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Stages of Economic Development in an Innovation-Education Growth Model

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Abstract

Physical capital accumulation, knowledge formation and R&D-based technological progress are considered the three main sources of growth. The common view is that they characterize, in a temporal order, the three phases that a typical advanced economy passes through in its development process. Recently it has been argued, however, that an innovation-education sequence could agree better than an education-innovation transition with the empirical fact that the rise in formal education to the masses follows rather than precedes the process of industrialization. Accordingly, this paper devises an endogenous growth model with physical capital, human capital and R&D that, unlike previous related work, is able to generate adjustment dynamics in which the innovative stage precedes knowledge formation, consistent with empirical evidence.

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1 Introduction

Physical capital accumulation, knowledge formation and R&D-based technological progress are considered the three main sources of growth. Although the bulk of the literature has treated them as alternative rather than complementary explanations, Funke and Strulik (2000) (FS henceforth) have combined them into an endogenous growth model with physical capital, human capital and R&D. Funke and Strulik (2000) conjecture that a typical advanced economy evolves through three stages of development. At the first stage —the standard neoclassical model—, physical capital is the only factor being accumulated; at the second stage —a knowledge economy in the Uzawa-Lucas framework—, human capital is also being accumulated, and at the third stage —the fully industrialized economy—, research is actively being conducted as well, which results in an increasing variety of goods. Funke and Strulik (2000) also present some simulation results that exemplify this development sequence.

However, there is broad consensus in the literature that formal education did not play a significant role in the British Industrial Revolution. Historical evidence reviewed, e.g, by Galor (2005) and Galor and Moav (2006), shows that in the first phase of industrialization, educational requirements in the production process were minimal, and education served religious, social, or national goals. Thus, Mitch (1993, p. 307) states that "education was not a major contributing factor to England's economic growth during the Industrial Revolution," and Mokyr (1990, p. 240) concludes that "If England led the rest of the world in the Industrial Revolution, it was despite, not because of, her formal education system." In the second phase of the Industrial Revolution, however, the increasing pace of technological progress ultimately brought about an industrial demand for human capital, because skills became necessary for production, which stimulated human capital formation. According with this evidence, Iacopetta (2010) argues that an innovation-education sequence would agree better than an education-innovation transition with the empirical fact that the rise in formal education to the masses follows rather than precedes the process of industrialization. Then, Iacopetta shows that the FS framework can actually generate a richer set of development scenarios; in particular, one in which the transition dynamics is characterized by an innovative stage —without knowledge accumulation— followed by a fully industrialized phase.[1]

[1]It should be noted that the role of human capital in the industrialization process could be different in the technological leader nation, England, than in the technological follower nations, like Germany. In this sense, recent evidence on the Prussian Industrial Revolution provided by Becker, Hornung, and Woessmann (2011) suggests that, unlike the British Industrial Revolution, basic education could have played a crucial role in the industrial catch-up of the technological follower countries because it is a key ingredient for the adoption of new technologies.

However, the simulation results reported by Iacopetta (2010) —like the ones previously reported by Funke and Strulik (2000) (see Gómez, 2005)— are seriously flawed. The reason is twofold. First, in three out of Iacopetta's four experiments the steady-state equilibrium of the last development stage —the fully industrialized economy— is instable, because the system that describes the dynamics of the model has more unstable roots than jump variables. Thus, it is not possible to make the system stable for arbitrary initial values of the predetermined variables, and the economy would start in the stable manifold only by coincidence. The most realistic Iacopetta's experiments suffer from this drawback; in particular, the one that illustrates that the FS model can generate an innovation-education sequence. Second, the numerical transition paths are incorrectly calculated. The implementation of the backward integration method (Brunner and Strulik, 2002) used to compute the simulations has a fatal error whose effect is to lengthen the transition over a longer period than the real one. Once the transition dynamics is correctly computed, the simulation results are strongly at odds with data. Hence, the question on whether the FS model is able to adequately replicate the process of development remains open.

The purpose of this paper is twofold. First, we re-examine the ability of the FS model to describe the development process. The main problems that we detect in previously reported simulations with the FS model are the instability issue, too fast convergence, and unrealistic highly oscillatory dynamics. All these features have to do with the stable roots of the dynamic system that leads the economy. Therefore, we perform a detailed sensitivity analysis of the stable roots of the last development stage —the fully industrialized economy—. Our numerical results show that saddle-path stability is achieved for a relatively small set of parameter values. Furthermore, the two stable roots are more likely to be complex conjugate, and even when the real part is low enough to accommodate a sufficiently long transition, the imaginary part is relatively high, which entails that the model features unrealistic highly oscillatory dynamics. These results suggest that numerical simulations with the FS model could hardly be reconciled with data.

Second, we devise an extension of the FS model that is capable of generating a more realistic adjustment dynamics in which the innovative stage precedes human capital formation. To this end, we add an externality in R&D associated to the duplication and overlap of research effort —a "stepping on toes" effect (e.g., Dasgupta and Maskin, 1987, Jones, 1995a, Stokey, 1995)— to the FS model. Intuitively, the larger the number of people searching for ideas is, the more likely it is that duplication of research would occur, so doubling the number of researchers will less than double the number of unique ideas or discoveries.[2] Furthermore, according with empirical evidence (e.g., Heckman, 1976, Haley, 1976), we allow for education to be subject to diminishing returns to effective time at the private level. In this case, an externality would restore constant returns to scale at the social level, which is a requirement for balanced growth (Hendricks, 1999). The transition dynamics of the model is represented by a two-dimensional stable manifold and, despite the complexity of the dynamic system, we provide a sufficient condition for stability —although the instability outcome cannot be ruled out—. We present some numerical results showing that the introduction of duplication externalities significantly increases the ability of the model to generate a realistic innovation-education sequence in which innovation and education time rise jointly along the transition.

The rest of this paper is organized as follows. Section 2 describes the model, and Section 3 analyzes its equilibrium dynamics. Section 4 examines the ability of the FS model to describe the development process. Section 5 presents some numerical results using the extended model. Section 6 concludes.

2 The model

Consider a closed economy inhabited by a constant population, normalized to one, of identical individuals who derive utility from consumption, C, according to

$$\int_0^\infty e^{-\rho t}(C^{1-\theta}-1)/(1-\theta)dt, \quad \rho > 0. \tag{2.1}$$

Individual's time, which is normalized to unity, can be devoted to production, u_P, education, u_E, or innovation, $u_I = 1 - u_P - u_E$. Human capital, H, is accumulated according to

$$\dot{H} = \xi(u_E H)^{\varepsilon}(\overline{u_E H})^{1-\varepsilon}, \quad \xi > 0, \quad 0 < \varepsilon \leq 1, \tag{2.2}$$

where $\overline{u_E H}$ expresses a sector-specific externality associated to average effective time devoted to education. Therefore, we allow for the presence of diminishing returns to effective learning time in education at the private level, combined with an external effect that restores constant returns to scale at the social level.[3]

[2] Empirical evidence of diminishing returns caused by duplicative research has been reported by Kortum (1993). Lambson and Phillips (2007) found that the probability of duplication is significant for most industries, and Griliches (1990) reviewed some evidence of diminishing returns found in the patent literature.

[3] Note that constant returns to scale at the social level is a requirement for endogenous growth to arise.

The budget constraint faced by the representative individual is

$$\dot{A} = rA + w(1 - u_E)H - C, \tag{2.3}$$

where w is the wage rate per unit of employed human capital, and r is the return per unit of aggregate wealth A. Let g_x denote x's growth rate, $g_x = \dot{x}/x$. The individual maximizes her intertemporal utility (2.1), subject to the budget constraint (2.3) and the knowledge accumulation technology (2.2). The first order conditions yield

$$g_C = (r - \rho)/\theta, \tag{2.4}$$

and

$$r - g_w = \varepsilon\xi \quad \text{and} \quad u_E > 0, \tag{2.5}$$

in an equilibrium with education, or

$$r - g_w > \varepsilon\xi \quad \text{and} \quad u_E = 0. \tag{2.6}$$

Output, Y, is produced with a Cobb-Douglas technology

$$Y = BK^{\beta} D^{\eta} (u_P H)^{1-\beta-\eta}, \quad B > 0, \quad \beta > 0, \quad \eta > 0, \quad \beta + \eta < 1, \tag{2.7}$$

where K is the physical capital stock, and D is an index of intermediate goods, $D = \left(\int_0^n x(i)^{\alpha} di\right)^{1/\alpha}$, $0 < \alpha < 1$, where $x(i)$ is the amount used for each one of the n intermediate goods. The market for final goods is perfectly competitive and the price for final goods is normalized to one. Profit maximization delivers the factor demands

$$r = \beta Y/K, \tag{2.8}$$

$$w = (1 - \beta - \eta)Y/(u_P H), \tag{2.9}$$

$$p(i) = \eta Y x(i)^{\alpha-1}/D^{\alpha}, \tag{2.10}$$

where $p(i)$ represents the price of intermediate i.

Invention of new intermediates is determined according to

$$\dot{n} = \delta(u_I H)(\overline{u_I H})^{\omega-1}, \quad \delta > 0, \quad 0 < \omega \leq 1, \tag{2.11}$$

where $\overline{u_I H}$ represents average effective time devoted to innovation. This specification allows for the presence of a duplication externality of research effort.[4]

[4]Funke and Strulik (2000) and Iacopetta (2010) consider the particular case in which $\varepsilon = 1$ and $\omega = 1$.

There is monopolistic competition in the intermediate-goods sector, and an intermediate good costs one unit of Y to produce. Facing the price elasticity of demand for the intermediates $1/(1-\alpha)$, firms maximize operating profits, $\pi(i) = (p(i)-1)x(i)$, by charging a constant markup price $p(i) = 1/\alpha$. Since both technology and demand are the same for all intermediates, the equilibrium is symmetric: $x(i) = x$, $p(i) = p$. Hence, the quantity of intermediates employed is $xn = \alpha\eta Y$, firms profits are

$$\pi = (1-\alpha)\eta Y/n, \tag{2.12}$$

and $D = xn^{1/\alpha} = n^{(1-\alpha)/\alpha}\alpha\eta Y$. Substituting this expression into (2.7) yields

$$Y^{1-\eta} = B(\alpha\eta)^{\eta}K^{\beta}n^{(1-\alpha)\eta/\alpha}(u_P H)^{1-\beta-\eta}. \tag{2.13}$$

The value of an innovation υ is the present value of the stream of monopoly profits, $\upsilon(t) = \int_t^{\infty} e^{-\bar{r}(\tau,t)}\pi(\tau)d\tau$, with $\bar{r}(\tau,t) = \int_t^{\tau} r(s)ds$. Differentiating this expression with respect to time yields the no-arbitrage equation

$$g_{\upsilon} = r - \pi/\upsilon. \tag{2.14}$$

Finally, free-entry into R&D requires[5]

$$w = \delta(\overline{u_I H})^{\omega-1}\upsilon \quad \text{and} \quad u_I > 0, \tag{2.15}$$

in an equilibrium with innovation, or[6]

$$w > \delta(\overline{u_I H})^{\omega-1}\upsilon \quad \text{and} \quad u_I = 0. \tag{2.16}$$

Henceforth we shall take into account that $\overline{u_E H} = u_E H$ and $\overline{u_I H} = u_I H$ in equilibrium. Let $\chi \equiv C/K$ denote the consumption to physical capital ratio, and $\psi \equiv H^{\omega}/n$, the knowledge-ideas ratio. Physical capital and claims to innovative firms are the assets in the economy. Aggregate wealth is then $A = K + n\upsilon$. From (2.3), (2.8)–(2.12) and (2.14) we can get the economy resource constraint, $\dot{K} = (1-\alpha\eta)Y - C$, which can be expressed as

$$g_K = \frac{1-\alpha\eta}{\beta}r - \chi. \tag{2.17}$$

Using (2.4) and (2.17), we get

$$g_{\chi} = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right)r + \chi - \frac{\rho}{\theta}. \tag{2.18}$$

[5] Here, R&D should be interpreted as research that takes the fruits of accumulated science and tries to find application, so there are no fixed setup costs.

[6] Note that this last case cannot occur if $\omega < 1$.

Some equations will be needed for solving the model. Log-differentiating the expressions for r in (2.8), w in (2.9), and Y in (2.13), and eliminating g_Y, we get

$$g_r = -\frac{1-\beta-\eta}{\beta} g_w + \frac{(1-\alpha)\eta}{\alpha\beta} g_n, \tag{2.19}$$

$$g_{u_P} = -\frac{1-\eta}{\beta} g_w + \frac{(1-\alpha)\eta}{\alpha\beta} g_n + g_K - g_H. \tag{2.20}$$

Log-differentiating (2.11) yields

$$g_{g_n} = \omega(g_{u_I} + g_H) - g_n. \tag{2.21}$$

3 Equilibrium dynamics

This section presents the dynamic systems that lead the different phases that an evolving economy can pass through.

3.1 The neoclassical growth model

The dynamics of the neoclassical growth model ($u_E = u_I = 0$) in terms of the variables r and χ is given by

$$g_r = -\frac{(1-\beta-\eta)(1-\alpha\eta)}{\beta(1-\eta)} r + \frac{1-\beta-\eta}{1-\eta}\chi, \tag{3.1}$$

$$g_\chi = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right) r + \chi - \frac{\rho}{\theta}. \tag{3.2}$$

Here, (3.1) results from (2.19) and (2.20), using (2.17) and that $g_{u_P} = g_n = g_H = 0$.

3.2 The knowledge economy

The dynamics of the knowledge economy ($u_E = 1 - u_P > 0$ and $u_I = 0$) in terms of the variables r, χ and u_P is described by the following system:

$$g_r = -\frac{1-\beta-\eta}{\beta}(r - \varepsilon\xi), \tag{3.3}$$

$$g_\chi = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right) r + \chi - \frac{\rho}{\theta}, \tag{3.4}$$

$$g_{u_P} = \frac{(1-\alpha)\eta}{\beta} r - \chi - \xi(1-u_P) + \frac{(1-\eta)\varepsilon\xi}{\beta}. \tag{3.5}$$

Eqs. (3.3) and (3.5) result from (2.19) and (2.20), using (2.5) to substitute for g_w, (2.17) and (2.2) to substitute for g_K and g_H, and using that $g_n = 0$.

3.3 The innovative economy

The dynamics of the innovative economy ($u_E = 0$ and $u_I = 1 - u_P > 0$) in terms of the variables r, χ, u_P and ψ is determined by the following system:

$$g_r = -\frac{1-\beta-\eta}{\beta} r + \frac{(1-\alpha)\eta(1-u_P+\alpha u_P)}{\alpha\beta(1-u_P)^{1-\omega}}\delta\psi - \frac{(1-\beta-\eta)(1-\omega)u_P}{\beta(1-u_P)} g_{u_P}, \tag{3.6}$$

$$g_\chi = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right) r + \chi - \frac{\rho}{\theta}, \tag{3.7}$$

$$g_{u_P} = \frac{(1-\alpha)\eta(1-u_P)}{\beta(1-u_P)+(1-\eta)(1-\omega)u_P} \times \left\{ r - \frac{\beta}{\eta(1-\alpha)}\chi + \left[\frac{(1-\eta)u_P}{(1-\beta-\eta)(1-u_P)} + \frac{1}{\alpha}\right]\delta(1-u_P)^\omega\psi \right\}, \tag{3.8}$$

$$g_\psi = -\delta(1-u_P)^\omega\psi, \tag{3.9}$$

where g_{u_P} in (3.6) should be substituted with (3.8).

The former system is obtained as follows. Log-differentiating the free-entry condition (2.15), we have $g_w = (\omega-1)(g_{u_I}+g_H)+g_\upsilon$. Substituting g_υ from (2.14), π from (2.12), w from (2.9), and υ from (2.15), we get

$$g_w = r + (\omega-1)(g_{u_I}+g_H) - \frac{(1-\alpha)\eta}{(1-\beta-\eta)u_I} u_P g_n. \tag{3.10}$$

Now, Eqs. (3.6) and (3.8) result from (2.19) and (2.20), using (3.10) to substitute for g_w, (2.17) and (2.11) to substitute for g_K and g_n, and taking into account that $g_H = 0$, $u_I = 1 - u_P$, $g_{u_I} = -g_{u_P} u_P/(1-u_P)$ and $g_n = \delta(1-u_P)^\omega\psi$. Eq. (3.9) results from $g_\psi = -g_n$.

3.4 The fully industrialized economy

If $\omega < 1$, the dynamics of the fully industrialized economy ($u_E > 0$ and $u_I > 0$) in terms of the variables r, χ, u_P, g_n and ψ is determined by

$$g_r = \frac{1-\beta-\eta}{\beta}(\varepsilon\xi - r) + \frac{(1-\alpha)\eta}{\alpha\beta} g_n, \tag{3.11}$$

$$g_\chi = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right) r + \chi - \frac{\rho}{\theta}, \tag{3.12}$$

$$g_{u_P} = \frac{(1-\alpha)\eta}{\beta} r - \chi - \xi \quad 1 - u_P - \frac{g_n^{1/\omega}}{\delta^{1/\omega}\psi^{1/\omega}}\Bigg) + \frac{(1-\alpha)\eta}{\alpha\beta} g_n + \frac{(1-\eta)\varepsilon\xi}{\beta}, \tag{3.13}$$

$$g_\psi = \omega\xi \quad 1 - u_P - \frac{g_n^{1/\omega}}{\delta^{1/\omega}\psi^{1/\omega}}\Bigg) - g_n, \tag{3.14}$$

$$g_{g_n} = -\frac{\omega(1-\alpha)\eta\delta^{1/\omega} u_P g_n^{1-1/\omega}\psi^{1/\omega}}{(1-\beta-\eta)(1-\omega)} - g_n + \frac{\omega\varepsilon\xi}{1-\omega}. \tag{3.15}$$

We have used that $u_E = 1 - u_P - u_I$ and $u_I = [g_n/(\delta\psi)]^{1/\omega}$. Eq. (3.11) results from (2.19) and (2.5). From Eqs. (2.20) and (2.5), using (2.17) and (2.2), we get (3.13). From $g_\psi = \omega g_H - g_n$, using (2.2), we obtain (3.14). Finally, Eq. (3.15) results from (2.21), (3.10) and (2.5), using (2.2).

If $\omega = 1$, from (3.10) and (2.5), using (2.11), we get

$$u_P = \frac{(1-\beta-\eta)\varepsilon\xi}{(1-\alpha)\eta\delta\psi}. \tag{3.16}$$

Hence, $g_{u_P} = g_n - g_H$, which combined with (2.20), (2.17) and (2.5) entails that

$$g_n = \frac{\alpha[(1-\alpha)\eta r - \beta\chi + (1-\eta)\varepsilon\xi]}{\alpha\beta - (1-\alpha)\eta}. \tag{3.17}$$

Thus, if $\omega = 1$ the evolution of the economy is described by (3.11), (3.12) and (3.14), where u_P and g_n should be replaced with (3.16) and (3.17), respectively.

Appendix A proves the following proposition.

Proposition 1. *Let $\varepsilon\xi > \rho$. The economy has a unique positive steady-state equilibrium with positive long-run growth, in which the interest rate is*

$$\hat{r} = \frac{(1+M)\theta\varepsilon\xi - \rho}{(1+M)\theta - 1}, \tag{3.18}$$

the ratio of consumption to physical capital is

$$\hat{\chi} = \left(\frac{1-\alpha\eta}{\beta} - \frac{1}{\theta}\right)\hat{r} + \frac{\rho}{\theta}, \tag{3.19}$$

the long-run growth rate of intermediates is

$$\hat{g}_n = \frac{\omega M(\varepsilon\xi - \rho)}{(1+M)\theta - 1}, \tag{3.20}$$

the long-run growth rate of human capital is

$$\hat{g}_H = \hat{g}_n/\omega, \tag{3.21}$$

the share of labor devoted to production and R&D can be obtained from

$$\hat{u}_I = \frac{(1-\alpha)\eta\hat{g}_n}{(1-\beta-\eta)(\varepsilon\xi - \hat{g}_H + \hat{g}_n) + (1-\alpha)\eta\hat{g}_n}\left(1 - \frac{\hat{g}_H}{\xi}\right), \tag{3.22}$$

$$\hat{u}_P = 1 - \hat{u}_I - \frac{\hat{g}_H}{\xi}, \tag{3.23}$$

the knowledge-ideas ratio is

$$\hat{\psi} = \hat{g}_n/\left(\delta\hat{u}_I^{\omega}\right), \tag{3.24}$$

and the long-run growth rate of income, consumption, and physical capital is

$$\hat{g}_Y = \hat{g}_C = \hat{g}_K = (1+1/M)\hat{g}_H, \tag{3.25}$$

where $M = \alpha(1-\beta-\eta)/[(1-\alpha)\eta\omega]$, *if and only if*

$$\theta > \frac{1+M[1-\rho/(\varepsilon\xi)]}{1+M}. \tag{3.26}$$

A simple sufficient condition for (3.26) to hold is $\theta \geq 1$. We shall now analyze the equilibrium stability in the neighbourhood of the steady state. If $\omega = 1$ (and $\varepsilon = 1$), the stability analysis has been made by Gómez (2005, Theorem 2), who shows that locally saddle-path stability requires two stable roots.[7] Therefore, we shall consider the case in which $\omega < 1$. Appendix A proves the following proposition.

Proposition 2. *Let* $\omega < 1$, *and assume that condition (3.26) in Proposition 1 holds.*
a) The steady-state equilibrium is either saddle-path stable or unstable.
b) A sufficient condition to rule out the instability outcome is

$$\alpha\beta \geq (1-\alpha)\eta\omega. \tag{3.27}$$

[7]The stability analysis in Gómez (2005) can be readily extended to the case $\varepsilon < 1$.

The following example shows that the instability outcome cannot be ruled out, although extensive numerical experimentation shows that the steady state is saddle-path stable for a much wider combination of parameters than that implied by condition (3.27).[8]

Example. *The parameterization* $\beta = 0.35$, $\eta = 0.3$, $\alpha = 0.08$, $\xi = 0.06$, $\rho = 0.023$, $\theta = 2$, $\delta = 0.1$, $B = 1$, $\omega = 0.987$ *and* $\varepsilon = 1$ *yields the (feasible) steady state:* $\hat{r} = 0.0907$, $\hat{\chi} = 0.2190$, $\hat{u}_P = 0.9101$, $\hat{u}_E = 0.0527$, $\hat{u}_I = 0.0373$, $\hat{g}_n = 0.0031$, $\hat{\psi} = 0.7952$, $\hat{g}_H = 0.0032$ *and* $\hat{g}_Y = \hat{g}_K = 0.0338$. *The eigenvalues of the linearized system are* $0.0876 \pm 0.2588i$, $0.0050 \pm 0.1391i$ *and* 0.0568 *and, therefore, the steady state is unstable.*

4 Transition dynamics in the FS model

This section re-examines the ability of the FS model to generate realistic transitional dynamics. Simulation results with the FS model have been reported by Funke and Strulik (2000), Gómez (2005) and Iacopetta (2010). We first show that the simulations made by Funke and Strulik (2000) and Iacopetta (2010) are seriously flawed. Next, we show that Gómez (2005) simulations do not generate a realistic adjustment either. Finally, we perform a detailed sensitivity analysis which suggests that numerical simulations with the FS model could hardly be reconciled with data.

4.1 Previous simulation results

Gómez (2005) shows that saddle-path stability of the steady state of the fully industrialized economy in the FS model requires the existence of two stable roots, rather than only one as had been argued by Funke and Strulik (2000). As a consequence, Gómez shows that the simulations reported by Funke and Strulik (2000) are flawed because the system that describes the dynamics of the last development stage — the fully industrialized economy— has too many unstable roots, i.e., the number of unstable roots exceeds the number of jump variables. Hence, the steady-state equilibrium is unstable and, therefore, it is not possible to make the system stable

[8] We have computed the stable roots for a grid of values of β between 0.1 and 0.8 with a step of 0.1, η between 0.1 and $0.9-\beta$ with a step of 0.1 (so that $1-\beta-\eta \geq 0.1$), α between 0.15 and 0.95 with a step of 0.1, ω between 0.05 and 0.95 with a step of 0.15, θ between 1 and 4 with a step of 0.25, ξ between 0.03 and 0.15 with a step of 0.02, ε between 0.3 and 1 with a step of 0.1, and ρ between 0.02 and $\varepsilon\xi - 0.01$ with a step of 0.02 (so that $\varepsilon\xi - \rho \geq 0.01$). We have found that the steady state is saddle-path stable for all the parameterizations considered. Instability seems to require a extremely high value of ω combined with a low value of α; i.e., a high markup.

Table 1: Parameter values considered by Iacopetta (2010) and implied stable roots

Cases	β	η	α	ρ	θ	ξ	δ	ω	ε	stable root(s)
1	0.36	0.36	0.54	0.023	2	0.050	0.1	1	1	-0.1485
2	0.65	0.20	0.62	0.023	2	0.050	0.1	1	1	-0.0203
3	0.20	0.56	0.75	0.025	2	0.042	0.1	1	1	-0.2005
4	0.23	0.70	0.70	0.023	2	0.040	0.1	1	1	$-0.0587 \pm 0.1131i$

for arbitrary initial values of the predetermined variables. The dynamic system that leads the economy has 'too many' initial conditions, and would start in the stable manifold only by coincidence.

The simulation results reported by Iacopetta (2010) are also flawed for two reasons. First, in three out of Iacopetta's four experiments the steady state is unstable. The parameter values considered by Iacopetta (2010), together with the corresponding stable root(s) of the fully industrialized economy, are displayed in Table 1.[9] It shows that in Cases 1, 2 and 3 there is only one stable root and, therefore, the steady state is unstable. Only in Case 4, the steady state is saddle-path stable. However, Iacopetta shows that this case generates non-monotonic (oscillatory) dynamics, which is highly at odds with data.[10] Second, the adjustment time paths are incorrectly calculated. To gain insight on the problem at hand, let us focus on Iacopetta's Case 3, although the following discussion is applicable to all his experiments. As shown in Table 1, the value of the unique stable root at the steady state of the fully industrialized economy is -0.2005. The implied value of the asymptotic convergence speed entails that the half life of convergence is about 3.4 years,[11] and that 90 percent of the difference between the initial point and the steady state is eliminated in about 11.4 years. This is clearly incompatible with the long transition observed in Iacopetta's Figure 4 —more than one and a half century—, and shows that there is an error in the code used to compute the numerical simulations. Similar appreciations can be made regarding his cases 1, 2 and 4. The problem is that the

[9] The parameter values displayed in Cases 3 and 4 of Table 1 differ slightly from those shown in Iacopetta's Tables 3.A and 4.A, respectively, because there are some errata in his paper. The values reported by Iacopetta do not generate the steady-state values displayed in Iacopetta's Tables 3.B and 4.B that can indeed be obtained by using the parameter values shown in our Table 1. In any case, using the actual values reported in Iacopetta's Tables 3.A and 4.A we would obtain similar results.

[10] Below we show that, once corrected the computation of the adjustment paths, Case 4 is even further from representing adequately the observed data.

[11] Half life of convergence is the time that it takes for half the initial gap between steady state and actual value to be eliminated.

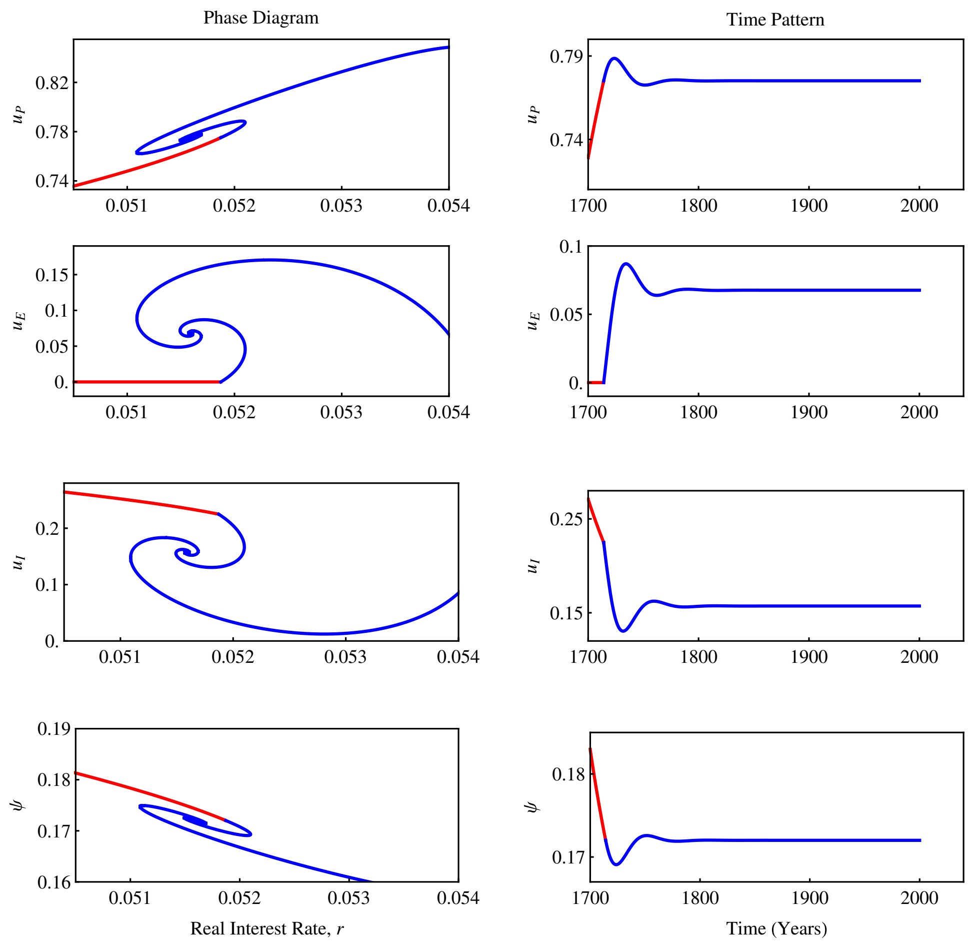

Figure 1: Corrected transition dynamics in Case 4 in Iacopetta (2010).
Note: The left side graphs are —from top to bottom— the phase diagrams of production time, education time, innovation time and the knowledge-ideas ratio. The right-hand side graphs show the time paths of the same variables. The simulation is extended backward to include an innovation phase without education, $u_E = 0$. The red line corresponds to the innovation phase, and the blue line to the fully industrialized phase. Parameter values are shown in Case 4 of Table 1.

second time reversal to transform the trajectory back into forward-looking time is not correctly made.

Figure 1 illustrates the corrected transition dynamics in Iacopetta's Case 4 —the only one that does not suffer from the instability problem—, which is depicted in his Figure 5.[12] The simulation is extended backward to include an innovation phase without education. The phase diagrams displayed in the left side of Figure 1 are exactly the same as those depicted in Iacopetta's Figure 5, because the problem with the time reversal affects the transitional time paths and not the phase diagrams. Looking at the corrected time paths of the variables, it can be noted that convergence is too fast to represent in an adequate manner the observed historical data —even if they are shifted forward by one or two centuries—. The behaviour of time devoted to innovation is specially at odds with data because of its implausibly high values —above 25 percent of total time— at initial stages of development, and its counterfactual oscillatory dynamics thereafter (see, e.g. Jones, 1995b, 2002).

Similar appreciations can be made regarding the simulation results reported by Gómez (2005, Figure 4), which feature unrealistic highly oscillatory dynamics for variables that show a monotonic behaviour in data, as education and innovation time. Furthermore, they move in opposite directions, although they show a joint expansion in data. In summary, we can conclude that previous numerical results with the FS model do not support its adherence to data.

4.2 Sensitivity analysis

Our former results entail that the question on whether the FS model is able to adequately replicate the development process remains open. The main problems detected were the instability issue, too fast convergence speed, and unrealistic highly oscillatory adjustment. All these features have to do with the stable roots of the dynamic system. Hence, in order to assess the ability of the FS model to generate realistic adjustment dynamics, we perform a detailed sensitivity analysis of its stable roots. Specifically, we have computed the stable roots of the fully industrialized economy for a grid of values of β between 0.2 and 0.7 with a step of 0.1, η between 0.2 and $0.9-\beta$ with a step of 0.1 (so that $1-\beta-\eta \geq 0.1$), and α between 0.4 and 0.9 with a step of 0.1. The starting value of α implies a markup as high as 2.5, which is well-above most estimates (e.g., Norrbin, 1993, Basu, 1996, Basu and Fernald, 1997). The values of the parameters ρ, θ and δ are those of Cases 1, 2 and 4 in Table 1,[13] whereas the value of ξ is adjusted so as to keep the long-run growth rate of output and capital, $\hat{g}_Y$, constant at 1.5 percent. This is done so in order to

[12] The transition dynamics has been obtained by means of the backward integration method (Brunner and Strulik, 2002) implemented with Mathematica 7.0. Given that there are two stable roots, the stable manifold is two-dimensional. Therefore, the choice of the initial point to start the backward integration is crucial to determine the transition dynamics.

[13] As Proposition 1 shows, the parameter δ does not affect the steady state and, therefore, the convergence speed either.

Table 2: Stable roots in the FS model for different parameter values

		$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$
$\beta = 0.2$	$\eta = 0.2$	$-0.0128 \pm 0.3013i$	–	–	–
	$\eta = 0.3$	$-0.0001 \pm 0.1831i$	$-0.0208 \pm 0.2683i$	–	–
	$\eta = 0.4$	–	$-0.0075 \pm 0.1768i$	$-0.0446 \pm 0.2776i$	–
	$\eta = 0.5$	–	$-0.0031 \pm 0.1336i$	$-0.0208 \pm 0.1766i$	$-0.2593 \pm 0.4127i$
	$\eta = 0.6$	–	$-0.0009 \pm 0.1063i$	$-0.0128 \pm 0.1274i$	$-0.0645 \pm 0.1952i$
	$\eta = 0.7$	–	–	$-0.0089 \pm 0.0955i$	$-0.0367 \pm 0.1207i$
$\beta = 0.3$	$\eta = 0.3$	$-0.0128 \pm 0.1757i$	–	–	–
	$\eta = 0.4$	$-0.0022 \pm 0.1189i$	$-0.0340 \pm 0.1798i$	–	–
	$\eta = 0.5$	–	$-0.0141 \pm 0.1175i$	$-0.1400 \pm 0.2092i$	–
	$\eta = 0.6$	–	$-0.0075 \pm 0.0873i$	$-0.0446 \pm 0.1166i$	–
$\beta = 0.4$	$\eta = 0.3$	$-0.0764 \pm 0.2225i$	–	–	–
	$\eta = 0.4$	$-0.0128 \pm 0.1122i$	–	–	–
	$\eta = 0.5$	$-0.0037 \pm 0.0808i$	$-0.0473 \pm 0.1158i$	–	–
$\beta = 0.5$	$\eta = 0.4$	$-0.0446 \pm 0.1171i$	–	–	–

eliminate the effect that differences in the long-run growth rate could induce on the value of the stable roots.[14]

Table 2 reports the stable roots for the cases in which the economy exhibits saddle-path stability; i.e., when there are two stable roots. In the other cases, which are not displayed, the steady state is unstable because there is either only one stable root or none. Three conclusions can be derived from Table 2. First, for many plausible combinations of the parameter values, the steady state is unstable. In particular, stability is more unlikely to occur the higher the value of the parameter α; i.e., the lower the markup $1/\alpha$. Thus, if $\alpha \geq 0.8$, no considered combination of parameters yields saddle-path stability, although the implied markup of 1.125 seems realistic (Norrbin, 1993, Basu, 1996, Basu and Fernald, 1997). Second, stable roots are much more likely to be complex conjugate than real —although Gómez (2005) shows that this last case can also occur—. Hence, the fully industrialized economy converges to its steady state through damped oscillations, where the damping factor depends on the real part of the eigenvalues and the frequency depends on their imaginary part. Third, whereas the real part of the stable roots displayed in Table 2 can have a low magnitude (in absolute value) which implies a low rate of decay and, therefore, a long transition, the imaginary part is relatively high. This entails that the fully industrialized economy displays markedly oscillatory dynamics which are

[14]Similar results —available upon request— can be obtained if ξ is set to its value 0.05 in Cases 1 and 2 of Table 1 and it is ρ that adjusts so as to keep $\hat{g}_Y$ constant at 1.5 percent; if both ξ and ρ are kept constant at their values in Cases 1 and 2 of Table 1, or if different values of θ are considered.

Table 3: Parameter and steady-state values

β	η	α	ρ	θ	ξ	δ	ω	ε
0.65	0.2	0.62	0.023	2	0.05	0.1	0.75	1
$\hat{r}$	$\hat{g}_n$	$\hat{g}_H$	$\hat{g}_K$	$\hat{u}_P$	$\hat{u}_E$	$\hat{u}_I$	stable roots	
0.0563	0.0078	0.0103	0.0167	0.7327	0.2067	0.0607	$-0.0180 \pm 0.0080i$	

difficult to match with data. These results cast doubts on the ability of the FS model to describe the development process in a realistic fashion.

5 Simulation results with the extended model

This section shows that the introduction of duplication externalities, as well as private diminishing returns in education, improves the ability of the model to generate a realistic development process through stages in which innovation precedes knowledge accumulation. For the sake of comparison, we first consider the parameter values in Case 2 in Iacopetta (2010, Table 2), along with a duplication externality of $\omega - 1 = -0.25$, which is the value considered by Jones (2001). Table 3 displays the parameter values, the steady-state values, and the stable roots.

Figure 2 depicts the transition dynamics of the economy. Several features of the displayed adjustment deserve attention. First, the model generates transitional dynamics in which innovation precedes human capital formation. The economy starts with a positive but low time devoted to innovation, so that the behaviour of the economy resembles that of the neoclassical model without knowledge formation and innovation. Innovation time increases steadily and, ultimately, the economy enters the fully industrialized stage, with knowledge accumulation as well. This transition accords with the historical evidence examined by Galor (2005) and Galor and Moav (2006) showing that in the first phase of the British Industrial Revolution, human capital played a limited role in the production process. In the second phase of industrialization, however, the increasing pace of technological progress ultimately brought about an industrial demand for human capital that stimulated human capital formation. Accordingly, as argued by Iacopetta (2010), an innovation-education sequence would agree better than an education-innovation transition with the empirical fact that the rise in formal education to the masses follows rather than precedes the process of industrialization in England. Thus, the beginning of formal education in Figure 2 is set around 1850, after the first phase of the British Industrial Revolution, when formal education started to grow in England as a result of the increasing

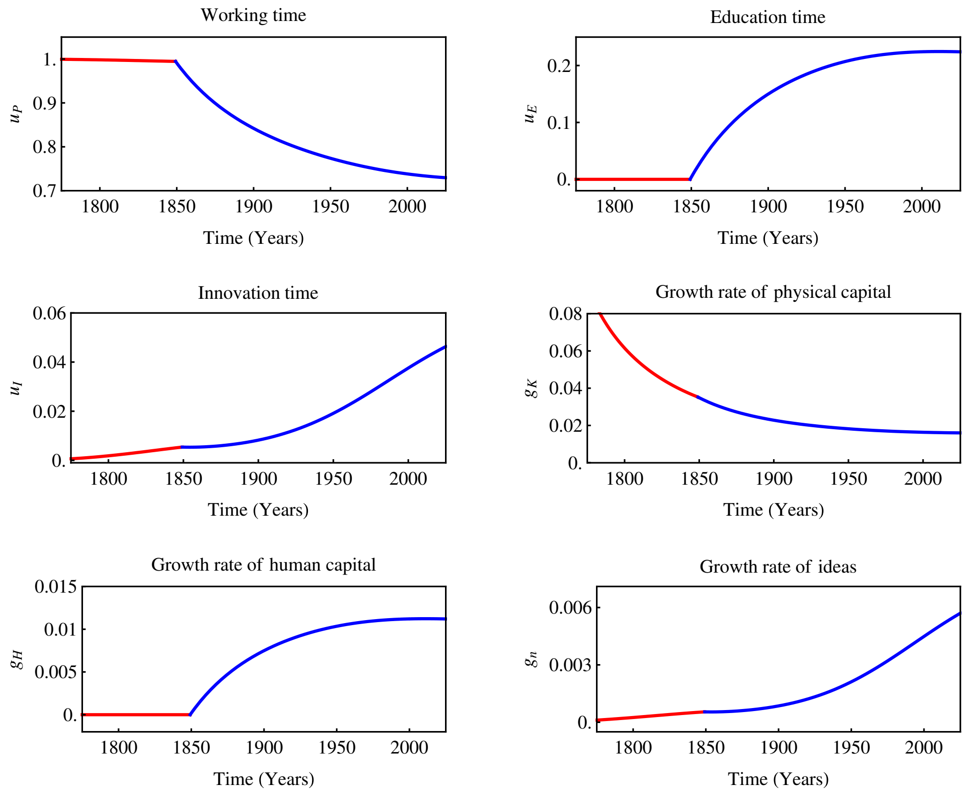

Figure 2: Transition paths for relevant variables in the extended model.
Note: The simulation is extended backward to include an innovation phase without education, $u_E = 0$. The red line corresponds to the innovation phase, and the blue line to the fully industrialized phase. Parameter and steady-state values are displayed in Table 3.

requirements of the industrialization process (see, e.g., Galor, 2005, Figure 28).[15] Second, innovation time increases steadily, first at a slow pace and sharply after the WWII. Furthermore, from the onset of knowledge accumulation, innovation time expands hand in hand with educational time. This behaviour accords with empirical evidence reported, e.g., by Jones (1995b, 2002). Third, in spite of the presence of complex stable eigenvalues, adjustment paths are practically monotonic given the low magnitude of the imaginary part of the stable roots. This agrees with the monotonic behaviour shown by these variables in data.

[15]The beginning of education in Iacopetta (2010, Figures 3 and 4) starts, instead, in 1875, when several countries started to discuss education reforms according to Galor, Moav, and Vollrath (2009).

Table 4: Parameter and steady-state values

β	η	α	ρ	θ	ξ	δ	ω	ε
0.45	0.4	0.82	0.023	2	0.064	0.1	0.7	0.75
$\hat{r}$	$\hat{g}_n$	$\hat{g}_H$	$\hat{g}_K$	$\hat{u}_P$	$\hat{u}_E$	$\hat{u}_I$	stable roots	
0.0523	0.0073	0.0104	0.0146	0.7775	0.1621	0.0604	$-0.0182 \pm 0.0080i$	

To further illustrate the ability of the extended model to generate realistic transition dynamics, we perform another simulation using the parameterization reported in Table 4. The elasticity of physical capital in final goods production, β, is reduced to 0.45, and the elasticity of intermediates, η, is increased to 0.4. There is a mild negative externality associated to human capital in the production of new ideas, $\omega - 1 = -0.3$. Furthermore, we assume that there are diminishing returns to effective learning time in education at the private level, $\varepsilon = 0.75$. The markup is 1.22, which is in line with the estimates reported, e.g., by Norrbin (1993) and Basu (1996). The values of θ and ρ are standard, and similar to those chosen by Iacopetta (2010).

Table 4 shows that the steady-state shares of time devoted to working, studying and innovation, as well as the long-run growth rate of income and the interest rate, have plausible values. The stable roots are complex conjugate, so the economy evolves through damped oscillations. The real part of the stable eigenvalues is relatively small in absolute value, which entails a low rate of decay and a long period of transition, whereas the small value of the imaginary part entails a very low frequency of oscillations so that the economy converges to its steady state in a practicallly monotonic fashion.

Figure 3 depicts the transition dynamics of the economy for the parameter values displayed in Table 4. According with the data examined by Galor (2005), the model generates transitional dynamics in which innovation precedes human capital formation. In early stages of development, innovation time is positive though small. As the economy evolves, technological progress increases steadily and, eventually, the economy enters the fully industrialized phase, with knowledge formation as well as R&D. After education sets in, innovation time expands hand in hand with educational time. The growth rate of ideas increases monotonically, first at a slow pace and sharply after the WWII. Finally, the growth rate of physical capital and so, the interest rate, has remained practically constant in the last century. This behaviour broadly agrees with empirical evidence.

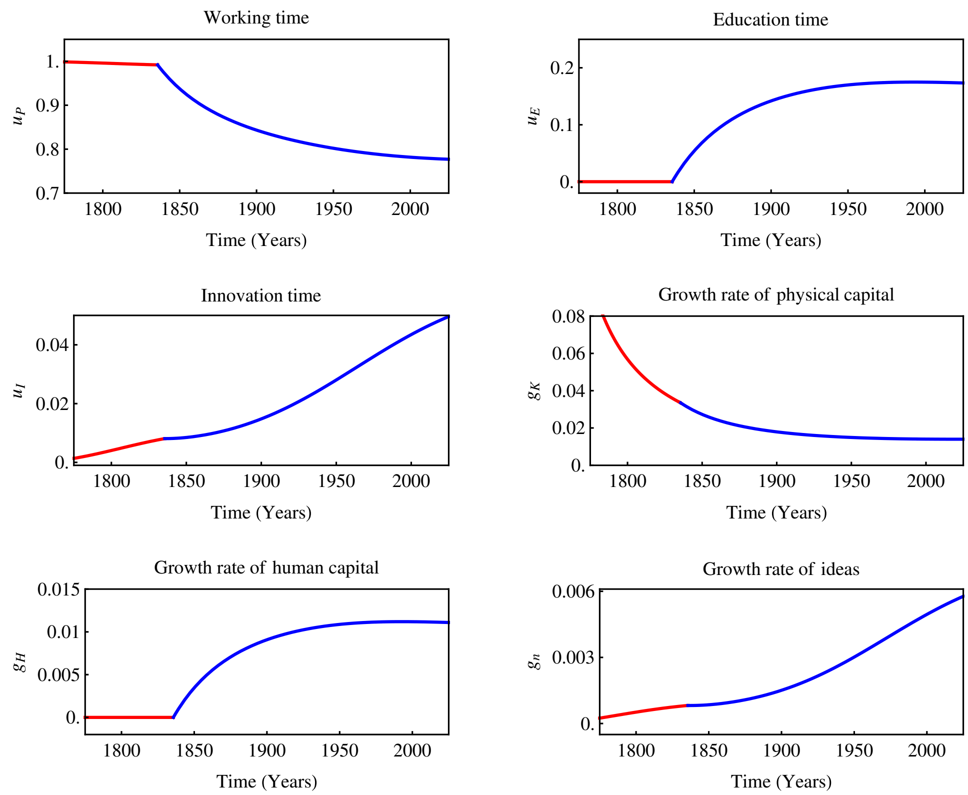

Figure 3: Transition paths for relevant variables in the extended model.
Note: The plots in this figure are similar to those in Figure 2. Parameter and steady-state values are displayed in Table 4.

6 Conclusions

In this paper, we have extended the Funke and Strulik (2000) endogenous growth model with physical capital, human capital and innovation, to add an externality in R&D associated to the duplication of research effort. Furthermore, we allow for knowledge formation to be subject to diminishing returns to effective time at the private level. We present some numerical results showing that the incorporation of duplication externalities significantly increases the ability of the model to generate a realistic innovation-education sequence in which innovation and education time rise jointly along the transition. This behaviour agrees with historical data showing that human capital played a limited role in the production process until the second phase of the British Industrial Revolution, when the acceleration of technological progress increased the demand for skilled labor in the industrial sector, which in turn stimulated human capital formation, and thus further technological progress.

Appendix

A Proofs

Proof of Proposition 1. Constancy of $\hat{g}_C$ implies, by (2.4), constancy of r, i.e., $\hat{g}_r = 0$. Therefore, $\hat{g}_Y = \hat{g}_K$, from (2.19), and χ is also constant in the steady state, $\hat{g}_\chi = 0$, from (3.12). Hence, $\hat{g}_Y = \hat{g}_C = \hat{g}_K$. Evaluating (2.2), (3.12) and (2.21) at the steady state we obtain (3.23), (3.19) and (3.21), respectively. Log-differentiating (2.13) with respect to time, using (3.21), we get (3.25). Combining (2.5) and (2.19), we get $\hat{g}_n = \alpha(1-\beta-\eta)(\hat{r}-\varepsilon\xi)/[(1-\alpha)\eta]$, which using (2.4), (3.21) and (3.25), yields (3.18) and (3.20). Finally, (3.22) results from (3.10), using (2.5) and (3.23), and (3.24) is obtained from (2.11).

The transversality condition associated with aggregate wealth, A, is equivalent to $-\hat{r}+\hat{g}_K < 0$ which, using (3.25), can be rewritten as $(\theta-1)\hat{r}+\rho > 0$. The transversality condition associated to H is equivalent to $-\varepsilon\xi+\hat{g}_H < 0$. After simplification, both conditions can be equivalently expressed as (3.26).

For the interior steady state to be feasible, we must have $0 < \hat{u}_P$, $0 < \hat{u}_I$, $\hat{u}_P+\hat{u}_I < 1$, $\hat{r} > 0$, $\hat{\chi} > 0$ and $\hat{\psi} > 0$. Eqs. (3.21) and (3.20) entail that condition $0 < \hat{u}_P+\hat{u}_I = (\xi-\hat{g}_H)/\xi < 1$ is satisfied if (3.26) holds. Since (3.26) entails that $\theta > 1/(1+M)$, Eqs. (3.18) and (3.20) entail that $\hat{r} > 0$ and $\hat{g}_n > 0$ if $\varepsilon\xi > \rho$. Furthermore, $\hat{u}_P$ and $\hat{u}_I$ are positive because $\varepsilon\xi-\hat{g}_H > 0$. Hence, Eq. (3.24) entails that $\hat{\psi} > 0$. Finally, the ratio of consumption to capital can be expressed as $\hat{\chi} = (1-\alpha\eta)\hat{r}/\beta-\hat{g}_K > (1-\alpha\eta-\beta)\hat{r}/\beta > 0$ if the transversality condition is met. Hence, the steady state is feasible, which completes the proof. □

Proof of Proposition 2. It will be useful to rewrite the dynamics of the economy in terms of the variables r, χ, g_n, $z \equiv \delta^{1/\omega}\psi^{1/\omega}u_P$, and u_P. Using that $g_z = g_{u_P} + (1/\omega)g_\psi$, we get that the dynamics of the economy is driven by the system (3.11), (3.12), and

$$g_{g_n} = -\frac{\omega(1-\alpha)\eta g_n^{1-1/\omega}z}{(1-\beta-\eta)(1-\omega)} - g_n + \frac{\omega\varepsilon\xi}{1-\omega}, \tag{A.1}$$

$$g_z = \frac{(1-\alpha)\eta}{\beta}r - \chi + \left[\frac{(1-\alpha)\eta}{\alpha\beta} - \frac{1}{\omega}\right]g_n + \frac{(1-\eta)\varepsilon\xi}{\beta}, \tag{A.2}$$

$$g_{u_P} = \frac{(1-\alpha)\eta}{\beta}r - \chi - \xi\left(1-u_P-\frac{u_P g_n^{1/\omega}}{z}\right) + \frac{(1-\alpha)\eta}{\alpha\beta}g_n + \frac{(1-\eta)\varepsilon\xi}{\beta}. \tag{A.3}$$

Note that the knowledge-ideas ratio, $\psi = H^{\omega}/n$, is a predetermined variable. Hence, once the jump variable u_P places on its saddle-path stable trajectory, $z(0) = \delta^{1/\omega}\psi(0)^{1/\omega}u_P(0)$ and

$$r(0) = \beta[B(\alpha\eta)^{\eta}]^{1/(1-\eta)}n(0)^{(1-\alpha)\eta/[\alpha(1-\eta)]}[u_P(0)H(0)/K(0)]^{(1-\beta-\eta)/(1-\eta)}$$

are uniquely determined by the initial values of the predetermined variables K, H and n. Hence, the system (3.11), (3.12), (A.1), (A.2), (A.3) features three jump-like variables —χ, g_n and u_P— and two predetermined-like variables —z and r—, so that saddle-path stability requires two stable roots.

Linearization around the steady state yields

$$\begin{pmatrix} \dot{r} \\ \dot{\chi} \\ \dot{g}_n \\ \dot{z} \\ \dot{u}_P \end{pmatrix} = \begin{pmatrix} J_{11} & 0 & J_{13} & 0 & 0 \\ J_{21} & J_{22} & 0 & 0 & 0 \\ 0 & 0 & J_{33} & J_{34} & 0 \\ J_{41} & J_{42} & J_{43} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & J_{55} \end{pmatrix} \begin{pmatrix} r-\hat{r} \\ \chi-\hat{\chi} \\ g_n-\hat{g}_n \\ z-\hat{z} \\ u_P-\hat{u}_P \end{pmatrix} = J \cdot \begin{pmatrix} r-\hat{r} \\ \chi-\hat{\chi} \\ g_n-\hat{g}_n \\ z-\hat{z} \\ u_P-\hat{u}_P \end{pmatrix},$$

where dots replace those elements that are irrelevant for the analysis, and

$$J_{11} = -\frac{1-\beta-\eta}{\beta}\hat{r} < 0, \qquad J_{13} = \frac{\eta(1-\alpha)}{\alpha\beta}\hat{r} > 0,$$

$$J_{21} = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right)\hat{\chi}, \qquad J_{33} = -\left[1 - \frac{(1-\alpha)\eta\hat{g}_n^{-1/\omega}\hat{z}}{1-\beta-\eta}\right]\hat{g}_n,$$

$$J_{22} = \hat{\chi} > 0, \qquad J_{34} = -\frac{(1-\alpha)\eta\omega}{(1-\beta-\eta)(1-\omega)}\hat{g}_n^{2-1/\omega} < 0,$$

$$J_{41} = \frac{(1-\alpha)\eta}{\beta}\hat{z} > 0, \qquad J_{42} = -\hat{z} < 0,$$

$$J_{43} = \left[\frac{\eta(1-\alpha)}{\alpha\beta} - \frac{1}{\omega}\right]\hat{z}, \qquad J_{55} = \xi\left(1 + \hat{g}_n^{1/\omega}/\hat{z}\right)\hat{u}_P > 0.$$

The eigenvalues of J are the four eigenvalues of its upper left 4×4 submatrix (say, $\bar{J}$) and its last diagonal element, $J_{55} = \xi(1+\hat{g}_n/\hat{z})\hat{u}_P > 0$. Therefore, the number of stable roots of J is equal to that of $\bar{J}$.

The characteristic equation of the matrix $\bar{J}$ is

$$p(\lambda) = \lambda^4 - \Delta_3\lambda^3 + \Delta_2\lambda^2 - \Delta_1\lambda + \Delta_0 = 0,$$

where Δ_0 is the determinant of $\bar{J}$, $\Delta_0 = \det(\bar{J})$; Δ_3 is the trace of $\bar{J}$, $\Delta_3 = \mathrm{tr}(\bar{J})$; Δ_2 is the sum of all 2×2 leading minors of $\bar{J}$, and Δ_1 is the sum of all 3×3 leading minors of J. After simplification, we can obtain that

$$\Delta_0 = \det(\bar{J}) = \frac{(1-\alpha)\eta(\varepsilon\xi-\rho)\omega}{\beta\theta(1-\omega)}\hat{r}\hat{\chi}\hat{z}\hat{g}_n^{1-1/\omega}, \tag{A.4}$$

$$\Delta_1 = J_{11}J_{22}J_{33} + J_{13}J_{41}J_{34} - (J_{11}+J_{22})J_{34}J_{43}, \tag{A.5}$$

$$\Delta_2 = (J_{11}+J_{22})J_{33} + J_{11}J_{22} - J_{34}J_{43}, \tag{A.6}$$

$$\Delta_3 = \mathrm{tr}(\bar{J}) = J_{11}+J_{22}+J_{33}. \tag{A.7}$$

Using the Routh-Hurwitz theorem, the number of roots of the characteristic equation with negative real parts is equal to the number of variations of sign in the scheme

$$1 \qquad \mathrm{tr}(\bar{J}) \qquad \Psi \qquad \Pi \qquad \det(\bar{J}) \tag{A.8}$$

where

$$\Psi \equiv \Delta_2 - \Delta_1/\mathrm{tr}(\bar{J}),$$
$$\Pi \equiv \Delta_1 - [\mathrm{tr}(\bar{J})\det(\bar{J})/\Psi].$$

The determinant of $\bar{J}$ is positive, $\Delta_0 = \det(\bar{J}) > 0$. Using (2.17) and (A.1) evaluated at the steady state, we can obtain that

$$J_{11}+J_{22} = \frac{(1-\alpha)\eta}{\beta}\hat{r} + \hat{r} - \hat{g}_K > 0,$$
$$J_{33} = \varepsilon\xi - \hat{g}_H > 0,$$

and, therefore, $\Delta_3 = \mathrm{tr}(\bar{J}) > 0$. Given the positivity of the determinant and the trace, there can be at most two variations of sign in the scheme (A.8). Hence, the matrix $\bar{J}$ may have 0 or 2 roots with negative real parts. This proves part a).

To prove part b), we first show that a sufficient condition to rule out the case of none stable roots is that $\Delta_1 < 0$. If $\Psi < 0$, there are two variations in sign in (A.8) —irrespective of the sign of Π—. If $\Psi > 0$ then $\Pi < 0$ and, therefore, there are two variations in sign in (A.8). If $\Psi = 0$, we substitute it by $\Psi = \kappa > 0$, and so, $\Pi = \Delta_1 - \mathrm{tr}(\bar{J})\det(\bar{J})/\kappa$. Taking the limit as $\kappa \to 0$, we have that $\Pi \to -\infty$ and, therefore, there are two variations in sign in (A.8). Given that $J_{11}+J_{22} > 0$, $J_{33} > 0$ and $J_{13}J_{41}J_{34} < 0$, Eq. (A.5) entails that a sufficient condition for $\Delta_1 < 0$ is that $J_{43} \leq 0$; i.e., that condition (3.27) holds. This proves part b). □

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