Studies in Nonlinear Dynamics & Econometrics

Volume 11, *Issue* 2 2007 *Article* 2

Equilibrium Efficiency in the Ramsey Model with Habit Formation

Manuel A. Gómez*

*University of A Coruña, mago@udc.es

Equilibrium Efficiency in the Ramsey Model with Habit Formation*

Manuel A. Gómez

Abstract

This paper analyzes the equilibrium efficiency in a Ramsey model with habit formation. Uniqueness and saddle-path stability of the steady state is proved analytically. The competitive equilibrium is efficient at the steady state. However, the presence of externalities arising from average past consumption renders the competitive equilibrium inefficient off the steady state because agents do not take (fully) into account the indirect effect that consumption has in utility through its influence on habits. The efficient equilibrium can be decentralized by means of a consumption tax that converges to an arbitrary constant value, or by means of an income tax that converges to zero.

*Please address correspondence to: Manuel A. Gómez Suárez, Facultad de Ciencias Económicas y Empresariales, Campus de Elviña, 15071 A Coruña, Spain. Tel: +34 981167000. Fax: +34 981167070. E-mail: mago@udc.es. I would like to thank, without implicating, Natalia Gomez Lopez, one anonymous referee and the editor for helpful comments and suggestions. Financial support from the Spanish Ministry of Education and Science and FEDER through Grant SEJ2005-00901, and from the Xunta de Galicia through Grant PGIDIT06PXIB100074PR is gratefully acknowledged.

1. Introduction

The role of consumption externalities and habit formation in dynamic equilibrium models has recently attracted a great attention in the literature. For the most part, the objective has been to improve the predictions of time separable models in different fields in order to account for empirical facts that cannot be explained under more traditional specifications of preferences.[1] The introduction of habits formed from some external benchmark taken as given by agents raises the question of whether the competitive equilibrium is efficient and, if it is not so, calls for the design of an optimal fiscal policy capable of internalizing the spillovers.

This paper analyzes the equilibrium efficiency in a Ramsey model with habit formation, and devises a tax policy capable of decentralizing the socially planned solution. Habits enter utility in the multiplicative way (e.g., Abel, 1990, Carroll et al., 1997, 2000, Fuhrer, 2000, Alvarez-Cuadrado et al., 2004), in which agents' utility depends on both their absolute current consumption and their current level of consumption relative to a reference consumption level determined by habits. A fairly general specification of the habit formation process is used in which the reference consumption level is formed as an exponentially declining average of past consumption, and habits depend on both own past consumption levels and economy-wide average past consumption levels in the economy. Such specification comprises the particular cases of internal and external habit formation. In the internal habit formation model,[2] individual habits depend only on own past consumption levels, whereas in the external habit formation (or catching-up with the Joneses) model,[3] habits arise from the economy-wide average past consumption levels in the economy. The presence of externalities arising from average past consumption levels may render the competitive equilibrium inefficient. Therefore, we analyze the efficiency of the competitive equilibrium and, when it is not efficient, devise an optimal tax policy capable of

---

[1] For example, Abel (1990), Constantinides (1990) and Campbell and Cochrane (1999) try to explain the equity premium puzzle; Carroll et al. (2000) study the observed relationship between savings and growth, Lettau and Uhlig (2000) and Boldrin et al. (2001) try to fit some stylized facts of business cycles; Fuhrer (2000) studies monetary policy, and Díaz et al. (2003) analyze the determination of precautionary savings and the shape of the wealth distribution.

[2] This specification has been used, e.g., by Constantinides (1990), Fuhrer (2000) and Boldrin et al. (2001). Empirical evidence for internal habits preferences is provided, e.g., by Ferson and Constantinides (1991), Heaton (1995), Fuhrer (2000), Fuhrer and Klein (2006) and Chen and Ludvigson (2006).

[3] This specification has been used, e.g., by Abel (1990), Campbell and Cochrane (2000) and Lettau and Uhlig (2000). Empirical evidence for external habits preferences is provided, e.g., by Campbell and Cochrane (1999), Menzly et al. (2004), Korniotis (2005) and Wachter (2006).

decentralizing the optimal growth path attainable by a central planner. Unlike previous works, we do not restrict the efficiency analysis to the polar cases of internal and external habit formation, but also analyze the plausible intermediate case in which individual habits depend on both own and average past consumption levels.

We first analyze the dynamics of the market and the centrally planned economy in terms of four real variables: the growth rate of consumption, the ratio of consumption to habits, the capital stock, and the habits stock per effective unit of labor. Alvarez-Cuadrado et al. (2004) have recently analyzed the equilibrium dynamics of this model. However, their stability analysis relies on numerical simulations, as they presume that an analytic analysis may be intractable. Thus, this paper provides an analytic proof that the steady state exhibits saddle-path stability. The transitional dynamics of the model is represented by a two-dimensional stable saddle-path. This provides a much richer dynamics for the transition paths relative to the standard Ramsey model without habits (e.g., Barro and Sala-i-Martin, 1995) or the Ak endogenous growth model with habit formation (e.g., Carroll et al. 1997, 2000) that feature a single stable root and a one-dimensional stable saddle-path.

Our analysis shows that the competitive equilibrium is efficient at the steady state regardless of the specification of the habit formation process. However, the competitive equilibrium is efficient off the steady state if and only if habits are formed in an internal way. Hence, the presence of consumption externalities when the formation of the habits stock depends on the economy-wide average past consumption renders the competitive equilibrium dynamically inefficient, and calls for the government intervention to internalize these spillovers. Inefficiency arises off the steady state because consumption spillovers cause the law of motion of the consumption growth rate in the market economy to differ from that in the centralized economy. Thus, decentralization of the optimal growth path requires that the market economy replicates the efficient growth rate of consumption. We show that the optimal growth path can be decentralized by means of either a consumption tax or an income tax irrespective of the specification of the habit formation process. The optimal income and consumption tax rates converge to zero and to a constant value, respectively, since no inefficiencies appear at a steady state. There is a degree of arbitrariness because the stationary value of the optimal consumption tax rate or, alternatively, its initial value, can be chosen in an arbitrary manner.

Unfortunately, the complexity of the expressions involved makes rather difficult to attain some intuition. Thus, we turn to express the dynamics of the economy in terms of three real variables –consumption, capital stock, and habits stock per effective unit of labor– and one shadow price –the shadow cost of habits relative to the shadow price of capital–.[4] This allows us to derive simpler expressions for the optimal tax policy, and provide some intuition. However, the greater simplicity and intuition attainable by using this set of variables comes at a price: a full characterization of the optimal tax policy can be obtained in the case of external habits, but not in the intermediate case in which habits arise from both own and average past consumption levels.

In the external habit formation case, agents do not take into account the indirect effect that present consumption has on future utility through its effect on the habits stock. The efficient equilibrium can be decentralized by means of either a consumption tax at an increasing (decreasing) rate or an income tax (subsidy) if the efficient shadow cost of habits grows at a greater (smaller) rate than the efficient shadow price of capital. Intuitively, if the shadow cost of habits increases at a greater rate than the shadow price of capital, agents in the market economy overvalues the benefit of future consumption relative to the efficient solution because they do not take into account the indirect effect of the rising habits stock on future utility. Hence, agents' willingness to shift present consumption to the future would be suboptimally high along the efficient solution. Equilibrium efficiency can be achieved by taxing consumption with a tax rate increasing over time. This tax policy increases the relative price of future consumption and discourages individuals to postpone consumption. Alternatively, equilibrium efficiency can be achieved by taxing income, which has the same effect because this policy increases the cost of shifting resources to future periods.

In the intermediate case, in which individual habits depend on both own and average past consumption levels, agents take partially into account the indirect effect that present consumption has on future utility through its effect on the habits stock. Unlike the case of external habit formation, agents in the market economy also choose in an optimal way the stock of habits. We show that the socially planned equilibrium can be decentralized if the market economy replicates the efficient path of the relative shadow cost of habits, as well as the efficient path of consumption. We find that a particular optimal tax policy consists on taxing consumption at an increasing (decreasing) rate if the efficient shadow cost of habits grows at a greater (smaller) rate than the shadow price of capital, while income is kept untaxed. Thus, a complete characterization of the optimal tax policy cannot be obtained in the intermediate case using this set of variables. The reason is that replication of the efficient paths of the real variables of the economy does not really require replication of the efficient path of the relative shadow cost of habits. However, if this replication is enforced, it entails one additional constraint on the optimal tax policy.

[4] These are also the variables considered by Alvarez-Cuadrado et al. (2004) in their dynamical analysis.

Finally, we present some numerical results to illustrate the adjustment process in the face of two types of shocks: i) a destruction in the initial stock of capital, and ii) an increase in the rate of productivity growth. Since the shock makes the economy leave its steady state, the presence of consumption spillovers renders the competitive equilibrium inefficient. Thus, we compute optimal tax policies aimed at restoring efficiency. We find that consumption externalities may have asymmetric effects on the optimal consumption and income taxes depending on the shock faced; i.e., on the initial conditions. Furthermore, we illustrate the possibility that the transitional path of the optimal consumption tax be non-monotonic, and that the optimal income tax be positive and negative along the transition. This fact could have practical implications if the (realistic) constraint that taxes be non-negative is imposed. Since the optimal income tax could be negative during some phases of transition, it would be infeasible if such a non-negativity constraint is imposed. On the contrary, even if the optimal consumption tax exhibits a non-monotonic behavior, its (arbitrary) stationary value or, alternatively, its initial value, could be chosen so that the entire path of the optimal consumption tax is non-negative.

Related work has been recently made. Ljungqvist and Uhlig (2000) analyze a model without capital accumulation, and Alonso-Carrera et al. (2004), a Ramsey model, in which habit formation is introduced by using an additive functional specification.[5] However, as Carroll (2000) argues, this specification has the problem that the utility function may be not well-defined for plausible calibrations. Alonso-Carrera et al. (2005) analyze a one-sector endogenous growth model where habits are formed in a multiplicative way. However, their efficiency analysis is restricted to the polar cases of internal and external habit formation. Furthermore, Alonso-Carrera et al. (2004, 2005) assume that the agent's reference consumption stock is determined solely by previous period's consumption. As Alvarez-Cuadrado et al. (2004) argues, such a specification cannot explain any empirical regularity requiring slow-moving habits.[6] Fisher and Hof (2000) and Liu and Turnovsky (2005) also analyze a Ramsey model with consumption externalities, but they do not consider habit formation, and externalities arise from contemporaneous consumption.

The paper runs as follows. Section 2 analyzes the equilibrium dynamics of the market economy, and Section 3, the equilibrium dynamics of the centralized economy. Section 4 analyzes equilibrium efficiency and derives the optimal tax policy. Section 5 derives simpler expressions for the optimal policy and provides some intuition. Section 6 presents some numerical results. Section 7 concludes.

---

[5] This specification has also been used, e.g., by Constantinides (1990), Campbell and Cochrane (1999), and Lettau and Uhlig (2000).

[6] As an example, they cite the behavior of savings after World War II.

2. The Decentralized Economy

Consider an economy populated by N identical infinitely-lived representative agents that grows at the exogenous rate $\dot{N}/N = n$. The intertemporal utility derived by the agent is represented by

$$\Omega \equiv \frac{1}{1-\varepsilon}\int_0^\infty \left[\frac{C_i}{H_i^\gamma}\right]^{1-\varepsilon} \mathrm{e}^{-\beta t}dt = \frac{1}{1-\varepsilon}\int_0^\infty \left[C_i^{1-\gamma}\left(\frac{C_i}{H_i}\right)^\gamma\right]^{1-\varepsilon} \mathrm{e}^{-\beta t}dt, \qquad (2.1)$$

with $\beta > 0$, $\varepsilon > 1$ and $0 < \gamma < 1$. Here, C_i and H_i are agent's consumption and reference consumption level (habits), respectively, γ reflects the importance of habits in utility,[7] β is the rate of time preference, and $1/\varepsilon$ is the elasticity of intertemporal substitution in the time-separable case ($\gamma = 0$). According to (2.1), instantaneous utility depends on absolute consumption, C_i, and consumption relative to the habits stock, C_i/H_i.

The reference consumption level is formed as an exponentially declining average of past consumption according to

$$H_i(t) = \rho\int_{-\infty}^t \mathrm{e}^{\rho(s-t)}C_i(s)^\phi \overline{C}(s)^{1-\phi}ds \qquad 0 \le \phi \le 1,\ \rho > 0, \qquad (2.2)$$

where $\overline{C} = \int_0^N C_i di \Big/ N$ denotes the economy-wide average consumption of agents. Setting $\phi = 1$ corresponds to the internal habit formation case, in which the reference stock is formed as an exponentially declining average of past own consumption. Setting $\phi = 0$ corresponds to the external habit formation case, in which the reference stock is formed as an exponentially declining average of past economy-wide average consumption. The case $0 < \phi < 1$ corresponds to an intermediate case, in which the reference stock is formed as an exponentially declining average of own and average past consumption levels.

Throughout the paper, a dot over a variable will denote its time derivative; i.e., $\dot{x} = dx/dt$. Differentiating (2.2) with respect to time, the rate of adjustment of the reference stock is

$$\dot{H}_i = \rho(C_i^\phi \overline{C}^{1-\phi} - H_i). \qquad (2.3)$$

Individual output, Y_i, is determined by the Cobb-Douglas technology

$$Y_i = \alpha(AL_i)^\sigma K_i^{1-\sigma} \qquad 0 < \sigma < 1, \qquad (2.4)$$

where K_i is the individual's capital stock, and L_i is the level of inelastically supplied labor. We shall assume that labor productivity grows at the exogenous rate $\dot{A}/A = g$.

---

[7] The case $\gamma = 0$ corresponds to the conventional time-separable case in which utility depends only on agent's consumption, C_i.

The rate of return on capital is denoted *r*, and the wage rate, *w*. The government taxes income at a rate τ^y, and consumption at a rate τ^c. The income raised is rebated as lump-sum transfers, S_i. The agent's budget constraint is, then,

$$\dot{K}_i = (1-\tau^y)(rK_i + wAL_i) - (1+\tau^c)C_i - (n+\delta)K_i + S_i, \tag{2.5}$$

where δ is the rate of depreciation of capital.

Profit maximization by competitive firms implies that labor and capital are used up to the point at which marginal product equates marginal cost:

$$r = (1-\sigma)Y_i/K_i = (1-\sigma)\alpha(AL_i)^{\sigma}K_i^{-\sigma}, \tag{2.6a}$$

$$w = \sigma Y_i/(AL_i) = \sigma\alpha(AL_i)^{\sigma-1}K_i^{1-\sigma}. \tag{2.6b}$$

We shall assume that the government runs a balanced budget:

$$\tau^y(rK_i + wAL_i) + \tau^c C_i = S_i. \tag{2.7}$$

The agent maximizes her intertemporal utility (2.1) subject to her budget constraint (2.5) and the constraint on the accumulation of the habits stock (2.3). For simplicity, L_i will be normalized to unity. Let $c \equiv C_i/A$, $k \equiv K_i/A$ and $h \equiv H_i/A$, and let $u(c,h) = (ch^{-\gamma})^{1-\varepsilon}/(1-\varepsilon)$. The agent's optimization problem can be equivalently expressed as

$$\max \int_0^{\infty} u(c,h)\mathrm{e}^{-\ \ t}dt$$

$$\text{s. t.:} \quad \dot{k} = (1-\tau^y)(rk+w) - (1+\tau^c)c - (n+\delta+g)k + s,$$

$$\dot{h} = \rho(c^{\phi}\bar{c}^{1-\phi} - h) - gh,$$

where $\ \ = \beta + (\varepsilon-1)(1-\gamma)g$, $s = S_i/A$, and $\bar{c} = \int_0^N C_i di/(AN)$.

Let *J* be the current value Hamiltonian of this problem, and let λ and μ be the shadow prices of the capital stock and the habits stock, respectively:

$$J = u(c,h) + \lambda\,[(1-\tau^y)(rk+w) - (1+\tau^c)c - (n+\delta+g)k + s] + \mu[\rho(c^{\phi}\bar{c}^{1-\phi} - h) - gh].$$

Let $u_c(c,h)$ and $u_h(c,h)$ denote the partial derivatives of $u(c,h)$ with respect to *c* and *h*, respectively. The first-order conditions for an interior optimum are[8]

$$u_c(c,h) + \mu\rho\phi c^{\phi-1}\bar{c}^{1-\phi} - (1+\tau^c)\lambda = 0, \tag{2.8a}$$

$$\dot{\mu} = (\ \ + \rho + g)\mu - u_h(c,h), \tag{2.8b}$$

[8] In the time-separable case ($\gamma = 0$), and in the time non-separable case ($0 < \gamma < 1$) with external habit formation ($\phi = 0$) –in which *h* is an externality–, the utility function $u(c,h)$ is concave in *c*. Given that the constraints are concave, the necessary first-order conditions are also sufficient for a maximum. When $0 < \phi \leq 1$, $u(c,h)$ is not concave in *c* and *h*, and so, the first-order conditions may fail to characterize the maximum. In this case, Alonso-Carrera et al. (2005) argue that the interior solution will indeed be a maximum if $\varepsilon > 1$, as empirical evidence suggests.

$$\dot{\lambda} = \quad \lambda - ((1-\tau^y)r - n - \delta - g)\lambda, \qquad (2.8c)$$

plus the transversality condition

$$\lim_{t\to\infty} e^{-\ \ t}\lambda k = \lim_{t\to\infty} e^{-\eta t}\mu h = 0. \qquad (2.8d)$$

Henceforth, the equilibrium condition $\bar{c} = c$ will be taken into account. Eq. (2.8a) can be expressed as

$$u_c(c,h) + \rho\phi\mu - (1+\tau^c)\lambda = 0. \qquad (2.9)$$

Now, we solve out for the costate variables λ and μ from the first-order conditions. Differentiating (2.9) with respect to time, we get

$$\dot{u}_c(c,h) + \rho\phi\dot{\mu} - (1+\tau^c)\dot{\lambda} - \dot{\tau}^c\lambda = 0,$$

which, substituting $\dot{\mu}$ for (2.8b) and $\dot{\lambda}$ for (2.8c), and using (2.9) to eliminate μ from the resulting equation, can be expressed as

$$\dot{u}_c(c,h) - (\quad + \rho + g)u_c(c,h) - \rho\phi u_h(c,h) + \Psi\lambda = 0, \qquad (2.10)$$

where

$$\Psi = (1+\tau^c)((1-\tau^y)r - n - \delta + \rho) - \dot{\tau}^c. \qquad (2.11)$$

Differentiating (2.10) with respect to time, and substituting $\dot{\lambda}$ for (2.8c), we get

$$\ddot{u}_c(c,h) - (\quad + \rho + g)\dot{u}_c(c,h) - \rho\phi\dot{u}_h(c,h) = -\dot{\Psi}\lambda - \Psi\dot{\lambda}$$
$$= [-\dot{\Psi} + \Psi((1-\tau^y)r - n - \delta - g - \quad)]\lambda.$$

Using (2.10) to eliminate λ from the former equation yields

$$\ddot{u}_c(c,h) - (\quad + \rho + g)\dot{u}_c(c,h) - \rho\phi\dot{u}_h(c,h)$$
$$= [\dot{u}_c(c,h) - (\quad + \rho + g)u_c(c,h) - \rho\phi u_h(c,h)]$$
$$\times\left[\frac{\dot{\Psi}}{\Psi} - ((1-\tau^y)r - n - \delta - g - \quad)\right]. \qquad (2.12)$$

Let $v \equiv \dot{c}/c$ denote the growth rate of consumption per effective unit of labor, and $z \equiv c/h$, the ratio of consumption to habits. Thus, we can obtain that

$$u_c(c,h) = c^{-\varepsilon}h^{-\gamma(1-\varepsilon)}, \qquad (2.13a)$$

$$u_h(c,h) = -\gamma c^{1-\varepsilon}h^{-\gamma(1-\varepsilon)-1} = -\gamma z u_c(c,h), \qquad (2.13b)$$

$$\dot{u}_c(c,h) = -(\varepsilon v + \gamma(1-\varepsilon)(\dot{h}/h))u_c(c,h), \qquad (2.13c)$$

$$\dot{u}_h(c,h) = -\gamma\dot{z}u_c(c,h) - \gamma z\dot{u}_c(c,h). \qquad (2.13d)$$

$$\ddot{u}_c(c,h) = -(\varepsilon\dot{v} + \gamma(1-\varepsilon)\rho\dot{z})u_c(c,h) - (\varepsilon v + \gamma(1-\varepsilon)(\dot{h}/h))\dot{u}_c(c,h), \qquad (2.13e)$$

where the fact that $\dot{h}/h = \rho(z-1) - g$, and so, $d(\dot{h}/h)/dt = \rho\dot{z}$, has been used to derive (2.13e).

The dynamics of the economy can be expressed in terms of the variables v, z, h and k as follows:[9]

$$\dot{v} = \frac{1}{\varepsilon}\left\{\gamma\rho(\varepsilon-1+\phi)\dot{z} + \left[\varepsilon v + \beta + g(\gamma+\varepsilon(1-\gamma)) + \rho(1-\gamma\phi z) - \gamma(\varepsilon-1)\frac{\dot{h}}{h}\right]\right.$$
$$\times\left[\varepsilon v - \gamma(\varepsilon-1)\frac{\dot{h}}{h} + \frac{\dot{\Psi}}{\Psi}\right.$$
$$\left.\left. - ((1-\tau^{y})\alpha(1-\sigma)k^{-\sigma} - n - \delta - \beta - g(\gamma+\varepsilon(1-\gamma)))\right]\right\}, \quad (2.14a)$$

$$\dot{h} = \rho(z-1)h - gh, \quad (2.14b)$$

$$\dot{z} = z(v - \rho(z-1) + g), \quad (2.14c)$$

$$\dot{k} = \alpha k^{1-\sigma} - zh - (n+\delta+g)k, \quad (2.14d)$$

where Ψ is defined by (2.11), and so,

$$\frac{\dot{\Psi}}{\Psi} = \frac{\dot{\tau}^{c}}{1+\tau^{c}} - \frac{(1-\tau^{y})\alpha(1-\sigma)\sigma k^{-\sigma-1}\dot{k} + \dot{\tau}^{y}\alpha(1-\sigma)k^{-\sigma} - (\dot{\tau}^{c}/(1+\tau^{c}))^{2} + \ddot{\tau}^{c}/(1+\tau^{c})}{(1-\tau^{y})\alpha(1-\sigma)k^{-\sigma} - n - \delta + \rho - \dot{\tau}^{c}/(1+\tau^{c})}.$$

Eq. (2.14c) is obtained from $\dot{z} = z(\dot{c}/c - \dot{h}/h)$. Substituting (2.13a)–(2.13e) and (2.6a) –which can be expressed as $r = (1-\sigma)\alpha k^{-\sigma}$ – into (2.12), and solving for $\dot{v}$, using that $= \beta + (\varepsilon-1)(1-\gamma)g$, we get (2.14a). It should be noted that a consumption tax at a constant rate, so that $\ddot{\tau}^{c} = \dot{\tau}^{c} = 0$, does not appear in the dynamical system (2.14). Thus, it has not effect on the equilibrium dynamics of the economy, and so, a consumption tax at a constant rate is non-distorting irrespective of the specification of the habit formation process; i.e., irrespective of the value of ϕ. However, an income tax at a constant rate does affect the equilibrium dynamics.

Now, we focus on an interior steady state at which the tax rates τ^{y} and τ^{c} are stationary, i.e., $\tau^{y} = \hat{\tau}^{y} < 1$, so that $\dot{\hat{\tau}}^{y} = 0$, and $\tau^{c} = \hat{\tau}^{c} > -1$, so that $\ddot{\hat{\tau}}^{c} = \dot{\hat{\tau}}^{c} = 0$. A hat over a variable will denote its steady-state value in the market economy. We can state the following Propositions.

---

[9] Carroll et al. (1997) express the dynamics of the economy in terms of the growth rate of consumption, the ratio of consumption to habits, and the ratio of capital to habits in an Ak endogenous growth model with internal and external habit formation.

Proposition 1. *The decentralized economy has a unique steady-state equilibrium:*

$$\hat{k} = \left(\frac{(1-\hat{\tau}^y)\alpha(1-\sigma)}{n+\beta+\delta+g(\gamma+\varepsilon(1-\gamma))} \right)^{1/\sigma}, \tag{2.15a}$$

$$\begin{aligned}\hat{h} &= \frac{\rho}{g+\rho}(\alpha\hat{k}^{1-\sigma} - (n+g+\delta)\hat{k}) \\ &= \frac{\rho(\beta+g(\varepsilon-1)(1-\gamma)+(n+\delta+g)(\sigma(1-\hat{\tau}^y)+\hat{\tau}^y))}{(g+\rho)(1-\hat{\tau}^y)(1-\sigma)}\hat{k},\end{aligned} \tag{2.15b}$$

$$\hat{v} = 0, \tag{2.15c}$$

$$\hat{z} = \frac{g+\rho}{\rho}. \tag{2.15d}$$

Proof. See Appendix.

Proposition 2. *The steady state of the decentralized economy described by* (2.15a)–(2.15d) *is locally saddle-path stable. Furthermore, there exists a two-dimensional differentiable stable manifold M containing* $(\hat{v},\hat{h},\hat{z},\hat{k})$ *that is invariant under the flow of system* (2.14) *and such that for any point in M the solution through this point converges to the steady state.*
Proof. See Appendix.

Recently, Alvarez-Cuadrado et al. (2004) have analyzed the equilibrium dynamics of this model. However, their stability analysis relied on numerical simulations, as they presume that an analytic analysis may be intractable. Propositions 1 and 2 prove analytically that there exists a unique saddle-path stable steady state. The transitional dynamics of the model are represented by a two-dimensional stable saddlepath. This provides a much richer dynamics for the transition paths relative to the standard Ramsey model without habits (e.g., Barro and Sala-i-Martin, 1995) or the Ak endogenous growth model with habit formation (e.g., Carroll et al. 1997, 2000) that feature a single stable root and a one-dimensional stable manifold.[10] As Alvarez-Cuadrado et al. (2004, Table 3) have shown, the two stable eigenvalues may be complex conjugate or real and, therefore, the economy may exhibit a non-monotonic behavior throughout the transition to the steady state (see also Section 6 below). Their numerical results suggest, however, that real roots are more likely than complex roots for plausible parameter values, although complex roots may also arise for empirically relevant parameter values.

---

[10] The endogenous growth model of Alonso-Carrera et al. (2005), in which output is produced with the hybrid neoclassical-Ak function, also features a two-dimensional stable saddle-path.

3. The Centrally Planned Economy

The central planner possesses complete information and chooses all quantities directly, taking all the relevant information into account. The central planner solves the problem

$$\begin{aligned} \max \quad & \int_0^\infty u(c,h)\mathrm{e}^{-\ \,t}dt \\ \text{s.t.}: \quad & \dot{k} = \alpha k^{1-\sigma} - c - (n+\delta+g)k, \\ & \dot{h} = \rho(c-h) - gh. \end{aligned}$$

Let J be the current value Hamiltonian of this problem, and let λ and μ be the shadow prices of the capital stock and the habits stock, respectively:

$$J = u(c,h) + \lambda\,[\alpha k^{1-\sigma} - c - (n+\delta+g)k] + \mu[\rho(c-h) - gh].$$

The first-order conditions for an optimum are

$$u_c(c,h) + \mu\rho - \lambda = 0, \tag{3.1a}$$

$$\dot{\mu} = (\ \ \ + \rho + g)\mu - u_h(c,h), \tag{3.1b}$$

$$\dot{\lambda} = \ \ \lambda - ((1-\sigma)\alpha k^{-\sigma} - n - \delta - g)\lambda, \tag{3.1c}$$

plus the transversality condition

$$\lim_{t\to\infty} e^{-\ \,t}\lambda k = \lim_{t\to\infty} e^{-\eta t}\mu h = 0. \tag{3.1d}$$

Note that the conditions (3.1a)–(3.1d) are identical to (2.8a)–(2.8d) with $\phi = 1$ and no taxes. Hence, the system that drives the dynamics of the centrally planned economy can be readily obtained simply by setting $\phi = 1$, $\tau^y = \dot{\tau}^y = 0$ and $\tau^c = \dot{\tau}^c = \ddot{\tau}^c = 0$ into (2.14) as

$$\begin{aligned} \dot{v} = \frac{1}{\varepsilon}\Bigg\{ & \gamma\rho\varepsilon\dot{z} + [\,\varepsilon v + \beta + g(\gamma + \varepsilon(1-\gamma)) + \rho(1-\gamma z) - \gamma(\varepsilon - 1)(\dot{h}/h)\,] \\ & \times\Bigg[\varepsilon v - \gamma(\varepsilon-1)\frac{\dot{h}}{h} - \frac{\alpha(1-\sigma)\sigma k^{-\sigma-1}\dot{k}}{\alpha(1-\sigma)k^{-\sigma} - n - \delta + \rho} \\ & - (\alpha(1-\sigma)k^{-\sigma} - n - \delta - \beta - g\,(\gamma + \varepsilon(1-\gamma)))\Bigg]\Bigg\}, \end{aligned} \tag{3.2a}$$

$$\dot{h} = \rho(z-1)h - gh, \tag{3.2b}$$

$$\dot{z} = z(v - \rho(z-1) + g), \tag{3.2c}$$

$$\dot{k} = \alpha k^{1-\sigma} - zh - (n+\delta+g)k. \tag{3.2d}$$

A bar over a variable will denote its steady state in the centralized economy. Propositions 1 and 2 can be specialized to cover the case of the centrally planned economy.

Proposition 3. *The centrally planned economy has a unique and locally saddle-path stable steady-state equilibrium:*

$$\bar{k} = \left(\frac{\alpha(1-\sigma)}{n+\beta+\delta+g(\gamma+\varepsilon(1-\gamma))} \right)^{1/\sigma}, \quad (3.3a)$$

$$\bar{h} = \frac{\rho}{g+\rho}(\alpha\bar{k}^{1-\sigma} - (n+g+\delta)\bar{k})$$
$$= \frac{\rho(\beta+(n+\delta+g)\sigma+g(\varepsilon-1)(1-\gamma))}{(g+\rho)(1-\sigma)}\bar{k}, \quad (3.3b)$$

$$\bar{v} = 0, \quad (3.3c)$$

$$\bar{z} = (g+\rho)/\rho. \quad (3.3d)$$

Furthermore, there exists a two-dimensional differentiable stable manifold M containing $(\bar{v},\bar{h},\bar{z},\bar{k})$ *that is invariant under the flow of* (3.2) *and such that for any point in M the solution through this point converges to the steady state.*

4. Equilibrium Efficiency and Optimal Tax Policy

This section analyzes the efficiency of the competitive equilibrium and devises an optimal tax policy capable of decentralizing the optimal growth path attainable by a central planner.

The competitive equilibrium is obviously efficient in the (externality-free) internal habit formation case, $\phi = 1$, both at the steady state and off the steady state. However, when $0 \leq \phi < 1$, the presence of externalities arising from average past consumption levels renders the competitive equilibrium inefficient off the steady state. It can be readily observed that, in this case, the system (2.14) that drives the dynamics of the market economy –with no taxes– does not coincide with the system (3.2) that drives the dynamics of the centralized economy, because the law of motion of the growth rate of consumption per effective unit of labor in the market economy (2.14a) differs from the corresponding one in the centralized economy (3.2a). Comparing the steady-state values of *v*, *z*, *h* and *k* in the market economy given by (2.15), with $\hat{\tau}^y=0$ and $\dot{\hat{\tau}}^c=0$ (i.e., $\hat{\tau}^c=\hat{\tau}$), we see that they coincide with their counterparts in the centrally planned economy given by (3.3), irrespective of the specification of habit formation; i.e., irrespective of the value of ϕ. Therefore, we can state the following Proposition.

Proposition 4. *i) The competitive equilibrium is efficient at the steady state irrespective of the specification of the habit formation process.*

ii) The competitive equilibrium is efficient off the steady state if and only if habits are formed in an internal way, $\phi = 1$.

Henceforth, we shall derive an optimal tax policy capable of decentralizing the optimal growth path when there are consumption spillovers; i.e., when $0 \leq \phi < 1$. The dynamics of the market economy is driven by the system (2.14), and the dynamics of the centralized economy, by the system (3.2). First, note that (2.14b), (2.14c) and (2.14d), which describe the evolution of z, h and k in the market economy, coincide with their counterparts in the centrally planned economy (3.2b), (3.2c) and (3.2d). Replication of the optimal growth path requires then that taxes be set so that the law of motion of the growth rate of consumption per effective unit of labor in the market economy (2.14a) coincides with its counterpart in the centralized economy (3.2a). Thus, the optimal tax rates must be set according to

$$\frac{\dot{\tau}^c}{1+\tau^c} - \frac{\dot{\tau}^y \alpha(1-\sigma)k^{-\sigma} - (\dot{\tau}^c/(1+\tau^c))^2 + \ddot{\tau}^c/(1+\tau^c)}{(1-\tau^y)\alpha(1-\sigma)k^{-\sigma} - n - \delta + \rho - \dot{\tau}^c/(1+\tau^c)} = -\tau^y \alpha(1-\sigma)k^{-\sigma}$$
$$+ \frac{\rho\gamma(1-\phi)[\dot{z} - z(\varepsilon v - \gamma(\varepsilon-1)(\dot{h}/h))]}{\Delta} - \frac{(\Delta - \rho\gamma(1-\phi)z)\alpha(1-\sigma)\sigma k^{-\sigma-1}\dot{k}}{\Delta(\alpha(1-\sigma)k^{-\sigma} - n - \delta + \rho)}$$
$$+ \frac{(1-\tau^y)\alpha(1-\sigma)\sigma k^{-\sigma-1}\dot{k}}{(1-\tau^y)\alpha(1-\sigma)k^{-\sigma} - n - \delta + \rho - \dot{\tau}^c/(1+\tau^c)}$$
$$+ \frac{\rho\gamma(1-\phi)z}{\Delta}[\alpha(1-\sigma)k^{-\sigma} - n - \delta - \beta - g(\gamma + \varepsilon(1-\gamma))], \tag{4.1}$$

where

$$\Delta = \varepsilon v + \beta + g(\gamma + \varepsilon(1-\gamma)) + \rho(1-\gamma\phi z) - \gamma(\varepsilon-1)(\dot{h}/h), \tag{4.2}$$

$\dot{z}$ is given by (3.2b), $\dot{h}$ is given by (3.2c), and $\dot{k}$ is given by (3.2d).

Eq. (4.1) shows that the optimal growth path can be decentralized by means of a consumption tax or an income tax. In the absence of income taxation, the optimal consumption tax must be set according to

$$\frac{\dot{\tau}^c}{1+\tau^c} - \frac{-(\dot{\tau}^c/(1+\tau^c))^2 + \ddot{\tau}^c/(1+\tau^c)}{\alpha(1-\sigma)k^{-\sigma} - n - \delta + \rho - \dot{\tau}^c/(1+\tau^c)}$$
$$= \frac{\rho\gamma(1-\phi)[\dot{z} - z(\varepsilon v - \gamma(\varepsilon-1)(\dot{h}/h))]}{\Delta}$$
$$+ \frac{\alpha(1-\sigma)\sigma k^{-\sigma-1}\dot{k}}{\alpha(1-\sigma)k^{-\sigma} - n - \delta + \rho - \dot{\tau}^c/(1+\tau^c)} - \frac{(\Delta - \rho\gamma(1-\phi)z)\alpha(1-\sigma)\sigma k^{-\sigma-1}\dot{k}}{\Delta(\alpha(1-\sigma)k^{-\sigma} - n - \delta + \rho)}$$
$$+ \frac{\rho\gamma(1-\phi)z}{\Delta}[\alpha(1-\sigma)k^{-\sigma} - n - \delta - \beta - g(\gamma + \varepsilon(1-\gamma))]. \tag{4.3}$$

When the optimal consumption tax is implemented, the dynamics of the economy is driven by the system (3.2a)–(3.2d) and (4.3). The steady state of the optimal consumption tax is obtained by making $\ddot{\tau}^c = \dot{\tau}^c = 0$ in (4.3), and substituting v, z, h

and k by their stationary values given by (3.3a)–(3.3d). Performing these substitutions, Eq. (4.3) is trivially satisfied, and so, the stationary value of the optimal consumption tax may be set arbitrarily.[11] Alternatively, enforcing the condition that the optimal consumption tax be constant at the steady state; i.e., $\dot{\tau}^c = 0$, its initial value may be set in an arbitrary manner.

In the absence of consumption taxation, the optimal tax on income must be set so that the following condition is satisfied:

$$\frac{\alpha(1-\sigma)k^{-\sigma}}{(1-\tau^y)\alpha(1-\sigma)k^{-\sigma}-n-\delta+\rho}\dot{\tau}^y$$
$$=\tau^y\alpha(1-\sigma)k^{-\sigma}-\frac{\rho\gamma(1-\phi)[\dot{z}-z(\varepsilon v-\gamma(\varepsilon-1)(\dot{h}/h))]}{\Delta}$$
$$-\frac{(1-\tau^y)\alpha(1-\sigma)\sigma k^{-\sigma-1}\dot{k}}{(1-\tau^y)\alpha(1-\sigma)k^{-\sigma}-n-\delta+\rho}+\frac{(\Delta-\rho\gamma(1-\phi)z)\alpha(1-\sigma)\sigma k^{-\sigma-1}\dot{k}}{\Delta(\alpha(1-\sigma)k^{-\sigma}-n-\delta+\rho)}$$
$$-\frac{\rho\gamma(1-\phi)z}{\Delta}[\alpha(1-\sigma)k^{-\sigma}-n-\delta-\beta-g(\gamma+\varepsilon(1-\gamma))]. \qquad (4.4a)$$

When the optimal income tax is implemented, the dynamics of the economy is driven by the system (3.2a)–(3.2d) and (4.4a). The steady state of the optimal income tax is obtained by making $\dot{\tau}^y = 0$ in (4.4a), and substituting v, z, h and k by their stationary values given by (3.3a)–(3.3d). Thus, the stationary value of the optimal income tax is found to be zero,

$$\bar{\tau}^y = 0. \qquad (4.4b)$$

It should be noted that substituting $\bar{\tau}^y = 0$ into the steady state of the decentralized economy $(\hat{v},\hat{z},\hat{h},\hat{k})$ given by (2.15) does yield the optimal steady state $(\bar{v},\bar{z},\bar{h},\bar{k})$ given by (3.3). Since $\partial\dot{\tau}^y/\partial\tau^y = \beta+\rho+g(\gamma+\varepsilon(1-\gamma)) > 0$, the steady state $(\bar{v},\bar{z},\bar{h},\bar{k},\bar{\tau}^y)$ of the system (3.2) and (4.4a) is locally saddle-path stable, and the stable differentiable manifold is two-dimensional. The following Proposition summarizes the former findings.

Proposition 5. *The efficient equilibrium can be decentralized by means of i*) *a consumption tax set according to* (4.3) *that converges to an arbitrary stationary value, or ii*) *an income tax set according to* (4.4) *that converges to zero.*

Proposition 5 characterizes the optimal consumption and income taxes designed to decentralizing the efficient equilibrium. Unfortunately, the

[11] This result agrees with that obtained in Section 2, where we showed that a consumption tax at a constant rate is not distorting at the steady state (and off the steady state).

complexity of the expressions involved makes rather difficult to obtain some intuition. Thus, in the next section we shall express the dynamics of the economy in terms of a different set of variables. This will allow us to derive simpler expressions for the optimal tax policy, and provide some intuition. However, the greater simplicity and intuition comes at the price that a complete characterization of the optimal tax policy cannot be obtained in the intermediate case where $0 < \phi < 1$.

5. Another Specification of the Optimal Tax Policy

In this section, we turn to express the dynamics of the economy in terms of three real variables –consumption, capital stock, and habits stock per effective unit of labor– and one shadow price –the shadow cost of habits relative to the shadow price of capital–, which are also the variables considered by Alvarez-Cuadrado et al. (2004). We shall first analyze the dynamics of the market and the centralized economies, and then derive the optimal tax policy and provide some intuition.

5.1 Dynamics of the market economy

Defining $q \equiv -\mu/\lambda$ as the ratio of the shadow cost of habits to the shadow price of capital, and using (2.9) we get

$$\lambda = u_c(c,h)/(1+\tau^c+\rho\phi q), \tag{5.1a}$$

$$\mu = -u_c(c,h)q/(1+\tau^c+\rho\phi q). \tag{5.1b}$$

Substituting $u_h(c,h)$ for (2.13b) into (2.8b), and then substituting μ for (5.1b), we get

$$\dot{\mu}/\mu = \quad +\rho+g+\gamma z u_c(c,h)/\mu = \quad +g+\rho-(1+\tau^c+\rho\phi q)\gamma z/q. \tag{5.2}$$

Substituting $u_c(c,h)$, $u_h(c,h)$ and $\dot{u}_c(c,h)$ for (2.13a), (2.13b) and (2.13c), and λ for (5.1a) into (2.10), using that $v \equiv \dot{c}/c$ and $\dot{h}/h = \rho(z-1)-g$, we get

$$-\left[\varepsilon\frac{\dot{c}}{c}+\gamma(1-\varepsilon)(\rho(z-1)-g)\right]-(\quad +\rho+g)+\rho\phi\gamma z+\frac{\Psi}{1+\tau^c+\rho\phi q}=0, \tag{5.3}$$

where Ψ is given by (2.11), and $\quad = \beta+(\varepsilon-1)(1-\gamma)g$.

The system that drives the dynamics of the economy in terms of the variables c, k, q and h is

$$\dot{c} = \frac{c}{\varepsilon}\left\{\frac{(1+\tau^c)[(1-\tau^y)(1-\sigma)\alpha k^{-\sigma}-\delta-n+\rho]-\dot{\tau}^c}{1+\tau^c+\rho\phi q}\right.$$
$$\left.+\rho\gamma\left[(1-\varepsilon)\left(1-\frac{c}{h}\right)+\phi\frac{c}{h}\right]-\beta-\rho-\varepsilon g\right\}, \tag{5.4a}$$

$$\dot{k} = \alpha k^{1-\sigma} - c - (n+\delta+g)k\,, \tag{5.4b}$$

$$\dot{q} = q\left[(1-\tau^{y})(1-\sigma)\alpha k^{-\sigma} - \delta - n + \rho - \gamma\frac{c}{h}\left(\frac{1+\tau^{c}+\rho\phi q}{q}\right)\right], \tag{5.4c}$$

$$\dot{h} = \rho(c-h) - gh\,. \tag{5.4d}$$

Eq. (5.4a) is obtained by solving (5.3) for $\dot{c}$, and substituting z for c/h. Eq. (5.4c) is obtained from $\dot{q}/q = \dot{\mu}/\mu - \dot{\lambda}/\lambda$, using (2.8c) and (5.2). Furthermore, the fact that $r = (1-\sigma)\alpha k^{-\sigma}$ has been taken into account. It should be stressed that in the model with external habit formation, $\phi = 0$, (5.4a), (5.4b) and (5.4d) form an autonomous system in c, k and h –independent of q– that drives the dynamics of the economy.

Now, we focus on an interior steady state at which the tax rates are stationary; i.e., $\tau^{y} = \hat{\tau}^{y} < 1$ and $\tau^{c} = \hat{\tau}^{c} > -1$, so that $\dot{\hat{\tau}}^{c} = 0$. We can state the following Propositions.

Proposition 6. *The decentralized economy has a unique steady-state equilibrium:*[12]

$$\hat{k} = \left(\frac{(1-\hat{\tau}^{y})\alpha(1-\sigma)}{n+\beta+\delta+g(\gamma+\varepsilon(1-\gamma))}\right)^{1/\sigma}, \tag{5.5a}$$

$$\hat{c} = \alpha\hat{k}^{1-\sigma} - (n+g+\delta)\hat{k} = \frac{(\beta+g(\varepsilon-1)(1-\gamma)+(n+\delta+g)(\sigma(1-\hat{\tau}^{y})+\hat{\tau}^{y}))}{(1-\hat{\tau}^{y})(1-\sigma)}\hat{k}\,, \tag{5.5b}$$

$$\hat{q} = \frac{\gamma(1+g/\rho)(1+\hat{\tau}^{c})}{\beta+(1-\gamma\phi)\rho+(\gamma(1-\phi)+\varepsilon(1-\gamma))g}\,, \tag{5.5c}$$

$$\hat{h} = \frac{\rho\hat{c}}{g+\rho}\,. \tag{5.5d}$$

Proof. *See Appendix.*

---

[12] Alvarez-Cuadrado et al. (2004) find, instead, two steady states. However, the second steady state reported in their paper, with $\tilde{k} = (\alpha(1-\sigma))^{1/\sigma}(n+\delta-\rho)^{-1/\sigma}$ (see their Eq. (A.6)), is not feasible. Setting $\hat{\tau}^{y} = \hat{\tau}^{c} = \dot{\hat{\tau}}^{c} = 0$, from (5.4d) we get $\tilde{c}/\tilde{h} = (g+\rho)/\rho$, which substituted, toghether with the expression for $\tilde{k}$, into (5.4c), yields $\tilde{q} = -1/(\rho\phi)$. However, (5.4a) is not well-defined for $q = \tilde{q}$, since a division by zero occurs.

Proposition 7. *The steady state of the decentralized economy described by* (5.5a)–(5.5d) *is locally saddle-path stable. Furthermore, there exists a two-dimensional differentiable stable manifold M containing* $(\hat{c},\hat{k},\hat{q},\hat{h})$ *that is invariant under the flow of system* (5.4) *and such that for any point in M the solution through this point converges to the steady state.*
Proof. *See Appendix.*

5.2 Dynamics of the centrally planned economy

The dynamics of the centrally planned economy can be obtained simply by setting $\phi = 1$ and $\tau^y = \tau^c = \dot{\tau}^c = 0$ into the system (5.4) as

$$\dot{c} = \frac{c}{\varepsilon}\left\{ \frac{(1-\sigma)\alpha k^{-\sigma} - \delta - n + \rho}{1+\rho q} + \rho\gamma\varepsilon\frac{c}{h} + \rho\gamma(1-\varepsilon) - \beta - \rho - \varepsilon g \right\}, \quad (5.6a)$$

$$\dot{k} = \alpha k^{1-\sigma} - c - (n+\delta+g)k\,, \quad (5.6b)$$

$$\dot{q} = q\left[(1-\sigma)\alpha k^{-\sigma} - \delta - n + \rho - \gamma\frac{c}{h}\left(\frac{1+\rho q}{q}\right)\right], \quad (5.6c)$$

$$\dot{h} = \rho(c-h) - gh\,. \quad (5.6d)$$

Propositions 6 and 7 can be specialized to cover the case of the centralized economy.

Proposition 8. *The centrally planned economy has a unique and locally saddle-path stable steady-state equilibrium:*

$$\bar{k} = \left(\frac{\alpha(1-\sigma)}{n+\beta+\delta+g(\gamma+\varepsilon(1-\gamma))} \right)^{1/\sigma}, \quad (5.7a)$$

$$\bar{c} = \alpha\bar{k}^{1-\sigma} - (n+g+\delta)\bar{k} = \frac{(\beta+(n+\delta+g)\sigma+g(\varepsilon-1)(1-\gamma))}{1-\sigma}\bar{k}\,, \quad (5.7b)$$

$$\bar{q} = \frac{\gamma(1+g/\rho)}{\beta+(1-\gamma)(\rho+\varepsilon g)}, \quad (5.7c)$$

$$\bar{h} = \frac{\rho}{g+\rho}\bar{c}\,. \quad (5.7d)$$

Furthermore, there exists a two-dimensional differentiable stable manifold M containing $(\bar{c},\bar{k},\bar{q},\bar{h})$ *that is invariant under the flow of system* (5.6) *and such that for any point in M the solution through this point converges to the steady state.*

5.3 Optimal tax policy

We shall first derive an optimal tax policy capable of decentralizing the optimal growth path in the case of external habit formation, $\phi = 0$. In this case, the dynamics of the market economy is driven by the system (5.4a), (5.4b) and (5.4d), and that of the centralized economy, by the system (5.6a)–(5.6d). First, note that (5.4b) and (5.4d) coincide with their counterparts in the centrally planned economy (5.6b) and (5.6d), respectively. Comparing (5.4a) and (5.6a), and using (5.6c), we find that decentralization of the efficient equilibrium requires that taxes be set so that

$$\tau^{y}(1-\sigma)\alpha k^{-\sigma} + \dot{\tau}^{c}/(1+\tau^{c}) = \rho\dot{q}/(1+\rho q), \tag{5.8}$$

where $\dot{q}$ is given by (5.6c).

Imposing $\tau^{y}=0$ into (5.8), we observe that the efficient equilibrium can be decentralized by means of a consumption tax solely, that must be set according to

$$\dot{\tau}^{c}/(1+\tau^{c}) = \rho\dot{q}/(1+\rho q). \tag{5.9}$$

Eq. (5.9) shows that the optimal consumption tax tends to a constant whatever the given initial value of the consumption tax rate is, since q tends to its steady state $\overline{q}$. Furthermore, the optimal consumption tax is increasing (decreasing) when the relative price q increases (decreases).

Imposing the condition $\dot{\tau}^{c}=0$ (i.e., $\tau^{c}=\overline{\tau}^{c}$) into (5.8), the efficient equilibrium can be decentralized by means of an income tax set according to

$$\tau^{y} = \rho\dot{q}/[(1+\rho q)(1-\sigma)\alpha k^{-\sigma}], \tag{5.10}$$

where $\dot{q}$ is given by (5.6c). Eq. (5.10) shows that the optimal income tax tends to zero, since q tends to its steady state $\overline{q}$. Furthermore, the optimal income tax is positive (negative) when the relative price q increases (decreases).

Propositions 7 and 8 showed that the transitional dynamics of the model is represented by a two-dimensional stable saddle-path, and so, non-monotonic dynamics may arise. Henceforth, the optimal taxes could exhibit a non-monotonic behavior, and the optimal income tax rate could change from positive to negative or vice versa during the transitional phase. Section 6 illustrates this possibility. The next Proposition summarizes the former findings.

Proposition 9. *In the external habit formation case* ($\phi = 0$)*, the efficient equilibrium can be decentralized or by means of i) a consumption tax set according to* $\dot{\tau}^{c}/(1+\tau^{c}) = \rho\dot{q}/(1+\rho q)$, *that converges to a constant for any given arbitrary initial value* $\tau^{c}(0)=\tau^{c}_{0}$, *or ii) an income tax set according to*

$$\tau^{y} = \rho\dot{q}/[(1+\rho q)(1-\sigma)\alpha k^{-\sigma}],$$

that converges to zero, where $\dot{q}$ is given by (5.6c).

Integrating (5.9) with respect to time between zero and t, or between t and infinity, the optimal tax on consumption can be expressed as a simple function of the relative price q, as the following Corollary shows.

Corollary 10. *In the external habit formation case* ($\phi = 0$)*, the efficient equilibrium can be decentralized by taxing consumption at a rate*

$$1+\tau^c = \frac{1+\tau_0^c}{1+\rho q_0}(1+\rho q),$$

where $\tau^c(0)=\tau_0^c$ *is the arbitrary initial value of the optimal consumption tax, or equivalently,*

$$1+\tau^c = \frac{1+\bar{\tau}^c}{1+\rho \bar{q}}(1+\rho q),$$

where $\bar{\tau}^c = \lim_{t\to\infty}\tau^c(t)$ *is the arbitrary stationary value of the optimal consumption tax.*

Some intuition for the optimal consumption tax given by (5.9) may be given. Comparing (2.9),

$$u_c(c,h) = \lambda(1+\tau^c),$$

with $\tau^c = 0$, and the corresponding condition (3.1a) in the centrally planned economy,

$$u_c(c,h) = \lambda - \mu\rho = \lambda(1+\rho q),$$

we observe that agents in the decentralized economy do not take into account the (negative) effect that an additional unit of present consumption has in future utility through its influence on a greater stock of habits (the term $\rho\mu$). Note that q is the ratio of the shadow cost of habits, $-\mu$, to the shadow price of capital, λ, and so, q is positive. If the efficient shadow cost of habits increases at a greater rate than the shadow price of capital (i.e., $\dot{q} > 0$), agents overvalue the benefit of future consumption relative to the efficient solution because they do not take into account the (negative) indirect effect of the rising habits stock on future utility. Hence, agents' willingness to shift present consumption to the future would be suboptimally high along the efficient solution. Equilibrium efficiency can be achieved by taxing consumption with a tax rate increasing over time. This tax policy drives the after-tax price of future consumption in terms of present consumption up and, therefore, discourages individuals to postpone consumption. Moreover, the optimal rate on consumption converges to a constant value since no inefficiencies appear at a steady state. Similarly, if the shadow cost of habits increases at a smaller rate than the shadow price of capital (i.e., $\dot{q} < 0$), then agents undervalue the benefit of future consumption relative to the efficient solution. Hence, agents' willingness to shift present consumption to the future would be suboptimally low along the efficient solution. Equilibrium efficiency can be achieved by taxing consumption at a decreasing rate. This tax policy decreases the after-tax price of future consumption in terms of present consumption and, therefore, encourages agents to shift consumption from the present to the future.

The intuition behind (5.10) is similar to that of the case of the optimal consumption tax, since and income tax (subsidy) is similar to a consumption tax at an increasing (decreasing) rate. Taxing income increases the relative price of future consumption because this policy increases the cost of shifting resources to future periods and, therefore, discourages individuals to postpone consumption. A subsidy on output reduces the relative price of future consumption because this policy reduces the cost of shifting resources to future periods and, therefore, encourages individuals to shift consumption from the present to the future.

We shall now analyze the intermediate case in which $0 < \phi < 1$. Now, the dynamics of the market economy is driven by the system (5.4a)–(5.4d), and that of the centralized economy, by the system (5.6a)–(5.6d), so that the dynamics of the relative shadow cost of habits, q, must be taken into account as well. First, note that Eqs. (5.4b) and (5.4d) that describe the evolution of k and h in the market economy coincide with their counterparts in the centrally planned economy (5.6b) and (5.6d), respectively. Hence, the optimal growth path can be decentralized by setting taxes so that the market economy replicates the efficient paths of consumption, c, and the relative shadow cost of habits, q; i.e., so that (5.4a) and (5.4c) coincide with their counterparts in the centralized economy (5.6a) and (5.6c), respectively. Using (5.6c), we find that the tax rates on consumption and income must be set so that

$$\dot{\tau}^c = (1+\tau^c-\phi)\frac{\rho\dot{q}}{1+\rho q} + \frac{\gamma c}{hq}(1+\tau^c+\rho\phi q)(\tau^c-(1-\phi)\rho q), \qquad (5.11a)$$

and

$$\tau^y = -\frac{(\tau^c-(1-\phi)\rho q)\gamma c}{(1-\sigma)\alpha k^{-\sigma} hq}. \qquad (5.11b)$$

When the optimal tax policy (5.11a) and (5.11b) is implemented, the dynamics of the economy is driven by the system (5.6a)–(5.6d) and (5.11a). Since in the steady state we have that $\dot{q}=0$, (5.11a) shows that there are two steady-state solutions to the equation $\dot{\tau}^c=0$:

$$\overline{\tau}_1^c = (1-\phi)\rho\overline{q} > 0,$$
$$\overline{\tau}_2^c = -(1+\rho\phi\overline{q}) < -1.$$

Since $\overline{\tau}_2^c = -(1+\rho\phi\overline{q}) < -1$, this solution is not feasible. Furthermore, the corresponding tax on income would be

$$\bar{\tau}_2^y = \frac{(1+\rho\bar{q})\gamma\bar{c}}{(1-\sigma)\alpha\bar{k}^{-\sigma}\bar{h}\bar{q}} \neq 0 .$$

However, this tax rate cannot be optimal at the steady state since, substituting it into the steady state of the market economy $\hat{k}$, $\hat{c}$ and $\hat{h}$ given by (5.5a), (5.5b) and (5.5d), it does not yield the steady state of the centralized economy $\bar{k}$, $\bar{c}$, and $\bar{h}$, given by (5.7a), (5.7b) and (5.7d). Hence, the steady state of the optimal consumption tax is the first one,

$$\bar{\tau}^c = (1-\phi)\rho\bar{q} , \tag{5.12a}$$

and the corresponding steady state of the optimal income tax is

$$\bar{\tau}^y = 0 . \tag{5.12b}$$

Substituting $\hat{\tau}^c$ for $\bar{\tau}^c = (1-\phi)\rho\bar{q}$ and $\hat{\tau}^y$ for $\bar{\tau}^y = 0$ into the steady state of the decentralized economy $(\hat{k}, \hat{c}, \hat{q}, \hat{h})$ given by (5.5a)–(5.5d) does yield the optimal steady state $(\bar{k}, \bar{c}, \bar{q}, \bar{h})$ given by (5.7a)–(5.7d).

It can be shown by direct substitution, recalling (5.6c), that the solution to (5.11a) and (5.12a) is simply

$$\tau^c = (1-\phi)\rho q , \tag{5.13a}$$

and, therefore, (5.11b) entails that income should be untaxed at any time,

$$\tau^y = 0 . \tag{5.13b}$$

Thus, in the intermediate case, decentralization of the efficient equilibrium can be achieved by setting the income tax to zero, and taxing consumption according to (5.13a). The following Proposition summarizes the former results.

Proposition 11. *If $0 < \phi < 1$, the efficient equilibrium can be decentralized by taxing consumption according to $\tau^c = (1-\phi)\rho q$, and keeping income untaxed.*

Proposition 11 shows that the higher the value of ϕ and, therefore, the smaller the inefficiency brought about by the consumption spillovers, the lower is the optimal tax rate on consumption needed to restore efficiency.

The intuition for (5.13a) is similar to that for the case of external habits. Comparing (2.9),

$$u_c(c,h) = (1+\tau^c)\lambda - \rho\phi\mu = \lambda(1+\tau^c + \rho\phi q) ,$$

with $\tau^c = 0$, and the corresponding condition (3.1a) in the centrally planned economy,

$$u_c(c,h) = \lambda - \rho\mu = \lambda(1+\rho q) ,$$

we observe that agents in the decentralized economy do take into account only a fraction ϕ of the (negative) effect that an additional unit of present consumption has in future utility through its influence on a greater habits stock. If the efficient shadow cost of habits increases at a greater rate than the shadow price of capital (i.e., $\dot{q}>0$), agents overvalue the benefit of future consumption relative to the efficient solution because they do not take fully into account the negative effect of the rising habits stock on future utility. Equilibrium efficiency can be achieved by taxing consumption at an increasing rate, which discourages individuals to postpone consumption. On the contrary, if the efficient shadow cost of habits increases at a smaller rate than the shadow price of capital (i.e., $\dot{q}<0$), consumption spillovers can be internalized by means of a consumption tax at a decreasing rate, which encourages individuals to postpone consumption.

Differently to the external habit formation case, we find that income should be untaxed in the intermediate case, $0 < \phi < 1$. The reason is that now agents in the decentralized economy must take into account both the shadow cost of habits and the shadow price of capital. Therefore, the growth rate of the shadow price of capital in the market economy (2.8c),

$$\dot{\lambda}/\lambda = -((1-\tau^{y})(1-\sigma)\alpha k^{-\sigma} - n - \delta - g),$$

must replicate the corresponding growth rate in the centrally planned economy (3.1c),

$$\dot{\lambda}/\lambda = -((1-\sigma)\alpha k^{-\sigma} - n - \delta - g),$$

which yields (5.13b); and the growth rate of the shadow cost of habits in the market economy (2.8b), which can be expressed as (5.2),

$$\dot{\mu}/\mu = +g + \rho - \gamma(c/h)(1+\tau^{c} + \rho\phi q)/q,$$

must replicate the corresponding growth rate in the centrally planned economy, which is given by (5.2) with $\tau^{c}=0$ and $\phi=1$; i.e.,

$$\dot{\mu}/\mu = +g + \rho - \gamma(c/h)(1+\rho q)/q,$$

which yields (5.13a).

Proposition 5 showed that the efficient equilibrium can be decentralized by means of a consumption tax or an income tax, irrespective of the specification of the habit formation process. However, Proposition 11 derives a (particular) optimal tax policy in the intermediate case that relies solely on consumption taxes while income should be untaxed. The explanation for this apparent contradiction lies in the fact that replication of the efficient paths of the real variables of the economy does not really require replication of the efficient path of the relative shadow cost of habits, q. However, when this replication is imposed, it results on one additional constraint on the optimal tax policy. Thus, the greater simplicity and intuition of the optimal tax policy derived in this section by using a different set of variables to express the dynamics of the economy comes at the price that a complete characterization of the optimal tax policy cannot be obtained in the intermediate case where $0 < \phi < 1$.

6. Numerical Analysis

This section presents some numerical results to get an insight on the optimal tax policy. To this end, we perform an exercise similar to that in Alvarez-Cuadrado et al. (2004), who study the effect of two types of shocks: i) a destruction in the initial stock of capital, and ii) an increase in the rate of productivity growth. Since the shock makes the economy leave its steady state, Proposition 4 shows that the presence of consumption externalities renders the competitive equilibrium inefficient. Thus, we compute optimal tax policies aimed at restoring efficiency after the shock takes place. To calibrate the model, we follow Alvarez-Cuadrado et al. (2004, Table 1) and consider the following benchmark parameter values: $\alpha = 1$, $\sigma = 0.65$, $\delta = 0.05$, $g = 0$, $\beta = 0.04$, $\varepsilon = 2.5$, $\gamma = 0.5$, $\phi = 0$, $\rho = 0.1$, $n = 0.015$.

Figure 1. Dynamics of consumption after a 10 percent destruction in capital

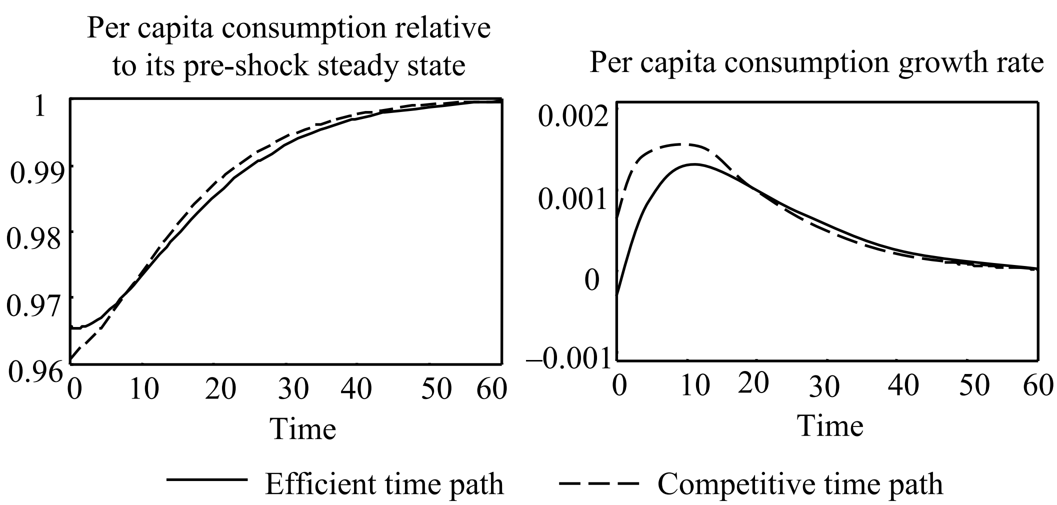

Figure 1 displays the dynamics of consumption after a 10 percent destruction of the stock of capital of an economy that was initially at its steady state.[13] The left panel depicts the time path of per capita consumption relative to its pre-shock steady-state value –which is equal to its stationary post-shock level–, and the right panel depicts the growth rate of per capita consumption. The solid line corresponds to the efficient time path, and the dashed line to the competitive time path. As Alonso-Carrera et al. (2005) argue, a reduction in the capital stock may

[13] See Alvarez-Cuadrado et al. (2004, Figure 2) for an illustration of the dynamics of other variables as well.

cause either an increase or a decrease in the growth rate depending on the initial habits stock due to the interplay of two competing effects. On the one hand, a reduction in the capital stock causes an increase in the rate of return, which has a positive effect on the growth rate. On the other hand, the reduction in the capital stock –and, therefore, in output– relative to the habits stock has a negative effect on the growth rate, because it would force agents to choose a consumption level too large to be sustainable in the long run and, therefore, consumption would have to fall in the future.[14] The combination of these two opposite forces also accounts for the possibility of non-monotonic transitional dynamics to arise.

Figure 2. Optimal tax policy after a 10 percent destruction of capital

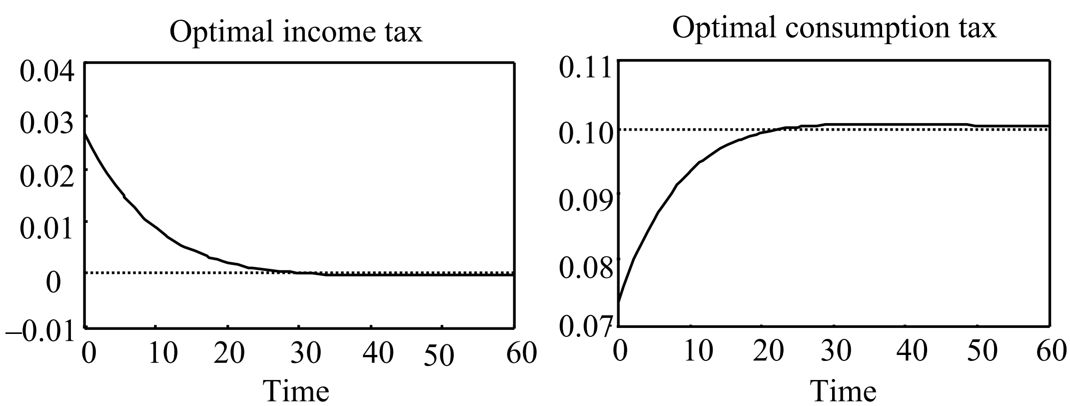

Figure 2 displays optimal taxes on income and consumption capable of decentralizing the efficient path. Since the steady-state level of the optimal consumption tax can be set in an arbitrary manner (see Proposition 5), it has been chosen to be 10 percent.

Alvarez-Cuadrado et al. (2004) show that the time paths for the centralized economy (i.e., the economy with internal habits) and those of the market economy (i.e., the economy with external habits) are similar (see their Figure 2). A destruction of capital causes an initial reduction in per capita consumption followed by a subsequent monotonic increase towards its stationary post-shock level. The agent in the market economy does not take into account the fact that a reduction in current consumption lowers the future stock of habits. Hence, the transition is characterized by initial under-consumption relative to the efficient solution, followed by subsequent over-consumption during later phases of the

[14] Alvarez-Cuadrado et al. (2004) term these effects the "rate of return effect" and the "status effect", respectively.

transition. Accordingly, the competitive consumption growth rate is initially higher than the efficient one, and then catches up and eventually undertakes the efficient one. A consumption tax at an increasing rate encourages shifting consumption to the present from the future, and so, has a depressing effect on the competitive growth rate. A consumption tax at a decreasing rate has the opposite effect. Hence, as Figure 2 shows, decentralizing the efficient path requires imposing a consumption tax at an increasing rate during the first phases of transition, followed by a consumption tax at a decreasing rate during latter phases of transition. The consumption tax rate converges to a constant because no inefficiency arise at the steady state. An income tax (subsidy) is similar to a consumption tax at an increasing (decreasing) rate. Thus, decentralization of the optimal growth path can be alternatively achieved by imposing initially an income tax followed during latter phases of transition by an income subsidy that converges to zero. Propositions 2 and 3 have stated that the stable manifold is two-dimensional, and so, non-monotonic adjustment may occur, as shown in Figure 1. The possibility that the optimal taxes exhibit a non-monotonic behavior is also illustrated in Figure 2.[15]

Figure 3. Dynamics of consumption after an increase in the rate of productivity growth from 0 percent to 2 percent

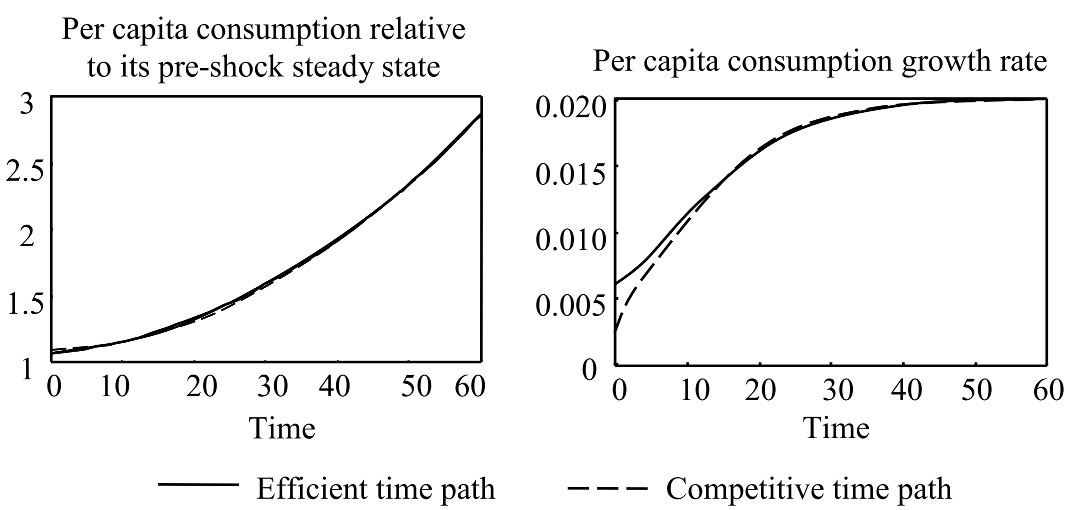

Figure 3 displays the dynamics of per capita consumption after a shock that increases the rate of productivity growth from 0 percent to 2 percent in an economy that was initially on its steady state. The left panel depicts the time path

[15] See also Figure 4 below.

of per capita consumption relative its pre-shock steady state value, and the right panel, the growth rate of per capita consumption. Differently to the case of a 10 percent destruction in the stock of capital, this shock is non-stationary in the sense that per capita variables will tend in the long run to their respective balanced growth paths, rather than to a constant steady state, along which they will grow at a rate of 2 percent.

Figure 4. Optimal tax policy after an increase in the rate of productivity growth from 0 percent to 2 percent

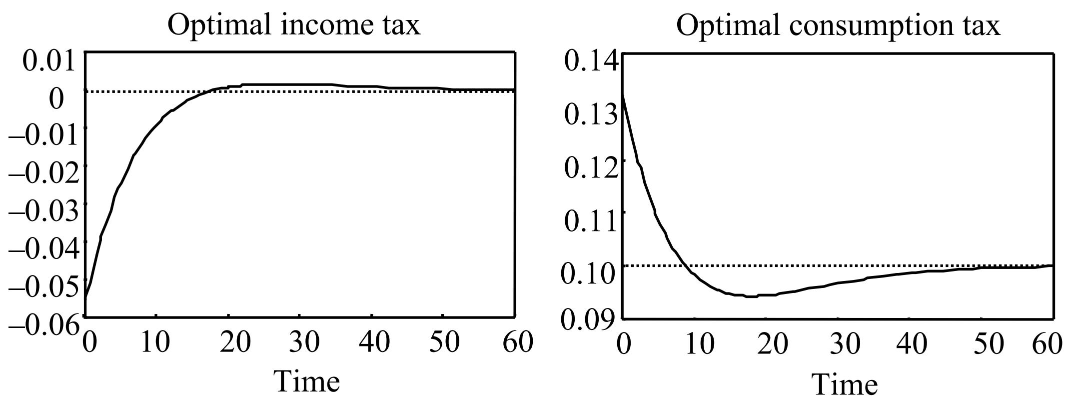

Figure 4 displays optimal taxes on income and consumption. The (arbitrary) stationary value of the optimal tax on consumption has been set again to 10 percent.

Alvarez-Cuadrado et al. (2004) show that the efficient and competitive transitional paths after the shock are quite similar (see their Figure 1). However, in the market economy with consumption spillovers, the agent does not take into account the negative effect that current consumption has in her future utility through its effect on a greater stock of habits. Thus, the transition in this case is characterized by initial over-consumption relative to the efficient solution followed by subsequent under-consumption, during later phases of the transition. The transition of the consumption growth rate is then characterized by an initial competitive consumption growth rate lower than the efficient one followed by the competitive growth rate of consumption catching up and eventually overtaking the efficient one. A consumption tax rate at a decreasing tax rate encourages shifting consumption to the future and, therefore, has a positive effect on the competitive growth rate of consumption. A consumption tax rate at a decreasing rate has the opposite effect. Accordingly, as Figure 4 shows, decentralizing the optimal growth path requires imposing a consumption tax at a decreasing rate during the first phases of transition, followed by a consumption tax at an increasing rate during latter phases of transition. The consumption tax rate converges to a constant because no inefficiency arise at the steady state. Alternatively, decentralization of the optimal growth path can also be achieved by imposing initially an income subsidy followed during latter phases of transition by an income tax that converges to zero.

Figures 2 and 4 illustrate the asymmetric effects that consumption externalities may have on the transitional paths of the optimal consumption and income taxes –which reflect the dynamics of the shadow cost of habits relative to the shadow price of capital, q, via Eqs. (5.9) and (5.10)– depending on the shock faced or, equivalently, depending on the initial conditions. After a destruction of capital, the optimal consumption tax rate is increasing in the first phases of transition, and decreasing during latter phases of transition. However, the dynamics of the optimal consumption tax after an increase in the productivity growth rate is quite opposite; the optimal tax rate is decreasing at the first stages, and increasing during latter phases of transition. This different behavior can also be observed in the transitional dynamics of the optimal income tax rate. Figures 2 and 4 also illustrate the possibility that the optimal income tax be negative; i.e., a subsidy. Taxing income at a negative rate; i.e., subsidizing income, might be termed as unrealistic in practice (see, e.g., Coleman, 2000). Since the optimal income tax might be negative during some phases of transition, it would be infeasible if such a non-negativity constraint is imposed. In contrast, even though the optimal consumption tax exhibits a non-monotonic behavior, its (arbitrary) stationary value –or, alternatively, its initial value– could be chosen so that the entire path of the optimal consumption tax is non-negative, and so, feasible.

7. Conclusions

This paper has analyzed the equilibrium efficiency in a Ramsey growth model with habit formation. A fairly general specification of the habit formation process has been used in which the reference consumption level is formed as an exponentially declining average of own past consumption levels and economy-wide average past consumption levels in the economy. Such specification comprises the particular cases of internal and external habit formation. We have analyzed the equilibrium dynamics of the economy, and proved analytically that there is a unique and saddle-path stable steady state. The equilibrium efficiency has been studied not only for the cases of internal and external habit formation, but also for the intermediate case in which the reference consumption level is formed from both own and average economy-wide past consumption.

The competitive equilibrium is efficient at the steady state irrespective of the specification of the habit formation process. However, the presence of consumption externalities renders the competitive equilibrium dynamically inefficient because agents do not take (fully) into account the indirect effect that consumption has in utility through its influence on the habits stock. The efficient equilibrium can be decentralized by means of a consumption tax that converges to a constant value that may be set arbitrarily, or by means of an income tax that converges to zero. Numerical results are presented to analyze the adjustment process in the face of a destruction in the initial stock of capital, and an increase in the rate of productivity growth. We illustrate the possibility that the transitional path of the optimal consumption tax be non-monotonic, and that the optimal income tax be positive and negative along the transition. Furthermore, we show that consumption externalities may have asymmetric effects on the optimal consumption and income taxes depending on the shock faced.

Appendix

Proof of Proposition 1. Imposing $\dot{h}=0$ and $\dot{z}=0$ in (2.14b) and (2.14c), we get (2.15c) and (2.15d). Imposing $\dot{k}=0$ in (2.14d) and using (2.15c), we get $\hat{h}=\rho[\alpha\hat{k}^{1-\sigma}-(n+g+\delta)\hat{k}]/(g+\rho)$, which substituted along with (2.15c) and (2.15d) into $\dot{v}=0$ allows obtaining (2.15a), and then (2.15b). The transversality condition (2.8d) can be easily shown to be equivalent to $\beta+g(\varepsilon-1)(1-\gamma)>0$, which is satisfied given that $\varepsilon>1$. *Q.E.D.*

Proof of Proposition 2. Linearizing the system (2.14) around its steady state $(\hat{v},\hat{h},\hat{z},\hat{k})$ we get

$$\begin{pmatrix} \dot{v} \\ \dot{h} \\ \dot{z} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ 0 & 0 & \rho\hat{h} & 0 \\ \hat{z} & 0 & -\rho\hat{z} & 0 \\ 0 & -\hat{z} & -\hat{h} & d_{44} \end{pmatrix} \begin{pmatrix} v-\hat{v} \\ h-\hat{h} \\ z-\hat{z} \\ k-\hat{k} \end{pmatrix} = D \begin{pmatrix} v-\hat{v} \\ h-\hat{h} \\ z-\hat{z} \\ k-\hat{k} \end{pmatrix}, \tag{A.1}$$

where

$$d_{11} = \beta+\rho+g(\gamma+\varepsilon(1-\gamma))+\frac{\gamma(g+\rho)(\varepsilon-1)(1-\phi)}{\varepsilon} > 0,$$

$$d_{12} = \frac{\alpha(1-\sigma)\sigma(1-\hat{\tau}^{y})(\beta+\rho(1-\gamma\phi)+g(\gamma(1-\phi)+\varepsilon(1-\gamma)))}{\varepsilon(\beta+\rho+g(\gamma+\varepsilon(1-\gamma)))}\hat{k}^{-\sigma-1}\hat{z} > 0,$$

$$d_{13} = \frac{\hat{h}}{\hat{z}}d_{12}$$

$$-\frac{\gamma\rho}{\varepsilon}\{(\varepsilon-1)(\beta+\rho+g(\gamma+\varepsilon(1-\gamma)))+(g+\rho)\phi+(\varepsilon-1)(g+\rho)(1-\gamma\phi)\},$$

$$d_{14} = \left(\rho + g - \frac{(n+\beta+\delta+g(\gamma+\varepsilon(1-\gamma)))\hat{\tau}^y}{1-\hat{\tau}^y} \right) \frac{d_{12}}{\hat{z}},$$

$$d_{44} = \frac{\beta+(n+\delta+g)\hat{\tau}^y + (1-\gamma)(\varepsilon-1)g}{1-\hat{\tau}^y} > 0.$$

The characteristic equation for the matrix D is

$$p(\lambda) = \lambda^4 + \pi_3\lambda^3 + \pi_2\lambda^2 + \pi_1\lambda + \pi_0 = 0,$$

where π_3 is the opposite of the trace of the matrix D, $\pi_3 = -\operatorname{tr}(D)$; π_2 is the sum of all the leading principal minors of order 2 of the matrix D; π_1 is the opposite of the sum of all the leading principal minors of order 3 of the matrix D, and π_0 is the determinant of the matrix D, $\pi_0 = \det(D)$. It can be proved by direct computation that

$$\pi_0 = \det(D) = d_{12}(g+\rho)(\beta+\rho+g(\gamma+\varepsilon(1-\gamma)))\hat{h} > 0,$$

$$\pi_1 = (\beta+\rho(1-\gamma\phi)+g(\gamma(1-\phi)+\varepsilon(1-\gamma)))(g+\rho)(\gamma+\varepsilon(1-\gamma))d_{44}/\varepsilon + (\beta+g(\varepsilon-1)(1-\gamma))d_{12}\hat{h} > 0,$$

$$\pi_2 = -(\beta+\rho(1-\gamma\phi)+g(\gamma(1-\phi)+\varepsilon(1-\gamma)))(g+\rho)(\gamma+\varepsilon(1-\gamma))/\varepsilon + d_{44}(\operatorname{tr}(D)-d_{44}) - d_{12}\hat{h},$$

$$\pi_3 = -\operatorname{tr}(D) = -(\beta+g(\varepsilon-1)(1-\gamma))(2-\hat{\tau}^y)/(1-\hat{\tau}^y) - (n+g+\delta)\hat{\tau}^y/(1-\hat{\tau}^y) - \gamma(\varepsilon-1)(\rho+g)(1-\phi)/\varepsilon < 0.$$

It should be noted that the sign of π_2 is not needed to perform the subsequent analysis.

The number of roots of the characteristic equation with negative real parts is equal to the number of the roots of the polynomial

$$p(-\lambda) = \lambda^4 - \pi_3\lambda^3 + \pi_2\lambda^2 - \pi_1\lambda + \pi_0$$

with positive real parts. Using the Routh-Hurwitz theorem (e.g., Gantmacher, 1959), the number of roots of the characteristic equation with negative real parts is then equal to the number of variations of sign in the scheme

$$1 \qquad -\pi_3 \qquad \psi_1 \qquad \psi_2 \qquad \pi_0$$

where $\psi_1 = (\pi_2\pi_3 - \pi_1)/\pi_3$ and $\psi_2 = -\pi_1 + \pi_3^2\pi_0/(\pi_2\pi_3-\pi_1) = -\pi_1 + \pi_3\pi_0/\psi_1$. If $\psi_1 > 0$ then $\psi_2 < 0$, and so, we have the scheme

$$+ \qquad + \qquad + \qquad - \qquad +$$

Since there are two variations in sign, the matrix D has two (stable) roots with negative real parts. If $\psi_1 < 0$ we have the configuration

$$+ \qquad + \qquad - \qquad ? \qquad +$$

where a question mark represents an unknown sign, which could be even zero. Irrespective of the unknown sign (even if it is zero), there are two variations in sign, so that the matrix D has two (stable) roots with negative real parts. If $\psi_1 = 0$, we substitute ψ_1 for a positive constant ε than tends to zero, and we obtain the following configuration

$$+ \quad + \quad 0 \quad - \quad +$$

Since the sign of the entry to the left of the zero is different to that to the right of it, this indicates a change of sign. Hence, there are two variations in sign, so there are two (stable) roots with negative real parts. In any case, as the matrix D has two stable roots and the system (2.14) features two predetermined variables, h and k, the number of stable roots is equal to the number of predetermined variables, and so, the steady state $(\hat{v}, \hat{h}, \hat{z}, \hat{k})$ is locally saddle-path stable. The *Stable Manifold Theorem* (e.g., Guckenheimer and Holmes, 1983) entails that there exists a two-dimensional differentiable stable manifold M containing $(\hat{v}, \hat{h}, \hat{z}, \hat{k})$ tangent to the stable space of (A.1), such that for any point (v,h,z,k) in M the solution through this point converges to the steady state. *Q.E.D.*

Proof of Proposition 6. Let

$$\hat{R} = (1-\hat{\tau}^y)\hat{r} = (1-\hat{\tau}^y)\alpha(1-\sigma)\hat{k}^{-\sigma}. \tag{A.2}$$

Imposing the stationary conditions $\dot{c} = \dot{k} = \dot{q} = \dot{h} = 0$, the steady state $(\hat{c}, \hat{k}, \hat{q}, \hat{h})$ of (5.4) is the solution of the system:

$$(1+\hat{\tau}^c)(\hat{R}-\delta-n+\rho)\big/(1+\hat{\tau}^c+\rho\phi\hat{q}) + \rho\gamma(\hat{c}\big/\hat{h})(\phi+\varepsilon-1) + \rho\gamma(1-\varepsilon) - (\beta+\rho) - \varepsilon g = 0, \tag{A.3}$$

$$\hat{c} = \alpha\hat{k}^{1-\sigma} - (n+\delta+g)\hat{k}, \tag{A.4}$$

$$\hat{R} - \gamma(\hat{c}\big/\hat{h})((1+\hat{\tau}^c+\rho\phi\hat{q})\big/\hat{q}) + \rho - \delta - n = 0, \tag{A.5}$$

$$\rho(\hat{c}-\hat{h}) - g\hat{h} = 0. \tag{A.6}$$

Eq. (A.6) entails that

$$\hat{c}\big/\hat{h} = (\rho+g)/\rho. \tag{A.7}$$

Eq. (A.5) allows obtaining that

$$(\hat{R}-\delta-n+\rho)\big/(1+\hat{\tau}^c+\rho\phi\hat{q}) = (\gamma/\hat{q})(\hat{c}\big/\hat{h}) = \gamma(\rho+g)/(\rho\hat{q}). \tag{A.8}$$

Substituting (A.8) and (A.7) into (A.3), and simplifying, we get

$$\gamma(\rho+g)(1+\hat{\tau}^c)\big/(\rho\hat{q}) - g(\gamma(1-\phi)+\varepsilon(1-\gamma)) - \rho(1-\gamma\phi) - \beta = 0. \tag{A.9}$$

From (A.9) we obtain the expression (5.5c) for $\hat{q}$. Then, from (A.8) we get

$$\hat{R} = \beta + \delta + n + g(\gamma + \varepsilon(1-\gamma)). \tag{A.10}$$

From (A.2) and (A.10) we get the expression (5.5a) for $\hat{k}$. Eq. (5.5b) is obtained from (A.4) and (5.5a). Finally, (5.5d) comes from (A.7). The transversality condition (2.8d) is equivalent to $\beta + g(\varepsilon-1)(1-\gamma) > 0$, which is satisfied given that $\varepsilon > 1$. *Q.E.D.*

Proof of Proposition 7. Linearizing the system (5.4) around its steady state $(\hat{c}, \hat{k}, \hat{q}, \hat{h})$ we get

$$\begin{pmatrix} \dot{c} \\ \dot{k} \\ \dot{q} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ -1 & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} \\ \rho & 0 & 0 & -(g+\rho) \end{pmatrix} \begin{pmatrix} c-\hat{c} \\ k-\hat{k} \\ q-\hat{q} \\ h-\hat{h} \end{pmatrix} = B \begin{pmatrix} c-\hat{c} \\ k-\hat{k} \\ q-\hat{q} \\ h-\hat{h} \end{pmatrix}, \quad \text{(A.11)}$$

where

$$b_{11} = \frac{\gamma(g+\rho)(\varepsilon-1+\phi)}{\varepsilon} \geq 0,$$

$$b_{12} = -\frac{\sigma(1+\hat{\tau}^c)(1-\hat{\tau}^y)\hat{c}}{\varepsilon(1+\hat{\tau}^c+\rho\phi\hat{q})\hat{k}}\hat{r} < 0,$$

$$b_{13} = \frac{\rho\phi(1+\hat{\tau}^c)(\beta+g(\gamma+\varepsilon(1-\gamma))+\rho)\hat{c}}{\varepsilon(1+\hat{\tau}^c+\rho\phi\hat{q})^2} > 0,$$

$$b_{14} = -\frac{\gamma(g+\rho)^2(\varepsilon-1+\phi)}{\varepsilon\rho} \leq 0,$$

$$b_{22} = \frac{\beta+g(\varepsilon-1)(1-\gamma)+(n+g+\delta)\hat{\tau}^y}{1-\hat{\tau}^y} > 0,$$

$$b_{31} = \frac{\gamma(g+\rho)(1+\hat{\tau}^c+\rho\phi\hat{q})}{\rho\hat{c}} > 0,$$

$$b_{32} = -\frac{\sigma(1-\hat{\tau}^y)\hat{q}\hat{r}}{\hat{k}} < 0,$$

$$b_{33} = \beta+g(\varepsilon(1-\gamma)+\gamma(1-\phi))+\rho(1-\gamma\phi) > 0,$$

$$b_{34} = -\frac{\gamma(g+\rho)^2(1+\hat{\tau}^c+\rho\phi\hat{q})}{\rho^2\hat{c}} < 0,$$

with $\hat{r} = (\beta+n+\delta+g(\gamma+\varepsilon(1-\gamma)))/(1-\hat{\tau}^y)$.

The characteristic equation for B is

$$p(\lambda) = \lambda^4+\pi_3\lambda^3+\pi_2\lambda^2+\pi_1\lambda+\pi_0 = 0,$$

where π_3 is the opposite of the trace of the matrix B, $\pi_3 = -\mathrm{tr}(B)$; π_2 is the sum of all the leading principal minors of order 2 of the matrix B; π_1 is the opposite of the sum of all the leading principal minors of order 3 of the matrix B, and π_0 is the determinant of the matrix B, $\pi_0 = \det(B)$. It can be proved by direct computation that the characteristic equation for the matrix B is identical to the characteristic equation for the matrix D defined in (A.1). Hence, the proof of Proposition 2 shows that the matrix B has two stable roots. Since the number of stable roots is equal to the number of predetermined variables, the steady state $(\hat{c}, \hat{k}, \hat{q}, \hat{h})$ is locally saddle-path stable. The *Stable Manifold Theorem* (e.g., Guckenheimer and Holmes, 1983) entails that there exists a two-dimensional differentiable stable manifold *M* containing $(\hat{c}, \hat{k}, \hat{q}, \hat{h})$ tangent to the stable space of (A.11), such that for any point (c,k,q,h) in *M* the solution through this point converges to the steady state. *Q.E.D.*

References

Abel, A. (1990): "Asset prices under habit formation and catching up with the Joneses," *American Economic Review*, 80, 38-42.

Alonso-Carrera, J., J. Caballé and X. Raurich (2004): "Consumption externalities, habit formation and equilibrium efficiency," *Scandinavian Journal of Economics*, 106, 231-251.

Alonso-Carrera, J., J. Caballé and X. Raurich (2005): "Growth, habit formation, and catching-up with the Joneses," *European Economic Review*, 49, 1665-1691.

Alvarez-Cuadrado, F., G. Monteiro and S.J. Turnovsky (2004): "Habit formation, catching-up with the Joneses, and economic growth," *Journal of Economic Growth*, 9, 47-80.

Barro, R.J. and X. Sala-i-Martin (1995): *Economic Growth*. New York: McGraw-Hill.

Boldrin, M., L.J. Christiano and J.D.M. Fisher (2001): "Habit persistence, asset returns and the business cycle," *American Economic Review*, 91, 149-166.

Campbell, J. and J. Cochrane (1999): "By force of habit: a consumption-based explanation of aggregate stock market behaviour," *Journal of Political Economy*, 107, 205-251

Campbell, J. and J. Cochrane (2000): "Explaining the poor performance of consumption-based asset pricing models," *Journal of Finance*, 55, 2863-2878.

Carroll, C.D. (2000): "Solving consumption models with multiplicative habits," *Economics Letters*, 68, 67-77.

Carroll, C.D., J. Overland and D.N. Weil (1997): "Comparison utility in a growth model," *Journal of Economic Growth*, 2, 339-367.

Carroll, C.D., J. Overland and D.N. Weil (2000): "Saving, growth and habit formation," *American Economic Review*, 90, 341-355.

Chen, X. and S.C. Ludvigson (2006): "Land of addicts? An empirical investigation of habit-based asset pricing models," working paper, New York University.

Coleman, II W.J. (2000): "Welfare and optimum dynamic taxation of consumption and income," *Journal of Public Economics*, 76, 1-39.

Constantinides, G.M. (1990): "Habit formation: A resolution of the equity premium puzzle," *Journal of Political Economy*, 98, 519-543.

Díaz, A., J. Pijoan-Mas and J.V. Ríos-Rull (2003): "Precautionary savings and wealth distribution under habit formation preferences," *Journal of Monetary Economics*, 50, 1257-1291.

Ferson, W.E. and G.M. Constantinides (1991): "Habit persistence and durability in aggregate consumption," *Journal of Financial Economics*, 29, 199-240.

Fisher, W. and F. Hof (2000): "Relative consumption, economic growth, and taxation," *Journal of Economics*, 72, 241-262.

Fuhrer, J.C. (2000): "Habit formation in consumption and its implications for monetary policy models," *American Economic Review,* 90, 367-390.

Fuhrer, J.C. and M.W. Klein (2006): "Risky habits: on risk sharing, habit formation and the interpretation of international consumption correlations," *Review of International Economics*, 14, 722-740.

Gantmacher, F.R. (1959): *The Theory of Matrices*. New York: Chelsea Publishing Company.

Guckenheimer, J. and P. Holmes (1983): *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. New York: Springer-Verlag.

Heaton, J. (1995): "An empirical investigation of asset pricing with temporally dependent preference specifications," *Econometrica*, 63, 681-717.

Korniotis, G.M. (2005): "Does external habit explain expected returns when markets are incomplete?," working paper, University of Notre Dam.

Lettau, M. and H. Uhlig (2000): "Can habit formation be reconciled with business cycles facts?," *Review of Economic Dynamics*, 3, 79-99.

Liu, W.-F. and S.J. Turnovsky (2005): "Consumption externalities, production externalities, and long-run macroeconomic efficiency," *Journal of Public Economics*, 89, 1097-1129.

Ljungqvist, L. and H. Uhlig (2000): "Tax policy and aggregate demand management under catching up with the Joneses," *American Economic Review*, 90, 356-366.

Menzly, L., T. Santos and P. Veronesi (2004): "Understanding predictability," *Journal of Political Economy*, 112, 1-47.

Wachter, J. (2006): "A consumption-based model of the term structure of interest rates," *Journal of Financial Economics*, 79, 365-399.