PROGRAMA DE DOCTORADO EN ECONOMÍA

TESIS DOCTORAL:

COPULA-BASED METHODS AND THEIR APPLICATION TO MULTIDIMENSIONAL POVERTY ANALYSIS

Presentada por **César García Gómez** para optar al grado de Doctor por la Universidad de Valladolid

Dirigida por:

Ana Pérez Espartero y Mercedes Prieto Alaiz

Acknowledgements

First and foremost I am extremely grateful to my supervisors, Dr. Ana Pérez Espartero and Dr. Mercedes Prieto Alaiz, for their invaluable advice, constant support and patience throughout my PhD. I feel very fortunate to have benefited from their knowledge and incredible enthusiasm for scientific research.

I owe my deepest gratitude to Dr. Fabrizio Durante, for his hospitality and precious guidance during my PhD visiting period at the Department of Economic Science of the University of Salento. His treasured suggestions and insightful comments have greatly contributed to the development and improvement of this thesis. I wish also to express my sincere thanks to Dr. José Luis García Lapresta, for his helpful advice and invaluable support, and to Dr. Carmelo García and Dr. Marina Romaguera, for their help with the data.

I would also like to acknowledge the faculty of the Department of Applied Economics of the University of Valladolid, for giving me the opportunity to spend five years in such a warm and stimulating environment.

Finally, I would like to express my heartfelt thanks to my parents, my brother and my sister, for their love and strong support. Without them, I would never have become who I am today. To them I dedicate this thesis.

Resumen

Actualmente, está ampliamente aceptado el carácter multidimensional de la pobreza, que involucra no solo la renta, sino también otros aspectos como la educación o la salud. En este entorno multidimensional, es importante analizar la dependencia entre las dimensiones, ya que un alto grado de dependencia podría exacerbar la pobreza. De hecho, esta dependencia interdimensional es el aspecto clave de cualquier análisis multivariante y, por lo tanto, cualquier evaluación certera de la pobreza multidimensional debe tener en cuenta la asociación multivariante entre las dimensiones. Sin embargo, tradicionalmente, la literatura sobre pobreza multidimensional no ha tenido suficientemente en cuenta este crucial aspecto. De hecho, la mayoría de los índices de pobreza multidimensional propuestos hasta ahora, especialmente los más ampliamente utilizados, no son suficientemente sensibles al grado de asociación multivariante entre las diferentes dimensiones de la pobreza.

En esta tesis, proponemos utilizar la metodología basada en cópulas para analizar la dependencia entre las dimensiones de la pobreza. Este enfoque, que ha sido recientemente introducido en el ámbito de la Economía del Bienestar, se centra en las posiciones de los individuos en las dimensiones, en lugar de en los valores específicos que esas dimensiones toman para tales individuos, y es particularmente útil cuando se mide la dependencia en contextos multivariantes, posiblemente no gaussianos y posiblemente no lineales, como los que solemos encontrar en los análisis multidimensionales de pobreza o bienestar. En particular, consideramos varios conceptos de dependencia multivariante basados en cópulas que son especialmente adecuados para el estudio de la pobreza multidimensional, a saber, los conceptos de concordancia multivariante, *orthant dependence* (dependencia en el ortante) y *tail dependence* (dependencia en las colas) multivariante.

En primer lugar, consideramos varias extensiones multivariantes del coeficiente rho de Spearman, basadas en los conceptos de *orthant dependence* y concordancia multivariante. Entre estas extensiones, el coeficiente de *average lower orthant dependence*, que mide, por término medio,

el nivel de dependencia en el ortante inferior, es especialmente relevante cuando se analiza la pobreza multidimensional, ya que captura la probabilidad media de estar simultáneamente mal posicionado en todas las dimensiones de la pobreza. A continuación, nos centramos en el concepto de *tail dependence* multivariante, con especial énfasis en la dependencia multivariante en la cola inferior, que es particularmente importante en un análisis de pobreza multidimensional, ya que captura la probabilidad de que un individuo que está extremadamente mal posicionado (es extremadamente "pobre") en una dimensión también lo esté en las otras dimensiones consideradas. A pesar de su atractivo teórico a la hora de analizar la pobreza desde un punto de vista multidimensional, el concepto de *tail dependence* multivariante nunca ha sido aplicado, hasta donde sabemos, en este campo. Por lo tanto, esta tesis proporciona una contribución pionera. En particular, proponemos, por primera vez, la *tail concentration function* (TCF) multivariante, una herramienta gráfica que permite analizar el grado de dependencia multivariante entre las dimensiones de la pobreza en las colas de la distribución conjunta y permite, al mismo tiempo, representar dicha dependencia en un plano cartesiano, independientemente del número de dimensiones de la pobreza considerado.

En la aplicación empírica de esta tesis, utilizamos estos conceptos de dependencia basados en cópulas para cuantificar el grado de dependencia multivariante entre las tres dimensiones de la tasa AROPE en los países de la Unión Europea (UE) y su evolución durante el período 2008-2018. Del análisis de *orthant dependence* podemos concluir que, en la UE, valores bajos (altos) de la renta tienden a ocurrir simultáneamente con valores bajos (altos) de las otras dos dimensiones de la pobreza consideradas. Además, en la gran mayoría de los países de la UE, la probabilidad media de estar simultáneamente mal posicionado en todas las dimensiones de la pobreza tiende a ser más alta que la probabilidad media de estar simultáneamente bien posicionado en todas las dimensiones. También encontramos, para la mayoría de los países, un aumento de la dependencia entre las dimensiones de la pobreza durante el período 2008-2014, tras la Gran Recesión. Es decir, la probabilidad de tener un ranking bajo (alto) simultáneamente en todas las dimensiones de la pobreza era mayor en 2014 que en 2008. Por el contrario, durante el período de recuperación (2014-2018) observamos que, en muchos países de la UE, el grado de dependencia multivariante entre las dimensiones de la pobreza se mantuvo bastante estable, con indicios de una disminución en algunos países. Sin embargo, en la mayoría de los países de la UE, la dependencia multivariante entre las dimensiones de la pobreza era todavía mayor en 2018 que en 2008.

A partir del análisis de *tail dependence* multivariante, podemos concluir que existe dependencia multivariante tanto en la cola inferior como en la superior de la distribución conjunta. Además, en la gran mayoría de los países de la UE, la dependencia en la cola inferior tiende a ser mayor que en la cola superior. También encontramos que, entre 2008 y 2014, en la mayoría de los países de la UE hubo un aumento de la dependencia multivariante en la cola inferior de la distribución conjunta de las dimensiones de la pobreza. Por el contrario, en el período 2014-2018 encontramos que, en la mayoría de los países, dicha dependencia se mantuvo bastante estable. Además, si consideramos todo el período analizado, únicamente en tres de los países de la UE la dependencia multivariante entre las dimensiones de la pobreza en la cola inferior de su distribución conjunta parece ser claramente más baja en 2018 que en 2008.

Summary

It is widely recognized that poverty is a multidimensional phenomenon involving not only income, but also other aspects such as education or health. In this multidimensional setting, analysing the dependence between dimensions becomes an important issue, since a high degree of dependence could exacerbate poverty. In fact, this interdimensional dependence is the key aspect of any multivariate analysis and, therefore, any sound multidimensional poverty assessment must appropriately account for the multivariate association between the dimensions. However, the literature on multidimensional poverty has traditionally overlooked this crucial aspect, to the point that most of the multidimensional poverty indices proposed so far, and especially those most widely applied, are not sufficiently sensitive to the degree of multivariate association between the different dimensions of poverty.

In this thesis, we propose to use the copula methodology to deal with the dependence between the dimensions of poverty. This approach, which has recently gained attention in Welfare Economics, focuses on the positions of the individuals across dimensions, rather than on the specific values that those dimensions attain for such individuals, and is particularly useful when measuring dependence in multivariate, possibly non-Gaussian and possibly non-linear contexts, such as the ones we usually face in multidimensional poverty or welfare analyses. In particular, we consider several copula-based concepts of multivariate dependence which are especially suitable for the study of multidimensional poverty, namely the concepts of multivariate concordance, orthant dependence and multivariate tail dependence.

We first consider several multivariate extensions of Spearman's rho, based on the concepts of orthant dependence and multivariate concordance. Among these extensions, the coefficient of average lower orthant dependence becomes especially relevant when analysing multidimensional poverty, as it captures the average probability of being simultaneously low-ranked in all dimensions of poverty. After that, we focus on the concept of multivariate tail dependence, with especial emphasis on multivariate lower tail dependence, which is particularly important in a

multidimensional poverty analysis, since it captures the probability that an individual who is extremely low-ranked (extremely "poor") in one dimension is also extremely poor in the other dimensions considered. Despite its theoretical appeal when it comes to analysing poverty from a multidimensional point of view, the concept of multivariate tail dependence has never been applied, to the best of our knowledge, in this field. Hence, this thesis provides a pioneering contribution. In particular, we propose, for the first time, the multivariate tail concentration function (TCF), a graphical tool that allows to analyse the degree of multivariate dependence between the dimensions of poverty in the tails of the joint distribution and allows, at the same time, to represent such dependence in a unit square, regardless of the poverty dimensions considered.

In the empirical application of this thesis, we apply these copula-based dependence concepts to quantify the degree of multivariate dependence between the three dimensions of the AROPE rate in the European Union (EU) countries and its evolution over the period 2008-2018. From the orthant dependence analysis, we can conclude that in the EU, low (high) values of income tend to occur simultaneously with low (high) values of the other two dimensions of poverty considered. Furthermore, in the vast majority of the EU countries, the average probability of being simultaneously low-ranked in all poverty dimensions tends to be higher than the average probability of being simultaneously high-ranked in all dimensions. We also find, for most countries, an increase in orthant dependence between poverty dimensions over the period 2008-2014. That is, the probability to be simultaneously low (high) ranked in all poverty dimensions was higher in 2014 than in 2008. By contrary, over the post-crisis period of 2014-2018, we observe that in many EU countries the degree of multivariate orthant dependence between poverty dimensions remained rather stable, with evidence of a decrease in some countries. Nevertheless, in the majority of the EU countries, multivariate orthant dependence between the dimensions of poverty was still higher in 2018 than in 2008.

From the multivariate tail dependence analysis, we can conclude that there is multivariate tail dependence both in the lower and in the upper joint tails of the distribution. Moreover, in the vast majority of the EU countries dependence in the lower tail tends to be higher than in the upper tail. We also find that, between 2008 and 2014, in most ot the EU countries there was an increase in multivariate lower tail dependence. By contrary, in the post-crisis period of 2014-2018 we find that, in most of the countries, the degree of multivariate lower tail dependence between poverty dimensions remained rather stable. Moreover, if we consider the whole period analysed, only in three of the EU countries multivariate lower tail dependence between poverty dimensions seems to be clearly lower in 2018 than in 2008.

Contents

Acknowledgements **i**

Resumen **ii**

Summary **v**

Contents **viii**

List of Figures xii

List of Tables xv

Introduction **1**

1 The measurement of poverty: a literature review **9**

1.1 Introduction 9

1.2 Unidimensional poverty 11

1.2.1 The identification of the poor 13

1.2.2 The aggregation of unidimensional poverty 15

1.2.2.1 Axioms 15

1.2.2.2 Measures of unidimensional poverty 18

1.2.3 Stochastic dominance techniques and unidimensional poverty 25
1.2.3.1 The concept of stochastic dominance 25
1.2.3.2 Application of stochastic dominance to poverty analysis 28
1.3 Multidimensional poverty 32
1.3.1 The selection of poverty dimensions 36
1.3.2 The identification of the multidimensionally poor 38
1.3.3 The aggregation of multidimensional poverty 41
1.3.3.1 Axioms 41
1.3.3.2 Measures of multidimensional poverty 47
1.3.4 Multivariate stochastic dominance and multidimensional poverty 54
1.3.4.1 The concept of multivariate stochastic dominance 55
1.3.4.2 Application to multidimensional poverty measurement 56

2 Copula-based measures of multivariate dependence 67
2.1 Introduction 67
2.2 A review of copulas: concept and main properties 70
2.3 Concepts of multivariate dependence 75
2.3.1 Multivariate concordance 76
2.3.2 Orthant dependence 78
2.4 Multivariate extensions of Spearman's rho 81
2.4.1 Bivariate Spearman's rho 81
2.4.2 Multivariate generalisations 84
2.4.2.1 Definition and main properties 84

2.4.2.2 Nonparametric estimation 90

2.5 Tail dependence 95

2.5.1 Bivariate tail dependence measures 96

2.5.2 Multivariate tail dependence measures 106

Appendix 2.A 116

Appendix 2.B 119

Appendix 2.C 122

3 Empirical application: multivariate dependence patterns between dimensions of poverty in Europe 125

3.1 Introduction 125

3.2 Data description and estimation procedure 127

3.2.1 Data and variables 127

3.2.2 Estimation procedure 129

3.3 A primer look at the multivariate data 132

3.4 Multivariate orthant dependence analysis 136

3.5 Multivariate tail dependence analysis 151

Conclusions 166

Bibliography 172

List of Figures

1.1 First-order stochastic dominance of G over F 26

1.2 First-order stochastic dominance of G over F 27

2.1 Graphs of basic copulas in the bivariate case 74

2.2 10000 realisations from two different trivariate distributions with the same marginal distributions and the same ρ_3, but with different degree of lower and upper orthant dependence 89

2.3 2000 realisations from different bivariate copulas. 99

2.4 TCFs for some bivariate parametric copulas. 102

2.5 Trivariate TCFs for different trivariate parametric copulas. 111

3.1 Scatter plots of scaled ranks for Bulgaria (2008) and Romania (2008) 133

3.2 Scatter plots of scaled ranks for Spain (2008, 2014 and 2018) 134

3.3 Evolution of $\hat{\rho}_3^-$ and their bootstrap standard 95% confidence intervals in the EU-28 countries over the period 2008-2018. 138

3.5 Cross-country differences in the level of $\hat{\rho}_3^-$ for years 2008, 2014 and 2018. 141

3.6 Evolution of $\hat{\rho}_{income,work}$ (red), $\hat{\rho}_{income,no-deprivation}$ (green) and $\hat{\rho}_{work,no-deprivation}$ (blue) and their bootstrap standard 95% confidence intervals for the EU-28 countries over the period 2008-2018. 148

3.7 Relationship between AROPE rate and $\hat{\rho}_3^-$ for EU-28 countries and years 2008 (left), 2014 (center) and 2018 (right). 150

3.8 Trivariate TCF for the EU-28 countries and years 2008 (black line), 2014 (red line) and 2018 (green line) with bootstrap confidence intervals. 152

3.9 Evolution of pairwise lower tail dependence ($\lambda_L^2(0.1)$) between income and work intensity (red), income and no-material deprivation (green) and work intensity and no-material deprivation (blue) and their bootstrap standard 95% confidence intervals for the EU-28 countries over the period 2008-2018. 164

3.10 Evolution of pairwise lower tail dependence ($\lambda_L^2(0.2)$) between income and work intensity (red), income and no-material deprivation (green) and work intensity and no-material deprivation (blue) and their bootstrap standard 95% confidence intervals for the EU-28 countries over the period 2008-2018. 165

List of Tables

1.1 Unidimensional poverty measures and axioms 24

1.2 Multidimensional poverty measures and axioms 53

3.1 Share of households in the main diagonal of the unit cube $[\mathbf{0}, \mathbf{1}]^{\mathbf{3}}$ in some selected EU-28 countries and some selected years. 135

3.2 Coefficient of trivariate lower orthant dependence, $\hat{\rho}_3^-$, between the dimensions of the AROPE rate in EU-28 countries (2008-2018). 145

3.3 Coefficient of trivariate upper orthant dependence, $\hat{\rho}_3^+$, between the dimensions of the AROPE rate in EU-28 countries (2008-2018). 146

3.4 Coefficient of trivariate average orthant dependence, $\hat{\rho}_3$, between the dimensions of the AROPE rate in EU-28 countries (2008-2018). 147

3.5 Coefficient of trivariate lower tail dependence $\hat{\lambda}_L^3(0.1)$ between the dimensions of the AROPE rate in EU-28 countries (2008-2018) 160

3.6 Coefficient of trivariate lower tail dependence (with $\hat{\lambda}_L^3(0.2)$) between the dimensions of the AROPE rate in EU-28 countries (2008-2018) 161

Introduction

Eradicating poverty remains one of the main objectives of public policy. In fact, one of the Sustainable Development Goals adopted by the United Nations in 2015 is precisely *"to end poverty in all its forms by 2030"*. In this context, how we measure poverty can determine how we analyse it and therefore how we implement public policies to reduce it. Over the last years, a consensus has emerged on the multidimensional nature of poverty. That is, it is now widely accepted that it does not depend only on income, but also on other non-monetary aspects such as health, education or labour status, for instance. This has given rise to a large and still growing literature on the measurement of poverty from a multidimensional perspective; see Alkire et al. (2015), Aaberge and Brandolini (2015) or Chakravarty (2018) for comprehensive reviews.

This literature has been mainly focused on the construction of multidimensional poverty indices, whose aim is to summarise in one figure the level of multidimensional poverty that exists in a given society. For instance, the United Nation Development Program (UNDP) adopted, in 2010, the Multidimensional Poverty Index (MPI), which considers three dimensions: education, health and standard of living. Furthermore, the European Union (EU) monitors poverty in Europe through the AROPE (At Risk Of Poverty or social Exclusion) rate, which also considers three poverty dimensions (income, work intensity and material deprivation). However, this approach has traditionally overlooked an important aspect of multidimensional poverty, namely the degree of dependence between the dimensions. In fact, as it has been argued by Duclos and Tiberti (2016) or Seth and Santos (2019), most of the multidimensional poverty indices proposed so far, and especially those most widely applied, are not sufficiently sensitive to this crucial aspect.

To understand the importance of multivariate dependence when analysing multidimensional poverty or well-being, consider the example put forward by Decancq (2014) and imagine a society in which there is one individual who is top-ranked in all dimensions of well-being, another individual who is second-ranked in all dimensions, and so on. As Decancq (2014) argues, *"this society, reminiscent of a feudal or caste system, is arguably less equitable than another society with exactly the same distributional profile in each dimension but where some individuals are performing relatively well on some dimensions and other individuals on others"*. In other words, for two different societies, the marginal distributions of the achievements may be the same but the joint distributions may differ, leading to different degrees of interdimensional association, which, as Chakravarty (2018, ch. 1) points out, is an intrinsic characteristic of the notion of multivariate analysis. Hence, in order to appropriately account for the multivariate association between the poverty dimensions, the analysis must be fully sensitive to the joint distribution of dimensional achievements.

One way to account for the association between dimensions when analysing multidimensional poverty is the use of the multivariate stochastic dominance techniques proposed, for instance, by Duclos et al. (2006) or Bourguignon and Chakravarty (2009), which were conceived to obtain robust multidimensional poverty orderings to address the multiplicity poverty index problem. However, and despite their theoretical appeal, the empirical implementation of these techniques can become extremely difficult, especially when the number of dimensions considered is high. Accordingly, multivariate stochastic dominance techniques have been scarcely applied in the multidimensional poverty literature.

An alternative methodology to deal with the dependence between the dimensions of poverty or welfare has recently gained attention, namely the copula methodology. This approach focuses on the positions of the individuals across dimensions, rather than on the specific values that those dimensions attain for such individuals, and has several advantages when it comes to measuring the multivariate association between poverty dimensions. First, it enables the decomposition of the joint distribution function of all dimensions into its univariate marginals and the dependence structure, which is captured by the copula. Second, the copula methodology allows to study well-known scale-free measures of bivariate association that capture other types of dependence beyond linear correlation, which is only appropriate for measuring bivariate linear relationships in the context of elliptical distributions. And third, this approach also permits the construction of multivariate generalisations of these bivariate measures. Hence, the copula methodology is particularly useful when measuring dependence in multivariate, possibly non-Gaussian and possibly non-linear contexts, such as the ones we usually face in multidimensional poverty or welfare analyses.

Despite these advantages, the application of copula-based approaches in Welfare Economics is still scarce. Early contributions by Dardanoni and Lambert (2001), Quinn (2007) and Bø et al. (2012) were limited to the bidimensional case; see also Aaberge et al. (2018) for a more recent contribution also in the bidimensional framework. The first application of the copula-based methodology in a fully multidimensional setting is that of Decancq (2014), who analysed the multivariate dependence between income, health and education in Russia. A similar approach was followed by Pérez and Prieto (2015) to analyse, for Spain, the multivariate dependence between the three dimensions (income, work intensity and material deprivation) of the AROPE rate, which is the measure of reference to monitor poverty in Europe. This analysis was recently extended by García-Gómez et al. (2021) for 28 EU countries between 2008 and 2014. Also, Pérez and Prieto-Alaiz (2016a) used three copula-based measures of multivariate association to analyse the dependence between the three dimensions of the Human Development Index using data from 187 countries. Other recent examples of the application of copula-based methods in Welfare Economics can be found in Decancq (2020), Matkovskyy (2020), Terzi and Moroni

(2020) and Tkach and Gigliarano (2020).

In this thesis, we contribute to this growing literature by applying copula-based methods to measure multivariate dependence between the different dimensions of poverty in the EU. Our contribution is twofold. First, from a theoretical point of view, we consider several copula-based concepts of multivariate dependence which are especially suitable for the study of multidimensional poverty. In particular, in addition to the measures of orthant dependence used by García-Gómez et al. (2021), we also consider the concept of multivariate tail dependence. This concept, which has not been previously applied in the literature on multidimensional well-being, is very appealing, since it captures the probability that an individual who is extremely poor (rich) in one dimension is also extremely poor (rich) in the other dimensions considered. The second contribution of this thesis is empirical, as we apply these copula-based dependence concepts to quantify the degree of multivariate dependence between the dimensions of poverty in the EU countries and its evolution over the period 2008-2018. In doing so, we can gain a better insight into the effects of the Great Recession and the subsequent recovery period on the poverty structure in the EU.

The thesis is structured as follows. First, in Chapter 1 we provide an extensive review of the literature on both unidimensional and multidimensional poverty measurement. In both cases, we will first review the different axioms and indices proposed to measure poverty. After that, the stochastic dominance techniques and their application to obtain robust poverty orderings will be reviewed. As we previously pointed out, multivariate stochastic dominance techniques account for the association between dimensions when analysing multidimensional poverty, since they are based on the joint distribution of the achievements. However, as we will see, this approach is not exempt of limitations, since its empirical implementation can become extremely difficult, especially when the number of dimensions considered is high. Moreover, the multivariate stochastic dominance techniques does not allow us to quantify the degree of multivariate dependence. To do so, in this thesis we propose the use of copula-based methods.

Chapter 2 is precisely devoted to the copula methodology and its application to study multivariate dependence. We begin reviewing the concept of copula and its main properties. After that, we will focus on the concepts of multivariate dependence that are specially valuable when analysing multidimensional poverty. Thus, we will first review the concepts of multivariate concordance and orthant dependence, discussing several copula-based measures of multivariate association which stem from those dependence concepts, namely several multivariate extensions of the well-known Spearman's rank correlation coefficient. Among these extensions, the coefficient of average lower orthant dependence becomes especially relevant when analysing multidimensional poverty, as it measures multivariate dependence from a "downward" perspective and captures the average probability of being simultaneously low-ranked in all dimensions of poverty as compared to what this would be were the dimensions independent. Then, we will turn the attention to the concept of tail dependence, which focuses on the extreme values of the poverty dimensions. We will analyse different proposals to measure it both in the bivariate and in the multivariate case. In particular, we put especial emphasis on multivariate lower tail dependence, which is particularly important in a multidimensional poverty analysis, since it captures the probability that an individual who is extremely low-ranked (extremely "poor") in one dimension is also extremely poor in the other dimensions considered. The concept of multivariate tail dependence, although very appealing when it comes to analysing poverty from a multidimensional point of view, has never been applied, to the best of our knowledge, in this field. Indeed, this thesis provides a pioneering contribution by proposing, for the first time, the multivariate tail concentration function (TCF), a graphical tool that allows to analyse the degree of multivariate dependence between the dimensions of poverty in the tails of the joint distribution. As we will see, this function has an important advantage because, in spite of its multivariate nature, it allows to represent in a unit square the degree of multivariate dependence in the joint tails of a distribution, regardless of the number of poverty dimensions considered.

In Chapter 3 we perform an application of the copula-based dependence concepts introduced in Chapter 2 to analyse the evolution of multivariate dependence between the dimensions of poverty in the EU over the period 2008-2018, using data from the EU-Statistics on Income and Living Conditions (EU-SILC) survey. In particular, we analyse the multivariate dependence between the three dimensions of the AROPE rate, namely income, work intensity and material needs. We begin by applying the multivariate extensions of Spearman's rho based on orthant dependence concepts. We find that, in the EU, low (high) values of income tend to occur simultaneously with low (high) values of the other two dimensions of poverty considered. Furthermore, in the vast majority of the EU countries, the average probability of being simultaneously low-ranked in all poverty dimensions tends to be higher than the average probability of being simultaneously high-ranked in all dimensions. We also find that, between 2008 and 2014 there was, in most of the EU countries, an increase in both average lower orthant dependence and average upper orthant dependence between dimensions of poverty. By contrary, over the post-crisis period of 2014-2018, we observe that in many EU countries the degree of multivariate orthant dependence between poverty dimensions remained rather stable, with evidence of a decrease in some countries. Nevertheless, in the majority of the EU countries, multivariate orthant dependence between the dimensions of poverty was still higher in 2018 than in 2008.

Also in Chapter 3, we complement this analysis based on the multivariate extensions of Spearman's rho by focusing on the concept of multivariate tail dependence. In particular, we use the multivariate TCF proposed in Chapter 2 to analyse the evolution of multivariate tail dependence between the three dimensions of the AROPE rate in the EU over the period 2008-2018, putting special emphasis on the evolution of multivariate lower tail dependence. As we argued above, to the best of our knowledge, this constitutes the first application of multivariate tail dependence in Welfare Economics. With this analysis, we find that there is multivariate tail dependence both in the lower and in the upper joint tails of the distribution. That is, there is a positive probability that a household that is poor (rich) in one of the dimensions of the AROPE rate is also simultaneously poor (rich) in the other two dimensions. Moreover, in the vast majority of the EU countries dependence in the lower tail tends to be higher than in the upper tail, which means that the probability that a household that is poor in one of the dimensions is also poor in the other two dimensions tends to be higher than the probability that a household that is rich in one of the dimensions is also rich in the other two dimensions. We also find that, between 2008 and 2014, in most ot the EU countries there was an increase in multivariate lower tail dependence, that is, the probability that a household that is poor in one dimension is also simultaneously poor in the other two dimensions of poverty was higher in 2014 than in 2008. By contrary, in the post-crisis period of 2014-2018 we find that, in most of the countries, the degree of multivariate lower tail dependence between poverty dimensions remained rather stable. Nonetheless, if we compare the situation in 2018 with that in 2008 we find that, in the vast majority of the countries, the degree of multivariate lower tail dependence between the dimensions of poverty was either higher in 2018 than in 2008 or the situation in both years was similar. In fact, only in three of the EU countries multivariate lower tail dependence between poverty dimensions seems to be clearly lower in 2018 than in 2008.

Finally, we conclude this thesis with a summary of the main conclusions and with some ideas for further research.

Chapter 1

The measurement of poverty: a literature review

1.1 Introduction

In order to analyse a phenomenon such as poverty, it is first necessary to understand its meaning. As Gordon (2006) points out, poverty does not have a single meaning. In fact, Spicker (1999) proposes twelve alternative concepts of poverty, and diverse institutions and authors consider a different definition of the phenomenon. The World Bank, for example, considers that "poverty is pronounced deprivation in well-being", a definition which is in line with that proposed by Ravallion (1992):

> *"Poverty can be said to exist in a given society when one or more persons do not attain a level of material well-being deemed to constitute a reasonable minimum by the standards of that society."*

The question that immediately arises is which is the appropriate way to measure well-being. It is in the answer to this question where two concepts of poverty can be distinguished. The first concept has been called "monetary poverty" or "unidimensional poverty". This has been the concept traditionally applied in the economic literature and entails comparing the level of income, consumption or wealth of individuals with a minimum threshold (a poverty line). If income, consumption or wealth is below this threshold, then the individual is considered poor. The International Labour Organization suggests a definition of poverty which is consistent with this unidimensional concept (ILO, 1995):

> *"At the simplest level, individuals or families are considered poor when their level of living, measured in terms of income or consumption, is below a particular standard."*

The second concept considers that poverty, like other phenomena such as well-being or inequality, has a multidimensional character; see, for example, the seminal contributions of Kolm (1977) and Sen (1985, 1987). This concept of poverty goes beyond the monetary approach and analyses whether individuals are also deprived in other dimensions of welfare such as health, education, shelter or working conditions, among others. Based on this conception, the United Nations (1995) sees poverty as:

> *"a condition characterized by severe deprivation of basic human needs, including food, safe drinking water, sanitation facilities, health, shelter, education and information. It depends not only on income but also on access to services."*

Similarly, in the European Union the concept of social exclusion has become prominent when analysing poverty. In fact, the European Community (1985) argues that

> *"The poor shall be taken as to mean persons, families and groups of persons whose resources (material, cultural and social) are so limited as to exclude them from the minimum acceptable way of life in the Member State in which they live."*

Accordingly, Eurostat provides the AROPE (At Risk of Poverty and Social Exclusion) rate, which identifies individuals as poor taking into consideration their income, material deprivation and work intensity status, and which is nowadays the measure of reference to monitor poverty in Europe.

Over the last decades, the multidimensional approach to poverty measurement has gained more and more attention in the literature and a consensus has emerged that sees poverty as a multidimensional phenomenon. In this context, an important number of both theoretical and empirical contributions has emerged trying to set the basis for a correct measurement of poverty.

In this chapter, a review of the literature on both unidimensional and multidimensional poverty measurement will be performed. Section 1.2 focuses on unidimensional or monetary poverty, reviewing its main methodological aspects for its measurement. Section 1.3, in turn, reviews the different approaches and methods proposed for the measurement of multidimensional poverty. In both cases, we will put special attention to the different axioms and measures proposed to measure poverty and also to the application of stochastic dominance techniques to obtain poverty orderings.

1.2 Unidimensional poverty

As we previously argued, to analyse poverty, the concept traditionally applied in the economic literature has been that of unidimensional or monetary poverty. To measure poverty in this unidimensional setting, Sen (1976) distinguishes two steps: the identification step, which consists of identifying who is poor; and the aggregation step, in which the information about the poverty status of the individuals is used to obtain an aggregate figure that reflects the level of poverty in the society. But before the identification and aggregation are performed, some methodological issues must be taken into account.

It is first necessary to choose a variable that captures the level of well-being of the individuals. Several variables can be used but, in empirical applications, income, consumption and wealth have called the attention of economists. However, and due to the difficulties that the measurement of wealth poses, the majority of poverty analyses relies on household surveys data that provide information on consumption and/or income at the household level. Thus, poverty of an individual or household is usually defined as a lack of income or as a shortfall in consumption. However, income and consumption are different concepts. In periods of low income, it is possible for some people to borrow or make use of savings. Also, in periods with high income individuals can opt for saving part of it. In both cases, income and consumption will differ. Advantages and disadvantages of each of these variables as measures of well-being can be found in the literature and the debate about which of them is more adequate for poverty measurement is still open; see Gradin et al. (2008).[1]

Once the variable used to identify the poor is defined, the unit of analysis must be established. In this respect, poverty can be measured for individuals or for households. In most surveys, income and consumption data are collected at a household level. In this case, one possibility is to calculate household income or expenditure per capita, dividing the total income or consumption of the household by the number of its members. However, this method has been criticised because it does not take into account that adults and children have different needs and that there can be economies of scale within the household. Thus, it is essential to account for differences in household composition, for which it is a common practice to use equivalence scales, obtaining the income or consumption per equivalent adult. Coulter et al. (1992) and Deaton and Zaidi (2002) provide an insightful discussion on equivalence scales; see also Morelli et al. (2015) for a review of different equivalence scales proposed in the literature.

[1]This debate goes beyond the scope of this work. For detailed discussions on these issues see Deaton and Zaidi (2002) and Morelli et al. (2015).

1.2.1 The identification of the poor

The identification of individuals or households as poor or non-poor relies on the determination of a poverty line, that is, a minimum threshold of welfare below which they are considered to be poor; see Duclos and Araar (2006) for a detailed discussion on poverty lines and their estimation. Different types of poverty lines have been considered in the literature. First, it is possible to distinguish between objective and subjective poverty lines. The subjective approach consists of asking individuals in a survey which income level they consider minimal to satisfy their basic needs. Then a subjective poverty line can be estimated with the information provided by the survey. Although subjective poverty lines have not been widely used in applied research, some examples can be found in Hagenaars and de Vos (1988), Pradhan and Ravallion (2000) and Lokshin et al. (2006).

Objective poverty lines are the most widely used in empirical studies, and two main types have been traditionally distinguished: absolute and relative poverty lines; see Callan and Nolan (1991) and Jäntti and Danziger (2000). An absolute poverty line is independent of the welfare indicator and can be defined in different ways. The standard one consists in defining a bundle of goods regarded necessary to reach a minimum level of welfare and determining its cost. For instance, the 1 $/day line used by the World Bank is an example of this type of lines. Obviously, absolute poverty lines must be updated in order to account for changes in the cost of living. In this respect, it is possible to calculate the cost of the same basket of goods in the different periods, but a more common method is to update the poverty line by using an appropriate price index, adjusting for inflation.

Alternatively, a relative poverty line is based on a "relativist" approach that sees poverty in terms of the shortfall of the individuals with respect to the standard of living of the society; see Sen (1983) or Townsend (1985) for the main theoretical arguments in favour of and against this relative conception. Relative poverty lines are generally set as a function of the distribution of income or consumption, usually as a proportion of the median or the mean. As Ruiz-

Castillo (2009) points out, the absolute approach to poverty measurement is widely used in the developing world, whereas in developed countries the common practice has been the use of relative poverty lines, with the exception of the United States. In the European Union, for example, the poverty line is defined as 60 % of the median income level.

Furthermore, Ravallion and Chen (2011) have defined the relative concept of poverty as "strongly relative", and have proposed a new approach called "weakly relative poverty" in which the elasticity of the poverty line to mean or median income is less than unity but at the same time rises with the income level of the country. Similarly, Foster (1998) puts forward the concept of "hybrid" poverty lines, which would have an absolute and a relative component and incorporate explicitly the elasticity of the poverty line with respect to income, which could be estimated. Previously, Hagenaars and van Praag (1985) had also proposed a parametrised definition of the poverty line that includes both absolute and relative definitions and depends on the perception of poverty in the population.

Finally, when the interest lies on observing the poverty trends within a given society, the idea of an "anchored" poverty line has been proposed and gained recognition. With this method, a relative poverty line is applied in the first year and then the trend in relative poverty is observed updating the value of the initial poverty line in order to account for price trends.

Once the poverty line is established, it is possible to identify the poor as those individuals whose income or consumption is below this threshold. Then, this information is usually aggregated by means of a poverty measure to give a figure representing the level of deprivation in a society. The most popular unidimensional poverty measures as well as the axioms characterising them are reviewed in the next section.

1.2.2 The aggregation of unidimensional poverty

The proposal of poverty measures has been traditionally based on the axiomatic approach introduced by Sen (1976). Following this approach, the axioms that a desirable measure should fulfil are established and then the researcher proposes a poverty measure consistent with these axioms. Moreover, since the seminal contribution of Sen (1976), it has been assumed that the measurement of poverty involves the analysis of three different aspects: incidence, intensity and inequality, known as "the three I's of poverty" (Jenkins and Lambert, 1997; Villar, 2017). The incidence of poverty refers to the number of individuals that are identified as poor in a society. The intensity captures the degree of deprivation of these individuals, that is, how poor they are. Finally, the inequality aspect of poverty captures the degree of income disparities that exists among the poor. In the literature, many poverty measures have been proposed and the axioms that these measures fulfil determine whether they capture each of these three aspects or not.

As preliminary notation, let $x = (x_1, x_2, \ldots, x_n)$ be the income distribution in a population, being n the population size, and let z be the poverty line. Then, the i^{th} individual is identified as poor if $x_i < z$. That is, the poor are defined in the weak sense; see Donaldson and Weymark (1986) and Zheng (1997). A poverty measure $P(x; z)$ indicates the level of poverty in a population with income distribution x for a given poverty line z.

1.2.2.1 Axioms

Next, we present those unidimensional poverty axioms that have been considered in the literature as the most relevant ones; see Zheng (1997), Foster (2006) and Chakravarty (2009) for a more detailed discussion.

A1. Symmetry. This axiom is also known as the anonymity principle, as it establishes that, for the measurement of poverty, it does not matter the identity of the individuals, but only their level of income. Formally, whenever x' is obtained from x by a permutation of incomes between two individuals, that is, $x'_i = x_j$ and $x'_j = x_i$, then $P(x'; z) = P(x; z)$.

A2. Population Replication Invariance. This axiom establishes that, when the population is replicated, then the value of the poverty measure will remain unchanged. That is, given two income distributions x and x', if x' is obtained from x by a replication of the population in such a way that $n' = k * n$, being n and n' the population sizes of distributions x and x' respectively, for some positive integer k, then $P(x'; z) = P(x; z)$.

With a poverty measure satisfying A1 and A2, two income distributions with the same cumulative distribution function would present the same level of poverty (Foster, 2006).

A3. Focus. The fulfilment of this axiom implies that the poverty measure is independent of the income distribution of the non-poor individuals. Formally, given two income distributions x' and x, if $x'_i = x_i$ for all $i \neq j$ and $x'_j > x_j$ being the j^{th} individual non-poor ($x_j \geq z$), then $P(x'; z) = P(x; z)$.

A4. Scale Invariance. This axiom establishes that, if all incomes and the poverty line are scaled by the same factor (λ), then the value of the poverty measure does not change. That is, $P(\lambda x; \lambda z) = P(x; z)$.

As Chakravarty (2009) and Zheng (1997) point out, all relative indices traditionally used in poverty analyses satisfy this axiom. By contrary, absolute poverty indices satisfy a different axiom, called translation invariance (Blackorby and Donaldson, 1980; Zheng, 1994), which requires that the poverty index does not change after equal absolute changes in all the incomes and the poverty line.[2]

A5. Continuity. This axiom requires that $P(x; z)$ is continuous as a function of x for any given z.

A6. Monotonicity axioms.

[2] Although in this review we will focus on relative poverty indices, proposals of absolute poverty indices can be found in Blackorby and Donaldson (1980), Chakravarty (1983a) and Vaughan (1987).

Following Zheng (1997), we can distinguish between two monotonicity axioms, namely:

- A6.1. Weak Monotonicity. It was proposed by Sen (1976) and requires that a decrease in the level of income of a poor individual, with the rest of the incomes remaining unchanged, increases the level of poverty. That is, if x' is obtained from x in such a way that $x'_i = x_i$ for all $i \neq j$ and $x'_j < x_j$, being the j^{th} individual poor ($x_j < z$), then $P(x'; z) > P(x; z)$.
- A6.2. Strong Monotonicity. This version of the monotonicity axiom requires that, if x' is obtained from x in such a way that $x'_i = x_i$ for all $i \neq j$ and $x'_j > x_j$, with $x_j < z$, then $P(x'; z) < P(x; z)$.

Noticeably, as Zheng (1997) points out, strong monotonicity implies weak monotonicity, but the reverse is not always true.

A7. Transfer axioms.

The original version of the transfer axiom, introduced by Sen (1976), requires that, given other things, the poverty measure increases after a transfer of income from a poor individual to anyone who is richer. However, different versions of this axiom can be found in the literature; see, for example, Donaldson and Weymark (1986) and Zheng (1997). We focus on two particular transfer axioms, denoted by Zheng (1997) as weak transfer and regressive transfer, the latter being a stronger version of the former.

Before discussing these two axioms, it is first necessary to introduce the concept of regressive transfer. In particular, an income distribution x' is obtained from x by a regressive transfer if, for some i and j with $x_i \leq x_j$, we have $x'_i < x_i \leq x_j < x'_j$ while $x'_k = x_k$ for all $k \neq i, j$. Then, the weak and regressive transfer axioms are as follows:

- A7.1. Weak Transfer. This axiom establishes that, if x' is obtained from x by a regressive transfer with at least the donor being poor and with no one crossing the poverty line as a result of the transfer, then the poverty measure should increase ($P(x'; z) > P(x; z)$).

- A7.2. Regressive Transfer. This is the original transfer axiom introduced by Sen (1976) and requires that $P(x';z) > P(x;z)$ whenever x' is obtained from x by a regressive transfer with at least the donor being poor. Notice that, in this case, individuals can cross the poverty line as a result of the transfer. As Zheng (1997) points out, regressive transfer implies weak transfer, but the reverse is not necessary true.

A8. Subgroup consistency. When the population is divided in subgroups, and if the sizes of these subgroups are fixed, an increase of poverty in one subgroup without a fall in any of the rest should produce an increase in the value of the poverty measure; see Foster and Shorrocks (1991) for a detailed discussion on this axiom.

A9. Additive decomposability. The overall level of poverty of a society is a population-weighted average of poverty in the subgroups.

1.2.2.2 Measures of unidimensional poverty

There is an extensive literature related to the characteristics, advantages and disadvantages of different poverty measures. Over the next lines, a review of the most widely used measures of aggregate poverty is provided. A summary of the measures discussed and the axioms that they fulfil is included in Table 1.1 in page 24; see Zheng (1997) or Foster (2006), among others, for more detailed discussions on poverty measures.

Probably the most widely used measure of poverty is the **Headcount Ratio**, which, given an income distribution x and a poverty line z, gives the proportion of poor in a given population of size n:

$$H(x;z) = \frac{q}{n}, \tag{1.1}$$

where $q = q(x;z)$ gives the number of poor individuals, that is, the number of individuals whose income falls below the poverty line z. This measure is not continuous and does not take into account neither the intensity of poverty nor the level of inequality among the poor, as it assigns the same value to every poor, regardless of how poor they are. In axiomatic terms, it does not satisfy neither the monotonicity nor the transfer axioms; see Sen (1976). However, and despite its shortcomings, the Headcount Ratio is still one of the most widely reported measures of poverty. It is used, for example, by institutions such as The World Bank for monitoring poverty in the developing world.

Another poverty measure is the **Income Gap Ratio**, which gives the average shortfall of the income of the poor with respect to the poverty line, and it is defined as:

$$I(x;z) = 1 - \frac{\mu_q(x;z)}{z}, \tag{1.2}$$

where $\mu_q(x;z)$ is the average income of those individuals identified as poor. This measure fulfils the weak monotonicity axiom, but it does not satisfy strong monotonicity. Furthermore, it does not fulfil the transfer axioms, being insensitive to inequality among the poor. Moreover, it is not decomposable.

Another proposal to measure poverty is the **Poverty Gap Ratio**, defined as:

$$HI(x;z) = \frac{1}{n}\sum_{i=1}^{n} g_i, \tag{1.3}$$

where g_i is the censored gap ratio of individual i, defined as:

$$g_i = \max\left\{\frac{z - x_i}{z}, 0\right\}. \tag{1.4}$$

Thus, for the non-poor $g_i = 0$. The Poverty Gap Ratio quantifies the per capita differences between the poverty line and the income of the poor. This measure satisfies weak and strong monotonicity, since the poorer the individual the higher the weight that he has. However, it violates the transfer axioms, since a regressive transfer does not change the value of the index. Thus, the Poverty Gap Ratio does not take into account the level of inequality among the poor, being insensitive to redistribution. As Zheng (1997), Chakravarty (2009) and Villar (2017) point out, the Poverty Gap Ratio can be calculated as the product of the Headcount Ratio and the Income Gap Ratio.

The first distribution-sensitive poverty measure that was proposed in the literature is the **Watts Index**, introduced by Watts (1968) and defined as:

$$W(x;z) = \frac{1}{n}\sum_{i=1}^{n}[\ln z - \ln \tilde{x}_i], \tag{1.5}$$

where $\tilde{x}_i$ is the censored income, defined as

$$\tilde{x}_i = \min(x_i, z). \tag{1.6}$$

That is, for all poor individuals ($x_i < z$), $\tilde{x}_i = x_i$, whereas for all non-poor persons ($x_i \geq z$), $\tilde{x}_i = z$. The Watts index satisfies all the axioms reviewed above; see Zheng (1997). Furthermore, this index can also be rewritten in terms of the Mean Logarithmic Deviation Index of inequality proposed by Theil (1967); see Blackburn (1989).

In order to jointly consider the effects of the incidence, depth and inequality of poverty, Sen (1976) also proposed a measure that is sensitive to inequality among the poor. In particular, the **Sen Index** can be expressed as

$$S(x;z) = \frac{2}{(q+1)n}\sum_{i=1}^{n} g_i(q+1-i), \tag{1.7}$$

where g_i is the censored gap ratio in (1.4). This measure satisfies weak and strong monotonicity and the weak transfer axiom. However, it does not fulfil neither replication invariance nor continuity. Moreover, this index does not satisfy the regressive transfer axiom; see Shorrocks (1995) and, more recently, Aristondo and Ciommi (2016) for a detailed discussion on this issue. Finally, as Zheng (1997) points out, the Sen measure cannot be additively decomposed and does not satisfy the subgroup consistency axiom.

When $n \to \infty$, the Sen Index can be expressed in its approximate form, namely:

$$S'(x;z) = H(x;z)[I(x;z) + (1 - I(x;z))G_p], \tag{1.8}$$

where $H(x;z)$ is the Headcount Ratio in (1.1), $I(x;z)$ is the Income Gap Ratio in (1.2) and G_p is the Gini coefficient of the income distribution of the poor. It is important to notice that, although this approximate version also fails to fulfil the regressive transfer axiom, continuity, additive decomposability and subgroup consistency, it satisfies the replication invariance axiom; see Shorrocks (1995), Zheng (1997) and Aristondo and Ciommi (2016). Other generalisations and variations of the Sen Index can be found in Hamada and Takayama (1977), Takayama (1979), Blackorby and Donaldson (1980), Kakwani (1980) and Giorgi and Crescenzi (2001); see also Zheng (1997) and Chakravarty (2009) for a more extensive discussion on these proposals.

The **Sen-Shorrocks-Thon (SST) Measure** is also a modified version of the Sen Index based on the works of Thon (1979) and Shorrocks (1995). This measure can be defined as

$$SST(x;z) = \frac{1}{n}\sum_{i=1}^{n} \frac{2(n + 0.5 - i)}{n} g_i, \tag{1.9}$$

where g_i is the censored gap ratio in (1.4). Moreover, Osberg and Xu (2000) show that the SST measure can also be expressed as

$$SST(x;z) = H(x;z)I(x;z)(1 + G_{PG}), \tag{1.10}$$

where $H(x;z)$ is the Headcount Ratio in (1.1), $I(x;z)$ is the Income Gap Ratio in (1.2) and G_{PG} is the Gini coefficient of the censored gap ratios g_i defined in (1.4).

The SST measure satisfies weak and strong monotonicity and the transfer axioms, but it is not decomposable in the usual way (across subgroups) and violates the subgroup consistency axiom. Nevertheless, the SST measure can be decomposed showing three different aspects of poverty: the number of poor, the severity of poverty and the inequality among the poor; see Osberg and Xu (2000).

In a very influential contribution, Foster et al. (1984) proposed a family of poverty measures, known as the **FGT family of poverty measures**, with the following general expression:

$$FGT_\alpha(x;z) = \frac{1}{n}\sum_{i=1}^{n}(g_i)^\alpha \quad \text{for} \;\; \alpha > 0 \tag{1.11}$$

and $FGT_0(x;z) = H(x;z)$. The parameter α indicates the degree of poverty aversion and g_i are the normalised poverty gaps defined in (1.4). All the FGT poverty measures are additively decomposable and fulfil the subgroup monotonicity axiom, which provides an important advantage for studying the poverty profile of a country or a region. Moreover, depending on the value of α we have different measures of poverty. In particular, when $\alpha = 1$, the FGT index becomes the Poverty Gap Ratio in (1.3), i.e., $FGT_1(x;z) = HI(x;z)$. On the other hand, when $\alpha = 2$ we have the **Squared Poverty Gap** measure:

$$FGT_2(x;z) = \frac{1}{n}\sum_{i=1}^{n}(g_i)^2. \tag{1.12}$$

This measure satisfies both the monotonicity and transfer axioms. Therefore, it takes into account the level of inequality among the poor. However, it has the disadvantage of being difficult to interpret, so it is not very widely used in applied research.

Another proposal of a family of poverty measures can be found in Clark et al. (1981). In particular, they first propose the following measure:

$$C_1(x;z,\alpha) = \frac{q}{nz}\left[\frac{1}{q}\sum_{i=1}^{n}(z-\tilde{x}_i)^\alpha\right]^{\frac{1}{\alpha}}, \tag{1.13}$$

where $\alpha \geq 1$ and $\tilde{x}_i$ is the censored income defined in (1.6). This measure satisfies weak and strong monotonicity and weak transfer. However, it does not fulfil the regressive transfer axiom. Moreover, it is not continuous and does not satisfy subgroup consistency.

Clark et al. (1981) also proposed a subgroup consistent and continuous poverty measure, defined as:

$$C_2(x; z, \beta) = 1 - \frac{1}{z}\left[\frac{1}{n}\sum_{i=1}^{n}(\tilde{x}_i)^{\beta}\right]^{\frac{1}{\beta}}, \tag{1.14}$$

where $\beta < 1$ and $\tilde{x}_i$ is the censored income defined in (1.6). This second measure satisfies all the axioms discussed above except for additive decomposability. Furthermore, Zheng (1997) shows that this measure is related to another decomposable poverty measure proposed by Chakravarty (1983b).

It is important to notice that none of the poverty measures is perfect. Each of them satisfies some particular axioms and has different characteristics, capturing diverse aspects of the phenomenon. There is an extensive literature related to the characteristics, advantages and disadvantages of different poverty measures. Some authors such as Atkinson (1987) or Hagenaars (1987) have provided an evaluation of different measures of poverty and their characteristics when it comes to analysing this phenomenon. In the same line, Zheng (1997) evaluates the satisfaction of the axioms by different poverty indices. Table 1.1 summarises the axioms satisfied by each of the measures reviewed above.

To close this section, we want to point out that, in the analysis of poverty, researchers are generally interested in comparing poverty levels in two situations, for example the poverty levels of a country in two different periods or the level of poverty in two different countries. As poverty measures are estimated based on sample data, it is necessary to test whether the observed differences in poverty levels are statistically significant. Davidson and Flachaire (2007) distinguish two statistical inference methods for poverty measures: asymptotic methods and bootstrap methods. With respect to the first option, Kakwani (1993) derived the asymptotic distribution of different poverty measures: the FGT family, the Watts Index and the family introduced by Clark et al. (1981). In the same line, Bishop et al. (1995) develop statistical tests for decomposable poverty measures and Bishop et al. (1997) also derived the asymptotic distribution for the Sen Index and provided consistent estimators for its components (Headcount Ratio, Income Gap Ratio and the Gini coefficient of the poor). Alternatively, Biewen (2002) proposes the use of the bootstrap method for the estimation of the FGT family of poverty measures, and Davidson and Flachaire (2007) point out the good performance of standard bootstrap procedures for these measures; see Efron and Tibshirani (1994) for a comprehensive review of bootstrap methods.

	H	I	HI	W	S	SST	FGT_2	C_1	C_2
A1. Symmetry	✓	✓	✓	✓	✓	✓	✓	✓	✓
A2. Population Replication Invariance	✓	✓	✓	✓	✗	✓	✓	✓	✓
A3. Focus	✓	✓	✓	✓	✓	✓	✓	✓	✓
A4. Scale Invariance	✓	✓	✓	✓	✓	✓	✓	✓	✓
A5. Continuity	✗	✗	✓	✓	✗	✓	✓	✗	✓
A6.1. Weak Monotonicity	✗	✓	✓	✓	✓	✓	✓	✓	✓
A6.2. Strong Monotonicity	✗	✗	✓	✓	✓	✓	✓	✓	✓
A7.1. Weak Transfer	✗	✗	✗	✓	✓	✓	✓	✓	✓
A7.2. Regressive Transfer	✗	✗	✗	✓	✗	✓	✓	✗	✓
A8. Subgroup Consistency	✓	✓	✓	✓	✗	✗	✓	✗	✓
A9. Additive Decomposability	✓	✗	✓	✓	✗	✗	✓	✗	✗

✓: the axiom is fulfilled

✗: the axiom is not fulfilled

Table 1.1: Unidimensional poverty measures and axioms

1.2.3 Stochastic dominance techniques and unidimensional poverty

So far our discussion has focused on the construction of poverty measures that are informative of the level of deprivation that exists in a given society. However, as several authors point out, specific choices on the functional form of the measure and the poverty line may lead to different rankings of distributions in terms of poverty. Thus, when making poverty comparisons based on particular poverty indices and specific poverty lines, some doubts about the robustness of the results may arise; see Atkinson (1987), Foster and Shorrocks (1988) and Zheng (1997). To overcome this drawback, some authors have proposed techniques that can be applied to obtain robust poverty orderings. For instance, Atkinson (1987) and Foster and Shorrocks (1988), among others, have proposed the use of stochastic dominance techniques. Alternatively, other authors, such as Jenkins and Lambert (1997), have proposed the use of which they denominate TIP ("Three I's of Poverty") Curves, which allow to analyse the three dimensions of unidimensional poverty cited by Sen (1976): incidence, intensity and inequality. In this section we will just focus on the first approach, that is, on the use of stochastic dominance techniques to obtain robust poverty orderings.[3] We refer those interested in the use of TIP curves to Jenkins and Lambert (1997) and Del Río and Ruiz-Castillo (2001).

First, we focus on the concepts of first-order and higher order stochastic dominance. After that, the relationship between the stochastic dominance conditions and poverty orderings is reviewed.

1.2.3.1 The concept of stochastic dominance

We begin by briefly reviewing the concept of stochastic dominance.[4] Let F and G be two cumulative distribution functions (CDF) with support in the non-negative real line, for instance

[3]Dominance methods have also been proposed to study other aspects in the field of Welfare Economics, such as inequality. See, for example, Butler and McDonald (1987) and the references therein.

[4]See Sriboonchitta et al. (2010) for an extensive treatment of stochastic dominance techniques both in the univariate and multivariate cases.

the income distributions of a country in two different periods or the income distributions of two different countries. Then, it is said (Hadar and Russell, 1969) that G first-order stochastically dominates F, denoted as $G \succeq_1 F$, if and only if

$$G(x) \leq F(x) \;\; \forall x \;\; \in \;\; \mathbb{R}, \tag{1.15}$$

with strict inequality for at least one x. That is, if G is never above F; see Figure 1.1.

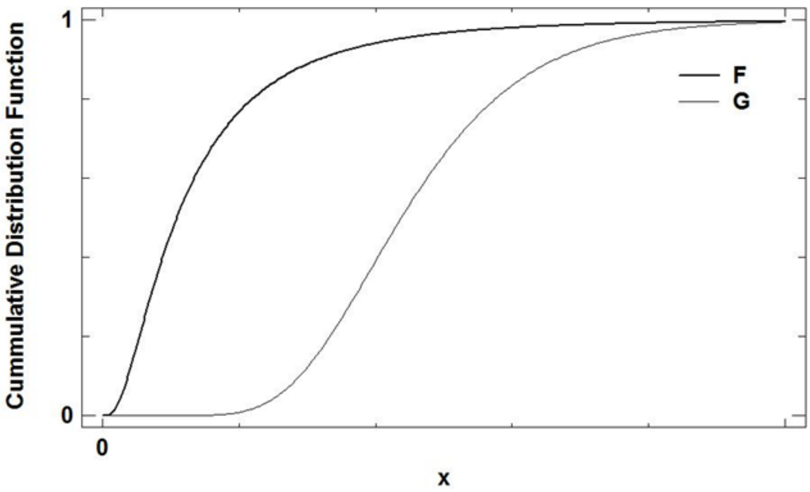

Figure 1.1: First-order stochastic dominance of G over F

Intuitively, if x denotes an income level, this condition means that the proportion of "poor" people (people with an income smaller than x) in distribution F is at least as great as the proportion of "poor" people in distribution G for any x.

It is also possible to define first-order stochastic dominance in terms of survival functions. Let $\bar{F}$ be the survival function associated to F, defined as $\bar{F}(x) = 1 - F(x)$, and let $\bar{G}$ be defined in a similar way. Then, the following condition holds:

$$G \succeq_1 F \Leftrightarrow G(x) \leq F(x) \;\; \forall x \;\; \in \;\; \mathbb{R} \Leftrightarrow \bar{G}(x) \geq \bar{F}(x) \;\; \forall x \;\; \in \;\; \mathbb{R}. \tag{1.16}$$

Roughly speaking, if x denotes an income level, then this condition means that the proportion of "rich" people (people with an income greater than x) in distribution G is at least as great as the proportion of "rich" people in distribution F for any x; see Figure 1.2.

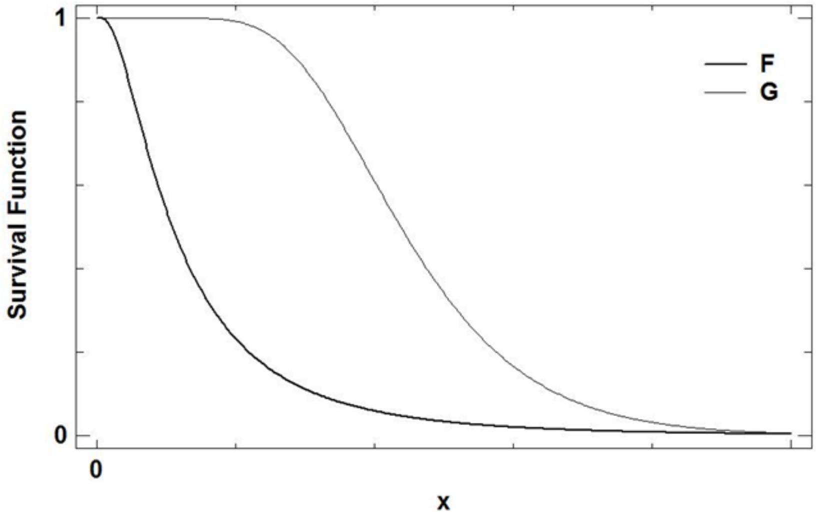

Figure 1.2: First-order stochastic dominance of G over F

First-order dominance can also be written in terms of quantiles. Let $F^{-1}(l)$ be the quantile function, defined as $F^{-1}(l) = \inf\{x : F(x) \geq l\}$ for a given $l \in [0,1]$, then: $G \succeq_1 F \Leftrightarrow G^{-1}(l) \geq F^{-1}(l)\ \forall l \in [0,1]$, with strict inequality for at least one l. Actually, this definition is equivalent to the concept of rank dominance introduced by Saposnik (1981).

When the cumulative distribution functions cross, it is not possible to establish a rank order between them and it is necessary to resort to second-order dominance. Following Hadar and Russell (1969), we say that G second-order stochastically dominates F, denoted as $G \succeq_2 F$, if and only if

$$\int_{t=0}^{x} G(t)\ dt \leq \int_{t=0}^{x} F(t)\ dt\ \forall x \in \mathbb{R}, \tag{1.17}$$

with strict inequality for at least one x. That is, when the area under G is smaller than that under F. Shorrocks (1983) shows that this condition is equivalent to Generalised-Lorenz dominance.

Second-order stochastic dominance can also be expressed in terms of quantiles. In particular,

$G \succeq_2 F$ if and only if $\int_{t=0}^{l} G^{-1}(t)\ dt \geq \int_{t=0}^{l} F^{-1}(t)\ dt\ \forall l \in [0,1]$, with strict inequality for at least one l. That is, distribution G second-order stochastically dominates distribution F when the area under G^{-1} is bigger than that under F^{-1}.

Higher orders of stochastic dominance can be defined by means of the general notation used by Davidson and Duclos (2000), which is based on the following functions:

$$D^1(x) = F(x), \quad D^\alpha(x) = \int_{t=0}^{x} D^{\alpha-1}(t)\ dt, \quad \text{for } \alpha = 2, 3, \ldots \tag{1.18}$$

As a general rule, G stochastically dominates F at order α, denoted as $G \succeq_\alpha F$, if and only if $D_G^\alpha(x) \leq D_F^\alpha(x)\ \forall x \in \mathbb{R}$, with strict inequality for at least one x. Thus, when $\alpha = 1$ the condition of stochastic dominance of first order in (1.15) comes up. In turn, when $\alpha = 2$, the concept of second-order stochastic dominance in (1.17) comes up, and so on.

The definition above makes it clear that first-order dominance is the strongest condition and implies dominance of all higher orders. In particular, second-order dominance is implied by, but does not imply, first-order dominance; third-order dominance is implied by, but does not imply, first- and second-order dominance, and so on. Then, lower order dominance is a stronger condition than higher order dominance.

1.2.3.2 Application of stochastic dominance to poverty analysis

Once we have reviewed the concept of stochastic dominance, we now focus on how it can be applied in a poverty analysis. Let X be a measure of income with CDF F and let z be the poverty line so that an individual is considered poor if her or his income is lower than z. Atkinson (1987) considers a general class of poverty measures $P_F(z)$, with $z > 0$, defined as:

$$P_F(z) = \int_{x=0}^{z} p(x; z)\ dF(x), \tag{1.19}$$

where $p(x; z)$ is a differentiable individual poverty function such that $p(x; z) = 0$ for all $x \geq z$. Here onwards we assume that $p(x; z)$ is differentiable with respect to x to the required degree. The general class of poverty measures given by $P_F(z)$ includes several of the well-known indices described in Section 1.2.2.2, such as those proposed by Watts (1968), Clark et al. (1981) and Foster et al. (1984) defined in (1.5), (1.13) and (1.11), respectively.

In order to compare the level of poverty of two distributions F and G, the following difference should be evaluated:

$$P_G(z) - P_F(z) = \int_{x=0}^{z} p(x; z)dG(x) - \int_{x=0}^{z} p(x; z)dF(x).$$

Integrating by parts above the following result comes up:

$$P_G(z) - P_F(z) = -\int_{x=0}^{z} p_x(x; z)[G(x) - F(x)] \; dx, \qquad (1.20)$$

where $p_x(x; z)$ is the first derivative of the individual poverty function with respect to x. Then, as Atkinson (1987) and Foster and Shorrocks (1988) showed, we have:

$$G \succeq_1 F \quad \Leftrightarrow \quad P_G(z) \leq P_F(z) \;\; \forall p(x; z)|p_x(x; z) \leq 0 \;\; \forall z. \qquad (1.21)$$

That is, the level of poverty in G is not higher than that in F for all individual poverty measures that are decreasing in the level income ($p_x(x; z) \leq 0$) and for all poverty lines if and only if the proportion of people with an income smaller than x ("poor" people) is at least as great in distribution F as in distribution G ($G \succeq_1 F$). Therefore, first-order stochastic dominance constitutes the condition under which all poverty measures satisfying $p_x(x; z) \leq 0$ rank distributions equally in terms of poverty. In particular, this condition is satisfied by well-known poverty measures such as the indices proposed by Watts (1968), Thon (1979), Clark et al. (1981) and Chakravarty (1983a); see Atkinson (1987) and Zheng (1999, 2000).

Despite its theoretical appeal, first-order stochastic dominance is a very demanding condition that many distributions may fail to satisfy, making it difficult to check in empirical applications and thus not leading to robust poverty orderings. To overcome this drawback, it is possible to resort to second-order dominance. With this approach, establishing poverty orderings requires an additional condition on the individual poverty function $p(x;z)$ that fewer poverty measures will be able to fulfil. In particular, Atkinson (1987) shows that:

$$G \succeq_2 F \;\; \Leftrightarrow \;\; P_G(z) \leq P_F(z) \;\; \forall p(x;z)|p_x(x;z) \leq 0, \;\; p_{xx}(x;z) \geq 0 \;\; \forall z, \tag{1.22}$$

where $p_{xx}(x;z)$ is the second derivative of the individual poverty function with respect to x. Then, $G \succeq_2 F$ if and only if the level of poverty in G is not higher than that in F for all poverty lines according to all poverty measures that are decreasing in the level of income ($p_x(x;z) \leq 0$) and convex ($p_{xx}(x;z) \geq 0$). This latter property implies that poverty must increase after a regressive transfer between two poor individuals. Hence, as Atkinson (1970) and Zheng (1999, 2000) point out, the poverty ordering implied by second-order dominance will be respected by well-known poverty measures such as the indices proposed by Watts (1968) and Thon (1979), among others.

When dominance analysis is not conclusive for first and second order, it is possible to resort to higher order dominance conditions based on expression (1.18), but such conditions require more assumptions on $p(x;z)$, reducing even more the number of poverty measures that will respect the poverty orderings obtained. However, the measures fulfilling such conditions are more sensitive to changes in the lower part of the income distribution, which some authors regard as a desirable property; see, for instance, Zheng (1997).

On the other hand, as Foster and Shorrocks (1988) point out, the stochastic dominance orderings are equivalent to the poverty orderings based on the FGT family of poverty measures defined in equation (1.11). In particular, these measures can be expressed as:

$$FGT_{\alpha-1}(z) = \frac{1}{z^{\alpha-1}}(\alpha-1)!D^{\alpha}(z) = \frac{1}{z^{\alpha-1}}\int_{x=0}^{z}(z-x)^{\alpha-1}dF(x) \quad \text{for } \alpha = 1,2,3,\ldots \tag{1.23}$$

where the function $D^{\alpha}(z)$ above is the function defined in equation (1.18) evaluated at the poverty line, which can be alternatively written as:

$$D^{\alpha}(z) = \frac{1}{(\alpha-1)!}\int_{x=0}^{z}(z-x)^{\alpha-1}dF(x) \quad \text{for } \alpha = 1,2,3,\ldots \tag{1.24}$$

Hence, in general, the following equivalences can be established:

$$G \succeq_{\alpha} F \;\Leftrightarrow\; D_G^{\alpha}(z) \leq D_F^{\alpha}(z)\ \forall z \;\Leftrightarrow\; FGT_{G,\alpha-1}(z) \leq FGT_{F,\alpha-1}(z)\ \forall z. \tag{1.25}$$

In particular, for $\alpha = 1$, and taking into account that $FGT_0(z)$ is the headcount ratio, the relationship in (1.25) implies that $G \succeq_1 F$ if and only if the headcount ratio in population G is unambiguously lower than in F for all possible values of the poverty line z. Similarly, taking $\alpha = 2$ in (1.23), $FGT_1(z)$ comes up. Then, as $FGT_1(z)$ is the poverty gap ratio, the relationship in (1.25) implies that $G \succeq_2 F$ if and only if the poverty gap ratio is lower in population G than in population F for all poverty lines z.

So far we have presented the theoretical results regarding the relationship between stochastic dominance conditions and robust poverty orderings. As it has been previously discussed, establishing poverty orderings amounts to comparing stochastic dominance for two distributions of income (or any other variable representing the level of welfare). However, in empirical applications, such distributions are not available for the whole population and it is necessary to work with sample data. Therefore, comparisons must be based on statistical functionals of the empirical distribution functions and statistical inference must be applied to determine whether the results of such comparisons are statistically significant. However, facing this task requires overcoming a number of difficulties. The two main challenges are, first, to establish the null and alternative hypotheses of the test for stochastic dominance and second, to find an appropriate testing procedure, taking into account that dominance conditions involve the comparison of distributions over a large number of points. The different approaches followed to overcome these two challenges have led to the proposal of different tests for stochastic dominance. Among them, we can highlight the proposals of McFadden (1989), Bishop et al. (1989), Bishop et al. (1992), Davidson and Duclos (2000) or Barrett and Donald (2003). A detailed review of these and other tests for univariate stochastic dominance and some relevant empirical applications of them can be found in Maasoumi (2001) and, more recently, in García-Gómez et al. (2019).

1.3 Multidimensional poverty

Over the last decades, a consensus has emerged in the literature considering necessary to study poverty from a multidimensional perspective, in the same way as other phenomena such as well-being or inequality (Kolm, 1977; Stiglitz et al., 2009). The concept of multidimensional poverty is strongly linked to the view that well-being does not only have a monetary dimension, an idea that dates back to the contribution of Streeten et al. (1981). These authors founded the theoretical arguments for the Basic Needs Approach, which considers that, for economic development, the aim should be the fulfilment of a series of basic needs such as minimum quantities of food, health, clothing or shelter, among others. Therefore, by this conception, welfare (and hence poverty) is recognised as a multidimensional phenomenon.

An essential contribution on this area was made by Sen (1985, 1992), who considers poverty not only as a deprivation in monetary terms but also as a deprivation of capabilities, which he links to the capacity of individuals to choose freely what they want to be. In this sense, the deprivation in terms of dimensions such as health or education is considered an essential feature of poverty.

Nowadays, there is little discussion on the multidimensional nature of poverty. For instance, Ravallion (1996) argues that monetary variables do not provide a complete picture of the phenomenon. Moreover, Tsui (2002) points out that, although an increase in income could improve the capacity of a person to satisfy their needs, the existence of market imperfections can be an impediment to the provision of basic goods. Thorbecke (2007) also considers that it is possible that individuals with income above the poverty line fail to allocate these resources to the purchasing of basic goods, being poor in other important dimensions.

In this multidimensional context, it is first necessary to select the dimensions to be considered, as well as the relative importance given to each of them. These issues will be discussed in Section 1.3.1. Once the dimensions are selected, the problem that arises is how to measure multidimensional poverty based on these dimensions. Two approaches can be distinguished. The first one consists of aggregating first, for each dimension, the information across individuals and then aggregating across dimensions. With this approach, individuals are not identified as multidimensionally poor. Thus, it is possible that there are individuals deprived in many different dimensions and this is not captured because the aggregation is performed for dimensions and not for individuals. An example of this method can be found in Anand and Sen (1997), who construct the Human Poverty Index (HPI). As Alkire et al. (2011) argue, this approach, usually denominated "marginal approach", does not reflect the joint distribution of achievements. Moreover, Alkire and Foster (2011b) point out that this approach ignores the information on the interrelations between dimensions, implying that, although the measure can inform about the overall extent of deprivation in a country, it says nothing about how deprivations are distributed among the population.

Alternatively, it is possible to aggregate first, for each individual, the information across dimensions and then aggregate across individuals. To do this, several possibilities come up. One consists of aggregating the achievements of each individual in the different dimensions to obtain an aggregate indicator of individual well-being. Then, the identification of multidimensionally poor individuals is performed as in the unidimensional case, that is, comparing the level of well-being with a threshold, the minimum level of welfare necessary to be considered non-poor. To aggregate the achievements of individuals, some multivariate statistical techniques such as Principal Components Analysis, Multiple Correspondence Analysis or Factor Analysis can be used; see Alkire et al. (2015) for an extensive review of these techniques. This approach has a serious drawback, as the information on the shortfalls in the different dimensions is lost. In fact, this approach resembles the unidimensional case and does not give importance to each deprivation, losing valuable information for the analysis of multidimensional poverty. An alternative is the widely used counting approach, which consists of counting the number of dimensions in which an individual is deprived. In this case, individuals or households are first identified as deprived or not in each of the dimensions using specific deprivation cut-offs. Then it is possible to identify those individuals that can be considered as multidimensionally poor based on the number of dimensions in which the person is deprived. Once the multidimensionally poor individuals are identified, this information is aggregated by means of a multidimensional poverty measure. This issues will be reviewed in Sections 1.3.2 and 1.3.3.

As expected, there is not an agreement on the appropriate method to measure multidimensional poverty. Some authors (Ravallion, 1996) have supported the use of marginal methods, while others (Alkire and Foster, 2011a; Bourguignon and Chakravarty, 2003) have proposed the use of multidimensional poverty indices that aggregates first across dimensions and then across individuals.

Moreover, Ferreira and Lugo (2013) point out that the key aspect of multidimensional poverty analyses is the interdependence between the different dimensions. However, and despite its importance, the measurement of the degree of association between poverty dimensions has been traditionally overlooked in the literature. On one hand, the marginal methods explained above only focus on the marginal distributions, thus not taking into consideration the dependency structure. On the other hand, the multidimensional poverty measures mentioned above are not fully sensitive to the degree of multivariate dependence between the different dimensions, thus not paying enough attention to this key aspect; see Duclos et al. (2006), Duclos and Tiberti (2016) and Seth and Santos (2019). Given this, Ferreira and Lugo (2013) consider some alternative methodologies that allow to focus on the structure of dependence.

For instance, multivariate stochastic dominance techniques have been proposed by Duclos et al. (2006) and by Bourguignon and Chakravarty (2009) as a methodological option for analysing poverty appropriately accounting for the correlation between dimensions. These techniques, which are extensions of those univariate stochastic dominance methods assessed in Section 1.2.3, will be reviewed in Section 1.3.4. This method has also some drawbacks. First, it is clearly constrained by the number of dimensions considered in the analysis. That is, multivariate stochastic dominance techniques are of limited use when the number of dimensions is large. Another shortcoming of this approach is that, when there is no dominance, the researcher cannot make unambiguous comparisons of poverty, which complicates its use in empirical applications. Furthermore, although multivariate stochastic dominance techniques can establish multidimensional poverty orderings taking into account the interdependence between dimensions, they do not allow to quantify the degree of multivariate dependence.

Another alternative to deal with the interdependence between the dimensions of poverty, also mentioned by Ferreira and Lugo (2013), is the approach based on copulas, which is the one that we will adopt in this thesis. With this method, which has been employed by Decancq (2014) and Pérez and Prieto-Alaiz (2016b) for multidimensional welfare and by Pérez and Prieto (2015) and, more recently, by García-Gómez et al. (2021), for multidimensional poverty, the joint distribution function of a set of dimensions is decomposed into the marginal distribution of each of the dimensions and a copula function that gives the information on the degree of interdependence between dimensions. As we will see throughout Chapters 2 and 3, the measurement of multidimensional poverty provides an interesting field for the application of copula-based methods for several reasons. First, these methods allow to quantify the interdependence between dimensions of poverty, an important aspect that has not been traditionally taken into account when measuring this phenomenon from a multidimensional perspective. Second, copula-based measures of dependence allow to handle concepts of dependence that go beyond linear correlation, a concept that is only appropriate for linear relationships with normally distributed data, which are not common when treating with welfare variables. Moreover, copula-based measures of multivariate dependence allow for multivariate extensions of well-known bivariate rank-based measures of dependence, such as the Spearman's rank correlation coefficient. The copula methodology will be extensively discussed in Chapter 2, where we will introduce several copula-based measures of multivariate association that will be applied in Chapter 3 to analyse the degree of multivariate dependence between the dimensions of poverty in the EU-28.

1.3.1 The selection of poverty dimensions

Given the multidimensional nature of poverty, there is a wide range of aspects that can be thought to be important when it comes to measuring this phenomenon. Thus, any empirical analysis requires the selection of a certain number of dimensions. Thorbecke (2007) considers this as a key step and argues that it would be rational to consider those basic needs presented by Streeten et al. (1981) as a point of departure. Additionally, he points out other potential dimensions such as freedom of expression and religion, but acknowledges the difficulties in their measurement. Alkire (2002) and Stiglitz et al. (2009) consider aspects such as material living standards, health, education, job status, social connections and relationships and the quality of natural environment to be important for personal well-being.

Ramos and Silber (2005) showed that the selection of the dimensions is not an irrelevant issue. In particular, they perform an efficiency analysis considering some of these lists of dimensions and argue that different groups of variables can lead to different multidimensional poverty figures. Alkire (2007) provide a review of the different alternatives for selecting the dimensions to be included in a multidimensional poverty analysis, pointing out that the selection can be based on the available or existing data, public consensus (based on the existence of universal human rights, for instance), assumptions about the value people give to each aspect, deliberative participatory processes or empirical evidence regarding people's values.

Another aspect that requires the attention of the researcher is the weighting of the dimensions, that is, the relative contribution of each of them to overall well-being. Decancq and Lugo (2013) provide a detailed discussion on the importance of weights and distinguish three main approaches to set them: data-driven procedures, normative weights and hybrid approaches.[5] Next, we briefly summarise them.

- Data-driven weights. With this approach, the weights are selected based on the data used by the researcher. One possibility is to assume that there is an inverse relationship between the frequency of deprivation in a dimension and its weight. Hence, a weighting scheme is set giving higher weights the smaller is the proportion of individuals with deprivation in that dimension; see, for instance, Desai and Shah (1988) and Deutsch and Silber (2005). Another possibility is to use multivariate statistical techniques such as Principal Component Analysis or Factor Analysis (Klasen, 2000; Maasoumi and Nickelsburg, 1988). However, as Aaberge and Brandolini (2015) argue, with the application of these techniques the independent meaning of the weights is lost. Decancq and Lugo (2013) also point out the lack of transparency of these methods.

- Normative weights. In this case, the weighting scheme is not selected as a function of the data, but in base of value judgements. One possibility is to weight dimensions equally, as in the case of the Human Development Index calculated by the United Nations Development Program. Alternatively, it is possible to give different relevance to each of the dimensions; see Atkinson et al. (2002) and Marlier and Atkinson (2010). To avoid arbitrariness when it comes to setting the different weights in a normative context, one possibility consists of resorting to the opinion of experts, but in this case it should be

[5] Although we follow here this classification, a review of several methods to determine weights can also be found in Aaberge and Brandolini (2015).

ensured that these experts are selected in an unbiased way. Another alternative, although not popular in the multidimensional poverty literature, is to use price-based weights computing implicit prices for the variables reflecting the different dimensions of poverty.

- Hybrid weights. This approach is a combination of data-driven and normative methods. In this framework it is possible to distinguish between stated preference weights and hedonic weights. The first ones are based on the opinions, retrieved for a sample of individuals, on the importance of the different dimensions; see, for example, Halleröd (1995) and, more recently, Bossert et al. (2009). In a similar fashion, hedonic weights are calculated by means of the information on the level of life satisfaction reported by the individuals.

Therefore, there are several possibilities to provide weights to the different dimensions included in a multidimensional poverty analysis, but none of these methods is exempt from flaws. It should also be pointed out that the choice of the particular procedure is arbitrary and that different weighting schemes may lead to different conclusions. In this respect, robustness checks and sensitivity analyses can help to determine if the results are driven by the specific choice of weights.

1.3.2 The identification of the multidimensionally poor

Once the poverty dimensions have been selected, the next question is how we can measure multidimensional poverty. As in the case of unidimensional poverty measurement, two steps can also be distinguished in the multidimensional case when the objective is to obtain a figure reflecting the level of deprivation in a given society: first, the identification of multidimensionally poor individuals and second, the aggregation of multidimensional poverty. However, the identification step is more complex in the multidimensional case, since the researcher must decide how many dimensions an individual should be deprived in to be considered poor.

Following Alkire et al. (2015), two approaches can be distinguished for the identification of multidimensionally poor individuals. The first one is the so-called "unidimensional approach" or "aggregate achievement approach", which consists of aggregating the achievements of the individuals in the different dimensions and comparing this aggregated value with a threshold in such a way that those individuals with a level of well-being below this threshold are considered as multidimensionally poor.

The second procedure is the widely used "counting approach", based on the calculation of the number of dimensions in which an individual is deprived. A deprivation cut-off is defined for each dimension and the dimensional achievement of the individuals is compared with this threshold. The number of dimensions in which an individual is deprived is therefore obtained. After this, the key question is to decide whether it is reasonable to consider as multidimensionally poor those individuals who are identified as deprived in any dimension (union approach) or those who do not reach the minimum established threshold in all of them (intersection approach), or, alternatively, to use an intermediate procedure.

The union and intersection criteria are reviewed by Bourguignon and Chakravarty (2003) and, as Alkire and Foster (2011a) point out, both of them have drawbacks. In the case of the union approach, it is possible that when the number of dimensions is high most of the population is identified as poor, not being possible to concentrate on those individuals most extensively deprived. All the contrary occurs when using the intersection criterion, not being possible to identify as poor those who are extensive but not universally deprived. Given this, Alkire and Foster (2011a) consider these extreme criteria as inadequate and call for the use of an intermediate approach that can be seen as a generalization of them.

This intermediate approach is known as the dual cut-off method, and is one of the bases of the Alkire-Foster (AF) methodology. This method consists of employing a dimensional threshold to identify individuals deprived in each dimension and a poverty cut-off that states the number of dimensions in which an individual must be deprived in order to be considered as multidimensionally poor. The notation will be as follows. Let X be the $(n \times d)$ matrix of achievements in d dimensions in a population of size n, such that the element x_{ij} denotes the achievement of the i^{th} individual in the j^{th} dimension, for all $i = 1, \ldots, n$ and $j = 1, \ldots, d$. Let $\mathbf{z}$ be the d-dimensional vector of deprivation cut-offs, that is, $\mathbf{z} = (z_1, z_2, \ldots, z_d)$, where z_j is the deprivation threshold for dimension j, so that a person i is identified as deprived in dimension j whenever $x_{ij} < z_j$.

Alkire and Foster (2011a) also consider the possibility of giving different importance to each of the dimensions with the introduction of a vector of weights $\mathbf{w}$ in which w_j is the weight of dimension j and $\sum_{j=1}^{d} w_j = 1$. The deprivation score (c_i) gives the weighted sum of deprivations of individual i, namely:

$$c_i = \sum_{j=1}^{d} w_j I_{ij} \text{ for } i = 1, \ldots, n, \tag{1.26}$$

where $I_{ij} = I_{\{x_{ij} < z_j\}}$ is an indicator function that takes the value 1 if individual i is deprived in dimension j and takes the value 0 otherwise.

The second cut-off (k) is employed to identify the set of multidimensionally poor individuals. In particular, an individual i is considered multidimensionally poor if its weighted sum of deprivations, c_i, is at least k. This information is retrieved by the following indicator function:

$$I_i = I_{\{c_i \geq k\}} = \begin{cases} 1, & \text{if } c_i \geq k, \\ 0, & \text{otherwise.} \end{cases} \tag{1.27}$$

That is, I_i takes the value 1 if individual i is deprived in at least k dimensions and takes the value 0 otherwise.

One of the main advantages of the AF methodology is its flexibility. In this sense, any particular choice of weights is a particular application of this method. Of course, equally weighted dimensions are another possibility within this framework. Furthermore, both the union and intersection approaches can be seen as special cases of the poverty cut-off method employed by the AF methodology. In particular, when $k = \min(w_1, w_2, \ldots, w_d)$, the AF poverty cut-off implies the use of the union approach, and when $k = \sum_{j=1}^{d} w_j$, the intersection approach comes up.

1.3.3 The aggregation of multidimensional poverty

Once multidimensionally poor individuals are identified, it is possible to aggregate the information into a single number that reflects the overall level of multidimensional poverty in the society. Given the matrix of achievements X and the vector of deprivation cut-offs $\mathbf{z}$ previously defined, let $P(X; \mathbf{z})$ be a multidimensional poverty measure. In the literature, several measures of multidimensional poverty have been proposed, each of them satisfying different axioms.

As we did in Section 1.2.2 for unidimensional poverty, we will present next the main multidimensional poverty axioms, reviewing later some of the multidimensional poverty measures proposed in the literature so far; see Aaberge and Brandolini (2015) and Alkire et al. (2015) for an extensive treatment of multidimensional poverty axioms and indices.

1.3.3.1 Axioms

As in the case of monetary poverty, the literature on the axiomatic characteristics of multidimensional poverty measures is extensive and has been highly influential. Some of the axioms are simple extensions of those on unidimensional poverty already reviewed in Section 1.2.2.1, but others are specific of the multidimensional case. Alkire et al. (2015), Datt (2013), Tsui (2002) and Bourguignon and Chakravarty (2003) provide an extensive treatment of the different axioms that have been proposed. Here we only review those that can be considered as the most relevant ones.

AM1. Symmetry. Only deprivations matter and not the identity of the individual. This means that permutations of the rows of the matrix X should not affect $P(X, \mathbf{z})$. In other words, swapping achievement vectors across people should not result in a change in overall poverty.

AM2. Replication Invariance. The value of the poverty measure is insensitive to a replication of the population. This ensures that the index does not depend on the size of the population. Formally, if an achievement matrix X' is obtained from a finite number replication of the achievement matrix X, then $P(X';\mathbf{z}) = P(X;\mathbf{z})$. This is a necessary property in order to compare poverty across societies with different population sizes.

AM3. Scale Invariance. According to this axiom, $P(X;\mathbf{z})$ should not vary if attributes and thresholds are subjected to a scale transformation. That is, the level of poverty should not be affected by a change in the scale of the indicators.

AM4. Continuity. This axiom requires that $P(X;\mathbf{z})$ does not change abruptly with marginal changes in the achievements. That is, a poverty measure should be continuous on the achievements.

AM5. Poverty Focus. The value of the poverty index is independent of the attribute level of any non-poor individual. That is, for the measurement of poverty, only the achievements of poor individuals are relevant. Formally, when an achievement matrix X' is obtained from another one X such that $x'_{lk} > x_{lk}$, being l a non-poor individual and with $x'_{ij} = x_{ij}\ \forall (i,j) \neq (l,k)$, then $P(X';\mathbf{z}) = P(X;\mathbf{z})$.

AM6. Deprivation Focus. It requires that an increase in the attainment in a dimension in which a person is not considered to be deprived does not affect the level of poverty, even if that individual is poor. Formally, if an achievement matrix X' is obtained from another one X such that $x'_{lk} > x_{lk}$ with $x_{lk} \geq z_k$ and $x'_{ij} = x_{ij}\ \forall (i,j) \neq (l,k)$, then $P(X';\mathbf{z}) = P(X;\mathbf{z})$.

Alkire and Foster (2011a) point out that there is a relationship between AM5 and AM6. In particular, when the union identification is applied deprivation focus implies poverty focus and when the intersection identification is used poverty focus implies deprivation focus.

AM7. Weak Monotonicity. According to this axiom, poverty should not increase with an increase in any person's achievement. Formally, when X' is obtained from X in such a way that $x'_{lk} > x_{lk}$ and $x'_{ij} = x_{ij} \ \forall (i,j) \neq (l,k)$, then $P(X';\mathbf{z}) \leq P(X;\mathbf{z})$.

AM8. Strong Monotonicity. It requires that poverty decreases after an increase in the achievement of a poor person in a deprived dimension. Formally, if X' is obtained from X such that $x'_{lk} > x_{lk}$ being individual l poor and deprived in dimension k, and $x'_{ij} = x_{ij} \ \forall (i,j) \neq (l,k)$, then $P(X';\mathbf{z}) < P(X;\mathbf{z})$

Alkire and Foster (2011a) introduced an additional monotonicity axiom that is spectific to the multidimensional context, namely dimensional monotonicity, which is defined as follows.

AM9. Weak Dimensional Monotonicity. This axiom states that, if a poor individual becomes deprived in an additional dimension, then poverty should not decrease. In formal terms, when X' is obtained from X such that $x'_{lk} < z_k \leq x_{lk}$ being individual l poor, and $x'_{ij} = x_{ij} \ \forall (i,j) \neq (l,k)$, then $P(X';\mathbf{z}) \geq P(X;\mathbf{z})$. The strong version of this axiom requires that $P(X';\mathbf{z}) > P(X;\mathbf{z})$.

A poverty measure that satisfies this axiom takes into consideration not only the number of poor in a society but also the extent to which poor individuals are deprived in multiple dimensions. Moreover, a measure that satisfies monotonicity also satisfies dimensional monotonicity.

AM10. Weak Transfer. According to this axiom, when achievements among the poor become more equal, poverty should not increase. Formally, if X' is obtained from X by a reduction of the inequality of the achievements of poor individuals that leaves the average achievement among the poor unchanged, then $P(X';\mathbf{z}) \leq P(X;\mathbf{z})$. The strong version of the transfer axiom requires that $P(X';\mathbf{z}) < P(X;\mathbf{z})$.

It should also be noticed that deprivation focus and strong transfer cannot be simultaneously satisfied; see Tsui (2002), Alkire et al. (2015) and Seth and Santos (2019).

AM11. Rearrangement Axioms. Let i and t be two poor individuals with $x_{ij} > x_{tj} \ \forall j = 1, 2, \ldots, d$. A matrix of achievements X' is obtained from X by an "association-decreasing rearrangement" among the poor if $x'_{ij} = x_{tj}$ and $x'_{tj} = x_{ij}$ for at least one j, with $x'_{ik} = x_{ik}$ and $x'_{tk} = x_{tk}\ \forall k \neq j$.

In other words, suppose that a poor individual i is at least as well off in all dimensions as other poor individual t and that there is an exchange of the level in one or more dimensions, but not in all of them, between these two individuals. Suppose also that, after this transformation, individual i is no longer equal or better off in all the attributes than individual t, but his achievements are equal to or higher than those of individual t only in some of the dimensions. Note that we assume that the achievements of everybody else do not change at all. This is the definition of an association-decreasing rearrangement, as the association between the level of achievement in the different dimensions has decreased.

Should an association-decreasing rearrangement lead to an increase or a decrease in the level of multidimensional poverty? Different positions can be distinguished in the literature in this respect. Tsui (2002) argues that poverty should decrease because the transformation seems to reduce inequality among the poor, as the number of deprivations in which a poor individual is more deprived than another decreases. Bourguignon and Chakravarty (2003), by contrary, argue that the answer to that question depends on the relationship between the dimensions. That is, the change in the poverty index as a result of an association-decreasing rearrangement depends on whether the dimensions are considered substitutes or complements; see also Atkinson and Bourguignon (1982). Two dimensions of poverty are considered ALEP substitutes when a higher level in one of them allows individuals to compensate for a deprivation in the other.[6] That is, the increase in well-being that is consequence of an increment in one dimension decreases with the value of the other. For example, an increase in the level of education generates more utility for a poor person (in terms of income) than for an individual whose income is higher. By contrary, two dimensions of poverty are ALEP complements when the increase

[6] As Yalonetzky (2014) points out, the acronym ALEP comes from Auspitz-Lieben-Edgeworth-Pareto. ALEP substitutability implies that the cross-partial derivative of the utility function with respect to a pair of goods is negative, with this derivative being positive for ALEP complementarity. A detailed discussion on these concepts can be found in Kannai (1980).

in welfare derived from an increase in one of them increases with the level of the other, so that a high level in one aspect of welfare complements a high level in other aspect. For example, education can be thought to complement nutritional status because well-nourished children take more advantage of the education opportunities than badly nourished children.

Depending on the relationship between the dimensions, Alkire et al. (2015) distinguish four different rearrangement axioms, namely:

- AM11.1. Weak Rearrangement. When dimensions are considered ALEP substitutes, the association-decreasing rearrangement would help individuals to compensate the low achievement in one dimension with a higher achievement in another. As a result, the level of poverty should not increase. This axiom is also known as "non-decreasing poverty under a correlation-increasing switch", henceforth NDP (Silber, 2007), and implies that, when dimensions are considered substitutes, an increase in the association between them must not have as a consequence a decrease in the level of poverty.
- AM11.2. Strong Rearrangement. It is the strong version of AM11.1 and requires that poverty decreases after an association-decreasing rearrangement.
- AM11.3. Converse Weak Rearrangement. When dimensions are ALEP complements, poverty should not decrease under an association-decreasing rearrangement, as the ability of one of the individuals to achieve a certain level of well-being by combining the achievement levels in the dimensions is reduced. This axiom is also known as "non-increasing poverty under a correlation-increasing switch", henceforth NIP (Silber, 2007).
- AM11.4. Converse Strong Rearrangement. It is the strong version of AM11.3 and requires that poverty increases after an association-decreasing rearrangement.

The weak versions of the rearrangement axioms simply ensure that a measure should not move to an undesired direction due to changes in association between dimensions. However, they do not guarantee that the measure will be strictly sensitive to these changes, as the strong versions do. Noticeably, as Seth and Santos (2019) point out, there is an incompatibility between the deprivation focus axiom and the strong versions of the rearrangement axioms. Thus, a measure that satisfies the deprivation focus axiom cannot simultaneously satisfy the strong (converse) rearrangement axiom.

As we will see later, and as it is also argued by Seth and Santos (2019), most of the measures of multidimensional poverty proposed so far satisfy deprivation focus, thus failing to being strictly sensitive to changes in the association between dimensions, a key aspect when it comes to analysing multidimensional poverty.

Alkire et al. (2015, ch.2) define an alternative set of rearrangement properties that are consistent with the deprivation focus principle. To do so, they consider rearrangement transformations that occur only among the deprived dimensions of the poor.

AM12. Deprivation Rearrangement Axioms. An association-decreasing deprivation rearrangement is defined as an association-decreasing rearrangement that occurs only among the deprived dimensions of poor individuals (Alkire et al., 2015). As in the case of the rearrangement axioms, four different versions of the deprivation rearrangement axiom can be distinguished:

- AM12.1. Weak Deprivation Rearrangement. When dimensions are considered ALEP substitutes, poverty should not increase after an association-decreasing deprivation rearrangement.
- AM12.2. Strong Deprivation Rearrangement. It is the strong version of AM12.1 and requires that poverty decreases after an association-decreasing deprivation rearrangement.
- AM12.3. Converse Weak Deprivation Rearrangement. If dimensions are considered ALEP complements, then with an association-decreasing deprivation rearrangement poverty should not decrease.
- AM12.4. Converse Strong Deprivation Rearrangement. It is the strong version of AM12.3

and requires that poverty increases after an association-decreasing deprivation rearrangement.

AM13. Subgroup Decomposability. Overall poverty is the population-share weighted average of the level of poverty in different subgroups of the population. This is an important property when it comes to analysing the determinants of poverty such as geography, gender, age, or work status, among others. Formally, and denoting ζ as a given subgroup of the population and m as the total number of subgroups:

$$P(X;\mathbf{z}) = \sum_{\zeta=1}^{m} \left(\frac{n^{\zeta}}{n}\right) P(X^{\zeta};\mathbf{z}),$$

where n^{ζ} is the population in subgroup ζ, n is the total population and $P(X^{\zeta};\mathbf{z})$ is the level of poverty in subgroup ζ.

AM14. Subgroup Consistency. According to this axiom, if poverty increases in one subgroup of the population remaining constant in the rest, overall poverty must increase. Formally, if $P(X'^{\zeta};\mathbf{z}) > P(X^{\zeta};\mathbf{z})$ for $\zeta = \zeta_1$ and $P(X'^{\zeta};\mathbf{z}) = P(X^{\zeta};\mathbf{z})$ for all $\zeta \neq \zeta_1$, then $P(X';\mathbf{z}) > P(X;\mathbf{z})$.

There is a relationship between AM13 and AM14, as any poverty measure that is subgroup decomposable satisfies also subgroup consistency.

AM15. Dimensional Breakdown. Overall poverty can be expressed as a weighted sum of dimensional deprivations. With this property, it is possible to analyse the contribution of each dimension to the overall level of multidimensional poverty.

1.3.3.2 Measures of multidimensional poverty

In the literature, several multidimensional poverty measures have been proposed. None of them satisfies all the axioms previously discussed. Given this, in order to choose a particular measure, it is necessary to assess which of the properties are more relevant when it comes to measuring multidimensional poverty. Here we review those multidimensional poverty indices that have been considered as the most relevant ones.[7] We also highlight some of the axioms that are fulfilled by each of them, but we refer to Table 1.2 for a complete account of them.

Regardless of the identification method used, the **Multidimensional Headcount Ratio** is the most basic measure, defined as:

$$H_d(X; \mathbf{z}) = \frac{q_d}{n}, \tag{1.28}$$

where q_d is the number of individuals that are considered to be multidimensionally poor by means of any identification method, and n is the total number of individuals. The Multidimensional Headcount Ratio reflects the incidence of multidimensional poverty. Alkire et al. (2015) point out that this measure satisfies, among other axioms, weak dimensional monotonicity (AM9), weak monotonicity (AM7), weak transfer (AM10), and weak rearrangement (AM11.1), but the strong versions of these axioms are not fulfilled; see Table 1.2.

As we have previously pointed out, the Multidimensional Headcount Ratio can be calculated independently of the identification method used. However, most of the multidimensional poverty measures proposed in the context of an axiomatic approach are designed to be implemented after identifying the poor by means of the counting approach. We focus on these types of indices over the rest of this section.

Let the normalised gap of individual i in dimension j be defined as

$$g_{ij} = \max\left\{\frac{z_j - x_{ij}}{z_j}, 0\right\}, \tag{1.29}$$

so that if individual i is poor in dimension j ($x_{ij} < z_j$), $g_{ij} = \frac{z_j - x_{ij}}{z_j}$, whereas if he is non-poor

[7] See Maasoumi and Lugo (2008) and Chakravarty and D'Ambrosio (2013) for other measures not reviewed in this work.

$(x_{ij} \geq z_j)$, $g_{ij} = 0$. Also, let $\tilde{x}_{ij}$ be the so-called censored achievements, defined as

$$\tilde{x}_{ij} = \min\{x_{ij}, z_j\}, \tag{1.30}$$

so that $\tilde{x}_{ij} = x_{ij}$ if individual i is poor in dimension j and $\tilde{x}_{ij} = z_j$ if individual i is non-poor in dimension j.

The first multidimensional poverty index that was presented in the literature was introduced by Chakravarty et al. (1998), and can be expressed as:

$$P_{CMR}(X; \mathbf{z}) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d} w_j f(g_{ij}), \tag{1.31}$$

where $w_j > 0$ is the weight of dimension j, and f is a function that takes higher values the higher is the level of deprivation. When an individual is not deprived in dimension j, then $g_{ij} = 0$ and $f(0) = 0$. This measure satisfies strong monotonicity (AM8), strong dimensional monotonicity, weak transfer (AM10), and weak deprivation rearrangement (AM12.1), among other axioms; see Table 1.2.

Tsui (2002) proposed a measure that is a **multidimensional generalization of the Watts Index** discussed in Section 1.2.2. Following Alkire et al. (2015), and given $\alpha_j > 0$, this measure can be expressed as:

$$P_T(X; \mathbf{z}) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d} \alpha_j \ln\left(\frac{z_j}{\tilde{x}_{ij}}\right). \tag{1.32}$$

In this case, $\sum_{j=1}^{d} \alpha_j$ does not necessarily sum up to one, but $\frac{\alpha_j}{\sum_{j=1}^{d} \alpha_j}$ can be seen as the relative weight of each dimension. This index satisfies the same properties as the P_{CMR} index in (1.31).

Bourguignon and Chakravarty (2003) propose a **multidimensional extension of the FGT measures** of poverty reviewed in Section 1.2.2, which is based on the normalised gaps and can be expressed as:[8]

$$P_{BC1}(X;\mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d} w_j g_{ij}^{\theta_j}, \tag{1.33}$$

with $\theta_j > 1$ and where w_j is the weight given to dimension j and g_{ij} is the normalised gap defined in (1.29). The family of indices in (1.33) satisfies the same properties as the P_{CMR} index in (1.31); see Alkire et al. (2015) and Aaberge and Brandolini (2015).

Another proposal of Bourguignon and Chakravarty (2003), also based on the normalised gap in (1.29), is a family of multidimensional poverty measures of the following form:

$$P_{BC2}(X;\mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{d} w_j g_{ij}^{\beta}\right]^{\frac{\alpha}{\beta}}, \tag{1.34}$$

where $\beta > 1$ and $\alpha \geq 0$. These parameters β and α determine the type of relationship between dimensions. As Alkire et al. (2015) point out, when dimensions are considered substitutes, $\alpha > \beta$ and P_{BC2} satisfies weak rearrangement. On the other hand, when complementarity between dimensions is assumed, $\alpha < \beta$ and P_{BC2} satisfies converse weak rearrangement.

In their influential contribution where they proposed the dual cut-off counting approach to multidimensional poverty measurement, Alkire and Foster (2011a) also presented a family of measures that can be seen as an extension of the FGT family of indices to the multidimensional case. The general form of the **AF family of multidimensional poverty measures** is the following:

$$M_\alpha(X;\mathbf{z}) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{d} g_{ij}^{\alpha} I_i}{nd}, \text{ for } \alpha \geq 0, \tag{1.35}$$

where $I_i = I_{\{c_i \geq k\}}$ is the indicator function defined in (1.27), c_i is the deprivation score defined in (1.26) and α is a parameter that indicates the degree of aversion to inequality among the poor. All the measures of the AF family in (1.35) satisfy subgroup decomposability, subgroup consistency, dimensional breakdown, dimensional monotonicity, symmetry, replication and scale

[8]See Lasso de la Vega and Urrutia (2011) for an axiomatic characterization of this family of measures.

invariance, poverty and deprivation focus and weak deprivation rearrangement; see Table 1.2.

Some particular measures of the AF family are worth mentioning. In particular, when $\alpha = 0$ in expression (1.35), we have the **Adjusted Headcount Ratio**, defined as:

$$M_0 = \frac{\sum_{i=1}^{n} c_i I_i}{nd}. \tag{1.36}$$

This index can also be expressed as $H_d \times A$, where H_d is the multidimensional headcount ratio in (1.28) and A is the average deprivation share across the poor, defined by Alkire and Foster (2011a) as

$$A = \frac{\sum_{i=1}^{n} c_i I_i}{d \sum_{i=1}^{n} I_i},$$

where $\sum_{i=1}^{n} I_i = q_d$. The introduction of the average deprivation share among the poor (A) allows the Adjusted Headcount Ratio (M_0) to increase when a poor individual becomes deprived in an additional dimension, satisfying dimensional monotonicity. However, M_0 does not satisfy strong monotonicity. Finally, it should be pointed out that M_0 has an important advantage, as it can be calculated by using ordinal indicators. This is not possible with any other measure of the AF family with $\alpha > 0$.

When $\alpha = 1$ in (1.35), the **Adjusted Poverty Gap Ratio** comes up, defined as:

$$M_1 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij} I_i}{nd}. \tag{1.37}$$

This index can also be expressed as $M_0 \times G$, where G is the average poverty gap, namely:

$$G = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij} I_i}{\sum_{i=1}^{n} c_i I_i}.$$

When a poor individual becomes more deprived in a dimension, then, the sum of normalised poverty gaps increases, G rises and the Adjusted Poverty Gap (M_1) increases, reflecting a greater intensity of poverty. Therefore, M_1 satisfies strong monotonicity. It also satisfies weak transfer.

When $\alpha = 2$ in (1.35), the **Adjusted Squared Poverty Gap** comes up, which is defined as:

$$M_2 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^2 I_i}{nd}. \tag{1.38}$$

This index can also be expressed as $M_0 \times S$, where S is the average severity, defined by Alkire and Foster (2011a) as:

$$S = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d} g_{ij}^2 I_i}{\sum_{i=1}^{n} c_i I_i}.$$

M_2 satisfies strong monotonicity and weak transfer, among other axioms; see Table 1.2.

The AF measures of multidimensional poverty discussed above are very popular in applied research; see, for example, the empirical applications in Batana (2013) for Sub-Saharan Africa, Santos and Villatoro (2016) for Latin America, or Alkire and Seth (2015) for India, among others. Furthermore, the MPI (Multidimensional Poverty Index) calculated every year by the United Nations Development Programme is a version of the Adjusted Headcount Ratio (M_0). However, and despite its popularity, AF measures are not exempt of critics. In particular, Duclos and Tiberti (2016) point out that AF measures can increase after rearrangements of attributes across individuals that decrease the incidence of multiple deprivation without altering the marginal distributions of attributes. Thus, the AF family of measures does not satisfy what Duclos and Tiberti (2016) define as “sensitivity to multiple deprivation”.

Finally, we want to point out that most of the measures reviewed so far have been designed for handling cardinal indicators. For a proposal of a multidimensional poverty measure applicable to ordinal data, see Bossert et al. (2013).

	H_d	P_{CMR}	P_T	P_{BC1}	$P_{BC2}^{(\alpha<\beta)}$	$P_{BC2}^{(\alpha>\beta)}$	M_0	M_1	M_2
Symmetry	✓	✓	✓	✓	✓	✓	✓	✓	✓
Pop. Invariance	✓	✓	✓	✓	✓	✓	✓	✓	✓
Scale Invariance	✓	✓	✓	✓	✓	✓	✓	✓	✓
Continuity	✗	✓	✓	✓	✓	✓	✗	✓[2]	✓[3]
Poverty Focus	✓	✓	✓	✓	✓	✓	✓	✓	✓
Deprivation Focus	✓[1]	✓	✓	✓	✓	✓	✓	✓	✓
Weak Monotonicity	✓	✓	✓	✓	✓	✓	✓	✓	✓
Strong Monotonicity	✗	✓	✓	✓	✓	✓	✗	✓	✓
Weak Dim. Monotonicity	✓	✓	✓	✓	✓	✓	✓	✓	✓
Strong Dim. Monotonicity	✗	✓	✓	✓	✓	✓	✓	✓	✓
Weak Transfer	✓	✓	✓	✓	✓	✓	✗	✓	✓
Strong Transfer	✗	✗	✗	✗	✗	✗	✗	✗	✗
Weak Rearr.	✓	✓	✓	✓	✗	✓	✓	✓	✓
Strong Rearr.	✗	✗	✗	✗	✗	✗	✗	✗	✗
Conv. Weak Rearr.	✗	✗	✗	✗	✓	✗	✗	✗	✗
Conv. Strong Rearr.	✗	✗	✗	✗	✗	✗	✗	✗	✗
Weak Depriv. Rearr.	✓	✓	✓	✓	✗	✓	✓	✓	✓
Strong Depriv. Rearr.	✗	✗	✗	✗	✗	?	✗	✗	✗
Conv. Weak Depriv. Rearr.	✗	✗	✗	✗	✓	✗	✗	✗	✗
Conv. Strong Depriv. Rearr.	✗	✗	✗	✗	?	✗	✗	✗	✗
Subgroup Consistency	✓	✓	✓	✓	✓	✓	✓	✓	✓
Subgroup Decomposability	✓	✓	✓	✓	✓	✓	✓	✓	✓
Dimensional Breakdown	✗	✓	✓	✓	✗	✗	✓	✓	✓

✓: The axiom is fulfilled.
✗: The axiom is not fulfilled.
?: Not characterised. Further research is needed
[1]If the union criterion is not used
[2]If the union criterion is used
[3]If the union criterion is used

Table 1.2: Multidimensional poverty measures and axioms

1.3.4 Multivariate stochastic dominance and multidimensional poverty

As in the unidimensional case, the literature of multidimensional poverty has mainly been focused on the construction of multidimensional poverty indices (see previous section) that are informative of the level of deprivation that exists in a given society. However, the choice of different measures of multidimensional poverty may lead to different results, thus generating doubts about the robustness of the analysis. Moreover, these indices, and especially those most widely used in empirical applications, are not sensitive enough to the degree of multivariate dependence between the different dimensions of poverty, which is the key aspect of the multidimensional poverty analysis. To overcome these drawbacks, multivariate stochastic dominance techniques have been proposed. This approach allows to obtain robust multidimensional poverty orderings. Furthermore, as it is based on the joint distribution of the achievements, this method also allows to capture the possible relationship between the dimensions of poverty.

Over the next lines, the use of multivariate stochastic dominance techniques in the poverty literature will be discussed.[9] First, we will introduce the concept of multivariate stochastic dominance, reviewing later its application for obtaining robust multidimensional poverty orderings. As we will see, the type of relationship between the different dimensions of poverty is relevant and plays an important role on the determination of the relationship between stochastic dominance conditions and poverty orderings.

[9]Although we focus on multidimensional poverty orderings based on the influential contribution made by Atkinson and Bourguignon (1982), other proposals can be found in the literature. Alkire and Foster (2011a) propose dominance tests based on the possible values of the poverty cut-off (k) to check the robustness of multidimensional poverty orderings based on their family of measures. Lasso de la Vega (2010) also proposes dominance conditions for analysing the robustness of poverty orderings to changes in the identification cut-off in the counting framework. Yalonetzky (2014) also proposes the use of stochastic dominance techniques to obtain conditions for robust multidimensional poverty orderings based on popular counting measures.

1.3.4.1 The concept of multivariate stochastic dominance

Following the notation introduced by O'Brien and Scarsini (1991), let $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ be a random d-dimensional vector and $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ a real vector in $\mathbb{R}^d_+$.[10] Also, let $L(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^d_+ \text{ such that } \mathbf{y} \leq \mathbf{x}\}$, where $\mathbf{y} \leq \mathbf{x}$ denotes component-wise inequalities, namely $y_i \leq x_i \ \forall i$. Additionally, let F be the d-dimensional joint cumulative distribution function of $\mathbf{X}$ and let $\succeq^M_\alpha$ denote multidimensional stochastic dominance of order α.

The functions defined in (1.18) can be extended to the multidimensional case as:

$$D^1(\mathbf{x}) = F(\mathbf{x}), \quad D^\alpha(\mathbf{x}) = \int_{L(\mathbf{x})} D^{\alpha-1}(\mathbf{t}) \ d\mathbf{t}, \text{ for } \alpha = 2, 3, \ldots \tag{1.39}$$

Following O'Brien and Scarsini (1991), given two d-dimensional joint cumulative distribution functions F and G, the condition for multidimensional stochastic dominance at order α of G over F is written as:

$$G \succeq^M_\alpha F \ \Leftrightarrow \ D^\alpha_G(\mathbf{x}) \leq D^\alpha_F(\mathbf{x}) \ \forall \mathbf{x} \ \in \ \mathbb{R}^d_+. \tag{1.40}$$

To provide the fundamental results of multidimensional stochastic dominance, and for simplification purposes, here onwards we limit our attention to the bidimensional case ($d = 2$).

From (1.39) and (1.40), the condition of first-order dominance ($\alpha = 1$) comes up, namely:

$$G \succeq^M_1 F \ \Leftrightarrow \ G(x_1, x_2) \leq F(x_1, x_2) \ \forall (x_1, x_2) \in \mathbb{R}^2_+. \tag{1.41}$$

That is, there is stochastic dominance of first order of G over F if G is never above F and it is below for some values. Noticeably, if $G \succeq^M_1 F$, then there is also dominance in both univariate

[10] As in the unidimensional case, we have assumed for concreteness that the variables of interest can take any non-negative real value as this is the domain relevant for the measurement of poverty, but it is possible to extended the domain to include any real number.

marginal distributions of F and G, that is $G_1 \succeq_1 F_1$ and $G_2 \succeq_1 F_2$, since:

$$G_1(x_1) = \lim_{x_2 \to +\infty} G(x_1, x_2) \leq \lim_{x_2 \to +\infty} F(x_1, x_2) = F_1(x_1) \ \forall x_1, \tag{1.42}$$

$$G_2(x_2) = \lim_{x_1 \to +\infty} G(x_1, x_2) \leq \lim_{x_1 \to +\infty} F(x_1, x_2) = F_2(x_2) \ \forall x_2. \tag{1.43}$$

As in the unidimensional case, when no conclusions can be reached analysing first-order dominance, it is necessary to resort to second-order stochastic dominance. In this case, taking $\alpha = 2$ in (1.39) and (1.40), it turns out that

$$G \succeq_2^M F \Leftrightarrow \int_{s_1=0}^{x_1} \int_{s_2=0}^{x_2} G(s_1, s_2) ds_1 ds_2 \leq \int_{s_1=0}^{x_1} \int_{s_2=0}^{x_2} F(s_1, s_2) ds_1 ds_2 \quad \forall (x_1, x_2) \in \mathbb{R}_+^2. \tag{1.44}$$

That is, G second-order stochastically dominates F if and only if the volume under G is not greater than that under F. Conditions for higher dimensions ($d > 2$) and higher orders ($\alpha > 2$) can be worked out in a similar fashion.

1.3.4.2 Application to multidimensional poverty measurement

This subsection is devoted to the application of these stochastic dominance techniques to establish poverty orderings. Only the bidimensional case is addressed, as this is the case considered by the vast majority of applications that can be found in the literature. Special attention will be paid to the role that the type of relationship between dimensions plays in establishing such orderings.

To begin with, it is important to highlight that bidimensional poverty analyses focus on the lowest part of the joint distributions and hence the domain of interest is that including the values of X_1 and X_2 up to the poverty lines for these variables. Let $\mathbf{z} = (z_1, z_2)$ be the vector of these poverty lines and let P_F be an additive multidimensional poverty measure given by:

$$P_F(\mathbf{z}) = \int_{x_1=0}^{z_1} \int_{x_2=0}^{z_2} p(x_1, x_2; \mathbf{z}) dF(x_1, x_2), \tag{1.45}$$

where $p(x_1, x_2; \mathbf{z})$ is an individual poverty function that is differentiable up to the required degree and takes the value 0 if the individual is not poor and a positive value when the individual is considered poor.

Once the poverty measure in (1.45) is defined, poverty orderings between F and G are established by computing the difference in the level of poverty of the two distributions as follows:

$$P_G(\mathbf{z}) - P_F(\mathbf{z}) = \int_{x_1=0}^{z_1} \int_{x_2=0}^{z_2} p(x_1, x_2; \mathbf{z}) dG(x_1, x_2) - \int_{x_1=0}^{z_1} \int_{x_2=0}^{z_2} p(x_1, x_2; \mathbf{z}) dF(x_1, x_2). \tag{1.46}$$

Integrating by parts above, the following equation is obtained:

$$P_G(\mathbf{z}) - P_F(\mathbf{z}) = -\int_{x_1=0}^{z_1} p_1(x_1, z_2; \mathbf{z})[G_1(x_1) - F_1(x_1)]dx_1 - \int_{x_2=0}^{z_2} p_2(z_1, x_2; \mathbf{z})[G_2(x_2) - F_2(x_2)]dx_2$$
$$+ \int_{x_1=0}^{z_1} \int_{x_2=0}^{z_2} p_{12}(x_1, x_2; \mathbf{z})[G(x_1, x_2) - F(x_1, x_2)]dx_2 dx_1, \tag{1.47}$$

where p_1 and p_2 denote the partial derivative of the individual poverty function p with respect to x_1 and with respect to x_2, respectively, and p_{12} denotes the cross-partial derivative of the individual poverty function.

Alternatively, it is possible to rewrite (1.47) in terms of survival functions as follows:

$$P_G(\mathbf{z}) - P_F(\mathbf{z}) = \int_{x_1=0}^{z_1} p_1(x_1, 0; \mathbf{z})[\bar{G}_1(x_1) - \bar{F}_1(x_1)]dx_1 + \int_{x_2=0}^{z_2} p_2(0, x_2; \mathbf{z})[\bar{G}_2(x_2) - \bar{F}_2(x_2)]dx_2$$
$$+ \int_{x_1=0}^{z_1} \int_{x_2=0}^{z_2} p_{12}(x_1, x_2; \mathbf{z})[\bar{G}(x_1, x_2) - \bar{F}(x_1, x_2)]dx_2 dx_1, \tag{1.48}$$

where $\bar{F}$ is defined as $\bar{F}(x_1, x_2) = Pr(X_1 > x_1, X_2 > x_2)$ and $\bar{G}$ is defined similarly.

Notice that the difference $P_G(\mathbf{z}) - P_F(\mathbf{z})$ in expressions (1.47) and (1.48) depends on three terms. The first two terms involve the univariate marginal distributions and thus they are based on the marginal behaviour of the variables, as in a unidimensional approach to poverty measurement. But there is also a third term, based on the joint distribution of the variables, which is the cornerstone when it comes to measuring poverty from a bidimensional perspective. Actually, the sign of cross-partial derivative $p_{12}(x_1, x_2; \mathbf{z})$ determines the effect of the type of relationship between the dimensions on the level of poverty; see Bourguignon and Chakravarty (2003). In fact, this has to do with the concepts of substitutability and complementarity in the ALEP sense to be explained below; recall also the discussion in page 45.

The ALEP relationships were originally defined in terms of the properties of a utility function $u(x_1, x_2)$ which is differentiable to the required degree; see Kannai (1980). In particular, it is said that two dimensions are ALEP substitutes if $u_{12} \leq 0$, where u_{12} is the cross-derivative of $u(x_1, x_2)$. Intuitively, with this property the increase in utility that follows an increment in one dimension, decreases with the value of the other. In our setting, this condition leads to the following condition $p_{12}(x_1, x_2; \mathbf{z}) \geq 0$; see Atkinson (2003). That is, the magnitude of the decrease in poverty that follows an increment in one dimension decreases with the level of the other dimension. For example, an increase in the level of education generates more utility for a poor person (in terms of income) than for an individual whose income is higher.

On the other hand, two dimensions x_1 and x_2 are ALEP complements if $u_{12} \geq 0$. Intuitively, with ALEP complementarity the increase in utility that follows an increment in one dimension increases with the value of the other. Notice that, in our setting, a poverty function satisfying ALEP complementarity requires $p_{12}(x_1, x_2; \mathbf{z}) \leq 0$; see Atkinson (2003). That is, the magnitude of the decrease in poverty that follows an increment in one dimension increases with the level of the other dimension. For example, education can be thought to complement nutritional status because well-nourished children take more advantage of the education opportunities than badly nourished children.

Finally, it is also possible to consider that the dimensions are ALEP neutral. In this case, we have $u_{12} = 0$, which means that the increase in utility that follows an increment in one dimension does not depend on the level of the other dimension. In our setting, a poverty function satisfying neutrality requires $p_{12}(x_1, x_2; \mathbf{z}) = 0$. This would mean that the decrease in poverty that is consequence of an increment in one dimension does not depend on the level of the other dimension. Noticeably, in this case, the difference $P_G(\mathbf{z}) - P_F(\mathbf{z})$ in expressions (1.47) and (1.48) would only depend on the behaviour of the marginal distributions.

Next, we will review the relationship between stochastic dominance conditions and poverty orderings in the bidimensional case focusing on first-order dominance and second-order dominance, the cases most frequently used in the literature.

FIRST-ORDER DOMINANCE

The relationship between first-order stochastic dominance conditions and multidimensional poverty orderings can be established from equations (1.47) and (1.48). As we will see, depending on which one we use, different conclusions will come out, which are related to whether the dimensions are considered as complementary, substitutives or neutral.

First, considering both equations (1.41) and (1.47), the following condition holds:

$$G \succeq_1^M F \;\;\Leftrightarrow\;\; P_G(\mathbf{z}) \leq P_F(\mathbf{z}) \;\; \forall p \;\mid\; p_1(x_1, x_2; \mathbf{z}) \leq 0,\;\; p_2(x_1, x_2; \mathbf{z}) \leq 0,\;\; p_{12}(x_1, x_2; \mathbf{z}) \geq 0 \;\; \forall \mathbf{z} \in \mathbb{R}^2_+. \tag{1.49}$$

Condition (1.49) states that if $G \succeq_1^M F$, then the level of poverty in G is not higher than that in F for all bidimensional poverty measures that are decreasing in both arguments ($p_1(x_1, x_2; \mathbf{z}) \leq 0$, $p_2(x_1, x_2; \mathbf{z}) \leq 0$) and assume ALEP substitutability between dimensions ($p_{12}(x_1, x_2; \mathbf{z}) \geq 0$) for all vectors of poverty lines $\mathbf{z}$.[11] The latter property is linked to the axiom of weak rearrangement (AM11.1); see section 1.3.3.

Noticeably, taking into account that $F(x_1, x_2) = \Pr_F(X_1 \leq x_1, X_2 \leq x_2)$, condition (1.49) has

[11] In practice, and similarly to the unidimensional case, when analysing multidimensional poverty, not all values of the poverty line vector $\mathbf{z}$, are relevant and restricted stochastic dominance analysis can be applied, determining whether there is dominance of one distribution over another for a certain range of poverty lines.

an interpretation in terms of the probability of the intersection and it is thus linked to the intersection approach in multidimensional poverty measurement. Recall that in the bivariate case, this approach considers an individual as poor if he is simultaneously deprived in both dimensions. Therefore, $G \succeq_1^M F$ states that, regardless of the poverty lines, the probability of being poor according to the intersection approach is lower in distribution G than in distribution F. Hence, condition (1.49) states that when this happens, then bidimensional poverty in G is lower than in F according to all additive bidimensional poverty measures that are decreasing and whose arguments are ALEP substitutes. For example, if X_1 is income and X_2 is the level of education, when the probability of being simultaneously poor in income and education is lower in G than in F, then multidimensional poverty in G is lower than in F. This poverty ordering will be respected by all additive multidimensional poverty measures that are decreasing and whose arguments are ALEP substitutes. In particular, this class of indices includes the P_{CMR} measure of Chakravarty et al. (1998), the P_T index proposed by Tsui (2002), the AF family of multidimensional poverty measures by Alkire and Foster (2011a) and the measures P_{BC1} and $P_{BC2}^{(\alpha>\beta)}$ in Bourguignon and Chakravarty (2003); see also the discussion in Chakravarty (2018, ch.3).

On the other hand, from (1.48) a poverty ordering based on the survival functions can be established, namely:

$$\begin{aligned} &\bar{G}(x_1,x_2) \geq \bar{F}(x_1,x_2) \;\; \forall (x_1,x_2) \in \mathbb{R}_+^2 \;\; \Leftrightarrow \\ &\quad \Leftrightarrow \;\; P_G(\mathbf{z}) \leq P_F(\mathbf{z}) \;\; \forall p \;\mid\; p_1(x_1,x_2;\mathbf{z}) \leq 0, \;\; p_2(x_1,x_2;\mathbf{z}) \leq 0, \;\; p_{12}(x_1,x_2;\mathbf{z}) \leq 0 \;\; \forall \mathbf{z} \in \mathbb{R}_+^2 \end{aligned} \tag{1.50}$$

Condition (1.50) states that, if the joint survival function $\bar{G}$ lies entirely, or partly, above $\bar{F}$ for all (x_1, x_2), then $P_G \leq P_F$ for all poverty lines according to all additive multidimensional poverty measures that are decreasing in both arguments ($p_1(x_1, x_2; \mathbf{z}) \leq 0$, $p_2(x_1, x_2; \mathbf{z}) \leq 0$) and assume ALEP complementarity between dimensions ($p_{12}(x_1, x_2; \mathbf{z}) \leq 0$). The latter property is linked to the axiom of converse weak rearrangement (AM11.3); see Section 1.3.3.

Moreover, since the survival function can be expressed in terms of the probability of the union as $\bar{F}(x_1, x_2) = 1 - \mathrm{Pr}_F(\{X_1 \leq x_1\} \cup \{X_2 \leq x_2\})$, condition (1.50) is related to the union approach in multidimensional poverty measurement. According to this approach, an individual is considered as poor in a bidimensional setting if he is deprived in at least one of the two dimensions. In particular, since

$$\bar{G}(x_1, x_2) \geq \bar{F}(x_1, x_2) \Leftrightarrow Pr_G(\{X_1 \leq x_1\} \cup \{X_2 \leq x_2\}) \leq Pr_F(\{X_1 \leq x_1\} \cup \{X_2 \leq x_2\}),$$

condition (1.50) states that, regardless of the poverty line, if the probability of being poor according to the union approach is lower in G than in F, then $P_G \leq P_F$ according to all additive bidimensional poverty measures that are decreasing in both arguments and assume ALEP complementarity between dimensions, for all vectors of poverty lines $\mathbf{z}$. For instance, if variable X_1 is the level of education and X_2 is the nutritional status, when the probability of being poor in education or being poor in nutrition is lower in G than in F, multidimensional poverty in G is lower than in F.

Additionally, Atkinson and Bourguignon (1982) show that:

$$\begin{aligned} &G(x_1, x_2) \leq F(x_1, x_2) \;\; \forall (x_1, x_2) \in \mathbb{R}_+^2 \text{ and } \bar{G}(x_1, x_2) \geq \bar{F}(x_1, x_2) \;\; \forall (x_1, x_2) \in \mathbb{R}_+^2 \;\; \Leftrightarrow \\ &\Leftrightarrow \;\; P_G(\mathbf{z}) \leq P_F(\mathbf{z}) \;\; \forall p \;\mid\; p_1(x_1, x_2; \mathbf{z}) \leq 0, \;\; p_2(x_1, x_2; \mathbf{z}) \leq 0, \;\; \forall \mathbf{z} \in \mathbb{R}_+^2 \end{aligned} \quad (1.51)$$

That is, according to condition (1.51), poverty in G is not higher than that in F for all bidimensional poverty measures that are decreasing in both arguments, for all vectors of poverty lines $\mathbf{z}$, if and only if there exists bivariate stochastic dominance of G over F both in the joint CDF and in the joint survival function.

Finally, from equation (1.47) we also have the following result:

$$G_1(x_1) \leq F_1(x_1) \ \forall x_1 \in \mathbb{R}_+ \text{ and } G_2(x_2) \leq F_2(x_2) \ \forall x_2 \in \mathbb{R}_+ \ \Leftrightarrow$$
$$\Leftrightarrow \ P_G(\mathbf{z}) \leq P_F(\mathbf{z}) \ \forall p \mid p_1(x_1, x_2; \mathbf{z}) \leq 0, \ p_2(x_1, x_2; \mathbf{z}) \leq 0, \ p_{12}(x_1, x_2; \mathbf{z}) = 0 \ \forall \mathbf{z} \in \mathbb{R}^2_+$$

This means that, if both marginal distribution functions of G lie below those of F, then poverty is lower in distribution G than in distribution F for all additive multidimensional poverty measures that are decreasing in both arguments ($p_1(x_1, x_2; \mathbf{z}) \leq 0$, $p_2(x_1, x_2; \mathbf{z}) \leq 0$) and assume ALEP neutrality between dimensions ($p_{12}(x_1, x_2; \mathbf{z}) = 0$). Hence, in this case the poverty ordering only depends on the marginal distributions. In other words, when the probability of being poor in dimension X_1 and the probability of being poor in dimension X_2 are both lower in G than in F for all possible poverty lines, then poverty in distribution G is lower than poverty in distribution F. Moreover, this ordering is respected by all additive multidimensional poverty measures that are decreasing in their arguments and assume ALEP neutrality.

At this point, an important difference between unidimensional and multidimensional stochastic dominance must be introduced. In the unidimensional case, there was an equivalence between first-order dominance definition in terms of cumulative distribution functions and in terms of survival functions, since $G(x) \leq F(x) \Leftrightarrow \bar{G}(x) \geq \bar{F}(x)$; see equation (1.16). However, this equivalence is no longer true in the multidimensional case. In particular, since $\bar{F}(x_1, x_2) = 1 + F(x_1, x_2) - F_1(x_1) - F_2(x_2)$ and the same relationship holds for G, it turns out that:

$$G(x_1, x_2) \leq F(x_1, x_2) \Leftrightarrow \bar{G}(x_1, x_2) + G_1(x_1) + G_2(x_2) \leq \bar{F}(x_1, x_2) + F_1(x_1) + F_2(x_2). \quad (1.52)$$

Therefore, if we deal with two bivariate distributions F and G , the conditions $G(x_1, x_2) \leq F(x_1, x_2)$ and $\bar{G}(x_1, x_2) \geq \bar{F}(x_1, x_2)$ can hardly hold simultaneously, as they do hold in the univariate case. In particular, if we have two distinct joint distributions ($G(x_1, x_2) \neq F(x_1, x_2)$ for some (x_1,x_2)) with identical marginals ($F_1 = G_1$ and $F_2 = G_2$), conditions (1.49) and (1.50) cannot be satisfied simultaneously, and therefore condition (1.51) cannot hold.

As it happened in the unidimensional case, first-order dominance results in the bidimensional case impose strong requirements on the distributions, making them difficult to check in empirical applications and thus not leading, in some cases, to robust poverty orderings; see Atkinson and Bourguignon (1982). Hence, it is necessary to resort to second-order dominance, imposing further conditions on the function $p(x_1, x_2; \mathbf{z})$.

SECOND-ORDER DOMINANCE

Atkinson and Bourguignon (1982) proved the relationship between second-order dominance conditions and bidimensional welfare orderings. Based on this result, it is possible to work out the relationship between second-order conditions and bidimensional poverty orderings. This requires the assumption of further conditions about the poverty measure, which will limit the number of poverty measures that will respect the orderings.

At this point, some further notation should be introduced. Let $p_{ij}(x_1, x_2; \mathbf{z}) = \frac{\partial^2 p(x_1,x_2;\mathbf{z})}{\partial x_i \partial x_j}$ with $i = 1, 2$ and $j = 1, 2$ and $p_{ijk}(x_1, x_2; \mathbf{z}) = \frac{\partial^3 p(x_1,x_2;\mathbf{z})}{\partial x_i \partial x_j \partial x_k}$ with $i = 1, 2$; $j = 1, 2$ and $k = 1, 2$, and so on. Then, for poverty functions satisfying: $p_1(x_1, x_2; \mathbf{z}) \leq 0$, $p_2(x_1, x_2; \mathbf{z}) \leq 0$, $p_{12}(x_1, x_2; \mathbf{z}) \geq 0$, $p_{11}(x_1, x_2; \mathbf{z}) \geq 0$, $p_{22}(x_1, x_2; \mathbf{z}) \geq 0$, $p_{112}(x_1, x_2; \mathbf{z}) \leq 0$, $p_{122}(x_1, x_2; \mathbf{z}) \leq 0$, and $p_{1122}(x_1, x_2; \mathbf{z}) \geq 0$, the condition $P_G(\mathbf{z}) \leq P_F(\mathbf{z})$ holds if and only if:

$$\int_{x_1=0}^{z_1} G_1(x_1)dx_1 \leq \int_{x_1=0}^{z_1} F_1(x_1)dx_1 \;\; \forall z_1 \in \mathbb{R}_+, \tag{1.53a}$$

$$\int_{x_2=0}^{z_2} G_2(x_2)dx_2 \leq \int_{x_2=0}^{z_2} F_2(x_2)dx_2 \;\; \forall z_2 \in \mathbb{R}_+, \tag{1.53b}$$

$$\int_{x_1=0}^{z_1} \int_{x_2=0}^{z_2} G(x_1, x_2)dx_2dx_1 \leq \int_{x_1=0}^{z_1} \int_{x_2=0}^{z_2} F(x_1, x_2)dx_2dx_1 \;\; \forall \mathbf{z} \in \mathbb{R}^2_+. \tag{1.53c}$$

According to the previous conditions, $P_G \leq P_F$ if and only if distribution G second-order stochastically dominates distribution F both in the joint CDF and in the marginal distributions.

On the other hand, for poverty functions satisfying $p_1(x_1, x_2; \mathbf{z}) \leq 0$, $p_2(x_1, x_2; \mathbf{z}) \leq 0$, $p_{12}(x_1, x_2; \mathbf{z}) \leq 0$, $p_{11}(x_1, x_2; \mathbf{z}) \geq 0$, $p_{22}(x_1, x_2; \mathbf{z}) \geq 0$, $p_{112}(x_1, x_2; \mathbf{z}) \geq 0$, $p_{122}(x_1, x_2; \mathbf{z}) \geq 0$ and $p_{1122}(x_1, x_2; \mathbf{z}) \leq 0$, the condition $P_G(\mathbf{z}) \leq P_F(\mathbf{z})$ holds if and only if conditions (1.53a) and (1.53b) are satisfied and also the following condition on the joint survival function holds:

$$\int_{x_1=0}^{z_1} \int_{x_2=0}^{z_2} \bar{G}(x_1, x_2) dx_2 dx_1 \geq \int_{x_1=0}^{z_1} \int_{x_2=0}^{z_2} \bar{F}(x_1, x_2) dx_2 dx_1 \;\; \forall \mathbf{z} \in \mathbb{R}^2_+. \tag{1.54}$$

Therefore, $P_G \leq P_F$ if and only if distribution G second-order stochastically dominates distribution F in the marginals and in the joint survival functions.

It could also be possible to establish the relationship between higher-order dominance ($\alpha > 2$) and poverty orderings by imposing more demanding conditions on the individual poverty function. However, to check that these conditions are fulfilled becomes a difficult task and so the empirical applications for higher-order dominance are scarce; see the contribution of Crawford (2005).

We have reviewed the relationship between bivariate stochastic dominance conditions and bidimensional poverty orderings. Although it is possible to extend these results to more than two dimensions, the literature on this issue is very scarce due to the problems that stochastic dominance involves in a d-dimensional setting with $d > 2$. First, stochastic dominance methods suffer from the curse of dimensionality, that is, as the number of dimensions increases so does the difficulty to compare distributions as well as the possibility to obtain conclusive results. Furthermore, the relationship between stochastic dominance and poverty orderings in the d-dimensional case with $d > 2$ relies on the sign of higher-order partial derivatives of the individual poverty function, and these signs are difficult to interpret, making the analysis cumbersome. For the conditions of stochastic dominance when we have more than two dimensions, we refer the interested reader to Crawford (2005) and, more recently, García-Gómez et al. (2019).

To conclude this section, we want to point out, as we did in the unidimensional case, that we have presented the theoretical results regarding the relationship between bivariate stochastic dominance conditions and robust poverty orderings. However, in empirical applications, it is necessary to work with sample data and thus statistical inference procedures are required. In the multivariate framework, the literature on statistical inference for stochastic dominance analyses is scarce, but there have been some proposals based on extending some of the tests proposed for the univariate case; see García-Gómez et al. (2019) and the references therein.

As we have argued in this chapter, the key aspect of the multidimensional poverty analysis is the multivariate association between the dimensions, whose study requires focusing on the joint distribution of the dimensional achievements. However, the multidimensional poverty indices, which constitute the traditional approach to measure poverty from a multidimensional perspective, are not sensitive enough to this crucial aspect. One way to account for the association between poverty dimensions is the use of the multivariate stochastic dominance techniques reviewed in this section. However, as we have pointed out, this methodology is difficult to implement empirically, especially when the number of dimensions considered is high. Moreover, although this approach allows to obtain multidimensional poverty orderings taking into account the interdependence between the dimensions, as it is based on the joint distribution, it does not allow to quantify the degree of multivariate association. In order to do so, the copula methodology has been proposed to complement the analysis based on multidimensional poverty indices, and this is the approach that we use in this thesis. The next chapter will be devoted to the copula approach and its application to study and quantify multivariate dependence.

Chapter 2

Copula-based measures of multivariate dependence

2.1 Introduction

As we argued in the previous chapter, the dominant approach to study poverty in a multivariate framework involves the application of multidimensional poverty indices, many of which were reviewed in Section 1.3.3.2. Actually, some international institutions have already adopted this approach. For example, the European Union (EU) calculates a multidimensional poverty index that is the main reference to monitor poverty in Europe, namely the AROPE (At Risk Of Poverty or social Exclusion) rate. Also, the United Nations Development Program (UNDP) adopted, in 2010, the Multidimensional Poverty Index (MPI), which is based on the Alkire and Foster (2011a) methodology reviewed in the previous chapter.

However, many of the multidimensional poverty indices proposed so far, like those mentioned above, are not sufficiently sensitive to the possible interrelations between the dimensions of poverty; see Duclos and Tiberti (2016). Hence, these indices could miss an important part of the picture, since, as several authors argue, a higher degree of dependence between poverty dimensions, that is, a stronger alignment of the positions of the individuals in the different dimensions, can make overall poverty worse. Consider, for instance, a society in which there is one individual who is top-ranked in all dimensions, another individual who is second-ranked in all dimensions, and so on. As Decancq (2014) points out, this society is less equitable than another society with the same marginal distributions for each dimension but with a different joint distribution in such a way that, in this case, some individuals are relatively well-ranked on some dimensions and other individuals on others. Therefore, any sound multidimensional poverty analysis must take into consideration this crucial aspect; see Atkinson and Bourguignon (1982), Bourguignon and Chakravarty (2003), Duclos et al. (2006), Seth (2013), Ferreira and Lugo (2013) and Chakravarty (2018, ch.1). Nonetheless, and despite its relevance, the problem of measuring the dependence between the dimensions of poverty has been scarcely addressed in the literature. A possible solution to deal with this issue is the use of the multivariate stochastic dominance techniques reviewed in Section 1.3.4, since they are based on the joint distribution, which, as Chakravarty (2018, ch.1) points out, is the key factor of multidimensional evaluation. However, as we have seen in the previous chapter, this approach is difficult to implement when many dimensions are involved and, in addition, it does not allow to quantify the degree of multivariate dependence.

The main goal of this thesis is precisely to quantify the multivariate dependence between the dimensions of poverty. In order to do so, several issues must be taken into consideration. First, the measurement of multivariate dependence, that is, the dependence between more than two variables, is challenging and requires special care, since some bivariate dependence properties are not preserved in higher dimensions; see Durante et al. (2014). Furthermore, in general, information about the pairwise dependence is not sufficient to infer global dependence. Second, some of the poverty and welfare dimensions, such as income, for instance, usually do not follow Gaussian distributions; see Kleiber and Kotz (2003) and the references therein. And third, there can be non-linear relationships between the different dimensions of poverty or welfare.

Hence, we face the problem of measuring dependence in a multivariate, possibly non-Gaussian and possibly non-linear context.

In this setting, it is required a methodology that goes beyond the well-known Pearson's linear correlation coefficient, since this measure is only appropriate for measuring bivariate linear relationships in the context of elliptical distributions; see Embrechts et al. (2001). The copula methodology constitutes a suitable alternative. This approach focuses on the positions of the individuals across dimensions, rather than on the specific values that those dimensions attain for such individuals, and has several advantages when it comes to measuring the multivariate association between poverty dimensions. First, it enables the decomposition of the joint distribution of the dimensions into the marginals and the dependence structure. Second, the copula methodology allows to study well-known scale-free measures of bivariate association that capture other types of dependence beyond linear correlation. And third, this approach also permits the construction of multivariate generalisations of these bivariate measures. For these reasons, in this study we propose the use of copula-based methods to measure the multivariate dependence between dimensions of poverty.

This chapter is devoted to the copula approach and its application to study multivariate dependence. We begin reviewing, in Section 2.2, the concept of copula and its main properties. After that, the attention will be focused on the usefulness of this methodology to measure dependence in multivariate contexts. In particular, we will pay attention to those concepts of multivariate dependence that are especially valuable to analyse multidimensional poverty, where it is essential to measure to what extent the positions of the individuals in the different dimensions of poverty are aligned, that is, to measure how likely it is that an individual is simultaneously low-ranked in all poverty dimensions. Thus, in Section 2.3, we will introduce two copula-based concepts of multivariate dependence, namely multivariate concordance and orthant dependence, which are particularly useful in this context. Then, in Section 2.4, we will discuss several copula-based measures of multivariate association which stem from those dependence concepts. In particular, we will review several multivariate extensions of the well-known Spearman's rank correlation coefficient, which measures "departure" of our data from independence. Finally, Section 2.5 will be devoted to another relevant concept of dependence, namely tail dependence, and the different proposals to measure it both in the bivariate and in the multivariate case. The concept of lower tail dependence, which relates to the degree of dependence in the joint lower tail of a multivariate distribution, is highly valuable in our multidimensional poverty context, as it captures the probability that an individual who is extremely poor in one dimension is also extremely poor in the rest of the dimensions. Despite its theoretical appeal when it comes to analysing poverty from a multidimensional point of view, the concept of multivariate tail dependence has never been applied, to the best of our knowledge, in this field. Indeed, this thesis provides a pioneering contribution by proposing, for the first time, the multivariate tail concentration function (TCF), a graphical tool that allows to analyse the degree of multivariate dependence between the dimensions of poverty in the tails of the joint distribution and which allows, at the same time, to represent such dependence in a unit square, regardless of the poverty dimensions considered.

2.2 A review of copulas: concept and main properties

In this section, the fundamentals of copulas are reviewed. However, the objective is not to provide an exhaustive discussion of copula theory, which can be found in other works, such as those of Nelsen (2006), Joe (2014) and Durante and Sempi (2015). The scope of this section is just to introduce the essentials on which the concepts that will be handled to measure the multivariate dependence between dimensions of poverty are based.

We begin introducing the definition of a copula. A d-dimensional copula is a function from $\mathbf{I}^d = [0,1]^d$ to $\mathbf{I} = [0,1]$ which fulfils certain properties in such a way that it is a multivariate distribution function with standard uniform margins. More precisely, a d-dimensional copula

C is a multivariate distribution function $C : \mathbf{I}^d \to \mathbf{I}$ defined for every $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbf{I}^d$ as

$$C(\mathbf{u}) = p(\mathbf{U} \leq \mathbf{u}) = p(U_1 \leq u_1, \ldots, U_d \leq u_d),$$

where U_i is $U(0,1)$, for $i = 1, ..., d$. Equivalent definitions can be found in Nelsen (2006, p.45) and Durante and Sempi (2010, p. 9-10).

From an statistical point of view, the most important result of the theory of copulas is given by **Sklar's theorem** (Sklar, 1959), which establishes the relationship between copulas and distribution functions of random variables. This theorem states that, given a $d-$dimensional random vector $\mathbf{X} = (X_1, ..., X_d)$ with joint distribution function $F(\mathbf{x}) = F(x_1, ..., x_d) = p(X_1 \leq x_1, ..., X_d \leq x_d)$ and univariate marginal distribution functions $F_i(x_i) = p(X_i \leq x_i)$, for $i = 1, ..., d$, then there exists a copula $C : \mathbf{I}^d \to \mathbf{I}$ such that, for all $(x_1, \ldots, x_d) \in R^d$,

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)). \tag{2.1}$$

Conversely, if C is a $d-$copula and $F_1, ..., F_d$ are univariate distribution functions, the function F defined in (2.1) is a joint distribution function with margins $F_1, ..., F_d$. If $F_1, ..., F_d$ are all continuous, the copula C in (2.1) is unique. Otherwise, C is uniquely determined on Range $F_1 \times ... \times$ Range F_d. Unless otherwise stated, throughout the rest of this chapter we will assume that the marginal distributions $F_1, ..., F_d$ are all continuous. A detailed discussion on the pitfalls related to non-continuity of the marginal distributions can be found in Genest and Nešlehová (2007) and the references therein.

According to (2.1), copulas are functions defined in the unit d-cube, $\mathbf{I}^d$, that join or "couple" multivariate distribution functions to their one-dimensional marginal distribution functions. In fact, Sklar's theorem decomposes the joint distribution into the marginals and the copula, the latter only representing the association between the random variables $X_1, \ldots, X_d$. Actually, if there is only information about the marginal distributions it will only be possible to retrieve the joint distribution function in the case of independence between the variables (in which case the joint distribution is the product of the marginal distributions). In any other case, this is not possible, as it is necessary to know the structure of dependence between the variables, which is precisely given by the copula C.

One of the noteworthy features of the copula-based methodology is that it is based on the transformation of the original random vector $\mathbf{X} = (X_1, \ldots, X_d)$ into the vector of $U = (U_1, \ldots, U_d)$ via the so-called *probability integral transformation*, that is, $U_1 = F_1(X_1), \ldots, U_d = F_d(X_d)$. In the multidimensional poverty framework, this transformation makes it possible to focus on the positions of individuals across dimensions rather than on the values that the variables take for such individuals. In particular, if the random vector $\mathbf{X}$ represents the relevant d dimensions of poverty for a population, the transformed variables $U_i = F_i(X_i)$, with $i = 1, \ldots, d$, attach to each individual in the population its relative position in all dimensions. The position is a real number between 0 and 1 and the vector of positions $\mathbf{u} = (u_1, \ldots, u_d)$ for a given individual gives his/her positions in all poverty dimensions. For instance, an individual with position vector $(1, \ldots, 1)$ will be top-ranked in all dimensions, i.e., he/she will be the "richest" one in terms of income, health, education, etc. On the contrary an individual with position vector $(0, \ldots, 0)$ will be bottom-ranked in all dimensions, i.e., he/she will be the "poorest" one in terms of income, health, education, etc. Moreover, from probability theory, the transformed variables $U_1, \ldots, U_d$ are standard uniform random variables $U(0, 1)$ and the joint distribution of the position vector $\mathbf{U} = (U_1, ..., U_d)$ is the copula C defined above. Therefore, for a given real vector $\mathbf{u} \in \mathbf{I}^d$, the value $C(\mathbf{u})$ represents the proportion of individuals in the population with positions outranked by $\mathbf{u}$. For instance, $C(0.25, ..., 0.25)$ will represent the probability that a randomly selected individual is simultaneously in the 1^{st} quartile ("low ranked") in all dimensions, i.e., in our setting, it will be the probability that he/she is simultaneously "poor" in all dimensions. As we will see along this chapter, the copula functions, as the distribution functions of the position vector of individuals, will play a central role in measuring dependence between poverty dimensions.

Some basic examples of copulas are the following:

- The **independent copula**, defined as $\Pi(\mathbf{u}) = u_1 \times \cdots \times u_d$, which accounts for the case of independence between the variables $X_1, \ldots, X_d$. In this case, the position of an individual in one dimension does not depend on his/her position in any other dimension.

- The **commonotonic copula**, $M(\mathbf{u}) = \min(u_1, \ldots, u_d)$, which represents perfect positive dependence, i.e, the case where each of the random variables $X_1, ..., X_d$ is almost surely a strictly increasing function of any of the others (the outcomes in all dimensions are ordered in the same way). That is, a distribution function with a commonotonic copula describes a society in which one individual is top-ranked in all dimensions, another is second-ranked in all dimensions, and so forth.

Any copula C satisfies the **Fréchet-Hoeffding bounds** inequality, attributed to Hoeffding (1940) and Fréchet (1951), and given by

$$W(\mathbf{u}) \leq C(\mathbf{u}) \leq M(\mathbf{u}), \tag{2.2}$$

for every $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbf{I}^d$, where $W(\mathbf{u})$ is a function defined as $W(\mathbf{u}) = \max(u_1 + \cdots + u_d - d + 1, 0)$ and $M(\mathbf{u})$ is the commonotonic copula defined above. The function W is only a copula when $d = 2$. In that case, it represents perfect negative dependence, that is, the case in which each of X_1 and X_2 is almost surely a decreasing function of the other, and it is named countermonotonic copula.[1]

In the bivariate case, it is possible to represent the graph of each bivariate copula as a continuous surface within the unit cube $\mathbf{I}^3$. Given the Fréchet-Hoeffding bounds inequality in (2.2), the graph of any copula C lies between the graphs of W and M. Hence, the independent copula Π,

[1]Notice that, for $d > 2$, perfect negative dependence is not a clear concept.

as any other copula, fulfils inequality (2.2) and its graph will lie between those of W and M; see, for an illustration, Figure 2.1, which displays the graphs of the basic copulas W, Π and M in this bivariate case.

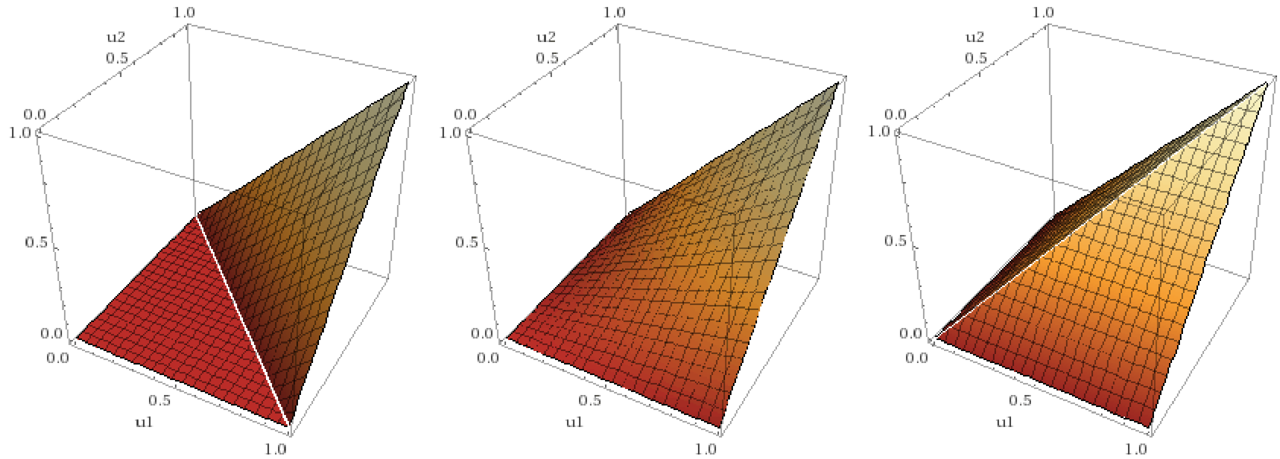

Figure 2.1: Graphs of basic copulas W (left), Π (center) and M (right) in the bivariate case.

In this setting, one intuitive possibility for measuring dependence between the variables $(X_1, \ldots, X_d)$ would be to quantify the volume between their copula, say C, and the independent copula Π, as compared to that between M and Π, so that the greater the volume, the higher the level of dependence. In fact, the measures of multivariate dependence that will be discussed in Section 2.4 do so, by measuring "how far" the distribution of the data being analysed (represented by the copula C) is from the situation of independence (represented by the independent copula Π) as compared to the "distance" between the commonotonic copula M, which represents perfect positive dependence, and the independence copula Π.

Noticeably, the volumes between the independent copula Π and M and between Π and W, respectively, for $d \geq 2$, are given by the following expressions:

$$a_d = \int_{\mathbf{I}^d} [M(\mathbf{u}) - \Pi(\mathbf{u})] d\mathbf{u} = \frac{1}{(d+1)} - \frac{1}{2^d}, \quad (2.3)$$

$$b_d = \int_{\mathbf{I}^d} [\Pi(\mathbf{u}) - W(\mathbf{u})] d\mathbf{u} = \frac{1}{2^d} - \frac{1}{(d+1)!}. \quad (2.4)$$

In the bidimensional case ($d = 2$), $a_2 = b_2 = 1/12$, that is, Π is equidistant from both M and W. However, when more than two dimensions are involved ($d > 2$) this symmetry does not hold and the independent copula Π is closer to W than to M. In fact, it can be shown that $\lim_{d\to\infty} \frac{b_d}{a_d} = 0$; see Wolff (1980).

Another important function, related to the copula C, is the survival function $\bar{C}$, which, in general, is not a copula. If $\mathbf{U} = (U_1, ..., U_d)$ is a random vector of variables $U(0, 1)$ whose joint distribution function is the copula C, the survival function of C, $\bar{C} : \mathbf{I}^d \to \mathbf{I}$, is defined as:

$$\bar{C}(\mathbf{u}) = p(\mathbf{U} > \mathbf{u}) = p(U_1 > u_1, \ldots, U_d > u_d). \tag{2.5}$$

When $U_1, ..., U_d$ are independent random variables, then their survival function is $\bar{\Pi}(\mathbf{u}) = (1-u_1) \times \cdots \times (1-u_d)$. Moreover, when there is perfect positive dependence, then the survival function is $\bar{M}(\mathbf{u}) = \min(1 - u_1, \ldots, 1 - u_d)$. In our setting, the survival function will be very important, as it accounts for the probability of being simultaneously "rich" in all dimensions. For instance, $\bar{C}(0.75, \ldots, 0.75)$ will represent the probability that a randomly selected individual is simultaneously in the 4^{th} quartile ("high ranked") in all dimensions.

Over the rest of this chapter, we will focus our attention on the role played by copulas in the measurement of multivariate association and their adequacy to capture the multivariate dependence between the dimensions of poverty. Then, in Chapter 3, we will carry out an application of these concepts to measure multivariate dependence between dimensions of poverty in Europe over the period 2008-2018.

2.3 Concepts of multivariate dependence

As we explained in Chapter 1, there is a wide consensus that poverty is a multidimensional phenomenon, involving not only the level of income, but also the attainments of individuals in other aspects such as education, health or labour status. In this multidimensional framework, measuring the dependence between the different dimensions is an important issue, since a high degree of dependence could exacerbate poverty. However, as Durante et al. (2014) point out, the study of dependence in a multivariate context is challenging, since some bivariate dependence properties are not preserved in the higher-dimensional case. Therefore, possible generalisations of the most widely used bivariate dependence concepts are not straightforward; see Malevergne and Sornette (2006, ch.4). In this section, we briefly review some multivariate dependence concepts that constitute the basis of the copula-based measures of multivariate association that will be discussed later. In particular, we review the concepts of multivariate concordance and orthant dependence, which, as we will see, have an appealing interpretation when it comes to measuring the multivariate dependence between dimensions of poverty. Another relevant concept of dependence, namely tail dependence, will be discussed in Section 2.5. This concept is particularly relevant in our setting, as it focuses on the lower and upper tails of the joint distribution, which in a multidimensional poverty framework means focusing on both the poorest and richest individuals. For an exhaustive treatment of these and other copula-based concepts of dependence, see Joe (2014, ch.2).

2.3.1 Multivariate concordance

Roughly speaking, two random variables are concordant if large (small) values of one tend to be associated with large (small) values of the other. On the other hand, two random variables are discordant if large (small) values of one tend to be associated with small (large) values of the other; see Kruskal (1958). More precisely, two observations (x_1, x_2) and (x_1', x_2') from a bidimensional random vector are concordant if ($x_1 < x_1'$ and $x_2 < x_2'$) or if ($x_1 > x_1'$ and $x_2 > x_2'$), that is, if $(x_1 - x_1')(x_2 - x_2') > 0$. On the other hand, they are discordant if ($x_1 < x_1'$ and $x_2 > x_2'$) or ($x_1 > x_1'$ and $x_2 < x_2'$), that is, if $(x_1 - x_1')(x_2 - x_2') < 0$. In a multidimensional poverty setting, measuring concordance is important because if, for instance, income and education are concordant variables, this will mean that an individual with a low level of income is likely to also be an individual with a low level of education, and this could make overall poverty worse.

Based on these concepts, Nelsen (1998) defines a concordance function Q as the difference between the probability of concordance and discordance of two vectors (X_1, X_2) and (X_1', X_2') of continuous random variables with (possibly) different joint distributions F and F' and copulas C and C' but with common marginal distributions F_1 (of X_1 and X_1') and F_2 (of X_2 and X_2'). In particular, this function Q is defined as:

$$Q = Pr[(X_1 - X_1')(X_2 - X_2') > 0] - Pr[(X_1 - X_1')(X_2 - X_2') < 0]. \tag{2.6}$$

Moreover, Nelsen (1998) shows that Q can be expressed in terms of the bivariate copulas C and C' as:

$$Q = Q(C, C') = 4 \int_{\mathbf{I}^2} C'(u_1, u_2) dC(u_1, u_2) - 1. \tag{2.7}$$

This function Q is symmetric in its arguments, that is, $Q(C, C') = Q(C', C)$, and it can be shown that it takes the following values for pairs of the copulas W, Π and M (Nelsen, 2006):

$$\begin{gathered} Q(M, M) = 1; \;\; Q(M, \Pi) = 1/3; \;\; Q(M, W) = 0; \\ Q(W, \Pi) = -1/3; \;\; Q(W, W) = -1; \;\; Q(\Pi, \Pi) = 0. \end{gathered} \tag{2.8}$$

When moving to the multivariate case, the concepts discussed above do not generalise so straightforwardly. In particular, the concept of concordance generalises as follows: two observations $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{x}' = (x_1', \ldots, x_d')$ from a d-dimensional random vector are concordant if $\mathbf{x} < \mathbf{x}'$ or $\mathbf{x} > \mathbf{x}'$, where this notation means component-wise inequality, that is, $x_i < x_i'$ or $x_i > x_i'$ for all $i = 1, \ldots, d$, respectively. However, the concept of discordance does not generalise to the multivariate case. Hence, in order to obtain a generalisation of the concordance function, Nelsen (2002) considers the probability of concordance alone, rather than the difference of the probabilities of concordance and discordance. In particular, given two independent d-dimensional continuous random vectors $\mathbf{X}$ and $\mathbf{X}'$ with common univariate margins and copulas C and C', respectively, he introduces the probability of multivariate concordance Q'_d as:

$$Q'_d = Pr(\mathbf{X} > \mathbf{X}') + Pr(\mathbf{X} < \mathbf{X}'). \tag{2.9}$$

Moreover, Nelsen (2002) shows that this probability can be written in terms of copulas as follows:

$$Q'_d = Q'_d(C, C') = \int_{\mathbf{I}^d} \big(C(\mathbf{u}) + \bar{C}(\mathbf{u})\big) dC'(\mathbf{u}), \tag{2.10}$$

where $\bar{C}$ is the survival function associated with the copula C defined in (2.5). This function Q'_d is symmmetric in its arguments, that is, $Q'_d(C, C') = Q'_d(C', C)$, and is easily evaluated for pairs of the d-copulas M and Π. In fact, it can be shown that

$$Q'_d(M, M) = 1; \;\; Q'_d(M, \Pi) = \frac{2}{d+1}; \;\; Q'_d(\Pi, \Pi) = \frac{1}{2^{d-1}}. \tag{2.11}$$

As we will see in the next section, multivariate concordance is related to another concept of dependence, namely orthant dependence.

2.3.2 Orthant dependence

Another important concept of dependence for multivariate distributions, which will be essential for defining the measures of multivariate association presented in Section 2.4, is orthant dependence, which is a multivariate generalisation of quadrant dependence.

Following Nelsen (1996), two random variables X_1 and X_2 with joint distribution function F, marginals F_1 and F_2, copula C and survival function $\bar{C}$, are positive quadrant dependent if the probability that they are simultaneously small is at least as great as it would be in the case of independence, that is, if $F(x_1, x_2) \geq F_1(x_1)F_2(x_2)$ for each $(x_1, x_2) \in \mathbb{R}^2$, or, in terms of copulas, if

$$C(u_1, u_2) \geq \Pi(u_1, u_2) \tag{2.12}$$

for each $(u_1, u_2) \in \mathbf{I}^2$. Moreover, in the bivariate case, it turns out that

$$\bar{C}(u_1, u_2) - \bar{\Pi}(u_1, u_2) = C(u_1, u_2) - \Pi(u_1, u_2). \tag{2.13}$$

Hence, condition (2.12) above is equivalent to

$$\bar{C}(u_1, u_2) \geq \bar{\Pi}(u_1, u_2). \tag{2.14}$$

Thus, two random variables X_1 and X_2 are said to be positive quadrant dependent if either (2.12) or (2.14) holds. That is, two variables are positive quadrant dependent if the probability that they are simultaneously small (or, equivalently, large) is at least as great as it would be in the case of independence. However, this convenient equivalence is limited to the bivariate setting and does not hold in the general multivariate case ($d > 2$), since the equality in (2.13) does no longer hold in higher dimensions. The corresponding negative concept, that is, negative quadrant dependence, is defined by reversing the sense of the inequalities in (2.12) and (2.14).

The concept of orthant dependence is a generalization of quadrant dependece (see Nelsen, 2006) that captures how “far” from independence the joint distribution of our data is from a downward and an upward perspective, that is, when looking from both the lower and upper parts of the joint distribution. Hence, in a multidimensional poverty setting, where the downward (upward) perspective has to do with individuals that are simultaneously low-ranked (high-ranked) in all dimensions, orthant dependence becomes a very appealing concept. To formally define this concept, let $\mathbf{X} = (X_1, ..., X_d)$ be a d-dimensional continuous random variable with joint distribution function F, marginals $F_1, ..., F_d$, copula C and survival function $\bar{C}$. Then, it is said that (Nelsen, 1996):

(1) $\mathbf{X}$ is positively *lower orthant dependent* (PLOD) if $C(\mathbf{u}) \geq \Pi(\mathbf{u})$, for each $\mathbf{u} \in \mathbf{I}^d$, that is, if the probability that the variables $X_1, ..., X_d$ are simultaneously small is at least as great as it would be in the case of independence;

(2) $\mathbf{X}$ is positively *upper orthant dependent* (PUOD) if $\bar{C}(\mathbf{u}) \geq \bar{\Pi}(\mathbf{u})$, for each $\mathbf{u} \in \mathbf{I}^d$, that is, if the probability that the variables $X_1, ..., X_d$ are simultaneously large is at least as great as it would be in the case of independence;

(3) $\mathbf{X}$ is positively *orthant dependent* (POD) if both previous inequalities hold.

The corresponding negative concepts (NLOD, NUOD and NOD) are defined by reversing the sense of the inequalities above.

In our multidimensional poverty context, the existence of positive lower orthant dependence (PLOD) would mean that the probability that the individuals are simultaneously low-ranked ("poor") in all dimensions of poverty is at least as great as it would be if the dimensions were independent. In turn, the existence of positive upper orthant dependence (PUOD) would mean that the probability that the individuals are simultaneously high-ranked ("rich") in all dimensions of poverty is at least as great as it would be if the dimensions were independent.

Notice that, in the bivariate case, and given (2.13), conditions (1), (2) and (3) above are equivalent and reduce to quadrant dependence defined above, a reduction that does not hold in more than two dimensions. This is an illustration of how a bivariate concept does not uniquely generalise to a multivariate setting and the care that should be taken when moving from the bivariate to the multivariate case.

Now, the question arises on how to move from these theoretical concepts to the construction of measures of orthant dependence that can be useful in practical applications. Notice that, geometrically, PLOD means that the graph of $C(\mathbf{u})$ lies above the graph of the independent copula $\Pi(\mathbf{u})$. Hence, following Nelsen (1996), the quantity $[C(\mathbf{u}) - \Pi(\mathbf{u})]$ measures "local" positive (or negative) lower orthant dependence. Thus, averaging this difference over all position vectors $\mathbf{u}$, a measure of "average" lower orthant dependence could be constructed. Similarly, the quantity $[\bar{C}(\mathbf{u}) - \bar{\Pi}(\mathbf{u})]$ can be thought of as a measure of "local" positive (or negative) upper orthant dependence and a measure of "average" upper orthant dependence could be constructed by averaging this difference over all position vectors $\mathbf{u}$. These measures of average orthant dependence will be introduced in the next section.

2.4 Multivariate extensions of Spearman's rho

As we said in the Introduction to this chapter, in order to address the measurement of the dependence between dimensions of poverty it is necessary to go beyond Pearson's linear correlation coefficient, since we face the problem of measuring dependence in a multivariate, possibly non-Gaussian and possibly non-linear context. In this section, we will discuss several copula-based multivariate generalisations of Spearman's rank correlation coefficient, which is one of the best-known alternatives to Pearson's linear correlation coefficient. As we will see, these generalisations are closely linked to the concepts of orthant dependence and multivariate concordance introduced in the previous section. After defining the different measures and describing their main properties, we will also discuss the proposals that can be found in the literature to estimate them in a non-parametric way. To start with, we will review the bivariate Spearman's rho and its definition and properties in terms of copulas.

2.4.1 Bivariate Spearman's rho

In the bivariate framework, one of the best-known alternatives to Pearson's linear correlation coefficient is the rank correlation coefficient, known as **Spearman's rho**. This measure is the Pearson's correlation coefficient between the ranks of two random variables X_1 and X_2, that is, the correlation coefficient between the position vector components $U_1 = F_1(X_1)$ and

$U_2 = F_2(X_2)$:

$$\rho_S = \text{Cor}(U_1, U_2) = \frac{\text{Cov}(U_1, U_2)}{\sqrt{\text{Var}(U_1)\text{Var}(U_2)}}.$$

As Embrechts et al. (2001) point out, while Pearson's correlation coefficient only measures the degree of linear dependence, ρ_S is a measure of the degree of monotonic dependence between X_1 and X_2. When the position of individuals in both variables coincide, ρ_S takes the value 1, and it takes the value -1 when the positions in both variables are the opposite. If the variables are independent, $\rho_S = 0$.

Nelsen (1991) shows that ρ_S can be expressed in terms of copulas. In particular, as the copula C is the joint distribution function of the position vector $\mathbf{U} = (U_1, U_2)$, with $U_i \sim U(0,1)$ for $i = 1, 2$, it follows that:

$$\rho_S = \frac{E(U_1U_2) - E(U_1)E(U_2)}{\sqrt{\text{Var}(U_1)\text{Var}(U_2)}} = \frac{\int_{\mathbf{I}^2} u_1u_2dC(u_1, u_2) - \frac{1}{4}}{\frac{1}{12}}$$

and thus

$$\rho_S = 12\int_{\mathbf{I}^2} u_1u_2dC(u_1, u_2) - 3. \tag{2.15}$$

Furthermore, Nelsen (1991) also shows that an equivalent expression for ρ_S is given by:

$$\rho_S = 12\int_{\mathbf{I}^2} C(u_1, u_2)du_1du_2 - 3. \tag{2.16}$$

Based on these expressions, we will explain next how ρ_S can be regarded both as a measure of quadrant dependence and a measure of concordance. To start with, taking into account that $\int_{\mathbf{I}^2} u_1u_2du_1du_2 = \frac{1}{4}$, expression (2.16) can be rewritten as:

$$\rho_S = 12\int_{\mathbf{I}^2}[C(u_1, u_2) - u_1u_2]du_1du_2. \tag{2.17}$$

Additionally, using the result in (2.13), the following equivalent expression for (2.17) turns out:

$$\rho_S = 12 \int_{\mathbf{I}^2} [\bar{C}(u_1, u_2) - \bar{\Pi}(u_1, u_2)] du_1 du_2. \tag{2.18}$$

From (2.17) and (2.18) it can be seen that, as Nelsen (1996) argues, ρ_S is a measure of average quadrant dependence. In particular, when X_1 and X_2 are positively quadrant dependent (see equation (2.12)), then $\rho_S \geq 0$. On the contrary, when X_1 and X_2 are negatively quadrant dependent, then $\rho_S \leq 0$. Moreover, in the case of perfect positive dependence of our data ($C = M$), and using equation (2.3), then $\int_{\mathbf{I}^2}[M(u_1, u_2) - \Pi(u_1, u_2)]du_1 du_2 = 1/12$ and thus $\rho_S = 1$. By contrast, when there is perfect negative dependence ($C = W$), and using equation (2.4), then $\int_{\mathbf{I}^2}[W(u_1, u_2) - \Pi(u_1, u_2)]du_1 du_2 = -1/12$ and thus $\rho_S = -1$. Finally, in the case of independence ($C = \Pi$), $\rho_S = 0$.

On the other hand, bivariate Spearman's rho (ρ_S) can also be regarded as a measure of concordance. In particular, from (2.7) and (2.15), it is straightforward to show that Spearman's rho can be written as:

$$\rho_S = 3Q(C, \Pi); \tag{2.19}$$

see Nelsen (1998, 2006). Thus, ρ_S is proportional to the difference between the probabilities of concordance and discordance for two independent pairs of random variables, one with copula C and the other being a pair of independent random variables with copula Π. That is, ρ_S allows us to compare the probabilities of concordance and discordance of our data (represented by any copula C) with respect to those of the independence case (represented by copula Π). Moreover, since $Q(M, \Pi) = \frac{1}{3}$ and $Q(W, \Pi) = -\frac{1}{3}$ (see (2.8)), it turns out that in the case of perfect positive dependence ($C = M$), $\rho_S = 1$ and in the case of perfect negative dependence ($C = W$), $\rho_S = -1$. Furthermore, in the case of independence ($C = \Pi$), $\rho_S = 0$, since $Q(\Pi, \Pi) = 0$; see (2.8). Thus, $\rho_S \in [-1, 1]$. Finally, as Nelsen (2006) points out, ρ_S is a measure of concordance according to the axiomatic definition of Scarsini (1984).

2.4.2 Multivariate generalisations

2.4.2.1 Definition and main properties

In the multivariate framework ($d \geq 3$), several extensions of Spearman's rho have been proposed and, as we will see throughout this section, they are also related to the concepts of multivariate concordance and orthant dependence introduced in Section 2.3. To begin with, we recall (see Section 2.3.2) that the differences $[C(\mathbf{u})-\Pi(\mathbf{u})]$ and $[\bar{C}(\mathbf{u})-\bar{\Pi}(\mathbf{u})]$ can be regarded as measures of "local" lower and upper orthant dependence, respectively. Hence, based on these differences, two multivariate extensions of bivariate Spearman's rho can be obtained as the "natural" extensions of (2.17) and (2.18), respectively, to the multivariate setting.

The first of these extensions, due to Wolff (1980) and Nelsen (1996), focuses on the difference $[C(\mathbf{u}) - \Pi(\mathbf{u})]$ and is the "natural" multivariate generalisation of (2.17), defined as

$$\rho_d^- = \frac{\int_{\mathbf{I}^d}[C(\mathbf{u}) - \Pi(\mathbf{u})]d\Pi(\mathbf{u})}{\int_{\mathbf{I}^d}[M(\mathbf{u}) - \Pi(\mathbf{u})]d\Pi(\mathbf{u})}. \tag{2.20}$$

Notice that, in the above expression, the denominator represents the maximum value of the numerator, i.e, its value in the case of maximal dependence ($C = M$). Thus, as Nelsen (1996) points out, ρ_d^- can be seen as a rescaled measure of average lower orthant dependence. In fact, ρ_d^- assesses, to some extent, the similarity between our multivariate data (represented by its copula C) and the situation of independence (represented by the copula Π) in the lower orthant.

Moreover, using (2.3) and taking into account that $\int_{\mathbf{I}^d} \Pi(\mathbf{u})d\Pi(\mathbf{u}) = \frac{1}{2^d}$, ρ_d^- can be rewritten as:

$$\rho_d^- = \frac{(d+1)}{2^d - (d+1)}\left[2^d \int_{\mathbf{I}^d} C(\mathbf{u})d\Pi(\mathbf{u}) - 1\right], \tag{2.21}$$

which is a multivariate generalisation of (2.16).

In a similar fashion, Nelsen (1996) defined a second generalisation of Spearman's rho, derived from average upper orthant dependence. This measure focuses on the difference $[\bar{C}(\mathbf{u}) - \bar{\Pi}(\mathbf{u})]$

and is a generalisation of (2.18) to the multivariate case, defined as:

$$\rho_d^+ = \frac{\int_{\mathbf{I}^d} [\bar{C}(\mathbf{u}) - \bar{\Pi}(\mathbf{u})] d\Pi(\mathbf{u})}{\int_{\mathbf{I}^d} [\bar{M}(\mathbf{u}) - \bar{\Pi}(\mathbf{u})] d\Pi(\mathbf{u})}. \tag{2.22}$$

Thus, ρ_d^+ can be seen as a rescaled measure of average upper orthant dependence which assesses, to some extent, the similarity of our data (represented by $\bar{C}$) and the situation of independence (represented by $\bar{\Pi}$) in the upper orthant.

Furthermore, as Nelsen (1996) points out, $\int_{\mathbf{I}^d} [\bar{M}(\mathbf{u}) - \bar{\Pi}(\mathbf{u})] d\Pi(\mathbf{u}) = \frac{1}{d+1} - \frac{1}{2^d}$ and thus, applying Lemma 3.1 in Dolati and Úbeda-Flores (2006) in the numerator in expression (2.22), ρ_d^+ can also be expressed as a multivariate generalisation of (2.15) as follows:[2]

$$\rho_d^+ = \frac{(d+1)}{2^d - (d+1)} \left[2^d \int_{\mathbf{I}^d} \Pi(\mathbf{u}) dC(\mathbf{u}) - 1 \right]. \tag{2.23}$$

Hence, to sum up, in the multivariate case it is possible to define a multivariate generalisation of Spearman's rho based on the concept lower orthant dependence (ρ_d^-), which focuses on the difference $[C(\mathbf{u}) - \Pi(\mathbf{u})]$, and another one based on the concept of upper orthant dependence (ρ_d^+), which focuses on the difference $[\bar{C}(\mathbf{u}) - \bar{\Pi}(\mathbf{u})]$. In particular, ρ_d^- measures multivariate orthant dependence from a "downward" perspective, somehow capturing the average probability that the individuals are simultaneously low-ranked in all dimensions, whereas ρ_d^+ measures multivariate orthant dependence from an "upward" perspective, somehow capturing the overall probability that the individuals are simultaneously high-ranked in all dimensions. In this setting, Dolati and Úbeda-Flores (2006) propose another measure of multivariate association which is based on an average of the two differences highlighted above, and which is defined as follows:

$$\rho_d = \frac{2^{d-1}(d+1)}{2^d - (d+1)} \Big[\int_{\mathbf{I}^d} \big(\bar{C}(\mathbf{u}) - \bar{\Pi}(\mathbf{u}) + C(\mathbf{u}) - \Pi(\mathbf{u}) \big) d\Pi(\mathbf{u}) \Big]. \tag{2.24}$$

[2] Lemma 3.1 in Dolati and Úbeda-Flores (2006) establishes that, for any pair of (C, C') copulas, the following equality holds: $\int_{\mathbf{I}^d} C(\mathbf{u}) dC'(\mathbf{u}) = \int_{\mathbf{I}^d} \bar{C}'(\mathbf{u}) dC(\mathbf{u})$.

That is, ρ_d can be regarded as a rescaled measure of average orthant dependence with respect to independence. In fact, this coefficient is a particular case of the general family of Average Orthant Dependence (AOD) measures proposed by Dolati and Úbeda-Flores (2006); see Appendix 2.A.

Noticeably, ρ_d is also the average of the coefficients ρ_d^- and ρ_d^+ in (2.21) and (2.23), i.e.,

$$\rho_d = \frac{\rho_d^- + \rho_d^+}{2}. \tag{2.25}$$

Moreover, this coefficient was previously proposed by Nelsen (2002) as a measure of multivariate concordance. In particular, ρ_d can be expressed as a function of the probability of multivariate concordance Q'_d introduced in (2.10) as follows:

$$\rho_d = \frac{(d+1)}{2^d - (d+1)}[2^{d-1}Q'_d(C,\Pi) - 1], \tag{2.26}$$

Hence, in our context, ρ_d can be interpreted as the normalised probability that a randomly selected individual from the distribution $\mathbf{X}$ outranks or is outranked by a randomly selected individual from a reference distribution in which the dimensions are independent. Therefore, the more dependence there is in our data, according to the concordance function, the higher this normalized probability; see Decancq (2014).

Noticeably, in the bivariate case, ρ_d coincides with the well-known bivariate Spearman's rho in (2.19). In particular, since, for $d = 2$, $\bar{C}(u_1, u_2) = 1 + C(u_1, u_2) - u_1 - u_2$, then, taking expression (2.10) and, after some algebra, it turns out that:

$$Q'_2(C,\Pi) = \int_{\mathbf{I}^2} \big(C(u_1,u_2) + \bar{C}(u_1,u_2)\big) d\Pi(u_1,u_2) = 2\int_{\mathbf{I}^2} C(u_1,u_2) d\Pi(u_1,u_2).$$

Hence, replacing this expression in (2.26) for $d = 2$, the expression in (2.19) comes up, namely:

$$\rho_2 = 3\Big[4\int_{\mathbf{I}^2} C(u_1, u_2) - 1\Big] = 3Q(C, \Pi) = \rho_S.$$

Several properties of the three multivariate generalisations of Spearman's rho reviewed so far (ρ_d^-, ρ_d^+ and ρ_d) can be highlighted:

1. If $\mathbf{X}$ is PLOD (NLOD), then $\rho_d^- \geq 0$ ($\rho_d^- \leq 0$).

2. If $\mathbf{X}$ is PUOD (NUOD), then $\rho_d^+ \geq 0$ ($\rho_d^+ \leq 0$).

3. If $\mathbf{X}$ is POD (NOD), then $\rho_d \geq 0$ ($\rho_d \leq 0$).

4. When the distribution of $\mathbf{X}$ is radially symmetric, that is, when there is a point $\mathbf{a}$ in $\mathbf{R^d}$ such that $Pr(\mathbf{X} \leq \mathbf{a} - \mathbf{x}) = Pr(\mathbf{X} > \mathbf{a} + \mathbf{x})$, then $\rho_d^- = \rho_d^+ = \rho_d$.

5. When the copula of $\mathbf{X}$ is the commonotonic copula ($C = M$), $\rho_d^- = \rho_d^+ = \rho_d = 1$.

6. When the components of $\mathbf{X}$ are independent ($C = \Pi$), $\rho_d^- = \rho_d^+ = \rho_d = 0$.

7. A lower bound for the three measures is $\frac{[2^d-(d+1)!]}{d![2^d-(d+1)]}$.

8. In the bivariate case ($d = 2$) the three coefficients above, ρ_2^-, ρ_2^+ and ρ_2, reduce to bivariate Spearman's rho.

9. In the trivariate case ($d = 3$), ρ_3 can be expressed as the average of the three pairwise Spearman's rho coefficients, that is:

$$\rho_3 = \frac{\rho_3^- + \rho_3^+}{2} = \frac{\rho_{12} + \rho_{13} + \rho_{23}}{3}, \tag{2.27}$$

   where ρ_{ik} denotes the pairwise Spearman's rho coefficient for the bivariate random variable (X_i, X_k), with $1 \leq i < k \leq 3$; see Nelsen (1996).

10. ρ_d is a measure of multivariate concordance according to the axiomatic definitions proposed by Dolati and Úbeda-Flores (2006) and Taylor (2007).

The advantage of ρ_d^- and ρ_d^+ when compared with ρ_d is that they are capable of revealing some forms of dependence that ρ_d fails to detect. For example, for $d > 2$ it is possible to find a situation in which ρ_d and all the pairwise Spearman's coefficients are 0, which is a sign of independence, but at the same time ρ_3^- and ρ_3^+ are different from 0, indicating some degree of lower and upper orthant dependence; see, for instance, Example 1 in Nelsen and Úbeda-Flores (2012). Moreover, it is possible to find different multivariate distributions with the same marginals and the same value for ρ_d but very different behaviours in the lower and uppper orthants, thus leading to different values for ρ_d^- and ρ_d^+. To illustrate this point, Figure 2.2 represents 10000 realisations from two different trivariate distributions that share the same marginals and the same value of ρ_3, namely $\rho_3 = 0.7$, but with different structures of dependence, since they come from two different trivariate parametric copulas. In particular, the distribution represented in the left panel, which has a Clayton copula (see Appendix 2.C), has higher dependence in the lower orthant ($\rho_d^- = 0.734$) than in the upper orthant ($\rho_d^+ = 0.665$), since the density of points around the corner (0,0,0) is higher than around the corner (1,1,1). On the other hand, the distribution represented in the right panel, which has a Gumbel copula (see Appendix 2.C), displays higher dependence in the upper orthant than in the lower orthant ($\rho_d^- = 0.666$ and $\rho_d^+ = 0.733$).[3] Hence, unlike ρ_d, ρ_d^- and ρ_d^+ allow us to distinguish what happens in the lower and in the upper orthant. For instance, if the three dimensions of these graphs were income, health and education in two different countries, in the country represented in the left panel the probability of being simultaneously poor in all dimensions would be clearly higher than the probability of being simultaneously rich in all dimensions, whereas the contrary would occur in the country represented in the right panel, where the probability of being simultaneously rich

[3]The values of ρ_d^- and ρ_d^+ cannot be computed analytically for these particular parametric copulas and so they are approximated by Monte Carlo simulation as the average of their corresponding sample versions proposed by Pérez and Prieto-Alaiz (2016b), which will be reviewed later on, across 300 samples of size 500000.

in all dimensions would be clearly higher than the probability of being simultaneously poor in all dimensions.

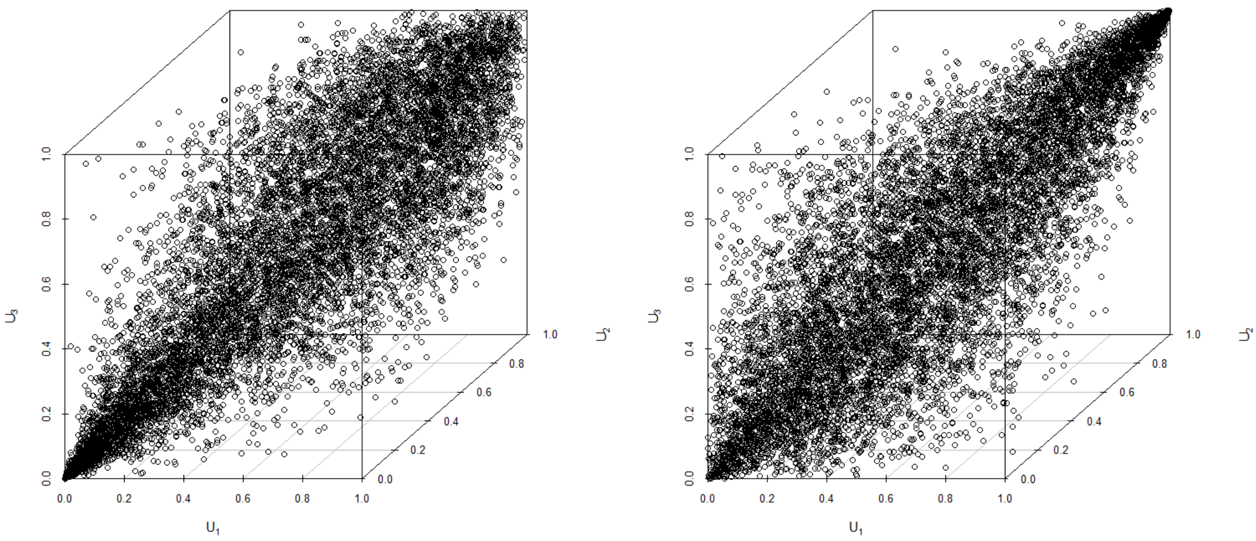

Figure 2.2: 10000 realisations from two different trivariate distributions with the same marginal distributions and the same $\rho_3 = 0.7$, but with different degree of lower and upper orthant dependence. The distribution in the left panel has a Clayton copula and $\rho_d^- = 0.734$ and $\rho_d^+ = 0.665$. The distribution in the right panel has a Gumbel copula and $\rho_d^- = 0.666$ and $\rho_d^+ = 0.733$.

Despite their advantages over ρ_d, ρ_d^+ and ρ_d^- may be still incapable of detecting other forms of dependence when they take values near zero. To overcome this drawback, Nelsen and Úbeda-Flores (2012) propose, in the trivariate case, new coefficients of what they call directional dependence, which are defined as follows. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_i \in \{-1, 1\}$, denote the eight vertexes of the cube $\mathbf{I}^3$ determining the eight directions in which we could measure dependence in trivariate distributions. For each direction α, a directional ρ-coefficient is defined as:

$$\rho_3^\alpha = \frac{\alpha_1\alpha_2\rho_{12} + \alpha_1\alpha_3\rho_{13} + \alpha_2\alpha_3\rho_{23}}{3} + \alpha_1\alpha_2\alpha_3\frac{\rho_3^+ - \rho_3^-}{2}. \tag{2.28}$$

Noticeably, $\rho_3^{(1,1,1)} = \rho_3^+$ and $\rho_3^{(-1,-1,-1)} = \rho_3^-$.

Based on these concepts, García et al. (2013) introduced the index of maximal dependence, $\rho_3^{\max}$, as the largest of the eight directional ρ-coefficients defined in (2.28) and they show that this index can be alternatively calculated as follows:

$$\rho_3^{\max} = \frac{2}{3}\max\{\rho_{12}, \rho_{13}, \rho_{23}, 3\rho_3\} - \min\{\rho_3^+, \rho_3^-\}. \tag{2.29}$$

It is worth noting that $0 \leq \rho_3^{\max} \leq 1$. Actually, $\rho_3^{\max}$ attains its maximum value, 1, when $C = M$ and it becomes 0 when $C = \Pi$. Moreover, if ρ_{12}, ρ_{13} and ρ_{23} are all positive, then $\rho_3^{\max}$ is equal to either ρ_3^- or ρ_3^+; see García et al. (2013).

2.4.2.2 Nonparametric estimation

So far, we have focused on the population versions of the different copula-based multivariate extensions of Spearman's rho. However, in practice, empirical versions of these measures are needed, as the copula C is unknown and the coefficients must be estimated from the data. Over the rest of this section, we review several proposals that can be found in the literature to estimate the different multivariate generalisations of Spearman's rho discussed above.

Let $\{(X_{1j}, ..., X_{dj})\}_{j=1,...,n}$ be a sample of n serially independent random vectors from the d-dimensional vector $\mathbf{X} = (X_1, ..., X_d)$ with associated copula C. Then, it is possible to estimate non-parametrically the copula C by the corresponding empirical copula, namely

$$\hat{C}_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{\widetilde{U}_{ij} \leq u_i\}}, \text{ for } \mathbf{u} = (u_1, ..., u_d) \in \mathbf{I}^d, \tag{2.30}$$

where $\mathbf{1}_A$ denotes the indicator function on a set A and $\widetilde{U}_{ij}$ are the transformed data to $[0,1]$ by scaling ranks, i.e.

$$\widetilde{U}_{ij} = R_{ij}/n, \tag{2.31}$$

where R_{ij} denotes the *rank* of X_{ij} among $\{X_{i1}, ..., X_{in}\}$, with $i = 1, ..., d$ and $j = 1, ..., n$.

Similarly, the survival function $\bar{C}$ can be estimated non-parametrically by its corresponding empirical version, given by

$$\hat{\bar{C}}_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{\widetilde{U}_{ij}>u_i\}}, \text{ for } \mathbf{u} = (u_1, ..., u_d) \in \mathbf{I}^d. \tag{2.32}$$

Schmid and Schmidt (2007b) propose the non-parametric estimation of ρ_d^- and ρ_d^+ by replacing the copula C in expressions (2.21) and (2.23), respectively, with the empirical copula defined in (2.30). However, this estimation procedure has an important drawback, since, as Pérez and Prieto-Alaiz (2016b) show, the resultant statistics are not proper estimators as they can take values out of the parameter space. Blumentritt and Schmid (2014) and Bedo and Ong (2014) proposed modifications of these estimators by using the so-called pseudo-observations, $U_{ij}^* = R_{ij}/(n+1)$, instead of $\widetilde{U}_{ij}$, in the computation of the empirical copula. Nevertheless, Pérez and Prieto-Alaiz (2016b) show that these estimators still suffer from some shortcomings, as they fail to achieve the maximum value of 1 when there is maximal dependence in the data and take a narrower range of values than they should.

To overcome these drawbacks, Pérez and Prieto-Alaiz (2016b) propose alternative non-parametric estimators of ρ_d^- and ρ_d^+ based on the ideas in Joe (1990). In particular, the estimation of ρ_d^+ is based on rewriting the expression of this coefficient in (2.22) by applying Lemma 3.1 in Dolati and Úbeda-Flores (2006) -see footnote in page 85-, so that the following expression arises:

$$\rho_d^+ = \frac{\int_{\mathbf{I}^d}\Pi(\mathbf{u})dC(\mathbf{u}) - \int_{\mathbf{I}^d}\Pi(\mathbf{u})d\Pi(\mathbf{u})}{\int_{\mathbf{I}^d}\Pi(\mathbf{u})dM(\mathbf{u}) - \int_{\mathbf{I}^d}\Pi(\mathbf{u})d\Pi(\mathbf{u})}.$$

Now, taking into account that $\Pi(\mathbf{u}) = u_1 \times \cdots \times u_d$, ρ_d^+ can be regarded as a scaled expected value of $U_1 \times \cdots \times U_d$, namely:

$$\rho_d^+ = \frac{E_C[U_1 \times \cdots \times U_d] - E_\Pi[U_1 \times \cdots \times U_d]}{E_M[U_1 \times \cdots \times U_d] - E_\Pi[U_1 \times \cdots \times U_d]} = \frac{c_0 - c_1}{c_2 - c_1}, \tag{2.33}$$

where $c_0 = E_C[U_1 \times \cdots \times U_d]$ is the expected value of $U_1 \times \cdots \times U_d$ with respect to copula C, $c_1 = E_\Pi[U_1 \times \cdots \times U_d] = \frac{1}{2^d}$ is the expected value of $U_1 \times \cdots \times U_d$ in the case of independence ($C = \Pi$) and $c_2 = E_M[U_1 \times \cdots \times U_d] = \frac{1}{(d+1)}$ is the expected value of $U_1 \times \cdots \times U_d$ in the case of maximal positive dependence ($C = M$).

The statistic proposed by Schmid and Schmidt (2007b) only estimates the expectation c_0 in (2.33), keeping the parameters c_1 and c_2 as known constants. By contrast, Pérez and Prieto-Alaiz (2016b) propose to estimate all the expectations in (2.33), that is, c_0, c_1 and c_2, by their corresponding sample means, leading to the following rank-based non-parametric estimator of ρ_d^+:

$$\widehat{\rho}_d^+ = \frac{\dfrac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\widetilde{U}_{ij} - \left(\frac{n+1}{2n}\right)^d}{\dfrac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^d - \left(\frac{n+1}{2n}\right)^d}. \tag{2.34}$$

For the estimation of ρ_d^- proposed by Pérez and Prieto-Alaiz (2016b), the same ideas apply. To start with, using again Lemma 3.1 in Dolati and Úbeda-Flores (2006), ρ_d^- in (2.20) can be rewritten as follows:

$$\rho_d^- = \frac{\int_{\mathbf{I}^d}\bar{\Pi}(\mathbf{u})dC(\mathbf{u}) - \int_{\mathbf{I}^d}\bar{\Pi}(\mathbf{u})d\Pi(\mathbf{u})}{\int_{\mathbf{I}^d}\bar{\Pi}(\mathbf{u})dM(\mathbf{u}) - \int_{\mathbf{I}^d}\bar{\Pi}(\mathbf{u})d\Pi(\mathbf{u})}$$

Hence, taking into account that $\bar{\Pi}(\mathbf{u}) = (1-u_1)\times\cdots\times(1-u_d)$, ρ_d^- can be regarded as a scaled expected value of $(1-U_1)\times\cdots\times(1-U_d)$, namely:

$$\rho_d^- = \frac{E_C[(1-U_1)\times\cdots\times(1-U_d)] - E_\Pi[(1-U_1)\times\cdots\times(1-U_d)]}{E_M[(1-U_1)\times\cdots\times(1-U_d)] - E_\Pi[(1-U_1)\times\cdots\times(1-U_d)]} = \frac{\bar{c}_0 - \bar{c}_1}{\bar{c}_2 - \bar{c}_1}, \tag{2.35}$$

where $\bar{c}_0 = E_C[(1-U_1)\times\cdots\times(1-U_d)]$, $\bar{c}_1 = E_\Pi[(1-U_1)\times\cdots\times(1-U_d)] = \frac{1}{2^d}$ is the expected value of $(1-U_1)\times\cdots\times(1-U_d)$ in the case of independence and $\bar{c}_2 = E_M[(1-U_1)\times\cdots\times(1-U_d)] = \frac{1}{(d+1)}$ is the expected value of $(1-U_1)\times\cdots\times(1-U_d)$ in the case of maximal positive dependence.

As in the case of ρ_d^+, the statistic proposed by Schmid and Schmidt (2007b) to estimate ρ_d^- only estimates the expectation $\bar{c}_0$ in (2.35), keeping $\bar{c}_1$ and $\bar{c}_2$ as known constants. By contrast, Pérez and Prieto-Alaiz (2016b) propose to estimate all the expectations in (2.35), that is, $\bar{c}_0$, $\bar{c}_1$ and $\bar{c}_2$, by their corresponding sample means, leading to the following rank-based estimator of ρ_d^-:

$$\widehat{\rho_d^-} = \frac{\frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\widetilde{\bar{U}}_{ij} - \left(\frac{n+1}{2n}\right)^d}{\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^d - \left(\frac{n+1}{2n}\right)^d}, \tag{2.36}$$

where $\widetilde{\bar{U}}_{ij} = \bar{R}_{ij}/n$ and $\bar{R}_{ij} = n + 1 - R_{ij}$.

The estimators defined in (2.34) and (2.36) overcome the drawbacks of those proposed by Schmid and Schmidt (2007b), Blumentritt and Schmid (2014) and Bedo and Ong (2014). In particular, they take values on the parameter space and, by construction, they achieve their maximum value of 1 in the case of maximal dependence in the data ($C = M$) and take the value of 0 in the case of independence ($C = \Pi$). Moreover, these estimators are consistent and share the same asymptotic Normal distribution as those in Schmid and Schmidt (2007b). Nonetheless, their asymptotic variances cannot be explicitly evaluated for the majority of known copulas (not even for $d = 2$) but, as shown in Schmid and Schmidt (2007b), they can consistently be estimated by nonparametric bootstrap methods. Therefore, in the empirical application (Chapter 3), bootstrap methods will be applied to estimate their standards errors and perform statistical inference.

Finally, to estimate the coefficient ρ_d in (2.26), we take into account the relationship between ρ_d, ρ_d^- and ρ_d^+ in (2.25) and propose the following plug-in estimator:

$$\widehat{\rho}_d = \frac{\widehat{\rho_d^-} + \widehat{\rho}_d^+}{2}, \tag{2.37}$$

where $\widehat{\rho_d^-}$ and $\widehat{\rho}_d^+$ are the estimators in (2.36) and (2.34), respectively. Noticeably, this estimator coincides with the estimator of ρ_d proposed by Dolati and Úbeda-Flores (2006) in the framework of AOD measures of multivariate concordance; see Proposition 1 in Appendix 2.B.

In the bivariate case, the estimators $\widehat{\rho_2^-}$, $\widehat{\rho}_2^+$ and $\widehat{\rho}_2$ defined above reduce to the well-known empirical version of the bivariate Spearman's rho. In the trivariate case ($d = 3$), the estimators $\widehat{\rho_3^+}$ and $\widehat{\rho_3^-}$ in (2.34) and (2.36) reduce to:

$$\widehat{\rho_3^+} = \frac{8}{n(n-1)(n+1)^2}\sum_{j=1}^{n} R_{1j}R_{2j}R_{3j} - \frac{n+1}{n-1}, \tag{2.38}$$

$$\widehat{\rho_3^-} = \frac{8}{n(n-1)(n+1)^2}\sum_{j=1}^{n} \overline{R}_{1j}\overline{R}_{2j}\overline{R}_{3j} - \frac{n+1}{n-1}; \tag{2.39}$$

see Pérez and Prieto-Alaiz (2016b). Moreover, it can be shown that the relationship between ρ_3, ρ_3^-, ρ_3^+ and the three pairwise coefficients, which is defined in expression (2.27), also holds for their empirical versions, that is,

$$\widehat{\rho}_3 = \frac{\widehat{\rho_3^-} + \widehat{\rho_3^+}}{2} = \frac{\widehat{\rho}_{12} + \widehat{\rho}_{13} + \widehat{\rho}_{23}}{3}, \tag{2.40}$$

where $\widehat{\rho}_{ik}$ denotes the bivariate sample Spearman's rho for the pair (X_i, X_k), with $1 \leq i < k \leq 3$; see Proposition 2 in Appendix 2.B. This means that, with $d = 3$, it is possible to easily calculate $\widehat{\rho}_3$ as the average of the three pairwise sample coefficients.

Finally, for the estimation of the coefficients ρ_3^α and $\rho_3^{\max}$, García et al. (2013) propose plug-in estimators consisting of replacing the population bivariate and trivariate Spearman's coefficients in (2.28) and (2.29) by their empirical versions. They also show that these plug-in estimators are asymptotically unbiased, consistent and asymptotically normally distributed.

An application of all the multivariate extensions of Spearman's rho explained so far will be carried out in Chapter 3, where we measure the multivariate dependence between those dimensions of poverty included in the AROPE rate for the EU28.

2.5 Tail dependence

The orthant dependence measures introduced so far are suitable to describe how likely it is, in average, that large (small) values of one variable appear with large (small) values of the other variables, as compared to what would happen were these variables independent. However, as Malevergne and Sornette (2006, ch. 4) argue, orthant dependence is a very strong property that imposes conditions for every point in the hypercube, since it requires the evaluation of $C(\mathbf{u})$ and $\Pi(\mathbf{u})$ for all $\mathbf{u} \in I^d$. In turn, one may be interested in only analysing dependence in some specific parts of the multivariate distribution, such as the tails. For instance, when studying multidimensional poverty, dependence in the joint lower tail of the multivariate distribution seems specially relevant, as it focuses on those individuals who are extremely poor in all dimensions. To address this issue, new concepts should be introduced.

In particular, this section is devoted to the concept of tail dependence, which relates to the degree of dependence in the joint (lower or upper) tail of a multivariate distribution, that is, the dependence between extreme events; see Schmid et al. (2010). As we will see, this concept captures the propensity of several random variables to simultaneously take extreme (very small or very large) values, or more specifically, the probability that some variables exceed a high (low) quantile given that others also exceed such quantile. In our setting, the concept of tail dependence becomes a very useful tool, since it would capture the probability of individuals to be extremely low(high)-ranked in some dimensions of poverty, provided that they are extremely low(high)-ranked in other dimensions. In particular, to analyse multidimensional poverty, the concept of lower tail dependence becomes specially relevant, since it captures the probability that an individual who is extremely "poor" in some dimensions is also extremely poor in other dimensions. In spite of its relevance, to the best of our knowledge, the application of the concept of tail dependence to poverty analysis has not yet been addressed in the literature. Hence, the contribution of this thesis is pioneering in this area and can pave the way for future contributions.

To face this goal, we will first discuss, in the next subsection, tail dependence in the bivariate case. After that, we will turn to the multivariate case, which has been scarcely addressed in the literature. As we will see, the step from the bivariate to the multivariate setting poses important difficulties both from a theoretical and a practical point of view, which reinforces the idea already mentioned in previous section that, when analysing dependence, moving from two to more than two dimensions is not straightforward. An application of these concepts to measuring multivariate tail dependence between dimensions of poverty in Europe will be performed in Chapter 3.

2.5.1 Bivariate tail dependence measures

In the bivariate case, tail dependence captures the degree of dependence in the lower-quadrant tail or the upper-quadrant tail of a bivariate distribution. Although several ways of measuring tail dependence have been proposed, the most commonly used measures are the so-called tail dependence coefficients introduced by Sibuya (1960), which are defined as follows. Given a bivariate random vector $\mathbf{X} = (X_1, X_2)$ with joint distribution function F and marginal distribution functions F_1 and F_2, the **lower tail dependence coefficient** of X_1 and X_2 is defined as

$$\lambda_L = \lim_{u \to 0^+} Pr\big(X_2 \leq F_2^{-1}(u) | X_1 \leq F_1^{-1}(u)\big), \tag{2.41}$$

and the **upper tail dependence coefficient** is given by

$$\lambda_U = \lim_{u \to 1^-} Pr\big(X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)\big), \tag{2.42}$$

provided that the limits above exist. In a probabilistic view, the lower tail dependence coefficient in (2.41) gives the asymptotic probability that a random variable becomes small, given that another random variable is also small. Similarly, the upper tail dependence coefficient in (2.42) gives the asymptotic probability that a random variable exceeds a high quantile, given that another random variable also exceeds such quantile. Thus, these coefficients become essential to understand the risk of simultaneous threshold crossing for two random variables. In our setting, if $\mathbf{X} = (X_1, X_2)$ represent two relevant dimensions of poverty for a population, say income and health, the lower tail dependence coefficient λ_L would measure the limit probability that an individual is extremely low-ranked (extremely poor) in one dimension (i.e, income) given that he/she is extremely low-ranked (extremely poor) in the other dimension (i.e, health). Similarly, the upper tail dependence coefficient λ_U would measure the limit probability that an individual is extremely high-ranked (extremely rich) in one dimension (i.e, income) given that he/she is extremely high-ranked (extremely rich) in the other dimension (i.e, health). Obviously, when the goal is the measurement of poverty, the lower tail dependence coefficient λ_L is the most relevant one.

When F_1 and F_2 are continuous, the tail dependence coefficients defined above only depend on the underlying copula C that links F to F_1 and F_2 and thus they are able to measure tail dependence independently on the marginal distributions. In particular, applying the probability integral transformations $U_1 = F_1(X_1)$ and $U_2 = F_2(X_2)$ into equations (2.41) and (2.42), respectively, the following alternative copula-based expressions for λ_L and λ_U come up (Joe, 2014):

$$\lambda_L = \lim_{u \to 0^+} Pr\big(U_2 \leq u | U_1 \leq u\big) = \lim_{u \to 0^+} \frac{Pr(U_1 \leq u, U_2 \leq u)}{u} = \lim_{u \to 0^+} \frac{C(u,u)}{u}, \tag{2.43}$$

and

$$\lambda_U = \lim_{u \to 1^-} Pr\big(U_2 > u | U_1 > u\big) = \lim_{u \to 1^-} \frac{Pr(U_1 > u, U_2 > u)}{1-u} = \lim_{u \to 1^-} \frac{\bar{C}(u,u)}{1-u}, \tag{2.44}$$

provided that the limits in (2.43) and (2.44) exist. If this is so, the copula C has an upper tail dependence coefficient λ_U and a lower tail dependence coefficient λ_L. By definition, the coefficients of tail dependence λ_L and λ_U are bounded between 0 and 1 inclusive. According to

Joe (2014), if $\lambda_L > 0$ (respectively, $\lambda_U > 0$), C has lower (respectively, upper) tail dependence. In turn, if $\lambda_L = 0$ (respectively, $\lambda_U = 0$), C has no lower (respectively, upper) tail dependence. Furthermore, it can be easily seen that, if $C = \Pi$, that is, if X_1 and X_2 are independent, then $\lambda_L = \lambda_U = 0$. Furthermore, if $C = M$, that is, if one variable is a strictly increasing function of the other, i.e. if the positions of individuals in both variables coincide, then $\lambda_L = \lambda_U = 1$. Theoretical values of the tail dependence coefficients have been derived for the most popular families of copulas; see, for example, Joe (1997), Malevergne and Sornette (2006, ch. 4) and Nelsen (2006).

To illustrate the behaviour of the tail dependence coefficients, Figure 2.3 displays 2000 realisations from two different bivariate parametric copulas, namely the bivariate Clayton copula and the bivariate Gumbel copula; see Appendix 2.C for a brief discussion on these families of copulas. In the two graphs at the top, the clouds of points reflects different degrees of lower tail dependence and no upper tail dependence, as they represent realisations from two Clayton copulas with different degrees of dependence. Noticeably, as the concentration of points in the lower tail of the distribution increases, so does the theoretical value of the lower tail dependence coefficient, going from $\lambda_L = 0.26$ in the top-left graph to $\lambda_L = 0.8$ in the top-right graph. In the two graphs displayed at the bottom of Figure 2.3, the clouds of points reflects some degree of upper tail dependence but no lower tail dependence, as they represent realisations from two Gumbel copulas with different degrees of dependence. Again, as the concentration of points in the upper joint tail increases, so does the theoretical value of the upper tail dependence coefficient, going from $\lambda_U = 0.35$ in the bottom-left graph to $\lambda_U = 0.7$ in the bottom-right.

The tail dependence coefficients defined in (2.43) and (2.44) give, by definition, an asymptotic approximation of the behaviour of the copula in the joint tails of the distribution. Some authors have argued that the study of tail dependence going beyond the asymptotic definition can be even more informative, and this has led to the distinction between tail dependence at asymptotic and sub-asymptotic levels. For instance Venter (2001), Manner and Segers (2011)

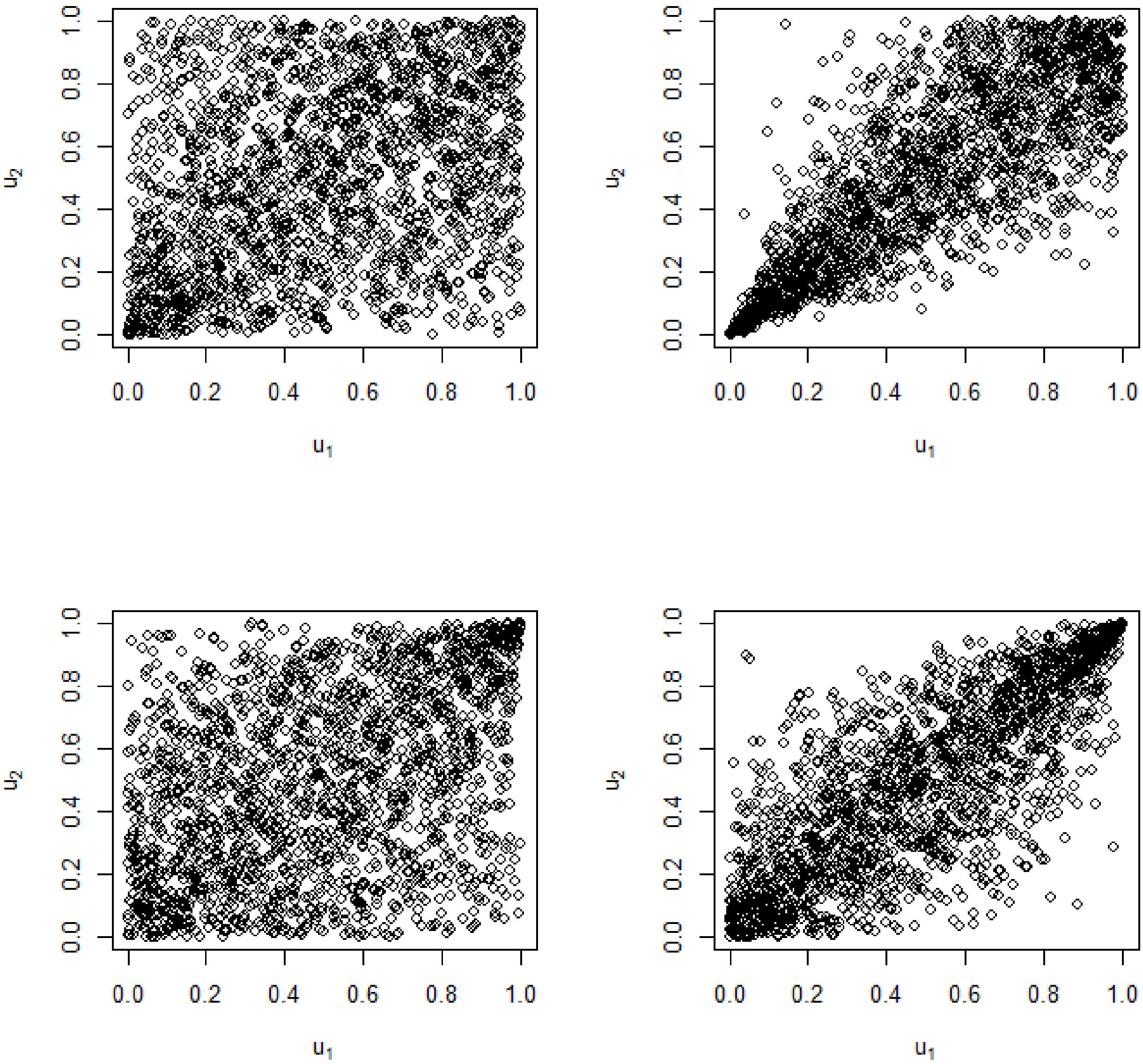

Figure 2.3: 2000 realisations from different bivariate copulas. The two distributions in the top come from a Clayton copula (lower tail dependent) with $\lambda_L = 0.26$ (left) and $\lambda_L = 0.8$ (right). The two distributions in the bottom come from a Gumbel copula (upper tail dependent) with $\lambda_U = 0.35$ (left) and $\lambda_U = 0.7$ (right).

and Sweeting and Fotiou (2013), propose a finite alternative to the asymptotic tail dependence coefficients in (2.43) and (2.44). In particular, they introduce, for any $u \in (0,1)$, the following functions:

$$\lambda_L(u) = \frac{C(u,u)}{u} \tag{2.45}$$

and

$$\lambda_U(u) = \frac{\bar{C}(u,u)}{1-u}. \tag{2.46}$$

Sweeting and Fotiou (2013) call these functions “finite tail dependence coefficients”, whereas they are denoted by Manner and Segers (2011) as “penultimate tail dependence coefficients”. The limits of these functions are the traditional tail dependence coefficients defined in (2.43) and (2.44), respectively, that is,

$$\lambda_L = \lim_{u \to 0^+} \lambda_L(u) \tag{2.47}$$

and

$$\lambda_U = \lim_{u \to 1^-} \lambda_U(u). \tag{2.48}$$

Venter (2001) had previously considered these finite tail dependence coefficients to analyse graphically the tail behaviour of the copula. In particular, he proposed to combine $\lambda_L(u)$ and $\lambda_U(u)$ into the so-called *tail concentration function* (TCF), further considered by Patton (2013) and Durante et al. (2015). The TCF is defined as $q_C : (0,1) \to \mathbf{I}$ given by

$$q_C(u) = \frac{C(u,u)}{u}\mathbf{1}_{(0,0.5]} + \frac{\bar{C}(u,u)}{1-u}\mathbf{1}_{(0.5,1)}. \tag{2.49}$$

This is a continuous function, since $C(0.5,0.5) = \bar{C}(0.5,0.5)$ and thus $q_C(0.5^-) = q_C(0.5^+)$. When $C = M$, $q_M(u) = 1\ \forall u$ and when $C = W$, $q_W(u) = 0\ \forall u$. Moreover, in the case of independence ($C = \Pi$), $q_\Pi(u) = u\mathbf{1}_{(0,0.5]} + (1-u)\mathbf{1}_{(0.5,1)}$.

As expected, the TCF is strictly related to the finite tail dependence coefficients in (2.45) and

(2.46), as it can be alternatively written as

$$q_C(u) = \begin{cases} \lambda_L(u), & \text{if } u \in (0, 0.5], \\ \lambda_U(u), & \text{if } u \in (0.5, 1). \end{cases}$$

Furthermore, it is related to the asymptotic tail dependence coefficients, since

$$\lim_{u \to 0^+} q_C(u) = \lambda_L \tag{2.50}$$

and

$$\lim_{u \to 1^-} q_C(u) = \lambda_U. \tag{2.51}$$

To better appreciate the usefulness of this function, in Figure 2.4 we present the TCF for three different bivariate parametric copulas (the Clayton copula in the left panel; the Frank copula in the central panel; and the Gumbel copula in the right panel) for different degrees of dependence; see Appendix 2.C for a brief discussion on these copulas and their tail dependence properties. Furthermore, in all panels, the TCF of the independent copula Π is represented in blue and the TCF of the commonotonic copula M is represented in red.

The first feature that can be observed in this figure is that the three copula models are related to different shapes of the TCF. In particular, Clayton copulas (left panel) are asymmetric, with observations in the lower tail being somewhat more dependent than observations in the upper tail, since $q_C(u) > q_C(1-u)$ for $0 < u < 0.5$. The contrary occurs in the case of Gumbel copulas (right panel), where observations in the upper tail are somewhat more dependent than observations in the lower tail, since $q_C(u) < q_C(1-u)$ for $0 < u < 0.5$. Finally, Frank copulas (central panel) show a symmetric behaviour, since $q_C(u) = q_C(1-u)$ for $0 < u < 0.5$. The second important feature that we can notice in Figure 2.4 is that, for the three models, as the degree of dependence increases (as the parameter θ increases), the TCFs move upwards and closer to the upper bound given by the maximal dependence copula M, whereas as θ decreases,

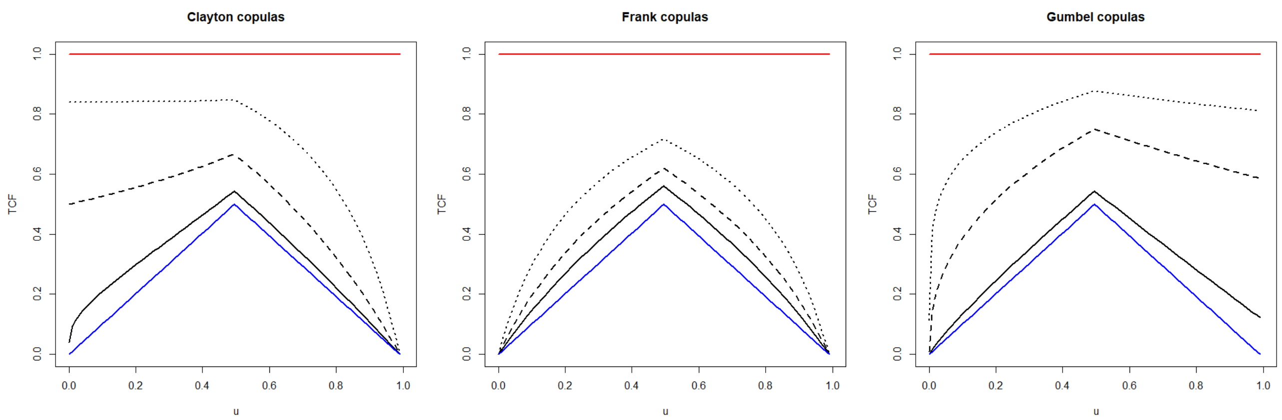

Figure 2.4: TCFs for some bivariate parametric copulas. Left panel represents Clayton copulas with $\theta = 0.1$ (solid), $\theta = 1$ (dashed) and $\theta = 4$ (dotted). Central panel represents Frank copulas with $\theta = 1$ (solid), $\theta = 2$ (dashed) and $\theta = 4$ (dotted). Right panel represents Gumbel copulas with $\theta = 1.1$ (solid), $\theta = 2$ (dashed) and $\theta = 4$ (dotted). In all panels, the TCF of the independent copula Π is represented in blue and the TCF of the commonotonic copula M is represented in red.

the TCF approaches the lower bound corresponding to the independent copula Π.

So far, we have focused on the population versions of the different measures of bivariate tail dependence. However, in practice, empirical versions of these measures are needed, as the copula C is unknown and they must be estimated from the data. As we have previously pointed out, the most-commonly used measures of bivariate tail dependence are the tail dependence coefficients in (2.43) and (2.44). However, since they are by definition asymptotic measures, the estimation of these coefficients is not straightforward. In fact, as Joe (2014) argues, the empirical measure of tail dependence for data does not really exist because of the limit; the best that can be done is to a apply estimation procedures, either parametric or nonparametric, such as those in Frahm et al. (2005). These authors point out that parametric procedures give biased estimators if the model is misspecified, something that is avoided with a non-parametric estimation, which, in turn, produces larger variances. In the following discussion, we will limit our attention to non-parametric procedures. For parametric techniques, we refer the interested reader to Frahm et al. (2005) and Supper et al. (2020), and the references therein.

There is an important number of proposals to estimate tail dependence in a non-parametric framework, specially in the case of upper tail dependence (Huang, 1992; Joe et al., 1992; Capéraà et al., 1997; Frahm et al., 2005). Here we focus on which probably are the most natural non-parametric estimators of the tail dependence coefficients λ_L and λ_U. In particular, as Schmid et al. (2010) argue, given a sample $\{(X_{1j}, X_{2j})\}_{j=1,\ldots,n}$ of n serially independent random vectors from the bidimensional vector $\mathbf{X} = (X_1, X_2)$ with associated copula C, a natural non-parametric estimator of λ_L is given by

$$\hat{\lambda}_L\left(\frac{k}{n}\right) = \frac{\hat{C}_n(\frac{k}{n}, \frac{k}{n})}{\frac{k}{n}}, \tag{2.52}$$

where $\hat{C}_n$ is the empirical copula defined in (2.30) and $k \in \{1, \ldots, n\}$ is a parameter that must be appropriately chosen. In fact, this is the estimator of λ_L proposed by Schmidt and Stadtmüller (2006), who proved it to be consistent and asymptotically normal under rather general assumptions.

In a similar fashion, a natural non-parametric estimator of λ_U is given by

$$\hat{\lambda}_U\left(\frac{k}{n}\right) = \frac{\hat{\bar{C}}(\frac{k}{n}, \frac{k}{n})}{1 - \frac{k}{n}}, \tag{2.53}$$

where $\hat{\bar{C}}$ is the empirical survival function defined in (2.32) and, again, $k \in \{1, \ldots, n\}$ is a parameter that must be appropriately chosen. This estimator also coincides with that proposed by Schmidt and Stadtmüller (2006), which is also proved to be consistent and asymptotically normal under rather general assumptions. The estimators $\hat{\lambda}_L$ and $\hat{\lambda}_U$ in (2.52) and (2.53) have been applied in multiple studies in finance (WeiB et al., 2014; Matkovskyy, 2019) or environmental sciences (Poulin et al., 2007; Aghakouchak et al., 2010; Ward et al., 2018). Furthermore, as we will see later, the estimators proposed by Schmidt and Stadtmüller (2006) allow for an intuitive extension to the multivariate case. For a comprehensive review of other non-parametric estimators of the tail dependence coefficients we refer the interested reader to Frahm et al. (2005).

An important point that must be taken into account when applying $\hat{\lambda}_L$ and $\hat{\lambda}_U$ is that an appropriate threshold k must be chosen. This is a feature shared by the vast majority of the non-parametric estimators of tail dependence proposed in the literature and is not a negligible issue, since the estimates of tail dependence can vary significantly depending on the threshold used, and the choice of this threshold is by no means trivial. Bücher et al. (2015) argue that a common approach in extreme-value theory to deal with this problem is to consider several different values of k. Alternatively, Caillault and Guégan (2005), Frahm et al. (2005) and Schmidt and Stadtmüller (2006) have proposed an algorithm to obtain an optimal k, which requires dealing with a variance-bias problem. A comparison of these two approaches can be found in Aghakouchak et al. (2010), who argue that, in any case, the choice of the threshold still warrants more in-depth research. Notice also that, with a parametric estimation, and for the most popular families of copulas, the problem of the choice of the threshold is avoided, since the tail dependence coefficients have close expressions as functions of the copula parameters. However, in that case there exists the specification problem that is inherent to any parametric inference procedure. As we pointed out before, in this thesis we do not consider parametric approaches.

As we argued before, the traditional bivariate tail dependence coefficients give, by definition, an asymptotic approximation of the behaviour of the copula in the joint tails of the distribution. In turn, the TCF in (2.49) allows to study tail dependence going beyond this asymptotic definition. Patton (2013) and Durante et al. (2015) propose to estimate this function in a non-parametric way. In particular, the empirical version of the TCF can be calculated, for $t \in (0,1)$, using the empirical versions of the copula and the survival function as follows:

$$\hat{q}_C(t) = \frac{\hat{C}_n(t,t)}{t}\mathbf{1}_{(0,0.5]} + \frac{\hat{\bar{C}}_n(t,t)}{1-t}\mathbf{1}_{(0.5,1)}. \tag{2.54}$$

Noticeably,

$$\hat{q}_C(t) = \begin{cases} \frac{\hat{C}_n(t,t)}{t} = \hat{\lambda}_L(t), & \text{if } t \in (0,0.5], \\ \frac{\hat{\bar{C}}_n(t,t)}{1-t} = \hat{\lambda}_U(t), & \text{if } t \in (0.5,1). \end{cases}$$

In our framework, that is, when analysing multidimensional poverty, this function allows to evaluate tail dependence using some specific thresholds, such as the first decile ($t = 0.1$) or the second decile ($t = 0.2$). Thus, if we consider income and health as the two dimensions of poverty, it is possible to analyse the probability that an individual that is in the first (second) decile in income is also in the first (second) decile in health.

Other alternatives to study extremal dependence going beyond the asymptotic nature of the tail dependence coefficients in (2.43) and (2.44) have been proposed in the literature. One possibility, widely used in finance, is the use of conditional versions of the linear correlation coefficient. This approach consists in calculating the linear correlation coefficient conditioned on values of the variables smaller or larger than a given threshold; see Malevergne and Sornette (2006, ch. 6). However, as we previously pointed out, the linear correlation coefficient is only appropriate when studying linear dependence in elliptical contexts. To measure extremal dependence in more general contexts, Charpentier (2003) suggests the use of conditional versions of the Spearman's rank correlation coefficient; see also Malevergne and Sornette (2006, ch. 6). However, as Sweeting and Fotiou (2013) point out, one potential complication with these conditional measures of dependence is that the thresholds at which they are measured must be chosen, and this choice is bound to be arbitrary. In fact, this is the same difficulty that we find when estimating the asymptotic tail dependence coefficients in a non-parametric way, and which have been already discussed.

So far we have focused our attention on the study of tail dependence in the bivariate case, which has been discussed extensively in the literature. In the next section we will review some of the contributions that have been made beyond the bivariate framework and we will make a new proposal to extend the TCF to a multivariate framework. As we will see, the bivariate tail dependence coefficients discussed above cannot be generalised to the general multivariate case ($d > 2$) in a straightforward and unique way. This is another proof of the difficulties that going from the bivariate to the multivariate framework entails; recall, again, the arguments put forward by Durante et al. (2014).

2.5.2 Multivariate tail dependence measures

As we have already seen in the previous section, the study of tail dependence, even for the bivariate case, is not a simple task. But things become even more difficult, both from a theoretical and from a practical point of view, when we turn to the scarcely addressed multivariate case. To start with, the presence of more than two components brings additional difficulties to the definition of tail dependence. As a result, several different proposals to measure multivariate tail dependence can be found in the literature. For instance, Frahm (2006) introduces the so-called "extremal dependence coefficient", which studies the asymptotic dependence structure of the minimum and the maximum of a random vector, whereas Schmid and Schmidt (2007a) define a measure of multivariate lower tail dependence based on a conditional version of the coefficient ρ_d^- discussed in Section 2.4.2. However, as Gijbels et al. (2020) argue, none of these measures are proper multivariate generalisations of the well-known bivariate tail dependence coefficients, since these measures do not reduce to λ_L and λ_U in (2.43) and (2.44), respectively, when $d = 2$. Hence, these coefficients can be better considered as different types of tail dependence measures, rather than multivariate generalisations of the traditional bivariate tail dependence coefficients previously discussed, and will not be further considered.

Over the rest of this section, we will limit our attention to the most natural multivariate extensions of the bivariate tail dependence coefficients λ_L and λ_U in (2.43) and (2.44), respectively. These extensions, which are considered by Schmidt (2002), Li (2008, 2009), Chan and Li (2008), De Luca and Rivieccio (2012), Di Bernardino and Rullière (2016) and Fernández-Sánchez et al. (2016), among others, are defined as follows.

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a d-dimensional continuous random vector with joint distribution function F, marginals $F_1, \ldots, F_d$, and associated copula C. Furthermore, let $J = \{1, \ldots, d\}$ and consider two non-empty subsets $J_h \subset J$ and $\bar{J}_h = J \setminus J_h$ with respective cardinal $h \geq 1$

and $d-h \geq 1$. Then, the coefficient of multivariate lower tail dependence is given by:

$$\begin{aligned}\lambda_L^{J_h|\bar{J}_h} &= \lim_{u\to 0^+} Pr\big(X_j \leq F_j^{-1}(u),\ \forall j \in J_h | X_i \leq F_i^{-1}(u),\ \forall i \in \bar{J}_h\big) = \\ &= \lim_{u\to 0^+} Pr\big(U_j \leq u,\ \forall j \in J_h | U_i \leq u,\ \forall i \in \bar{J}_h\big). \end{aligned} \tag{2.55}$$

Similarly, the coefficient of multivariate upper tail dependence is given by:

$$\begin{aligned}\lambda_U^{J_h|\bar{J}_h} &= \lim_{u\to 1^-} Pr\big(X_j > F_j^{-1}(u),\ \forall j \in J_h | X_i > F_i^{-1}(u),\ \forall i \in \bar{J}_h\big) = \\ &= \lim_{u\to 1^-} Pr\big(U_j > u,\ \forall j \in J_h | U_i > u,\ \forall i \in \bar{J}_h\big). \end{aligned} \tag{2.56}$$

Hence, these coefficients measure the probability that a group of h variables take simultaneously extreme (very small or very large) values, given that the remaining $d-h$ variables take simultaneously extreme (very small or very large) values. If, for some subset $J_h \subset J$, $\lambda_L^{J_h|\bar{J}_h} > 0$, then it is said that there is lower tail dependence, whereas there is upper tail dependence if, for some subset $J_h \subset J$, $\lambda_U^{J_h|\bar{J}_h} > 0$. If for all $J_h \subset J$, $\lambda_L^{J_h|\bar{J}_h} = 0$ ($\lambda_U^{J_h|\bar{J}_h} = 0$), then we say that there is lower (upper) tail independence. Noticeably, when $d = 2$, $\lambda_L^{J_h|\bar{J}_h}$ and $\lambda_U^{J_h|\bar{J}_h}$ reduce to λ_L and λ_U in (2.43) and (2.44), respectively.

Although the coefficients defined in (2.55) and (2.56) are very intuitive measures of multivariate tail dependence, an important drawback must be mentioned. In particular, for $d > 2$, several different coefficients can be considered, depending on the choice of the set J_h. Obviously, things become more difficult as the number of dimensions increases. Furthermore, when conditioning on more than one variable, the representation of the tail dependence coefficients through the copula function, which is clear-cut in the bivariate case, is not straightforward, since multivariate marginals of the copula are involved. This can be avoided by conditioning on just one variable, that is, by taking $h = d-1$. In doing so, the following multivariate tail dependence coefficients come up:

$$\begin{aligned}\lambda_L^d &= \lim_{u\to 0^+} Pr\big(X_j \le F_j^{-1}(u),\ \ \forall j \in J_{d-1}|X_i \le F_i^{-1}(u),\ \ \forall i \in \bar{J}_{d-1}\big) = \\ &= \lim_{u\to 0^+} Pr\big(U_j \le u,\ \ \forall j \in J_{d-1}|U_i \le u,\ \ \forall i \in \bar{J}_{d-1}\big) = \\ &= \lim_{u\to 0^+} \frac{C(u,\ldots,u)}{u} \end{aligned} \tag{2.57}$$

and

$$\begin{aligned}\lambda_U^d &= \lim_{u\to 1^-} Pr\big(X_j > F_j^{-1}(u),\ \ \forall j \in J_{d-1}|X_i > F_i^{-1}(u),\ \ \forall i \in \bar{J}_{d-1}\big) = \\ &= \lim_{u\to 1^-} Pr\big(U_j > u,\ \ \forall j \in J_{d-1}|U_i > u,\ \ \forall i \in \bar{J}_{d-1}\big) = \\ &= \lim_{u\to 1^-} \frac{\bar{C}(u,\ldots,u)}{1-u}. \end{aligned} \tag{2.58}$$

As we can see, these coefficients, which are considered by Alink et al. (2007), Joe et al. (2010) and Hua and Joe (2011), among others, have a simple representation through the copula function, and are the ones we adopt in this study. That is, we focus on the probability that a group of $d-1$ variables take simultaneously extreme (very small or very large) values, given that the remaining variable takes extreme (very small or very large) values. In a multidimensional poverty setting, the coefficient λ_L^d in (2.57) becomes specially relevant, since it measures the limit probability that an individual is simultaneously extremely low-ranked (extremely poor) in $d-1$ relevant dimensions of poverty given that he/she is extremely low-ranked (extremely poor) in the remaining dimension. Similarly, λ_U^d in (2.58) measures the limit probability that an individual is simultaneously extremely high-ranked (extremely rich) in $d-1$ relevant dimensions of poverty given that he/she is extremely high-ranked (extremely rich) in the remaining dimension. For example, if we consider that income, education and health are the three dimensions of poverty, λ_L^d measures the limit probability that an individual that is extremely poor in one dimension, say income, is also extremely poor in education and health. On the other hand,

λ_U^d measures the limit probability that an individual that is extremely rich in one dimension, say income, is also extremely rich in education and health.

In the case of independence, that is, if $C = \Pi$, $\lambda_L^d = \lambda_U^d = 0$. On the other hand, in the case of maximal dependence, that is, if $C = M$, $\lambda_L^d = \lambda_U^d = 1$. Moreover, Fernández-Sánchez et al. (2016) show that, for any $d \geq 3$, the coefficients λ_L^d and λ_U^d are non-increasing in d, that is, $\lambda_L^d \leq \lambda_L^{d-1}$ and $\lambda_U^d \leq \lambda_U^{d-1}$ for any $d \geq 3$.

As we argued when we considered the bivariate case, the tail dependence coefficients defined above give an asymptotic approximation of the behaviour of the copula in the joint tails of the distribution. Alternatively, one may be interested in analysing extremal dependence going beyond this asymptotic definition, analysing the tail behaviour at some finite points near the corners of the hypercube. To that aim, Sweeting and Fotiou (2013) consider the following multivariate extensions of the finite tail dependence coefficients defined in (2.45) and (2.46):

$$\lambda_L^d(u) = \frac{C(u, \ldots, u)}{u} \tag{2.59}$$

and

$$\lambda_U^d(u) = \frac{\bar{C}(u, \ldots, u)}{1-u}, \tag{2.60}$$

for $u \in (0, 1)$. These functions allow to study multivariate tail dependence at a subasymptotic level. Following this approach, we propose a natural multivariate generalisation of the tail concentration function introduced in (2.49), which integrates in a unique function information about the degree of multivariate dependence in both the lower and the upper tail of the joint distribution. The multivariate generalisation of the tail concentration function is defined as the function $q_C^d : (0, 1) \to \mathbf{I}$, given by:

$$q_C^d(u) = \frac{C(u, \ldots, u)}{u} \mathbf{1}_{(0,0.5]} + \frac{\bar{C}(u, \ldots, u)}{1-u} \mathbf{1}_{(0.5,1)}. \tag{2.61}$$

This function provides an important advantage when measuring dependence in a multivariate framework, since, in spite of its multivariate nature, it allows to represent in the plane the degree multivariate dependence (for any $d > 2$) in the joint tails of a distribution. This will make it easier to carry out graphical comparisons of multivariate tail dependence in different distributions, for instance, comparisons between different countries or different temporal moments for the same country.

The multivariate TCF, contrary to what happens in the bivariate case, is not necessarily continuous, since, with $d > 2$, $C(0.5, \ldots, 0.5)$ is not necessarily equal to $\bar{C}(0.5, \ldots, 0.5)$ and thus $q_C^d(0.5^-)$ does not necessarily coincide with $q_C(0.5^+)$. When $C = M$, that is, in the case of maximal dependence, $q_M^d(u) = 1\ \forall u$, and, in the case of independence, that is, when $C = \Pi$, then $q_\Pi^d(u) = u^{d-1}\mathbf{1}_{(0,0.5]} + (1-u)^{d-1}\mathbf{1}_{(0.5,1)}$. For instance, for d = 3, $q_\Pi^d(u) = u^2\mathbf{1}_{(0,0.5]} + (1-u)^2\mathbf{1}_{(0.5,1)}$. Noticeably, q_C^d can be alternatively defined in terms of the functions $\lambda_L^d(u)$ and $\lambda_U^d(u)$ as follows.

$$q_C^d(u) = \begin{cases} \lambda_L^d(u), & \text{if } u \in (0, 0.5], \\ \lambda_U^d(u), & \text{if } u \in (0.5, 1). \end{cases}$$

To better appreciate the usefulness of this function, in Figure 2.5 we present the trivariate TCF, q_C^3, for three different trivariate parametric copulas (the Clayton copula in the left panel; the Frank copula in the central panel; and the Gumbel copula in the right panel) for different degrees of dependence (given by the parameter θ); see Appendix 2.C for a brief discussion on these copulas. Moreover, in all panels, the trivariate TCF of the independent copula Π is represented in blue and the TCF of the commonotonic copula M is represented in red.

Several conclusions emerge from this figure. First, the three copula models are related to different shapes of the TCF. In particular, Clayton copulas (left panel) are asymmetric, with observations in the lower tail being somewhat more dependent than observations in the upper tail, since $q_C^3(u) > q_C^3(1-u)$ for $0 < u < 0.5$, that is, $\lambda_L^3(u) > \lambda_U^3(1-u)$. In a multivariate poverty setting, this means that the probability that an individual that is extremely poor in

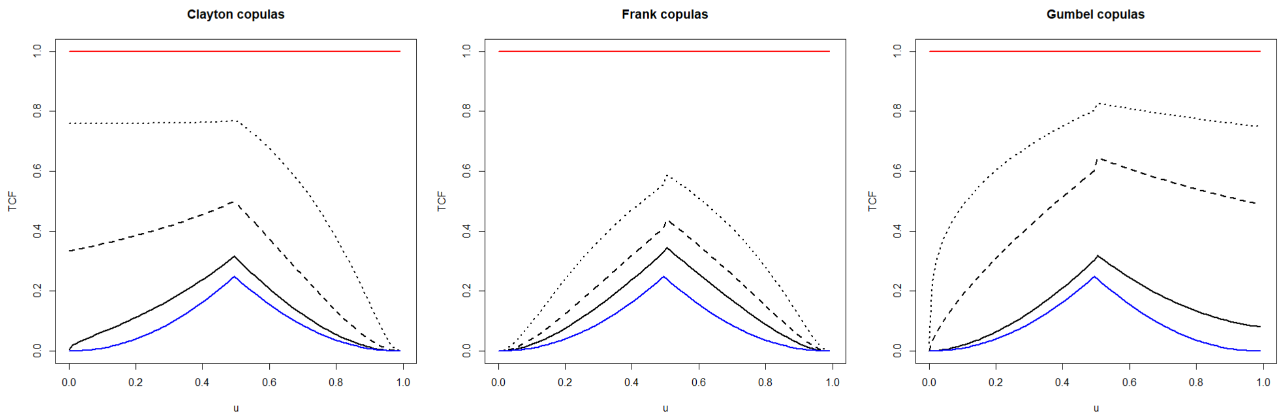

Figure 2.5: Trivariate TCFs for some trivariate parametric copulas. Left panel represents Clayton copulas with $\theta = 0.1$ (solid), $\theta = 1$ (dashed) and $\theta = 2$ (dotted). Central panel represents Frank copulas with $\theta = 1$ (solid), $\theta = 2$ (dashed) and $\theta = 4$ (dotted). Right panel represents Gumbel copulas with $\theta = 1.1$ (solid), $\theta = 2$ (dashed) and $\theta = 4$ (dotted). In all panels, the trivariate TCF of the independent copula Π is represented in blue and the TCF of the commonotonic copula M is represented in red.

one dimension is also extremely poor in the other two dimensions is higher than the probability that an individual that is extremely rich in one dimension is also extremely rich in the other two dimensions. The contrary occurs in the case of Gumbel copulas (right panel), where observations in the upper tail are somewhat more dependent than observations in the upper tail, since $q_C^3(u) < q_C^3(1-u)$ for $0 < u < 0.5$, that is, $\lambda_L^3(u) > \lambda_U^3(1-u)$. In a multidimensional poverty setting, this means that the probability that an individual that is extremely rich in one dimension is also extremely rich in the other two dimensions is higher than the probability that an individual that is extremely poor in one dimension is also extremely poor in the other two dimensions. Finally, Frank copulas (central panel) show a symmetric behaviour, since $q_C^3(u) = q_C^3(1-u)$ for $0 < u < 0.5$, that is, $\lambda_L^3(u) = \lambda_U^3(1-u)$, which in our setting means that the probability that an individual that is extremely poor in one dimension is also extremely poor in the other two dimensions is equal to the probability that an individual that is extremely rich in one dimension is also extremely rich in the other two dimensions. Furthermore, for the three models, as the degree of dependence increases (as the parameter θ increases), the trivariate TCF moves upwards and closer to the upper bound given by the maximal dependence copula

M, whereas as θ decreases, the trivariate TCF approaches the lower bound corresponding to the independent copula Π.

As in the bivariate case, other proposals to measure extremal dependence going beyond the asymptotic definition given by the multivariate generalisations of the tail dependence coefficients can be found in the literature, although these proposals are still scarce. One possibility is to apply conditional versions of multivariate measures of dependence such as the multivariate versions of Spearman's rank correlation coefficient. For instance, Schmid and Schmidt (2007a) propose a conditional version of the coefficient of average lower orthant dependence ρ_d^- defined in (2.20). This conditional version allows to measure lower orthant dependence focusing on the lowest part of the joint distribution. In fact, they also propose, based on this conditional version of ρ_d^-, a new measure of multivariate tail dependence that can be regarded as an alternative to the multivariate generalisations of the traditional tail dependence coefficients that have been discussed in this section.

Once we have specified the coefficients of multivariate tail dependence we are going to focus on, namely λ_L^d and λ_U^d, we next move to the problem of estimating them. As in the bivariate case, we will only consider non-parametric estimation. It is important to take into account that, while in the bivariate case there exist several proposals of non-parametric estimators of the tail dependence coefficients, the literature in the multivariate case is very scarce. Nonetheless, some proposals can be found. For example, Gijbels et al. (2020) propose to estimate the coefficients λ_L^d and λ_U^d non-parametrically by using the empirical counterparts of the copula and the survival functions. This approach has also been considered by Salazar and Ng (2015), who propose the following estimators:

$$\hat{\lambda}_L^d\left(\frac{k}{n}\right) = \frac{\hat{C}_n\left(\frac{k}{n}, \ldots, \frac{k}{n}\right)}{\frac{k}{n}}, \tag{2.62}$$

and

$$\hat{\lambda}_U^d\left(\frac{k}{n}\right) = \frac{\hat{\bar{C}}_n\left(\frac{k}{n}, \ldots, \frac{k}{n}\right)}{1 - \frac{k}{n}}, \tag{2.63}$$

where $\hat{C}_n$ and $\hat{\bar{C}}_n$ are the empirical copula and empirical survival function defined in (2.30) and (2.32), respectively, and $k \in \{1, \ldots, n\}$ is a parameter to be chosen. As Salazar and Ng (2015) show, these are multivariate extensions of the non-parametric estimators of bivariate tail dependence considered by Schmidt and Stadtmüller (2006) and defined in (2.52) and (2.53). They also show that these estimators are asymptotically normal and strongly consistent.

The estimators in (2.62) and (2.63) suffer from the same drawback as their bivariate counterparts, i.e., they depend on the choice of an appropriate threshold and, as we have already discussed in the previous section, this is not a trivial task; see Salazar and Ng (2015) for some proposals to deal with this issue. Moreover, in the multivariate case, additional difficulties arise. For instance, Krupskii and Joe (2019) point out that the non-parametric estimation of tail dependence is affected by the number of dimensions considered. In particular, as the number of dimensions increases, larger sample sizes are needed to have reliable estimators of tail dependence when k is small (in the case of lower tail dependence) or large (in the case of upper tail dependence), due to sparsity of data in the tail regions. Krupskii and Joe (2019) give some proposals to deal with this problem, although further research is surely needed.

The multivariate TCF defined in (2.61) allows to analyse multivariate tail dependence going beyond the asymptotic definition of the coefficients (2.57) and (2.58). By analogy to the bivariate case, we propose to estimate this multivariate extension of the TCF by replacing the copula and the survival function in (2.61) by their empirical counterparts. That is, the empirical version of the multivariate TCF that we propose is given, for $t \in (0, 1)$, by:

$$\hat{q}_C^d(t) = \frac{\hat{C}_n(t, \ldots, t)}{t}\mathbf{1}_{(0,0.5]} + \frac{\hat{\bar{C}}_n(t, \ldots, t)}{1-t}\mathbf{1}_{(0.5,1)}. \tag{2.64}$$

That is,

$$\hat{q}_C^d(t) = \begin{cases} \frac{\hat{C}_n(t,\ldots,t)}{t} = \hat{\lambda}_L^d(t), & \text{if } t \in (0, 0.5], \\ \frac{\hat{\bar{C}}_n(t,\ldots,t)}{1-t} = \hat{\lambda}_U^d(t), & \text{if } t \in (0.5, 1). \end{cases}$$

In our framework, that is, when analysing multidimensional poverty, this function allows to evaluate tail dependence using some specific thresholds, such as the first decile ($t = 0.1$) or the second decile ($t = 0.2$). Thus, if we consider income, health and education as the three dimensions of poverty, it is possible to analyse the probability that an individual that is in the first (second) decile in income is also simultaneously in the first (second) decile in both health and education.

To conclude this section, we want to highlight a particular characteristic of the tail dependence coefficients defined so far that has been the subject of some debate. In particular, the tail dependence coefficients in (2.43) and (2.44), as well as their natural multivariate generalisations in (2.57) and (2.58), evaluate dependence only in the diagonal section of the copula, i.e, at points $(u, \ldots, u) \in \mathbf{I}^d$. That is, all the variables considered are required to become extreme at the same rate. In fact, this characteristic is also shared by the non-asymptotic tool to measure extremal dependence such as the tail concentration function defined in (2.49) and (2.61). For the analysis of multidimensional poverty, focusing on the main diagonal of the copula allows to study if the positions of the individuals in all the poverty dimensions are aligned, which is a very relevant question, as we have repeatedly argued along this chapter. Hence, for our purpose, limiting our attention to measures of tail dependence based on the main diagonal of the copula does not constitute a problem. Nevertheless, some authors, such as Abdous et al. (2005) or Schmid and Schmidt (2007a), among others, argue that this is a limiting notion of dependence, as the tail behaviour of a distribution may be very different going to the copula's corners following an alternative route to the main diagonal. In fact, one could reasonably argue that a multidimensional poverty analysis would be enriched by an approach that allows to rigorously study tail dependence using different thresholds for each of the dimensions. In the bivariate case, the conditional versions of measures of dependence such as the Spearman's rank correlation coefficient, which were briefly mentioned at the end of Section 2.5.1, allow to do so. However, in the multivariate case, the literature on this issue is scarce. Nonetheless, some proposals can be found. In fact, the conditional version of ρ_d^- proposed by Schmid and Schmidt (2007a) and its associated measure of multivariate tail dependence, which have been previously mentioned, enable to study extremal dependence going beyond the main diagonal of the copula. Furthermore, Schmid and Schmidt (2007a) also propose a non-parametric estimator for this measure which is based on the empirical copula. However, as Gijbels et al. (2020) point out, this estimator suffer from the same problems of the estimator for ρ_d^- proposed in Schmid and Schmidt (2007b), and which have been discussed in Section 2.4.2.2. Finally, another, more recent, proposal to measure tail dependence going beyond the main diagonal of the copula can be found in Escanciano and Hualde (2019). We consider this as a promising field for further research.

Appendix 2.A

In this appendix, we briefly review the class of Average Orthant Dependence (AOD) measures, which were introduced by Dolati and Úbeda-Flores (2006) and include, as a particular case, the multivariate version of Spearman's rho defined in (2.26). To define these measures, we recall the concept of orthant dependence reviewed in Section 2.3.2. Then, taking into account that, as it has been previously argued, the differences $[C(\mathbf{u}) - \Pi(\mathbf{u})]$ and $[\bar{C}(\mathbf{u}) - \bar{\Pi}(\mathbf{u})]$ can be thought of as measures of "local" lower and upper orthant dependence, respectively, let us define the following function $D(C, C')$, which represents the C'-average orthant dependence:

$$D(C, C') = \int_{\mathbf{I}^d} (\bar{C}(\mathbf{u}) - \bar{\Pi}(\mathbf{u}) + C(\mathbf{u}) - \Pi(\mathbf{u}))dC'(\mathbf{u}).$$

Hence, this function measures average orthant dependence with respect to a reference copula C'. Based on this function, Dolati and Úbeda-Flores (2006) propose the general class of AOD measures of multivariate association, which is given by

$$\omega_d(C, C') = \frac{D(C, C')}{D(M, C')}. \tag{A1}$$

As Pérez and Prieto-Alaiz (2016a) show, an appealing property of the AOD measures is that they have a probabilistic interpretation in terms of multivariate concordance and include, as members of this family, some of the well-known multivariate measures of association, as we will see next. In fact, recalling the definition in (2.10), the family of AOD measures of multivariate association given by (A1) can be written in terms of the probability of multivariate concordance Q'_d as follows:

$$\omega_d(C, C') = \frac{Q'_d(C, C') - Q'_d(\Pi, C')}{Q'_d(M, C') - Q'_d(\Pi, C')}. \tag{A2}$$

Hence, ω_d is a normalised probability of concordance between a random vector $\mathbf{X}$ with copula C and a reference random vector with copula C'. Noticeably, different choices of C' in (A2) lead to different AOD measures which, at the same time, become multivariate extensions of well-known bivariate measures of association such as Spearman's rho, Spearman's footrule and Gini's gamma; see Nelsen (1998) for a detailed description of bivariate Speaman's footrule and Gini's gamma in terms of copulas and see Genest et al. (2010) for a further discussion of these measures in a multivariate framework.

In particular, when $C' = \Pi$ in (A2), the multivariate version of Spearman's rho ρ_d in (2.26) is obtained:

$$\rho_d = \omega_d(C, \Pi) = \frac{Q'_d(C,\Pi) - Q'_d(\Pi,\Pi)}{Q'_d(M,\Pi) - Q'_d(\Pi,\Pi)} = \frac{(d+1)}{2^d - (d+1)} \left[2^{d-1} Q'_d(C,\Pi) - 1\right]. \qquad \text{(A3)}$$

Hence, ρ_d can be interpreted as a measure of average orthant dependence with respect to independence and, as we pointed out in Section 2.4.2, is the average of the coefficients ρ_d^- and ρ_d^+ also previously discussed in Section 2.4.2.

On the other hand, as it was shown by Pérez and Prieto-Alaiz (2016a), when $C' = M$ in (A2), the multivariate version of Spearman's footrule proposed by Úbeda-Flores (2005) comes up, namely:

$$\varphi_d = \omega_d(C, M) = \frac{Q'_d(C,M) - Q'_d(\Pi,M)}{Q'_d(M,M) - Q'_d(\Pi,M)} = \frac{d+1}{d-1} Q'_d(C,M) - \frac{2}{d-1}. \qquad \text{(A4)}$$

Thus, φ_d is the normalised probability of concordance between a random vector $\mathbf{X}$ with copula C and maximal dependence, represented by the commmonotonic copula M. When there is maximal dependence ($C = M$), $\varphi_d = 1$ and when there is independence ($C = \Pi$), $\varphi_d = 0$. Moreover, the lower bound of the multivariate Spearman's footrule is $-\frac{1}{d}$. Noticeably, unlike ρ_d, φ_d is not a measure of multivariate concordance according to the axiomatic definitions of Dolati and Úbeda-Flores (2006) and Taylor (2007).

Finally, Pérez and Prieto-Alaiz (2016a) also show that another particular case of the family of AOD measures is the multivariate Gini's gamma, γ_d, proposed by Behboodian et al. (2007). In particular, this coefficient comes up when $C = A$ in (A2), that is,

$$\gamma_d = \omega_d(C, A) = \frac{Q'_d(C, A) - Q'_d(\Pi, A)}{Q'_d(M, A) - Q'_d(\Pi, A)}, \tag{A5}$$

where A is the average of the upper and lower Fréchet-Hoeffding bounds, i.e. $A = \frac{M+W}{2}$. Therefore, the multivariate Gini's gamma can be regarded as the normalised probability of concordance with respect to A. This coefficient takes the value 1 when there is perfect positive dependence ($C = M$) and the value 0 in the case of independence ($C = \Pi$). However, as far as we know, it is not settled yet if γ_d is a measure of multivariate concordance according to the axiomatic definition of Dolati and Úbeda-Flores (2006) and Taylor (2007), since A is not a copula.

For a review of these three coefficients (ρ_d, φ_d and γ_d) and the proposals to estimate them, see Pérez and Prieto-Alaiz (2016a), who also provide an empirical application of them to analyse the multivariate dependence between the dimensions of the Human Development Index (HDI) in the world using data from 187 countries. See also Genest et al. (2010) for a review of φ_d and γ_d.

Appendix 2.B

Proposition 1. *The plug-in estimator of* ρ_d *defined in (2.37), namely*

$$\widehat{\rho}_d = \frac{\widehat{\rho_d^-} + \widehat{\rho}_d^+}{2}, \tag{B1}$$

where $\widehat{\rho_d^-}$ *and* $\widehat{\rho}_d^+$ *are the estimators in (2.36) and (2.34), respectively, coincides with the estimator of* ρ_d *proposed by Dolati and Úbeda-Flores (2006) in the framework of AOD measures of multivariate concordance.*

Proof. The estimator of ρ_d proposed by Dolati and Úbeda-Flores (2006), that will be denoted as $\widehat{\rho}_d^{DUF}$, is as follows (see Example 5.1 in that paper):

$$\widehat{\rho}_d^{DUF} = \frac{\frac{1}{n}\sum_{j=1}^{n}\left[C'\left(\frac{R_{1j}}{n+1},\cdots,\frac{R_{dj}}{n+1}\right) + \overline{C}'\left(\frac{R_{1j}}{n+1},\cdots,\frac{R_{dj}}{n+1}\right)\right] - a_{d,n}}{b_{d,n} - a_{d,n}}, \tag{B2}$$

where

$$C'\left(\frac{R_{1j}}{n+1},\cdots,\frac{R_{dj}}{n+1}\right) = \prod_{i=1}^{d}\frac{R_{ij}}{n+1},\ \overline{C}'\left(\frac{R_{1j}}{n+1},\cdots,\frac{R_{dj}}{n+1}\right) = \prod_{i=1}^{d}\left(1 - \frac{R_{ij}}{n+1}\right), \tag{B3}$$

and

$$a_{d,n} = \frac{1}{2^{d-1}},\ b_{d,n} = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n+1}\right)^d + \frac{1}{n}\sum_{j=1}^{n}\left(1 - \frac{j}{n+1}\right)^d. \tag{B4}$$

Putting back (B3) and (B4) in (B2), the following expression comes up:

$$\widehat{\rho}_d^{DUF} = \frac{\frac{1}{n}\frac{1}{(n+1)^d}\sum_{j=1}^{n}\left[\prod_{i=1}^{d}R_{ij} + \prod_{i=1}^{d}(n+1-R_{ij})\right] - \frac{1}{2^{d-1}}}{\frac{1}{n}\frac{1}{(n+1)^d}\left[\sum_{j=1}^{n}j^d + \sum_{j=1}^{n}(n+1-j)^d\right] - \frac{1}{2^{d-1}}}. \tag{B5}$$

Now, multiplying both the numerator and the denominator of (B5) by $(n+1)^d$ and taking into account that $\sum\limits_{j=1}^{n} j^d = \sum\limits_{j=1}^{n}(n+1-j)^d$, we have:

$$\widehat{\rho}_d^{DUF} = \frac{\dfrac{1}{n}\sum\limits_{j=1}^{n}\left(\prod\limits_{i=1}^{d} R_{ij} + \prod\limits_{i=1}^{d} \overline{R}_{ij}\right) - \dfrac{(n+1)^d}{2^{d-1}}}{\dfrac{2}{n}\sum\limits_{j=1}^{n} j^d - \dfrac{(n+1)^d}{2^{d-1}}}, \tag{B6}$$

where $\overline{R}_{ij} = n+1-R_{ij}$. On the other hand, replacing $\widehat{\rho_d^-}$ and $\widehat{\rho}_d^+$ in (B1) by their expressions in (2.36) and (2.34), respectively, the expression in (B6) comes up. Hence, it turns out that $\widehat{\rho}_d = \widehat{\rho}_d^{DUF}$ and the result is proven. □

Proposition 2. *In the trivariate case ($d = 3$), the plug-in estimator of ρ_3 defined in (2.37) can be computed as the average of their corresponding pairwise sample coefficients, that is,*

$$\widehat{\rho}_3 = \frac{\widehat{\rho}_{12} + \widehat{\rho}_{13} + \widehat{\rho}_{23}}{3}, \tag{B7}$$

where $\widehat{\rho}_{ik}$ denotes the bivariate sample Spearman's rho for the pair (X_i, X_k), with $1 \leq i < k \leq 3$.

Proof. First, from equations (2.37)-(2.39), we obtain the following expression for $\widehat{\rho}_3$:

$$\widehat{\rho}_3 = \frac{\widehat{\rho_3^-} + \widehat{\rho}_3^+}{2} = \frac{1}{2}\left[\frac{8}{n(n-1)(n+1)^2}\sum_{j=1}^{n}(R_{1j}R_{2j}R_{3j} + \overline{R}_{1j}\overline{R}_{2j}\overline{R}_{3j}) - 2\frac{n+1}{n-1}\right]. \tag{B8}$$

Now, taking into account that $\overline{R}_{ij} = n+1-R_{ij}$, we have

$$\overline{R}_{1j}\overline{R}_{2j}\overline{R}_{3j} = (n+1)^3 - (n+1)^2\sum_{i=1}^{3} R_{ij} + (n+1)(R_{1j}R_{2j} + R_{1j}R_{3j} + R_{2j}R_{3j}) - R_{1j}R_{2j}R_{3j},$$

and so the sumation in (B8) becomes

$$\sum_{j=1}^{n}(R_{1j}R_{2j}R_{3j} + \overline{R}_{1j}\overline{R}_{2j}\overline{R}_{3j}) = n(n+1)^3 - (n+1)^2\sum_{i=1}^{3}\sum_{j=1}^{n} R_{ij} + (n+1)\sum_{j=1}^{n}(R_{1j}R_{2j} + R_{1j}R_{3j} + R_{2j}R_{3j}).$$

Now, since $\sum_{j=1}^{n} R_{ij} = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, the expression above becomes

$$\sum_{j=1}^{n}(R_{1j}R_{2j}R_{3j} + \overline{R}_{1j}\overline{R}_{2j}\overline{R}_{3j}) = -\frac{1}{2}n(n+1)^3 + (n+1)\sum_{j=1}^{n}(R_{1j}R_{2j} + R_{1j}R_{3j} + R_{2j}R_{3j}). \quad \text{(B9)}$$

Putting back (B9) in (B8), it turns out that

$$\widehat{\rho}_3 = \frac{4}{n(n^2-1)}\left(\sum_{j=1}^{n} R_{1j}R_{2j} + \sum_{j=1}^{n} R_{1j}R_{3j} + \sum_{j=1}^{n} R_{2j}R_{3j}\right) - \frac{3(n+1)}{(n-1)}. \quad \text{(B10)}$$

On the other hand, the average of the pairwise sample Spearman's coefficients is

$$\begin{aligned} \frac{\widehat{\rho}_{12} + \widehat{\rho}_{13} + \widehat{\rho}_{23}}{3} &= \frac{1}{3}\left[\frac{12}{n(n^2-1)}\left(\sum_{j=1}^{n} R_{1j}R_{2j} + \sum_{j=1}^{n} R_{1j}R_{3j} + \sum_{j=1}^{n} R_{2j}R_{3j}\right) - \frac{9(n+1)}{n-1}\right] = \\ &= \frac{4}{n(n^2-1)}\left(\sum_{j=1}^{n} R_{1j}R_{2j} + \sum_{j=1}^{n} R_{1j}R_{3j} + \sum_{j=1}^{n} R_{2j}R_{3j}\right) - \frac{3(n+1)}{(n-1)}. \end{aligned}$$

Hence, comparing this last equation with (B10), the result in (B7) comes up. □

Appendix 2.C

In the literature on copulas, several parametric families of copulas have been proposed for building stochastic models. In this appendix, we focus on some parametric copulas in the class of Archimedean copulas, which are especially popular in the literature and have been frequently employed in financial applications. For a more complete discussion on parametric families of copulas, we refer the interested reader to Nelsen (2006), Joe (2014) and Durante and Sempi (2015).

The Archimedean copulas, which have been extensively discussed in McNeil and Nešlehová (2009), are capable of capturing a wide range of different dependence structures. In the following, we focus on three particularly popular families of Archimedean copulas, namely the Gumbel, Clayton and Frank families, with special attention to their tail dependence structure.

Gumbel d-copula

The Gumbel family of d-copulas, introduced in Gumbel (1960), is given by the following expression:

$$C(u_1, \ldots, u_d) = \exp\left(-\Big(\sum_{i=1}^{d} (-\log(u_i))^{\theta} \Big)^{\frac{1}{\theta}} \right), \tag{C1}$$

where $\theta \geq 1$ is a parameter controlling the dependence. The case $\theta = 1$ gives the independent copula Π as a special case, and the limit of (C1) as $\theta \to +\infty$ is the comonotonicity copula M. That is, as the parameter θ increases, so does the degree of multivariate dependence.

Bivariate Gumbel copulas are positive quadrant dependent. Moreover, their tail dependence coefficients are:

$$\lambda_L = 0 \ \text{ and } \ \lambda_U = 2 - 2^{\frac{1}{\theta}}.$$

That is, bivariate Gumbel copulas present upper tail dependence for $\theta > 1$ but they are lower tail independent.

Clayton d-copula

The Clayton family of d-copulas, introduced in Clayton (1978) is given by the following expression:

$$C(u_1, \dots, u_d) = \max\left\{ \left(\sum_{i=1}^{d} u_i^{-\theta} - (d-1) \right)^{-\frac{1}{\theta}}, 0 \right\}, \tag{C2}$$

with $\theta \geq \frac{-1}{d-1}$ and $\theta \neq 0$, where θ is a parameter controlling the dependence. The limiting case $\theta \to 0$ gives the independent copula Π. Furthermore, for bivariate Clayton copulas, the tail dependence coefficients take the following form:

$$\lambda_L = \begin{cases} 2^{-\frac{1}{\theta}}, & \text{if } \theta > 0, \\ 0, & \text{if } \theta \in [-1, 0) \end{cases}, \qquad \lambda_U = 0.$$

That is, Clayton copulas can present lower tail dependence for $\theta > 0$ but they are upper tail independent.

Frank d-copula

The Frank family of d-copulas was introduced by Frank (1979). The standard expression of this family is:

$$C(u_1, \dots, u_d) = -\frac{1}{\theta} \log \left(1 + \frac{\prod_{i=1}^{d} (e^{-\theta u_i} - 1)}{(e^{-\theta} - 1)^{d-1}} \right), \tag{C3}$$

where $\theta > 0$ when $d \geq 3$. For the case d = 2, the parameter θ can be also extended to the case $\theta < 0$. The limiting case $\theta \to 0$ gives the independent copula Π. Moreover, the tail dependence coefficients of bivariate Frank copulas are

$$\lambda_L = 0 \text{ and } \lambda_U = 0.$$

That is, bivariate Frank copulas do not present neither lower nor upper tail dependence. Furthermore, as Mai and Scherer (2017) points out, the distinct property of the Frank family is that, in the bivariate case, it is the only Archimedean family which is radially symmetric.

Chapter 3

Empirical application: multivariate dependence patterns between dimensions of poverty in Europe

3.1 Introduction

The main goal of this thesis is to incorporate the measurement of the dependence between dimensions of poverty into the multidimensional poverty analysis, an important aspect that has been scarcely addressed in the literature. As we argued in the previous chapter, the copula methodology is highly useful in this framework, since it allows to measure dependence in multivariate, possibly non-Gaussian and possibly non-linear contexts, such as the ones we usually face in Welfare Economics.

However, and despite their popularity in other fields such as finance or environmental sciences, the application of copula-based approaches in Welfare Economics is scarce and mostly limited to the bidimensional setting; see Dardanoni and Lambert (2001), Quinn (2007), Bonhomme and Robin (2009), Bø et al. (2012) and Aaberge et al. (2018). In a multidimensional framework, the first contribution employing copula-based methods is due to Decancq (2014). He analysed the temporal evolution of well-being in Russia by means of the multivariate Kendall's tau and the multivariate version of Spearman's rho defined in (2.26) applied to the dimensions included in the Human Development Index (HDI). Also, Pérez and Prieto-Alaiz (2016a) analysed the multivariate dependence between the dimensions of the HDI using data from 187 countries and three copula-based measures of multivariate association: the Spearman's footrule in (A4), the Gini's gamma in (A5) and the Spearman's rho defined in (2.26). More recently, García-Gómez et al. (2021) analysed the evolution of multivariate dependence between the dimensions of poverty in Europe over the period 2008-2014 using the multivariate extensions of Spearman's rho introduced in Section 2.4. These coefficients had been previously used by Pérez and Prieto (2015) to study the evolution of the dependence between the poverty dimensions in Spain over the period 2009-2013. Other recent examples of the application of copula-based methods to poverty and welfare analyses can be found in Decancq (2020), Terzi and Moroni (2020) and Tkach and Gigliarano (2020).

In this chapter, we contribute to the literature by performing an application of the copula-based dependence concepts introduced in Chapter 2 to analyse the evolution of multivariate dependence between the dimensions of poverty in the European Union over the period 2008-2018 using data from the EU-Statistics on Income and Living Conditions (EU-SILC) survey. This is a very interesting period, since it includes both the Great Recession and the subsequent period of economic recovery. First, we focus on the concept of orthant dependence and update the application in García-Gómez et al. (2021). In particular, we apply the multivariate extensions of Spearman's rho reviewed in Section 2.4 to study the multivariate association between the three dimensions of the AROPE rate. Among these multivariate extensions, especial emphasis will be put on the coefficients ρ_d^- and ρ_d^+, which allow to distinguish multivariate orthant dependence from a "downward" and an "upward" perspective, respectively, and thus become particularly informative in our context. In fact, when analysing multidimensional poverty, the coefficient

ρ_d^- becomes especially relevant, as it measures multivariate dependence from a "downward" perspective and captures the average probability of being simultaneously low-ranked in all dimensions of poverty as compared to what this would be were those dimensions independent. After the analysis based on orthant dependence, we focus on the concept of multivariate tail dependence introduced in Section 2.5. Again, when analysing multidimensional poverty, the multivariate dependence in the lower tail of the joint distribution is particularly important, since it captures the probability that an individual who is extremely low-ranked (extremely poor) in one dimension is also extremely poor in the other dimensions considered. To the best of our knowledge, this is the first application of multivariate tail dependence in Welfare Economics and, more specifically, to analyse multidimensional poverty. Hence, the contribution of this thesis is pioneering in this area and can pave the way for future contributions.

The rest of the chapter is organised as follows. In Section 3.2 we describe the data, variables and estimation procedure used to carry out the empirical application. Section 3.3 provides a descriptive graphical analysis of some interesting patterns of multivariate dependence between the poverty dimensions that can be found in the European Union. The results of the empirical application for the EU-28 countries are presented in Sections 3.4 and 3.5, presenting first the results based on the multivariate extensions of Spearman's rho and then those based on the multivariate tail dependence analysis.

3.2 Data description and estimation procedure

3.2.1 Data and variables

The data we use comes from the EU-SILC survey, which is the key reference for data on income and living conditions in the EU. In particular, we use the cross-sectional surveys of all years of the period 2008-2018.

The dimensions of poverty we consider are those included in the AROPE rate, namely income, material needs and work intensity. The selection of these dimensions is based on the relevance of the AROPE rate in the European context, as it is the headline indicator to monitor and implement effective poverty-reduction policies in the framework of the Europe 2020 Strategy. In fact, one of the Europe 2020 headline targets established by the European Commission was to reduce, by 20 million, the number of people at risk of poverty and social exclusion.[1]

The three measures characterising the three dimensions of the AROPE rate, namely income, work intensity and material deprivation, are defined as follows:

- The measure of income is the equivalised disposable income, which is calculated as the total income of the household, after taxes and other deductions, divided by the equivalised household size.[2]
- The work intensity of a household is the ratio of the total number of months that all working-age household members have worked during the income reference year and the total number of months they could have theoretically worked during the same period.[3]
- Material deprivation is originally defined as the enforced lack in a number of essential items, namely: 1) the capacity of facing unexpected expenses; 2) one-week annual holiday away from home; 3) a meal involving meat, chicken or fish every second day; 4) an adequately warm dwelling; 5) a washing machine; 6) a colour television; 7) a telephone; 8) a car; 9) the capacity to pay their rent, mortgage or utility bills.

For ease of interpretation we transform material deprivation into a variable that indicates the number of no-deprivations out of the nine possible, so that the new variable takes the following

[1]Despite the importance of the AROPE rate from the public policy perspective, the choice of the dimensions involved in its calculation is not exempt of criticism; see, for instance, the discussion in Nolan and Whelan (2011, ch.11).

[2]The equivalised household size is defined according to the modified OECD scale, which gives a weight of 1 to the first adult, 0.5 to other household members aged 14 or over and 0.3 to household members aged less than 14.

[3]Eurostat considers that a working-age person is a person aged 18-59 years, excluding also the students aged 18-24 years.

values: 0 (having all the 9 possible deprivations), 1 (having eight out of the nine aforementioned deprivations), ..., 9 (having no deprivations). Thus, high values of the three variables considered (equivalised disposable income, work intensity, and number of no-deprivations) convey lower chance to be poor, while low values of each variable convey higher chance to be poor.

The unit of analysis is the household. We only work with subsamples of households for which we have complete information for all the three variables. In particular, in these subsamples, households composed only of children, of students aged 18-24 and/or people aged 60 or more are excluded, due to their missing values in the work intensity variable. In these subsamples, the sample sizes range from 2270 observations in Cyprus in 2009 to 14773 observations in Italy in 2008.[4]

3.2.2 Estimation procedure

As we have seen in Chapter 2, the copula methodology focuses on the positions of households in each variable and not on the values these variables take for these households. Thus, the use of copula-based methods requires ranking the households in each dimension. In doing so, ties could arise in one or multiple variables. In our case, for example, the work intensity and material deprivation variables are of non-continuous nature, thus leading to a considerable number of ties. The problem of having ties in a copula-based framework is that the copula in (2.1) is no longer unique. Therefore, the values of the different coefficients reviewed in the previous chapter can vary widely even based on the same joint distribution. Moreover, the non-parametric estimators of these measures cannot be applied directly, since there are not unique ranks.

Although the literature is still scarce, some proposals to deal with ties can be found. On one hand, Decancq (2014) proposed to use secondary variables to break the ties and thus obtain unique ranks. On the other hand, Quessy (2009), Mesfioui and Quessy (2010) and Genest et al.

[4]We do not have data for Croatia in 2008 and 2009 and for Ireland, Slovakia and the UK in 2018.

(2013) have proposed tie-corrected versions of the multivariate generalisations of Spearman's rho in (2.21), (2.23) and (2.25). An empirical application of these tie-corrected measures to multivariate poverty can be found in García-Gómez et al. (2021). However, to the best of our knowledge, there is not any proposal of tie-corrected versions of tail dependence measures, neither in the bivariate nor in the multivariate case. Hence, in order to be coherent over the entire empirical application, we have adopted the solution proposed by Decancq (2014). Moreover, García-Gómez et al. (2021) showed that, when applying the multivariate extensions of Spearman's rho, there are not substantial differences in the results when using one approach or the other. Hence, we break the ties using additional information from secondary variables so that we eventually get, for each of the three variables considered (income, work intensity and no material deprivation), unique ranks $\{1, 2, \ldots, n\}$, where n is the sample size (or the number of households in the sample). These unique ranks will be used in order to estimate non-parametrically the multivariate extensions of Spearman's rho discussed in Section 2.4 and the empirical version of the multivariate tail concentration function introduced in Section 2.5.2.

Over the next lines, we will explain in detail how we use additional information to break the ties and obtain unique ranks. Firstly, when a tie occurs in work intensity, households are ranked according to two secondary ranking variables measuring the intensity in both education and health of the household. The intensity of education is the sum of the highest ISCED (International Standard Classification of Education) level attained by all members of the household that are not currently in education divided by the highest possible value of this sum. The health intensity indicator is constructed in a similar way as the sum of the values of the self-assessed health indicator of all members of the household divided by the highest possible value of this sum. The choice of these secondary variables is not arbitrary. Both the relationships between educational and labour market outcomes and between health and labor market attainments are well documented in the literature; see, for example, Nickell (1979), Mincer (1991), Wolbers (2000), Farber (2004) and Riddell and Song (2011), regarding the former, and Chirikos

(1993), Ettner et al. (1997), Currie and Madrian (1999), Pelkowski and Berger (2004) and García-Gómez and López-Nicolás (2006), regarding the latter. As secondary ranking variable for material deprivation, we use the burden of the housing cost. An overburden of the housing cost can be seen as an indicator of financial stress (Whelan and Maître, 2012; Deidda, 2015) and as an indicator of vulnerability (Brandolini et al., 2013). We use both a dummy variable taking the value 1 if the housing cost is a burden for the household and the value of the housing cost itself. Thus, households for which the housing cost is a burden are assigned worse positions than those for which it is not. If a tie still exists for those households for which the housing cost is a burden they are ranked using the value of the housing cost. That is, the higher is the housing cost the worse is the position of the household. Both in the case of work intensity and material deprivation, if ties still exist after ranking households according to the secondary variables, the ties are broken at random. As we previously argued, with this procedure we end up having, for each variable, unique ranks $\{1, 2, \ldots, n\}$. Then, with these unique ranks we will estimate non-parametrically the multivariate extensions of Spearman's rho in (2.21), (2.23) and (2.25) and the multivariate tail concentration function in (2.61), using their corresponding sample versions in (2.34), (2.36), (2.37) and (2.64), respectively. We are aware that this solution is far from being completely satisfactory, as it is unclear the effect of using additional secondary variables on the concordance properties of the original variables. Nevertheless, as we argued before, García-Gómez et al. (2021) showed that, when applying the multivariate extensions of Spearman's rho, the procedure of breaking the ties using additional information and the use of the tie-corrected versions of the coefficients in Quessy (2009), Mesfioui and Quessy (2010) and Genest et al. (2013) give very similar results. On the other hand, with regard to the application of tail dependence in the presence of ties, further research is needed, since there is not, as far as we know, any proposal of tie-corrected versions of tail dependence measures, neither in the bivariate nor in the multivariate setting.[5]

[5]Despite this, we have tested the robustness of the results of our multivariate tail dependence application both by breaking the ties at random with 1000 replications and by applying the methods in Genest et al. (2017) to estimate the empirical copula with discrete margins. This is just a first approximation to address this

3.3 A primer look at the multivariate data

Before presenting the results on the estimated multivariate dependence measures, it is worth performing a preliminary graphical analysis of the multivariate data, since it allows to observe some interesting patterns of multivariate dependence between the three dimensions of the AROPE rate in the EU. As we will see later, these patterns will be precisely confirmed by the analyses provided in Section 3.4.

To begin with, and to illustrate cross-country comparisons, Figure 3.1 represents the unique ranks described above, rescaled to $[0, 1]$ as defined in (2.31), for the three dimensions of the AROPE rate in Bulgaria and Romania in 2008. As we can see, the points are not uniformly distributed over the unit cube, indicating departure from independence. Actually, in both countries we observe a positive association, as the points tend to concentrate around the main diagonal of the cube, that is, the three variables tend to be jointly large or small together. Moreover, both plots are denser around the vertexes $(0, 0, 0)$ and $(1, 1, 1)$, but in Bulgaria the concentration is higher around the former than around the latter, suggesting that dependence in the lower orthant is higher than in the upper orthant. This means that, in this country, the tendency of households to be simultaneously low-ranked in the three dimensions of poverty is higher than their tendency to be simultaneously high-ranked in the three dimensions of poverty. The contrary occurs in Romania, where there is a higher concentration of observations around the vertex $(1, 1, 1)$, suggesting that upper orthant dependence is higher than lower orthant dependence. That is, in Romania, the tendency of households to be simultaneously high-ranked in the tree dimensions of poverty is higher than their tendency to be simultaneously low-ranked in the three dimensions of poverty. As a matter of fact, these patterns are properly captured by the coefficients $\hat{\rho}_d^-$ and $\hat{\rho}_d^+$, which in the case of Bulgaria will fulfil the condition $\hat{\rho}_d^- > \hat{\rho}_d^+$ while they will behave the other way round in Romania. Furthermore, as we will

complex problem, but the preliminary results are very promising, since our main conclusions seem to be robust to the method used to deal with the ties.

see when analysing tail dependence, in Bulgaria lower tail dependence tends to be higher than upper tail dependence, whereas the contrary occurs in Romania.

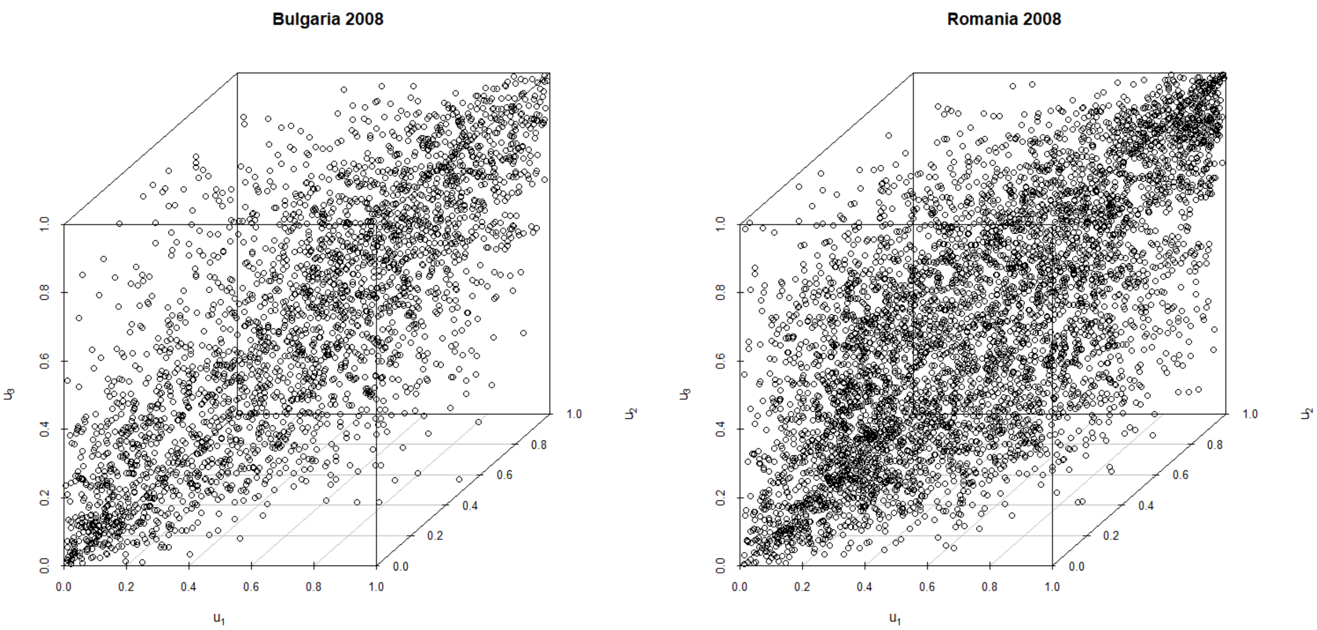

Figure 3.1: Scatter plots of scaled ranks for Bulgaria (2008) and Romania (2008)

To illustrate temporal comparisons, Figure 3.2 displays three scatter plots representing the scaled ranks for Spain in 2008, 2014 and 2018. As we can see, between 2008 and 2014 there was an increase in the multivariate dependence between dimensions of the AROPE rate in Spain, as the concentration of the observations around the main diagonal increased over that period. This suggests that, over the period 2008-2014, there was, in this country, an increase in the tendency of households to be simultaneously low or high-ranked in the tree dimensions of poverty. On the contrary, the concentration of observations around the main diagonal seems to remain rather stable or slightly decrease between 2014 and 2018. Thus, between those two years there seems to be a stabilisation or slight decrease in multivariate dependence between dimensions of poverty, that is, in the tendency of household to be simultaneously low or high-ranked in the tree dimensions of poverty. Moreover, in the three years considered, the concentration of points in the lower orthant seems to always be higher than in the upper orthant, which means that, in Spain, the tendency of households to be simultaneously low-ranked in the three dimensions of the AROPE rate is higher than their tendency to be simultaneously high-ranked in these dimensions. Hence, we would expect $\hat{\rho}_3^- > \hat{\rho}_3^+$, with an increase in the value of the coefficients between 2008 and 2014 and a decrease afterwards. Moreover, with regard to tail dependence, we would expect multivariate tail dependence between poverty dimensions to be higher in the lower tail than in the upper tail of the joint distribution, with an increase in the degree of tail dependence between 2008 and 2014 followed by a slight decrease afterwards. That is, we expect, first, the probability that a household that is extremely poor in one dimensions is also extremely poor in the other two dimensions to be higher than the probability that a household that is extremely rich in one dimension is also extremely rich in the other two dimensions. We also expect an increase in these probabilities over the period 2008-2014 and a decline afterwards. As we will see, the results in Sections 3.4 and 3.5 confirm these patterns.

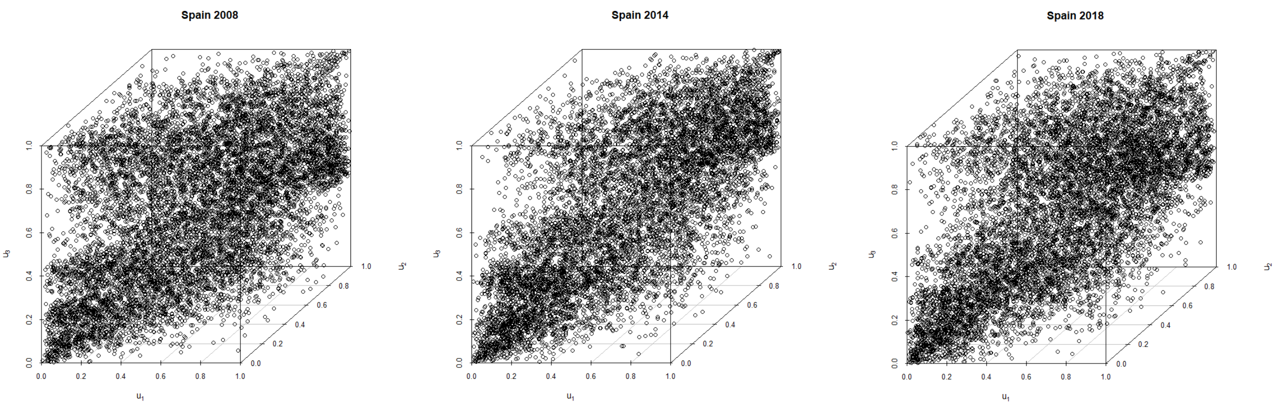

Figure 3.2: Scatter plots of scaled ranks for Spain (2008, 2014 and 2018)

To complement this graphical analysis, we have split the unit cube $[0,1]^3$ in 64 boxes of the same size and we have computed (see Table 3.1) the observed relative frequencies in the four boxes along the main diagonal for the same countries and years presented in Figures 3.1 and 3.2. The four boxes are denoted as $\{\mathbf{u} \leq 0.25,\ 0.25 < \mathbf{u} \leq 0.5,\ 0.5 < \mathbf{u} \leq 0.75,\ \mathbf{u} > 0.75\}$, where $\mathbf{u} \leq 0.25$ denotes the component-wise inequality, i.e. $u_i \leq 0.25$ for $i = 1, 2, 3$, and so this first box records the share of households being simultaneously in the 1^{st} quartile (low-ranked) in all dimensions. The other three boxes are defined similarly.

	$\mathbf{u} \leq 0.25$	$0.25 < \mathbf{u} \leq 0.5$	$0.5 < \mathbf{u} \leq 0.75$	$\mathbf{u} > 0.75$	Total
Bulgaria (2008)	11.06%	3.43%	3.30%	7.28%	25.07%
Romania (2008)	5.80%	2.34%	2.57%	9.10%	19.81%
Spain (2008)	7.23%	2.28%	2.35%	3.29%	15.15%
Spain (2014)	8.14%	2.77%	2.32%	5.27%	18.5%
Spain (2018)	7.68%	2.07%	2.17%	3.38%	15.3%

Table 3.1: Share of households in the main diagonal of the unit cube $[\mathbf{0}, \mathbf{1}]^\mathbf{3}$ in some selected EU-28 countries and some selected years.

If the three variables were independent, the proportion of points in each box would be the same and equal to 1.56%. However, in all the examples in Table 1, there is a larger proportion of points concentrated around the main diagonal, implying departure from independence. It is also worth noticing that, in all cases, the frequencies are higher in the extreme boxes ($\mathbf{u} \leq 0.25$ and $\mathbf{u} > 0.75$), suggesting positive orthant dependence, in agreement with the patterns displayed in Figures 3.1 and 3.2. Furthermore, in Bulgaria and Spain, the frequencies are higher in the lower box ($\mathbf{u} \leq 0.25$) than in the upper box ($\mathbf{u} > 0.75$), which suggest that multivariate dependence is higher in the lower part of the joint distribution than in the upper part. That is, the tendency of households to be simultaneously low-ranked in the three dimensions of the AROPE rate is higher than their tendency to be simultaneously high-ranked in these three dimensions. However, the contrary occurs in Romania, where dependence in the upper part of the joint distribution seems to be higher than in the lower part. Finally, Table 3.1 also shows the increase in multivariate dependence between poverty dimensions in Spain between 2008 and 2014, since the proportion of points around the main diagonal of the unit cube increased over that period, suggesting an increase in the tendency of households to be simultaneously low or high-ranked in the three dimensions of poverty. By contrary, this proportion decreased between 2014 and 2018, suggesting a decline in multivariate dependence in this period, a decrease that is specially relevant in the upper part of the joint distribution. As we will see later, all these patterns will be confirmed by the detailed analysis performed in the next sections.

3.4 Multivariate orthant dependence analysis

In this section, we analyse the evolution of the multivariate orthant dependence between the three dimensions considered in the AROPE rate in the EU-28 countries over the period 2008-2018 by applying the multivariate generalisations of Spearman's rho, ρ_d^-, ρ_d^+ and ρ_d discussed in Section 2.4. Recall that ρ_d^- and ρ_d^+ measure multivariate orthant dependence from a downward and an upward perspective, respectively. In particular, ρ_d^- assesses the similarity between our multivariate data and the situation of independence in the lower orthant, hence capturing somehow the probability, in average, that individuals are simultaneously "poor" in all dimensions of poverty. On the other hand, ρ_d^+ assesses the similarity between our multivariate data and the situation of independence in the upper orthant, and thus captures somehow the probability, in average, that individuals are simultaneously "rich" in all dimensions. Hence, the higher the values of these coefficients, the greater the departure from independence and the higher the degree of lower and upper orthant dependence, respectively.

To perform the empirical application, we compute the non-parametric estimators of ρ_d^- and ρ_d^+ defined in (2.36) and (2.34), respectively, applied to the unique ranks obtained as explained in Section 3.2.2. To estimate the coefficient of average orthant dependence, namely ρ_d, we use the plug-in estimator introduced in (2.37). As we pointed out in Section 2.4.2.2, the asymptotic variances of these estimators are complex. Therefore, we rely on a non-parametric bootstrap method to compute the bootstrap standard errors as the sample standard deviation of 1000 bootstrapped point estimates of the coefficients.

Figures 3.3 and 3.4 display, for the EU-28 countries and over the whole period analysed, the evolution of the values of $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$, respectively, together with the 95% standard confidence intervals using the bootstrap standard errors. The first conclusion that can be drawn from Figures 3.3 and 3.4 is that, for all countries and all years, both $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$ are positive, indicating a positive multivariate association between poverty dimensions both in the lower and in the upper orthant. This means that low (high) values of income tend to occur simultaneously with low (high) values of the other two dimensions.

Another relevant feature that can be highlighted is that, regardless of the year and the country, the value of $\hat{\rho}_3^-$ (in Figure 3.3) is greater than that of $\hat{\rho}_3^+$ (in Figure 3.4) in all countries except in Romania, where $\hat{\rho}_3^+ > \hat{\rho}_3^-$. This means that, in the vast majority of the EU countries, average lower orthant dependence tends to be higher than average upper orthant dependence, that is, on average, the probability of being simultaneously low-ranked in all poverty dimensions tends to be higher than the probability of being simultaneously high-ranked in all poverty dimensions.

However, Figures 3.3 and 3.4 show that there are significant cross-country differences in the evolution of multivariate dependence between poverty dimensions. In general, two different periods can be distinguished in these figures. First, between 2008 and 2014 we observe, in most of the EU countries, an increase in multivariate dependence both in the lower and in the upper orthant. This means that, following the financial crisis of 2008, most of the EU countries experienced an increase in the tendency of households to be simultaneously poor or simultaneously rich in all dimensions of poverty. In fact, in countries such as Spain, Cyprus, Denmark, Italy or the Netherlands, we observe a clear increasing trend in multivariate dependence over that period. Moreover, in some countries such as Greece and the UK there is not a clear trend, with several ups and downs over the period, but the multivariate dependence is clearly higher in 2014 than in 2008. By contrary, in countries like Austria, Germany or Sweden there is a considerable overlap in the confidence intervals for these two years and thus, just by looking at Figures 3.3 and 3.4, we cannot give meaningful conclusions on the variation of the dependence coefficients. Noticeably, no clear decreasing trend shows up in any country. That is, we do not find any country where the tendency of household to be simultaneously poor or simultaneously rich in the three dimensions of the AROPE rate decreased between 2008 and 2014.

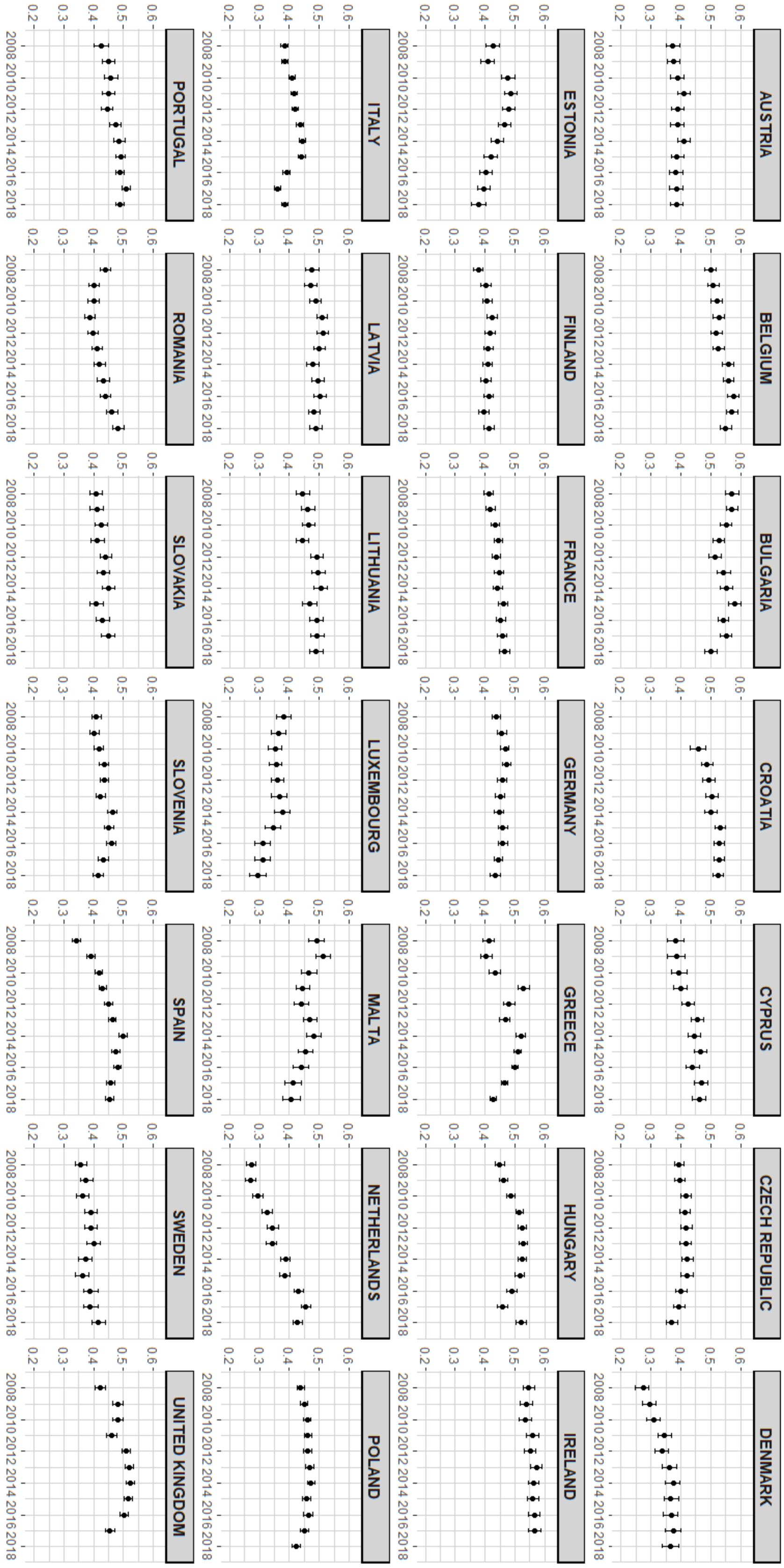

Figure 3.3: Evolution of $\hat{\rho}_3^-$ and their bootstrap standard 95% confidence intervals in the EU-28 countries over the period 2008-2018.

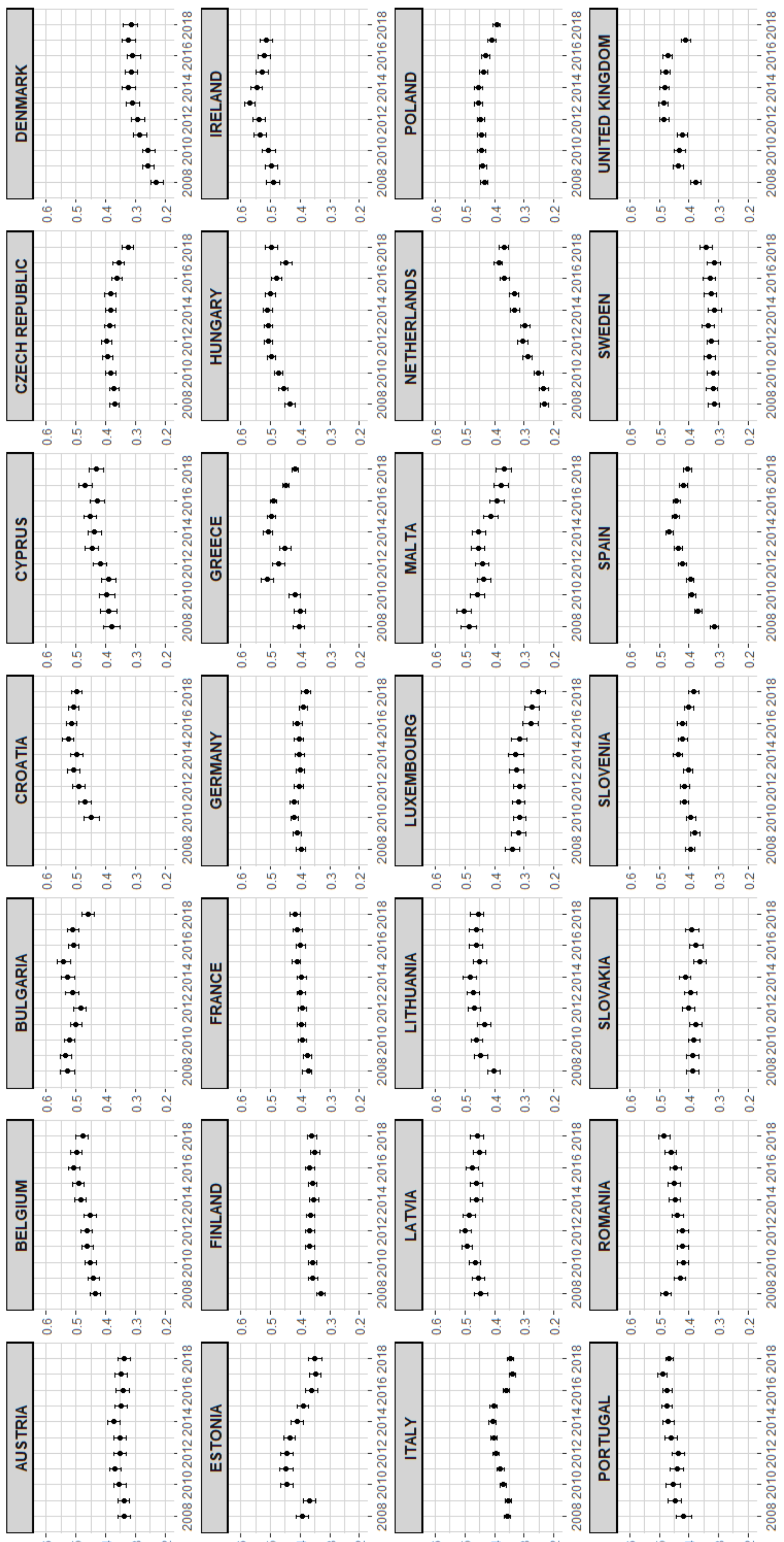

Figure 3.4: Evolution of $\hat{\rho}_3^+$ and their bootstrap standard 95% confidence intervals in the EU-28 countries over the period 2008-2018.

We also notice that, after the general increase in multivariate dependence observed between 2008 and 2014, there seems to have been a period of stabilisation or even, in some countries, decrease in orthant dependence. In particular, we find countries where the values of $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$ have remained rather stable over that period, such as Austria, Denmark or Finland. However, in other countries, we find a clear decreasing trend, such as in the case of Estonia, Greece, Luxembourg and Spain. In Italy and Hungary we also find this decreasing trend, but with an upturn at the end of the period. Noticeably, despite the stable or decreasing trends that are found in the vast majority of the countries over 2014-2018, there are some particular cases, namely France, Sweden and, especially, Romania and the Netherlands, in which $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$ increased over this period.

Finally, Figures 3.3 and 3.4 also highlight that, despite the declining trend in multivariate dependence observed in many EU countries between 2014 and 2018, the values of both $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$ seem to be still greater in 2018 than in 2008 in most of the countries. This feature is clear in some cases, such as Greece or Spain, whereas in others, say the Czech Republic or Slovenia, there is a considerable overlap in the confidence intervals and hence we cannot draw meaningful conclusions.

To get a better insight into the evolution of multivariate dependence over the whole period analysed (2008-2018) and the two sub-periods previously highlighted (2008-2014 and 2014-2018), Figure 3.5 displays, for each EU-28 country, the levels of $\hat{\rho}_3^-$ for years 2008, 2014 and 2018.[6] Furthermore, Tables 3.2 and 3.3 report point estimates of $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$, respectively, as well as their bootstrap standard errors (in parenthesis), for 2008, 2014 and 2018. Moreover, in columns 4, 5 and 6 of these tables we display the results of three two-independent sample t-tests with unequal variances, calculated using bootstrap standard errors. In particular, we perform a one-sided test to determine the significance of the variation of the corresponding coefficient over the two sub-periods 2008-2014 (in column 4) and 2014-2018 (in column 5), and

[6] We focus on $\hat{\rho}_3^-$ because it measures lower orthant dependence, which is arguably particularly relevant in multidimensional poverty analysis, but the figure is very similar if we consider $\hat{\rho}_3^+$.

over the whole period 2008-2018 (in column 6). The corresponding p-value (in parenthesis) is computed assuming asymptotic normality of the t-statistic. Table 3.4 displays the same results for $\hat{\rho}_3$, the coefficient of average orthant dependence.

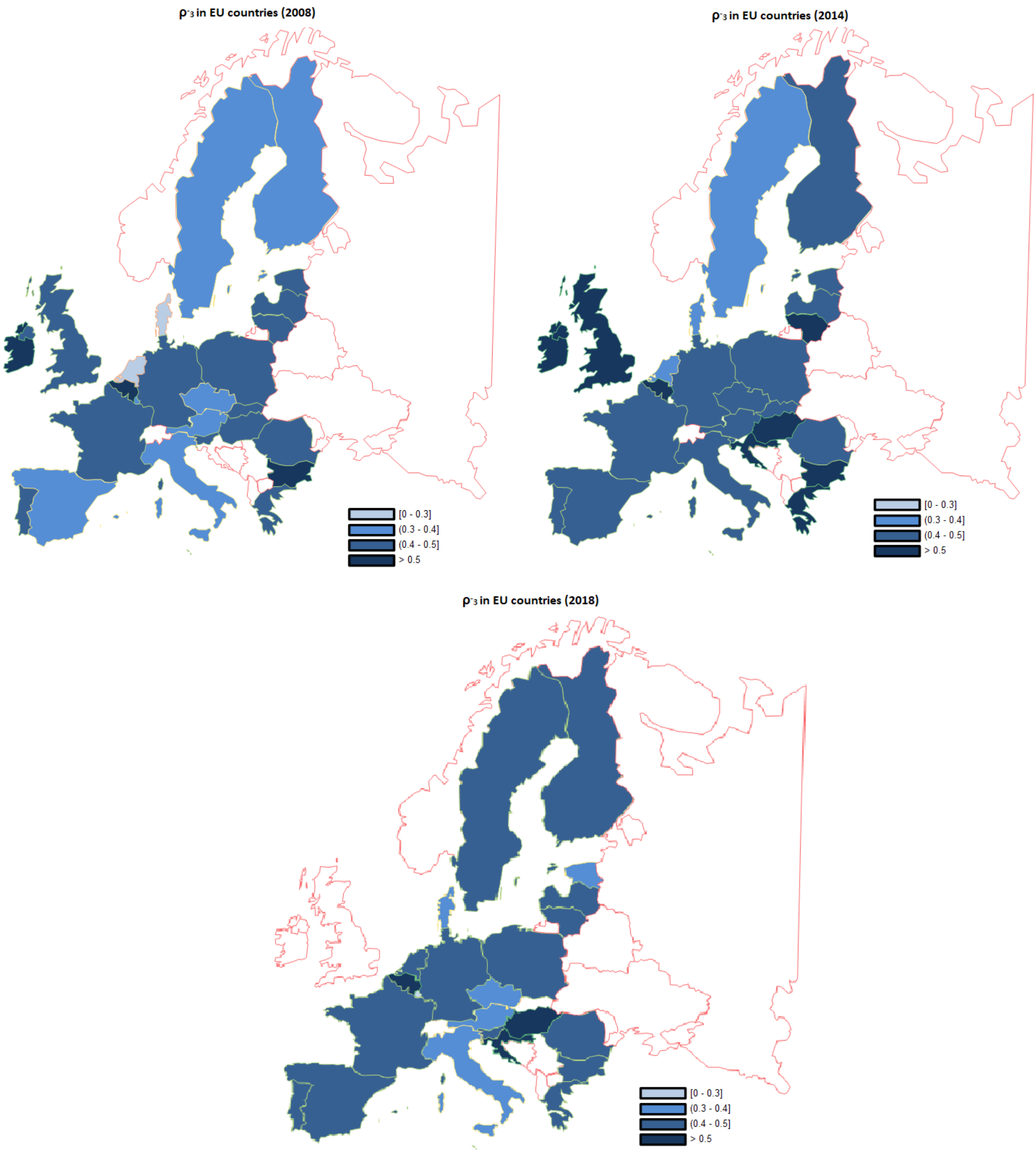

Figure 3.5: Cross-country differences in the level of $\hat{\rho}_3^-$ for years 2008, 2014 and 2018.

The first feature that can be noticed in Figure 3.5 is that, in 2008, the lowest values for $\hat{\rho}_3^-$ are found in Denmark and the Netherlands. Other Scandinavian countries, namely Sweden and Finland, had also relatively low levels of multivariate dependence, a feature that can also be found in countries from Central (Austria, Luxembourg and the Czech Republic) and Southern (Italy and Spain) Europe. By contrary, the highest values of $\hat{\rho}_3^-$ are found in Belgium, Bulgaria and Ireland. If we look now at the situation in 2014 as compared to 2008, we first observe a general increase in multivariate dependence between dimensions of poverty. This means that, after the financial crisis, there was in Europe a general increase in the tendency of households to be simultaneously poor in the three dimensions of the AROPE rate. We also observe a reduction in the cross-country differences, with those countries that were relatively better off coming closer to the situation of the rest of the countries. Furthermore, between 2014 and 2018 we observe a decrease in the level of $\hat{\rho}_3^-$ in some countries such as Italy, Austria or the Czech Republic, whereas in other countries the level of the coefficient has remained stable or has even increased (as in the case of Sweden). Thus, cross-country differences seem to have increased again over that period. In this sense, in 2018 we observe, on one hand, countries with relatively low levels of multivariate dependence (such as Luxembourg, Denmark or Austria, among others) and, on the other hand, countries where the dependence between poverty dimensions is particularly high (such as Croatia, Hungary or Belgium).

In accordance with Figure 3.5, the results reported in column 4 of Tables 3.2 and 3.3, which focus on the change of the coefficients in the sub-period 2008-2014, indicate that, in most EU countries, there was a significant increase in both $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$ between these two years. Thus, we can say that, over that period, there was a general increase in the multivariate dependence between dimensions of poverty in the EU. This means that, after the Great Recession, there was, in most of the EU countries, a significant increase in both the average rescaled probability of being simultaneously low-ranked and the average rescaled probability of being simultaneously high-ranked in all dimensions of poverty. Noticeably, the highest increase in both lower and upper orthant dependence is found in Spain, one of the countries most hardly hit by the economic crisis. Another country severely affected by the crisis, namely Greece, also experienced a substantial increase in these two types of dependence. Furthermore, the same conclusions are drawn if we focus on the coefficient of average orthant dependence, namely $\hat{\rho}_3$; see column 4 of Table 3.4.

Looking at column 5 of Tables 3.2 and 3.3, which focus on the change of the coefficients in the sub-period 2014-2018, the results displayed in Figure 3.5 are confirmed. In particular, we find, for the vast majority of EU-28 countries, either significant decreases in the coefficients $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$ (as it is the case of the Czech Republic, Greece, Spain or Poland, among others) or no significant changes in the level of multivariate dependence (as it is the case of Belgium, Denmark or Portugal). We also find four countries (France, the Netherlands, Romania and Sweden) where both coefficients increased over the period 2014-2018. This results suggest that,during the economic recovery period, in most of the EU countries, the average rescaled probability of being simultaneously low-ranked and the average rescaled probability of being simultaneously high-ranked in all dimensions of poverty decreased or remained stable. Again, the same conclusions are drawn if we focus on the coefficient of average orthant dependence, namely $\hat{\rho}_3$; see column 5 of Table 3.4.

Finally, from the results displayed in column 6 of Tables 3.2 and 3.3, which focus on the changes in the coefficients in the whole period analysed (2008-2018), we can conclude that, in the majority of the EU countries, the values of the coefficients were still significantly higher in 2018 than in 2008. This is the case, for example, of Belgium, Spain, France, Portugal or the Netherlands. Thus, in these countries, it was still more likely to be simultaneously low(high)-ranked in all dimensions of poverty in 2018 than in 2008. In other countries, such as Austria or Italy, we do not find significant differences between the values in 2018 and those in 2008. Finally, we also find countries (e.g, the Czech Republic, Estonia or Poland) where the values of $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$ were significantly lower in 2018 than at the beginning of the whole period analysed, which means that, in these countries, the average probability of being simultaneously low or high-ranked in all dimensions of poverty was lower in 2018 than in 2008. Moreover, as it can be seen in column 6 of Table 3.4, the same conclusions can be drawn regarding the coefficient of average orthant dependence, $\hat{\rho}_3$.

To complement the analysis of three-dimensional dependence, we have also analysed all possible pairwise relationships between the three dimensions of the AROPE rate. The results are displayed in Figure 4.5, which shows the evolution of the Spearman's rho coefficients of pairwise dependence between income and work intensity ($\hat{\rho}_{income,work}$, in red), income and no-material deprivation ($\hat{\rho}_{income,no-deprivation}$, in green) and work intensity and no-material deprivation ($\hat{\rho}_{work,no-deprivation}$, in blue), together with 95% bootstrap confidence intervals. Several features emerge from this figure. First, all the pairwise coefficients are always positive, thus confirming the positive relationship between the different dimensions of poverty in the EU. Second, in all countries and for all years, the highest level of dependence is found either between income and work intensity or between income and no-material deprivation. This means that the probability of being simultaneously low-ranked (or high-ranked) in income and work intensity and in income and no-material deprivation is clearly higher than the probability of being simultaneously low-ranked (or high-ranked) in work intensity and no-material deprivation. And third, the evolution between 2008 and 2018 is very similar for the three pairwise coefficients. In particular, as it happened with $\hat{\rho}_3^-$ and $\hat{\rho}_3^+$, we observe, in most of the EU countries, an increase in dependence after the Great Recession (between 2008 and 2014), followed by a stabilisation or decline over the last four years of the period analysed.

Moreover, and most interestingly, since the three pairwise coefficients are all positive, the coefficient of maximal dependence $\hat{\rho}_3^{max}$, defined in (2.29), should be equal to either $\hat{\rho}_3^-$ or $\hat{\rho}_3^+$; see page 90. Noticeably, in our case, $\hat{\rho}_3^{\max} = \hat{\rho}_3^-$ in all countries (except in Romania, where $\hat{\rho}_3^{\max} = \hat{\rho}_3^+$), indicating that the positions of the households in the three dimensions of poverty tend to be aligned around the corner $(0, 0, 0)$. In terms of poverty analysis, this fact

	$\hat{\rho}_3^-$					
	2008	**2014**	**2018**	**t-test Change 08-14**	**t-test Change 14-18**	**t-test Change 08-18**
AUSTRIA	0.373	0.412	0.386	2.458	-1.673	0.839
	(0.011)	(0.011)	(0.011)	(0.007)	(0.047)	(0.201)
BELGIUM	0.501	0.558	0.549	4.221	-0.673	3.474
	(0.010)	(0.010)	(0.010)	(0.000)	(0.251)	(0.000)
BULGARIA	0.572	0.553	0.500	-1.167	-3.482	-4.683
	(0.011)	(0.011)	(0.010)	(0.122)	(0.000)	(0.000)
CYPRUS	0.383	0.446	0.462	3.535	1.029	4.351
	(0.014)	(0.011)	(0.012)	(0.000)	(0.152)	(0.000)
CZECH REPUBLIC	0.394	0.421	0.369	2.097	-3.690	-2.008
	(0.008)	(0.010)	(0.010)	(0.018)	(0.000)	(0.022)
GERMANY	0.437	0.447	0.435	0.928	-1.065	-0.196
	(0.008)	(0.008)	(0.008)	(0.177)	(0.143)	(0.422)
DENMARK	0.276	0.374	0.365	5.911	-0.514	4.896
	(0.012)	(0.012)	(0.014)	(0.000)	(0.304)	(0.000)
ESTONIA	0.425	0.441	0.378	0.966	-3.948	-2.824
	(0.012)	(0.011)	(0.012)	(0.167)	(0.000)	(0.002)
GREECE	0.412	0.520	0.428	8.573	-9.570	1.470
	(0.010)	(0.008)	(0.005)	(0.000)	(0.000)	(0.071)
SPAIN	0.342	0.499	0.452	15.792	-4.699	10.669
	(0.007)	(0.007)	(0.007)	(0.000)	(0.000)	(0.000)
FINLAND	0.378	0.410	0.412	2.769	0.223	2.856
	(0.008)	(0.008)	(0.009)	(0.003)	(0.412)	(0.002)
FRANCE	0.412	0.441	0.464	2.596	2.045	4.575
	(0.008)	(0.008)	(0.008)	(0.005)	(0.020)	(0.000)
CROATIA	NA	0.502	0.524	NA	1.564	NA
	NA	(0.007)	(0.009)	NA	(0.059)	NA
HUNGARY	0.449	0.525	0.520	6.980	-0.405	5.701
	(0.008)	(0.009)	(0.009)	(0.000)	(0.343)	(0.000)
IRELAND	0.546	0.562	NA	1.144	NA	NA
	(0.010)	(0.006)	NA	(0.126)	NA	NA
ITALY	0.384	0.443	0.385	7.168	-7.172	0.119
	(0.006)	(0.011)	(0.006)	(0.000)	(0.000)	(0.452)
LITHUANIA	0.444	0.506	0.489	3.821	-1.036	2.704
	(0.012)	(0.013)	(0.012)	(0.000)	(0.150)	(0.003)
LUXEMBOURG	0.382	0.377	0.295	-0.277	-4.420	-4.737
	(0.013)	(0.011)	(0.013)	(0.391)	(0.000)	(0.000)
LATVIA	0.475	0.477	0.488	0.102	0.715	0.786
	(0.012)	(0.012)	(0.011)	(0.459)	(0.237)	(0.216)
MALTA	0.491	0.481	0.405	-0.590	-3.944	-4.285
	(0.013)	(0.008)	(0.015)	(0.277)	(0.000)	(0.000)
THE NETHERLANDS	0.272	0.387	0.426	9.497	3.242	12.610
	(0.009)	(0.007)	(0.009)	(0.000)	(0.001)	(0.000)
POLAND	0.438	0.472	0.422	3.868	-5.329	-1.767
	(0.006)	(0.010)	(0.007)	(0.000)	(0.000)	(0.039)
PORTUGAL	0.426	0.486	0.489	3.830	0.312	4.555
	(0.012)	(0.010)	(0.007)	(0.000)	(0.377)	(0.000)
ROMANIA	0.439	0.419	0.482	-1.481	4.705	3.355
	(0.009)	(0.012)	(0.009)	(0.069)	(0.000)	(0.000)
SWEDEN	0.357	0.375	0.413	1.166	2.300	3.626
	(0.010)	(0.008)	(0.012)	(0.122)	(0.011)	(0.000)
SLOVENIA	0.410	0.463	0.415	4.922	-4.175	0.451
	(0.008)	(0.011)	(0.009)	(0.000)	(0.000)	(0.326)
SLOVAKIA	0.408	0.451	NA	2.802	NA	NA
	(0.011)	(0.007)	NA	(0.003)	NA	NA
UNITED KINGDOM	0.423	0.522	NA	8.377	NA	NA
	(0.009)	(0.007)	NA	(0.000)	NA	NA

Note: Standard errors for the coefficients and p-values for the one-side t-test are displayed in parentheses.

Table 3.2: Coefficient of trivariate lower orthant dependence, $\hat{\rho}_3^-$, between the dimensions of the AROPE rate in EU-28 countries (2008-2018).

	$\hat{\rho}_3^+$					
	2008	**2014**	**2018**	**t-test Change 08-14**	**t-test Change 14-18**	**t-test Change 08-18**
AUSTRIA	0.338	0.372	0.338	2.262	-2.274	0.045
	(0.011)	(0.010)	(0.010)	(0.012)	(0.011)	(0.482)
BELGIUM	0.436	0.484	0.477	3.549	-0.482	3.039
	(0.009)	(0.010)	(0.010)	(0.000)	(0.315)	(0.001)
BULGARIA	0.528	0.529	0.461	0.046	-4.466	-4.305
	(0.012)	(0.011)	(0.010)	(0.482)	(0.000)	(0.000)
CYPRUS	0.379	0.437	0.431	3.326	-0.341	2.854
	(0.013)	(0.011)	(0.013)	(0.000)	(0.367)	(0.002)
CZECH REPUBLIC	0.370	0.383	0.325	1.065	-4.383	-3.745
	(0.008)	(0.009)	(0.009)	(0.143)	(0.000)	(0.000)
GERMANY	0.397	0.401	0.379	0.397	-2.012	-1.675
	(0.007)	(0.008)	(0.008)	(0.346)	(0.022)	(0.047)
DENMARK	0.229	0.325	0.315	6.381	-0.602	5.273
	(0.010)	(0.011)	(0.012)	(0.000)	(0.274)	(0.000)
ESTONIA	0.392	0.410	0.349	1.175	-4.003	-2.689
	(0.011)	(0.010)	(0.011)	(0.120)	(0.000)	(0.004)
GREECE	0.404	0.508	0.416	8.134	-9.204	1.019
	(0.010)	(0.008)	(0.005)	(0.000)	(0.000)	(0.154)
SPAIN	0.314	0.465	0.403	15.909	-6.440	9.285
	(0.007)	(0.007)	(0.007)	(0.000)	(0.000)	(0.000)
FINLAND	0.330	0.353	0.360	2.144	0.660	2.705
	(0.008)	(0.008)	(0.008)	(0.016)	(0.255)	(0.003)
FRANCE	0.373	0.395	0.416	2.014	1.828	3.807
	(0.008)	(0.008)	(0.008)	(0.022)	(0.034)	(0.000)
CROATIA	NA	0.497	0.499	NA	0.085	NA
	NA	(0.011)	(0.009)	NA	(0.466)	NA
HUNGARY	0.434	0.509	0.496	6.429	-1.009	4.602
	(0.009)	(0.008)	(0.010)	(0.000)	(0.157)	(0.000)
IRELAND	0.491	0.545	NA	3.748	NA	NA
	(0.011)	(0.010)	NA	(0.000)	NA	NA
ITALY	0.355	0.407	0.347	6.621	-7.634	-1.048
	(0.005)	(0.006)	(0.006)	(0.000)	(0.000)	(0.147)
LITHUANIA	0.402	0.483	0.456	5.239	-1.710	3.393
	(0.011)	(0.011)	(0.012)	(0.000)	(0.044)	(0.000)
LUXEMBOURG	0.339	0.329	0.254	-0.525	-4.132	-4.718
	(0.013)	(0.013)	(0.013)	(0.300)	(0.000)	(0.000)
LATVIA	0.445	0.460	0.459	0.976	-0.083	0.878
	(0.011)	(0.010)	(0.011)	(0.165)	(0.467)	(0.190)
MALTA	0.486	0.453	0.368	-1.874	-4.652	-6.136
	(0.013)	(0.012)	(0.014)	(0.030)	(0.000)	(0.000)
THE NETHERLANDS	0.232	0.331	0.366	8.949	3.121	11.967
	(0.008)	(0.008)	(0.008)	(0.000)	(0.001)	(0.000)
POLAND	0.434	0.454	0.392	2.082	-6.635	-4.799
	(0.006)	(0.007)	(0.006)	(0.019)	(0.000)	(0.000)
PORTUGAL	0.417	0.469	0.466	3.237	-0.307	3.358
	(0.013)	(0.010)	(0.007)	(0.001)	(0.379)	(0.000)
ROMANIA	0.478	0.448	0.483	-2.246	2.577	0.401
	(0.009)	(0.010)	(0.010)	(0.012)	(0.005)	(0.344)
SWEDEN	0.315	0.316	0.341	0.066	1.697	1.914
	(0.009)	(0.011)	(0.010)	(0.474)	(0.045)	(0.028)
SLOVENIA	0.395	0.437	0.384	4.019	-4.837	-1.027
	(0.007)	(0.007)	(0.008)	(0.000)	(0.000)	(0.152)
SLOVAKIA	0.387	0.412	NA	1.745	NA	NA
	(0.011)	(0.010)	NA	(0.040)	NA	NA
UNITED KINGDOM	0.377	0.482	NA	8.835	NA	NA
	(0.009)	(0.008)	NA	(0.000)	NA	NA

Note: Standard errors for the coefficients and p-values for the one-side t-test are displayed in parentheses.

Table 3.3: Coefficient of trivariate upper orthant dependence, $\hat{\rho}_3^+$, between the dimensions of the AROPE rate in EU-28 countries (2008-2018).

	$\hat{\rho}_3$					
	2008	2014	2018	t-test Change 08-14	t-test Change 14-18	t-test Change 08-18
AUSTRIA	0.355	0.392	0.362	2.442	-2.041	0.465
	(0.011)	(0.010)	(0.010)	(0.007)	(0.021)	(0.321)
BELGIUM	0.468	0.521	0.513	4.012	-0.596	3.372
	(0.009)	(0.009)	(0.010)	(0.000)	(0.276)	(0.000)
BULGARIA	0.550	0.541	0.480	-0.575	-4.110	-4.634
	(0.011)	(0.011)	(0.010)	(0.283)	(0.000)	(0.000)
CYPRUS	0.381	0.441	0.447	3.559	0.341	3.734
	(0.013)	(0.011)	(0.012)	(0.000)	(0.367)	(0.000)
CZECH REPUBLIC	0.382	0.402	0.347	1.653	-4.179	-2.964
	(0.007)	(0.009)	(0.009)	(0.049)	(0.000)	(0.002)
GERMANY	0.417	0.424	0.407	0.692	-1.578	-0.944
	(0.007)	(0.007)	(0.008)	(0.245)	(0.057)	(0.173)
DENMARK	0.252	0.349	0.340	6.340	-0.573	5.230
	(0.011)	(0.011)	(0.013)	(0.000)	(0.283)	(0.000)
ESTONIA	0.408	0.425	0.364	1.105	-4.103	-2.857
	(0.011)	(0.010)	(0.011)	(0.134)	(0.000)	(0.002)
GREECE	0.408	0.514	0.422	8.646	-9.715	1.288
	(0.009)	(0.008)	(0.005)	(0.000)	(0.000)	(0.099)
SPAIN	0.328	0.482	0.428	16.372	-5.726	10.316
	(0.007)	(0.007)	(0.007)	(0.000)	(0.000)	(0.000)
FINLAND	0.354	0.381	0.386	2.557	0.448	2.878
	(0.007)	(0.008)	(0.008)	(0.005)	(0.327)	(0.002)
FRANCE	0.393	0.418	0.440	2.387	2.006	4.342
	(0.008)	(0.008)	(0.008)	(0.008)	(0.022)	(0.000)
CROATIA	NA	0.500	0.511	NA	0.850	NA
	NA	(0.011)	(0.008)	NA	(0.198)	NA
HUNGARY	0.442	0.517	0.508	6.966	-0.747	5.344
	(0.008)	(0.007)	(0.009)	(0.000)	(0.228)	(0.000)
IRELAND	0.519	0.554	NA	2.579	NA	NA
	(0.010)	(0.009)	NA	(0.005)	NA	NA
ITALY	0.369	0.425	0.366	7.136	-7.682	-0.466
	(0.006)	(0.006)	(0.005)	(0.000)	(0.000)	(0.321)
LITHUANIA	0.423	0.494	0.472	4.673	-1.420	3.150
	(0.011)	(0.011)	(0.011)	(0.000)	(0.078)	(0.001)
LUXEMBOURG	0.360	0.353	0.275	-0.414	-4.430	-4.897
	(0.012)	(0.012)	(0.013)	(0.339)	(0.000)	(0.000)
LATVIA	0.460	0.469	0.474	0.552	0.333	0.860
	(0.011)	(0.010)	(0.011)	(0.290)	(0.370)	(0.195)
MALTA	0.488	0.467	0.387	-1.268	-4.442	-5.369
	(0.013)	(0.011)	(0.014)	(0.102)	(0.000)	(0.000)
THE NETHERLANDS	0.252	0.359	0.396	9.575	3.285	12.711
	(0.008)	(0.008)	(0.008)	(0.000)	(0.001)	(0.000)
POLAND	0.436	0.463	0.407	3.067	-6.206	-3.405
	(0.006)	(0.006)	(0.006)	(0.001)	(0.000)	(0.000)
PORTUGAL	0.421	0.477	0.477	3.652	0.002	4.085
	(0.012)	(0.009)	(0.006)	(0.000)	(0.499)	(0.000)
ROMANIA	0.458	0.433	0.482	-1.931	3.764	1.926
	(0.009)	(0.009)	(0.009)	(0.027)	(0.000)	(0.027)
SWEDEN	0.336	0.345	0.377	0.660	2.083	2.910
	(0.009)	(0.011)	(0.011)	(0.255)	(0.019)	(0.002)
SLOVENIA	0.402	0.450	0.399	4.664	-4.677	-0.287
	(0.007)	(0.007)	(0.008)	(0.000)	(0.000)	(0.387)
SLOVAKIA	0.397	0.432	NA	2.368	NA	NA
	(0.010)	(0.010)	NA	(0.009)	NA	NA
UNITED KINGDOM	0.400	0.502	NA	8.914	NA	NA
	(0.009)	(0.007)	NA	(0.000)	NA	NA

Note: Standard errors for the coefficients and p-values for the one-side t-test are displayed in parentheses.

Table 3.4: Coefficient of trivariate average orthant dependence, $\hat{\rho}_3$, between the dimensions of the AROPE rate in EU-28 countries (2008-2018).

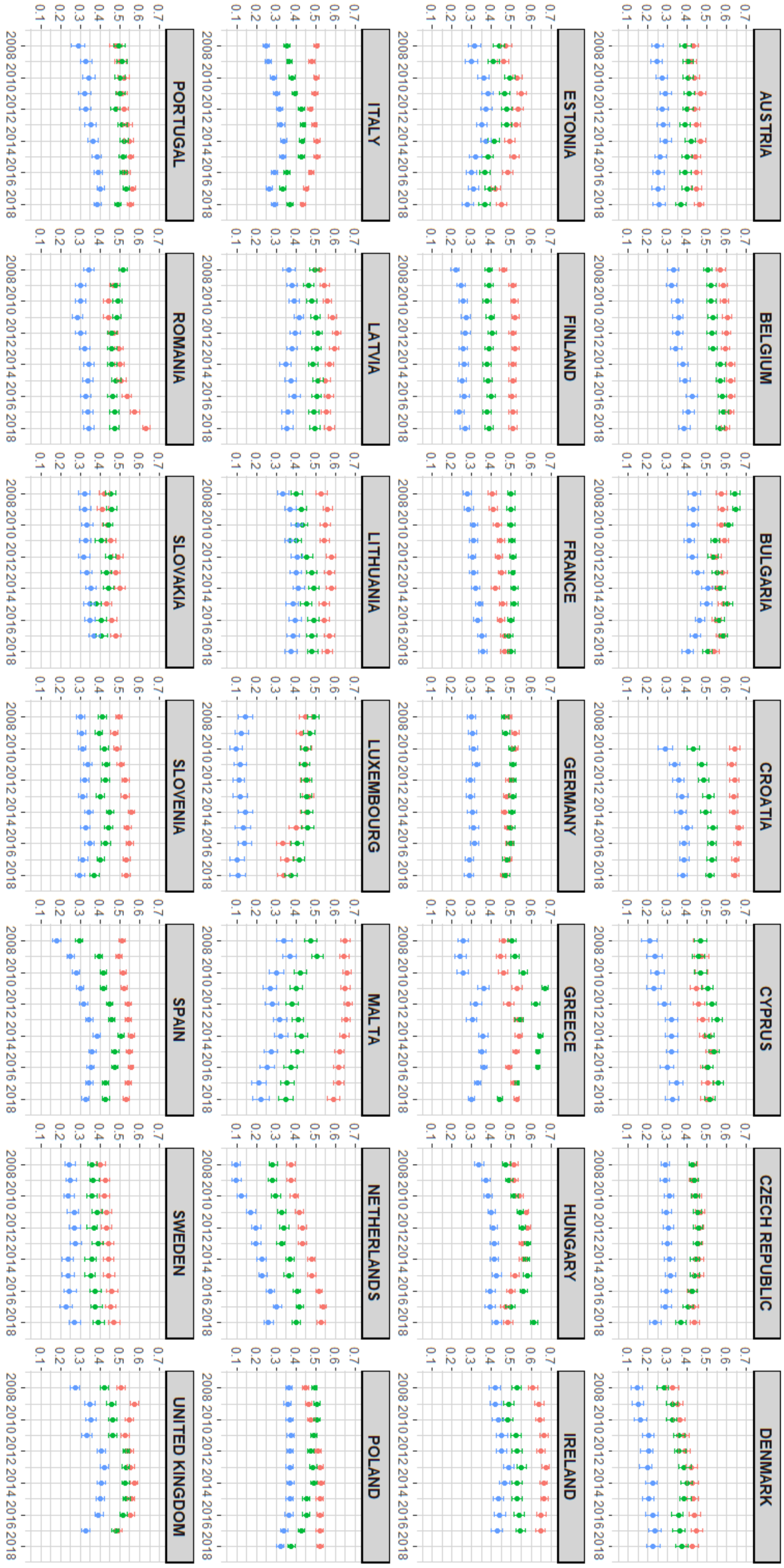

Figure 3.6: Evolution of $\hat{\rho}_{income,work}$ (red), $\hat{\rho}_{income,no-deprivation}$ (green) and $\hat{\rho}_{work,no-deprivation}$ (blue) and their bootstrap standard 95% confidence intervals for the EU-28 countries over the period 2008-2018.

is particularly relevant because this means that, in the EU, there is a general strong tendency of the three poverty dimensions (income, no-material deprivation and work intensity) to take low values together, so that households tend to be simultaneously low ranked in all dimensions, and this could make overall poverty worse.

To close this section, we wonder whether those countries with higher AROPE rates are also countries with high levels of dependence between its dimensions. To address this issue, Figure 3.7 contains three scatter plots showing the relationship between the AROPE rate and the coefficient $\hat{\rho}_3^-$ for the EU-28 countries in the years 2008, 2014 and 2018.[7] In all graphs, the horizontal and vertical reference lines represent the corresponding values for the EU-28 as a whole. We focus on $\hat{\rho}_3^-$ because it measures lower orthant dependence, which is arguably the most relevant concept when analysing multidimensional poverty. The main conclusions that can be drawn from this figure are the following. First, there is a positive relationship between the AROPE rate and lower orthant dependence, that is, countries with high incidence of multidimensional poverty tend to experience also a high degree of multivariate dependence between its dimensions in the lower orthant. This means that those countries with a high proportion of households at risk of poverty and social exclusion tend to be also the countries where, in average, a household is more likely to be simultaneously poor in all dimensions (income, work intensity and material deprivation). Second, those countries with either very low or very high values of $\hat{\rho}_3^-$ in 2008 came closer to the situation of the majority of the EU-28 countries in 2014. And third, looking at the evolution of both the scatter plot and the reference lines, we observe, in the EU-28 as a whole, an increase in both the AROPE rate and the multivariate dependence between its dimensions over the period 2008-2014, followed by a slight decrease afterwards. Hence, it becomes clear that the financial crisis of 2008 exacerbated poverty in terms of both the incidence (proportion of poor households) and the likelihood of being simultaneously poor in all dimensions.

[7]The AROPE rate is calculated here as the proportion of households in our sample that are poor in at least one of the three dimensions considered.

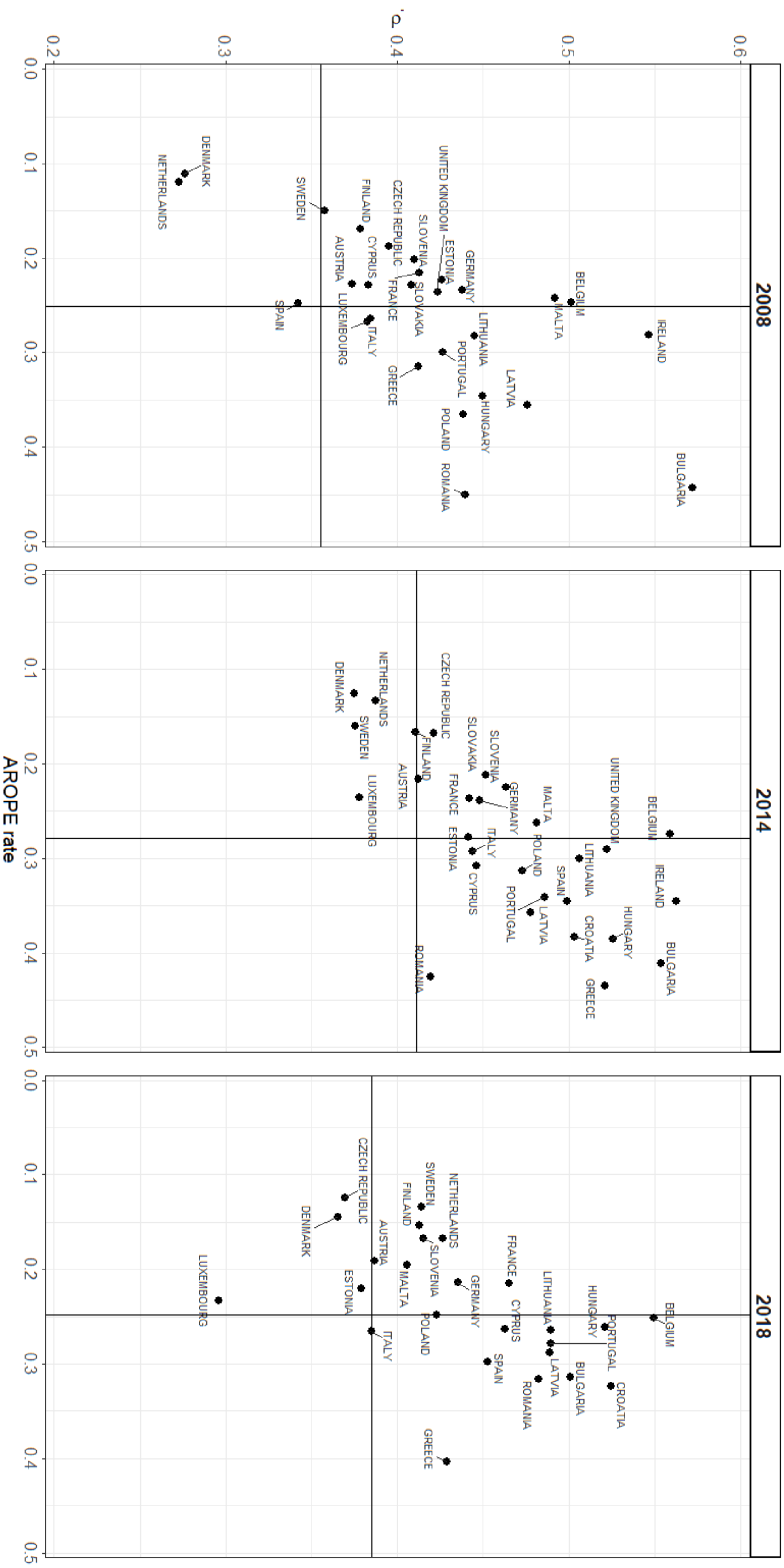

Figure 3.7: Relationship between AROPE rate and $\hat{\rho}_3^-$ for EU-28 countries and years 2008 (left), 2014 (center) and 2018 (right).

3.5 Multivariate tail dependence analysis

In this section, we complement the analysis of multivariate dependence between dimensions of poverty in the EU-28 performed in the previous section by focusing on the concept of tail dependence, which refers to the probability that, given that a variable takes extremely low (high) values, the other dimensions also take extremely low (high) values. In particular, we will analyse the evolution of multivariate tail dependence between the three dimensions of the AROPE rate in the countries of the EU-28 over the period 2008-2018. To do so, we use the multivariate version of the tail concentration function (TCF) that we proposed in Section 2.5.2, and which is defined in (2.61). We estimate this function non-parametrically using the empirical versions of the copula and the survival function as proposed in (2.64) applied to the unique ranks as explained in Section 3.2.2. Throughout this section, special emphasis will be given to multivariate lower tail dependence, which refers to the probability that a household that is extremely poor in one of the dimensions is also extremely poor in the rest of the dimensions considered.

Figure 3.8 displays, for the EU-28 countries, the trivariate TCF for years 2008 (black line), 2014 (red line) and 2018 (green line) together with 95% standard bootstrap confidence intervals using 1000 bootstrap replications. The choice of these years allows us to study the change in multivariate tail dependence between the tree dimensions of the AROPE rate after the Great Recession (2008-2014) and also in the period of economic recovery (2014-2018). The TCFs are calculated in all cases for $t \in [0.05, 0.95]$ over 100 points. Recall that the multivariate TCF allows to analyse multivariate tail dependence going beyond the asymptotic notion of dependence of the traditional tail dependence coefficients. In particular, the left part of the TCF (for $t \in (0, 0.5]$) measures dependence in the lower tail of the joint distribution, whereas its right part (for $t \in (0.5, 1)$) measures dependence in the upper tail of the joint distribution.

Several conclusions emerge from Figure 3.8. First, since the TCF take values different from 0 when t is near 0 and 1, there is tail dependence both in the lower and in the upper joint tails of

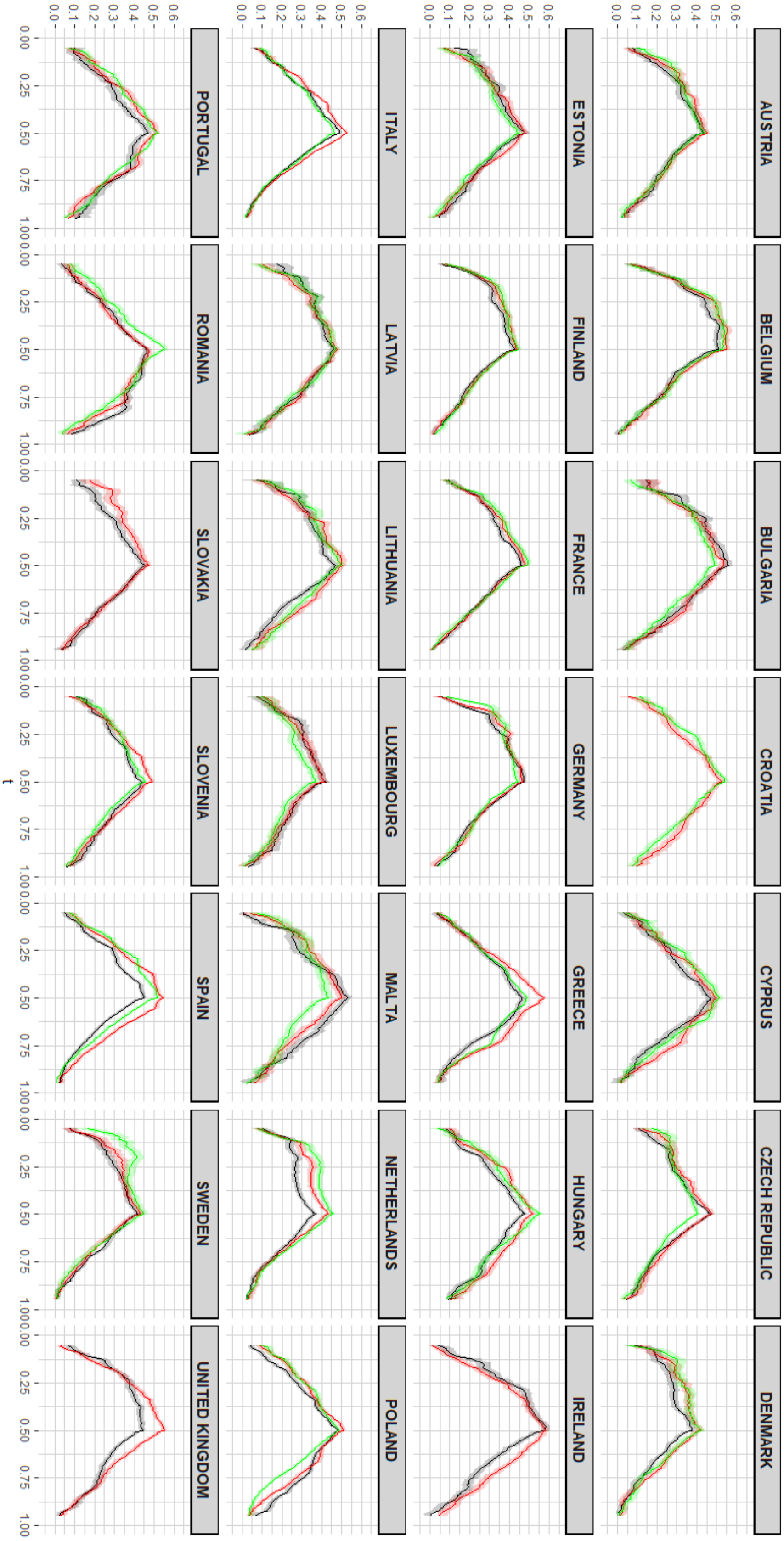

Figure 3.8: Trivariate TCF for the EU-28 countries and years 2008 (black line), 2014 (red line) and 2018 (green line) with bootstrap confidence intervals.

the distribution. That is, there is a positive probability that a household that is poor (rich) in one of the dimensions of the AROPE rate is also simultaneously poor (rich) in the rest of the two dimensions. Second, looking at the shapes of the TCF, we observe considerable cross-country differences. On one hand, there are countries, such as Croatia, Cyprus, Hungary or Poland, where the curves are rather symmetric. This means that, in these countries, dependence in the lower tail of the joint distribution seems to be similar to dependence in its upper tail. Take, for instance, the case of Poland in 2008, where there was a probability of $\hat{q}_C^3(0.25) = 0.28$ of being simultaneously in the first quartile in two poverty dimensions given that you are in the first quartile in the other dimension, and the probability of being simultaneously in the fourth quartile in two poverty dimensions given that you are in the fourth quartile in the other dimension was very similar, $\hat{q}_C^3(0.75) = 0.3$. On the other hand, we observe that, in many EU countries, the TCF is not symmetric, but observations in the lower tail are somewhat more dependent than observations in the upper tail, since $\hat{q}_C^3(t) > \hat{q}_C^3(1-t)$ for $0 < t < 0.5$. To better appreciate this feature, let us focus on the case of Belgium. In this country, in 2008, there was a probability of $\hat{q}_C^3(0.25) = 0.45$ of being simultaneously in the first quartile in two poverty dimensions given that you are in the first quartile in the other dimension, whereas the probability of being simultaneously in the fourth quartile in two poverty dimensions given that you are in the fourth quartile in the other dimension reduces to $\hat{q}_C^3(0.75) = 0.22$. As we pointed out before, this feature is observed in many countries, such as Austria, Belgium, Denmark, France, Germany, Finland, Italy, The Netherlands, Spain or Sweden. Interestingly, we find one country, namely Romania, where upper tail dependence is somewhat stronger than lower tail dependence, since $\hat{q}_C^3(t) < \hat{q}_C^3(1-t)$ for $0 < t < 0.5$. For instance, in this country in 2008 there was a probability of $\hat{q}_C^3(0.25) = 0.25$ of being simultaneously in the first quartile in two poverty dimensions given that you are in the first quartile in the other dimension, whereas the probability of being simultaneously in the fourth quartile in two poverty dimensions given that you are in the fourth quartile in the other dimension increases to $\hat{q}_C^3(0.75) = 0.35$. Therefore, we can conclude that, in most of the EU countries, the probability that a household that is poor in one of the three dimensions of the AROPE rate is also poor in the other two dimensions tends to be higher than the probability that a household that is rich in one of the three dimensions of the AROPE rate is also rich in the other two dimensions. These results are consistent with those obtained in the previous section, where we found that, for all EU countries except for Romania, multivariate orthant dependence between the dimensions of the AROPE rate tends to be higher when analysing it from a "downward" perspective than when the analysis is performed an "upward" perspective.

Further noteworthy features can be observed if we focus on the value of the TCF in the extremes, that is, for $t = 0.05$ and $t = 0.95$. In doing so, we can have an approximation to the asymptotic tail dependence coefficients λ_L^d and λ_U^d, respectively, defined in (2.57) and (2.58). We observe that, in many EU countries, dependence in the lower tail is higher than in the upper tail, since $\hat{q}_C^3(0.05)$ is higher than $\hat{q}_C^3(0.95)$. This is clearly the case of Belgium, Bulgaria, Czech Republic, Denmark, France, Italy, The Netherlands or Slovakia, among others. This means that, in all these countries, the probability that a household that is extremely poor (in the 5th percentile) in one dimension is also simultaneously extremely poor in the other dimensions is higher than the probability that a household that is extremely rich (in the 95th percentile) in one dimension is also simultaneously extremely rich in the other dimensions. For instance, look again at the case of Belgium. In this country, in 2008, there was a probability of $\hat{q}_C^3(0.25) = 0.08$ of a household to be simultaneously in the 5th percentile ($t = 0.05$) in two poverty dimensions given that it was in the 5th percentile in the other dimension, whereas the probability of a household to be simultaneously in the 95th percentile ($t = 0.95$) in two poverty dimensions given that it was in the 95th percentile in the other reduces to $\hat{q}_C^3(0.95) = 0.01$. By contrary, in other countries, there seems to be a symmetry between the value of the function in the lower and in the upper extremes, as $\hat{q}_C^3(0.05)$ is approximately equal to $\hat{q}_C^3(0.95)$; see, for instance, the cases of Croatia, Cyprus or Greece.

We will focus now on the evolution of multivariate tail dependence over the period analysed. As we pointed out before, we will mainly focus on lower tail dependence, since it is the most relevant concept in a multidimensional poverty analysis, capturing the probability that a household that is extremely poor in one of the dimensions of poverty is also extremely poor in the rest of the dimensions.

To begin with, in Figure 3.8 we observe that there are no significant differences in the shapes of the TCF in the different years considered, except in the case of Ireland, where the TCF seems to be symmetric in 2008 and asymmetric in 2014. Figure 3.8 also reveals, in the majority of the EU countries, an increase in multivariate lower tail dependence between 2008 and 2014, since the left side of the TCF of 2014 (red line) seems to be above that of 2008 (black line) in most cases. In fact, we only find some evidence of a decrease in multivariate lower tail dependence in Bulgaria and Ireland. This means that, after the Great Recession, most of the EU countries experienced an increase in the probability of a household to be poor in one of the dimensions given that it is poor in the rest of the dimensions. Nonetheless, and despite the general increase in multivariate lower tail dependence observed, it is possible to observe different cross-country patterns. First, we find countries where we observe a clear increase in multivariate lower tail dependence. This is the case, for instance, of countries such as Denmark, Greece, Hungary, Italy, Portugal, Spain or The Netherlands. Furthermore, this increase seems to be particularly remarkable in Spain, one of the countries most hardly hit by the Financial Crisis of 2008. However, it is worth pointing out that, in many of these countries, this increase in multivariate lower tail dependence in the period 2008-2014 is less clear when focusing in the lowest part of the joint distribution, that is, for small values of t. A subsequent analysis will throw more light into this issue. Second, Figure 3.8 also reveals that there are countries where multivariate lower tail dependence between poverty dimensions did not change much between 2008 and 2014. This is the case of Germany or Luxembourg, for instance. A rather stable behaviour can also be found in France (with some evidence of an increase in certain parts of the distribution) and Estonia (in this case with some evidence of a decrease in some parts of the distribution). Third, as we argued before, we only find evidence of a decrease in multivariate lower tail dependence in this period in Bulgaria and Ireland.

In the period of recovery from the crisis, namely the period 2014-2018, we observe that, for most of the countries, multivariate lower tail dependence between dimensions of poverty remained rather stable, as the left side of the multivariate tail concentration function barely moved between 2014 (red line) and 2018 (green line). This is the case of Belgium, Denmark, Finland, France, Hungary, Latvia or Poland, among others. This stable behaviour is also found in other countries such as Czech Republic, Slovenia or even Spain, but in these cases with some evidence of a decrease in multivariate lower tail dependence when focusing in certain parts of the distribution. In this period we also find countries (Romania or Sweden, for instance) where multivariate lower tail dependence between dimensions of poverty still increased. This increase is also found in other countries, such as Germany or The Netherlands, but only in certain parts of the distribution. Finally, there are also countries, such as Greece, Italy or Luxembourg, which display a decrease in multivariate lower tail dependence between poverty dimensions in that period. However, it is important to remark that this decrease is not so clear when focusing on the lowest part of the joint distribution, that is, when considering small values of t. Again, in a subsequent analysis we will try to throw more light into this issue.

To sum up, as a consequence of the Great Recession, over the period 2008-2014 we find evidence, in most of the EU countries, of an increase in the probability that a household which is poor in one of the three dimensions of the AROPE rate is also poor in the other two dimensions. By contrast, over the period of economic recoverty (2014-2018), this probability seemed to remain rather stable or even decrease in the majority of the countries.

Finally, if we look at the change in multivariate lower tail dependence between poverty dimensions over the whole period analysed, that is, if we compare the situation in 2008 (black line) and in 2018 (green line), some interesting conclusions can be drawn. In particular, in countries such as France, the Netherlands or Spain, for instance, the level of multivariate lower tail dependence between the dimensions of poverty was still higher in 2018 than in 2008. This means that, over that decade, these countries have experienced an increase in the probability of being simultaneously poor in two dimensions of poverty given that one is poor in the other dimension. We also find countries, such as Greece or Italy, which experienced an increase in lower tail dependence during the period of economic crisis (2008-2014) that was followed by a decrease afterwards. As a result, the level of multivariate lower tail dependence between poverty dimensions seems to be similar in 2018 to that observed at the beginning of the period analysed. Noticeably, we only find three countries (Bulgaria, Estonia and Luxembourg) where multivariate lower tail dependence between poverty dimensions seems to be clearly lower in 2018 than in 2008.

So far we have focused our analysis on the evolution of lower tail dependence, since this is the most relevant concept in a multidimensional poverty framework. Nonetheless, Figure 3.8 also allows us to analyse the evolution of multivariate upper tail dependence between poverty dimensions. The first feature that is worth mentioning is that, while multivariate lower tail dependence between dimensions of poverty increased in most of the EU countries over the period 2008-2014, with multivariate upper tail dependence we do not find this generalised increase. That is, over that period, there was a generalised increase in the probability that a household that is poor in one of the three dimensions of the AROPE rate is also poor in the other two dimensions. However, over that same period, we do not find evidence of a generalised increase in the probability that a household which is rich in one of the dimensions of poverty is also rich in the rest of the dimensions. In fact, in many of the EU countries we find a rather stable behaviour of multivariate upper tail dependence between 2008 and 2014; see, for example, the cases of France, Germany or Estonia. Nevertheless, we also find some countries, such as Greece, Hungary, Ireland or Spain, where there was a clear increase in multivariate upper tail dependence between dimensions of poverty. It is also worth mentioning that only in a few cases multivariate upper tail dependence seems to decrease over that period; see, for example, the case of Malta and Romania.

Moving to the change in multivariate upper tail dependence between 2014 and 2018, the first noteworthy feature is that we do not find evidence, for any country, of an increase in multivariate upper tail dependence between dimensions of poverty in that period. In particular, we find countries in which multivariate upper tail dependence did not seem to change, such as Denmark, France or Germany, and countries in which we find evidence of a decrease in upper tail dependence between poverty dimensions, such as Poland, Slovenia or Spain.

Finally, when we compare the situation at the beginning (2008, in the black line) and at the end (2018, in the green line) of the period analysed, we observe that, in most of the EU countries, multivariate upper tail dependence between poverty dimensions in 2018 seems to be either similar to or lower than in 2008. In particular, in countries such as France, Germany or Italy, multivariate tail dependence seems to be similar in 2018 to that observed at the beginning of the period, whereas in other cases, such as that of Poland, it seems to be lower in 2018 than in 2008. Finally, we also find some countries (see, for example, Spain or The Netherlands) where multivariate upper tail dependence between the dimensions of poverty seemed to be higher in 2018 than in 2008, although, this result is less clear when we focus on the highest end of the joint distribution, that is, when we consider values of t close to 1.

Next, to get a better insight into the evolution of multivariate dependence between poverty dimensions in the lowest part of the joint distribution, which is the most relevant one to analyse multidimensional poverty, we have estimated the coefficient of multivariate lower tail dependence introduced in (2.59) using the non-parametric estimator proposed in (2.62) for two particular thresholds, namely $t = 0.1$ and $t = 0.2$. These coefficients, namely $\lambda_L^3(0.1)$ and $\lambda_L^3(0.2)$, allow to analyse the probability that a household that is in the first (second) decile in one of the dimensions of the AROPE rate is also simultaneously in the first (second) decile in the other two dimensions. Tables 3.5 and 3.6 display the estimated values (with bootstrap standard errors in parenthesis) of $\lambda_L^3(0.1)$ and $\lambda_L^3(0.2)$, respectively, for three selected years, 2008, 2014 and 2018. These tables also include, in columns 4, 5 and 6, the results of a two-independent sample one-side t-test with unequal variances, calculated using bootstrap standard errors, to test the significance of the change in the coefficient in the periods 2008-2014, 2014-2018 and 2008-2018, respectively. The corresponding p-value (in parenthesis) is computed assuming asymptotic normality of the t-statistic.

We begin analysing the evolution of multivariate lower tail dependence by focusing on the first decile. To do so, we focus on the results provided by Table 3.5. Several conclusions can be drawn from these results. In particular, looking at column 6, we observe that, in the majority of the countries, $\hat{\lambda}_L^3(0.1)$ did not significantly change between 2008 and 2018. Nevertheless, in some countries, $\hat{\lambda}_L^3(0.1)$ was significantly higher in 2018 than in 2008. Thus, in these countries (Austria, Czech Republic, Germany, Denmark, Lithuania, Malta, Poland, Portugal, Romania and Sweden), there was a significant increase in the probability that a household that is extremely poor (in the first decile) in one dimension is also extremely poor in the other two dimensions. Noticeably, this increase in multivariate lower tail dependence is the result of different evolutions of the coefficient in the sub-periods 2008-2014 and 2014-2018. In particular, in Czech Republic, Malta and Poland the increase in multivariate lower tail dependence was experienced between 2008 and 2014, when we also observe a significant increase in $\hat{\lambda}_L^3(0.1)$ in Slovakia. On the other hand, in Germany, Lithuania, Romania and Sweden the increase in the coefficient is observed in the period 2014-2018. We also find two countries, namely Italy and Estonia, where the evolution of $\hat{\lambda}_L^3(0.1)$ is rather singular. In particular, in Italy this coefficient significantly increased between 2008 and 2014 and significantly decreased afterwards, whereas exactly the opposite occurred in Estonia. Finally, it is worth remarking that, in none of the EU countries, $\hat{\lambda}_L^3(0.1)$ was significantly lower in 2018 and 2008.

If we set the second decile as the threshold to estimate multivariate tail dependence, that is, if we focus on how likely it is that a household being in the second decile in one poverty dimension is also simultaneously in the second decile in the other two poverty dimensions,

	$\hat{\lambda}_L^3(0.1)$					
	2008	**2014**	**2018**	**t-test** Change 08-14	**t-test** Change 14-18	**t-test** Change 08-18
AUSTRIA	0.145 (0.017)	0.176 (0.018)	0.208 (0.019)	1.233 (0.109)	1.244 (0.107)	2.517 (0.006)
BELGIUM	0.205 (0.018)	0.212 (0.019)	0.199 (0.020)	0.280 (0.390)	-0.473 (0.318)	-0.209 (0.417)
BULGARIA	0.177 (0.022)	0.176 (0.021)	0.181 (0.020)	-0.022 (0.491)	0.150 (0.440)	0.124 (0.451)
CYPRUS	0.120 (0.020)	0.107 (0.017)	0.157 (0.020)	-0.511 (0.305)	1.930 (0.027)	1.301 (0.097)
CZECH REPUBLIC	0.169 (0.013)	0.235 (0.017)	0.264 (0.018)	3.144 (0.001)	1.161 (0.123)	4.271 (0.000)
GERMANY	0.152 (0.012)	0.181 (0.014)	0.299 (0.016)	1.604 (0.054)	5.621 (0.000)	7.522 (0.000)
DENMARK	0.193 (0.018)	0.204 (0.019)	0.245 (0.022)	0.415 (0.339)	1.412 (0.079)	1.839 (0.033)
ESTONIA	0.213 (0.021)	0.160 (0.018)	0.212 (0.018)	-1.917 (0.028)	2.012 (0.022)	-0.045 (0.482)
GREECE	0.103 (0.014)	0.108 (0.013)	0.112 (0.008)	0.227 (0.410)	0.288 (0.387)	0.547 (0.292)
SPAIN	0.111 (0.010)	0.128 (0.010)	0.129 (0.011)	1.208 (0.113)	0.021 (0.492)	1.212 (0.113)
FINLAND	0.206 (0.013)	0.223 (0.013)	0.224 (0.015)	0.901 (0.184)	0.076 (0.470)	0.917 (0.179)
FRANCE	0.167 (0.013)	0.167 (0.012)	0.187 (0.015)	-0.018 (0.493)	1.052 (0.146)	0.998 (0.159)
CROATIA	NA NA	0.166 (0.019)	0.202 (0.016)	NA NA	1.469 (0.071)	NA NA
HUNGARY	0.130 (0.013)	0.129 (0.012)	0.139 (0.017)	-0.065 (0.474)	0.449 (0.327)	0.381 (0.352)
IRELAND	0.115 (0.018)	0.084 (0.014)	NA NA	-1.318 (0.094)	NA NA	NA NA
ITALY	0.126 (0.008)	0.152 (0.009)	0.132 (0.009)	2.079 (0.019)	-1.564 (0.059)	0.519 (0.302)
LITHUANIA	0.174 (0.020)	0.179 (0.021)	0.237 (0.023)	0.175 (0.430)	1.891 (0.029)	2.066 (0.019)
LUXEMBOURG	0.145 (0.019)	0.155 (0.019)	0.150 (0.020)	0.390 (0.348)	-0.197 (0.422)	0.185 (0.426)
LATVIA	0.221 (0.019)	0.191 (0.019)	0.220 (0.024)	-1.101 (0.135)	0.948 (0.172)	-0.037 (0.485)
MALTA	0.091 (0.019)	0.171 (0.020)	0.198 (0.024)	2.858 (0.002)	0.849 (0.198)	3.501 (0.000)
THE NETHERLANDS	0.230 (0.014)	0.222 (0.014)	0.205 (0.015)	-0.418 (0.338)	-0.824 (0.205)	-1.226 (0.110)
POLAND	0.104 (0.009)	0.152 (0.010)	0.153 (0.010)	3.486 (0.000)	0.099 (0.461)	3.592 (0.000)
PORTUGAL	0.127 (0.018)	0.158 (0.016)	0.174 (0.012)	1.297 (0.097)	0.801 (0.212)	2.201 (0.014)
ROMANIA	0.087 (0.012)	0.083 (0.012)	0.131 (0.015)	-0.208 (0.418)	2.521 (0.006)	2.345 (0.010)
SWEDEN	0.200 (0.016)	0.227 (0.020)	0.326 (0.022)	1.048 (0.147)	3.343 (0.000)	4.697 (0.000)
SLOVENIA	0.177 (0.013)	0.170 (0.013)	0.201 (0.016)	-0.373 (0.355)	1.540 (0.062)	1.202 (0.115)
SLOVAKIA	0.181 (0.018)	0.293 (0.021)	NA NA	4.105 (0.000)	NA NA	NA NA
UK	0.148 (0.014)	0.133 (0.013)	NA NA	-0.797 (0.213)	NA NA	NA NA

Note: Standard errors for the coefficients and p-values for the one-side t-test are displayed in parentheses.

Table 3.5: Coefficient of trivariate lower tail dependence $\hat{\lambda}_L^3(0.1)$ between the dimensions of the AROPE rate in EU-28 countries (2008-2018)

	$\hat{\lambda}_L^3(0.2)$					
	2008	**2014**	**2018**	**t-test Change 08-14**	**t-test Change 14-18**	**t-test Change 08-18**
AUSTRIA	0.272	0.313	0.296	1.983	-0.752	1.197
	(0.014)	(0.015)	(0.015)	(0.024)	(0.226)	(0.116)
BELGIUM	0.373	0.406	0.422	1.574	0.724	2.218
	(0.015)	(0.015)	(0.016)	(0.058)	(0.235)	(0.013)
BULGARIA	0.364	0.335	0.362	-1.228	1.267	-0.084
	(0.017)	(0.017)	(0.014)	(0.110)	(0.103)	(0.467)
CYPRUS	0.219	0.212	0.231	-0.281	0.843	0.513
	(0.017)	(0.015)	(0.016)	(0.389)	(0.200)	(0.304)
CZECH REPUBLIC	0.258	0.278	0.275	1.198	-0.185	0.988
	(0.010)	(0.013)	(0.013)	(0.115)	(0.426)	(0.162)
GERMANY	0.315	0.351	0.386	2.489	2.237	4.719
	(0.010)	(0.011)	(0.011)	(0.006)	(0.013)	(0.000)
DENMARK	0.265	0.295	0.309	1.425	0.628	2.039
	(0.014)	(0.015)	(0.016)	(0.077)	(0.265)	(0.021)
ESTONIA	0.291	0.285	0.265	-0.309	-0.954	-1.246
	(0.015)	(0.015)	(0.015)	(0.379)	(0.170)	(0.106)
GREECE	0.219	0.221	0.208	0.119	-0.969	-0.809
	(0.012)	(0.012)	(0.007)	(0.453)	(0.166)	(0.209)
SPAIN	0.217	0.276	0.286	4.497	0.752	5.514
	(0.009)	(0.010)	(0.009)	(0.000)	(0.226)	(0.000)
FINLAND	0.317	0.338	0.358	1.365	1.320	2.622
	(0.011)	(0.011)	(0.011)	(0.086)	(0.093)	(0.004)
FRANCE	0.291	0.307	0.331	1.071	1.607	2.640
	(0.010)	(0.010)	(0.011)	(0.142)	(0.054)	(0.004)
CROATIA	NA	0.258	0.299	NA	2.072	NA
	NA	(0.015)	(0.012)	NA	(0.019)	NA
HUNGARY	0.257	0.288	0.311	1.977	1.262	2.990
	(0.011)	(0.011)	(0.014)	(0.024)	(0.103)	(0.001)
IRELAND	0.318	0.273	NA	-2.051	NA	NA
	(0.016)	(0.015)	NA	(0.020)	NA	NA
ITALY	0.228	0.282	0.230	5.084	-4.844	0.145
	(0.007)	(0.008)	(0.007)	(0.000)	(0.000)	(0.442)
LITHUANIA	0.308	0.298	0.349	-0.464	2.205	1.717
	(0.016)	(0.015)	(0.017)	(0.321)	(0.014)	(0.043)
LUXEMBOURG	0.300	0.271	0.246	-1.333	-1.083	-2.467
	(0.015)	(0.016)	(0.016)	(0.091)	(0.139)	(0.007)
LATVIA	0.339	0.323	0.339	-0.712	0.694	-0.010
	(0.016)	(0.015)	(0.016)	(0.238)	(0.244)	(0.496)
MALTA	0.261	0.316	0.319	2.220	0.128	2.186
	(0.018)	(0.017)	(0.019)	(0.013)	(0.449)	(0.014)
THE NETHERLANDS	0.275	0.323	0.374	3.252	3.433	6.671
	(0.010)	(0.011)	(0.011)	(0.001)	(0.000)	(0.000)
POLAND	0.232	0.260	0.266	2.230	0.525	2.858
	(0.008)	(0.009)	(0.008)	(0.013)	(0.300)	(0.002)
PORTUGAL	0.237	0.246	0.295	0.477	3.165	3.306
	(0.015)	(0.013)	(0.009)	(0.317)	(0.001)	(0.000)
ROMANIA	0.172	0.197	0.235	1.526	2.284	3.822
	(0.011)	(0.012)	(0.012)	(0.063)	(0.011)	(0.000)
SWEDEN	0.273	0.326	0.415	2.815	4.081	7.173
	(0.012)	(0.015)	(0.016)	(0.002)	(0.000)	(0.000)
SLOVENIA	0.262	0.283	0.292	1.427	0.557	1.975
	(0.010)	(0.011)	(0.011)	(0.077)	(0.289)	(0.024)
SLOVAKIA	0.235	0.325	NA	4.471	NA	NA
	(0.014)	(0.014)	NA	(0.000)	NA	NA
UK	0.329	0.328	NA	-0.108	NA	NA
	(0.012)	(0.011)	NA	(0.457)	NA	NA

Note: Standard errors for the coefficients and p-values for the one-side t-test are displayed in parentheses.

Table 3.6: Coefficient of trivariate lower tail dependence (with $\hat{\lambda}_L^3(0.2)$) between the dimensions of the AROPE rate in EU-28 countries (2008-2018)

the general picture does not change much, although the values of $\hat{\lambda}_L^3(0.2)$ (in Table 3.6) are greater than those of $\hat{\lambda}_L^3(0.1)$ (in Table 3.5). That is, the probability that a household that is in the second decile in one poverty dimension is also in the second decile in the other two poverty dimensions is greater than the probability that a household that is in the first decile in one poverty dimension is also in the first decile in the other two dimensions. Moreover, if we look at the results displayed in column 6 of Table 3.6, we observe that, in the majority of the EU countries, $\hat{\lambda}_L^3(0.2)$ significantly increased between 2008 and 2018. That is, there has been a significant increase in the probability that a household that is in the second decile in one poverty dimension is also in the second decile in the other two poverty dimensions. In most of these countries, the increase in multivariate lower tail dependence was observed over the period 2008-2014, that is, after the financial crisis of 2008. This is the case of Belgium, Germany, Spain, Hungary, Malta, The Netherlands, Poland, Romania and Sweden. Noticeably, in Germany, The Netherlands, Romania and Sweden we also observe a significant increase in $\hat{\lambda}_L^3(0.2)$ over the recovery period of 2014-2018. Again, Italy constitutes a very interesting case, as it experience a very significant increase in multivariate lower tail dependence between poverty dimensions over the period 2008-2014 that was followed by a significant decrease afterwards. Finally, it is worth mentioning that only in Luxembourg was $\hat{\lambda}_L^3(0.2)$ lower in 2018 than in 2008.

To complement the analysis of trivariate tail dependence, we have also computed all the possible pairwise lower tail dependence coefficients between the three dimensions of the AROPE rate. Again, we focus on lower tail dependence, which is the most relevant concept when analysing multidimensional poverty. In particular, Figure 3.9 displays the evolution, in all the EU-28 countries, of the estimated coefficients of finite bivariate lower tail dependence in the first decile, namely $\hat{\lambda}_L(0.1)$, between income and work intensity (in the red line), income and no-material deprivation (in the green line) and work intensity and no-material deprivation (in the blue line), together with 95% bootstrap confidence intervals. Figure 3.10 displays the same results but using the second decile as the threshold ($\hat{\lambda}_L(0.2)$).

Several conclusions can be drawn from these figures. First, as expected, we observe higher values than those observed for trivariate lower tail dependence (recall the theoretical result in page 109). Second, for the vast majority of countries, and both for $\hat{\lambda}_L(0.1)$ and $\hat{\lambda}_L(0.2)$, the highest bivariate tail dependence is between income and work intensity and the lowest is between no-material deprivation and work intensity. In fact, the gap between the red line and the others is very big in certain countries, such as Estonia, Finland or Slovenia. Nevertheless, there are some significant exceptions. In particular, in Luxembourg, Greece and Romania, the highest dependence is between income and no-material deprivation and, in the UK and for $\hat{\lambda}_L(0.1)$, is between work intensity and no-material deprivation. Notice that these results confirm the patterns already observed in the previous section, since when we applied the Spearman's rho coefficients of pairwise dependence, we observed that, in all countries, the highest level of dependence is found either between income and work intensity or between income and no-material deprivation. Third, the evolution is very similar for the three pairwise coefficients. In particular, if we focus on $\hat{\lambda}_L(0.1)$, in most countries lower tail dependence remained rather stable over the period, but in some of them, such as Czech Republic, Malta, Slovakia or Germany, we observe a slight increasing trend. With regard to $\hat{\lambda}_L(0.2)$, we observe, for the vast majority of the countries, a rather stable trend, although we also find slight increasing trends in some countries such as The Netherlands, Spain or Sweden.

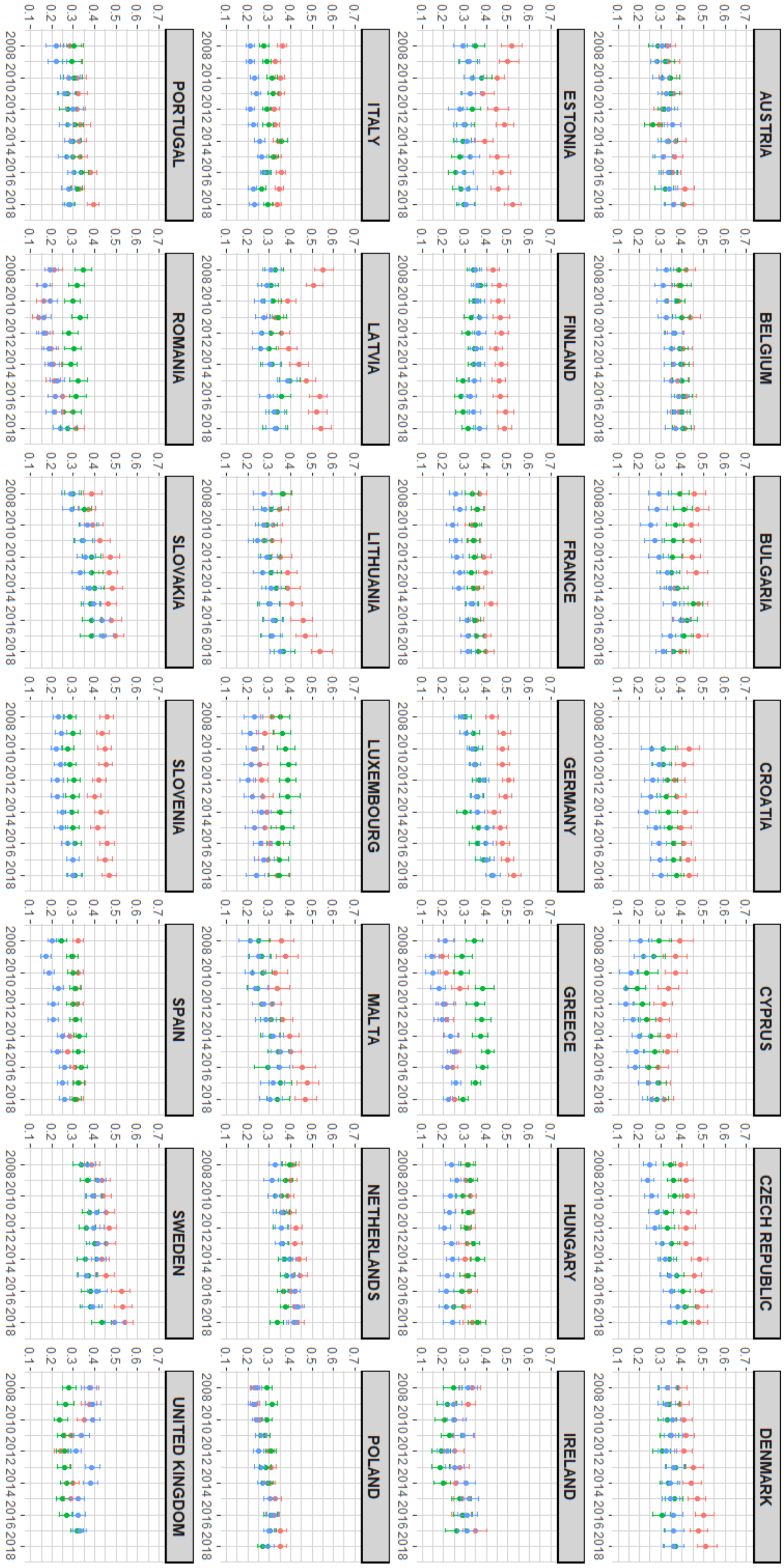

Figure 3.9: Evolution of pairwise lower tail dependence ($\lambda_L^2(0.1)$) between income and work intensity (red), income and no-material deprivation (green) and work intensity and no-material deprivation (blue) and their bootstrap standard 95% confidence intervals for the EU-28 countries over the period 2008-2018.

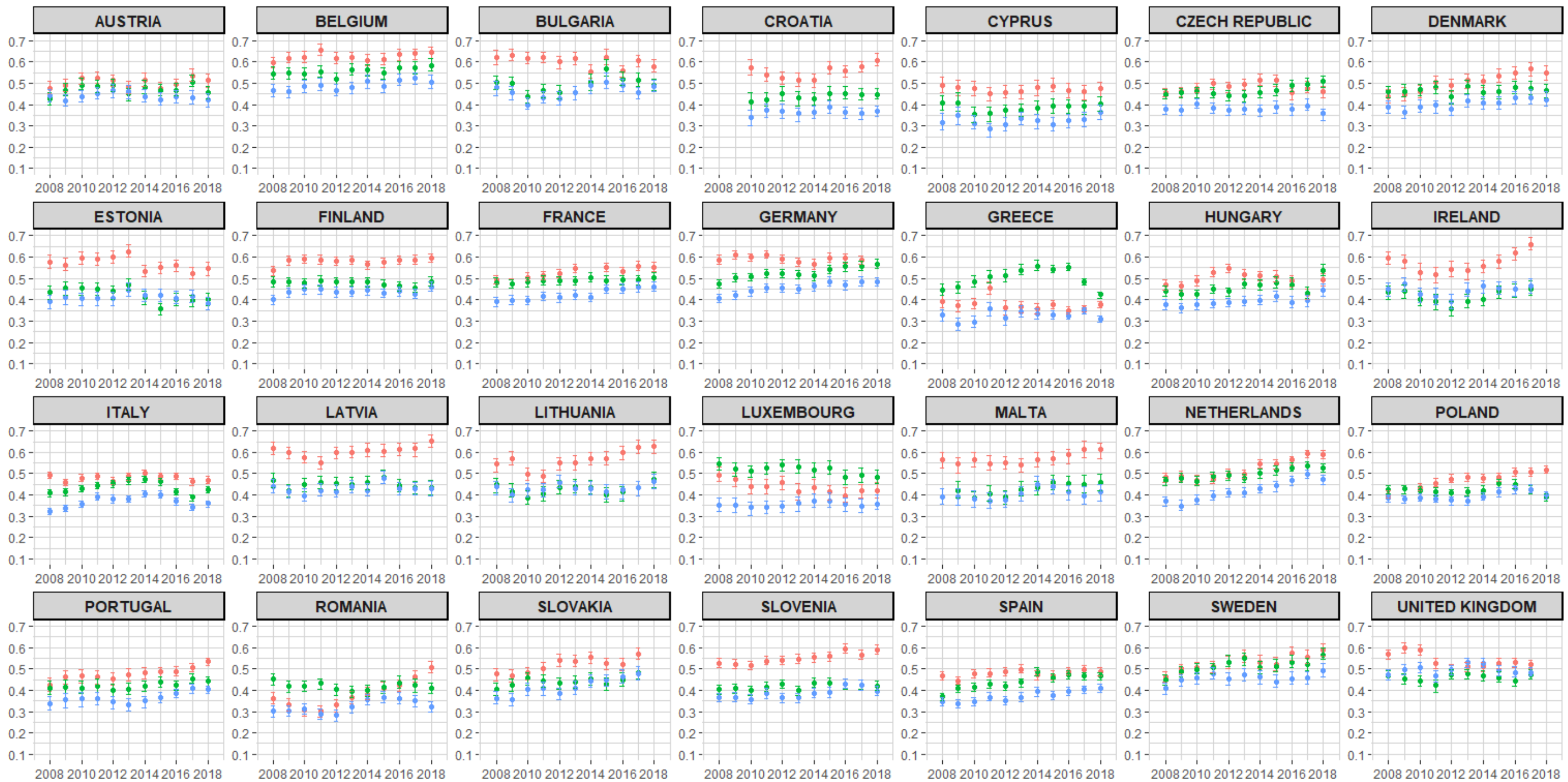

Figure 3.10: Evolution of pairwise lower tail dependence ($\lambda_L^2(0.2)$) between income and work intensity (red), income and no-material deprivation (green) and work intensity and no-material deprivation (blue) and their bootstrap standard 95% confidence intervals for the EU-28 countries over the period 2008-2018.

Conclusions

There is a consensus in the literature on the multidimensional nature of poverty, as it does not depend only on income, but also on other non-monetary aspects such as health, education or labour status, among others. Accordingly, over the last years, a large and still growing literature on multidimensional poverty measurement has emerged. However, this literature, which has mainly focused on the construction of multidimensional poverty indices, has traditionally overlooked a crucial aspect of multidimensional poverty, namely the dependence between its different dimensions. The aim of this thesis is precisely to incorporate the measurement of such dependence into the multidimensional poverty analysis, complementing the information provided by multidimensional poverty indices. In order to do so, several issues should be addressed. First, the measurement of multivariate dependence is challenging and requires special care, since some bivariate dependence properties are not preserved when we have more than two dimensions. Second, some of the dimensions usually considered in Welfare Economics, such as income, for instance, usually do not follow Gaussian distributions. And third, there can be non-linear relationships between the different dimensions.

Hence, when measuring dependence between poverty dimensions, we face the problem of measuring dependence in a multivariate, possibly non-Gaussian and possibly non-linear context. In this framework, we need a methodology that goes beyond the well-known Pearson's linear correlation coefficient, which is only appropriate for the measurement of bivariate linear relationships in elliptical contexts. The copula approach, which is based on the positions of the individuals across dimensions, constitutes a suitable methodology in this setting due to several reasons. First, it enables the decomposition of the joint distribution of the dimensions into the marginals and the dependence structure. Second, it allows to study well-known scale-free measures of bivariate association that capture other types of dependence beyond linear correlation. And third, this approach also permits the construction of multivariate generalisations of these bivariate measures.

In this thesis, we have proposed the measurement of multivariate dependence between the dimensions of poverty using the copula approach. In particular, we have discussed two copula-based concepts of multivariate dependence, namely multivariate concordance and orthant dependence, which allow to build up measures of overall association between poverty dimensions from both "downward" and "upward" perspectives. Furthermore, we have also studied the concept of multivariate tail dependence, which focuses on the extremes of the joint distribution and measures the probability that an individual who is extremely low(high)-ranked in one dimensions of poverty is also extremely low(high)-ranked in the other dimensions considered. In particular, we propose a new tool, the multivariate TCF, which allows to analyse multivariate tail dependence going beyond the asymptotic notion of dependence of the traditional tail dependence coefficients. This function is a very powerful tool, since it allows to represent in the plane the degree of multivariate dependence in the joint tails of a distribution, regardless of the number of dimensions considered. Moreover, we have applied these copula-based concepts to measure multivariate dependence between the different dimensions of poverty in the European Union. In particular, we have used data from EU-SILC to analyse the evolution of multivariate dependence between the dimensions of the AROPE rate (income, work intensity and material deprivation) in the EU-28 countries between 2008 and 2018.

In this empirical application, we have first applied several multivariate extensions of Spearman's rho, which are based on the concept of orthant dependence. Among these multivariate extensions, the coefficient of average lower orthant dependence, ρ_d^-, becomes especially relevant when analysing multidimensional poverty, as it captures the rescaled average probability of being simultaneously low-ranked in all dimensions of poverty as compared to what this would be were those dimensions independent. Several noteworthy results emerge from this analysis. First, there is a positive multivariate association between poverty dimensions in the EU both in the lower and in the upper orthant, which means that low (high) values of income tend to occur simultaneously with low (high) values of the other two dimensions. Second, in the vast majority of the EU countries, average lower orthant dependence tends to be higher than average upper orthant dependence, which means that the average probability of being simultaneously low-ranked in all poverty dimensions tends to be higher than the average probability of being simultaneously high-ranked in all poverty dimensions. Third, between 2008 and 2014, the degree of multivariate dependence between poverty dimensions increased in most of the EU countries both in the lower and in the upper orthant. Thus, over that period, there was, in most of the EU countries, a significant increase in the overall level of association between poverty dimensions from both "downward" and "upward" perspectives. By contrary, over the period 2014-2018 we observe that in many EU countries the degree of multivariate dependence between the three dimensions of the AROPE rate remained rather stable, with some evidence of a decrease in some countries. Nonetheless, in the majority of the EU countries, multivariate dependence between the dimensions of poverty was still significantly higher in 2018 than in 2008. Finally, we detect a positive relationship between the incidence of multidimensional poverty, measured by the AROPE rate, and the dependence between its dimensions. This means that countries with a high poverty incidence tend to experience also a higher degree of dependence between the dimensions of poverty.

We have complemented the analysis based on orthant dependence by focusing on the concept of multivariate tail dependence, which is very useful when it comes to analysing multidimensional poverty. In particular, the concept of multivariate lower tail dependence captures the probability that an individual who is extremely low-ranked (extremely "poor") in one dimension is also extremely poor in the other dimensions considered. We have analysed, using our proposed multivariate TCF, the evolution of multivariate tail dependence between the three dimensions of the AROPE rate in the countries of the EU-28 over the period 2008-2018, putting special emphasis on the evolution of multivariate lower tail dependence. To the best of our knowledge, this constitutes the first application of multivariate tail dependence in Welfare Economics. Several conclusions can be drawn from this analysis. First, there is multivariate tail dependence both in the lower and in the upper joint tails of the distribution, which means that there is a positive probability that a household that is poor (rich) in one of the dimensions of the AROPE rate is also simultaneously poor (rich) in the rest of the two dimensions. Second, in most EU countries dependence in the lower tail seems to be higher than in the upper tail, which means that the probability that a household that is extremely poor in one of the three dimensions of the AROPE rate is also extremely poor in the other two dimensions tends to be higher than the probability that a household that is extremely rich in one of the three dimensions of the AROPE rate is also extremely rich in the other two dimensions. Third, we observe, in the majority of the EU countries, an increase in multivariate lower tail dependence between 2008 and 2014. Thus, over this period, most of the EU countries experienced an increase in the probability that a household which is poor in one of the dimensions is also simultaneously poor in the rest of the dimensions. By contrary, between 2014 and 2018, we observe that, for most of the countries, multivariate lower tail dependence between dimensions of poverty remained rather stable. Moreover, if we compare the situation in 2008 and in 2018, we observe some countries where the level of multivariate lower tail dependence between the dimensions of poverty was higher in 2018 than in 2008 and others in which the level of multivariate lower tail dependence between poverty dimensions seems to be similar in 2018 to that observed at the beginning of the period analysed. Noticeably, we only find three countries (Bulgaria, Estonia and Luxembourg) where multivariate lower tail dependence between poverty dimensions seems to be clearly lower in 2018 than in 2008.

Finally, we have come across several topics that still deserve further research. First, more research is needed to analyse tail dependence in the presence of ties. In this thesis, which provides the first application of multivariate tail dependence in Welfare Economics, we have followed the approach suggested by Decancq (2014) to deal with the problem of ties, which consists in using the information from secondary variables to obtain continuous ranks. Nevertheless, further studies that provide tie-corrected versions of the tail dependence measures used in this work would be very valuable. Second, in this thesis we have proposed the multivariate TCF to measure multivariate tail dependence. However, this is just one possible approach to analyse multivariate tail dependence, and it is not exempt from limitations. In particular, the multivariate TCF, like the traditional tail dependence coefficients, only evaluates dependence in the diagonal section of the copula. Some authors have considered this as a limiting approach and have suggested to study tail dependence using different thresholds for each of the dimensions. We believe that this is a very promising idea. Unfortunately, in the multivariate framework the literature is still scarce. Schmid and Schmidt (2007a), for instance, propose the use of conditional versions of the multivariate extensions of Spearman's rho discussed in Section 2.4. More recently, Escanciano and Hualde (2019) have also proposed a measure to study tail dependence going beyond the main diagonal of the copula. The study and applicaton of these proposals constitute a very promising field for further research. Third, a more exhaustive study is needed in order to establish possible relationships between our proposed multivariate TCF and some copula-based measures of multivariate dependence also based on the diagonal section of the copula, such as the multivariate Blomqvist's beta. Finally, we believe that the application of the copula-based concepts and measures of multivariate dependence performed in this thesis can be extended to study multidimensional poverty at a regional level or to analyse the phenomenon in other geographic areas. Furthermore, and more importantly, this approach can also be applied to other fields of Economics that requires multivariate analyses.

Bibliography

Aaberge, R., Atkinson, A. B., and Königs, S. (2018). From classes to copulas: wages, capital, and top incomes. *The Journal of Economic Inequality*, 16(2):295–320.

Aaberge, R. and Brandolini, A. (2015). Multidimensional poverty and inequality. In Atkinson, A. B. and Bourguignon, F., editors, *Handbook of Income Distribution*, pages 141–216. Elsevier-North Holland.

Abdous, B., Fougères, A.-L., and Ghoudi, K. (2005). Extreme behaviour for bivariate elliptical distributions. *Canadian Journal of Statistics*, 33(3):317–334.

Aghakouchak, A., Ciach, G., and Habib, E. (2010). Estimation of tail dependence coefficient in rainfall accumulation fields. *Advances in Water Resources*, 33(9):1142 – 1149.

Alink, S., Löwe, M., and Wüthrich, M. V. (2007). Diversification for general copula dependence. *Statistica Neerlandica*, 61(4):446–465.

Alkire, S. (2002). Dimensions of human development. *World Development*, 30(2):181–205.

Alkire, S. (2007). *Choosing Dimensions: The Capability Approach and Multidimensional Poverty*, chapter 6, pages 89–119. Palgrave Macmillan UK, London.

Alkire, S. and Foster, J. E. (2011a). Counting and multidimensional poverty measurement. *Journal of Public Economics*, 95(7-8):476–487.

Alkire, S. and Foster, J. E. (2011b). Understandings and misunderstandings of multidimensional poverty measurement. *Journal of Economic Inequality*, 9(2):289–314.

Alkire, S., Foster, J. E., and Santos, M. E. (2011). Where did identification go? *Journal of Economic Inequality*, 9(3):501–505.

Alkire, S., Foster, J. E., Seth, S., Santos, M. E., Roche, J. M., and Ballon, P. (2015). *Multidimensional poverty measurement and analysis*. Oxford University Press.

Alkire, S. and Seth, S. (2015). Multidimensional poverty reduction in India between 1999 and 2006: Where and how? *World Development*, 72(Supplement C):93 – 108.

Anand, S. and Sen, A. (1997). *Concepts of Human Development and Poverty: A Multidimensional Perspective*. United Nations Development Programme, New York.

Aristondo, O. and Ciommi, M. (2016). The decompositions of rank-dependent poverty measures using ordered weighted averaging operators. *International Journal of Approximate Reasoning*, 76(Supplement C):47 – 62.

Atkinson, A. (1970). On the measurement of inequality. *Journal of Economic Theory*, 2(3):244–263.

Atkinson, A. B. (1987). On the measurement of poverty. *Econometrica*, 55(4):749–764.

Atkinson, A. B. and Bourguignon, F. (1982). The comparison of multi-dimensioned distributions of economic status. *The Review of Economic Studies*, 49(2):183–201.

Atkinson, A. B., Cantillon, B., Marlier, E., and Nolan, B. (2002). *Social indicators: The EU and social inclusion*. OUP Oxford.

Barrett, G. F. and Donald, S. G. (2003). Consistent tests for stochastic dominance. *Econometrica*, 71(1):71–104.

Batana, Y. M. (2013). Multidimensional measurement of poverty among women in sub-saharan Africa. *Social Indicators Research*, 112(2):337–362.

Bedo, J. and Ong, C. S. (2014). Multivariate Spearman's rho for aggregating ranks using copulas. *Journal of Machine Learning Research*, 17:1–30.

Behboodian, J., Dolati, A., and Úbeda-Flores, M. (2007). A multivariate version of Gini's rank association coefficient. *Statistical Papers*, 48(2):295–304.

Biewen, M. (2002). Bootstrap inference for inequality, mobility and poverty measurement. *Journal of Econometrics*, 108(2):317–342.

Bishop, J., Chakraborti, S., and Thistle, P. (1989). Asymptotically distribution-free statistical inference for generalized Lorenz curves. *The Review of Economics and Statistics*, 71(4):725–727.

Bishop, J., Formby, J., and Thistle, P. (1992). Convergence of the south and non-south income distributions, 1969-1979. *The American Economic Review*, 82(1):262–272.

Bishop, J. A., Chow, K. V., and Zheng, B. (1995). Statistical inference and decomposable poverty measures. *Bulletin of Economic Research*, 47(4):329–340.

Bishop, J. A., Formby, J. P., and Zheng, B. (1997). Statistical inference and the Sen index of poverty. *International Economic Review*, 38(2):381.

Blackburn, M. L. (1989). Poverty measurement: An index related to a Theil measure of inequality. *Journal of Business & Economic Statistics*, 7(4):475–481.

Blackorby, C. and Donaldson, D. (1980). Ethical indices for the measurement of poverty. *Econometrica*, 48(4):1053–1060.

Blumentritt, T. and Schmid, F. (2014). Nonparametric estimation of copula-based measures of multivariate association from contingency tables. *Journal of Statistical Computation and Simulation*, 84(4):781–797.

Bø, E. E., Lambert, P. J., and Thoresen, T. O. (2012). Horizontal inequity under a dual income tax system: principles and measurement. *International Tax and Public Finance*, 19(5):625–640.

Bonhomme, S. and Robin, J.-M. (2009). Assessing the equalizing force of mobility using short panels: France, 1990-2000. *The Review of Economic Studies*, 76(1):63–92.

Bossert, W., Chakravarty, S. R., and D'Ambrosio, C. (2009). Multidimensional poverty and material deprivation. Working Papers 129, ECINEQ, Society for the Study of Economic Inequality.

Bossert, W., Chakravarty, S. R., and D'Ambrosio, C. (2013). Multidimensional poverty and material deprivation with discrete data. *Review of Income and Wealth*, 59(1):29–43.

Bourguignon, F. and Chakravarty, S. (2003). The measurement of multidimensional poverty. *Journal of Economic Inequality*, 1(1):25–49.

Bourguignon, F. and Chakravarty, S. (2009). Multidimensional poverty orderings: Theory and applications. In Basu, K. and Kanbur, R., editors, *Arguments for a Better World: Essays in Honor of Amartya Sen.* Oxford University Press, Oxford.

Brandolini, M., Coroneo, F., Giarda, E., Moriconi, C., and See, S. G. (2013). Differences in perceptions of the housing cost burden among European countries. Working Paper 2010-01. Prometeia.

Bücher, A., Jäschke, S., and Wied, D. (2015). Nonparametric tests for constant tail dependence with an application to energy and finance. *Journal of Econometrics*, 187(1):154 – 168.

Butler, R. J. and McDonald, J. B. (1987). Interdistributional income inequality. *Journal of Business & Economic Statistics*, 5(1):13–18.

Caillault, C. and Guégan, D. (2005). Empirical estimation of tail dependence using copulas: application to asian markets. *Quantitative Finance*, 5(5):489–501.

Callan, T. and Nolan, B. (1991). Concepts of poverty and the poverty line. *Journal of Economic Surveys*, 5(3):243–261.

Capéraà, P., Fougères, A., and Genest, C. (1997). A nonparametric estimation procedure for bivariate extreme value copulas. *Biometrika*, 84(3):567–577.

Chakravarty, S. R. (1983a). Ethically flexible measures of poverty. *The Canadian Journal of Economics / Revue canadienne d'Economique*, 16(1):74–85.

Chakravarty, S. R. (1983b). A new index of poverty. *Mathematical Social Sciences*, 6(3):307–313.

Chakravarty, S. R. (2009). Inequality and income poverty. In *Inequality, Polarization and Poverty*, chapter 2, pages 47–83. Springer, New York,.

Chakravarty, S. R. (2018). *Analyzing multidimensional well-being: A quantitative approach.* Wiley, New York.

Chakravarty, S. R. and D'Ambrosio, C. (2013). A family of unit consistent multidimensional poverty indices. In Berenger, V. and Bresson, F., editors, *Poverty and Social Exclusion around the Mediterranean Sea*, pages 75–88. Springer US, Boston, MA.

Chakravarty, S. R., Mukherjee, D., and Ranade, R. (1998). On the family of subgroup and factor decomposable measures of multidimensional poverty. *Research on Economic Inequality*, 8:175–194.

Chan, Y. and Li, H. (2008). Tail dependence for multivariate t-copulas and its monotonicity. *Insurance: Mathematics and Economics*, 42(2):763 – 770.

Charpentier, A. (2003). Tail distribution and dependence measures. In *Proceedings ASTIN*, Berlin.

Chirikos, T. N. (1993). The relationship between health and labor market status. *Annual Review of Public Health*, 14(1):293–312.

Clark, S., Hemming, R., and Ulph, D. (1981). On indices for the measurement of poverty. *The Economic Journal*, 91(362):515.

Clayton, D. G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika*, 65(1):141–151.

Coulter, F. A. E., Cowell, F. A., and Jenkins, S. P. (1992). Equivalence scale relativities and the extent of inequality and poverty. *The Economic Journal*, 102(414):1067–1082.

Crawford, I. (2005). A nonparametric test of stochastic dominance in multivariate distributions. University of Surrey, Institute for Fiscal Studies.

Currie, J. and Madrian, B. C. (1999). Health, health insurance and the labor market. In Ashenfelter, O. and Card, D., editors, *Handbook of Labor Economics*, pages 3309–3416. Elsevier.

Dardanoni, V. and Lambert, P. (2001). Horizontal inequity comparisons. *Social Choice and Welfare*, 18(4):799–816.

Datt, G. (2013). Making every dimension count: Multidimensional poverty without the dual cut off. Monash University, Department of Economics.

Davidson, R. and Duclos, J.-Y. (2000). Statistical inference for stochastic dominance and for the measurement of poverty and inequality. *Econometrica*, 68(6):1435–1464.

Davidson, R. and Flachaire, E. (2007). Asymptotic and bootstrap inference for inequality and poverty measures. *Journal of Econometrics*, 141(1):141–166.

De Luca, G. and Rivieccio, G. (2012). Multivariate tail dependence coefficients for archimedean copulae. In Di Ciaccio, A., Coli, M., and Angulo Ibañez, J. M., editors, *Advanced Statistical Methods for the Analysis of Large Data-Sets*, pages 287–296, Berlin, Heidelberg. Springer Berlin Heidelberg.

Deaton, A. and Zaidi, S. (2002). Guidelines for constructing consumption aggregates for welfare analysis. World Bank Living Standards Measurement Study Working Paper 135.

Decancq, K. (2014). Copula-based measurement of dependence between dimensions of well-being. *Oxford Economic Papers*, 66(3):681–701.

Decancq, K. (2020). Measuring cumulative deprivation and affluence based on the diagonal dependence diagram. *Metron*, (78):103–117.

Decancq, K. and Lugo, M. A. (2013). Weights in multidimensional indices of wellbeing: An overview. *Econometric Reviews*, 32(1):7–34.

Deidda, M. (2015). Economic hardship, housing cost burden and tenure status: Evidence from EU-SILC. *Journal of Family and Economic Issues*, 36(4):531–556.

Del Río, C. and Ruiz-Castillo, J. (2001). TIPs for poverty analysis. The case of Spain, 1980-81 to 1990-91. *Investigaciones Económicas*, 25(1):63–91.

Desai, M. and Shah, A. (1988). An econometric approach to the measurement of poverty. *Oxford Economic Papers*, 40(3):505–522.

Deutsch, J. and Silber, J. (2005). Measuring multidimensional poverty: An empirical comparison of various approaches. *Review of Income and Wealth*, 51(1):145–174.

Di Bernardino, E. and Rullière, D. (2016). On tail dependence coefficients of transformed multivariate archimedean copulas. *Fuzzy Sets and Systems*, 284:89 – 112.

Dolati, A. and Úbeda-Flores, M. (2006). On measures of multivariate concordance. *Journal of Probability and Statistical Science*, 4(2):147–163.

Donaldson, D. and Weymark, J. A. (1986). Properties of fixed-population poverty indices. *International Economic Review*, 27(3):667.

Duclos, J.-Y. and Araar, A. (2006). Estimating poverty lines. In *Poverty and Equity*, pages 103–125. Springer US, Boston, MA.

Duclos, J.-Y., Sahn, D. E., and Younger, S. D. (2006). Robust multidimensional poverty comparisons. *Economic Journal*, 116(514):943–968.

Duclos, J.-Y. and Tiberti, L. (2016). Multidimensional poverty indices: A critical assessment. In *The Oxford Handbook of Well-Being and Public Policy*. Oxford University Press, Oxford.

Durante, F., Fernández-Sánchez, J., and Pappadà, R. (2015). Copulas, diagonals, and tail dependence. *Fuzzy Sets and Systems*, 264:22 – 41. Special issue on Aggregation functions at AGOP2013 and EUSFLAT 2013.

Durante, F., Nelsen, R. B., Quesada-Molina, J. J., and Úbeda-Flores, M. (2014). Pairwise and global dependence in trivariate copula models. In Laurent, A., Strauss, O., Bouchon-Meunier, B., and Yager, R. R., editors, *Information Processing and Management of Uncertainty in Knowledge-Based Systems*, pages 243–251, Cham. Springer.

Durante, F. and Sempi, C. (2010). Copula theory: An introduction. In Jaworski, P., Durante, F., Härdle, W. K., and Rychlik, T., editors, *Copula Theory and Its Applications*, pages 3–31, Berlin, Heidelberg. Springer.

Durante, F. and Sempi, C. (2015). *Principles of Copula Theory.* Chapman and Hall/CRC, Boca Raton, FL.

Efron, B. and Tibshirani, R. J. (1994). *An introduction to the bootstrap.* CRC press, New York, NY.

Embrechts, P., McNeil, A., and Straumann, D. (2001). *Correlation and dependency in risk management: Properties and pitfalls*, chapter 7, pages 176–223. Cambridge University Press, Cambridge.

Escanciano, J. C. and Hualde, J. (2019). Measuring asset market linkages: Nonlinear dependence and tail risk. *Journal of Business & Economic Statistics*, 0(0):1–25.

Ettner, S. L., Frank, R. G., and Kessler, R. C. (1997). The impact of psychiatric disorders on labor market outcomes. *ILR Review*, 51(1):64–81.

European Community (1985). On specific community action to combat poverty (Council Decision of 19.2.84). *85/8/EEC, Official Journal of the EEC.*

Farber, H. S. (2004). Job loss in the United States, 1981-2001. *Research in Labor Economics*, 23:69–117.

Fernández-Sánchez, J., Nelsen, R. B., Quesada-Molina, J. J., and Úbeda-Flores, M. (2016). Independence results for multivariate tail dependence coefficients. *Fuzzy Sets and Systems*, 284:129 – 137.

Ferreira, F. H. G. and Lugo, M. A. (2013). Multidimensional poverty analysis: Looking for a middle ground. *World Bank Research Observer*, 28(2):220–235.

Foster, J. E. (1998). Absolute versus relative poverty. *The American Economic Review*, 88(2):335–341.

Foster, J. E. (2006). Poverty Indices. In de Janvry, A. and Kanbur, R., editors, *Poverty, Inequality and Development*, pages 41–65. Springer US, Boston, MA.

Foster, J. E., Greer, J., and Thorbecke, E. (1984). A class of decomposable poverty measures. *Econometrica*, 52(3):761–766.

Foster, J. E. and Shorrocks, A. F. (1988). Poverty orderings. *Econometrica*, 56(1):173.

Foster, J. E. and Shorrocks, A. F. (1991). Subgroup consistent poverty indices. *Econometrica*, 59(3):687–709.

Frahm, G. (2006). On the extremal dependence coefficient of multivariate distributions. *Statistics & Probability Letters*, 76(14):1470 – 1481.

Frahm, G., Junker, M., and Schmidt, R. (2005). Estimating the tail-dependence coefficient: Properties and pitfalls. *Insurance: Mathematics and Economics*, 37(1):80 – 100.

Frank, M. (1979). On the simultaneous associativity of F(x, y) and x+y-F(x, y) . *Aequationes mathematicae*, 18:266–268.

Fréchet, M. (1951). Sur les tableaux de corrélation dont les marges sont données. *Annals University Lyon: Series A*, 14:53–77.

García, J. E., González-López, V., and Nelsen, R. B. (2013). A new index to measure positive dependence in trivariate distributions. *Journal of Multivariate Analysis*, 115:481–495.

García-Gómez, C., Pérez, A., and Prieto-Alaiz, M. (2019). A review of stochastic dominance methods for poverty analysis. *Journal of Economic Surveys*, 33(5):1437–1462.

García-Gómez, C., Pérez, A., and Prieto-Alaiz, M. (2021). Copula-based analysis of multivariate dependence patterns between dimensions of poverty in europe. *Review of Income and Wealth*, 67(1):165–195.

García-Gómez, P. and López-Nicolás, Á. (2006). Health shocks, employment and income in the Spanish labour market. *Health Economics*, 15(9):997–1009.

Genest, C., NeÅ¡lehovÃ¡, J. G., and RÃ©millard, B. (2017). Asymptotic behavior of the empirical multilinear copula process under broad conditions. *Journal of Multivariate Analysis*, 159:82–110.

Genest, C. and Nešlehová, J. (2007). A primer on copulas for count data. *ASTIN Bulletin*, 37(2):475–515.

Genest, C., Nešlehová, J., and Ben Ghorbal, N. (2010). Spearman's footrule and Gini's gamma: a review with complements. *Journal of Nonparametric Statistics*, 22(8):937–954.

Genest, C., Nešlehová, J., and Rémillard, B. (2013). On the estimation of Spearman's rho and related tests of independence for possibly discontinuous multivariate data. *Journal of Multivariate Analysis*, 117:214 – 228.

Gijbels, I., Kika, V., and Omelka, M. (2020). Multivariate tail coefficients: Properties and estimation. *Entropy*, 22(7):728.

Giorgi, G. and Crescenzi, M. (2001). A proposal of poverty measures based on the Bonferroni inequality index. *Metron - International Journal of Statistics*, 59(3-4):3–16.

Gordon, D. (2006). The concept and measurement of poverty. In Pantazis, C., Gordon, D., and Levitas, R., editors, *Poverty and Social Exclusion in Britain*, pages 29–69. The Policy Press, Bristol.

Gradin, C., Cantó., O., and Del Río, C. (2008). Inequality, poverty and mobility: Choosing income or consumption as welfare indicators. *Investigaciones Económicas*, 32:169–200.

Gumbel, E. J. (1960). Bivariate exponential distributions. *Journal of the American Statistical Association*, 55(292):698–707.

Hadar, J. and Russell, W. (1969). Rules for ordering uncertain prospects. *The American Economic Review*, 59(1):25–34.

Hagenaars, A. (1987). A class of poverty indices. *International Economic Review*, 28(3):583.

Hagenaars, A. and de Vos, K. (1988). The definition and measurement of poverty. *The Journal of Human Resources*, 23(2):211–221.

Hagenaars, A. J. M. and van Praag, B. M. S. (1985). A synthesis of poverty line definitions. *Review of Income and Wealth*, 31(2):139–154.

Halleröd, B. (1995). The truly poor: Direct and indirect consensual measurement of poverty in Sweden. *Journal of European Social Policy*, 5(2):111–129.

Hamada, K. and Takayama, N. (1977). Censored income distributions and the measurement of poverty. *Bulletin of the International Statistical Institute*, 47:617–630.

Hoeffding, W. (1940). Masstabinvariante korrelationstheorie: Schriften des matematischen instituts fur angewandte matematik der universitat berlin, 5, heft 3, p.179-233.

Hua, L. and Joe, H. (2011). Tail order and intermediate tail dependence of multivariate copulas. *Journal of Multivariate Analysis*, 102(10):1454 – 1471.

Huang, X. (1992). *Statistics of bivariate extreme values.* PhD thesis, Erasmus University Rotherdam.

ILO (1995). *The Framework of ILO Action Against Poverty.* ILO International Institute for Labour Studies, Geneva.

Jäntti, M. and Danziger, S. (2000). Income poverty in advanced countries. In Atkinson, A. and Bourguignon, F., editors, *Handbook of Income Distribution*, pages 309–378. Elsevier-North Holland, Amsterdam.

Jenkins, S. P. and Lambert, P. J. (1997). Three “I”s of poverty curves, with an analysis of UK poverty trends. *Oxford Economic Papers*, 49(3):317–327.

Joe, H. (1990). Multivariate concordance. *Journal of Multivariate Analysis*, 35(1):12–30.

Joe, H. (1997). *Multivariate models and multivariate dependence concepts.* Chapman and Hall/CRC.

Joe, H. (2014). *Dependence Modeling with Copulas.* Chapman and Hall, London, UK.

Joe, H., Li, H., and Nikoloulopoulos, A. K. (2010). Tail dependence functions and vine copulas. *Journal of Multivariate Analysis*, 101(1):252 – 270.

Joe, H., Smith, R. L., and Weissman, I. (1992). Bivariate threshold methods for extremes. *Journal of the Royal Statistical Society, Series B*, 54(1):171–183.

Kakwani, N. (1980). On a class of poverty measures. *Econometrica*, 48(2):437–446.

Kakwani, N. (1993). Statistical inference in the measurement of poverty. *The Review of Economics and Statistics*, 75(4):632.

Kannai, Y. (1980). The ALEP definition of complementarity and least concave utility functions. *Journal of Economic Theory*, 22(1):115 – 117.

Klasen, S. (2000). Measuring poverty and deprivation in South Africa. *Review of Income and Wealth*, 46(1):33–58.

Kleiber, C. and Kotz, S. (2003). *Statistical size distributions in economics and actuarial sciences*. John Wiley & Sons, Hoboken, NJ.

Kolm, S.-C. (1977). Multidimensional egalitarianisms. *The Quarterly Journal of Economics*, 91(1):1–13.

Krupskii, P. and Joe, H. (2019). Nonparametric estimation of multivariate tail probabilities and tail dependence coefficients. *Journal of Multivariate Analysis*, 172:147 – 161. Dependence Models.

Kruskal, W. H. (1958). Ordinal measures of association. *Journal of the American Statistical Association*, 53(284):814–861.

Lasso de la Vega, M. C. (2010). Counting poverty orderings and deprivation curves. In Bishop, J. A., editor, *Studies in Applied Welfare Analysis: Papers from the Third ECINEQ Meeting*, pages 153–172. Emerald.

Lasso de la Vega, M. C. and Urrutia, A. (2011). Characterizing how to aggregate the individuals' deprivations in a multidimensional framework. *The Journal of Economic Inequality*, 9(2):183–194.

Li, H. (2008). Tail dependence comparison of survival marshall–olkin copulas. *Methodology and Computing in Applied Probability*, 10(1):39–54.

Li, H. (2009). Orthant tail dependence of multivariate extreme value distributions. *Journal of Multivariate Analysis*, 100(1):243 – 256.

Lokshin, M., Umapathi, N., and Paternostro, S. (2006). Robustness of subjective welfare analysis in a poor developing country: Madagascar 2001. *The Journal of Development Studies*, 42(4):559–591.

Maasoumi, E. (2001). Parametric and nonparametric tests of limited domain and ordered hypotheses in economics. In Baltagi, B., editor, *A Companion to Theoretical Econometrics*, pages 538–556. Blackwell Publishing Ltd, Malden, MA, USA.

Maasoumi, E. and Lugo, M. A. (2008). The information basis of multivariate poverty assessments. In Kakwani, N. and Silber, J., editors, *Quantitative Approaches to Multidimensional Poverty Measurement*, pages 1–29. Palgrave Macmillan UK, London.

Maasoumi, E. and Nickelsburg, G. (1988). Multivariate measures of well-being and an analysis of inequality in the Michigan data. *Journal of Business & Economic Statistics*, 6(3):327–334.

Mai, J.-F. and Scherer, M. (2017). *Simulating copulas: stochastic models, sampling algorithms, and applications*. World Scientific, Singapore.

Malevergne, Y. and Sornette, D. (2006). *Extreme financial risks: From dependence to risk management*. Springer Science & Business Media.

Manner, H. and Segers, J. (2011). Tails of correlation mixtures of elliptical copulas. *Insurance: Mathematics and Economics*, 48(1):153 – 160.

Marlier, E. and Atkinson, A. B. (2010). Indicators of poverty and social exclusion in a global context. *Journal of Policy Analysis and Management*, 29(2):285–304.

Matkovskyy, R. (2019). Centralized and decentralized bitcoin markets: Euro vs USD vs GBP. *The Quarterly Review of Economics and Finance*, 71:270 – 279.

Matkovskyy, R. (2020). A measurement of affluence and poverty interdependence across countries: Evidence from the application of tail copula. *Bulletin of Economic Research*, 72(4):404–416.

McFadden, D. (1989). Testing for stochastic dominance. In Fomby, T. and Seo, T., editors, *Studies in the Economics of Uncertainty*, pages 113–134. Springer, New York.

McNeil, A. J. and Nešlehová, J. (2009). Multivariate Archimedean copulas, d-monotone functions and â1-norm symmetric distributions. *The Annals of Statistics*, 37(5B):3059 – 3097.

Mesfioui, M. and Quessy, J.-F. (2010). Concordance measures for multivariate non-continuous random vectors. *Journal of Multivariate Analysis*, 101(10):2398 – 2410.

Mincer, J. (1991). Education and unemployment. Working Paper 3838, National Bureau of Economic Research.

Morelli, S., Smeeding, T. M., and Thompson, J. P. (2015). Post-1970 trends in within-country inequality and poverty: Rich and middle income countries. In Atkinson, A. and Bourguignon, F., editors, *Handbook of Income Distribution*, pages 593–696. Elsevier-North Holland.

Nelsen, R. B. (1991). *Copulas and Association*, pages 51–74. Springer Netherlands, Dordrecht.

Nelsen, R. B. (1996). Nonparametric measures of multivariate association. In Rüschendorf, L., Schweizer, B., and Taylor, M. D., editors, *Distributions with Given Marginals and Related Topics*, volume 28, pages 223–232. Institute of Mathematical Statistics.

Nelsen, R. B. (1998). Concordance and Gini's measure of association. *Journal of Nonparametric Statistics*, 9(3):227–238.

Nelsen, R. B. (2002). Concordance and copulas: A survey. In Cuadras, C. M., Fortiana, J., and Rodriguez-Lallena, J. A., editors, *Distributions With Given Marginals and Statistical Modelling*, pages 169–177. Springer, Dordrecht.

Nelsen, R. B. (2006). *An introduction to copulas.* Springer-Verlag, New York.

Nelsen, R. B. and Úbeda-Flores, M. (2012). Directional dependence in multivariate distributions. *Annals of the Institute of Statistical Mathematics*, 64(3):677–685.

Nickell, S. (1979). Education and lifetime patterns of unemployment. *Journal of Political Economy*, 87(5):117–131.

Nolan, B. and Whelan, C. T. (2011). *Poverty and Deprivation in Europe.* OUP Catalogue. Oxford University Press.

O'Brien, G. L. and Scarsini, M. (1991). Multivariate stochastic dominance and moments. *Mathematics of Operations Research*, 16(2):382–389.

Osberg, L. and Xu, K. (2000). International comparisons of poverty intensity: Index decomposition and bootstrap inference. *The Journal of Human Resources*, 35(1):51.

Patton, A. (2013). Copula methods for forecasting multivariate time series. In Elliott, G. and Timmermann, A., editors, *Handbook of Economic Forecasting*, volume 2 of *Handbook of Economic Forecasting*, pages 899 – 960. Elsevier.

Pelkowski, J. M. and Berger, M. C. (2004). The impact of health on employment, wages, and hours worked over the life cycle. *The Quarterly Review of Economics and Finance*, 44(1):102–121.

Pérez, A. and Prieto, M. (2015). Measuring dependence between dimensions of poverty in Spain: An approach based on copulas. In *International Fuzzy Systems Association (IFSA) and European Society for Fuzzy Logic and Technology (EUSFLAT) International Joint Conference*, Gijón, Spain.

Pérez, A. and Prieto-Alaiz, M. (2016a). Measuring the dependence among dimensions of welfare: A study based on Spearman's footrule and Gini's gamma. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 24(Suppl. 1):87–105.

Pérez, A. and Prieto-Alaiz, M. (2016b). A note on nonparametric estimation of copula-based multivariate extensions of Spearman's rho. *Statistics & Probability Letters*, 112:41–50.

Poulin, A., Huard, D., Favre, A.-C., and Pugin, S. (2007). Importance of tail dependence in bivariate frequency analysis. *Journal of Hydrologic Engineering*, 12(4):394–403.

Pradhan, M. and Ravallion, M. (2000). Measuring poverty using qualitative perceptions of consumption adequacy. *Review of Economics and Statistics*, 82(3):462–471.

Quessy, J.-F. (2009). Tests of multivariate independence for ordinal data. *Communications in Statistics - Theory and Methods*, 38(19):3510–3531.

Quinn, C. (2007). The health-economic applications of copulas: Methods in applied econometric research. HEDG Working Papers 07/22.

Ramos, X. and Silber, J. (2005). On the application of efficiency analysis to the study of the dimensions of human development. *Review of Income and Wealth*, 51(2):285–309.

Ravallion, M. (1992). Poverty comparisons: A guide to concepts and methods. Living Standards Measurement Study Working Paper 88. Washington DC World Bank.

Ravallion, M. (1996). Issues in measuring and modelling poverty. *The Economic Journal*, 106(438):1328.

Ravallion, M. and Chen, S. (2011). Weakly relative poverty. *Review of Economics and Statistics*, 93(4):1251–1261.

Riddell, W. C. and Song, X. (2011). The impact of education on unemployment incidence and re-employment success: Evidence from the U.S. labour market. *Labour Economics*, 18(4):453–463.

Ruiz-Castillo, J. (2009). Pobreza relativa y absoluta: El caso de México (1992-2004). *El Trimestre Económico*, 76(301(1)):67–99.

Salazar, Y. and Ng, W. L. (2015). Nonparametric estimation of general multivariate tail dependence and applications to financial time series. *Statistical Methods & Applications*, 24(1):121–158.

Santos, M. E. and Villatoro, P. (2016). A multidimensional poverty index for Latin America. *Review of Income and Wealth*.

Saposnik, R. (1981). Rank-dominance in income distributions. *Public Choice*, 36(1):147–151.

Scarsini, M. (1984). On measures of concordance. *Stochastica*, 8(3):201–218.

Schmid, F. and Schmidt, R. (2007a). Multivariate conditional versions of Spearman's rho and related measures of tail dependence. *Journal of Multivariate Analysis*, 98(6):1123 – 1140.

Schmid, F. and Schmidt, R. (2007b). Multivariate extensions of Spearman's rho and related statistics. *Statistics & Probability Letters*, 77(4):407–416.

Schmid, F., Schmidt, R., Blumentritt, T., Gaißer, S., and Ruppert, M. (2010). Copula-based measures of multivariate association. In Jaworski, P., Durante, F., Härdle, W. K., and Rychlik, T., editors, *Copula Theory and Its Applications*, pages 209–236, Berlin, Heidelberg. Springer.

Schmidt, R. (2002). Tail dependence for elliptically contoured distributions. *Mathematical Methods of Operations Research*, 55(2):301–327.

Schmidt, R. and Stadtmüller, U. (2006). Non-parametric estimation of tail dependence. *Scandinavian Journal of Statistics*, 33(2):307–335.

Sen, A. (1976). Poverty: An ordinal approach to measurement. *Econometrica*, 44(2):219–231.

Sen, A. (1983). Poor, relatively speaking. *Oxford Economic Papers*, 35(2):153–169.

Sen, A. (1985). *Commodities and Capabilities*. North-Holland, Amsterdam.

Sen, A. (1987). *The Standard of Living*. Cambridge University Press, Cambridge.

Sen, A. (1992). *Inequality Reexamined*. Oxford: Clarendon Press.

Seth, S. (2013). A class of distribution and association sensitive multidimensional welfare indices. *The Journal of Economic Inequality*, 11(2):133–162.

Seth, S. and Santos, M. (2019). On the interaction between focus and distributional properties in multidimensional poverty measurement. *Social Indicators Research*, 145:503–521.

Shorrocks, A. F. (1983). Ranking income distributions. *Economica*, 50(197):3 – 17.

Shorrocks, A. F. (1995). Revisiting the Sen poverty index. *Econometrica*, 63(5):1225–1230.

Sibuya, M. (1960). Bivariate extreme statistics. *Annals of the Institute of Statistical Mathematics*, 11(3):195–210.

Silber, J. (2007). Measuring poverty: Taking a multidimensional perspective. *Hacienda Pública Española*, 182(3):29–73.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut Statistique de l'Université de Paris*, 8:229–231.

Spicker, P. (1999). Definitions of poverty: Twelve clusters of meaning. In Gordon, D. and Spicker, P., editors, *The International Glossary on Poverty*, pages 229–243. Zed Books, London.

Sriboonchitta, S., Wong, W.-K., and Nguyen, H. T. (2010). *Stochastic Dominance and Applications to Finance, Risk and Economics.* CRC Press, Boca Raton, Florida.

Stiglitz, J. E., Sen, A., and Fitoussi, J.-P. (2009). Report by the commission on the measurement of economic performance and social progress.

Streeten, P., Burki, S., Haq Ul, M., Hicks, N., and Stewart, F. (1981). *First Things First: Meeting Basic Human Needs in the Developing Countries.* Oxford University Press, New York.

Supper, H., Irresberger, F., and WeiÃ, G. (2020). A comparison of tail dependence estimators. *European Journal of Operational Research.*

Sweeting, P. and Fotiou, F. (2013). Calculating and communicating tail association and the risk of extreme loss. *British Actuarial Journal*, 18(1):13–83.

Takayama, N. (1979). Poverty, income inequality, and their measures: Professor Sen's axiomatic approach reconsidered. *Econometrica*, 47(3):747–759.

Taylor, M. D. (2007). Multivariate measures of concordance. *Annals of the Institute of Statistical Mathematics*, 59(4):789–806.

Terzi, S. and Moroni, L. (2020). Local concordance and some applications. *Social Indicators Research.*

Theil, H. (1967). *Economics and Information Theory.* North Holland, Amsterdam.

Thon, D. (1979). On measuring poverty. *Review of Income and Wealth*, 25(4):429–439.

Thorbecke, E. (2007). Multidimensional poverty: Conceptual and measurement issues. In Kakwani, N., editor, *The Many Dimensions of Poverty*, pages 3–19. Palgrave Macmillan UK, London.

Tkach, K. and Gigliarano, C. (2020). Multidimensional poverty index with dependence-based weights. *Social Indicators Research*.

Townsend, P. (1985). A sociological approach to the measurement of poverty- a rejoinder to Professor Amartya Sen. *Oxford Economic Papers*, 37(4):659–668.

Tsui, K. Y. (2002). Multidimensional poverty indices. *Social Choice and Welfare*, 19(1):69–93.

Úbeda-Flores, M. (2005). Multivariate versions of Blomqvist's beta and Spearman's footrule. *Annals of the Institute of Statistical Mathematics*, 57(4):781–788.

United Nations (1995). *World Summit for Social Development: The Copenhagen Declaration and Programme of Action*. United Nations, New York.

Vaughan, R. N. (1987). Welfare approaches to the measurement of poverty. *The Economic Journal*, 97:160–170.

Venter, G. (2001). Tails of copulas. In *Proceedings ASTIN*, pages 68–113, Washington, USA.

Villar, A. (2017). Poverty measurement. In *Lectures on Inequality, Poverty and Welfare*, pages 115–134. Springer.

Ward, P. J., Couasnon, A., Eilander, D., Haigh, I. D., Hendry, A., Muis, S., Veldkamp, T. I. E., Winsemius, H. C., and Wahl, T. (2018). Dependence between high sea-level and high river discharge increases flood hazard in global deltas and estuaries. *Environmental Research Letters*, 13(8):084012.

Watts, H. (1968). An economic definition of poverty. In Moynihan, D., editor, *On Understanding Poverty*, pages 316–329. Basic Books, New York.

WeiB, G. N., Neumann, S., and Bostandzic, D. (2014). Systemic risk and bank consolidation: International evidence. *Journal of Banking & Finance*, 40:165 – 181.

Whelan, C. T. and Maître, B. (2012). Understanding material deprivation: A comparative European analysis. *Research in Social Stratification and Mobility*, 30(4):489–503.

Wolbers, M. H. J. (2000). The effects of level of education on mobility between employment and unemployment in the Netherlands. *European Sociological Review*, 16(2):185–200.

Wolff, E. F. (1980). N-dimensional measures of dependence. *Stochastica*, 4(3):175–188.

Yalonetzky, G. (2014). Conditions for the most robust multidimensional poverty comparisons using counting measures and ordinal variables. *Social Choice and Welfare*, 43(4):773–807.

Zheng, B. (1994). Can a poverty index be both relative and absolute? *Econometrica*, 62(6):1453–1458.

Zheng, B. (1997). Aggregate poverty measures. *Journal of Economic Surveys*, 11(2):123–162.

Zheng, B. (1999). On the power of poverty orderings. *Social Choice and Welfare*, 16(3):349–371.

Zheng, B. (2000). Poverty orderings. *Journal of Economic Surveys*, 14(4):427–466.