

# Nash Solution as a Multi-criteria Decision Making technique for control problems

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## Abstract

*A control system problem can be viewed as a multi-objective problem due to the fact that there are many requirements to be satisfied. Nowadays, multi-objective optimization deals with this kind of problem by implementing optimization techniques, capable of searching for the Pareto set approximation, hereafter the designer needs to select the best solution that provides a good trade-off among the competitive objectives from the Pareto front approximation. Therefore, in this communication we address the problem of selecting the best trade-off between the conflictive objectives, this stage is called the Multi-Criteria Decision Making (MCDM). In this paper we propose to use the Nash solution as a tuning technique to select the design alternative to implement on the control system. This selection is compared with others Proportional-Integral (PI) tuning rules on the literature.*

**Keywords:** Multi-objective Optimization, Multi-criteria decision making, PID control

## 1 Introduction

It is well-known that the controller design is a challenge for the control engineer, satisfying a set of requirements or constraints such as performance, robustness, control effort usage, reliability and others is not an easy task. Sometimes, the improvement of one objective is at the expense of worsening another. These kinds of problems where the designer have to deal with the fulfillment of multiple objectives are known as Multi-Objective Problems. Such problems can be addressed using a simultaneous optimization of all targets (multi-objective optimization). This implies to seek for a Pareto optimal solution in which the objectives have been improved as possible without giving anything in exchange (select a design alternative).

There are two different approaches to solve an op-

timization statement for an Multi-objective problem (MOP) according to [17]; first, the Aggregate Objective Function (AOF) where the designer needs to describe all the trade-off at once and from the beginning of the optimization process, for example, the designer can use a weighting vector to indicate relative importance among the objective. Secondly, the Generate-First Choose-Later (GFCL) approach in which the target is to generate a set of Pareto optimal solutions and then the designer will select, a posteriori, the most preferable solution according to his/her preferences [16].

In order to generate such set of desirable solutions in the GFCL approach, the Multi-Objective Optimization (MOO) techniques might be used. Such techniques generate what is called the Pareto front approximation, where all the solutions are Pareto optimal and non-dominated solutions. It is important to mention that the true Pareto front is unknown, for this reason MOO techniques search for a discrete description of the Pareto set capable of generating a good approximation of the Pareto front, see Figure 1. Finally when the decision maker has been provided with a Pareto front approximation she/he will need to analyze the trade-off between the competitive objectives and select the most preferable solution for a particular situation.

In this paper, we propose the Nash solution as a MCDM technique to choose a unique point from the Pareto front approximation. The paper is organized as follows. Section 2 briefly describes some concepts related to Multi-objective optimization design whilst Section 3 the bargaining solutions are presented. The Nash Solution as a technique it is introduced in Section 4. Illustrative examples are in Section 5 and conclusions are drawn in Section 6.

## 2 Multi-objective Optimization Design

In order to incorporate the MOO process into any engineering design process, a Multi-objective Op-

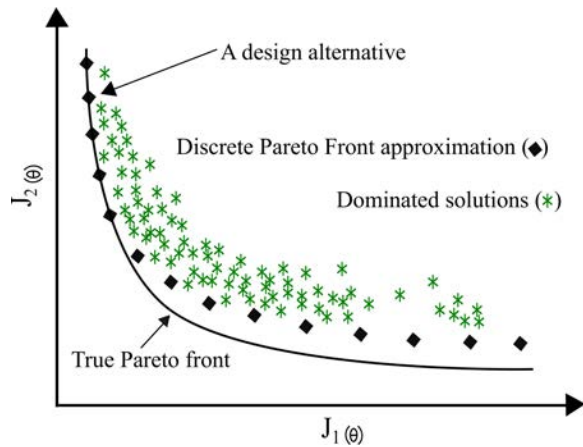


Figure 1: Pareto front concept for two objectives.

timization Design (MOOD) procedure should be carried out [10, 21]. The MOOD procedure have shown to be a valuable tool for control engineers, see [20, 22–24]. This procedure allows the designer to be more involved with the design process and to evaluate the performance exchange between the conflicting objectives, the follow step are needed: i) the MOP definition (objectives, decision variables and constraints), ii) the MOO process (search) and iii) the MCDM stage (analysis and selection). In Figure 2 a general framework is shown. This procedure would guarantee the possibility of obtaining a desirable trade-off among the design alternatives or for the trade-off analysis of the requirements of the controller tuning.

**Multi Criteria Decision Making stage**

All points within the Pareto front are equally acceptable solutions. Once the Pareto front approximation is provided, the designer needs to choose one of those points as the final solution to the MOP for the implementation phase. Several tools and methodologies are available, in order to facilitate the decision making stage [8, 9, 11, 14, 26], a review with different techniques for decision making analysis can be consulted in [12] and a taxonomy to identify the visualizations is presented in [19].

Somehow, the decision making can be undertaken by using two different approaches: i) by including additional criteria such that at the end only one point from the Pareto front satisfies all of them, and ii) by considering one point that represents a fair compromise between all used criteria. From a controller design point of view, the first option can be used to improve the control performance by introducing additional criteria. In other words, as the MOP establishes the search among the Pareto front for a compromise among a set of performance

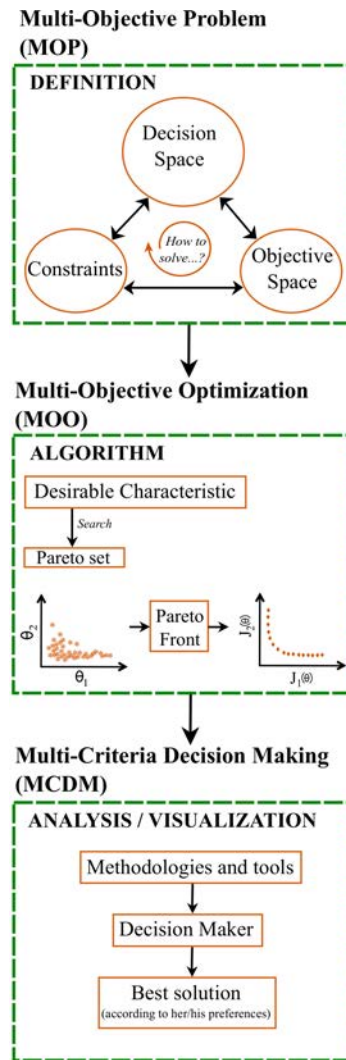


Figure 2: A Multi-objective Optimization Design (MOOD) procedure.

indices, and additional performance (probably of secondary importance) can be introduced. In this way, a new optimization problem will start with the search domain located in the Pareto set in order to find the best solution. The second option does not introduce more information for the decision making and a *fair* point should be selected in order to represent an appropriate trade-off among the different considered cost functions. In the context of finding a PID controller tuning rule, this second option has been preferred because it can be somehow somehow easily automated. It means that a single proposal for the controller design will be the outcome for the MOP. Obviously, the ideal setup would be to reach the utopia point. However, the utopia point is normally unattainable and does not belong to the Pareto front approximation. This is because it is not possible to optimize all individual objective functions independently and simultaneously. Thus, it is only possi-

ble to find a solution that is as close as possible to the utopia point. Such solution is called the *compromise solution* (CS) and is Pareto optimal. This approach however, starts from a neither attainable nor feasible solution. Therefore it is not very practical as it does not take into account what can be achieved for each one of the individual objectives functions. Another procedure to select a fair point is to use bargaining games [7]. This solution leads us to a practical procedure for choosing a unique point from the Pareto front, as it will be seen in the next section.

### 3 Bargaining solutions

In a transaction, when the seller and the buyer value a product differently, a surplus is created. A bargaining solution is then a way in which buyers and sellers agree to divide the surplus. There is an analogous situation regarding a controller design method that is facing two different cost functions for a system. When the controller locates the solution on the disagreement point (D), as shown in Figure 3, there is a way for the improvement of both cost functions. We can move within the feasible region towards the Pareto front in order to get lower values for both cost functions. Let  $\theta_1^*$  and  $\theta_2^*$  denote the values for the free parameter vector  $\theta$  that achieve the optimal values for each one of the cost functions  $f_1$  and  $f_2$ , respectively. Let these optimal values be  $f_1^* = f_1(\theta_1^*)$  and  $f_2^* = f_2(\theta_2^*)$ . On that basis, the utopia point will have coordinates  $f_1^*$  and  $f_2^*$  whereas the disagreement point will be located at  $(f_1(\theta_2^*), f_2(\theta_1^*))$ . As the utopia point (U) is not attainable, we need to analyze the Pareto front in order to obtain a solution. A fair point that represents an appropriate trade-off among the cost functions  $f_1$  and  $f_2$  is defined by the coordinates  $(f_1^{Pf}, f_2^{Pf}) = (f_1(\theta_1^{Pf}), f_2(\theta_2^{Pf}))$ , where the superindex *Pf* means Pareto front. On the basis of this formalism, we can identify, in economic terms the benefit of each one of the cost functions (buyer and seller) as the differences  $f_1(\theta_2^*) - f_1^{Pf}$  and  $f_2(\theta_1^*) - f_2^{Pf}$ . The bargaining solution will provide a choice for  $(f_1^{Pf}, f_2^{Pf})$  therefore a benefit for both  $f_1$  and  $f_2$  with respect to the disagreement. It is important to notice that the problem setup is completely opposite to the one that generates the compromise solution (CS) as the closest one to the utopia point.

Formally, a bargaining problem is denoted by a pair  $\langle S; d \rangle$  where  $S \in \mathbb{R}^2$ ,  $d \in S$  represents the disagreement point and there exists  $s = (s_1, s_2) \in S$  such that  $s_i < d_i$ . In our case,  $S$  is the shaded area shown in Figure 3 delimited by the Pareto front and its intersection with the axis corresponding to the coordinates of the disagreement

point. In Figure 3, different solutions for selecting a point from the Pareto front can be seen:

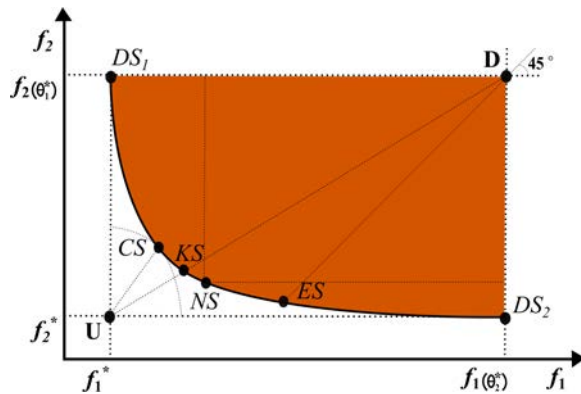


Figure 3: Location of the bargaining solutions into the Pareto front.

1. The disagreement solution (D): it is the solution associated to the disagreement point. Even, if it is not the preferred solution for none of the players, it is a well-defined solution.
2. The dictatorial solution for player 1 (DS1): it is the point that minimize the cost function for player 1. The same concept can be applied to player 2, yielding the dictatorial solution for player 2 (DS2).
3. The egalitarian solution (ES): This point coincides with the intersection of the 45° diagonal line that passes through the disagreement point with the Pareto front.
4. The Kalai-Smorodinsky solution (KS): This point correspond to the intersection of the Pareto front with the straight line that connects the utopia and the disagreement point.
5. The Nash Solution (NS): it selects the unique solution to the following maximization problem:

$$\begin{aligned} \max_{(f_1^{Pf}, f_2^{Pf})} & (f_1(\theta_2^*) - f_1^{Pf})(f_2(\theta_1^*) - f_2^{Pf}) \\ \text{s.t.} & f_1^{Pf} \leq f_1(\theta_2^*) \\ & f_2^{Pf} \leq f_2(\theta_1^*) \end{aligned}$$

In order to illustrate the location of the different solutions that can be selected from the Pareto front, using the bargaining concept can be seen in Figure 3.

### 4 Nash Solution

In his pioneering work on bargaining games, Nash in [15] established a basic two-person bargaining

framework between two rational players, and proposed an axiomatic solution concept which is characterized by a set of predefined axioms and does not rely on the detailed bargaining process of players. Nash proposed four axioms that should be satisfied by a reasonable bargaining solution:

- Pareto efficiency: none of the players can be made better off without making at least one player worse off.
- Symmetry: if the players are indistinguishable, the solution should not discriminate between them. The solution should be the same if the cost function axis are swapped.
- Independence of affine transformations: an affine transformation of the cost functions and of the disagreement point should not alter the outcome of the bargaining process.
- Independence of irrelevant alternatives: if the solution  $(f_1^{P_f}, f_2^{P_f})$  chosen from a feasible set  $A$  is an element of a subset  $B \in A$ , then  $(f_1^{P_f}, f_2^{P_f})$  must be chosen from  $B$ .

Nash proved that, under mild technical conditions, there is a unique bargaining solution called *Nash bargaining solution* satisfying the four previous axioms. Indeed, by considering the different options for selecting a point from the Pareto front, the NS is the only solution that satisfies these four axioms [15]. In fact, the Nash solution is simultaneously utilitarian (Pareto efficient) and egalitarian (fair). Also from a MOO point of view, by maximizing the product, we are maximizing the area of the rectangle that represents the set of solutions dominated by the NS. Actually, the NS provides the Pareto front solution that dominates the larger number of solutions, therefore being absolutely better (that is, with respect to both cost functions at the same time) than any one of the solutions of such rectangle. These are the reasons why the NS represents an appropriate choice for the (semi)-automatic selection of the fair point from the Pareto front.

## 5 Comparison Examples

In this section, the previous ideas will be used to locate well known tuning rules into the Pareto front and to show how the NS performs compared to existing tuning rules.

In order to evaluate the performance obtained with the different tuning methods, the following standard measures will be used:

- Output performance: the Integrated Absolute Error (*IAE*) is the most natural way

to measure performance is by minimizing the integrated-absolute-error.

$$IAE := \int_0^\infty |e(\tau)|d\tau \quad (1)$$

- Input performance: the Total Variation (*TV*) of control action, is a measure of the smoothness of control action [27]. In order to evaluate the manipulated input usage  $u(t)$ .

$$TV := \sum_{k=1}^\infty |u(k+1) - u(k)| \quad (2)$$

Therefore, the MOP will be formulated in order to find the parameters of the PI controller required to obtain the desired regulatory control performance

$$\min_{\theta_c} J(\theta_c) = [J_{IAE}(\theta_c), J_{TV}(\theta_c)] \quad (3)$$

$$\theta_c = [K_p, T_i],$$

where  $K_p$  is the proportional gain and  $T_i$  is the integral time constant. For the MOO process the Normal Normalized Constraint algorithm [18] is implemented, in order to generate the set of optimal solutions. Hereafter, the calculation of the Nash Solution is carried out to select the solution that offers the best trade-off among the competitive objectives.

Consider the fourth-order controlled processes proposed as benchmark in [5] and given by the transfer function:

$$P_\alpha(s) = \frac{1}{\prod_{n=0}^3 (\alpha^n s + 1)}, \quad (4)$$

with  $\alpha \in \{0.1, 1.0\}$ .

Using the three-point identification procedure *123c* [1] FOPDT models were obtained, whose parameters are listed in Table 1.

Table 1: Example -  $P_\alpha(s)$  FOPDT models.

$\alpha$	$K_p$	$T$	$L$	$\tau_o$
0.10	1	1.003	0.112	0.112
1.0	1	2.343	1.860	0.794

For comparison purposes the following PI tuning methods that in some extent considered the control system robustness into the design procedure were selected: the *Model-Reference Robust Tuning* (MoReRT) [2] that uses a model matching approach for smooth time responses and, at the same time, ensures a pre-specified, level of robustness

$M_s = \{1.4, 1.6, 1.8, 2.0\}$ ; the *Kappa-Tau* (K-T) [3] that uses an empirical closed-loop dominant pole design of 2DoF PI controllers for a batch of controlled processes and provides tuning relations for robustness levels of  $M_s = 2.0$  and  $M_s = 1.4$ ; the *Simple Internal Model Control* (SIMC) [25] that is an IMC-based tuning for 1DoF PI controllers to obtain a good trade-off between speed of response, disturbance rejection, robustness ( $M_s \approx 1.59$ ), and control effort requirements; the *Approximated MIGO* (AMIGO) [13] which is based on the loop shaping MIGO method [6] that maximizes the controller integral gain for the minimization of the integrated error to a step load disturbance, subject to a robustness constraint ( $M_s = 1.4$ ) for PI controllers, in particular the revised version of the AMIGO method in [4] will be used.

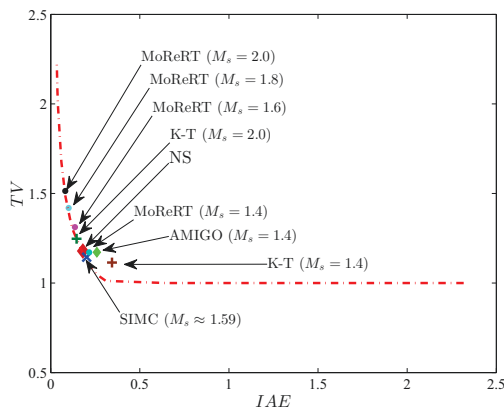


Figure 4: Pareto front for the process model corresponding to  $\alpha = 0.1$ .

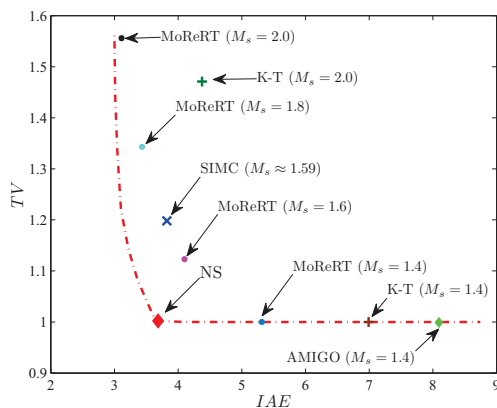


Figure 5: Pareto front for the process model corresponding to  $\alpha = 1.0$ .

The Pareto fronts corresponding to the process models corresponding to  $\alpha = 0.1$  and  $\alpha = 1.0$ , are shown in Figures 4 and 5, respectively. These examples provide a graphical comparison of the performance/robustness trade-off among the tuning

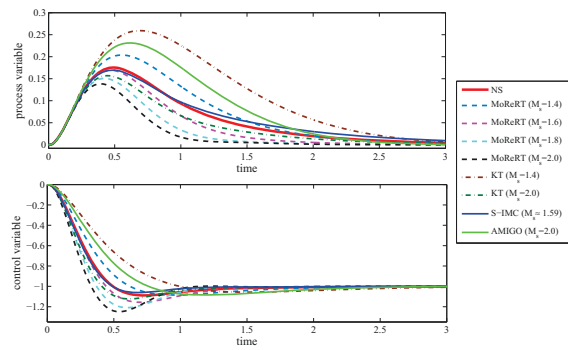


Figure 6: Load disturbance step responses for  $P_{\alpha=0.1}(s)$ .

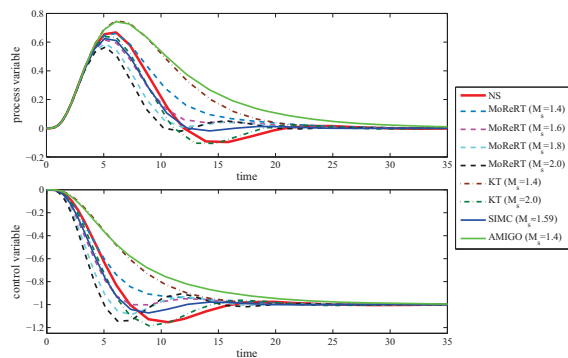


Figure 7: Load disturbance step responses for  $P_{\alpha=1.0}(s)$ .

rules aforementioned and the Nash Solution (NS). The achieved time responses when facing a step load disturbance are shown in Figures 6 and 7. It can be stated that the controller tuning suggested by the NS choice improves the time responses in comparison with the other tuning rules.

Note that the *MoReRT*, *SIMC* and *K-T* ( $M_s = 2.0$ ) tuning rules are almost located on the Pareto front, for the process model corresponding to  $\alpha = 0.1$ . However, for the second example  $\alpha = 1.0$ , it can be seen that some of the *MoReRT* and *K-T* ( $M_s = 1.4$ ) tuning rules are located on the Pareto front. Nevertheless, as it can be observed the *NS* is located in the Pareto front and very close to *MoReRT* ( $M_s = 1.6$ ) tuning rule, which means that the robustness level is  $M_s \approx 1.6$  (remember we can assimilate *TV* to  $M_s$ ).

## 6 Conclusions

In this communication, the Nash Solution (NS) [7] has been analyzed as an automatic way of selecting compromise solutions in a Multi-Criteria Decision Making problem. The scenario for design of PI controllers has been taken as a design scenario. The NS provides an automatic selection and a di-

rect approach for the choice of one point from the Pareto front approximation, this will generate a possibility for tuning a controller that it can generate better system outputs than existing tuning methods.

## 7 Acknowledgments

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