S_n = S$. In order to calculate these solutions, two major families of procedures have been proposed: quotient and remainder methods, and divisor methods.

First methods require to multiply $\bar{V}$ by a scalar $k > 0$ and to allocate the integer part of kV_i to every party i. After this, one additional seat is given to the parties with the largest remainders (fractional part) in the product kV_i, up to sum the S seats. The most significant method based on quotients is the Largest Remainder Method (or Hamilton), which uses as a factor the value $k = S/V$, and hence the quantity SV_i/V is q_i, the quota of party i.

On the other hand, divisor methods establish a way to round the fractions in every interval limited by two consecutive integer numbers, that is, they establish in each interval a threshold or signpost so that any fraction below that signpost is rounded downwards, any fraction above the signpost is rounded upwards, and any fraction that coincide with the signpost allows both types of rounding (tie situation).

Once the signposts are established, in order to apply the divisor method, we have to find a value for k so that the roundings of kq_i add up to S, the total number of seats to be assigned. Divisor method

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[2] We use the Anglo-Saxon denomination of these methods in the design of the disproportionality indexes introduced along this paper in order to avoid misunderstandings with already existing indexes (see footnote 8 for more terminology details).

allotments can be carried out, equivalently, generating a table with the quotients of the party votes divided by the established signposts. The S greatest quotients indicate the allotted seats to the corresponding parties. We describe some of the divisor methods:

1) *Jefferson (D'Hondt) method*
   The signpost for the [0, 1] interval is 1, for the interval [1, 2] is 2, and so on. That is, the signposts are: 1, 2, 3, 4, 5,… So, the Jefferson method assigns to the quotient kq_i its integer part.
2) *Webster (Sainte-Laguë) method*
   In this case, the signposts are the centre of the intervals, that is: 0.5; 1.5; 2.5; 3.5;…. So, each one of the fractions kq_i are assigned to the nearest integer.
3) *Adams method*
   For this method the signposts are: 0, 1, 2, 3,… That is, each fraction kq_i is assigned to the rounded up integer. Because of the first signpost, parties with just only one vote could obtain representation. Hence, the Adams method is not suitable for allotting seats to political parties, unless there are restrictions to the parties to get into the allotment. This is the reason why this method is not used in real practice[3].

2.3 *Analysing properties of PR methods*

One way to choose one PR method among all the theoretically possible is to consider properties that the methods satisfy and select the method that fulfils those deemed more relevant. Below we analyse the most significant properties (see again on this issue Balinski and Young [1], Pukelsheim [2], and Palomares et al. [10], as well as Niemeyer and Niemeyer [11]).

i) *Exactness.* If the quotas of all the parties are integer numbers, then the allotted seats to every party should be those quotas. This is an unquestionable property, and in fact, it is verified by all the aforementioned proportional representation methods.
ii) *House Size Monotonicity.* If the number of seats to be allotted increases, then no party should receive less seats than in the initial situation. The opposite behaviour is known as the *Alabama Paradox*. All the divisor methods are house size monotone. The Largest Remainder method is not.
iii) *Quota.* A method satisfies the quota property if no party obtains neither more seats than its rounded up quota, nor less seats than its rounded down quota. No divisor method verifies the quota property. Largest Remainder method does verifiy the quota property.
iv) *Lower Quota.* A method fulfils the lower quota property if no party receives less seats than the integer part of its quota. The only divisor method that fulfils this property is Jefferson (D'Hondt).
v) *Near Quota.* A method is 'near quota' if for any pair of parties it is impossible to take a seat of one party and give it to the other and simultaneously bring both of them nearer their quotas. The only divisor method that fulfils this property is Webster (Sainte-Laguë).
vi) *Schisms Penalization.* A PR method penalizes schisms, or equivalently, favours coalitions if, after a party splits into two, with the sum of the two parties votes being equal to the votes of the original party, and supposing the votes of the other parties remain unchanged, then the method assigns to the two new parties a number of seats equal to or less than the number of seats that the original party would have obtained. The only divisor method that satisfies this property is Jefferson.
vii) *Unbiasedness.* A method is unbiased if it does not systematically favour the most voted parties over the less voted, nor vice versa. Largest Remainder method and Webster are unbiased.
viii) *Coherence.* Once an apportionment has been undertaken, if the total number of seats obtained by a subset of political parties is reassigned among them using the same method, it is reasonable that they will receive the same number of seats. A method that guarantees this result, that is, that any part of a

[3] However, the Adams method can be very useful for distributing parliament seats among the circumscriptions in proportion to their populations, because it guarantees representation to each of them, no matter how little population they have.

proportional allotment is also proportional with such method, is said coherent or consistent. The only consistent PR methods are the divisor methods.

2.4 *Choosing suitable PR methods*

Unfortunately, as shown above, all desirable properties are not simultaneously satisfied by any method, the Balinski-Young impossibility theorem being the formal expression of this assert. We have to drop some properties when choosing a PR method. Also notice that we cannot give the same relevance to all properties.

Regarding the quota property, this demands very little in terms of proportionality to the less voted parties, and demands a lot to the most voted parties. For example, a party with 0.5 quota may lose its representation being assigned 0 seats, or it may obtain one seat, duplicating its quota; while another party with 7.5 quota will never obtain double its quota, that is 15 seats, but instead 8 at most.

The lower quota property is interesting because its fulfilment guarantees that each party would never receive fewer seats than its rounded down quota.

Regarding house size monotonicity, it is important to avoid the Alabama Paradox, but it is even more important to require consistency. Unlike the Alabama Paradox, inconsistency is a bad behaviour that in many cases can be directly perceived by the voters (on the theoretical importance of consistency, see Palomares et al. [10]).

It is also important to prevent high fragmentations of parliaments for the sake of governability, and consequently a method that encourages coalitions would be suitable. This behaviour is particularly interesting when the districts size is large and there are no electoral thresholds. Otherwise, an excessive atomization of the political parties' spectrum with parliamentary representation may appear.

Consequently, Jefferson becomes an appropriate PR method to assign seats to the parties, at least when electoral constituencies are big and there is no electoral thresholds. This method being house size monotone and consistent, is the only PR method that favours coalitions and guarantees to every party its lower quota. Moreover, it is the only divisor method that guarantees absolute majority of seats for absolute majority of votes, when the total number of seats is odd (Palomares and Ramírez [12]). These properties support the Jefferson method as one of the most used in real practice (Gallagher [3]; Karpov [13]; Nohlen [14]).

On the other hand, as a divisor method, Webster shares with Jefferson good features as house size monotonicity and consistency. Instead of lower quota, Webster fulfils the 'near quota' property, which is a very interesting transfer condition similar to Pareto optimality in economics, (as pointed out by Balinski and Young [1]). Even more, according to these authors, "of all divisor methods, Webster's is the least likely to violate quota". These reasons, jointly which unbiasedness, a relevant property, also make Sainte-Laguë (Webster) an advisable method for impartiality purposes.

3. Classic disproportionality indexes

In this section we describe some disproportionality indexes already appearing in the literature. We focus on those most used in real practice. Next, we present some desirable properties that such indexes could verify.

3.1 *Formal expressions*

In order to measure the disproportionality of an electoral outcome, more than twenty indexes have been proposed in the literature (Taagapera and Grofman [6]; Karpov [5]). All of them consider that there is disproportionality if, for some party, its fraction of votes does not coincide with the corresponding fraction of seats, that is if $s_i \neq v_i$ for some i. In that case, party i is overrepresented when $s_i > v_i$, and underrepresented when $s_i < v_i$.

None of the disproportionality indexes presented satisfies all the desirable properties (see subsection 3.2 below), and so "the perfect index" does not exist. But some of them, like Loosemore-Hanby and Gallagher, are more widely used.

The *Loosemore-Hanby* index (Loosemore and Hanby [15]), I_{LH}, is defined as

$$I_{LH} = \frac{1}{2}\sum_{i=1}^{n}|v_i - s_i| \quad (1)$$

It is divided by two. This is due to the fact that the seats over apportioned to some parties are compensated with the seats of other parties that received fewer seats than they deserve and we do not want to count these discrepancies twice. A variant of this index, due to Grofman, consists in changing the 2 divisor. Instead of dividing by two, it is divided by the effective number of parties[4] E, where

$$E = \frac{1}{\sum_{i=1}^{n} v_i^2} \text{ (see Karpov [5]).} \quad (2)$$

The I_{LH} index has a very clear meaning: it is the share of seats we would have to transfer from some parties to others in order to obtain the perfect proportionality ($s_i = v_i$ for all i). Nevertheless, in practice, such transfer is not possible because of the indivisible nature of seats.

Sometimes there is only one party j having its fraction of seats greater than its fraction of votes, $s_j > v_j$. In that case, the I_{LH} index takes the same value as the maximum deviation index, defined by

$$I_{MD} = \max_{i=1,\dots,n}|v_i - s_i| \quad (3)$$

If, instead considering only the maximum deviation, we take half of the sum of the two bigger deviations, we obtain the so called Lijphart index (see Karpov [5]).

The Gallagher index (Gallagher [3]), I_G, is defined by:

$$I_G = \sqrt{\frac{1}{2}\sum_{i=1}^{n}(v_i - s_i)^2} \quad (4)$$

This mathematical expression tries to reduce the effects of the small differences between votes and seats ratios. This is the reason why many authors defend its use. As a drawback, this index lacks an interpretation in terms of seats transfer.

There is a variant of this index proposed by Monroe [17], defined as

$$\sqrt{\frac{\sum_{i=1}^{n}(v_i - s_i)^2}{1 + \sum_{i=1}^{n} v_i^2}} \quad (5)$$

where the denominator decreases as the number of parties increases.

Gallagher [3] defined another index, called Sainte-Laguë *(I_{SL})* which is also analyzed by Goldenberg and Fisher [18], whose expression is:

$$I_{SL} = \sum_{i=1}^{n} v_i\left(\frac{s_i}{v_i} - 1\right)^2 \quad (6)$$

[4] Introduced by Laakso and Taagapera [16].

We next exemplify the use of I_{LH} and I_G. Table 1 includes votes and seats, in relative and absolute terms, of all parties in 1998 Danish elections, whose data have been obtained from [I].

Table 1. Votes/seats for parties in 1998 Danish elections

Parties	Votes	v_i	Seats	s_i
Social Democratic Party	1,223,620	0.3593	63	0.3600
Libearal Party of Denmark	817,894	0.2401	42	0.2400
Conservative People's Party	303,965	0.0892	16	0.0913
Socialist People's Party	257,406	0.0756	13	0.0743
Danish People's party	252,429	0.0741	13	0.0743
Center Democrats	146,802	0.0431	8	0.0457
Danish Social-Liberal Party	131,254	0.0385	7	0.0400
United List-Red-Green Alliance	91,933	0.0270	5	0.0286
Christian People's Party	85,656	0.0252	4	0.0229
Progress Party	82,437	0.0242	4	0.0229
Democratic Renewal	10,768	0.0032	0	0
Independent Candidates	1,833	0.0005	0	0
Total	3,405,997	1.0000	175	1.0000

From this data we obtain $I_{LH} = 0.0175$ and $I_G = 0.0042$. According to the first value, we observe that $0.0175 \cdot 175 = 3.0625$ would be the number of seats needed to be transferred in order to achieve exact proportionality; more concretely it would be needed to transfer "a little more than 3 seats" from Social Democratic Party, Conservative People's Party, Danish People's Party, Center Democrats, Danish Social-Liberal Party and United List Red-Green Alliance parties to the remaining ones. But this is impossible in real practice, because the number of seats transferred from one party to another one must be an integer number. Only by transferring a seat from Center Democrats to Democratic Renewal the value of the first index can decrease to $I_{LH} = 0.0174$. All in all, a total of $0.0174 \cdot 175 = 3.0447$ seats cannot be transferred to achieve exact proportionality. On the other hand, the I_G value is slightly smaller[5] than the previous one, and as previously noted this has no interpretation in terms of badly allotted seats.

3.2 Desirable properties

There are several properties that are reasonably required to any disproportionality index. According to Karpov [5], some of the most basic ones are the following.

1) *Anonymity*
   If we permute votes $(V_1, V_2, \dots, V_n)$ and seats $(S_1, S_2, \dots, S_n)$ in the same manner, the value of the index must not change.
2) *Principle of transfers*
   If we transfer a seat from an overrepresented party to an underrepresented one, the value of the disproportionality index should not increase.
3) *Homogeneity regarding the votes (scale invariance)*
   The value of the index must remain invariant if the votes change proportionally, that is, if we substitute V by kV for any $k > 0$. Therefore, it is irrelevant if votes are expressed in units, thousands, percentages, etc.
4) *Normalization*
   The index value must be between 0 and 1. Also, for any of the extreme values, there should be at least one distribution of seats that reaches such value.

[5] In fact, Borysiuk et al. [19] proves that always $I_G \leq I_{LH}$.

The first property is satisfied for any of the classic indexes. With regard to the principle of transfers, some authors like Karpov [5] affirm that both I_{LH} and I_G verify the property, except for unrealistic situations. However, real data from the previous example (Table 1) show that such argument is not true. Note that in the aforementioned example, the overrepresented parties are *SDP*, *Left* and *Liberal*. If we transfer one seat from any of these parties to any of the underrepresented ones, then both the values of I_{LH} and I_G increase, as one can easily check. Consequently, the definitions of seats transfer or overrepresentation have to be refined, or else no classic index is going to satisfy this principle. In fact, Goldenberg and Fisher [18] consider a more restrictive definition than the classic one in order to demonstrate that I_{SL} decreases when one seat is transferred from an overrepresented party to an underrepresented one.

Furthermore, in regards to normalization, although all the aforementioned indexes verify it, the maximum value can only be reached if all the seats are assigned to a party with zero votes, something unthinkable under any kind of allotment; furthermore, achieving the zero value requires that the allotment were exact, something very improbable in real practice. As it will be shown, this is not the case for the indexes introduced in the next section.

4. New disproportionality indexes

To motivate our proposal, we come back to Table 1, which contains the real outcome obtained by the main political parties in the 1998 Danish elections. We saw that, from this data, the Loosemore–Hanby and Gallagher indexes, as well as all the classical indexes, take non-zero values. However, for this particular case, only by transferring one seat from Center Democrats to Democratic Renewal the value of the I_{LH} index can decrease from 0.0175 to 0.0174, but not to zero; and this is similar for other indices. However, the same apportionment is obtained by the Largest Remainder method, the Webster method and many other of the parametric divisor methods (Balinski and Ramírez [20]) to proportionally allocate the seats, without corrections or interferences.

That is to say, for all these methods, it is not possible to transfer seats among parties in order to obtain a better allotment. In that sense, the appearing disproportionality is unavoidable with the methods used in real practice, and we will say that there is no other disproportionality than the one forced by the fact that the assigned seats to the parties ought to be whole numbers. As a consequence, our aim is to design disproportionality indexes that measure only the non-forced or avoidable disproportionality.

Usually, when applying different PR methods to a particular apportionment problem, they give different solutions. Hence, for each PR method we can define an index for measuring the non-forced disproportionality regarding this method[6].

The index associated with a method M will be calculated taking into account both the total seats allotted to the parties in a particular electoral process and the seats that they would be assigned by method M if it were applied just considering one electoral circumscription without thresholds or bonus. In this way, in the previous example, the index value associated with any of Webster (Sainte-Laguë), Largest Remainder (Hamilton) or many other methods, will be zero, this fact reflecting that the obtained apportionment is optimal.

4.1. *Disproportionality index associated with a PR method*

Let us suppose that $\bar{R} = (R_1, R_2, \dots, R_n) = M(\bar{V}, S)$ is the allotment obtained by applying the PR method M for assigning the S seats of the parliament in proportion to the total votes of the parties without interferences such as multiple circumscriptions, exclusion thresholds and/or bonus to the winner. Also, let

[6] This idea of non-forced disproportionality already appeared (in other context) in Martínez-Panero et al. [21], a paper where the reference for a disproportionality index was not a proportional allotment method, but the fulfilment of the quota condition.

$\bar{S} = (S_1, S_2, \ldots, S_n)$ be the actual allotment obtained on the elections. We say that a party i is *overrepresented regarding the PR method M* if $S_i > R_i$ and that the party is *underrepresented regarding the PR method*[7] if $S_i < R_i$.

Let M be a fixed PR method; then the value

$$d = \frac{1}{2}\sum_{i=1}^{n}|R_i - S_i| \tag{7}$$

is the number of seats allotted disproportionally to the parties in the election, in a non-forced way.

Once the integer number d is calculated, the disproportionality index associated with M, denoted by I_M, is given by $I_M = d/S$ (we divide by S for the sake of normalization).

Notice then that, given a method M, there is M-disproportionality if $I_M \neq 0$ (equivalent to $R_i \neq S_i$ for some party i). Due to the exactness property of PR methods, if there is M-disproportionality, there will also be disproportionality in the classic sense, but the converse is not true (see indexes obtained in the example corresponding to Table 1).

The new index can also be interpreted in terms of seats transfer. The product $S \cdot I_M$ is the integer number d, and this is the amount of seats to be transferred among parties to replicate the allotment which would be obtained by strictly applying the method M, without any kind of distortion.

For practical purposes, a drawback at the time of selecting an index associated with a PR method consists in choosing a particular one among the variety of possibilities. Nonetheless, the number of interesting PR methods is small, and thus the same happens with the disproportionality indexes associated with a method. In real practice, due to the suitable properties of the Jefferson (D'Hondt) method and Webster (Sainte-Laguë) method (see subsection 2.4), in this paper we consider the indexes associated with both of them[8], I_J and I_W, to measure disproportionality.

For illustrating our proposal, in Table 2 we calculate the value of I_J and I_W for the 2013 German elections (see [II] for sources). The electoral disproportionality in that election was the greatest in that country for the last 40 years because the *FDP* and *AfD* parties did not participate in the allotment, due to they stayed some decimals below the 5% electoral threshold. In total, the parties that obtained no representation added up near 7 million votes, which is equivalent to near the 16% of the total.

The fourth and sixth columns of Table 2 includes the 631 seats (which was the size of the Bundestag on that term) assigned under the Jefferson and Webster methods, respectively, using proportionality to the total parties' votes without considering any other restriction. The third column includes the results obtained by the German electoral system (considering the 5% barrier). These data allow us to calculate the value of I_J and I_W.

The Jefferson method would assign 94 seats to the parties that did not reach the 5% threshold. Such parties range from *FDP*, which would get 30 seats, to *pro Deutschland,* which would be assigned one seat. All of them were underrepresented in 2013 with regard the Jefferson method. Those 94 seats were assigned by the German electoral system to the five most voted parties, which were the overrepresented ones. So, $I_J = \frac{94}{631} = 0.14897$; in other words, 14.897% of the seats should be transferred to obtain the Jefferson allotment without distortions. Similarly, the value of I_W is $I_W = \frac{96}{631} = 0.1521$ (slightly more than I_J).

[7] In what follows, we use just "over/underrepresented" referring to the classic context of disproportionality (see Subsection 3.1), and "*M*-over/underrepresented*"* when dealing with this new approach of *M*-disproportionality.

[8] In the literature the terms of D'Hondt and Sainte-Laguë indexes are already coined, with a distinct meaning to that considered here (see, for example, Karpov [5]; Goldenberg and Fisher 2017). This reason why, from now on, we refer to Jefferson and Webster indexes as the ones *associated with* the Jefferson-D'Hondt and Webster-Sainte-Lagüe methods, respectively.

Table 2. 2013 Bundestag election.

Parties	$\bar{V}$ =Votes	$\bar{S}$=Seats	$\bar{R}_J$	$\lvert\bar{R}_J-\bar{S}\rvert$	$\bar{R}_W$	$\lvert\bar{R}_W-\bar{S}\rvert$
CDU	14,921,877	255	218	37	217	38
SPD	11,252,215	193	164	29	163	30
DIE LINKE	3,755,699	64	54	10	54	10
GRÜNE	3,694,057	63	54	9	54	9
CSU	3,243,569	56	47	9	47	9
FDP	2,083,533	0	30	30	30	30
AfD	2,056,985	0	30	30	30	30
PIRATEN	959,177	0	14	14	14	14
NPD	560,828	0	8	8	8	8
FREIE WÄHLER	423,977	0	6	6	6	6
Tierschutzpartei	140,366	0	2	2	2	2
ÖDP	127,088	0	1	1	2	2
REP	91,193	0	1	1	1	1
Die PARTEI	78,674	0	1	1	1	1
pro Deutschland	73,854	0	1	1	1	1
BP	57,395	0	0	0	1	1
Volksabstimmung	28,654	0	0	0	0	0
RENTNER	25,134	0	0	0	0	0
Rest (12 parties)	152,581	0	0	0	0	0
Total	43,726,856	631	631	188	631	192

For this election, $I_{LH} = 0.15678$ and $I_G = 0.0783$; thus the value of the first index doubles the second one. The I_{LH} value means that the 15.678% of the seats have to be transferred to achieve the exact proportionality. Notice that, in this example, I_{LH} is very similar to I_J and I_W, while the value of the Gallagher index is far from the other three indexes.

Likewise, in order to show here other comparisons, the maximum deviation index is $I_{MD} = 0.0628$ something lower to the Gallagher index; an even lower is the Lijphart index which equals 0.0557. A similar computation to the previous ones is given by the Grofman index, which is 0.0652. Hence, those indexes of Gallagher, maximum deviation, Lijphart and Grofman give values which raise no suspicions that two parties (*FDP* and *AfD*), gathering almost the 10% of the total votes, obtained no representation. In this case, the Monroe index is 0.1007 and it reflects something better than the four previous indexes the disproportionality of this election; and I_{SL}=0.1861, which is equivalent to three percentage points more than the exact proportionality required by I_{LH}.

4.2 *Interpretation and properties of the index associated with a method M*

- The I_M value is easy to calculate, and has a clear and transparent meaning, as it represents the fraction of seats that have to be transferred to cancel the non-forced disproportionality regarding such method.

- The value of I_M is zero whenever there is only one electoral circumscription, the PR method used is M, and there is no other requirements for the parties to enter the seats allotment[9]. There is also the possibility to reach the zero value with more than one electoral circumscription[10].
- The value of I_M is not affected by the existence of parties which obtain no seats in an election because they do not have enough votes (insufficient to obtain a seat when the M method is applied nationwide).
- If we transfer a seat from an M-overrepresented party to another M-underrepresented party, the d value decreases one unit, and the value of I_M decreases $\frac{1}{S}$.
- I_M could take any value in the set $\{0, \frac{1}{S}, \frac{2}{S}, \ldots, \frac{S-1}{S}, 1\}$. Besides, while the classical disproportionality indexes only reach value 1 in situations in which seats are assigned to parties with zero votes, I_M equals 1 in more realistic contexts: For example, if all the circumscriptions are single-seat constituencies and in all of them the seats are won by a local party with insufficient votes to get it if the allotment were undertaken nationwide.
- Evidently, I_M is anonymous and homogeneous in regards to the votes.

5. Empirical application to different elections and countries

Due to the reasons argued in Subsection 2.4, we are now going to select as reference methods those of Jefferson and Webster in order to compare the behaviour of their associated disproportionality indexes in contrast with Gallagher, Loosemore-Hanby and Sainte-Laguë ones for some electoral outcomes in five countries with very different electoral systems: Germany, Sweden, Spain, Portugal and United Kingdom. We notice that the Adams method is interesting in order to assign seats to the constituencies, but not to political parties without restrictions, and this is the reason why it is not considered here.

- Germany uses a mixed member electoral system. 299 seats are elected in single-member constituencies; but in addition, at least the same number of seats is assigned to the parties exceeding a 5% threshold proportionally to their number of votes. This considerably corrects the disproportionality coming from the majoritarian electoral system.
- Sweden elects 310 seats on 29 multi-nominal circumscriptions of different sizes which create some disproportionality but, then, other 39 so-called compensatory seats are used to correct this and to achieve high proportionality among the parties exceeding the 4% nationwide. The same procedure is used in other Baltic countries.
- Spain has a Parliament with 350 seats (one more than Sweden) elected in 52 circumscriptions of different sizes, being the medium sized ones much less populated than in Sweden. An exclusion threshold of 3% in every circumscription is also considered. The size of the constituencies can cause a significant disproportionality and there are no seats to correct this behaviour like in Germany or Sweden.
- Portugal elects 230 seats in 22 constituencies. The deputies are chosen in each constituency using the Jefferson method with no legal electoral threshold. As in Spain, there are no provisions to correct the global disproportionalities caused by the 22 independent elections.

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[9] Usually an electoral system includes more than one electoral circumscription and in many occasions also an electoral threshold, to prevent the less voted parties to enter in the allotment. Even in some cases, as in Greece, the electoral system sets aside a certain number of seats to give a bonus to the winner party (Bedock and Sauger [22]). In these circumstances, if the allotment is carried out by method M, the index associated with this method shows the non-forced or unavoidable disproportionality due to such interferences.

[10] It is possible to achieve a zero value for I_M even when there are several electoral circumscriptions with pre-established sizes, because we can use method M to assign seats $(R_1, R_2, \ldots, R_n)$ to the parties, and afterwards a biproportional apportionment (Balinski and Pukelsheim [23]; Ramírez et al. [24]) to distribute the seats to the different circumscriptions.

- United Kingdom chooses the 650 deputies of the House of Commons using a majority system, which generates a very high disproportionality.

Thus, the five selected countries illustrate a very wide spectrum of electoral systems. Besides, in every one of them we will calculate the disproportionality indexes values for the outcomes of several elections, 55 altogether.

5.1 *Application to the German Bundestag*

Table 3 shows the percentage values of I_G, I_J, I_W, I_{LH} and I_{SL} for the German Bundestag elections held from 1976 to the most recent in 2017. Notice that the size of this parliament (*S*) varies along electoral calls. The following data have been obtained from [II].

Table 3. Disproportionality indexes. Germany elections 1976-2017.

Election	I_G	I_J	I_W	I_{LH}	I_{SL}	S
2017	1.95	3.67	4.80	5.00	5.26	709
2013	7.83	14.90	15.21	15.69	18.61	631
2009	3.40	4.66	5.95	6.01	6.68	622
2005	2.28	3.09	3.75	4.33	4.56	614
2002	3.83	5.97	6.47	6.70	6.89	672
1998	2.75	3.59	4.33	4.72	5.00	669
1994	2.22	2.83	3.42	3.61	3.78	672
1990	4.62	7.40	8.01	8.05	8.78	662
1987	0.71	0.80	1.01	1.35	1.38	497
1983	0.50	0.20	0.80	0.79	0.53	498
1980	1.41	1.41	1.81	1.98	2.02	497
1976	0.57	0.40	0.81	0.94	0.89	496
Mean	2.67	4.08	4.70	4.93	5.37	

The 2013 data are an exception (parties under the electoral threshold obtained approximately 16% of the votes), and negatively affect the means in the last row. As we noted in the previous section, the values of I_J, I_W and I_{LH} reflect this situation better than the Gallagher index. Notice that, with the exception of 1983 elections, I_{SL} reaches the highest value among the five considered indexes.

In 1990 The Greens and The Republicans obtained no representation, adding together the 5.9% of the votes. In 2002, the *SPD* got 4% of the votes, and they also were left out the parliament. In these two occasions the values of the indexes show how disproportionality has increased, but much less than in 2013. In almost all the elections the values of the five indexes have followed the ordering $I_G < I_J < I_W < I_{LH} < I_{SL}$, unless when the disproportionality is very low. In such cases, I_J is usually nearer the Gallagher index; even twice (1976 and 1983) it turns out to become lower than I_G.

5.2 *Application to the Swedish Riskdag*

Table 4 shows the values of I_G, I_J, I_W, I_{LH} and I_{SL} for all the Riskdag elections held in Sweden from 1998 to 2014. In [III] there are the electoral outcomes regarding the elections.

Table 4. Disproportionality indexes. Sweden elections 1998-2014 (S=349)

Election	I_G	I_J	I_W	I_{LH}	I_{SL}
2014	2.64	3.44	3.44	4.04	4.26
2010	1.25	1.43	2.01	2.07	1.49
2006	3.31	4.58	6.30	6.87	7.39
2002	1.55	2.01	2.58	2.85	2.93
1998	1.27	1.43	2.58	2.52	2.59
Mean	2.04	2.58	3.38	3.67	3.73

As we note above, the Swedish electoral system causes a high proportionality unless one or several political parties had a vote percentage somewhat lower the electoral threshold, which currently is 4%. That happened in 2014 as the *Feminist Initiative* party got the 3.1% of the votes, and also in 2006 because *Sweden Democrats* got 2.9%. In this case, the ordering $I_G < I_J < I_W < I_{LH} < I_{SL}$ is satisfied for all elections except that of 2010, the most proportional one. Then, I_{SL} reached the lowest value, while all the other indexes kept their relationships.

5. 3 *Application to the Spanish Congreso de los Diputados*

Table 5 contains the values of I_G, I_J, I_W, I_{LH} and I_{SL} in all the parliamentary elections held in Spain from 1977 to 2019a (April). In this case the electoral data have been taken from [IV].

Table 5. Disproportionality indexes. Spanish elections 1977-2019a (S=350)

Election	I_G	I_J	I_W	I_{LH}	I_{SL}
2019a	5.52	8.57	8.57	9.64	6.98
2016	5.25	6.29	7.71	7.85	5.04
2015	5.94	9.43	10.29	10.54	8.71
2011	6.92	9.14	10.57	11.30	9.92
2008	4.51	4.86	7.14	8.09	7.58
2004	4.63	4.86	7.43	7.96	6.79
2000	5.61	6.00	7.43	8.59	7.54
1996	5.33	5.71	7.43	8.08	5.69
1993	6.82	9.43	11.14	12.01	10.61
1989	8.97	12.29	14.29	15.10	13.72
1986	7.35	10.00	12.00	12.69	11.28
1982	8.17	10.57	12.86	13.87	12.67
1979	10.56	14.57	16.86	17.66	17.94
1977	10.40	14.57	16.29	18.14	18.06
Mean	6.86	9.02	10.71	11.54	10.18

As we can see, with all the indexes that we are considering, the Spanish electoral system has a significantly higher disproportionality than the German and Swedish ones. This happens because in Spain the Jefferson-D'Hondt method is used to assign the seats, and because there are 52 circumscriptions, mainly of medium or small size. We notice again that all four first indexes keep the same ordering in every election: $I_G < I_J < I_W < I_{LH}$. However, I_{SL} ranges all positions, from staying below Gallagher in 2016 to being over I_{LH} in 1979.

5.4 *Application to Assembleia da República Portuguesa.*

Table 6 contains the values of I_G, I_J, I_W, I_{LH} and I_{SL} indexes in every parliamentary elections held in Portugal from 1975 to 2015. Electoral data have been obtained from [V].

Portugal has an electoral system with closed and blocked lists, and uses Jefferson-D'Hondt method to carry out the apportionment, as in Spain, but the mean size of the circumscription is greater than 10 seats, while the mean size in Spain is 7 seats. Moreover, the three biggest circumscriptions in Portugal constitute more than half of the Assembleia seats. This largely justifies the higher proportionality of the electoral outcomes in comparison with that of Spain. Important deviations arise due to the most voted parties. For example, in each of the four last elections, among both of them have received about 21 seats of surplus (more than 9% of all 226 seats), and such disproportionality is far from what I_G and I_{SL} measure, even somehow far from I_J.

Table 6. Disproportionality. Portugal elections 1975-2015

Election	I_G	I_J	I_W	I_{LH}	I_{SL}
2015	5.41	7.52	10.62	10.20	8.76
2011	5.42	6.64	9.29	9.29	6.93
2009	5.41	7.52	8.85	9.27	5.89
2005	5.82	7.52	9.29	8.98	5.68
2002	4.65	6.19	7.96	7.64	4.42
1999	4.34	5.31	7.08	7.09	4.32
1995	4.54	6.19	6.64	7.38	4.57
1991	5.60	7.08	7.96	8.55	6.88
1987	6.13	7.32	8.54	8.97	7.91
1985	3.63	4.87	6.10	5.76	4.49
1983	2.97	4.07	4.88	5.22	4.01
1980	3.96	4.47	6.10	5.98	5.60
1979	3.78	4.87	6.10	5.91	5.68
1976	3.70	4.63	6.56	6.50	6.05
1975	5.71	8.91	10.12	9.91	7.47
Mean	4.74	6.21	7.74	7.78	5.91

5.5 *Application to the House of Commons in United Kingdom*

Finally, Table 7 shows the behaviour of the five indexes in elections which use a majoritarian system; specifically they are computed for all the United Kingdom House of Commons elections from 1983, whose data have been obtained from [VI].

Table 7. Disproportionality. United Kingdom elections 1983-2015 (S=650)

Election	I_G	I_J	I_W	I_{LH}	I_{SL}
2017	6.36	9.55	9.71	9.68	11.45
2015	14.92	23.11	23.42	23.38	31.55
2010	14.94	21.57	21.88	21.87	23.31
2005	16.57	20.16	20.31	20.14	23.06
2001	17.44	21.09	21.09	21.10	20.70
1997	16.30	20.52	20.52	20.56	20.47
1992	13.33	16.59	16.74	16.74	17.70
1987	17.68	20.15	20.31	20.30	23.70
1983	20.50	23.23	23.54	23.48	29.26
Mean	15.34	19.55	19.72	19.69	22.36

As could be expected, due to the majoritarian nature of its electoral system, the disproportionality in United Kingdom is much higher than in the previously considered countries. It affects near 20% of the seats in almost all the elections.

Note that here the value of I_J and I_W are almost identical to that of I_{LH} in all the elections. However, I_G always stays several points below them, and, with few exceptions, I_{SL} usually reaches the highest value, sometimes exaggeratedly, as it happened in 2015, taking a value of 0.3155 while I_{LH} was 0.2338.

5.6 *Discussion*

1) The indexes I_J, I_W and I_{LH} have a very clear interpretation and their computation is very simple. In addition, the calculation of I_J and I_W does not require to take into account the parties with small quota (significantly less than 1 and 0.5, respectively), if they have not received seats in the corresponding election.
2) The value of I_G is usually lower than those of I_J, I_W and I_{LH}, and does not have a specific meaning. In most cases, but not always, these four indexes are ordered as follows $I_G < I_J < I_W < I_{LH}$.
3) The value of I_{SL} is less sensitive to differences in representation of large parties rather than in medium and less voted ones; that is because it uses the quotient between fraction of votes and seats. Its value does not inform us if we are far or near the exact proportionality. From its value of 0.3155 in 2015 UK election, we cannot imagine that the disproportionality was then below 25%, nor from the value of 0.0568 in 2005 Portugal election we can know that the disproportionality was around 9%. In addition its value depends on the parties' size with disproportionality in the representation. Therefore, in some countries it is below I_G and in others it is the one that takes the highest value.

Table 8 contains, in a comprehensive way, the average values of the considered indexes in the analysed elections for each country.

Table 8. Indexes averages in the five countries

Country	I_G	I_J	I_W	I_{LH}	I_{SL}
Germany	2.67	4.08	4.70	4.93	5.37
Sweden	2.04	2.58	3.38	3.67	3.73
Spain	6.86	9.02	10.71	11.54	10.18
Portugal	4.74	6.21	7.74	7.78	5.91
UK	15.34	19.55	19.72	19.69	22.36

6. Conclusions

In this paper we consider a new scheme to measure electoral disproportionality. Contrary to the classic approach, which takes as reference the unattainable exact proportionality, our proposal takes into account deviations from feasible values: The seats distributions obtained when a prefixed proportional allotment method is carried out without interferences such as the existence of several circumscriptions, thresholds of exclusion and/or bonus for the winner. In this way, we quantify the discrepancies between the actual seat allotment and that obtained by the considered method.

The indexes associated with a method, proposed in this work, are easy to calculate and their values have a very clear meaning. When they are multiplied by the size of the parliament, they show the number of seats needed to be transferred, from *M*-overrepresented to *M*-underrepresented parties, in order to get a proportional allotment with regard to the chosen method. Thus, these indexes quantify the distortions produced by electoral thresholds, circumscriptions and direct bonus to the winning party. However, they do not measure the forced disproportionality due to the fact that the seats allotment has to consist of whole numbers.

It is important to note that it is possible to design an electoral system with zero disproportionality, in the manner introduced in this paper, even maintaining several circumscriptions. This can be accomplished if the seats are assigned to the parties in proportion to the total votes, and then, those seats are distributed to the circumscriptions using a biproportional method, so that every electoral circumscription will get the previously established seats.

Due to the positive properties of the Jefferson (D'Hondt) and Webster (Sainte-Laguë) methods, we advocate for the suitability of the indexes associated with these methods (I_J and I_W). We have contrasted such indexes with classic ones: Gallagher, Loosemore-Hanby and Sainte-Laguë, (I_G, I_{LH} and I_{SL}), in 55 electoral processes held in the last decades in countries with very different electoral systems: Germany, Sweden, Spain, Portugal and United Kingdom. In most of the cases $I_G < I_J < I_W < I_{LH}$, being I_W and I_{LH} near values, while I_J is closer to I_W than I_G. However, the relationship among I_{SL} and the other indexes varies: 23 times reaches the highest position, but usually reaches intermediate positions and sometimes stays below I_G. This is because the same difference between votes and seats is reflected in this index in a different way depending of the size of the party.

From the arguments exposed along this paper, it is possible to conclude that the indexes associated with the Jefferson and Webster methods are valid alternatives to the hitherto considered indexes to measure the disproportionality of electoral outcomes.

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Appendix A. Electoral databases

I. *Denmark*
http://electionresources.org/dk/

II. *Germany*
https://www.bundeswahlleiter.de/bundestagswahlen/2017/publikationen.html

III. *Sweden*
http://electionresources.org/se/
https://data.val.se/val/val2018/slutresultat/R/rike/index.html

IV. *Spain*
http://www.infoelectoral.mir.es/infoelectoral/min/

V. Portugal
http://electionresources.org/pt/

VI. *United Kingdom*
http://electionresources.org/uk/

CAPÍTULO 4: *Nueva propuesta de financiamiento público a partidos políticos en México*

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Nueva propuesta de financiamiento público a partidos políticos en México

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Resumen

El financiamiento público a los partidos políticos es uno de los temas que generan polémica y discusión entre la ciudadanía, en particular en el electorado mexicano (población de estudio en este trabajo, periodo 2008 al 2020). El sistema actual de la administración mexicana, con respecto a la repartición de recursos a los partidos políticos, ignora algunos elementos importantes para alcanzar un trato equitativo entre los diversos partidos políticos, tal como la barrera a la entrada del reparto de los recursos disponibles para éstos. El objetivo central de esta investigación es proponer una metodología para estimar y distribuir el financiamiento público a los partidos políticos en México. Dicha metodología utiliza el modelo de Promedios Móviles Ponderados para estimar la Votación Total Emitida. Asimismo, se propone una nueva función para la barrera a la entrada de reparto de recursos entre partidos. Se determinó que la tasa de participación del voto es alrededor del 55%, otorgándose en general, a los partidos políticos, un 45% más de los recursos que deberían de obtener; tomando en cuenta el número de votos obtenidos por cada uno de estos. Se concluyó, que se generarían ahorros por encima del 50% en el gasto del erario público.

Palabras clave: Sistemas electorales; recursos; partidos políticos; financiamiento; erario público.

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New proposal for public financing of political parties in Mexico

Abstract

Public financing to political parties is one of the issues that generate controversy and discussion among citizens, specifically in the Mexican electorate (population of study in this research, period 2008-2020). The actual system of Mexican administration, ignores some important elements to reach an equitable treatment between the diversity of political parties, such as the threshold of resources' distribution for the political parties. The main objective of this research is to propose a methodology for the estimation and distribution public financing to Mexican political parties. The proposed methodology uses the Weighted Moving Average Model to estimate the Total Vote Cast. Additionally, it is proposed a new function to estimate the threshold of resources' distribution for political parties. It is determined that the vote cast is around 55%, granting generally to political parties 45% more resources that they should have obtained, taking into account the number of votes earned by each of these. It is also concluded that the proposal would generate over 50% savings in the public spending.

Keywords: Electoral systems; resources; political parties; financing; public treasury.

Introducción

Actualmente, ciudadanos y diversas organizaciones sociales se han manifestado en contra del financiamiento público que reciben los partidos que integran el sistema político mexicano; o al menos han hecho visible su interés en conocer cómo se designa y se define instrumentalmente el presupuesto que se otorga a estas instituciones.

Uno de los argumentos principales que se esgrime es que el monto de esta prerrogativa debe disminuir porque es superior al que se concede a dependencias más relevantes para el desarrollo social y económico; pero, y quizá no sea tan palpable, la democracia, como bien público, requiere que el Estado encuentre o establezca las formas de financiar a los partidos políticos, puesto que son entidades de interés colectivo que promueven la competencia política y la democracia, lo cual también coadyuva en los cambios necesarios para el desarrollo integral de una sociedad. Si el aparato estatal instituye las normas de financiamiento de los partidos, se evita que los intereses privados, y el afán de lucro de grupos empresariales, constituyan autocracias contrarias al interés general.

Bajo este contexto de insatisfacción ciudadana, es decir, el financiamiento a los partidos políticos, resulta importante considerar que la disponibilidad de modelos para analizar las variables que determinan el adecuado monto y reparto de la subvención a los partidos —con base en distintos criterios para maximizar la equidad entre los que participan en los procesos electorales— es limitada. La situación, entonces, no garantiza la existencia de un sistema democrático moderno, pero presenta la oportunidad de entablar una discusión de donde emerja un nuevo tipo de soluciones.

La presente investigación se enfoca en la falta de un modelo de financiamiento público a los partidos políticos, desarrollado con base en la proporcionalidad entre el porcentaje de votos y el de recursos otorgados a éstos, el reparto equitativo del monto calculado, y de sus efectos en la competencia electoral.

Bajo esa perspectiva, este estudio tiene como objetivo presentar un modelo para determinar el financiamiento público que incluya las variables mencionadas y una barrera de entrada al reparto de éste; además de compararlo con el modelo vigente, para evidenciar los posibles ahorros y demostrar que el financiamiento público a los partidos políticos puede reducirse en función de una nueva propuesta metodológica.

El modelo para determinar la disminución del financiamiento público a los partidos políticos se justifica porque introduce variables que modifican razonablemente el volumen de recursos monetarios y aumentan la equidad en su distribución, con resultados positivos para el desarrollo de la democracia.

En ese sentido, este artículo se compone en la primera parte, por el fundamento teórico referente a un recorrido a través del tiempo del financiamiento de partidos políticos en México; donde se definen, además, algunos de sus elementos sustanciales. En la siguiente sección se desarrolla la metodología empleada para elaborar la nueva propuesta de asignación de recursos; para ello se toma en cuenta una estimación de la Votación Total Emitida, se recurre al método de Promedios Móviles Ponderados y se propone una nueva forma de estimar la barrera a la entrada de reparto de recursos financieros. Finalmente, se presentan los resultados y conclusiones respectivas.

1. Fundamento teórico

1.1. Sistemas de financiamiento de los partidos políticos

Históricamente, se puede exponer se han constituido tres sistemas para financiar a los partidos políticos, estos son de carácter: Privado, público y mixto. De acuerdo con Zovatto (2017), el sistema de financiamiento privado, comprende las aportaciones en dinero o en especie de particulares; sin distinguir si estos son militantes o simpatizantes. El sistema público, es equivalente al concepto de financiamiento legal otorgado por el Estado y que de hecho forma parte de su mismo presupuesto; este es el que se abordará en este trabajo. Respecto al sistema de financiamiento mixto, este se estructura por el registro y aportación que ocurre como una combinación de fondos públicos y privados.

Resulta importante destacar que, existe un concepto de financiamiento más: El financiamiento político, que se refiere a la parte del dinero existente en el proceso político, y comprende "el financiamiento (legal e ilegal) de las actividades en curso de los partidos políticos y de las campañas electorales" (Ohman, 2015, p.2). El financiamiento político, de acuerdo con Zovatto (2017), resulta de la política de ingresos y egresos de los partidos políticos, y difiere del concepto de financiamiento público porque este:

> Se refiere al empleo de recursos públicos en beneficio de los partidos y las campañas. Puede hacerse de manera directa, transfiriendo a los partidos dinero del presupuesto nacional, o de manera indirecta, mediante la concesión de ventajas, prerrogativas, aportes en especie o subvenciones a favor de los partidos o de las campañas electorales. (p.19)

Si se contextualiza lo que se dice en Ohman (2015) y Zovatto (2017), se entiende que: La subvención de los partidos es el conjunto de recursos económicos que estos obtienen de fuentes públicas y privadas, legales e ilegales, para financiar sus funciones públicas. El financiamiento político se divide en legal e ilegal. En ambos conceptos se incluye, además, la subvención privada.

Como financiamiento legal, se va a entender al conjunto de prerrogativas que el Estado autoriza a los partidos políticos, porque éste considera a las instituciones partidistas como entidades de interés público. Este tipo de financiamiento se divide en directo e indirecto. Entre esas exenciones se encuentra el financiamiento público, identificado comúnmente como financiamiento público directo. En este sentido, Aparicio y Pérez (2007) entienden por financiamiento público "el empleo del dinero público en beneficio de los partidos" (p.3).

La enunciación anterior hace notar que existe una relación compleja entre dinero y democracia. Se podría considerar determinante que el dinero invertido en los partidos políticos convierte en realidad a un Estado democrático. Si bien, los fondos públicos destinados a los partidos se pueden significar como el combustible que hará funcionar la maquinaria democrática, se hace necesario implementar distintos modelos de administración, basados en modelos matemáticos, para que esta maquinaria cumpla funcionalmente la tarea para la que ha sido dispuesta.

Es indudable que los países han implementado distintos modelos, con diferentes virtudes y defectos, para asignar adecuadamente el financiamiento que proviene de las arcas públicas y han optado, por diversas circunstancias, por los sistemas que combinan el financiamiento público con el privado, es decir, los sistemas mixtos. Gutiérrez y Zovatto (2011), observan que en la actualidad, más partidos políticos participan en elecciones, y que las formas de financiamiento son más competitivas y especializadas. Aunque no se aclara, es probable que los tipos de financiamiento, su combinatoria o evolución, tengan algún tipo de incidencia en la práctica, pero ello no se ha revelado del todo. Sin embargo, sí se evidencia que mientras los sistemas de financiamiento se especialicen, los resultados se hacen más tangibles.

Por ejemplo, para las elecciones nacionales de Uruguay en 2014, el financiamiento fue mixto (Pérez y Piñeiro, 2016); un caso similar, según un estudio realizado por el Instituto de Investigaciones Jurídicas de la UNAM y la Organización de los Estados Americanos en 2011, refiere que Chile tiene un financiamiento mixto, el cual está establecido en la Constitución (Fuentes, 2011).

1.2. Financiamiento de partidos políticos

Oppo (1976), comenta que la figura de los partidos políticos se origina hacia la primera mitad del siglo XIX, en Europa y Estados Unidos, lugares en los que, al afirmarse el poder de la clase burguesa, se adopta primero la democracia representativa y las instituciones parlamentarias para consolidarla.

Asimismo, en Inglaterra, Oppo (1976) dice que los partidos políticos hacen su aparición con la Reforma Constitucional de 1832 que, ampliando el sufragio, permitió que los estratos industriales y comerciales del país participaran junto a la aristocracia en la gestión de los negocios públicos. Antes de esa fecha, no puede hablarse en Inglaterra de partidos políticos propiamente dichos: Los dos grandes partidos de la aristocracia, surgidos en el siglo XVIII y presentes desde entonces en el parlamento, no tenían fuera del mismo ninguna relevancia y ningún tipo de organización; se trataba de simples etiquetas detrás de las cuales estaban los representantes de un estrato homogéneo no dividido por conflictos de interés.

En la historia, los partidos políticos han desempeñado funciones que permiten dar mayor validez a los procesos democráticos, dan pie a la pluralidad y posibilitan la transformación continua del poder, son de suma importancia en todo Estado democrático en el que los procesos electorales son fundamentales. Sin embargo, no siempre ha existido la regulación del derecho a financiamiento de los partidos políticos (Gómez, 1996; López y López, 2006).

En México, el financiamiento político-legal se autoriza en la Constitución Política de los Estados Unidos Mexicanos (CPEUM) (Congreso de la República de México, 2020). Este financiamiento público a los partidos políticos inicia en 1977 con la constitucionalización de éstos. En la literatura no se cuestiona en general el grado de inequidad establecido en la fórmula para distribuir los fondos. La desigualdad se presenta en la forma de determinar los partidos que tienen derecho al reparto de los recursos disponibles, es decir, con la barrera de entrada, que para el caso de México actualmente es del 3% de la Votación Total Emitida según la Ley General de Partidos Políticos de 2014 (Congreso de los Estados Unidos Mexicanos, 2014). Una vez obtenido el porcentaje citado, queda patente la regla general para distribuir los recursos a los partidos políticos, Castro (2014) la describe de la siguiente manera:

> Un procedimiento de distribución entre los distintos partidos políticos a través

> del cual, una parte minoritaria del dinero (30%) se distribuye de manera igualitaria entre todos, y la otra mayoritaria (70%) de forma desigual, en proporción al peso de los votos obtenidos en la última elección de diputados federales. (p.32)

El Instituto Belisario Domínguez (IBD), institución que depende directamente de la Cámara Alta en México, afirma que "las fórmulas de cálculo producen una importante desigualdad entre partidos políticos de nueva creación y partidos consolidados que compiten en elecciones" (IBD, 2017, p.32).

Ugalde (2017), propone las siguientes vías para reducir el financiamiento público a los partidos políticos nacionales: Cambiar la fórmula de financiamiento para actividades ordinarias de los partidos del 65% de la Unidad de Medida y Actualización (UMA) al 40%; utilizar en la fórmula la lista nominal del padrón electoral, compuesta por quienes efectivamente cuentan con credencial para votar; emplear el número de votantes de la última elección en lugar del total de empadronados.

En este sentido, el financiamiento público a los partidos políticos es un privilegio que el Estado mexicano entrega en efectivo. No es un gasto, sino una inversión que favorece el interés público mediante el incremento en la oferta de un servicio público, necesario para fortalecer la democracia.

1.3. Reformas en el Sistema Electoral de México

Las reformas electorales hechas a la Constitución Política de los Estados Unidos Mexicanos (CPEUM), encuadran la historia del financiamiento público recibido por los partidos nacionales durante más de 30 años. En este trabajo de investigación se analiza el tema del financiamiento público a los partidos políticos, que los legisladores introdujeron en las siete reformas electorales del sistema político mexicano entre 1977 y 2020. El análisis se concentró particularmente en las reformas aplicadas del 2008 al 2020; las cuales fueron promulgadas de los años 2007 al 2014. Las reformas legisladas en ese periodo corresponden a los sexenios que se indican en la Tabla 1.

Tabla 1
Reformas electorales 2007-2014

Reforma	Sexenio	Presidente
1977	1976-1982	José López Portillo
1987	1982-1988	Miguel de la Madrid Hurtado
1990	1988-1994	Carlos Salinas de Gortari
1993	1988-1994	Carlos Salinas de Gortari
1996	1994-2000	Ernesto Zedillo Ponce de León
2007	2006-2012	Felipe de Jesús Calderón Hinojosa
2014	2012-2018	Enrique Peña Nieto

Fuente: Elaboración propia, 2020.

Los presidentes publicaron dichas reformas electorales en el Diario Oficial de la Federación, en cuyo análisis se utiliza el concepto de modelo, que el Diccionario de la Lengua Española de la Real Academia Española (RAE, 2020), define como "el esquema teórico de un sistema, o de una realidad compleja, expresado generalmente en forma matemática". Dichos modelos, para el caso de México según la Ley General de Partidos Políticos de 2014 (Congreso de los Estados Unidos Mexicanos, 2014), se integran con dos factores: El procedimiento correspondiente para estimar el monto total a distribuir entre todos los partidos políticos; y, la fórmula para calcular el monto del financiamiento público a distribuir a cada uno de estos, de acuerdo con los votos obtenidos.

El conjunto de las reformas electorales en estudio, se divide en dos grupos. Comparativamente, se advierte que el primero, se enfoca en el financiamiento público destinado a captar la máxima cantidad de votos en las elecciones federales; y el segundo, básicamente, en el financiamiento de las actividades ordinarias de los partidos políticos.

Según la reforma de 1977 de la Constitución Política de los Estados Unidos Mexicanos (CPEUM) referente al Sistema Electoral que deriva en la Ley Federal de Organizaciones Políticas y Procesos Electorales 1977 (Congreso de los Estados Unidos Mexicanos, 1977), fija esa tendencia al determinar que los partidos contarían, en forma equitativa, con un mínimo de elementos para captar votos en elecciones federales. Sin embargo, la reforma de 1987 en el Código Federal Electoral 1987 (Congreso de los Estados Unidos Mexicanos, 1987), no precisa en qué tipo de actividades se invertirían los recursos. Se infiere que se aplicarían al objetivo de acrecentar la votación de los partidos en los procesos electorales, porque las variables para calcular el monto del financiamiento lo sugieren: El costo mínimo de una campaña para diputado, el número de candidatos a diputados de mayoría relativa, el número de votos válidos, la votación efectiva para determinar el importe unitario por voto, entre otras.

Las reformas electorales de 1990: Código Federal de Instituciones y Procedimientos Electorales de 1990 (Congreso de los Estados Unidos Mexicanos, 1990), y de 1993: Código Federal de Instituciones y Procedimientos Electorales de 1993 (Congreso de los Estados Unidos Mexicanos, 1993), manifiestan expresamente financiamiento en las siguientes vertientes: a) Por financiamiento de la actividad electoral, en primer lugar; y b) Por actividades generales como entidades de interés público, en segundo lugar. En ese orden se incluyen: c) Por subrogación del Estado de las contribuciones que los legisladores habrían de aportar para el sostenimiento de sus partidos; d) Por actividades específicas como entidades de interés público; y, e) Para el desarrollo de los partidos políticos.

En sentido opuesto, la reforma electoral de 1996: Código Federal de Instituciones y Procedimientos Electorales de 1996 (Congreso de los Estados Unidos Mexicanos, 1996), opta por la alternativa de asignar recursos en efectivo bajo el siguiente orden: a) Para el sostenimiento de actividades ordinarias permanentes; b) Para gastos de campaña y, c) Para actividades específicas como entidades de interés público.

La reforma electoral 2007 que se exhibe en el Código Federal de Instituciones y Procedimientos Electorales de 2008 (Congreso de los Estados Unidos Mexicanos, 2008), mantiene la tendencia de aplicar el financiamiento público referido en la de 1996: a) Para el sostenimiento de actividades ordinarias permanentes; b) Para gastos de campaña; y, c) Por actividades específicas como entidades de interés público. La reforma electoral de 2014 a través de la Ley General de Partidos Políticos (Congreso de los Estados Unidos Mexicanos, 2014), dispone el siguiente orden: a) Para financiar las actividades ordinarias permanentes; b) Para gastos de Campaña; y, c) Por actividades específicas como entidades de interés público.

La cantidad de variables que las reformas electorales de 1987, 1990, 1993 y 1996 emplearon para calcular el monto de actividades a financiar, apuntalaron un incremento de los recursos otorgados. Con base en la Ley Federal de Organizaciones Políticas y Procesos Electorales que conllevo al Código Federal Electoral 1987 (Congreso de los Estados Unidos Mexicanos, 1987), el modelo de reforma electoral de 1987, utilizó las variables de costo mínimo de una campaña electoral para diputado y número de candidatos a diputados de mayoría relativa. Según el Código Federal de Instituciones y Procedimientos Electorales de 1990 (Congreso de los Estados Unidos Mexicanos, 1990) y el Código Federal de Instituciones y Procedimientos Electorales de 1993 (Congreso de los Estados Unidos Mexicanos, 1993), los modelos de 1990 y 1993, aplicaron las variables de costo mínimo de una campaña para diputado y una para senador, número de candidatos propietarios a diputados de mayoría relativa y número de candidatos propietarios a senadores.

El modelo electoral de 1996 operaba con las variables: Costos mínimos de una campaña para diputado, una para senador y otra para presidente, número de diputados a elegir, y de partidos políticos con representación en las Cámaras del Congreso. Las reformas electorales de los años 2007 y 2014 introdujeron un modelo de financiamiento que, en teoría, disminuiría la cantidad de recursos públicos que se otorgaban a los partidos políticos.

Reyes (2007), estimó que el ahorro que se generaría en las finanzas públicas por la reducción del financiamiento público derivado de la reforma, en materia de la fórmula de financiamiento público para los partidos, sería de dos mil 542 millones de pesos, durante el periodo 2008-2009. Las variables correspondientes a la fórmula del modelo son: El padrón electoral y el salario mínimo diario vigente para el Distrito Federal, reemplazado el 29 de febrero de 2016 por la Unidad de Medida y Actualización (UMA) que publica diariamente el Banco de México.

Por otra parte, de acuerdo con Hernández y Chumaceiro (2018) la democracia “ha sido un proceso de definición tanto coyuntural como estructural, caracterizada por las perspectivas ideológicas del momento, determinada por elites, por grupos de interés corporatizados en función de la representatividad y/o participación y de los estatus quo dominantes” (p.57). Asimismo, se sabe que aunque existen diversos modelos de democracia, algunas veces no es posible alcanzar con ellos los mecanismos para generar la participación ciudadana que puedan ser aplicables en el entorno en el que se desarrollan los regímenes democráticos (Contreras y Montecinos, 2019); sin embargo, debido a que los partidos políticos nacionales han sido constitucionalmente entidades de interés público, el tema de su financiamiento no es coyuntural. Por ello, el debate social y político persistirá en el mediano y largo plazo.

Por ese motivo, es importante considerar que la disponibilidad de modelos para analizar las variables que determinan el adecuado monto y reparto de financiamiento a los partidos —con base en distintos criterios para maximizar la equidad entre los que participan en los procesos electorales— es limitada. Esta situación no garantiza la existencia de un sistema democrático moderno.

En concreto, las instituciones competentes que se encargan de organizar, arbitrar y vigilar las contiendas electorales, carecen de una metodología equitativa o justa, por decirlo de un modo más sencillo, hay una falta de metodología para la asignación de los recursos económicos a partidos políticos, considerando variables significativas que garanticen la transparencia de los recursos otorgados, de tal forma que dé un trato igualitario a los partidos políticos.

Debido a la falta de un modelo de financiamiento público a los partidos políticos, este estudio se enfoca en desarrollar una propuesta orientada a generar un mayor ahorro en el erario público, así como una mayor equidad en el trato a dichos partidos, con base en la proporcionalidad entre el porcentaje de votos y el de recursos otorgados a los mismos.

2. Metodología

Como se comentó en la sección anterior, el objetivo principal de este estudio fue proponer un modelo que permita estimar el recurso que debe otorgarse a cada partido político, evitando por un lado, el trato inequitativo a los partidos políticos y por el otro, estimar el recurso global con base en la tasa de participación electoral; tomando en cuenta únicamente el número de votos recibidos por cada uno de los partidos, siempre y cuando estos hayan superado la barrera a la entrada en el reparto de escaños.

Se propuso una función continua para la estimación de la barrera a la entrada, con el fin de determinar qué partidos políticos tienen derecho al reparto de recursos financieros. Dicha función depende del número de votos que cada partido político obtuvo a nivel nacional para elegir diputados federales de representación proporcional, y la resta de un 3% de la Votación Total Emitida, a cada uno de los partidos políticos que han sido parte de la contienda electoral en cuestión. En ese sentido, se analiza el recurso asignado a los partidos políticos en el periodo de 2008 a 2020. Se contempló este periodo de tiempo pues es aquel en el que se aplicó la ley vigente para tal efecto. Finalmente, se estimó un modelo que permite dar un pronóstico de la estimación de electores que ejercerán su derecho a voto; para este caso se utilizará un modelo de Promedios Móviles Ponderado.

La Votación Total Emitida (VTE), vista a través del tiempo ha tenido un comportamiento similar, no ha presentado grandes cambios en las distintas elecciones efectuadas en México. Así, desde el punto de vista de las series de tiempo la VTE es una variable que presenta una serie de tiempo estable. El modelo de series de tiempo de Promedios Móviles Ponderados es propio para datos que tienen el comportamiento de series de tiempo estables, sin grandes fluctuaciones o cambios abruptos en las observaciones (Hanke y Reitsh, 1996). Es por ello, que se sugiere el uso del modelo citado para la estimación de la VTE.

3. Resultados

En este apartado se muestran los resultados de las metodologías y procedimientos propuestos, los cuales se aplicaron a datos reales para el caso de México a nivel federal, con la finalidad de mostrar una nueva metodología para el financiamiento público a los partidos políticos, desarrollado con base en la proporcionalidad entre el porcentaje de votos y el de recursos otorgados a los partidos, el reparto equitativo del monto calculado, y de sus efectos en la competencia electoral; además de considerar como barrera a la entrada, una función continua que evite la inequidad entre los recursos otorgados a los partidos políticos. Asimismo, se aplicará la nueva propuesta y se presentarán los posibles ahorros monetarios.

3.1. Nueva propuesta de metodología para la asignación de recursos públicos a los partidos políticos en México

Esta nueva propuesta contempla en primer lugar, el procedimiento para estimar la Votación Total Emitida, considerada como la participación electoral. Enseguida, un procedimiento para la barrera a la entrada de reparto de recursos públicos. Finalmente, se presenta la nueva propuesta de asignación de presupuesto para financiamiento público de partidos políticos.

a. Procedimiento propuesto para estimar la participación electoral (Votación Total Emitida, VTE)

Para estimar la tasa de participación electoral y/o votación total, se utilizó el modelo de series de tiempo conocido como Promedios Móviles Ponderados. Se consideró estimar la votación total emitida (VTE) mediante este método, dado que es adecuado para datos que no registran variaciones abruptas en el tiempo, es decir, el número total de electores tuvo un comportamiento que se ha mantenido estable (Hanke y Reitsh, 1996), en términos del porcentaje de electores que emiten el voto. En este caso la VTE tiene un comportamiento de esta forma.

Además, este método le asignó a cada observación una importancia diferenciada a cada uno de los datos que componen dicho promedio, es decir, en el promedio móvil ponderado se asignan pesos a cualquier dato del promedio (siempre que la sumatoria de las ponderaciones sean equivalentes al 100%). Entre mayor sea el peso de la observación, mayor será la importancia dada a la misma. Así, este método permite dar especial relevancia a la elección inmediata anterior.

El modelo de Promedios Móviles, considera un número constante de datos para hacer la estimación, una vez que se tiene nueva información, de observaciones nuevas de la variable en cuestión; los valores más antiguos son sustituidos y ponderados en la medida determinada a priori. La fórmula matemática de este modelo es:

$$VTE_{Estimada\ (t+1)} = \sum_{t=1}^{n} C_t\, VTE_t$$

Dónde:
$VTE_{Estimada\ (t+1)}$: Votación total emitida estimada en el periodo $t+1$
C_t: Ponderación de la VTE en el periodo t
n: Número de periodos considerados
Además, la sumatoria:

$$\sum_{t=1}^{n} C_t = 100\%$$

Se estimó el promedio de las últimas tres observaciones dando un mayor peso al último dato, es decir, se tiene en cuenta que la elección más próxima anterior tiene especial importancia.

b. Procedimiento propuesto para determinar la barrera a la entrada de reparto de recursos públicos

La propuesta disminuye la inequidad entre partidos políticos que resulta de las barreras electorales que se aplican comúnmente (partido político o coalición que no satisface un porcentaje específico de votos, es excluido del reparto de escaños de representación proporcional y por tanto, del reparto de recursos públicos) se presenta en Ramírez (2013), quienes han propuesto utilizar una barrera a la entrada de reparto de escaños que sea una función continua (que depende del número de votos) y no como una función discontinua. El salto o discontinuidad se da exactamente en el valor de la barrera a la entrada.

Como se ha comentado, actualmente los partidos políticos que no reciben el 3% o más de votos respecto de la Votación Total Emitida (VTE), no tienen derecho a reparto de recurso público (y de escaños por representación proporcional). De esta manera, si el partido A obtiene $2.99\bar{9}\%$ de la VTE, no recibe financiamiento. Por otra parte, si un partido político B, cuenta con un solo voto más de los que obtiene el partido A, y alcanza el 3% de la VTE, este partido B obtiene el 3% del total de los recursos disponibles.

Lo anterior, implica que la función de asignación de recursos actualmente presenta una discontinuidad en el valor de la barrera a la entrada. Además, otra desventaja de esta discontinuidad generada este tipo de barrera, es que contribuye a la presencia de desproporcionalidad electoral, es decir, diferencias entre el porcentaje de votos y el de escaños recibidos por los partidos políticos participantes (Martínez-Panero, et al., 2019). Para este caso se propone restar el 3% de los votos a todos los partidos políticos, y a partir de los votos restantes, se otorgará la proporción de recursos financieros que corresponda. Así habrá trato igual para todos los partidos políticos, sin importar si son grandes o pequeños (en términos de votos).

Formalmente, lo anterior es: A los partidos se les disminuye la misma cantidad que representa el porcentaje *r* (barrera a la entrada) de la votación total emitida. Así para cada partido i la cantidad de votos que se toma en cuenta para el reparto de recursos públicos será:

$$V_i' = V_i - r * V$$

Este será un mínimo número de votos que deberá tener el partido para acceder a los fondos públicos de financiamiento. Aquí se ha formulado y adecuado la función del umbral de partidos con derecho al reparto de escaños por representación proporcional, contemplada por Ramírez (2013). Dicha función del umbral de exclusión, evita, por ejemplo, el caso extremo en el que por diferencia de un voto un partido político recibe porcentaje de escaños fijo (el 3% para el caso de México) y otro partido con un voto menos obtenga cero escaños. A partir de lo anterior, se hace la siguiente propuesta para el reparto de recursos por partidos políticos:

Sea $V' = \sum_{i=1}^{n} V_i'$, $V_i' = V_i - r * VTE$. Tomando en cuenta el actual sistema electoral de México de acuerdo con la Ley General de Partidos Políticos de 2014 (Congreso de los Estados Unidos Mexicanos, 2014) y la nueva propuesta de umbral de exclusión, el monto F_i' de financiamiento a cada partido político será dada por la siguiente función:

$$F_i' = \begin{cases} 0 & si \quad V_i' < r * VTE \\ F * \frac{V_i':}{VTE} & si \quad V_i' \geq r * VTE \end{cases}$$

Dónde:

VTE: Es la votación total emitida de acuerdo a la Ley General de Partidos Políticos (2014) (participación electoral).

V_i': Porcentaje de votos considerados para el reparto de recursos públicos (una vez que se ha restado la cantidad de votos $r * VTE$).

F: Financiamiento total a repartir entre los partidos políticos que tengan derecho.

Es de hacer notar que, aquí solo se está esquematizando la función general del reparto del financiamiento público, es decir, solo se establece qué partidos tendrían derecho al mismo y cuáles no. La cantidad exacta del monto se determinará según indique la fórmula que se establezca para este fin.

c. Nueva propuesta para asignación de presupuesto para financiamiento público de partidos políticos

Para la determinación de qué partidos políticos tienen derecho a la obtención de recursos públicos, de acuerdo a lo que establece la ley, se considera el límite (barrera a la entrada) de votos que deben pasar los partidos para tal fin. Para ello, se estimó el número de votos -$VTE_{Estimada\ (t+1)}$- que se utiliza para el cálculo de la cantidad total de los recursos económicos que se distribuiría entre todos los partidos que tuvieran derecho a ello. Como ya se expuso, este procedimiento es una nueva propuesta que contempla el modelo Promedios Móviles Ponderados.

Una vez que se obtuvieron los resultados de los dos puntos anteriores, se aplicó el procedimiento indicado por la Ley General de Partidos Políticos 2014 (Congreso de los Estados Unidos Mexicanos, 2014), excepto que en lugar de usar el número total de ciudadanos inscritos en el padrón electoral federal para los partidos nacionales, o local para los partidos locales, según sea el caso, a la fecha de corte de julio de cada año, se utiliza la $VTE_{Estimada\ (t+1)}$ propuesta en este trabajo. El procedimiento indicado por la Ley General de Partidos Políticos del año 2014 (Congreso de los Estados Unidos Mexicanos, 2014) es:

a. El número total de ciudadanos inscritos en el Padrón Electoral Federal para los partidos nacionales, o local para los partidos locales, según sea el caso, a la fecha de corte de julio de cada año.

b. El monto del salario mínimo vigente para el Distrito Federal para los partidos políticos nacionales, o el salario mínimo de la región en la cual se encuentre la entidad federativa, para el caso de los partidos políticos locales.

c. Conociendo el valor de ambas variables, lo que sigue es multiplicar el número total de los inscritos en el padrón por el 65% del monto del salario mínimo.

d. El resultado de la operación señalada en el inciso anterior constituye el financiamiento público anual a los partidos políticos por sus actividades ordinarias permanentes.

e. El 30% de la cantidad que resulte de acuerdo a lo señalado anteriormente, se distribuirá entre los partidos políticos en forma igualitaria y el 70% restante de acuerdo con el porcentaje de votos que hubieren obtenido en la elección de diputados inmediata anterior.

f. En el año de la elección en que se renueven el Poder Ejecutivo Federal o local y las dos Cámaras del Congreso de la Unión o el Congreso de alguna entidad federativa, a cada partido político nacional o local, en su caso, se le otorgará para gastos de campaña un monto equivalente al 50% del financiamiento público que para el sostenimiento de sus actividades ordinarias permanentes le corresponda en ese año.

g. En el año de la elección en que se renueve solamente la Cámara de Diputados federal o los Congresos de las entidades federativas, a cada partido político nacional o local, respectivamente, se le otorgará para gastos de campaña un monto equivalente al 30% del financiamiento público que para el sostenimiento de sus actividades ordinarias permanentes le corresponda en ese año.

3.2. Resultados de la implementación de la nueva metodología para la asignación de recursos públicos a los partidos políticos en México

En este apartado, se analizó el recurso asignado a los partidos políticos en el periodo de 2008 a 2020. Se contempló este periodo de tiempo pues es aquel en el que se aplicó la ley vigente para tal efecto. La tasa de participación del voto fue alrededor del 55% en dicho periodo, en este sentido, se otorgó en términos generales a los partidos políticos un 45% más de los recursos que realmente deberían recibir considerando su número de votos.

Por otra parte, una vez establecido el monto total de los recursos públicos asignados a los partidos políticos, existieron variaciones entre la proporción de recursos públicos que recibieron y la proporción de votos que obtuvieron. Esto debido a que se reparte el 100% de los recursos disponibles entre los partidos que tienen derecho a ello, es decir, aquellos que pasan la barrera a la entrada. Con base en lo anterior, se sabe que se hace una distribución de los recursos totales a aquellos partidos que se encuentran en ese supuesto, sin cuantificar los votos nulos y votos obtenidos por los candidatos independientes.

Si se efectúa una comparación entre el presupuesto otorgado actualmente a los partidos políticos en México y el asignado mediante la metodología propuesta en este trabajo, se evidencian diferencias significativas. En la primera, cada partido depende en gran medida de la cantidad de votos que obtiene, así como de una Votación Total Estimada; en la segunda, se contempla el número real de electores, los cuales en su mayoría, no acuden a las urnas.

En México, una parte del financiamiento público (70%), se reparte de manera proporcional a los partidos políticos según el porcentaje de votos de acuerdo con la Votación Válida Emitida (Votación Total Emitida menos los votos recibidos por los candidatos(as) independientes y los nulos), y la otra, se reparte de forma equitativa (30%).

Con la finalidad de exponer la nueva metodología propuesta en este artículo, se explican y analizan las cifras que se obtuvieron para el financiamiento, para lo cual se tomó en cuenta el año en el que se dio el proceso electoral más reciente (2018); en las estimaciones para 2018, se retoma la votación del año 2015.

Cabe resaltar el hecho de que el Partido del Trabajo estuvo en el límite de la votación que no le permitió recibir recursos públicos, con un 2,9% de la Votación Total Emitida, así cualquier partido que haya obtenido el 3% o poco más de esta votación, recibió una cantidad importante de dinero con una diferencia de menos del 0,1%, esta inequidad no sucedería con la función propuesta para la barrera a la entrada en el reparto de recursos disponibles para los partidos políticos.

Los resultados de la aplicación de la propuesta y su comparación con los montos obtenidos bajo las leyes aplicables para el periodo de tiempo contemplado en este trabajo, se concentran en la Tabla 2. En la columna A, el monto real asignado a los partidos políticos; en la columna B, el monto que corresponde a la estimación considerando la propuesta presentada, el ahorro que se habría derivado en pesos mexicanos; y, la última columna, presenta el ahorro en porcentaje.

Tabla 2
Estimación del ahorro en financiamiento a partidos 2008-2020

AÑO	A: Real Asignado	B: Propuesta	Ahorro (A-B)	% ahorro
2020	$5.239.001.651	$2.908.540.636	$2.894.629.285	44,4
2019	$4.870.560.864	$2.774.628.657	$2.095.932.207	43,0
2018	6.444.499.869	3.044.503.702	3.399.996.167	52,8
2017	4.059.213.905	1.983.049.886	2.076.164.019	51,2
2016	3.953.658.321	1.903.228.327	2.050.429.994	51,9
2015	5.199.695.918	3.138.313.770	2.061.382.148	39,6
2014	3.925.109.677	2.028.031.361	1.897.078.316	48,3
2013	3.670.843.753	1.951.780.517	1.719.063.236	46,8
2012	5.142.514.888	2.126.506.024	3.016.008.864	58,7
2011	3.212.932.809	1.373.920.783	1.839.012.026	57,2
2010	2.997.358.834	1.319.717.288	1.677.641.546	56,0
2009	3.633.067.352	2.042.636.216	1.590.431.136	43,8
2008	$2.690.888.670	$1.319.717.288	$1.371.171.382	51,0

Fuente: Elaboración propia, 2020.

Para el financiamiento público a los partidos políticos de 2019, la votación nacional emitida que se tomó como referencia fue de 40.7 millones de votos. Esta votación correspondiente a las elecciones previas, que beneficiaron en mayor medida al Partido Morena (Morena), al Partido Acción Nacional (PAN) y al Partido de la Revolucionario Institucional (PRI), los cuales concentraron 79,5% del total de votos emitidos. El financiamiento público anual para el sostenimiento de sus actividades ordinarias permanentes (SAOP), que resulta de multiplicar el 65% de la UMA por el padrón electoral total en ese año (90´259,589), fue de 4.728,7 millones de pesos. De acuerdo con lo expuesto en los apartados anteriores, el 70% de dicho monto (3.310,1 millones de pesos) se distribuyó entre los partidos políticos según el porcentaje de la votación total emitida alcanzada, mientras que el 30% restante (1.418,6 millones de pesos) se repartió de forma igualitaria.

A lo anterior se debe sumar un monto equivalente al 50% del financiamiento público anual para el SAOP que se distribuye igualmente según la participación de cada partido en la votación, lo que da como resultado el financiamiento total a los partidos. De esta manera, en 2019 Morena, el PAN y el PRI concentraron 3.339,5 millones de pesos del presupuesto que se destinó a los partidos en ese año; en sentido opuesto, los partidos que menos recursos recibieron fueron el Partido del Trabajo (PT) y Movimiento Ciudadano (MC).

Cabe señalar que el monto disponible en la propuesta es muy difícil que sea igual al monto que finalmente se otorga a los partidos políticos. La razón, como se ha dicho, es que para estimar el monto disponible se emplea el método de Promedios Móviles Ponderados, utilizando las últimas tres elecciones para diputados locales. En dicha estimación, se toma en cuenta los votos nulos, los votos de partidos políticos que no han pasado la barrera a la entrada y los candidatos independientes, si es pertinente. En la propuesta, a diferencia de la ley vigente para asignación de recursos públicos a partidos políticos, este recurso no se redistribuyó entre los partidos que sí pasaron la barrera a la entrada y la propuesta es que se regrese el recurso a la federación.

Así, por ejemplo para el año 2019, se podría disponer de un monto de $2.774.628.657,00 pero solo se distribuiría un monto de $2.236.435.232 (Ver Tabla 2) pues los candidatos no registrados, votos nulos y candidatos independientes, no obtienen recursos aún si hubiesen pasado la barrera a la entrada de financiamiento.

En ese sentido, es posible apreciar que considerar un promedio móvil sobre la votación nacional emitida, implica una base menor para calcular el financiamiento para los partidos políticos, con diferencias porcentuales que van desde 39,6% para 2015 hasta 59,7% en 2012; de esta manera, el financiamiento propuesto implica en promedio un gasto público menor en 50,1% para el periodo 2008-2020.

Además, las diferencias entre el porcentaje de votos y el de recursos obtenidos por los partidos políticos a través de la propuesta, son notablemente menores. Cabe señalar, que estas diferencias se deben principalmente a que en la propuesta a todos los partidos políticos (no solo a los que obtuvieron 3% o menos de la VTE) se les resta una cantidad fija de votos (el 3% de la VTE, en este caso).

Conclusiones

El tema del reparto de los recursos financieros a los partidos políticos siempre será un fenómeno que genere polémicas y disyuntivas tanto en las instituciones participantes como entre los ciudadanos. Para salvar esta controversia es necesario presentar de una manera transparente una propuesta metodológica clara y concisa, además de justa y bien argumentada, sobre la asignación de recursos a los partidos políticos dependiendo de las variables bajo las que se han construido como instancias políticas y del desarrollo de un proceso de competencia electoral.

El sistema actual de administración ignora variables importantes en la repartición de recursos, factores que pueden preverse cualitativa y cuantitativamente si se recurriera al estamento científico. Por ejemplo, una barrera a la entrada que genere trato equitativo a los partidos políticos. Probablemente esto no sería relevante si no fuese porque el patrocinio de los partidos políticos corre a cargo de los recursos públicos, es decir, fondos que tienen su origen en el dinero de los contribuyentes. Este es un motivo trascendental por lo que el sistema debería ser evaluado y modificado por las instancias pertinentes.

Actualmente, las competencias electorales se encuentran bajo el foco de la desconfianza por parte de los ciudadanos, principalmente porque los partidos políticos no solo rebasan casi siempre los topes de campaña, sino porque considerando los presupuestos asignados a distintos rubros, como por ejemplo, los apoyos al campo, el financiamiento recibido por los partidos políticos parece ser excesivo. Cabe señalar que para 2018, periodo de elecciones, se otorgó 6.573,4 millones de pesos. Este presupuesto es casi el doble de lo que se presupuesta en educación para grupos vulnerables y casi igual a lo que se les asigna en salud en todo el país; y representa también casi el triple de lo que se presupuestó para el desarrollo rural en todo México en el año 2019.

Como se ha expuesto, los ahorros que se generarían de aplicar la propuesta de financiamiento público a los partidos políticos, es de un poco más del 50% de lo que actualmente reciben, con lo que evidentemente, podrían verse beneficiadas instancias de desarrollo social y económico desprotegidos en México.

Resulta trascendental no solo transparentar el gasto y los recursos de los partidos políticos, sino tener un método más adecuado en la administración de los recursos que se les asignan. El ahorro que es posible obtener en su financiamiento podría ser trasladado a rubros como la educación y la salud, temas que en esta coyuntura se han visto fuertemente afectados, lo que es bien sabido necesitan ser apuntalados para que, como nación y sociedad, se pueda responder ante situaciones no previstas, por ejemplo la pandemia del COVID-19. Podría decirse que el gasto en mejorar una democracia nunca es excesivo, si el último fin es evolucionar para conseguirla o mantenerla. Sin embargo, es posible plantearse más de una solución para lograrlo de una manera más económica, en condiciones más igualitarias, justas, legales, que sean de la aprobación de los ciudadanos.

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Capítulo 5: *Mathematical political districting taking care of minority groups*

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ORIGINAL - OR MODELING/CASE STUDY

Mathematical political districting taking care of minority groups

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Abstract

Political districting (PD) is a wide studied topic in the literature since the 60s. It typically requires a multi-criteria approach, and mathematical programs are frequently suggested to model the many aspects of this difficult problem. This implies that exact models cannot be solved to optimality when the size of the territory is too large. In spite of this, an exact formulation can also be exploited in a heuristic framework to find at least a sub-optimal solution for large size problem instances. We study the design of electoral districts in Mexico, where the population is characterized by the presence of minority groups ("indigenous community") who have a special right to be represented in the Parliament. For this, the Mexican electoral law prescribes that a fixed number of districts must be designed to support the representation of the indigenous community. We formulate mixed integer linear programs (MILP) following these two principles, but also including the basic PD criteria of contiguity and population balance. The district map is obtained in two stages: first we produce the fixed number of indigenous districts established by the Law; then we complete the district map by forming the non-indigenous districts. This two-phase approach has two advantages: a dedicated objective function can be formulated in Phase 1 to form indigenous districts at best; in the second phase the instance size is reduced (both in the number of territorial units and in the number of districts) so that the computational effort to solve the problem is reduced as well. We test our procedure on the territory of Chiapas in Mexico and on some fictitious problem instances in which the territory is represented by a grid graph. We also compare our district map with the Institutional one currently adopted in Chiapas.

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Keywords Political districting · Minorities' representation · Respect of administrative boundaries · Graph partitioning · Mixed integer linear programming

1 Introduction

Designing electoral districts has been a wide studied subject of research in the literature of the last 60 years. Since the first Multi-Kernel Growth procedure introduced by Vickrey (1961), and the location-allocation model provided by Hess et al. (1965), the challenging task of obtaining a fair division of the territory, with respect to specific electoral criteria, has motivated the production of a variety of Political Districting (PD) models and methods which follow different, exact and heuristic, optimization approaches. A general picture of the state-of-the-art on this subject is provided in Ricca et al. (2013, 2017) and Ricca and Scozzari (2020) were the main models and criteria for PD are surveyed and discussed. Other interesting survey papers on the topic are those by Duque et al. (2007) and Kalcsics et al. (2005) and the very recent volume edited by Rios-Mercado (2020). All provide surveys of methods and models for the general Territory Design problem which includes PD.

Typically PD models are integer or mixed integer mathematical programs. Differences among models depend on which objective function is minimized (or maximized) and on the number and type of restrictions imposed in each formulation. They follow from the particular features deemed relevant in each specific model, according to the principles established in the electoral law of each country. The complexity of the solution process also derives from such elements, according to which of them are considered in the PD problem and how they are actually formulated in the corresponding mathematical model. The classical PD criteria are *population equality* (or *balance*), *contiguity*, *compactness*, and *conformity to administrative boundaries*, see Grilli di Cortona et al. (1999). All these criteria pursue desirable properties of the districts, but it is generally impossible to include all of them in an algebraic model, since, due to the binary nature of the decision variables, this may lead to PD models which cannot be solved in reasonable computational time.

Other criteria have been suggested in the literature on PD, referring to particular aspects which may be considered valid only in very special cases. In this view, George et al. (1997) propose conformity to the natural configuration of the territory, that is, a criterion which requires to avoid that electoral districts' boundaries cross rivers, high mountains, and all such natural barriers of the territory which may prevent reachability between two sites located inside the same district. This criterion is generally well-accepted, but frequently it is discarded in order to avoid that the model complexity increases too much.

A different issue is related to the idea of trying to avoid the split of *communities of common interests* (also referred to as *respect of minority groups*, or *communities representation*).

This last criterion is widely discussed in the literature and, although there are authors that strongly criticize its inclusion in a PD model, others support it. Morril (1987) mentions the importance of considering respect of minority groups in PD and Williams (1995) discusses some examples where the lack of communities' representation produces an unfair solution. Even more, the "Erice Decalogue" (unanimously signed in 2005 by relevant researchers in this area) explicitly established in its eighth point that: "A system using electoral districts should respect existing communities of interest", see Simeone and Pukelsheim (2006). In this sense, in Cameron et al. (1996), the authors suggest non linear estimation techniques and simulate districting strategies trying to solve the under-representation of minority interests in the political process. On the other side, using surveys that sampled minorities in the U.S. and New Zealand, both these countries modified their electoral systems, so that the minority groups will have sufficient power to elect their representatives. In Banducci et al. (2004) it is showed that giving this option to minorities in U.S. these groups also improves their knowledge about representation and the contact with their representatives; in the case of New Zealand this also increases the electoral participation.

In our opinion, the representation of minority groups must be included in the PD model only when the electoral law of a country explicitly prescribes this kind of requirement, and given that the communities under consideration are well defined and can be easily located over the territory. This is in fact a way to pursue a better representation of the minorities in the Parliament, with positive implications on public policies concerning socio-economic problems. This is the case of our application to Mexico, where the resident population is characterized by the presence of community of *indigenous* people having very well defined cultural and social characteristics which allow to identify with a high level of precision the community to represent.

According to the Mexican law, seven criteria must be considered in the political districting process. They are: 1. balancing districts' populations, 2. guaranteeing a prefixed number of indigenous districts in which the indigenous population is at least 40% of the total (respect of the indigenous minorities), 3. guaranteeing integrity of municipalities (respect of administrative boundaries), 4. optimize compactness, 5. minimizing traversing time from one municipality to another within the same district, 6. contiguity, 7. respecting socio-economic factors and geographic features. To be more precise, the first two criteria have a priority over the others, since they are acknowledged by the Constitution.[1]. Population balance is established in Articles 53 and 116, paragraph II of the Constitution, while the respect of indigenous minorities is established in Article 2. Differently, the other five criteria are stated in the Mexican Electoral Law.[2] For the two main principles stated above some precise rules are given: for population balance the requirement is that each district has a population which does not differ more than the 15% from the average one; for respect of indigenous minorities, a prefixed number of districts must be formed by collecting together the indigenous municipalities.

In the paper we propose an automatic procedure which solves the PD problem in two successive stages, both adopting integer mathematical models, which can be equivalently formulated as Mixed Integer Linear Programs (MILP), and by applying ordinary optimization techniques. We consider the classical, PD criteria, such as population balance and districts' contiguity. To follow the principles stated in the Mexican law, we also include respect of minority groups, but, to give it a major importance, we model it as an objective, with the aim of collecting together as many indigenous municipalities as possible in the established-by-the-law indigenous districts. Besides this, also respect of municipal boundaries is considered, which, together with the specific purpose for which it is conceived, is also useful to avoid the splitting of indigenous municipalities. This was possible in our model since respect of municipal boundaries was considered as a hard constraint. We point out that this is an innovative aspect for PD in Mexico, since even if this criterion is clearly stated by the law, it seems to be ignored in the institutional district map.

The methodology proposed in this paper is motivated by the idea that optimization procedures can be a valid decision support for lawmakers in the delicate process of designing

[1] The Constitución Política de México, available in Spanish at http://www.ordenjuridico.gob.mx/Documentos/Federal/wo14166.doc and in English https://www2.juridicas.unam.mx/constitucion-reordenada-consolidada/en/vigente.

[2] Article 214, paragraph 1 of the Ley General de Instituciones y Procedimientos Electorales, Acuerdo INE/CG59/2017, available in Spanish at https://repositoriodocumental.ine.mx/xmlui/handle/123456789/92303.

electoral districts. The automatic nature of the method is also a guarantee of “fairness”, provided that decision makers agree on formulating the PD problem by a mathematical model. Electoral districts satisfying many desirable criteria simultaneously can be obtained efficiently by a super-partes automated tool, which can be easily implemented in the computers of an electoral office. If, on the one hand, the solution procedures (which, in any case, are a matter of the optimization solver) are not within everyone’s reach, on the other hand, MILP models can be easily understood by everyone, also with the possibility of evaluating the output solution and, possibly, modifying it, if this is deemed necessary. In this view, basic principles of electoral problems, like simplicity, transparency, and impartiality, can be pursued while providing useful tools to solve efficiently problems which are technically difficult. Since considering too many criteria in a MILP leads to the impossibility of solving it in reasonable times, not all PD criteria can be included in the model, and, in our approach, we chose to give a secondary role to compactness. There are different reasons supporting this choice which we illustrate in what follows. The first one is that unlikely a country imposes compactness explicitly in the electoral law. As in Mexico, it is generally recommended that electoral districts have a “round” shape, but it is well-known that translating this into a formal constraint is difficult and it is generally impossible to model the many facets of compactness, see Horn et al. (1993) or Grilli di Cortona et al. (1999) and the references therein. A second problem is related to the fact that it is generally not correct to judge the compactness of the districts looking directly at their geographical shape on the map, since the actual compactness also depends on the distribution of the population over the territory. Also the geographical orientation of the state may affect the compactness level which can be actually reached in a district map. Therefore, even if compactness is a deterrent for gerrymandering, when an automatic procedure is adopted, fair district maps can be obtained also without imposing compactness, provided that the procedure is not biased a priori.

Finally, in the particular case of our PD problem in Mexico, a priority is respecting municipal boundaries, that is, municipalities must not be divided into more than one district. Since the Mexican municipalities have highly non regular geographical shapes, it is very hard to find districts with regular shapes without dividing them in smaller parts. This means that geographical compactness and respect of municipal boundaries are conflicting criteria and, between the two, we gave priority to the second. This choice also depends on an additional consideration, which is very important when coping with the delicate criterion of respect of minorities. Following the principle that indigenous representation must be guaranteed in the Parliament, the idea of this criterion is that indigenous must be collected in the same districts to gain representation power. Depending on where the indigenous are actually located in the country’s territory, this task may become really hard. Imposing compactness in this framework may produce conflicts in the model tasks, and this may prevent the formation of suitable indigenous districts. For this reason we do not consider compactness when designing the indigenous districts in Phase 1 of our procedure.

In our second phase the situation is completely different, since we have just to produce non-indigenous districts, and we do not care any more about the criterion related to the minorities. Therefore, in this case, compactness could be considered, even if, together with population balance and contiguity, this results in a very hard model. To manage this aspect, in our second phase we include compactness and try to solve the model. As we will see, the model becomes impossible to solve to optimality in short times. For this reason we resorted to a successive divisions approach by applying the districting model with compactness and dividing the territory into two parts in successive steps. This is a classical approach in territory design and in PD literature, see, for example, Forrest (1964).

In the paper we test our procedure on Chiapas and on some additional artificial data sets with size ranging from 100 to 200 territorial units (municipalities). For Chiapas and the other territories of similar size (120 units) we manage to solve the problem by applying our MILP models, while for the cases with 210 units the computational time for solution becomes too long and an heuristic procedure is required.

The paper is organized as follows. Section 2 outlines the main characteristics of the classical PD problem and the traditional PD criteria. Section 3 introduces the notation and illustrates the mathematical programs used in our procedure to solve the problem exactly. On the other hand, Sect. 4 illustrates an alternative heuristic approach which can be used when the exact solution becomes a hard task. Section 5 reports on the results of the application to the Mexican state of Chiapas and to a set of fictitious territories with size similar to Chiapas or larger. Finally, Sect. 6 draws some conclusions and final thoughts, as well as possible extensions of our approach.

2 Criteria

Classical criteria in political districting are population equality (or balance), conformity to administrative boundaries, contiguity and compactness. In our model we also include the respect of minority groups. We briefly recall each of them in what follows.

Population equality Under the assumption that the electoral system is a plurality one, all districts should have the same proportion of representation (according to the "one person, one vote" principle); in particular, in case of single member districts, they should have nearly the same population. This criterion could be implemented as a constraint in the model, by fixing a maximum tolerance on the deviation of the population of each district from the average district population, and imposing each district population to be within the tolerance thresholds. The ideal situation would be having exactly the same population in each district, but this is obviously impossible to get, and typically the maximum absolute deviation is fixed as a percentage of the average district population. In this way, the model rigidly bounds the districts' populations from above and below.

As for any other criteria, in a mathematical program, population balance can be pursued also by means of the objective function. In this case, a measure of the total deviation of population over all districts must be established (for example one can compute the L_1-norm or L_2-norm of the districts' populations), and the model tries to minimize it. This leads to different mathematical models according to the algebraic form of the objective function.

Conformity to administrative boundaries According to this criterion, one should avoid as much as possible that already existing administrative or normative zones (such as counties or municipalities) are split among two or more electoral districts. Actually, splitting a municipality into two or more parts does not affect conformity to its administrative boundaries if each of these parts corresponds to a single district. In fact, even if in this case the integrity of the municipality is broken, its separated parts are not included in a district together with portions of other municipalities. This last situation must be avoided instead. On the other hand, if integrity of municipalities is imposed, the districts automatically satisfy conformity to municipal boundaries, too. This is in fact the way in which we pursued this criterion at best in our model.

Compactness Compactness is a very intuitive concept, but, unfortunately, a rigorous definition of this notion does not exist. A district may be considered compact if it spans a round region, without straggling or rambling. Deviation from compactness has been classified by Taylor (1973) according to the shape of the district in the following four categories: elongation, indentation, separation, and puncturedness. Notice that Taylor's definition of compactness is very strong, since it implies both contiguity (non-separation) and absence of holes (non-puncturedness). Caution should be taken when dealing with this criterion because every attempt to define a correct measure of compactness turns out to be strongly related to only one of the above categories, and each indicator is able to detect only some types of non-compactness. An idea could be to consider more than one compactness measure at the same time. This implies a careful measurement of compactness at the expense of a very complicated mathematical model which could become unmanageable if there are also other criteria. This is why, in the end, the most practical thing to do is to measure compactness, as every other criterion, with a single index. This is the typical approach in the literature on PD where compactness is frequently measured as the total inertia of the district map, see Grilli di Cortona et al. (1999).

Contiguity A district is contiguous if it is possible to reach any place in the district from any other one in the same district without leaving it. In any territorial districting problem, contiguity is a crucial aspect, since it provides a structural feature aimed at guaranteeing well defined responsibility areas (whichever the activity related to the district: political vote, commercial activities, the distribution of a service). The difficulty with this criterion is that it is hard to be implemented in an algebraic model. For these reasons, the PD problem is frequently modeled via the use of a connected graph, called *contiguity graph*, which represents the territory, see Ricca et al. (2008) and Ricca and Simeone (2008), among others. After a discretization of the territory into a fixed number of elementary territorial units, in the contiguity graph, nodes correspond to territorial units and an edge between node i and node j exists if i and j are neighbors. In this context, contiguity can be guaranteed by searching for a partition of the graph into connected components. An organized presentation of the main issues related to contiguity in territorial districting can be found in Duque et al. (2007) and in Ricca and Scozzari (2020).

In this paper we consider population balance, conformity to administrative boundaries, contiguity and respect of the Mexican indigenous community.

We basically control population balance by imposing lower and upper thresholds for the district populations.

Conformity to municipal boundaries is considered as a hard condition, and, in principle, our aim is to maintain integral the territory of all municipalities. Unfortunately, as we will see in the application to Chiapas reported in Sect. 5, this is not always possible, since some municipalities with population much larger than the average district population may exist. This evidently prevents perfect respect of municipal boundaries, but, in any case, our approach aims at avoiding as much as possible the division of a municipality into many pieces assigned to different districts.

Districts' contiguity is a main issue in the methodology proposed in this paper. We apply the approach provided by Shirabe (2009) who introduces a set of flow constraints able to enforce the contiguity of the districts. Even if this requires a polynomial number of constraints, the mathematical models for PD have binary variables, so that the solving the problem to optimality is computationally hard and only problems of small size can be solved in reasonable time.

We point out that there is a variety of approaches for managing contiguity when a graph representation of the territory is adopted. There are three typical approaches based on the following three ideas, see Duque et al. (2007) and Miyazawa et al. (2021): (1) imposing order constraints on the assigning variables related to nodes lying on specific paths; (2) exploiting tree structures to guarantee the contiguity of each district; (3) using networks flow formulations. Among these methods, flow constraints have the best performance since they are able to solve problems with up to 100 nodes. However, this is generally possible only for the basic problem of partitioning a graph into connected components (without other constraints). When, on the contrary, some additional constraints are included in the model even small problems become difficult to solve to optimality. Attempts to overcome this problem are present in the current literature on the subject, and some interesting results are provided for example in Validi et al. (2020), Rios-Mercado and Lopez-Perez (2013) and Oehlein and Haunert (2017). Unfortunately, efficiency in the design of contiguous district maps can be obtained at the price of using high sophisticated technical tools, such as *cutting planes* strategies, or adding extra constraints to formulate *separation problems*. On the contrary, a districting method has to be as simple as possible in order to be understood and used by politicians, legislators, public administrators, etc. This is a long-standing question in the mathematics of electoral systems, which always leads to the evaluation of the compromise between efficiency and user-friendlyness of the proposed procedures. In our approach we model contiguity as a hard constraint since we believe that it is a fundamental aspect of the problem, not easily manageable by non technicians. Including it in the model guarantees that lawmakers have a procedure which produces a balanced and contiguous district map, without any need of additional intervention for fixing possible non-contiguity.

Respect of minority groups Given a specific and recognizable type of community, which can be reasonably considered a minority w.r.t. the rest of the population of a country, when designing the electoral districts, one should collect as much as possible people from the minority group in the same districts. In other words, electoral districts' populations should be as much homogeneous as possible w.r.t. the group identity.

As already discussed, in the Mexican case, maintaining the identity and the representation power of the indigenous community is a main issue. This is the reason why, in our approach, we focus our attention on this aspect and try to design a model approach aimed at guaranteeing respect of communities.

Finally, as we will see in Sect. 3, we consider the compactness criterion in our second phase model by modeling it in the objective function.

3 A two-phase districting procedure for political districting with minority groups

The methodology proposed in this paper is tailored for PD in Mexico, but can be applied in general when the population of a given country is characterized by the presence of a well defined minority group and the law supports the representation of such minority. The method is based on Mathematical Programming, and works in two successive stages: in the first one, districts including the minority groups are designed; then the district map is completed in the second stage with the formation of non-indigenous districts. The method is tested on real-life data referring to the territory of Chiapas and on some fictitious territories represented by grid graphs with size similar to Chiapas or larger. A successive divisions approach is applied when the MILP cannot be solved to optimality. In these cases, even if the solutions of the models are not exact, we manage to obtain a good solution w.r.t. to all the PD criteria considered in the model. This provides a practical solution tool for political districting able to give automated decision support to lawmakers and to electoral offices. A heuristic variant of the method is also developed to solve problem instances of larger size.

The PD problem is the following one. A division of the territory into municipalities is given. Each municipality represents an elementary territorial unit and is characterized by its resident population. Municipalities are classified into two types according to how large is the number of residents belonging to the minority. A threshold τ (fixed in advance by the Government) is used to obtain this classification. A municipality is classified as a *minority municipality* if at least $\tau\%$ of its population belongs to the minority. We also consider a graph representation of the territory which is useful for visualizing the results on a map and helpful to formulate and understand the contiguity constraints, which in our model are implemented by means of network flow constraints, see Ahuja et al. (1993).

The connection structure of the graph is also exploited in the heuristic procedure that follows a constructive strategy of the districts and works by adding municipalities one at a time.

In the contiguity graph G, nodes represent municipalities and there exists an edge between nodes i and j if the corresponding municipalities share a common boundary (different from a single point). As we will see, in order to implement the flow constraints in the model, we have also to consider the network $\mathcal{G}$ which can be obtained from G in the ordinary way, i.e., by maintaining the set of nodes and replacing each edge $< i, j >$ of G by a pair of (directed) arcs (i, j) and (j, i).

The PD problem is formulated as partitioning the set of the municipalities (nodes of the graph G) into a prefixed number k of contiguous districts (connected components of G), taking into account the basic PD criteria. Aggregating nodes of G, the municipal boundaries are automatically respected by the fact that a municipality (node of G) is never split among more than one district. No other administrative boundaries are considered.

The special situation given by the existence of the indigenous minority also requires that only a part of the k districts contains the minority group population, so that a number k_I is given as an input parameter providing the number of *minority districts* to design.

3.1 Basic notation and parameters' definitions

We introduce here all the notation necessary to formulate the mathematical programming models presented in the next sections. Considering the Mexican case, we have indigenous and non-indigenous population.

Sets

M	Municipalities
M_I	Indigenous municipalities
M_{NI}	Non-indigenous municipalities
K	Districts
K_I	Indigenous districts
K_{NI}	Non-indigenous districts

Parameters

p_i	Population of municipality i
m	Number of municipalities
m_I	Number of indigenous municipalities
m_{NI}	Number of non-indigenous municipalities
k	Number of districts

k_I	Number of indigenous districts
k_{NI}	Number of non-indigenous districts;
$\bar{P} = \sum_{i \in M} p_i/k$	Average district population (target population)
P_h	Population of district h
α	Maximum absolute deviation for each district population to $\bar{P}(\%)$
d_{ij}	Distance between municipality i and municipality j (in km)

Decision variables

x_{ih}	: Equals 1 if municipality i is assigned to district h, 0 otherwise
w_{ih}	: Equals 1 if municipality i is chosen as the center of district h, 0 otherwise
y_{ijh}	: Amount of flow from municipality i to municipality j for district h

3.2 Mathematical programming models

In this section we introduce the mathematical programming models which we apply to build a PD map taking into account minority groups in Mexico. We propose Integer Programming formulations with non linear objective functions which are then linearized to obtain the corresponding Mixed Integer Linear Programs.

Our approach applies in two successive phases. The first one is dedicated to the design of the k_I indigenous districts, which should include all the indigenous municipalities. In the second phase, the remaining municipalities are considered to form the rest of the districts (k_{NI}). Note that in the first phase all municipalities are considered, but the model does not force those in M_{NI} to be assigned to some districts. This is motivated by the idea of giving flexibility to the model in the first phase, when population from the minority group must be collected together as much as possible, but provided that all PD constraints are satisfied.

$$\min_{x} \left(\max_{h \in K_I} \sum_{i \in M_{NI}} x_{ih} \right)$$

s.t.

$$\sum_{h \in K_I} x_{ih} = 1 \quad \forall i \in M_I \tag{1}$$

$$\sum_{h \in K_I} x_{ih} \leq 1 \quad \forall i \in M_{NI} \tag{2}$$

$$\sum_{i \in M} w_{ih} = 1 \quad \forall h \in K_I \tag{3}$$

$$w_{ih} \leq x_{ih} \quad \forall i \in M, \forall h \in K_I \tag{4}$$

$$\sum_{j|(i,j)\in A} y_{ijh} - \sum_{j|(j,i)\in A} y_{jih} \geq x_{ih} - [\,|M| - |K_I| + 1]\, w_{ih} \quad \forall h \in K_I, \forall i \in M \tag{5}$$

$$\sum_{j|(j,i)\in A} y_{jih} \leq [\,|M| - |K_I|\,] x_{ih} \quad \forall h \in K_I, \forall i \in M \tag{6}$$

$$(1-\alpha)\bar{P} \leq \sum_{i \in M} p_i x_{ih} \leq (1+\alpha)\bar{P} \quad \forall h \in K_I \tag{7}$$

$$y_{ijh} \geq 0 \quad \forall h \in K_I, \quad \forall (i,j) \in A$$

$$x_{ih},\ w_{ih} \in \{0,1\}, \quad \forall h \in K_I, \quad \forall i \in M$$

The formation of the k_I districts of the first phase is performed via restrictions (1) and (2) in the model, according to which a municipality $i \in M_I$ *should be* assigned to some districts in K_I (the constraint is an *equality*), and a municipality $i \in M_{NI}$ *could be* assigned to one of them (the constraint is an *inequality*). Constraints (3) state that each district in K_I should have exactly one center, while constraints (4) model the dependency between variables x_{ih} and w_{ih}. Restrictions (5) and (6) are the flow constraints proposed by Shirabe (2009) for districts' contiguity and involve variables y_{ijh} representing the flow from municipality i to municipality j related to contiguity for the hth district. Note that the use of these flow constraints requires the model to identify one center per district. The role of such centers is merely technical and in our application it has no specific political, territorial, or demographic meaning. Constraints (7) limit the population of the districts between two bounds whose level can be modulated by changing the percentage α that establishes the maximum and minimum absolute population deviations from the average district population $\bar{P}$. The respect of minority groups is considered in the objective function which tries to minimize the maximum number of non-indigenous municipalities belonging to the districts in K_I. This objective function can be linearized using some well-known techniques in order to obtain an equivalent MILP. A new variable $\lambda \geq 0$ is introduced representing the maximum number of non-indigenous municipalities in an indigenous district. Therefore, the objective of the model is to minimize λ under the following set of constraints which guarantees that λ is the above defined maximum:

$$\lambda \geq \sum_{i \in M_{NI}} x_{ih} \quad \forall h \in K_I$$

Once the indigenous districts have been formed, a second mathematical model is applied to complete the district map. At this stage only non-indigenous municipalities are involved and, in particular, only those which were not assigned to any district in the first phase. Even if they actually are a subset of M_{NI}, for the sake of simplicity, in the following we refer to this subset (of only non-indigenous municipalities) still as M_{NI}.

$$\max \left(\min_{h \in K_{NI}} \sum_{i \in M_{NI}} p_i x_{ih} \right)$$

s.t.

$$\sum_{h \in K_{NI}} x_{ih} = 1 \quad \forall i \in M_{NI} \tag{1}$$

$$\sum_{i \in M_{NI}} w_{ih} = 1 \quad \forall h \in K_{NI} \tag{2}$$

$$w_{ih} \leq x_{ih} \quad \forall i \in M_{NI}, \forall h \in K_{NI} \tag{3}$$

$$\sum_{j|(i,j) \in A} y_{ijh} - \sum_{j|(j,i) \in A} y_{jih} \geq x_{ih} - [\,|M_{NI}| - |K_{NI}| + 1]\, w_{ih} \quad \forall h \in K_{NI}, \forall i \in M_{NI} \tag{4}$$

$$\sum_{j|(j,i) \in A} y_{jih} \leq [\,|M_{NI}| - |K_{NI}|\,] x_{ih} \quad \forall h \in K_{NI}, \forall i \in M_{NI} \tag{5}$$

$$\sum_{i \in M_{NI}} p_i x_{ih} \leq (1 + \alpha) \bar{P} \quad \forall h \in K_{NI} \tag{6}$$

$$y_{ijh} \geq 0 \quad \forall h \in K_{NI}, \quad \forall (i, j) \in A$$
$$x_{ih},\ w_{ih} \in \{0, 1\} \quad \forall h \in K_{NI}, \quad \forall i \in M_{NI}$$

This second model has a structure similar to the previous one, but now the territorial units are all of the same type (all non-indigenous), so that the objective function can be exploited in order to reduce the number of constraints in the model. Therefore, now we bound districts' populations only by above, and we control them by below with a max–min objective that can be linearized as before, by introducing a new variable $\mu \geq 0$ representing the minimum district population, as well as the related set of constraints:

$$\mu \leq \sum_{i \in M_{NI}} p_i x_{ih} \quad \forall h \in K_{NI}$$

In order to take into account also compactness, an alternative model was considered in which both lower and upper bounds on population are included in the set of constraints, while *total inertia* is minimized as objective function. This leads to a Phase II model which analogous to the one by Hess et al. (1965), but including also the contiguity constraints. Using our model variables the inertia objective is formulated as a quadratic function:

$$\sum_{h \in K_{NI}} \sum_{i \in M_{NI}} \sum_{j \in M_{NI}} p_i d_{ij}^2 x_{ih} w_{jh}$$

and the Phase II model becomes a Quadratic Program. It is worth noting that, even if the objective function is quadratic, the constraints remain linear, and the model can be solved with the standard optimization solver for this type of programs, by applying a successive division approach, we were able to solve this quadratic program at each stage for the territories of smallest size.

To conclude this section, we point out that, after Phase I, some non-indigenous municipalities which were not assigned to a district may form *enclaves*, i.e., they can be isolated from the rest of non-indigenous municipalities which will be processed altogether in Phase II. This is a physiological consequence of the two-phase procedure which sometimes may arise. As we will see in Sect. 5, if this happens, only few territorial units remain isolated.

4 Heuristics for the two-phase PD approach

In the application of our exact procedure to Chiapas we encountered some problems of different nature. On the one hand, we detected some irregularities in the structure of the territory which prevented the possibility of exploiting the contiguity graph in the best way. On the other hand, some difficulties arose in the solution of some MILP due to the excessive computational effort required by the procedure, especially when solving Phase II.

The first problem is due to the preexisting territory subdivision into municipalities, which in our model correspond to the elementary territorial units (nodes of the contiguity graph). As already discussed, the model input should be given by elementary units having more or less similar populations. The huge population of Tuxtla in Chiapas shows that this condition is not satisfied in our real case. Since Tuxtla population is not enough to form two districts, we were forced to operate some manual preprocessing on this municipality. This kind of situations does not generally arise if the given administrative division of the territory into municipalities is population balanced. This was one of the reasons why, we decided to extend our experimental analysis and test our procedure also on fictitious territories with balanced municipalities' populations.

A second problem was detected during the solution of our programs, especially in Phase II, when considering the large-size problems for which the exact approach required too much computational time. This was probably due to the presence of connectivity constrains combined with those on population bounds. To overcome these problems efficiently we developed a heuristic procedure in which we tried to maintain the same two-phase approach as the exact one. It is a constructive procedure which aggregates municipalities one at a time to form the non-indigenous districts. The basic principles of the two-phase approach are still satisfied, but the heuristic procedure allows to solve the second phase faster than before.

As we will see in the section dedicated to the experiments, there is also an additional problem that may require a particular computational effort. It is related to the spatial distribution of the indigenous population on the territory. In fact, even if in most cases the minority groups are all located in a specific well defined part of the state, it may also happen that different communities are found far from each other. In our experiments on fictitious data we generated different configurations in which indigenous municipalities are concentrated, or they spread all over the territory. It is natural to expect that, when the communities are sparse, the districting problem becomes harder, but we will see that some difficulties arise also when they are concentrated in the same zone, but geographically located in the center of the territory.

In all cases in which the computational effort to solve the problem optimally becomes too heavy, we resort to the heuristic method. The empirical study of these cases permitted us to better understand the PD problem with minorities, and also to provide a reference scheme about which cases, for size and territory configuration, can be solved optimally and which necessarily require a heuristic approach.

4.1 The heuristic procedure

The procedure works on the contiguity graph by forming one district at a time. For each district a heavy node (in terms of population) is selected and the algorithm follows a sequential kernel-growth strategy. In order to avoid premature arrests and to satisfy district population bounds, a parameter P_{low} is introduced to control the amount of population of the districts under construction. The idea is to obtain first the kernels of the districts with a population which is smaller than the imposed lower bound. Next, in the growth step, these *partial districts* are completed by adding the unassigned municipalities to the districts with smallest population and taking into account contiguity.

At the end, all the municipalities are assigned to some districts. Thanks to the use of parameter P_{low}, which must be carefully calibrated, in our experiments we are able to obtain districts whose population is either in the range of the bounds imposed at the beginning (15%), or it is not too far from such bounds. Therefore, if some districts exceed the maximum possible population, the municipality in the district with the smallest population is removed and it is assigned to an adjacent district with the smallest population. Similarly, a district h may have a population size smaller than the minimum allowed, so that the district adjacent to h with the largest population is selected and, among its municipality adjacent to h, the one with the smallest population is removed from it and assigned to h. In our experiments this kind of situations arises, but, in all cases, they are solved by performing just few of these steps.

This is due to the fact that districts with population outside the imposed bounds are, in fact, not too far from such limits, and also thanks to the fact that elementary territorial units are population balanced.

The heuristic algorithm was implemented by using the software Mathematica.[3] We briefly illustrate the main steps of the procedure.

Algorithm: `HEURISTIC PD`

Set P_{low} equal to a percentage of $\bar{P}$ and $Ndistr$ equal to the number of districts to form

repeat:

(CONSTRUCTION OF DISTRICT h)

$P_h := 0$

select the non assigned municipality with the largest population, say p_i

set $P_h := P_h + p_i$

while $P_h < P_{low}$ **do**:

among all the municipalities adjacent to those already included in district h

select the one with the largest population, say j, and add it to district h

update $P_h := P_h + p_j$

end while

$Ndistr = Ndistr - 1$

until $Ndistr = 0$

for each non assigned municipality i

assign i to the the district with the smallest population among those adjacent to i

end for

The above heuristics is mainly thought for solving the second phase problem, since this is typically the computationally hardest. However, in the larger data sets we encountered computational problems also in solving the MILP model of Phase I, so that the heuristic approach was exploited also in this phase for the hardest problems.

In particular, we resorted to the heuristics for solving the PD problem on fictitious territories when indigenous municipalities have the configuration of being spread over the territory. In this case, the task of collecting all indigenous municipalities in the indigenous districts becomes, in fact, too challenging, and imposing this as a hard constraint may lead to an unfeasible model. This problem arises for all data sets when indigenous are spread-located and, in order to find the indigenous districts, in our Phase I model we have to relax into inequalities the assignment constraints for indigenous municipalities (which before were formulated as equalities) as follows:

$$\sum_{h \in K_I} x_{ih} \leq 1 \quad \forall i \in M_I \tag{1}$$

The MILP is then solved with these relaxed constraints and lowering the average district population to a value equal to $P_{low} < \bar{P}$. This implies that in Phase I only partial districts are produced, for which the population is maintained smaller than the final district population (typically, it is under the lower bound imposed in the model). The partial districts are then completed heuristically by adding adjacent municipalities in order to bring districts' populations within the prefixed bounds.

[3] https://www.wolfram.com/mathematica/.

A natural consequence of this procedure is that not necessarily all indigenous municipalities are assigned to some indigenous districts in Phase I. With indigenous population spread over the territory this is unavoidable, but thanks to the objective function of our Phase I MILP, we are still able to assign as many indigenous municipalities as possible in such districts.

The above illustrated strategies showed to be effective, since for all our grids we were able to obtain a final contiguous district map with all district populations within the imposed bounds. For spread indigenous configurations, it happened that a single indigenous municipality (or just few) is assigned to a non-indigenous district.

5 Application and experiments

The Mexican Chamber of Deputies is elected by a mixed electoral system. The Chamber of Deputies, is composed by 500 representatives: 300 are elected by a first-past-the-post method in single-member districts; the other 200 are elected through a proportional method in five electoral constituencies with 40 representatives each. The Mexican Electoral Institute recognizes the peculiarity of the Mexican population to have indigenous communities and a municipality is considered indigenous if at least 40% of its population is indigenous, according to INEGI (National Institute of Statistics and Geography, Mexico). We applied our model to design single-member districts in real territory of the municipality of Chiapas, but also to solve the same PD problem on a set of fictitious territories. It is worth mentioning that in the Mexican mixed electoral system the allocation of proportional seats depends on the results of the election by the plurality system. In fact, the votes obtained by a party in a single member district are also used to get seats in the electoral constituencies where the proportional method is applied.

We solved our models by using the AMPL modeling language[4] and the available solver CPLEX 12.6.1.0. All our runs could be solved within 15 min on a processor Intel(R) Core(TM) i7-4810MQ CPU @ 2.80 GHZ.

5.1 Application to the State of Chiapas

The state of Chiapas has a total population of 4.8 million people, and a considerable part of it is indigenous. The number of municipalities in Chiapas is 118 and the percentage of indigenous municipalities is 39.8 (47 municipalities).

The number of districts in Chiapas is 13, 5 of which should be indigenous according to the INE/CG59/2017 (total population data and indigenous population per municipality was taken from INEGI's website[5]). In order to visualize the solutions on the map, we used the open-source software R.

In order to implement our model on these real data, it is necessary to perform a preprocessing on those municipalities which show some structural features that do not fit our modeling framework. In Chiapas we detected three situations which need in fact a specific treatment. We illustrate them in the following special cases.

Tuxtla municipality

This municipality is the capital of the state of Chiapas with a population size equal to 555,374. The average district population in Chiapas is $\bar{P} = 368,969$. Hence, depending on the value fixed for the parameter α, this municipality alone may exceed the upper limit imposed on

4 https://ampl.com/.

5 www.INEGI.GOB.MX.

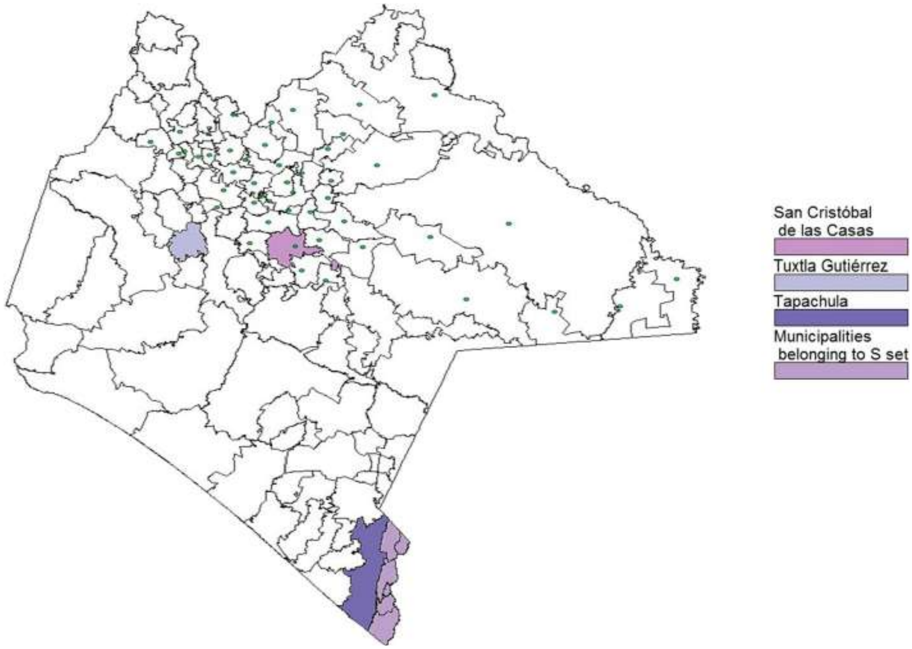


Fig. 1 Chiapas municipalities not fitting basic data requirements for the application of our models

the district population. Unfortunately, this already happens when considering a maximum percentage of deviation about 15%, but it still happens if it is increased up to 30%.

San Cristobal de las Casas municipality

This municipality is geographically divided into two separated parts. Therefore, it is structurally non contiguous by itself.

Tapachula municipality

Tapachula has a population of 320, 450 inhabitants. Therefore, it alone could form a district respecting the population bounds of a maximum deviation of 15% from the target $\bar{P} = 368,969$. In spite of this, municipality of Tapachula is geographically located in a position that isolates a group of six municipalities (Suchiate, Frontera de Hidalgo, Metapa, Tuxtla Chico, Cacahuatán and Unión Juárez). These municipalities can be connected among them, but obtaining a total population about 148, 391 which alone is not enough to produce a district. Alternatively, they can be connected to the rest of the Chiapas municipalities, but only through Tapachula. In fact, for this set of municipalities Tapachula represents a barrier of connection with the rest of the territory. Therefore, when it is left alone, it provides a feasible district, but this prevents the formation of other contiguous districts satisfying the population bounds.

The particular features of each of the three cases listed above (shown in Fig. 1) do not allow to satisfy the modeling prerequisites to apply our methodology. Therefore each situation must be treated in some way before the application of the models.

Since respect of municipal boundaries is a hard constraint in this models, it is not possible to split a municipality (elementary unit) among two or more electoral districts. If a municipality alone exceeds the upper bound on the district population, it could be divided in more than one part, provided that the exceeding population is sufficient to form another district (i.e., it reaches at least the lower bound). Since this is not possible for Tuxtla, perfect respect of municipal boundaries is not possible for Chiapas. To cope with this problem, without dividing Tuxtla into too many pieces, we split it a priori into two parts. One has population equal to the upper bound over the district population and forms a district completely included in Tuxtla; the other covers the rest of the population of this municipality and is assigned to another district. Given the structure of this big municipality, which in any case requires to split Tuxtla, this is a simple way to optimize respect of municipal boundaries since the municipality is divided only in two pieces rather than in many.

For San Cristobal de las Casas, which is an indigenous municipality, we have a strange situation in which the territory of the municipality covers two separated portions of land. These are in fact two elementary territorial units in our data set (obtained from the INEGI data base), labeled by number 78 and 120, respectively. In order to avoid unexpected and undesired districts configurations, in our data we maintained the two separated territorial units, but we included in the model an additional constraint which forces these two municipalities to be always in the same district. Therefore, given that x_{78k} and x_{120k} are the assignment variables for district k and municipality 78 and 120 respectively, the restrictions to impose are the following:

$$x_{78k} = x_{120k} \quad \forall k \in K_I \tag{2}$$

Only districts in K_I are involved in this set of constraints, since the two units belong to an indigenous municipality. Combined with contiguity, these additional constraints act to guarantee that in any output district map the two units are always included in the same (connected) district.

Finally, for Tapachula municipality we have another technical problem, since this municipality geographically configures as a physical barrier between a set of municipalities and the rest of the territory. The set, which we denote by S, includes six municipalities, i.e., Suchiate, Frontera de Hidalgo, Metapa, Tuxtla Chico, Cacahuatán and Unión Juárez, which in total do not reach the minimum district population when the threshold of 15% is adopted. On the other hand, by adding also Tapachula the district population upper limit is exceeded. It is then impossible to form a set of contiguous districts, all within the 15% population limit, without dividing Tapachula. Hence, we separated a priori a piece of territory from Tapachula to get one district formed by this part and the 6 municipalities. After this, the rest of the territory of Tapachula forms a new territorial unit to be included in some other districts. By this preprocessing of Tuxla and Tapachula, even if perfect respect of municipal boundaries cannot be reached in Chiapas, we manage to optimize it since only two municipalities are split, and in only two parts. Operating in this way produces two advantages: (1) the structural problems of Chiapas municipalities is overcome; (2) two districts are formed a priori and the number of districts to be formed by the MILP is reduced by 2.

Figure 2 shows the results obtained in the first phase of the proposed model. In this first phase all the 47 indigenous municipalities in Chiapas were assigned to some indigenous districts. After the application of the first phase, three non-indigenous municipalities remained non assigned. Since they formed enclaves enclosed in one of the indigenous districts, they were added a posteriori to that district.

Figure 3 shows the non-indigenous districts formed after Phase II is completed. First of all, one can visualize the two districts formed a priori. For Tapachula, one part is included in District 13, together with Suchiate, Frontera de Hidalgo, Metapa, Tuxtla Chico, Cacahuatán and Unión Juárez, while the rest formed a territorial unit which was then included in District

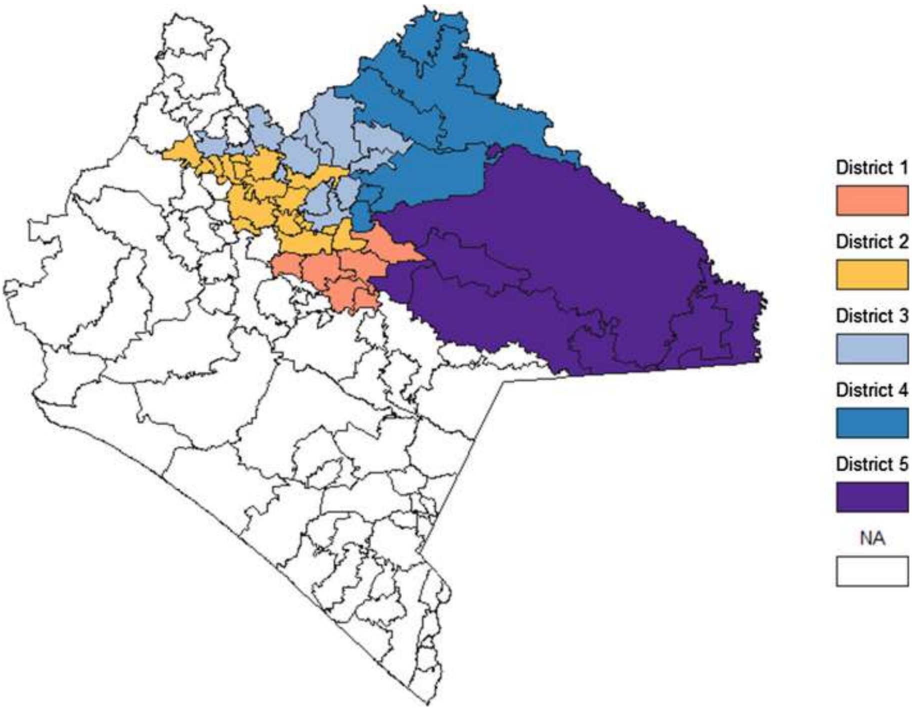


Fig. 2 Map of the indigenous districts in Chiapas formed after Phase I

6. On the other hand, one can easily visualize that District 12 is cut from the municipality of Tuxtla, the rest of which is included in District 11.

In this phase 6 districts were actually formed out of the total 8 non-indigenous districts, since, as already discussed, a part Tuxtla forms one district on its own, while another is given by a part of Tapachula and the 6 municipalities of the set S.

These 6 districts are the output of our successive division approach. We first divided the territory into 2 districts, and, successively, we further divided each of the two parts into 3 districts. Therefore we got in total 6 districts in which one can appreciate also the effect of considering compactness. In fact, 4 out of 6 districts have a quite compact shape, while the other two, namely district 7 and 10, have an elongated shape. We point out here that also the institutional districts map shown in Fig. 5 has elongated districts, although with a different shape. In Chiapas this is an unavoidable output for some districts, since some municipalities in the western part of the country have small population density, and elongation derives from the effort of collecting together a sufficient amount of residents to get a district. We point out in any case that, what is guaranteed by our approach is that mathematical programs do not pursue gerrymandering on their own, so that lack of shape compactness cannot be attributed to lack of fairness in the division process.

The final 13 districts obtained for Chiapas with our procedure are presented in Fig. 4. The population of all the districts satisfies the imposed bounds which, according to a maximum 15% absolute deviation from $\bar{P}$, correspond to $P_{min} = 313,624$ and $P_{max} = 424,314$. Figure 5 shows the institutional districts currently adopted in Chiapas.

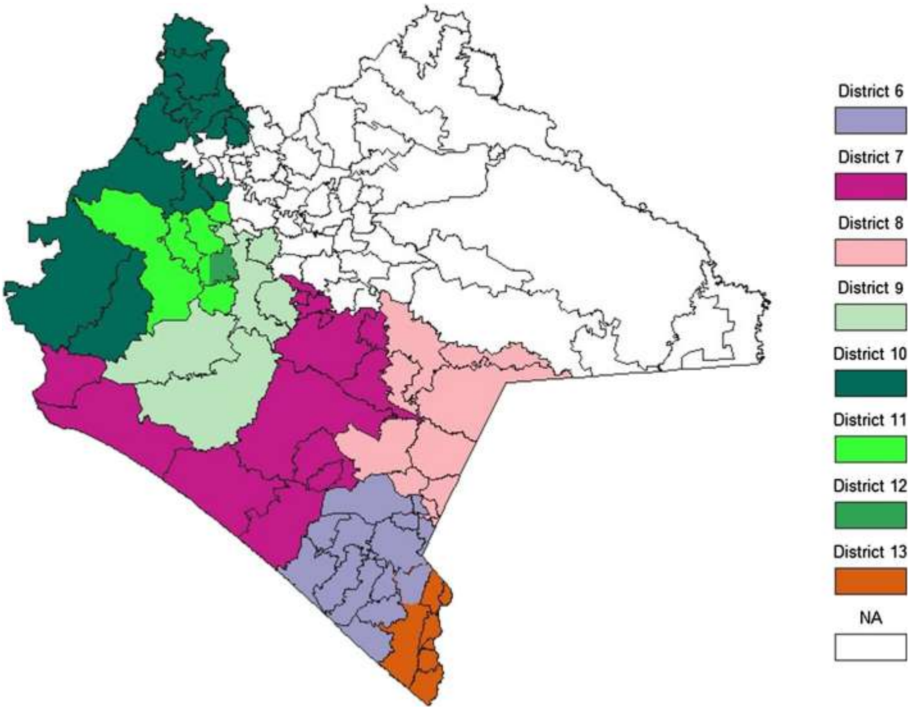


Fig. 3 Map of the non-indigenous districts in Chiapas formed after Phase II

Comparing the two maps, we see that both are contiguous and with balanced districts' populations. On the contrary, we guarantee that municipal boundaries are respected by the districts in Fig. 4, while this is not true for those showed in Fig. 5. Finally, the main criterion related to respect of minority groups is fully satisfied by districts in Fig. 4, since here indigenous municipalities are all collected inside the prefixed 5 indigenous districts. Differently, in the institutional map in Fig. 5 they spread outside such districts (see also Figs. 6 and 7).

In fact, as expected, the main difference between the two solutions appears in the formation of indigenous districts. Both maps satisfy the requirement that at least 40% of the population of these districts is indigenous, but in our map this aspect is optimized since it concentrates in these districts as many indigenous municipalities as possible. Figure 6 for our map and Fig. 7 for the institutional one illustrate the composition of the districts w.r.t. which municipalities they include, showing a clear distinction in our solution between indigenous and non-indigenous districts: indigenous municipalities are all collected inside the prefixed 5 indigenous districts and are well separated from the non-indigenous ones. On the contrary, in the institutional solution they spread outside such districts and district 11, which is non-indigenous, includes many indigenous municipalities.

The characteristics of the indigenous districts in the two maps can be further illustrated by computing the actual percentage of indigenous inhabitants in each of the five districts for both layouts, which, however may be a bit misleading. In fact, on the one hand, we see that the percentages of indigenous people in indigenous districts are high in both cases, and for both maps there is a district with a clearly lower percentage: District 1 in our solution (65.6%), and District 4 in the institutional one (58.8%). On the other hand, one can realize

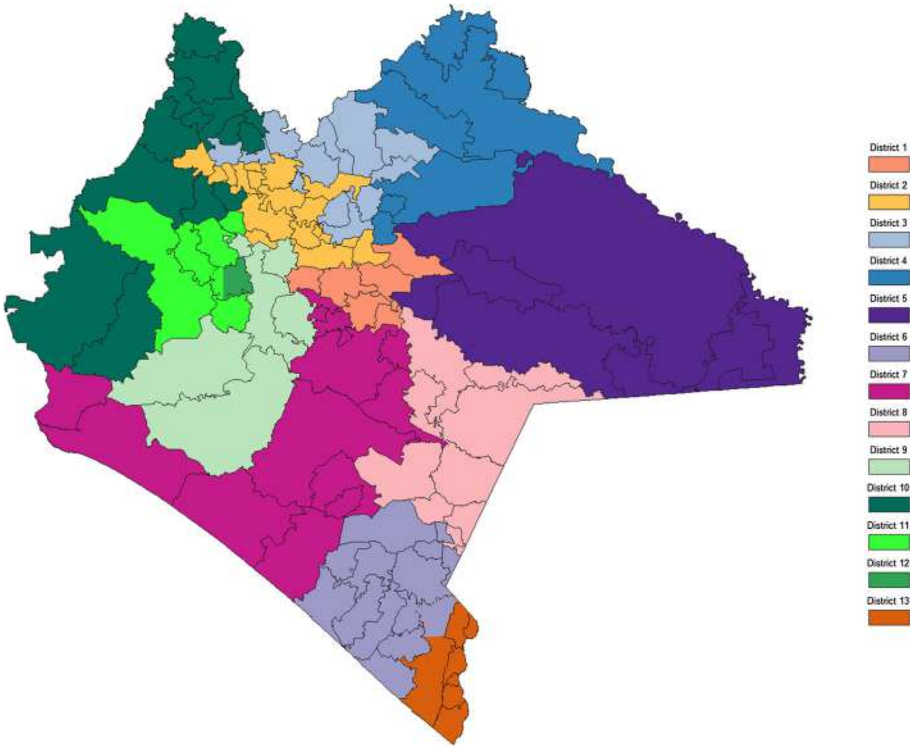


Fig. 4 Complete district map for Chiapas provided by our method

that this is due to some indigenous municipalities (those labeled as 7, 38, 64, 78, 94 appear in both low percentage indigenous districts) which themselves have probably a low percentage of indigenous. In the end, the result is that the configuration in Fig. 6 has indigenous districts with more homogeneous percentages of indigenous residents, and no indigenous people in the non-indigenous districts. On the contrary in Fig. 7 we see a big difference between the lowest (58.8%) and the highest (87.5%) percentage, and, most all, we have a number of indigenous included in a non-indigenous district (21.8% in District 6).

Basing on optimization, we can maximize the concentration of indigenous into their districts, without leaving any indigenous municipalities outside them. This is very important in view of the representation power which the law wants to guarantee to the indigenous population. Actually, the fact that indigenous population is concentrated in the same districts is meaningful, not only at the moment of the expression of the vote, but also under a territorial administrative viewpoint, when the representatives elected by the indigenous community become operational in the practical implementation of public policies.

5.2 Application to fictitious data

We test our exact and heuristic procedures also on fictitious territories. We choose territories with a very regular structure for which the corresponding contiguity graph is a rectangular grid. As known, and frequently exploited in the applications, see Ricca et al. (2008) and Miyazawa et al. (2021), grids resemble real territories, since they are planar graphs with

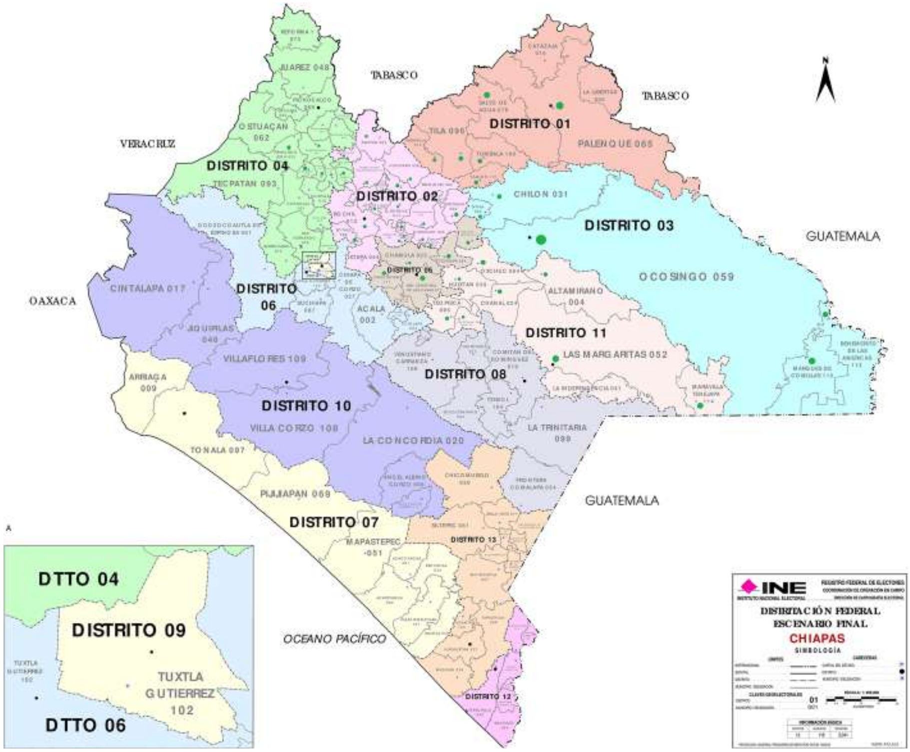


Fig. 5 Institutional district map for Chiapas

District	Population	Indigenous population	%	Corresponding municipalities
1	333,598	219,006	65.6	7, 38, 64, 78, 94, 110
2	361,069	296,619	82.2	13, 14, 23, 33, 47, 49, 56, 60, 67, 72, 73, 81, 85, 90, 91, 93, 112, 118
3	299,024	263,317	88.1	5, 22, 25, 26, 39, 42, 66, 76, 95, 99, 108, 117
4	343,124	260,066	75.8	16, 31, 50, 65, 77, 82, 111
5	389,632	276,622	71.0	4, 24, 52, 59, 113, 114, 115

6	421,932	1, 3, 32, 36, 37, 40, 53, 54, 57, 70, 71, 80, 89, 102
7	423,065	8, 9, 20, 28, 51, 58, 69, 75, 83, 96, 97, 105, 109, 116
8	424,035	6, 10, 11, 19, 30, 34, 41, 98, 103
9	317,954	2, 27, 44, 63, 106, 107
10	321,110	17, 18, 21, 43, 45, 46, 48, 62, 68, 74, 84, 88, 92
11	314,785	12, 29, 61, 79, 86, 100
12	422,950	100
13	468,842	15, 35, 55, 87, 89, 101, 104

Fig. 6 Indigenous and non-indigenous Chiapas municipalities distribution in the districts provided by our method

District	Population	Indigenous Population	%	Corresponding municipalities
1	352,655	261,035	74.0	16 50 65 76 77 95 99 108
2	342,658	257,515	75.2	5 13 14 22 26 39 44 42 47 49 56 66 72 81 85 112 117 118
3	349,838	294,911	84.3	31 59 82 113 115
4	409,034	240,549	58.8	4 7 24 38 41 52 64 78 94 114
5	262,272	229,389	87.5	23 78 93 109 110 111
6	352,590	76,747	21.8	12 18 21 25 29 33 43 45 48 60 62 63 67 68 73 74 79 84 88 90 91 92
7	784,808			2 27 28 61 86 97 100
8	372,348			1 3 9 32 40 51 69 71 96
9	403,116			19 34 58 75 83 98 103 105
10	553,374			100
11	366,637			8 17 20 46 106 107 116
12	308,616			15 35 55 87 89 101 104
13	471,874			6 10 11 30 36 37 53 54 57 70 80 89 102

Fig. 7 Indigenous and non-indigenous Chiapas municipalities distribution in the institutional districts

Table 1 Fictitious problem size

m	m_I	m_{NI}	k	k_I	k_{NI}
120	46	74	13	5	8
210	86	124	25	10	15

node degree between 2 and 4. In the pictures shown in the rest of the paper, the elementary units of the grids are represented as adjacent cells.

In our experimental plan we consider rectangular grids of different size m (number of nodes/municipalities), and the corresponding subdivision into a number (m_I) of indigenous municipalities and the rest (m_{NI}) of non-indigenous ones. We also consider different values for the number of districts (k, k_I, k_{NI}). Table 1 shows the numerical characteristics of our test problems. It can be seen that we test first fictitious territories of size similar to that of Chiapas, then we try to solve the problem on larger size data sets. Results on even larger graphs are not reported since, already with 210 nodes we had some difficulties which forced us to switch from the exact procedure to the (partially) heuristic one. We generate randomly the population of the municipalities of the test problems by choosing populations with size ranging between 100 and 350.

To take into account the existence of indigenous and non-indigenous municipalities, we consider three different configurations for the indigenous distribution over the territory. Thus, in each grid graph, for the location of the m_I nodes corresponding to the indigenous municipalities we have three situations: (1) all indigenous nodes are inside a smaller subgrid located in the *center* of the original grid; (2) all are in a smaller subgrid located at a *corner* of the original grid; (3) all indigenous nodes randomly *spread* all over the original grid.

District		Population
1	I	1684
2	I	1745
3	I	1730
4	I	1565
5	I	1399
6	NI	1516
7	NI	1490
8	NI	1700
9	NI	1441
10	NI	1324
11	NI	1345
12	NI	1427
13	NI	1547

Fig. 8 Case 1: $M = 120$, indigenous located in a corner

In the pictures showing our fictitious territories the subgrids where indigenous are located are visualized within a rectangle (in bold), and the indigenous municipalities are labeled by white numbers (on the contrary, the black numbers refer to non-indigenous). In each figure we also report a table showing the populations of indigenous and non-indigenous districts.

First we illustrate the results obtained on the first three grids with $M = 120$ municipalities, among which $M_{NI} = 46$ are indigenous. In this data sets we have 13 districts, 5 indigenous and 8 non-indigenous. With respect to population, we have $\bar{P} = 1532$ and, considering a value $\alpha = 0.15$, $P_{min} = 1302$ and $P_{max} = 1762$. Computational times required for the solution of the optimization models of Phase I for the grid with 120 nodes and the indigenous communities located in a corner were similar to those for Chiapas. As for Chiapas, we were also able to consider compactness in Phase II and to obtain a good map in reasonable times by applying a successive division approach (see Fig. 8).

Figures 8, 9 and 10 show the results on grids with 120 nodes. One can verify that the bounds on the district population are fully satisfied in all cases. With respect to the distribution of the indigenous municipalities in the territory, with the first two configurations the procedure was able collect all of them within the indigenous districts. On the contrary, when the indigenous municipalities are spread, as expected, in the final solution some indigenous municipalities belong to non-indigenous districts (see Fig. 10). When indigenous are sparse over the territory, in Phase I we are forced to relax the assignment constraint for indigenous and to use the MILP model only to find partial districts which are then completed heuristically. The spread configuration produces some additional difficulties also for solving Phase II, since, after the formation of indigenous districts, in this case the remaining municipalities to be still assigned are sparse as well. In our experiments, this implied that only the heuristic procedure could be applied to complete the district map.

A similar problem was encountered with the configuration in which the indigenous municipalities are located in the center. In this case, for the 120 node grid we were able to apply the exact approach in Phase I, and assign all indigenous municipalities to indigenous districts, but, after this, the remaining non-indigenous municipalities had a ring configuration, and this again made hard the task of solving the MILP model of Phase II.

For the grid with 210 nodes we have $\bar{P} = 1876$, and, using $\alpha = 0.15$, we have $P_{min} = 1594$ and $P_{max} = 2157$. The results for this case are shown in Figs. 11, 12, and 13. It can be seen that, when indigenous population is spread, relaxing the assignment constraints in the model

District		Population
1	I	1739
2	I	1580
3	I	1698
4	I	1694
5	I	1436
6	NI	1698
7	NI	1681
8	NI	1631
9	NI	1399
10	NI	1358
11	NI	1315
12	NI	1355
13	NI	1326

Fig. 9 Case 2: $M = 120$, indigenous located in the center

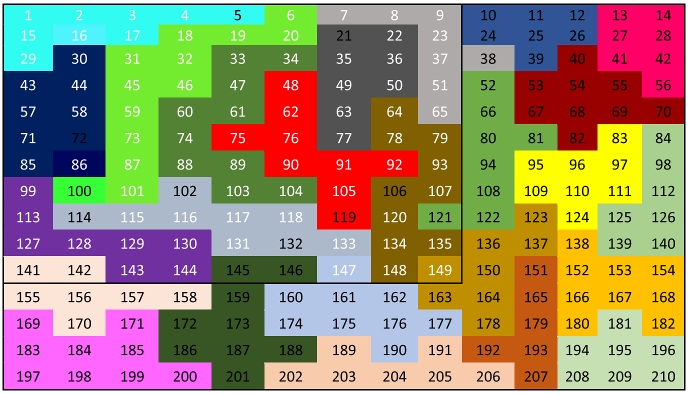
District		Population
1	NI	1584
2	I	1537
3	NI	1321
4	NI	1331
5	I	1473
6	I	1592
7	NI	1597
8	I	1682
9	NI	1586
10	NI	1499
11	NI	1565
12	NI	1506
13	NI	1520

Fig. 10 Case 3: $M = 120$, indigenous spread

of Phase I unavoidably implies that some indigenous municipalities are assigned to a non-indigenous district (Fig. 13).

For all cases with 210 nodes we had computational problems in solving the MILP models exactly. Therefore, it was necessary to exploit the heuristic procedure to get solutions in reasonable times by constructing partial (kernel) districts in Phase I and then using the heuristic to complete them, and then also to produce the non-indigenous ones. In any case, for all grids we were able to obtain a contiguous district map with all district populations within the imposed bounds. In addition, thanks to the integral nature of the municipalities, represented as nodes of the graph, perfect conformity is always guaranteed.

The increasing difficulties found in the solution of PD on territories when the number of units becomes large suggests that it is not possible to apply our MILP models further. Our experiments made us understand up to which size our MILP models can be exploited at best. We also tried to solve the PD problem on territories with 500 municipalities, but, as expected, in these cases only a heuristic approach can be applied.



District		Population
1	I	2057
2	I	2111
3	I	2069
4	I	2146
5	I	2128
6	I	1972
7	I	1968
8	I	2149
9	I	1936
10	I	1919
11	NI	1770
12	NI	1741
13	NI	1699
14	NI	1978
15	NI	1770
16	NI	2097
17	NI	1737
18	NI	1614
19	NI	1682
20	NI	1867
21	NI	1603
22	NI	1810
23	NI	1600
24	NI	1670
25	NI	1803

Fig. 11 Case 4: $M = 210$, indigenous located in the corner

District		Population
1	I	2078
2	I	2156
3	I	1651
4	I	2034
5	I	2149
6	I	1738
7	I	2018
8	I	1861
9	I	1944
10	I	1962
11	NI	1857
12	NI	1863
13	NI	1756
14	NI	1685
15	NI	1699
16	NI	1936
17	NI	1815
18	NI	1843
19	NI	2103
20	NI	2058
21	NI	1609
22	NI	1943
23	NI	1754
24	NI	1697
25	NI	1687

Fig. 12 Case 5: $M = 210$, indigenous located in a center

District		Population
1	I	2117
2	I	1629
3	I	2136
4	I	1776
5	I	1797
6	I	1655
7	I	1678
8	I	1597
9	I	1652
10	I	1564
11	NI	1990
12	NI	1890
13	I	1925
14	NI	1944
15	NI	1975
16	NI	1963
17	NI	2202
18	NI	1900
19	NI	2222
20	NI	2000
21	NI	1873
22	NI	1891
23	NI	1970
24	NI	1611
25	NI	2201

Fig. 13 Case 6: $M = 210$, indigenous spread

The results in this section provide district maps which satisfy population equality, contiguity, and guarantee perfect conformity to municipal boundaries. In addition, they optimize the criterion of respecting minority groups. This goal is fully reached for the small-size territories, for which all indigenous municipalities are assigned to indigenous districts, while it is optimized in the larger data sets, or when indigenous are spread, by collecting as much as possible indigenous in their districts.

6 Conclusions

In this paper we propose an optimization approach for Political Districting in Mexico. The aim is to obtain a procedure for designing electoral districts taking into account that in this country there exist minority groups which, according to the law, have a special right to be represented in the Parliament. The approach is tailored for the Mexican case, but it is applicable to any PD problem in which respect of minority groups is a main issue. It was motivated by the observation that in Mexico the law considers the existence of the minority groups and prescribes explicitly that a fixed number of districts should be designed with the precise aim of supporting their representation. For this reason, we focus our attention on minorities and suggest a procedure which collects together in the prescribed number of indigenous districts as many indigenous municipalities as possible. We also consider the basic PD criteria of contiguity, which is treated as a strict constraint, together with population equality and conformity to municipal boundaries, which are satisfied at best. In particular, population equality is satisfied according to the bounds on district populations imposed by the law, while, by considering municipalities as elementary (indivisible) territorial units, conformity to municipal boundaries is automatically satisfied by the districts. This is one aspect of the more general conformity to administrative boundaries criterion which, in our opinion, is fundamental in any territorial zoning problem with an institutional role.

The solution procedure is performed in two stages: the first one is devoted to the design of indigenous districts, the second is aimed at completing the map with the construction of the non-indigenous districts. Our experimental experience showed the pros and cons of the procedure, but the general performance is good, especially for the results w.r.t. respect of indigenous minorities and conformity to municipal boundaries.

In particular, for the solution of the specific Mexican PD problem, we have the following results:

The application of our method to Chiapas produces a district map which matches the law requirements of balancing population and having a fixed number of indigenous districts; under this point of view, our solutions outperforms the institutional district map, since in such map indigenous municipalities are not concentrated only in the indigenous districts, but they spread also in some non-indigenous ones. In our solution they are all collected in the K_I indigenous districts, providing a map in which the separation between the two types of districts is clear.

The districts map provided by our procedure outperforms the institutional one also w.r.t. conformity to municipal boundaries, which is one of the most important criteria according to the Mexican law. Actually, in the institutional district map, several municipalities are split between more than one district, and this is an evident consequence of pursuing population balance without caring of municipalities' borders. This splitting does not appear in our map, except for only two cases in which the territory of a municipality is divided between only two districts. This is, in fact, unavoidable for Chiapas, due to some structural problems in the already existing administrative division of its territory into municipalities. If population equality must be satisfied within the bounds imposed by the law on district populations, at least two splits must be performed.

The good performance of the exact two-phase districting procedure is confirmed when applied to our artificial data sets, but only if the size of the problem is within a maximum which corresponds more or less to the size of Chiapas. For these problems, as for Chiapas, we also managed to take into account the compactness criterion when designing the non-indigenous districts, without loosing anything in conformity to municipalities boundaries. On the contrary, we discarded compactness in the formation of the indigenous districts, basically because it is conflicting with respect of minority groups. These results suggest that the method can be successfully used for practical application of PD problems.

On territories of larger size good district maps can be still obtained in reasonable times by resorting to the heuristic procedure. The two-phase approach shows some difficulties when the distribution of the indigenous population over the territory has a very particular (non-concentrated) configuration. Unfortunately, in PD with minority groups, if this is the case, the problem becomes harder for any procedure, and typically this requires ad hoc intervention and additional computational effort. In our approach, we were able to cope with this by applying our kernel-growth procedure.

We finally point out that our procedure is conceptually simple and easy to apply (of course, with suitable computer aid) by the electoral offices who have to implement the operations prescribed by the law. We think that this is an important aspect when mathematical models and methods are applied in the social science context and, in our opinion, this should be promoted also in view of the principles of transparency and simplicity which are at the basis of the electoral law in any country.

We conclude the paper by meditating on our heuristics which is clearly very simple and straightforward, but this was mainly due to the fact that it was developed when experiments demonstrated the impossibility of applying our MILP for large size problems. We think that our future work may be headed towards developing a most sophisticated heuristic procedure which is still able to exploit the two-phase approach proposed in this work.

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Conclusiones

Es innegable el interés de la temática electoral para la ciudadanía, como se puede comprobar especialmente en los periodos de campaña de los partidos políticos a la finalización de las legislaturas. En tales ocasiones, es usual que sean debatidos aspectos tales como los que han sido objeto de esta tesis: acusada desproporcionalidad votos/escaños, hasta el punto de llegar a la falta de representatividad de algunos partidos con un número relevante de votos debido al diseño de circunscripciones o a la regla de reparto empleada, que puede propiciar el denominado "voto útil"; un tratamiento desigual de partidos grandes y pequeños por las mismas razones; e incluso discordancias en el reparto de escaños de partidos con similar número de votos, de manera que es posible que algunos de ellos obtengan menor número de escaños con mayor número de votos. Lo anterior también tiene repercusiones en el correcto funcionamiento del sistema de partidos, pues puede dificultar e incluso impedir el acceso a financiación legal institucionalizada de organizaciones políticas con cierta relevancia incluso a nivel nacional.

Por todo ello creemos que, pese a la inercia que existe para cambiar sistemas y leyes electorales (ya que, en ocasiones, se necesitan reformas constitucionales que requieren amplias mayorías), investigaciones y trabajos de tipo normativo y empírico-comparativo, como el que nos ha ocupado, son necesarios.

En este sentido, a la luz de los resultados de investigación de esta tesis doctoral, se concluye que, pese a las numerosas reformas de las que ha sido objeto el sistema electoral en México, no se han logrado subsanar paradojas, contradicciones y asimetrías aún vigentes. Por el contrario, nuestra propuesta, basada en el método bi-proporcional de reparto, evidencia una mayor simplicidad, claridad y tratamiento más justo de los partidos políticos que el sistema actual, lo que presumiblemente le otorgaría una mayor estabilidad.

Por otra parte, cabe señalar que la desproporcionalidad electoral no es únicamente un asunto matemático, sino también político, ya que se trata de una distorsión en las preferencias de la ciudadanía. Debido a su relevancia, hemos abordado este problema utilizando nuevos índices como herramientas para su cuantificación. Así, en primera instancia, se introduce el que denominamos índice de la cuota, con una clara interpretación en términos de transferencia escaños. Otra de sus ventajas es que la existencia de muchos partidos con una cuota inferior a la unidad y sin representación parlamentaria no incrementa su valor. Asimismo, hemos demostrado que este primer índice propuesto cumple muchas propiedades que se consideran deseables. A continuación, diseñamos toda una familia de índices asociados a los diferentes métodos proporcionales de asignación de escaños, con especial atención a los de D'Hondt y Sainte-Laguë. De la misma manera que ocurría con el índice anterior, también esta familia tiene una clara interpretación al cuantificar las discrepancias entre la asignación real de asientos y la obtenida por la pura aplicación, sin ningún tipo de interferencias, como son la existencia de circunscripciones, de umbrales de exclusión y/o de primas para el ganador, del método considerado.

Contrariamente a los índices clásicos, que toman como referencia para medir la desproporcionalidad la inalcanzable proporcionalidad exacta, los índices introducidos tienen en cuenta desviaciones respecto a otros valores factibles, lo que les hace más realistas. Además, una vez efectuado un estudio comparativo de propiedades y contrastado empíricamente su desempeño en distintos procesos electorales celebrados en las últimas décadas en países con sistemas electorales muy diferentes, hemos constatado su idoneidad. Las consideraciones anteriores nos permiten proponerlos como alternativas válidas a los índices clásicos al uso.

Puesto que los partidos políticos desempeñan un papel esencial en el ejercicio de la representación y en la canalización de las demandas de la ciudadanía, es de vital importancia que su financiación desde instancias estatales sea la correcta. Por tal motivo, otra línea de investigación plantea un mecanismo, más eficiente que el actual, para estimar y distribuir el presupuesto público destinado a los partidos políticos en México. Para repartir los recursos disponibles, nuestra propuesta tiene en cuenta la votación total emitida en vez del número de electores inscritos en el padrón electoral, como actualmente se contempla. Además, para evitar tratamientos no equitativos a partidos políticos con similar número de votos, se emplea una función continua para determinar la barrera a la entrada en el reparto.

Utilizando métodos documentales y descriptivos para el análisis de datos y obtención de resultados en los años del periodo 2008-2020, una simulación de la nueva propuesta de financiación a partidos políticos en México permite concluir que los ahorros que se generarían al aplicarla hubieran sido de un poco más del 50% de lo que actualmente reciben, con lo que, evidentemente, podrían verse beneficiadas políticas de desarrollo social y económico en ámbitos desprotegidos de México.

La tesis se cierra con una nueva propuesta de distritación electoral para el caso de México. Dado que en este país existe una población indígena significativa y que su propia ley electoral recoge el respeto a este tipo de minorías, presentamos un nuevo modelo de Programación Lineal Entera que toma en cuenta, además de los objetivos usuales, un criterio adicional que presta atención a tales grupos minoritarios. El modelo se aplicó en un territorio real, Chiapas (México) cuya población nativa es muy numerosa y se encuentra en regiones con niveles considerables de pobreza y marginación. La aplicación de nuestra propuesta permite concluir que, para dicho territorio, la distritación obtenida satisface los requerimientos establecidos en la ley. Adicionalmente, se mejora el diseño distrital vigente, ya que, a diferencia de éste último, evita la división de municipios y que estos sean asignados a distritos distintos.

Así mismo, testamos nuestra propuesta en territorios ficticios que se representaron por medio de redes cuadriculadas, comprobándose que para territorios con un mayor número de unidades aparecen problemas computacionales que hicieron necesario un enfoque heurístico. Los resultados obtenidos también permiten confirmar las ventajas de nuestro modelo. Para regiones de hasta 120 unidades (municipios) fue posible aplicarlo sin problemas computacionales; paralelamente, en territorios con un mayor número de unidades, el método heurístico proporciona mapas distritales óptimos en tiempos razonables.

Finalmente, el bagaje de esta tesis permite abordar futuras investigaciones. Así, nuestra experiencia en el diseño de índices de desproporcionalidad electoral puede ser utilizado como punto de partida para un análisis por géneros de la representación parlamentaria y para la propuesta de medidas de disparidad en tal sentido. Los distintos sistemas electorales deberían producir representaciones equitativas entre hombres y mujeres, y al mismo tiempo conseguir representaciones proporcionales para los partidos. De esta forma se propiciarían reformas de la democracia representativa que contribuirían a su mejora.

Por otro lado, los resultados obtenidos con nuestros modelos diseño distrital, respetuosos con la representación de minorías indígenas, sugieren que estos se pueden aplicar en contextos más generales de demarcación territorial, siempre que se pueda establecer la característica de grupo minoritario (no necesariamente indígena) sobre la unidad básica considerada. De este modo, tales métodos podrían ser usados exitosamente para fines prácticos de distritación, ya sea o no con fines electorales. Una ampliación de este trabajo de investigación, puede extenderse a la consideración del estudio de la democracia participativa que se presenta en los distritos electorales y su relación con una medida desigualdad social en éstos. De esta manera, se podrían contemplar propuestas de políticas públicas diversas que coadyuven a un impulsar la participación colectiva en la vida democrática de un país, considerando los distintos estratos sociales.

Conclusions

The interest of the electoral issue for citizens is undeniable, as it can be seen specially in the campaign periods of the political parties at the end of the legislatures. In such case, it is usual for aspects such as those that have been the subject of this thesis to be debated: accused disproportionality of votes/seats, to the point of reaching the lack of representativeness of some parties with a relevant number of votes due to the design of constituencies or the distribution rules used, which can lead to the so-called "useful vote"; unequal treatment of large and small parties for the same reasons; and even discrepancies in the distribution of seats in parties with a similar number of votes, so that it is possible that some of them obtain fewer seats with a greater number of votes. The latter has repercussions on the proper functioning of the party system as well, as it can hinder and even prevent access to institutionalized legal funding for political organizations with some relevance even at the national level.

These are reasons why we believe that, despite the inertia that exists to change electoral systems and laws (since, sometimes, constitutional reforms are needed and require large majorities), research and normative as well as empirical-comparative work, such as ours, are necessary.

In this sense, from the results of this Ph.D. thesis, it is concluded that, despite the numerous reforms that the electoral system in Mexico has undergone, it has not been possible to avoid paradoxes, contradictions and asymmetries still arising. On the contrary, our proposal, based on the bi-proportional distribution method, shows greater simplicity, clearness and fairer treatment of political parties than the current system, which presumably would give it greater stability.

On the other hand, it should be noted that electoral disproportionality is not only a mathematical issue, but also a political one, since it is a distortion in the preferences of the citizens. Due to its relevance, we have tackled this problem using new indices as tools for its quantification. Thus, at first instance, what we call the quota index is introduced, with an explicit interpretation in terms of seats transfer. Another of its advantages is that the existence of many parties with a lower quota than the unity and without parliamentary representation does not increase its value. We have also shown that this first proposed index satisfies many desiderable properties. In addition, we design a whole family of indices associated with the different proportional methods of seat allocation, with special attention to those of D'Hondt and Sainte-Laguë. In the same way that occurred with the previous index, this mentioned family has also a clear interpretation when quantifying the discrepancies between the real allocation of seats and that obtained by the pure application, without any type of interferences, such as the existence of districts, of exclusion and/or premium thresholds for the winner, of the considered method.

Contrary to classical indexes, which take the unattainable exact proportionality as a reference to measure disproportionality, the introduced indexes take into account deviations with respect to other feasible values, which make them more realistic. Furthermore, once we have carried out a comparative study of properties and empirically tested their performance in different electoral processes held in recent decades in countries with very different electoral systems, we have been able to verify their suitability. The previous considerations allow us to propose them as valid alternatives to the classic indexes in use.

Since political parties play an essential role in the exercise of representation and in channeling the citizens' demands, it is crucially important that they obtain an appropriate funding from state authorities. For this reason, another line of research proposes a mechanism, more efficient than the current one, to estimate and distribute the public budget destined to political parties in Mexico. To distribute the available resources, our proposal takes into account the total votes cast instead of the number of voters registered in the electoral roll, as is currently contemplated. Furthermore, to avoid unequal treatment of political parties with a similar number of votes, a continuous function is used to determine the barrier to enter the distribution.

By using documentary and descriptive methods to analyze data, and obtaining results in the period 2008-2020, a simulation of the new financing proposal for political parties in Mexico allows us to conclude that the savings that would be generated by applying it would have been more than 50% from what they currently receive, with which, obviously, social and economic development policies for under-resourced areas of Mexico could be benefited.

The present thesis ends with a new proposal of electoral district design intended for Mexico. Given that in this country exists a significant indigenous population and that its own electoral law includes respect for this type of minorities, we present a new model of Integer Linear Programming that takes into account, in addition to the usual objectives, an additional criterion that pays attention to such minority groups. The model was applied in a real territory, Chiapas (Mexico), whose native population is very large and is found in regions with considerable levels of poverty and marginalization. The application of our proposal allows us to conclude that, for this territory, the obtained district division satisfies the requirements established by the law. Additionally, the current district design is improved, since, unlike the latter, our approach avoids splitting municipalities to different districts.

Likewise, we tested our proposal in fictitious territories that were represented by means of squared networks, verifying that for territories with a greater number of units (municipalities), computational problems appeared that made a heuristic approach necessary. The results obtained also allow us to confirm the advantages of our model. For regions with up to 120 units, it was possible to apply it without computational problems; at the same time, in territories with a greater number of units, the heuristic method provides optimal district maps in reasonable times.

Finally, the background of this thesis allows future research to be addressed. Thus, our experience in the design of electoral disproportionality indexes can be used as a starting point for a gender analysis of parliamentary representation and for the proposal of disparity measures in this regard. The different electoral systems should produce equal representation between men and women, and at the same time achieve proportional representation for the parties. In this way, reforms of representative democracy would be promoted and would contribute to its improvement.

On the other hand, the results obtained with our district design models, concerning the representation of indigenous minorities, suggest that these arguments can be applied in more general contexts of territorial division, provided that the characteristic of a minority group (not necessarily indigenous) can be established on the considered basic unit. In this regard, such methods could be successfully used for practical districting purposes, whether or not in an electoral context.

An extension of this research can be lead to the consideration of the study of participative democracy that occurs in electoral districts and its relationship with a measure of social inequality in them. In this way, diverse public political proposals could be considered and contribute to promote collective participation in the democratic life of a country, taking into account different social strata.

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