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Numerical methods to predict human induced vibrations on low frequency stairs. Part 2: Evaluation by comparing with experimental data

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ABSTRACT

Nowadays, vibration serviceability criteria are becoming the governing factor in the design of most modern staircases, because their increasing susceptibility to human induced vibrations. Although more awareness have been raised to the dynamic design of new staircases, there are still few studies found in the literature that compare the different results of numerical methods for predicting vibrations with experimental data, in order to validate the same.

Hence, this paper employs the main existing numerical methods to an actual staircase with known liveness, by comparing the predicted results with the experimental data, to evaluate their accuracy when designing flexible staircases. This paper is Part 2 of a set of two papers. In Part 1, the different numerical methods are presented and details are given of how to apply them.

To accomplish this, an in-situ staircase dynamic characterisation and several walking tests are performed. The measured vibrations are initially compared with different proposed acceptable limits to confirm that the vibrations exceed the limits. The different numerical methods are then employed and the predicted results are compared with the experimental results. Lastly, the main findings of this work are discussed together with those of diverse researchers who also applied one of these procedures to estimate vibrations.

The results obtained showed that, with two of the numerical procedures applied (footfall force time histories and simplified vibration evaluation), it was possible to effectively predict the vibrations, while with the remaining two (Fourier series walking models and steady-state analysis), in general, overestimated responses were predicted.

1. Introduction

In contemporary design, it is becoming a notorious trend for many public buildings, such as hotels, libraries, shopping centres, restaurants, etc. to feature slender monumental staircases, which, from the ultimate limit state (ULS) perspective, do not raise any problems for structural engineers. However, from the serviceability limit states (SLS) viewpoint, these structures are usually very demanding due to the use of long and unsupported spans, often resulting in flexible staircases that are highly susceptible to human induced vibrations.

To avoid possible uncomfortable vibrations it is necessary during the design to estimate correctly the maximum accelerations of the staircase due to human walking. Currently, there are four main existing numerical methods to predict vibrations: i) footfall force time histories (GRFs), ii) Fourier series walking models, iii) steady-state analysis and iv) simplified vibration evaluation, these being the four methodologies presented in Part 1 of this paper. Since the evaluation of vibration limit states is becoming the governing criterion in the design of most new staircases, it is necessary to compare the predicted vibrations with experimental measurements, for calibration and validation of the different numerical procedures.

Some examples of researchers who have experimentally evaluated vibrations on monumental steel staircases and compared them with the vibrations estimated using different numerical methods are Davis and Murray [1], González [2,3] and Zhou et al. [4].

Davis and Murray [1] experimentally measured accelerations on a flexible steel staircase and compared them to predicted steady-state acceleration peaks for 3.05 Hz and 2.03 Hz step frequencies, the

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second and third submultiples of the staircase's fundamental frequency (6.1 Hz) respectively, so the maximum resonance accelerations could be obtained. With the steady-state analysis performed, Davis and Murray [1] obtained an average ratio of measured-to-predicted peak accelerations equal to 0.33.

González [2,3] predicted accelerations in two descents, at 1.60 Hz (walking) and at 3.88 Hz (running), by applying his measured GRFs footfall traces to a numerical staircase model, and subsequently compared the obtained accelerations with those experimentally measured on the actual staircase. For the descent at 1.60 Hz an experimental peak acceleration of 1.06 m/s^2 and a numerical peak of 0.06 m/s^2 were obtained, and for the descent at 3.88 Hz experimental and numerical peak accelerations of 1.25 m/s^2 and 0.11 m/s^2 were obtained respectively. The step frequencies do not correspond to any of the staircase's sub-multiples, therefore being difficult to perceive how accurately the accelerations would be predicted using step frequencies that exactly matched one of its sub-multiples.

Zhou et al. [4] analysed an indoor spiral steel staircase, with problems of relatively large vibration found in service, by numerically calculating the accelerations using Fourier series and then comparing these with experimentally measured accelerations. In the Fourier series definition, only the first two harmonics for ascents and descents at normal pace and the first harmonic for ascents and descents at fast pace were considered, these researchers stating that these harmonics comprise the main dynamic components of walking and running forces. For the normal ascents and descents, a step frequency of 1.60 Hz was selected and for the fast ascents and descents step frequencies of 3.5 Hz and 4.3 Hz respectively were selected. The predicted accelerations tended to be approximately the same as the experimental accelerations; however, these step frequencies do not correspond to submultiples of the staircase's fundamental frequency.

Other researchers, such as Eid et al. [5] and Setareh [6], only estimated vibrations on stairs numerically. Eid et al. [5] conducted a dynamic analysis on a numerical model of a steel staircase, taking into account only the information provided by the design guides SCI P354 [7] and AISC 11 [8]. Both design guides recommend the use of loads functions given by Fourier series to estimate the numerical accelerations, the main objective of the study being to compare the results obtained by the two guides. Due to the large difference between the results obtained, the authors concluded that further studies are needed for greater consensus among the various design guides.

Setareh [6] performed a steady-state analysis and applied the Fourier series methodology to a finite element model of a steel staircase with a fundamental frequency equal to 9.15 Hz, to assess the numerical accelerations of two descents with step frequencies of 2.29 Hz and 3.05 Hz, respectively, the third and fourth submultiples of the staircase's fundamental frequency, in order to obtain the maximum response in resonance. The predicted accelerations with both methods were approximated, but were not compared with any experimental results.

Setareh [9] in a more recent study, which had as its main objective the precise determination of staircase's dynamic properties, only briefly compared the acceleration magnitudes, from the frequency function responses (FRFs), numerically obtained and experimentally measured, demonstrating that the difference between both was equal to 23%.

Davis [10] experimentally measured the accelerations on six different floors: i) three laboratory specimens, ii) one full-scale mock-up, and iii) two buildings under construction, and subsequently compared these with the numerically predicted accelerations, applying GRFs and Fourier series footfall traces to FE models of the floors. Although the measured and predicted accelerations are referring to floors, since it is a very comprehensive and detailed work, load functions specifically for design purposes were obtained, and, due to the lack of works directly related to staircases, this is worth mentioning as a basis for comparison with the results obtained in this paper. It should be noted that the accelerations were numerically predicted, taking into account step frequencies within the range of 1.6 Hz to 2.2 Hz, which corresponded to one of the first four submultiples of the sample floors' fundamental frequencies. Considering all the tests performed using GRFs traces, Davis [10] obtained a predicted-to-measured peak acceleration average ratio of 1.47. According to this researcher, a number of tests were affected by inaccurate mode shape predictions, leading to over-prediction responses. If these are excluded, the average ratio of predicted-to-measured peak acceleration becomes equal to 1.18. Using Fourier series traces, an average predicted-to-measured peak acceleration ratio equal to 1.84 was obtained. Excluding the tests with an inaccurate mode shape prediction, a predicted-to-measured peak acceleration average ratio of 1.71 was obtained.

Davis and Avci [11,12], in a more recent work, also proposed a simplified vibration evaluation procedure to manually predict walking peak accelerations on staircases, without the need to perform a complex FE numerical analysis. Davis and Avci [11,12] experimentally measured vibrations in two different steel staircases, performing for the 2nd, 3rd and 4th submultiples of their fundamental frequencies a total of 10, 28 and 11 tests, respectively. Employing the proposed simplified vibration evaluation method and comparing to the results measured, an average ratio of experimental-to-predicted for the 2nd, 3rd and 4th sub-multiples equal to 0.48, 0.67 and 0.65, respectively, was obtained.

In the last few years, human-structure interaction (HSI) based models have gained notoriety, showing promising results. However, the studies found are mostly related to footbridges [13–18], being only few directly developed for staircases [19–22]. G. Busca et al. [20] observed a high increase of damping ratios and a slight decrease of natural frequencies when considering PGRF's, with the proposed model being able to predict changes in the modal parameters of the occupied structure, even when employing the average values of apparent masses from the literature [23]. A. Cappellini et al. [21] estimated the vibration response of two different steel staircases by adding AGRF's to the modified H-S system encompassing the PGRF's. The results indicated that applying AGRF's to the empty structure (classical approach) can lead to overestimated vibrations, while considering AGRF's and PGRF's showed good correlation with the measurements. M. Berardengo et al. [19] further extended the former work [21] using a statistical approach to compare the accelerations experimentally measured on a steel staircase and numerically obtained by four different models. The results obtained are in agreement with A. Cappellini et al. [21], i.e., only applying AGRF's led to overestimated responses and besides different complexity of the three models, vibrations levels were close to those experienced on a real staircase.

Although there is a growing need to design staircases with human–structure interaction (HSI) in mind to avoid potentially unwanted vibrations, there are still few works developed in this field and according with different authors a general and suitable model with adequate experimental validation is still needed [15,16,19]. Furthermore, HSI based models are complex for day-to-day basis employment and reckon on a much higher number of variables (position, posture, apparent masses, inter and intra-subject variability, subject's mass, stiffness and damping, etc.) than the procedures presented in this paper, which are difficult to predict and are not yet consensually accepted and widely available for design purposes.

Thus, this paper aims to evaluate the precision and viability of the existing numerical methods based on walking force models (classical approach) by comparing their predicted results with experimental results. This is Part 2 of a set of two papers. In Part 1 the numerical methods are introduced and a description is given of how to apply them.

With this purpose, an experimental campaign is carried out on a steel staircase, performing an in-situ dynamic characterisation and several walking tests to measure the levels of vibration. The measured vibrations are compared with the acceptance criteria of different design guides and researchers, to assess the staircase's serviceability. A detailed FE model of the staircase is built to compute the vibrations numerically. Then, a comparison is made between the results of the different numerical methods and the experimental measurements. Finally, the main findings of this study are presented, discussed and compared with the findings of diverse researchers who also applied the described numerical procedures in their work.

2. Experimental program

The steel staircase evaluated in this work is located inside a building in Funchal, Madeira, Portugal. This particular staircase, since the beginning of its construction, was the object of adverse comments by its users, in relation to the discomfort caused by its liveness, becoming of primary interest to the study developed in this paper.

2.1. Staircase description

The staircase is composed of four flights of steps, with identical geometry, which serve as a connection between the three floors of the building (see Fig. 1a)). Each flight of steps is supported by two stringers with a steel hollow structural section (HSS) 120 mm × 60 mm × 4 mm (see Fig. 1b)). The separation between each flight of steps is made by means of intermediate landings that are supported by the same HSS stringers of the flight of steps and two HSS steel beams with the same cross-section parallel to these. The steps have a length of 1.15 m and a width of 0.32 m and are composed of a metal plate with a thickness of 3 mm coated by a granite sheet stone 30 mm thick. The intermediate landings are also coated by a 30 mm thick granite stone sheet over a 3 mm metal plate. The staircase is supported on each floor by a European wide flange beam HEB180 and in the intermediate landings by three columns, also constituted by European wide flange beams HEB180, belonging to the steel structure that supports the whole building. The span between supports, taking into account the sum of the length of the flight of steps with the length of the intermediate landing, makes a total of 4.44 m.

The connection between the HSS 120 × 60 × 4 stringers of the flight of steps and HEB180 beam is made by means of an 8 mm steel plate and an M 20 × 100 mm screw (area indicated with circles in Fig. 1a). It is important to note that with this solution, in the connection between HEB180 beam and HSS stringers, rotational movement is possible, so the support could be assumed as pinned.

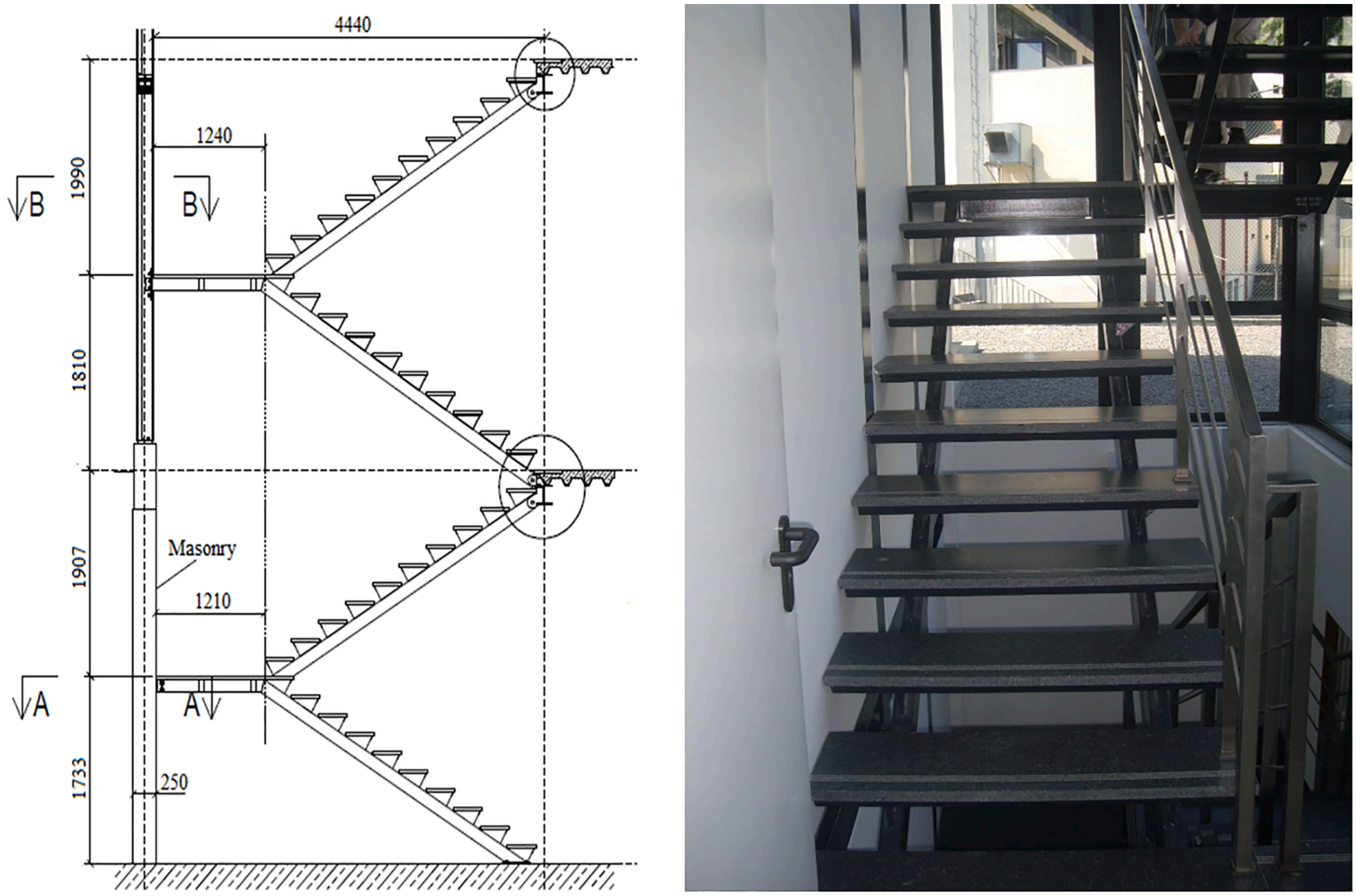
The staircase consists of four flights of steps, but for the purposes of the analysis only the two upper flights were considered, since their behaviour is independent of the two lower flights, due to the aforementioned screw connection between the HEB180 beam and HSS stringers.

2.2. Dynamic characterisation

The dynamic characteristics were determined by conducting a battery of experimental modal tests on the sample staircase. For experimental determination of the staircase vibration modes two accelerometers MMA8452Q (whose specifications are indicated in

Table 1
Accelerometers specifications.

Range	Frequency bandwidth	Sampling Frequency	Resolution	Noise
± 20m/s^2	1,6Hz – 800Hz	200Hz	0,01m/s^2	1 mm/s^2/$\sqrt{\text{Hz}}$



a)

b)

Fig. 1. Sample staircase: a) complete drawing of project (mm) and b) flight of steps top view.

Table 1), connected to the structure near the driving point and other locations of interest, and a hammer with a rubber edge were used. Fig. 2 displays the equipment used for determining the staircase vibration modes.

The damping, frequencies and corresponding vibration modes were obtained by recording accelerations in free vibration at several points and calculating the staircase's dynamic properties on a subroutine created specifically for this purpose in MATLAB.

To increase the staircase vibrations, several measurements were made with the application of strikes at different points of the stringers along the staircase. The graph of Fig. 3 shows the averaged normalised power spectrum density (ANPSD) obtained after performing numerous tests.

After obtaining the frequencies of the modes, their modal shapes were characterised. Fig. 4a) and 4b) display the shape, in their respective modal coordinates, of the first two vibration modes. Observing Fig. 4a) and 4b), it is verified that in the shape of the first mode the two flights of steps are in phase and in the shape of the second mode they are in counter-phase.

The damping was obtained by applying the half-power bandwidth method to the free vibration responses of the staircase, these being consistently estimated to be about 1.18% of critical, which is in agreement with various researchers [1,2,11,24], who obtained in their measurements on steel staircases a damping of approximately 1%.

2.3. Walking tests

After determining the dynamic properties, the vibrations to which the sample staircase was subjected were experimentally measured, so as to be able later to verify the effectiveness of the various numerical methods that will be presented throughout this paper. With this in mind, several walking tests were performed to estimate the staircase's acceleration response to ascents and descents. Considering that the fundamental frequency of the studied staircase is 13.9 Hz, this means that descending or ascending with a step frequency of 3.5 Hz (the 4th submultiple of the frequency), in the range of possible step frequencies on staircases (approximately 2.0 Hz to 4.5 Hz), is the only plausible scenario capable of producing resonant effects. Measurements were also made with pedestrians ascending and descending the staircase at step frequencies of 2.0 Hz and 2.5 Hz, respectively, which are more commonly used. Table 2 describes the experimental tests performed for a single pedestrian and a group of pedestrians, respectively.

A group of four pedestrians was used. Kerr [25] verified that the group enhancement effect was greater when pedestrians walk up or down the staircase at the same velocity applying dynamic forces at the same rate and with a minimum phase shift between them. Hence, there was an effort during the tests to have the group of pedestrians ascend and descend the staircase with the same step frequency and in phase.

Fig. 2. Experimental program equipment.

Experimental measurements of the accelerations were made using the two accelerometers described before, both being placed on the centre of the step located at midspan of the third flight of steps (the flight of steps that connects the second floor and the intermediate landing between the second and third floors) as seen in Fig. 5, since it is the location where higher accelerations are generated.

3. Experimental results

Despite the existing debate about which accelerations should be considered to better assess human induced vibrations on stairs – peak or RMS – it was decided on the scope of this paper to present peak rather than RMS accelerations since the same are conditioning and because performing a steady-state analysis and employing the simplified vibration evaluation procedure, only peak accelerations can be obtained. This facilitates and makes straightforward the comparison between the experimental accelerations and those obtained by the different numerical methods.

In this section, the peak accelerations measured for individual and group tests are compared with the acceptable limits proposed by the various design guides and researchers, in order to more easily perceive the vibration level to which the studied staircase is subjected.

The acceptance criteria given in the majority of the design guides are defined by multiplying a series of factors, which take into account different vibration environments in buildings and different types of structures, with frequency-weighted base curves. The only design guide that directly refers to the acceptance criteria of vibrations in staircases is SCI P354 [7]. Due to the lack of specific information for stairs, Bishop et al. [24] proposed factors of 24 and 32, respectively for heavily (e.g. public) and lightly (e.g. offices) used stairs, and 64 for a group of pedestrians, which should be multiplied by the frequency-weighted base curves, these being the factors given in SCI P354 [7]. Kim et al. [26] and Eid et al. [5] suggested that accelerations should be compared with the base curve of peak accelerations for indoor bridges given by AISC 11 [8]. Zhou et al. [4] and Davis et al. [1,11] were other researchers who, for similar reasons, proposed their own acceptable limits to be applied on stairs.

Zhou et al. [4] suggested the minimum value between 0.5 m/s^2 and $0.15\sqrt{f}$ m/s^2, with f representing the stair's natural frequency, for the limit of peak vertical accelerations. Although based on a work developed for an indoor spiral steel stair, there is not a clear explanation of why these should be the limits applied.

Davis and Murray [1] suggested an acceptable limit of 1.7% g (0.167 m/s^2) for individual ascents and descents at normal paces, i.e. those descending and ascending the stair below 2.5 Hz, and 4.6% g (0.451 m/s^2) for rapid individual descents and ascents and for a group of pedestrians, however, later modifying it to 3.0% g (0.294 m/s^2) for rapid walking individuals [11].

3.1. Single pedestrian

The single pedestrian was placed during the experimental tests performing 7 ascents and 8 descents (see Table 2), obtaining a total of 15 graphs with the measured accelerations, one for each individual passage on the staircase. Fig. 6 represents the peak accelerations obtained from midspan measurements of the flight of steps, for all ascents and descents at different step frequencies (see Table 2). The same graph also shows the acceptable limits proposed by the design guides and researchers SCI P354 [7] / Bishop et al. [24], AISC 11 [8], Davis et al. [1,11] and Zhou et al. [4].

It should be noted that for the limit proposed by SCI P354 [7] / Bishop et al. [24] a multiplying factor of 32 was used since the staircase is located in an office building where the occupancy level is lower than a

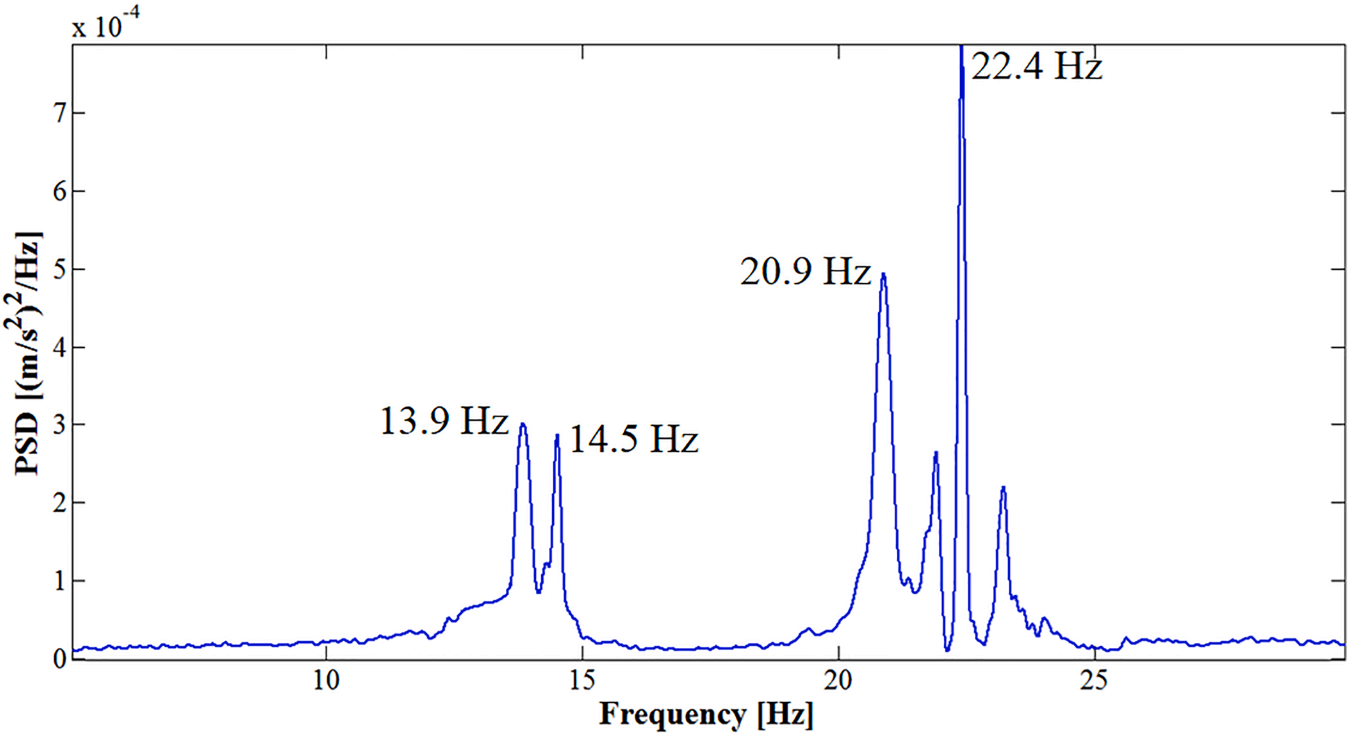


Fig. 3. Power spectrum density.

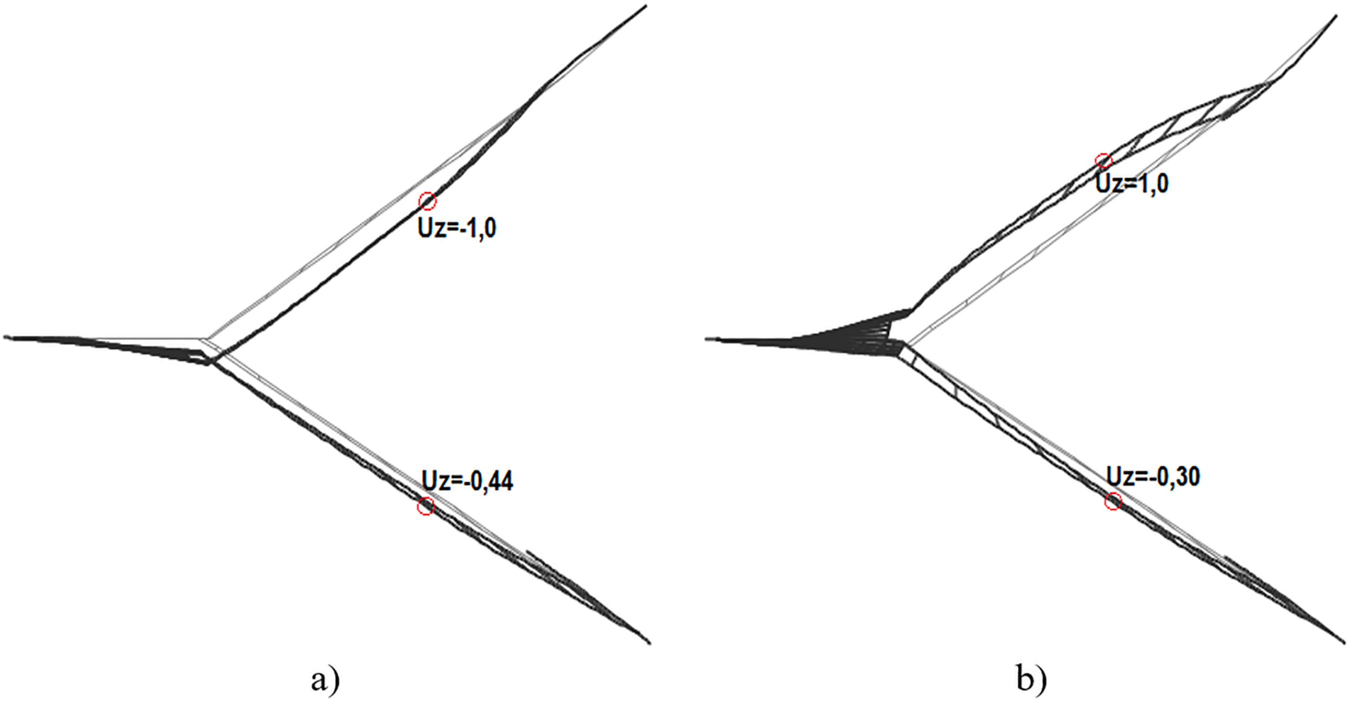


Fig. 4. Modal shapes: a) first mode (13.9 Hz) and b) second mode (14.5 Hz).

public building, so a higher vibration level can be allowed. For the limit suggested by Zhou et al. [4], 0.5 m/s^2 was considered because the staircase has a natural frequency of 13.9 Hz, this value being lower than $0.15\sqrt{f}$ m/s^2. The remaining limit values are those explained in the previous subsection.

Some observations can be made from Fig. 6:

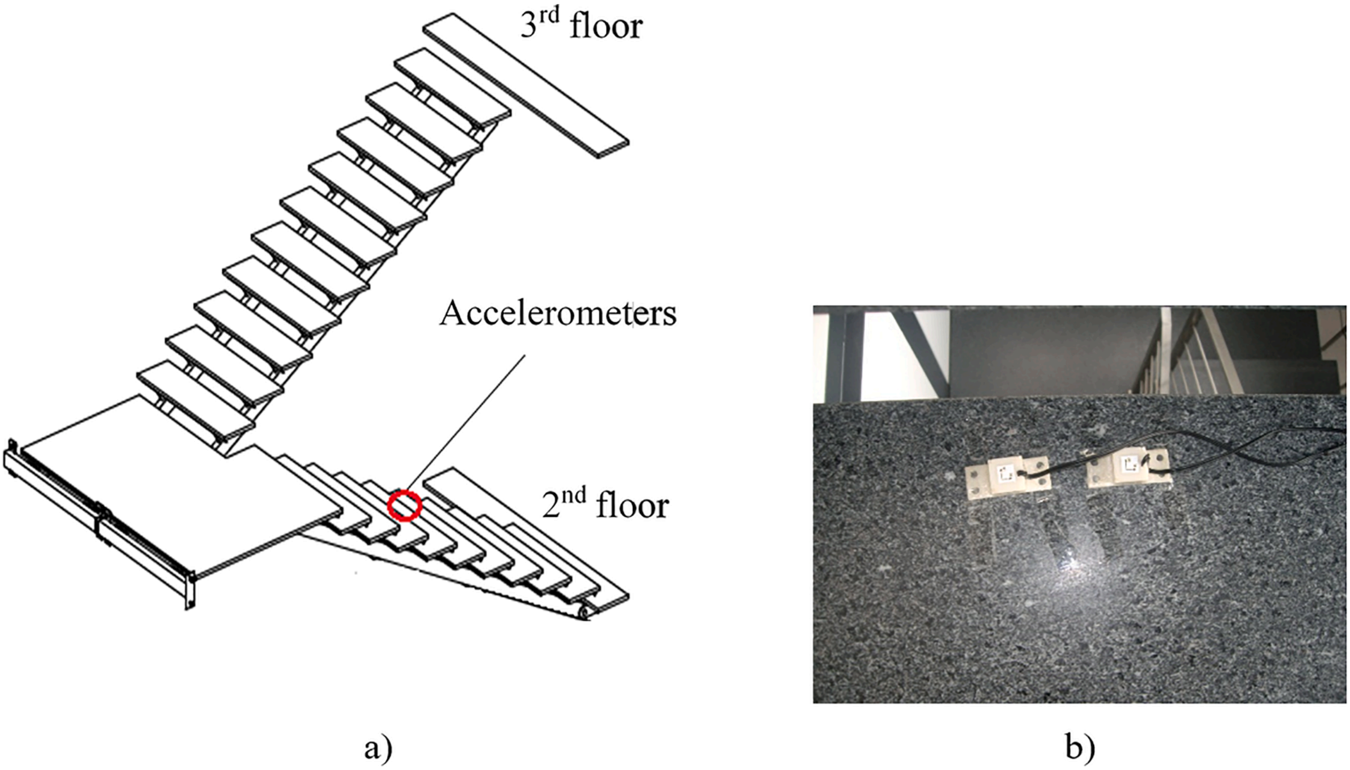
- Experimental peak accelerations, as would be expected, are higher for ascents and descents with step frequencies close to 3.5 Hz, as this is one of the submultiples of the stair's natural frequency (13.9 Hz), a resonance build-up being obtained with the consecutive application of the steps by the test subject.
- From the accelerations measured, it can be seen that vibrations increase with increasing step frequency and are higher for descents than for ascents, which is in agreement with the work performed on

Table 2
Description of walking tests.

Step frequency	Number of trials	
	Single pedestrian	Group of pedestrians
Descent 2.5 Hz	4	4
Descent 3.5 Hz	4	4
Ascent 2.0 Hz	3	6
Ascent 3.5 Hz	4	–

steel stairs by Bishop et al. [24], Davis et al. [1,11], González [2,3] and Setareh et al. [6,27].

- The experimental peak accelerations obtained for individual tests are much higher than the recommended acceptable limits proposed by the different design guides and researchers, both for ascents and descents, regardless of the step frequency employed by the test subject. Accelerations for step frequencies near to 3.5 Hz significantly exceed the recommended limits, especially for descents, ranging from about 1.0 m/s^2 to 2.0 m/s^2, clearly demonstrating the susceptibility of the sample staircase to an unacceptable level of



a)

b)

Fig. 5. Accelerometers location: a) midspan of flight of steps and b) centre of the step.

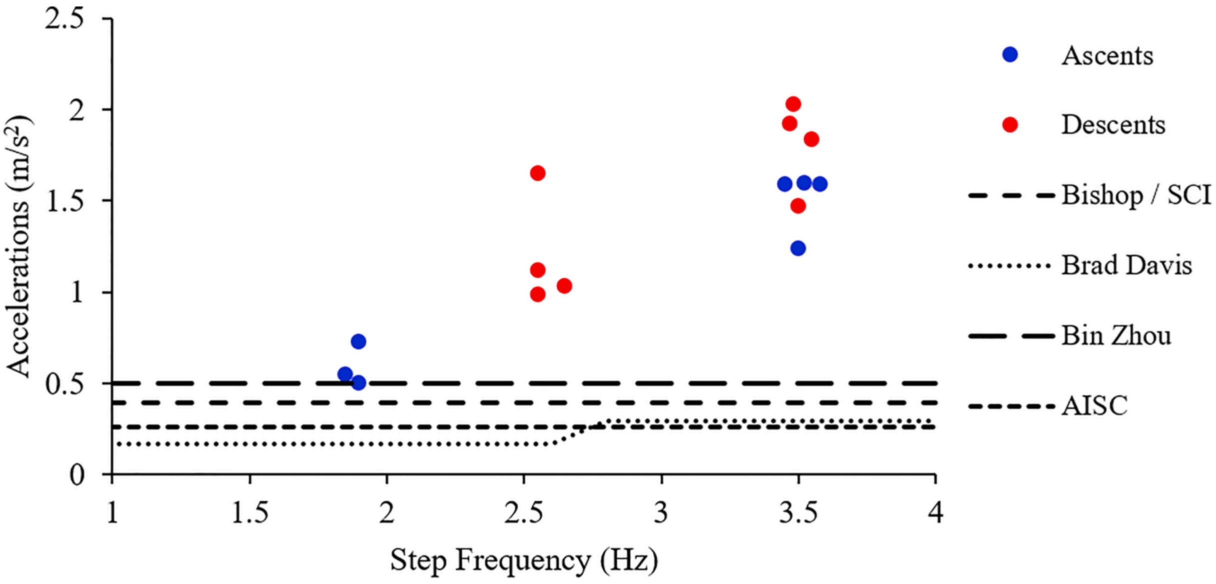


Fig. 6. Comparison between measured peak accelerations and acceptable limits for isolated ascents and descents.

vibrations and the reason for the feeling of insecurity by those who use it.

3.2. Group of pedestrians

In the experimental tests for a group of pedestrians, 14 graphs with the measured accelerations were obtained, one for each passage of the four individuals on the analysed staircase. Fig. 7 shows the peak accelerations obtained from experimental measurements for a group of 4 individuals ascending the staircase with step frequencies close to 2.0 Hz and descending with step frequencies close to 2.5 Hz and 3.5 Hz. The same graph also presents the different acceptable limits proposed. For the limits proposed by SCI P354 [7] / Bishop et al. [24] and Davis et al. [1,11], a multiplying factor of 64 and a value of 4.6%g (0.451 m/s^2) were applied, respectively, to take into account the accelerations corresponding to a group of pedestrians. The researchers Zhou et al. [4] and the design guide AISC 11 [8] refer to acceptable limits that vary with the structure's natural frequency, but making no distinction between a single or a group of pedestrians; therefore the same limit values as presented in the graph of Fig. 6 were employed for individual accelerations.

Observing Fig. 7, it can be verified that the peak accelerations for a group of pedestrians are of greater magnitude than the accelerations for a single pedestrian. Apart from this, the remaining conclusions are identical for both cases: peak accelerations are higher (as would be expected) for descents at 3.5 Hz, since it is the fourth submultiple of the staircase's natural frequency (13.9 Hz), increase as the step frequency increases, are higher for descents, and are larger than the proposed tolerance limits regardless of the step frequency, the difference between the limit values and peak accelerations being higher and quite significant for a step frequency of 3.5 Hz. Due to the vibration level measured on the staircase, great discomfort is expected if a group of 4 or more individuals are crossing it simultaneously, especially at higher step frequencies.

In this paper the enhancement factors obtained between individual and group accelerations are predominantly situated in the interval of 2.0 to 3.0 and are slightly higher for step frequencies of 3.5 Hz, correlating well with results verified by Kerr [25] for the same number of pedestrians, as referred to in Part 1 of this paper.

4. Staircase FE model

4.1. Model description

Before numerically evaluating the accelerations for subsequent comparison with the experimental measurements, it was necessary to create a numerical model of the sample staircase. The numerical model was created using the Finite Element software SAP2000. All the structural elements described in Subsection 2.1, steel hollow stringers HSS, European wide flange beams HEB180, metal plates and granite sheet coating of the stair steps and intermediate landing, were modelled using shell elements, beam elements being used only in the modelling of the guardrails. To the shell and beam elements were attributed the mechanical properties of the materials employed in its construction (steel S275 and granite). Fig. 8 represents the actual structure and the numerical model built.

4.2. Modal properties

The vibration modes and respective frequencies have been numerically predicted using the modal analysis type (Eigen Vectors) in SAP2000. Table 3 presents the comparison between the first six modes numerically obtained and experimentally measured (Fig. 3). From the 2nd vibration mode, the numerical natural frequencies begin to differ from the experimental natural frequencies, which can be explained, according to Davis [10], by the fact that FE models have a limited capacity to successfully predict the shapes and frequencies of all modes. That is, usually the first modes predicted with numerical models are comparable to those of the real structure, but as the number of modes increases, the shapes and natural frequencies obtained with FE models are quite different from those verified for a real structure. Nevertheless, it was possible to predict with close approximation the frequencies of the first two vibration modes, and for the higher modes the difference between the experimental and numerical results is not expected to significantly alter the level of vibrations.

4.3. Solution of the analysis method

SAP2000 provides two types of time-domain response history analysis that can be used to obtain numerical accelerations: direct integration and modal superposition. As described in Subsection 4.2, mentioning Davis [10], FE models of a structure have a limited capacity

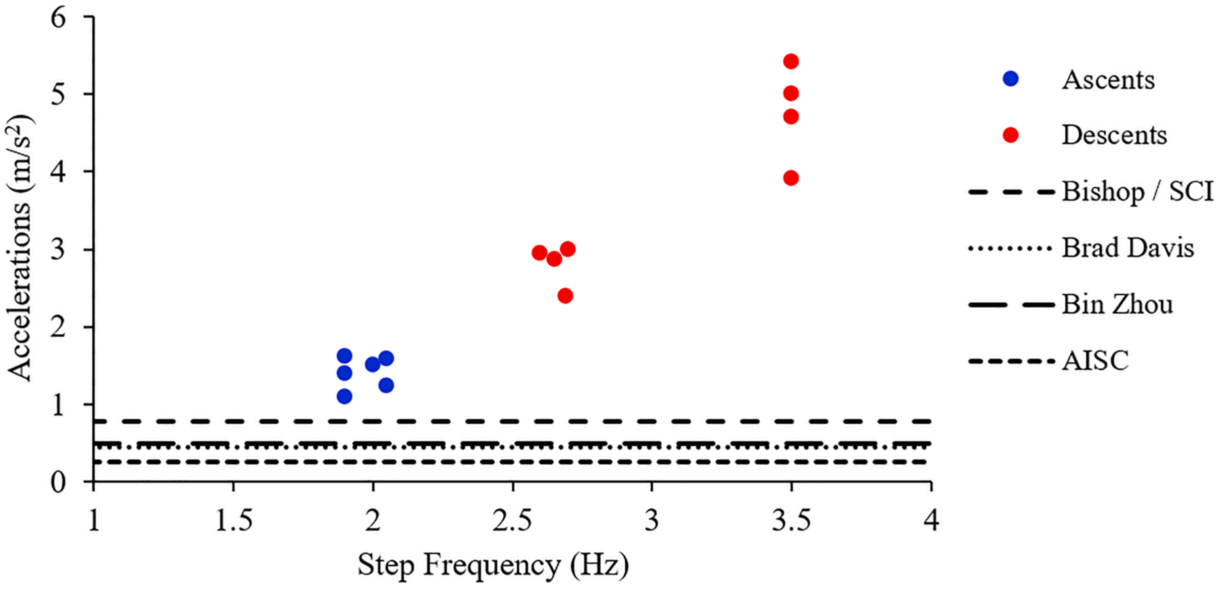


Fig. 7. Comparison between measured peak accelerations and acceptable limits for group ascents and descents.

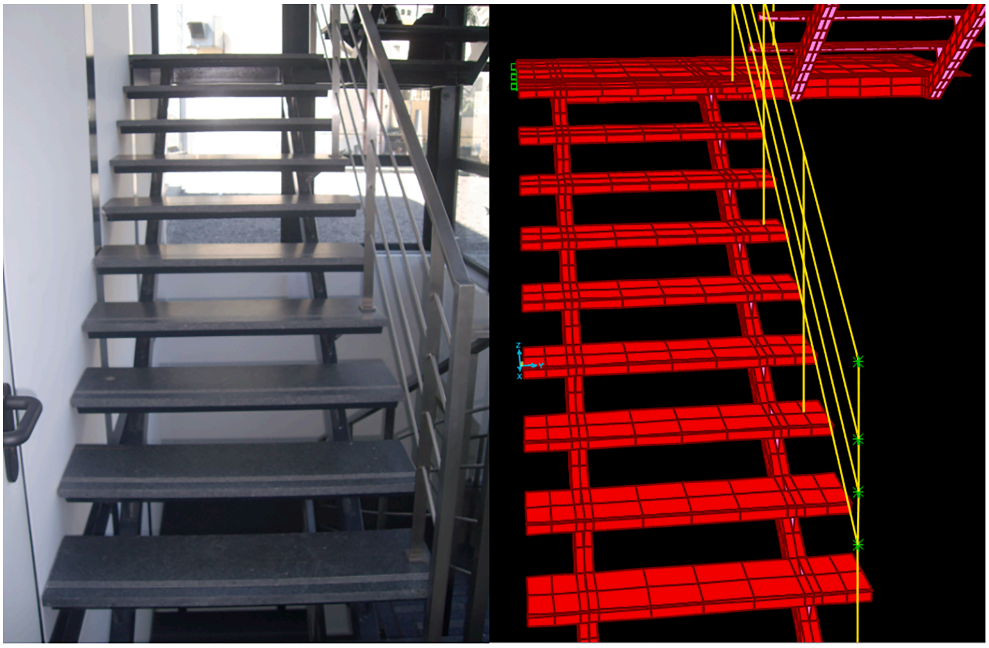


Fig. 8. Real steel staircase and FE staircase model.

Table 3
Numerical and experimental vibration modes.

Modes		Numerical Frequency (Hz)	Experimental Frequency (Hz)
N°	Shape		
1	Vertical	13,9	13,9
2	Vertical	14,9	14,5
3	Torsion	23,4	20,9
4	Torsion	26,5	21,9
5	Torsion	27,1	22,4
6	Torsion	29,3	23,2

to successfully predict the frequencies and shapes of all vibration modes. This means that using the direct integration method to compute accelerations could lead to unrealistic results, since its response takes into account the contribution of all the structure's modes, and numerous of these may not be comparable to those of the real structure. Moreover, according to Davis [10], the structure's response will be mostly governed by low frequency modes as these are in the frequency-band that is excitable by human walking. Hence, the use of modal superposition may be an advantage over direct integration, since it allows control over the number of modes to be considered, filtering out the contribution of higher modes which are not of interest.

Several analysis were performed in order to evaluate which of the two numerical methods, direct integration or modal superposition, should be employed to compute accelerations. Taking into account what was reported by Davis [10] and the fact that in the analysis performed it was observed that the direct integration method is significantly slower and overestimates in large-scale numerical accelerations, it was decided to use the modal superposition method in this work.

However, computing a large number of modes in modal superposition can lead to highly overestimated accelerations for the same reason as was verified with direct integration. In the analysis performed, comparing the numerical and experimental results, it was also found that the consideration of ten vibrations modes is suitable, without the loss of important information.

5. Comparison between experimental and numerical results

Part 1 of this paper reviews and describes the main existing numerical methods to predict accelerations when designing flexible staircases: i) footfall force time histories (GRFs), ii) Fourier series walking models, iii) steady-state analysis and iv) simplified vibration evaluation. The studied steel staircase has a fundamental frequency lower than the cut-off frequency of 16 Hz suggested by Santos et al. [28], referred to in Part 1, and can therefore be treated as an LFS with the possibility of applying the four methods. The different procedures were widely applied to this practical case, an extensive number of analyses being performed for each numerical method. All the analysis were employed reproducing the steps specified and comprehensively explained in Part 1. Also selected were step frequencies ranging from 1.90 Hz to 3.50 Hz, so numerical accelerations could be accurately and straightforwardly comparable with the experimental accelerations.

5.1. *Single pedestrian*

In order to facilitate the comparison between the numerical and experimental results for an individual pedestrian, two graphs were created (one for ascents and another for descents), encompassing all the peak accelerations at midspan obtained from experimental tests described in Subsection 3.1 and from the various numerical methodologies.

5.1.1. *Ascents*

The graph including all the peak accelerations obtained for the ascent is shown in Fig. 9.

Plotting all the peak accelerations, some important observations can be made regarding each of the numerical analysis performed:

5.1.1.1. GRFs footfall traces. As can be seen for ascents close to 1.90 Hz, the experimental accelerations correlate in general with those obtained numerically, it being possible to accurately simulate with GRFs traces the pedestrian's normal walking during the experimental tests. For a step frequency of 3.50 Hz, with all the GRFs traces employed a resonant response was obtained, but only with the GRF Kerr trace was it possible to obtain accelerations peaks close to the experimental ones. It is noteworthy that at 1.90 Hz (approximately), the GRF González force function generates a peak acceleration approximately the same as the experimental peaks and the GRF Kerr force function generates a peak

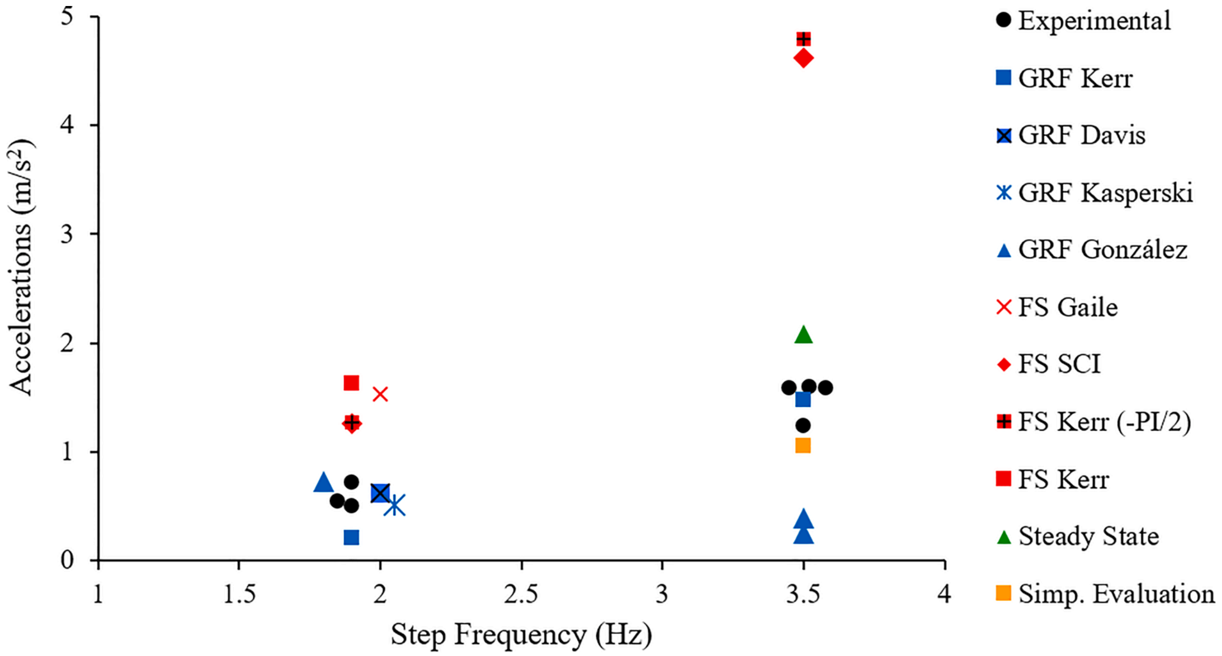


Fig. 9. Comparison between numerical and measured peak accelerations for isolated ascents.

acceleration lower than the measured peaks, while at 3.50 Hz (approximately) the exact opposite occurs.

5.1.1.2. Fourier series footfall traces. The numerical accelerations obtained by the Fourier series walking models are higher than the experimental accelerations for both ascents at 1.90 Hz and 3.50 Hz. Fourier series traces tend to overestimate the acceleration values, since they are only applied at the midspan of the staircase numerical model over time. For ascents at 3.50 Hz, the difference between the acceleration peaks obtained by the Fourier series traces and the experimental acceleration peaks is even more evident, far surpassing the measurements. This can be explained by the fact that Fourier series forces, in addition to acting consecutively at midspan, were applied with increments of 1/3.5 s, originating a resonant response.

5.1.1.3. Steady-state analysis. Although the peak acceleration obtained for an ascent at 3.50 Hz was slightly higher than the experimental peak accelerations, the results predicted with this method were relatively accurate, not being excessively conservative.

5.1.1.4. Simplified vibration evaluation. Even without the employment of any resonant build-up enveloped function and calibration factors, so the different numerical procedures could be compared on the same basis as explained in Part 1, the predicted peak acceleration for the ascent at 3.50 Hz was lower than the measured peak accelerations. Contrary to the steady-state analysis (also a simplified procedure), the predicted value was slightly unconservative.

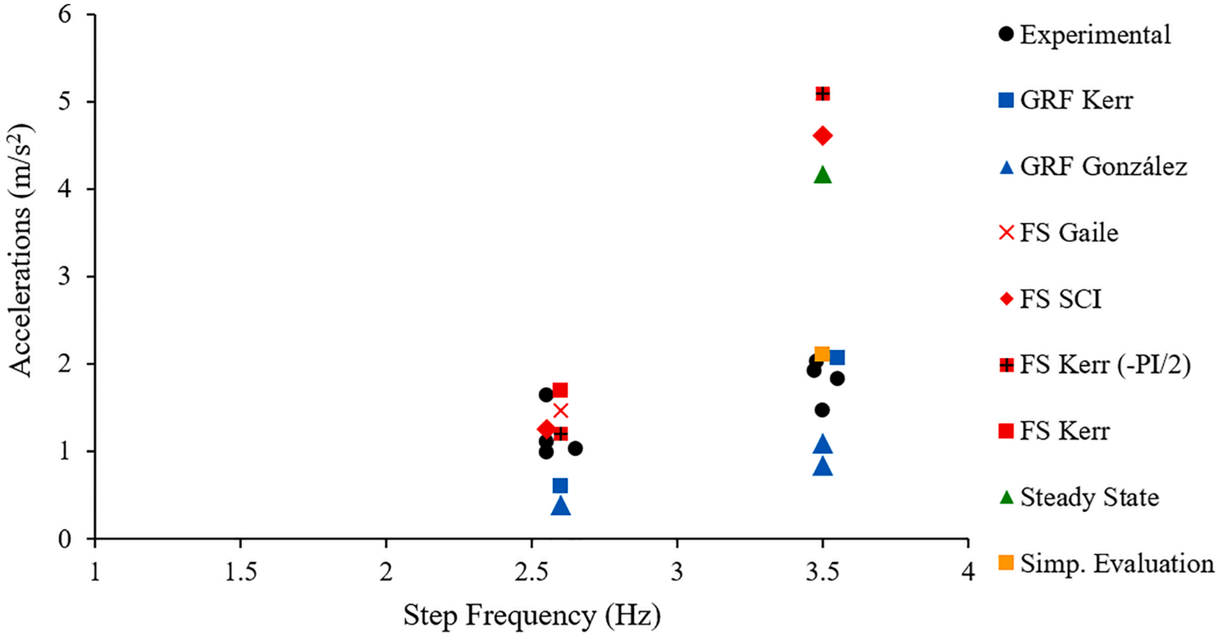


Fig. 10. Comparison between numerical and measured peak accelerations for isolated descents.

5.1.2. Descents

Fig. 10 presents the graph with all the peak accelerations obtained for descents with different step frequencies.

As for the ascents, some relevant observations can be drawn from the plot of peak accelerations for the descents, concerning the different numerical methodologies employed:

5.1.2.1. GRFs footfall traces. For normal descents with step frequencies of around 2.60 Hz, it was not possible with GRFs traces to obtain results as accurate as for the remaining step frequencies. None of the footfall forces (GRF González or GRF Kerr) applied were able to predict precise results. Presumably, it would be necessary to perform force plate tests in order to reach a correct form of the force function engaged by the individual who performed the descents at 2.60 Hz, or else it would be necessary to further find GRFs functions with step frequencies close to 2.60 Hz in the literature, but this was not possible. For rapid descents with a 3.50 Hz step frequency, the peak acceleration reached with the GRF Kerr trace is very close to the experimental peak accelerations. As would be expected, this is the most conditioning case, i.e. it generated a higher peak acceleration, since in addition to 3.5 Hz being one of the submultiples of the staircase's fundamental frequency, it was obtained for a descent.

It should be emphasised that, with the exception of the descents at approximately 2.60 Hz, for the majority of ascents and descents with different step frequencies it was feasible to predict the peak accelerations employing the first numerical method.

5.1.2.2. Fourier series footfall traces. The peak accelerations obtained by the Fourier series traces are approximately the same as the experimental measurements for a step frequency of 2.60 Hz. The level of accelerations obtained by applying this method is mainly associated with the harmonics amplitudes. If the considered values, for different step frequencies and for ascending and descending, yield overall similar maximum dynamic loads, the acceleration values will also be similar, as occurred when applying the Fourier series traces outside the interval of natural frequency submultiples, between ascents at 1.90 Hz and descents at 2.60 Hz.

As for ascents with a step frequency of 3.50 Hz, the numerical results obtained by the Fourier series traces for descents are substantially higher than the experimental results, being too conservative. As can be observed, when the step frequency matches one of the staircase's fundamental frequency submultiples, the resonant response obtained by the Fourier series force functions greatly overestimate the predicted accelerations, not being comparable to reality.

5.1.2.3. Steady-state analysis. Contrary to ascent, the peak acceleration obtained by the steady-state analysis for the descent at 3.50 Hz is substantially higher than the peak accelerations measured. The 4th harmonic force amplitude for the descent is twice the amplitude for the ascent (shown in Part 1 of this paper), thereby also giving rise to a peak acceleration for descending that is twice as high as for ascending. Although the experimentally obtained accelerations for descents at 3.50 Hz were higher than for ascents with the same step frequency, the difference of the acceleration peaks between the two types of motion was not near to double.

5.1.2.4. Simplified vibration evaluation. For coherence, the same harmonic force amplitudes used for the steady-state analysis were considered; hence, the peak acceleration computed with this method for the descent at 3.50 Hz is also double that of the ascent at 3.50 Hz. This resulted in a predicted peak acceleration that was approximately the same as the experimental peak accelerations for the descent at 3.50 Hz, being accurately estimated.

5.2. Group of pedestrians

For a group of pedestrians a summary graph was also created encompassing all the peak accelerations experimentally measured and numerically obtained with the different procedures, so that they could be more feasibly compared. Fig. 11 presents all the peak accelerations obtained at midspan, referring to group ascents and descents for different step frequencies.

From the acceleration peaks obtained for the group ascents and descents, it is important to note the following:

5.2.1. GRFs footfall traces

The peak accelerations obtained for ascents and descents were in general accurately predicted, regardless of the step frequency employed, demonstrating that, in both individual and group numerical analysis, this method was the one that most realistically simulated the participants' behaviour during the experimental tests performed on the sample staircase. For step frequencies near to 2.60 Hz, the GRFs traces also generated group accelerations close to the experimental ones, opposite to the individual analysis with the same step frequencies, as can be seen from Fig. 10.

Notably, as for the experimentally measured accelerations, for the numerical accelerations using GRFs functions, group enhancement factors ranging from approximately 2 to 3 were also obtained.

5.2.2. Fourier series footfall traces

The fact that four Fourier series force traces were used, all being applied only at midspan during the time required for the four individuals to walk the flight of steps, caused the values to be highly amplified, no longer making physical sense. Moreover, for descents at 3.50 Hz, a resonant response was obtained, with the acceleration values being clearly higher than for the other step frequencies.

5.2.3. Steady-state analysis and simplified vibration evaluation

With the two numerical methods it is not possible to directly predict the peak accelerations due to a group of pedestrians; hence, these values were obtained by multiplying the peak accelerations predicted for an individual pedestrian descending at 3.50 Hz (see Fig. 10) by an enhancement factor of 3, as suggested by Davis and Murray [1]. Although the peak acceleration referring to the steady-state analysis was not as overestimated as when applying Fourier series traces, it was still very conservative. However, this was as expected, since, for the single descent at 3.50 Hz, a much higher steady-state peak acceleration than the measured values was predicted. Considering the accurately predicted peak acceleration for an isolated pedestrian and an enhancement factor of 3, it was possible with the simplified vibration evaluation method to estimate a peak acceleration for a group of pedestrians that was approximately the same as the experimentally measured value.

6. Discussion

As previously mentioned, the amount of information available on predicting staircase vibration due to human induced loads is still limited. Although, an attempt was made to accurately predict accelerations with the main numerical methods existing in the literature, several shortcomings still need to be overcome, so that these methods can be correctly used in the design of staircases.

Besides that, in Section 5 a comparison between the experimental and predicted results by the four numerical methods was done for just one staircase, which is a limitation of this work. Therefore, this section discloses the main conclusions achieved by the authors and compares these with the findings of other researchers who have also predicted accelerations employing numerical methods.

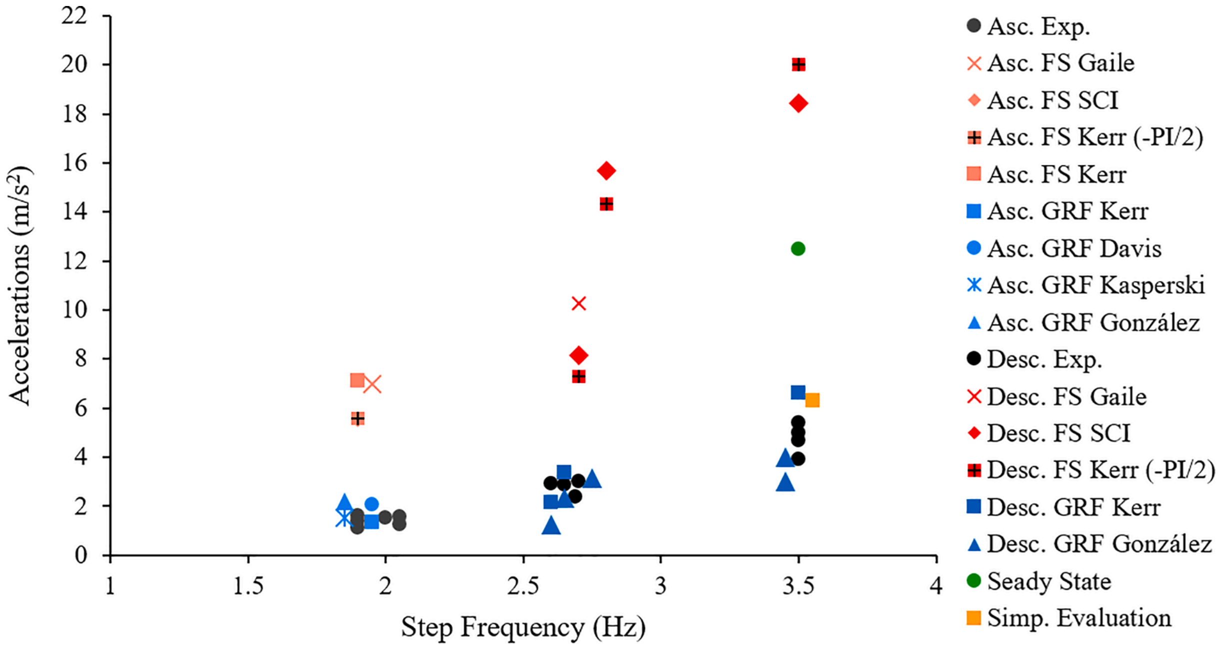


Fig. 11. Comparison between numerical and measured peak accelerations for group ascents and descents.

6.1. GRFs footfall traces

By applying GRFs load functions to the FE staircase model, in the majority of cases it was possible to accurately simulate the individuals walking on the actual staircase, this being the most reliable method used to predict accelerations. However, as shown in Part 1, this procedure presents some disadvantages, since GRFs traces with the desired step frequencies are not always found and it is required to perform a time history analysis for each ascent and descent, making the calculation process very time-consuming.

Davis [10] employed this method to evaluate numerical accelerations in six different types of floors. Although this work was developed for floors, it is compared with the findings of this study. Taking into account the low value of the average predicted-to-measured ratio obtained, it can be verified that his results are in agreement with the results of the work developed here. It should be mentioned that group accelerations were not obtained, so it is not possible to compare them with the responses obtained for this type of test.

The only researcher found in the literature to apply this numerical procedure directly on staircases, later comparing it with an experimental campaign, was González [2,3]. The step frequencies used in the analyses do not exactly correspond to submultiples of the staircase's fundamental frequency; however, once again the difference between the experimental and predicted peak accelerations, especially for the descent at 3.88 Hz, is not significant, being in concordance with the accelerations observed in Subsections 5.1.1 and 5.1.2. For the descent at 3.88 Hz, González [2,3] also measured experimentally and numerically simulated the accelerations generated by a group of four pedestrians, similar to the analysis performed in this work. This researcher obtained an enhancement factor of approximately 2.0, which also correlates well with that verified in this paper.

Simulating a group of pedestrians walking using GRFs traces, and subsequently performing a time history analysis is a very time-consuming procedure that the majority of structural engineers cannot afford. Therefore, taking into account the coherence between the enhancement factors obtained by Kerr [25], González [2,3] and in this work, it can be concluded that it apparently makes sense for pedestrian group accelerations to be predicted using GRFs footfall traces to simulate a single pedestrian and then multiply the resulting accelerations by an enhancement factor, i.e. a factor of 2 or 3.

6.2. Fourier series footfall traces

From the possibility of human walking forces being simply defined by the equation presented in Part 1, to the fact that harmonic amplitude values could be easily found in the literature, there are obvious and noticeable advantages in using load functions in terms of Fourier series. However, except for descents at 2.60 Hz, predicted accelerations using Fourier series walking models have been considerably overestimated, particularly in simulations performed with a 3.50 Hz step frequency (the 4th submultiple of the staircase's fundamental frequency), as detailed in Subsections 5.1.1 and 5.1.2. Furthermore, similar to the previous procedure, this method also presents the disadvantage of having a slow calculation process due to the need to perform a different time history analysis for each simulation.

Contrary to the results obtained in this work, Zhou et al. [4], applying this method to an indoor spiral steel staircase, were able to predict accelerations, in general, close to experimental accelerations. However, the step frequencies selected do not exactly match the staircase's fundamental frequency submultiples and no further explanation is given regarding only two harmonics being considered in the definition of the Fourier series traces, nor any reference to the origin of their amplitude values.

Davis [10], using this numerical method, predicted peak acceleration values nearly double the experimental values, even excluding the average predicted-to-measured peak predictions of the tests with an inaccurate mode shape prediction, these being significantly overestimated, which is in accordance with the predicted accelerations in this work employing the same numerical procedure.

According to Davis [10], this method results in over-prediction of accelerations due to the walking force being applied at midspan for the entire response history duration and because the Fourier series models "walk" at a perfect cadence. As a consequence, an adjustment factor is proposed by this researcher, which should be applied to predicted accelerations. Likewise, Eid et al. [5] also suggested an adjustment factor to be applied in the case of staircases, referring to the value specified for footbridges by the design guide AISC 11 [8]. Taking into account the reasons described above and the fact that, as explained in Part 1, the GRFs and Fourier series load functions only produce the same effect if applied at the same point, it seems reasonable that accelerations can only be successfully predicted using this numerical method if an adjustment factor is considered.

In this work, an attempt was also made to obtain an adjustment factor based on the average ratio between the measured and predicted peak accelerations. Table 4 presents the adjustment factors proposed by the different researchers and obtained in this work. The calculated adjustment factor is slightly lower than the factors suggested by Davis [10] and Eid et al. [5], being less conservative. However, from the variation between the different values there is a clear need to apply an adjustment factor if this numerical method is employed.

This method proved to be the most inaccurate in calculating numerical accelerations for a group of pedestrians. Of the previously mentioned researchers [5,6,10], only Zhou et al. [4] numerically estimated accelerations referring to a group of pedestrians using Fourier series walking models, but without any comparison with experimental data.

For this reason and the same mentioned for the first numerical method, in order to accurately predict group accelerations with Fourier series traces it seems more plausible to obtain the accelerations due to a single pedestrian and apply an adjustment factor and then, in a simplified manner, multiply for an enhancement factor, i.e. a value of 2 or 3.

6.3. Steady-state analysis

With the third method presented in this work, mixed results were obtained, a slightly higher peak acceleration being predicted than the experimental one for the 3.50 Hz ascent and a much more conservative one for the 3.50 Hz descent.

The most extensive work on steady-state analysis was performed by Davis and Murray [1]. Comparing the numerical and experimental peak accelerations obtained by these researchers, it is verified that there is a clear overestimation when performing the steady-state analysis. Therefore, it seems that this numerical method also requires the application of an adjustment factor. Davis and Murray [1] justify the need for an adjustment factor with the fact that pedestrians do not walk with a step frequency exactly matching one of the fundamental frequency submultiples, and that the peak accelerations were obtained by a load applied at midspan (the location of maximum accelerations), which does not occur in practice.

Setareh [6] states that these researchers' proposal for the adjustment factor may be unconservative, recommending a higher value. The average adjustment factor obtained in this work is also higher than the value suggested by Davis and Murray [1], being closer to that calculated by Setareh [6]. Table 5 presents the adjustment factors values referring to the researchers mentioned above and obtained here. Contrary to the second numerical method, the different factors vary within a broader range of values, making it difficult to assess which should be applied when performing a steady-state analysis.

Davis and Murray [1] did not attempt to experimentally measure group accelerations to further compare them with the predicted accelerations multiplied by the proposed enhancement factor of 3. Still, as mentioned previously, this value is in agreement with the results of this work.

6.4. Simplified vibration evaluation

Overall, with this simplified method, it was possible to calculate satisfactory results, predicting peak accelerations slightly lower than those measured for the ascent at 3.50 Hz and close to the experimental peak accelerations for the descent at 3.50 Hz.

Table 4
Fourier series adjustment factors.

Source	Adjustment Factor
Davis [10]	0,65
Eid et al. [5] / AISC 11 [8]	0,70
This work	0,50

Table 5
Proposed steady-state adjustment factors.

Source	Adjustment Factor
Davis and Murray [1]	0,35
Setareh [6]	0,84
This work	0,60

This simplified procedure was developed by Davis and Avci [11,12], and to date no other work is known to have been performed that employed it to predict peak accelerations and subsequently compare them with an experimental campaign, having the purpose of validating this methodology.

Davis and Avci [11,12] obtained peak accelerations 1.5 times higher than the measured values, which do not correlate well with the results observed here, considering the same harmonic amplitude. Davis reported that accelerations calculated with this simplified method tend to be overestimated due to the fact resonant build-ups do not last long enough to achieve a steady-state response and footsteps are not perfectly periodic, also proposing the application of several adjustment factors to the predicted response. However, the same was not verified in this study, being necessary to employ this procedure on a larger number of practical cases to evaluate if the predicted accelerations are consistently and continuously overestimated.

Davis and Avci [11,12] did not estimate the accelerations for a group of pedestrians, recommending an enhancement factor of 3, which, as aforementioned, is similar to the results obtained in this work.

7. Summary and conclusions

With the growing popularity of designing lighter and slender monumental steel staircases, where the susceptibility to high vibrations is significant, it is becoming gradually more important to study this phenomenon in order to provide users with adequate comfort levels and avoid any feeling of unsafety.

In this paper the effectiveness of the four existing methods in estimating the maximum vibrations in staircases was evaluated, by comparing the experimental results with the predicted results. To do this, an experimental campaign was initially developed, using people ascending and descending a real staircase and measuring the vertical accelerations. Furthermore, a finite element model of that staircase was also created, the loads from each method were applied to compute the responses, and the maximum vibrations were analysed.

From this work several conclusions can be drawn:

- The tested staircase presents levels of vibration that clearly exceed the acceptable limits recommended by the different design guides and researchers, especially for descents (higher responses) and for step frequencies that match one of the fundamental frequency submultiples (resonant build-up).
- The numerical model (FE) developed was able to adequately simulate the dynamic characteristics of the tested staircase (vibration modes and respective frequencies).
- Four numerical methods that can be used to predict vibrations were analysed: footfall force time histories (GRFs), Fourier series walking models, steady-state analysis and simplified vibration evaluation. GRFs and Fourier series footfall traces are a manner of representing pedestrian loads, while steady-state analysis and simplified vibration evaluation are straightforward methods for estimating numerical accelerations, being only applicable to structures where a resonance build-up is possible.
- Simulations using GRFs footfall traces were the numerical method that predicted accelerations closest to the experimental measurements. Although this method was the most accurate, there are some limitations when applying it, i.e. the scarcity of

GRFs footfall traces for staircases found in the literature and the slow process of obtaining numerical accelerations.
- Fourier series models and steady-state analysis, in the majority of cases, generated far more conservative results. Hence, when carrying out simulations using these methods, it is suggested to multiply the predicted accelerations by an adjustment factor.
- Employing the simplified vibration evaluation method, it was possible to predict results approximately the same as the experimental ones, especially for the descent at 3.50 Hz. Contrary to what was explained by the researcher's method, overestimated responses were not verified without considering the application of adjustments factors and a resonant build-up envelope function.
- The group effect was verified by obtaining enhancement factors between the experimental accelerations measured for a single pedestrian and for a group of pedestrians. For a group of pedestrians, enhancement factors ranging from 2 to 3 were obtained. To overcome the fact that directly obtaining numerical accelerations for a group of pedestrians is a very time-consuming procedure, it is suggested to multiply the individual numerical accelerations by an enhancement factor, thus feasibly predicting the group effect on staircases.
- Regarding the application of the different numerical methods to predict vibrations, there are still several limitations that need to be overcome in the future:

i) Improve the amount of information and specification concerning how these methods should be employed when designing flexible staircases.
ii) Reach a greater consensus on the acceptance criteria of vibrations for staircases, taking into account the scarcity and difference of acceptable limits proposed for this particular type of structure.
iii) Obtain numerous GRFs footfall time histories directly for staircases, with closely spaced intervals of step frequencies and indicated for design purposes, similar to the work by Davis [10] for floors.
iv) Define a more accurate and narrower interval of adjustment factors values based on a higher number of experimental campaigns on different staircases, to avoid excessively overestimated responses when employing Fourier series walking models and performing steady-state analysis.
v) Compare the predicted results using the simplified vibration evaluation with a broader range of experimental data on a more extensive number of real staircases, to assess if it is consistently giving precise response estimations.
vi) Develop a faster and more efficient numerical procedure to directly estimate the responses generated by a group of pedestrians or evaluate if the aforementioned enhancement factors are consistently within the same range of values, making it possible, in a valid manner, to indirectly predict the group effect.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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