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Residual Contraction

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Abstract In this paper, we propose and axiomatically characterize residual contractions, a new kind of contraction operators for belief bases. We establish that the class of partial meet contractions is a strict subclass of the class of residual contractions. We identify an extra condition that may be added to the definition of residual contractions, which is such that the class of residual contractions that satisfy it coincides with the class of partial meet contractions. We investigate the interrelations in the sense of (strict) inclusion among the class of residual contractions and other classes of well known contraction operators for belief bases.

Keywords Belief Change · Belief Bases · Contraction · Axiomatic Characterizations · Residual Contractions · Residuums

1 Introduction

The research area that studies the dynamics of knowledge is known as *belief change* (also called *belief revision* or *belief dynamics*). One of the main goals underlying this area is to model how a rational agent updates his/her set of beliefs when confronted with new information. When facing new information an agent can change his/her set of beliefs: he/she can acquire some new beliefs and revise or give up some old ones. The main objective of most of the works

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in this area is to investigate and model how these changes occur. One of the main contributions to the study of belief change is the so-called AGM model for belief change—named after the initial of its authors: Alchourrón, Gärdenfors and Makinson. This model was proposed in 1985 in [1] and gained the status of standard model of belief change. In that framework, beliefs are represented by sentences (of a propositional language $\mathcal{L}$), the belief state of an agent is modelled by a belief set—*i.e.* a logically closed set of (belief-representing) sentences—, and epistemic inputs are represented by single sentences. The AGM model considers three kinds of belief change operators, namely *expansion, contraction* and *revision.* Expansion is the simplest of the AGM operators. An expansion occurs when new information is added to the set of the beliefs of an agent. The expansion of a belief set $\mathbf{K}$ by a sentence α is the logical closure of $\mathbf{K} \cup \{\alpha\}$. A contraction occurs when information is removed from the set of beliefs of an agent. A revision occurs when new information is added to the set of the beliefs of an agent while retaining consistency if the new information is itself consistent. When performing a revision some beliefs may be removed in order to ensure consistency. Contractions and revisions can be defined one in terms of the other. Thus one of these operators can be considered primitive and the other one derived. In [1] some properties were proposed as being the characteristic properties of a contraction. These properties (which are recalled in Subsection 2.2) are commonly called the *AGM postulates for contraction* and an operator that satisfies them is designated by *AGM contraction.*

There are in the belief change literature several constructive methods for defining operators which satisfy all or at least some of the AGM postulates for contraction. Some of those models are the *(transitively relational) partial meet contractions* [1], *safe contraction* [3,22], *system of spheres-based contraction* [11] and *epistemic entrenchment-based contraction* [9,10].

Although the AGM model has quickly acquired the status of standard model of theory change, several researchers (for an overview see [5]) have pointed out its inadequateness in several contexts and proposed several extensions and generalizations to that framework. One criticism of the AGM framework is that it uses logically closed sets (or belief sets) to model the belief state of an agent. This can be considered undesirable for a number of reasons. Firstly, belief sets are very large entities (eventually even infinite), and its use is not adequate for computational implementations. The logical closure of belief sets raises other issues not related to computational implementations. Rott pointed out in [21] that the AGM theory is unrealistic in its assumption that epistemic agents are "ideally competent regarding matters of logic. They should accept all the consequences of the beliefs they hold (that is, their set of beliefs should be logically closed), and they should rigorously see to it that their beliefs are consistent". Furthermore, belief sets make no distinction between different inconsistent belief states. They also make no distinction between basic beliefs and those that are inferred from them.

Several of the existing models of contraction for beliefs sets have been adapted to the case when belief states are represented by belief bases: the partial meet contractions for belief bases were presented in [14,16,17]; the kernel contractions—which can be seen as a generalization of safe contractions—were introduced in [18]; the basic AGM-generated base contractions, a kind of contraction operators for belief bases defined by means of a contraction operator for belief sets was proposed in [6] and, the *ensconcement-based contractions* (of belief bases), which can be seen as adaptations to the case of belief bases of the *epistemic entrenchment-based contractions*, were introduced in [23] and axiomatically characterized in [4].

In this paper we propose and axiomatically characterize a new class of belief base contraction operators, called *residual contractions*, which is a more general class than that of the partial meet contractions. The definition of these new operators relies on the notion of *residuums*, which is similar to that of *remainders*—the basic constructs underlying the definition of partial meet contractions—, but which treats sets with the same closure as equals. Indeed, a strict subset of a remainder is not a remainder (and, consequently, is considered not usable in the process of contraction) even if its closure is identical to the closure of that remainder. On the other hand, given a residuum, any of its subsets that has the same consequences as the whole is also a residuum.

The rest of the paper is organized as follows: In Section 2 we introduce the notations and recall the main background concepts that will be needed throughout this article. In particular we recall the definitions of the belief change operators mentioned above and some of their axiomatic characterizations. In Section 3 we present the definition of residual contractions, obtain an axiomatic characterization for those operators, and show that the class of residual contractions strictly contains the class of partial meet contractions. Afterwards, in Section 4, we show that the class of residual contractions that are based on selection functions that satisfy a certain additional condition coincides with the class of partial meet contractions. Then, in Section 5 we study the interrelations between the class of residual contractions and the classes of other well known contraction operators for belief bases, namely (smooth) kernel contractions and basic AGM-generated base contractions. More precisely, we analyse whether each of those classes is or is not (strictly) contained in each of the remaining ones. Finally, in Section 6 we summarize the main contributions of the paper and briefly discuss their relevance. In the Appendix we provide proofs for all the original results presented.

2 Background

2.1 Formal preliminaries

We will assume a propositional language $\mathcal{L}$ that contains the usual truth functional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence). We shall make use of a consequence operation Cn that takes sets of sentences to sets of sentences and which satisfies the standard

Tarskian properties, namely *inclusion, monotony* and *iteration*. Furthermore we will assume that Cn satisfies *supraclassicality, compactness* and *deduction*. We will call $Cn(A)$ the *logical closure* of A. We will sometimes use $Cn(\alpha)$ for $Cn(\{\alpha\})$, $A \vdash \alpha$ for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$, $A \nvdash \alpha$ for $\alpha \notin Cn(A)$, $\nvdash \alpha$ for $\alpha \notin Cn(\emptyset)$. We will say that Cn is purely truth-functional when Cn is the consequence operation such that, for any sentence α and any set of sentences A, it holds that $\alpha \in Cn(A)$ if and only if α can be derived from A in the framework of classical propositional logic. The letters $\alpha, \beta, \dots$ (except for γ and σ) will be used to denote sentences of $\mathcal{L}$. Lowercase Latin letters such as $p, q, \dots$ will be used to denote atomic sentences of $\mathcal{L}$. $A, B, \dots$ shall denote sets of sentences of $\mathcal{L}$. $\mathbf{K}$ is reserved to represent a set of sentences that is closed under logical consequence (i.e. $\mathbf{K} = Cn(\mathbf{K})$) — such a set is called a *belief set* or *theory*. Given a set A we will denote the power set of A by $\mathcal{P}(A)$. Given $A \subseteq \mathcal{L}$, the expression "(contraction) operator (or function) on A" will be used for designating a function $- : \mathcal{L} \longrightarrow P(\mathcal{L})$ and, in that context, we shall represent by $A - \alpha$ the image of a sentence α by $-$.

2.2 AGM contractions

In this subsection we recall the AGM postulates for contraction and the concept of AGM contraction.

Definition 1 ([1]) Let $\mathbf{K}$ be a belief set. An operator $-$ on $\mathbf{K}$ is an AGM contraction if and only if it satisfies the following conditions:

$(\mathbf{K}-1)$ $\mathbf{K} - \alpha = Cn(\mathbf{K} - \alpha)$. (Closure)
$(\mathbf{K}-2)$ $\mathbf{K} - \alpha \subseteq \mathbf{K}$. (Inclusion)
$(\mathbf{K}-3)$ If $\alpha \notin \mathbf{K}$, then $\mathbf{K} - \alpha = \mathbf{K}$. (Vacuity)
$(\mathbf{K}-4)$ If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin \mathbf{K} - \alpha$. (Success)
$(\mathbf{K}-5)$ If $\alpha \leftrightarrow \beta \in Cn(\emptyset)$, then $\mathbf{K} - \alpha = \mathbf{K} - \beta$. (Extensionality)
$(\mathbf{K}-6)$ $\mathbf{K} \subseteq Cn((\mathbf{K} - \alpha) \cup \{\alpha\})$. (Recovery)
$(\mathbf{K}-7)$ $(\mathbf{K} - \alpha) \cap (\mathbf{K} - \beta) \subseteq \mathbf{K} - (\alpha \wedge \beta)$. (Conjunctive overlap)
$(\mathbf{K}-8)$ $\mathbf{K} - (\alpha \wedge \beta) \subseteq \mathbf{K} - \alpha$ whenever $\alpha \notin \mathbf{K} - (\alpha \wedge \beta)$. (Conjunctive inclusion)

These conditions, that are usually designated by *AGM postulates for contraction*. Postulates $(\mathbf{K}-1)$—$(\mathbf{K}-6)$ are called *basic AGM postulates for contraction* and an operator $-$ that satisfies those properties is called a *basic AGM contraction*. Postulates $(\mathbf{K}-7)$ and $(\mathbf{K}-8)$ are designated by *supplementary AGM postulates for contraction*.

2.3 Belief base contraction

We now recall some postulates for base contraction and also several constructive models of contraction functions on belief bases.

2.3.1 Postulates for base contraction

The following are some well known postulates for belief base contraction:[1]

Success: If $\nvdash \alpha$, then $A - \alpha \nvdash \alpha$.
Inclusion: $A - \alpha \subseteq A$.
Failure: If $\vdash \alpha$ then $A \div \alpha = A$.
Vacuity: If $A \nvdash \alpha$, then $A \subseteq A \div \alpha$.
Relative Closure: $A \cap Cn(A \div \alpha) \subseteq A \div \alpha$.
Relevance: If $\beta \in A$ and $\beta \notin A \div \alpha$ then there is some set A' such that $A \div \alpha \subseteq A' \subseteq A$ and $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$.
Core-retainment: If $\beta \in A$ and $\beta \notin A \div \alpha$ then there is some set A' such that $A' \subseteq A$ and $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$.
Disjunctive elimination: If $\beta \in A$ and $\beta \notin A \div \alpha$, then $A \div \alpha \nvdash \alpha \vee \beta$.
Extensionality: If $\vdash \alpha \leftrightarrow \beta$, then $A \div \alpha = A \div \beta$.
Uniformity: If it holds for all subsets A' of A that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$, then $A \div \alpha = A \div \beta$.

Success [1] states that the outcome of a contraction by a non-tautological sentence does not imply that sentence. *Inclusion* [1] states that the outcome of a contraction is subset of the contracted set. *Failure* [7] states that contracting by a tautology leaves the set to be contracted unchanged. *Vacuity* ensures that nothing is removed when contracting a set by a sentence that is not a logical consequence of that set. *Relative closure* [15] ensures that the original beliefs that are implied by the outcome of the contraction are kept. This postulate results of adapting the belief set contraction postulate of *closure* to the context of belief base contractions. The *relevance* postulate [13,16] states that if a sentence β is removed from A when contracting it by α, then β must contribute to deduce α from A. *Core-retainment* [15] is a weaker version of *relevance* since it does not require that $A - \alpha \subseteq A'$. *Disjunctive elimination* was proposed in [6] and states that if a sentence β is removed in the process of contracting A by another sentence α then the disjunction of α and β is not deducible from the outcome of that contraction. In the context of contractions of belief sets, *relevance*, *core-retainment* and *disjunctive elimination* are equivalent to *recovery* in the presence of the other basic AGM postulates for contraction [15,7, 6]. These three postulates as well as *failure*, *vacuity* and *relative closure* try to formalize the *minimal change criteria*, according to which unnecessary loss of information should be avoided. *Extensionality* states that the contraction of a set by logical equivalent sentences produces the same output. This postulate is a formalization of the *irrelevance of syntax criteria*, according to which the outcome of a change should not depend on the syntax/representation used. *Uniformity*, which was originally presented in [16], states that if α and β are two sentences implied by exactly the same subsets of A, then the result of

[1] For an overview of these postulates see [19,5].

contracting A by α is identical to the outcome of contracting A by β.

The following observation presents some relations between the belief base postulates for contraction mentioned above.

Observation 1 *Let A be a belief base and $\div$ an operator on A. Then:*

(a) [19] *If $\div$ satisfies relevance, then it satisfies relative closure and core-retainment.*
(b) [19] *If $\div$ satisfies inclusion and core-retainment, then it satisfies failure and vacuity.*
(c) [19] *If $\div$ satisfies uniformity, then it satisfies extensionality.*
(d) [6] *If $\div$ satisfies disjunctive elimination, then $\div$ satisfies relative closure. If $\div$ also satisfies inclusion then it satisfies failure.*
(e) [6] *If $\div$ satisfies relevance, then $\div$ satisfies disjunctive elimination.*

2.3.2 Partial meet contraction

In the rest of this section we recall some explicit definitions of contraction functions as well as axiomatic characterizations for them. The first kind of contraction operators that we will present are known as *partial meet contractions* and were originally presented in [1]. We start by recalling the concept of *remainder set*, that is a set of maximal subsets (of a given set) that fail to imply a given sentence. Formally:

Definition 2 ([2]) Let A be a belief base and α a sentence. The set $A\perp\alpha$ (A remainder α) is the set of sets such that $B \in A\perp\alpha$ if and only if:

1. $B \subseteq A$.
2. $B \nvdash \alpha$.
3. If $B \subset B' \subseteq A$, then $B' \vdash \alpha$.

$A\perp\alpha$ is called remainder set of A by α and its elements are called remainders (of A by α).

It follows from compacteness and Zorn's lemma (cf. [2, Proof of Observation 2.2]) that, given a set of sentences A and a sentence α, every subset D of A that does not imply α is contained in some remainder of A by α. This property is known as **upper bound property**:

If $D \subseteq A$ and $D \nvdash \alpha$, then there is some D' such that $D \subseteq D' \in A \perp \alpha$.

The partial meet contractions are obtained by intersecting some elements of the (associated) remainder set. The choice of those elements is performed by a *selection function*.

Definition 3 Given a set of sentences A and a set D such that $D \subseteq \mathcal{P}(\mathcal{P}(A))$ and $\emptyset \in D$, a function $\gamma : D \longrightarrow \mathcal{P}(\mathcal{P}(A))$ is called a selection function (for A) if and only if:

1. $\gamma(\emptyset) = \{A\}$;
2. For all $X \in D \setminus \{\emptyset\}$, $\emptyset \neq \gamma(X) \subseteq X$.

We note that the definition of *selection function* proposed above is more general than the one originally presented in [1]. The reason for presenting here this more general definition is the fact that in this paper we will use not only selection functions that receive remainder sets as arguments but also selection functions that receive other kinds of sets of sets of sentences.

Definition 4 ([1, 14]) The partial meet contraction operator on A based on a selection function $\gamma : \{A \perp \epsilon : \epsilon \in \mathcal{L}\} \longrightarrow \mathcal{P}(\mathcal{P}(A))$ is the operator $\div_\gamma$ such that for all sentences α:

$$A \div_\gamma \alpha = \cap\gamma(A \perp \alpha).$$

An operator $\div$ for a set A is a partial meet contraction if and only if there is a selection function γ for A such that $A \div \alpha = A \div_\gamma \alpha$ for all sentences α.

Hansson characterized partial meet contractions defined on belief bases in terms of postulates:

Observation 2 ([14]) *Let A be a belief base. An operator $\div$ on A is a partial meet contraction function for A if and only if $\div$ satisfies success, inclusion, uniformity and relevance.*

2.3.3 Kernel contraction

As we mentioned previously the partial meet contraction operators of a given set by a sentence α are based on a selection among the maximal subsets of that set that fail to imply α. Another different proposal consists of constructing a contraction operator based on a selection of elements of A that are fundamental in some deduction of α and then discarding them when contracting A by α. Following this approach, Hansson, in [18], introduced the *kernel contraction* operators, which can be seen as a generalization of the *safe contraction* defined by Alchourrón and Makinson in [3].[2]

Definition 5 ([18]) Let A be a set of sentences and α be a sentence. Then $A \perp\!\!\!\perp \alpha$ is the set such that $B \in A \perp\!\!\!\perp \alpha$ if and only if:

1. $B \subseteq A$.
2. $B \vdash \alpha$.
3. If $B' \subset B$ then $B' \nvdash \alpha$.

$A \perp\!\!\!\perp \alpha$ is called the kernel set of A with respect to α and its elements are the α-kernels of A.

When contracting a belief α from a set A we must give up sentences of each α-kernel, otherwise α would continue being implied by the outcome of the contraction. The so-called incision functions [18] select the beliefs to be discarded.

[2] For a deep study of safe contraction functions see [22].

Definition 6 ([18]) Let A be a set of sentences. Let $A \perp\!\!\!\perp \alpha$ be the kernel set of A with respect to α. An incision function σ for A is a function such that for all sentences α:

1. $\sigma(A \perp\!\!\!\perp \alpha) \subseteq \bigcup(A \perp\!\!\!\perp \alpha)$.
2. If $\emptyset \neq B \in A \perp\!\!\!\perp \alpha$, then $B \cap \sigma(A \perp\!\!\!\perp \alpha) \neq \emptyset$.

Definition 7 ([18]) Let A be a set of sentences and σ an incision function for A. The kernel contraction on A based on σ is the operator $\div_\sigma$ such that for all sentences α:

$$A \div_\sigma \alpha = A \setminus \sigma(A \perp\!\!\!\perp \alpha).$$

An operator $\div$ for a set A is a kernel contraction if and only if there is an incision function σ for A such that $A \div \alpha = A \div_\sigma \alpha$ for all sentences α.

Hansson presented in [18] an axiomatic characterization for kernel contractions defined on belief bases.

Observation 3 ([18]) *Let A be a belief base. An operator $\div$ on A is a kernel contraction if and only if it satisfies success, inclusion, uniformity and core-retainment.*

Sometimes, when contracting a set by means of a kernel contraction, some beliefs are removed without any apparent reason. For example if $\beta \in A$ and $\beta \in Cn(A \div \alpha)$, then it is natural to expect that β is also in $A \div \alpha$, *i.e.* it is reasonable to require that an operator of kernel contraction satisfies *relative closure*. For this reason, Hansson proposed in [18] a more conservative type of kernel contractions, that he designated by *smooth kernel contractions*. A *smooth kernel contraction* is a kernel contraction that is based on an incision function that satisfies the condition expressed in the following definition.

Definition 8 ([18]) An incision function σ for a set A is smooth if and only if it holds for all subsets A' of A that if $A' \vdash \beta$ and $\beta \in \sigma(A \perp\!\!\!\perp \alpha)$ then $A' \cap \sigma(A \perp\!\!\!\perp \alpha) \neq \emptyset$.
A kernel contraction is smooth if and only if it is based on a smooth incision function.

The following observation presents an axiomatic characterization for smooth kernel contractions.

Observation 4 ([18]) *Let A be a belief base. An operator $\div$ on A is a smooth kernel contraction if and only if it satisfies success, inclusion, uniformity, core-retainment and relative closure.*

2.3.4 Basic AGM-generated base contraction

We now recall the definition of another kind of base contraction functions as well as an axiomatic characterization for them.

Definition 9 ([6]) Let A be a belief base. An operator $\div$ for A is a basic AGM-generated base contraction[3] if and only if, for all $\alpha \in \mathcal{L}$:

$$A \div \alpha = (Cn(A) - \alpha) \cap A$$

where $-$ is a basic AGM contraction (i.e. an operator that satisfies the basic AGM postulates for contraction) on $Cn(A)$.

Observation 5 ([6]) *Let A be a belief base. An operator $\div$ on A is a basic AGM-generated base contraction if and only if it satisfies success, inclusion, vacuity, extensionality and disjunctive elimination.*

In the following observation we expose some interrelations among the different classes of contractions previously mentioned. These interrelations follow trivially from the axiomatic characterizations presented in Observations 2, 3, 4 and 5 and the interrelations among postulates that we recalled in Observation 1.

Observation 6 *Let A be a belief base and $\div$ be a contraction operator on A. Then:*

(a) If $\div$ is an operator of partial meet contraction, then it is an operator of smooth kernel contraction.
(b) If $\div$ is an operator of smooth kernel contraction, then it is an operator of kernel contraction.
(c) If $\div$ is an operator of partial meet contraction, then it is an operator of basic AGM-generated base contraction.

We note that, in general, the converse of the statements presented in the previous observation do not hold. To see this, it is enough to consider the counter-examples presented in [8, Section 6.4]. The counter-examples there presented allow us to conclude also the facts stated in the following observation.

Observation 7 *Let A be a belief base and $\div$ be a contraction operator on A. Then:*

(a) There are operators of basic AGM-generated base contractions that are not kernel base contractions (nor smooth kernel contractions).
(b) There are operators of (smooth) kernel base contractions that are not basic AGM-generated base contractions.

3 Residual Contractions

In this section we will present the definition and axiomatic characterization of a new type of contraction operators on belief bases, called *residual contractions*. We start by introducing the concept of *residuum* which will be the basic

[3] In [6] these operators were designated by basic related-AGM base contractions.

construct underlying the definition of *residual contractions*. Given a set A and a sentence α a residuum of A by α is a subset B of A that does not imply α and whose logical closure, $Cn(B)$, is maximal in the sense that any subset of A whose logical closure strictly contains the logical closure of B, implies α.

Definition 10 Let A be a belief base and α a sentence. The set $A\tilde{\perp}\alpha$ (A residuum α) is the set of sets such that $B \in A\tilde{\perp}\alpha$ if and only if:

1. $B \subseteq A$.
2. $B \nvdash \alpha$.
3. For all sets $B' \subseteq A$ if $Cn(B) \subset Cn(B')$, then $B' \vdash \alpha$.

$A\tilde{\perp}\alpha$ is called residuum set of A by α and its elements are called residuums (of A by α).

It follows immediately from the definition of A residuum α that if $\vdash \alpha$, then $A\tilde{\perp}\alpha = \emptyset$. It follows also that if $A \nvdash \alpha$, then $A \in A\tilde{\perp}\alpha$. However, it is not always the case that if $A \nvdash \alpha$, then $A\tilde{\perp}\alpha = \{A\}$. For example, if $A = \{p, p \vee q\}$, then $A\tilde{\perp}q = \{\{p\}, A\}$.

The following example clarifies the concept of *residuum*.

Example 1 Let $A = \{p, q, p \vee q\}$ and Cn be purely truth-functional. Hence $A \perp (p \wedge q) = \{\{p, p \vee q\}, \{q, p \vee q\}\}$ and $A\tilde{\perp}(p \wedge q) = \{\{p\}, \{p, p \vee q\}, \{q\}, \{q, p \vee q\}\}$.

The previous example shows that not all residuums are remainders. However, the following observation highlights that the converse is true.

Observation 8 *If $X \in A \perp \alpha$, then $X \in A\tilde{\perp}\alpha$.*

The following property, that will be designated by **residuum upper bound property**, follows immediately from the *upper bound property* and Observation 8.

If $X \subseteq A$ and $X \nvdash \alpha$, then there is some X' such that $X \subseteq X' \in A\tilde{\perp}\alpha$.

In the following two observations we present some relations between remainders and residuums.

Observation 9 *If $Y \in A\tilde{\perp}\alpha$, then there is some $X \in A \perp \alpha$ such that $Y \subseteq X$ and $Cn(X) = Cn(Y)$.*

Observation 10 *If $X \in A \perp \alpha$ and $Y \subseteq X$ is such that $Cn(X) = Cn(Y)$, then $Y \in A\tilde{\perp}\alpha$.*

In the following observation we present a result that will be useful in the proof of the representation theorem that we shall present further ahead.

Observation 11 *The following conditions are equivalent:*

1. *$A\tilde{\perp}\alpha = A\tilde{\perp}\beta$;*
2. *For all subsets B of A: $B \vdash \alpha$ if and only if $B \vdash \beta$.*

The following observation exposes that if $X \in A \wr \alpha$, then all subsets of $Cn(X) \cap A$ which contain X are elements of the residuum set of A by α.

Observation 12 *Let $X \in A \wr \alpha$. If $X \subseteq Y \subseteq Cn(X) \cap A$, then $Y \in A \wr \alpha$.*

We are now in conditions to present the definition of *residual contractions*. If $A \nvdash \alpha$, then contracting A by α through a residual contraction leaves the set to be contracted unchanged. Otherwise, the result of the contraction consists of the intersection of a selection among residuums.

Definition 11 The residual contraction on A based on a selection function $\gamma : \{A \wr \epsilon : \epsilon \in \mathcal{L}\} \longrightarrow \mathcal{P}(\mathcal{P}(A))$ is the operator $\div_\gamma$ such that for all sentences α:
(1) if $A \vdash \alpha$, then $A \div_\gamma \alpha = \bigcap \gamma(A \wr \alpha)$ and
(2) if $A \nvdash \alpha$, then $A \div_\gamma \alpha = A$.
An operator $\div$ for a set A is a residual contraction if and only if there is a selection function γ for A such that $A \div \alpha = A \div_\gamma \alpha$ for all sentences α.

At this point we remark that the main difference between residual contractions and partial meet contractions is the fact that the former are defined by means of selection of residuums and the latter make use of a selection of remainders. However, on a more technical note, we must mention that while partial meet contractions can be *simply* defined by $A \div \alpha = \bigcap \gamma(A \perp \alpha)$ (cf. Definition 4), this is not the case in what concerns the definition of residual contractions. In fact, the following example clarifies the necessity of condition (2) in the above definition.[4]

Example 2 Let $A = \{p, p \vee q\}$ and Cn be purely truth-functional. Then $A \wr q = \{\{p, p \vee q\}, \{p\}\}$. Note that if condition (2) were not included in Definition 11, then a possible outcome of contracting A by q through a residual contraction would be $\{p\}$. If this were the case, then the residual contractions would not satisfy the vacuity postulate.

In the following theorem we present an axiomatic characterization for residual contractions.

Theorem 1 *Let A be a belief base. An operator $\div$ on A is a residual contraction function for A if and only if $\div$ satisfies success, inclusion, uniformity, vacuity, failure and*
Weak relevance: *If $\beta \in A \setminus A \div \alpha$, then there exists $X \subseteq A \setminus \{\beta\}$ such that $A \div \alpha \subseteq X$ and $X \nvdash \alpha$ but for any $Y \subseteq A$ such that $Cn(X) \subset Cn(Y)$ it holds that $Y \vdash \alpha$.*

[4] We should note here also that, in [12, p.38], the following alternative definition for partial meet contraction (equivalent to Definition 4) was proposed:
1. If $\nvdash \alpha$, then $A \div_\gamma \alpha = \bigcap \gamma(A \perp \alpha)$ and
2. If $\vdash \alpha$, then $A \div_\gamma \alpha = A$,

where γ is a function such that $\gamma(A \perp \alpha) \subseteq A \perp \alpha$, and if $A \perp \alpha \neq \emptyset$ then $\gamma(A \perp \alpha) \neq \emptyset$.
This alternative definition is similar to the above definition of residual contraction, in the sense that for certain sentences (namely, for tautologies) the outcome of the contraction is defined explicitly (by A) without making use of the selection function γ.

Making use of the concept of residuum, the postulate of *weak relevance* included in the above representation theorem expresses that for every sentence β that is given up in the process of contracting A by α, there exists a residuum of A by α that contains the outcome of that contraction and does not contain β.

The following observation exposes that (as suggested by its designation) the postulate of weak relevance is implied by the (stronger) postulate of *relevance*.

Observation 13 *Let A be a belief base and $\div$ an operator on A. If $\div$ satisfies relevance, then it satisfies weak relevance.*

Having in mind the axiomatic characterizations presented for partial meet and residual contractions (in Observation 2 and Theorem 1, respectively) and Observations 1 and 13 we may conclude that any partial meet contraction is a residual contraction. On the other hand, the following example allows us to conclude that not every residual contraction is a partial meet contraction.

Example 3 Let A and Cn be as stated in Example 1. It follows that if $-$ is a partial meet contraction on A, then the outcome of $A - (p \wedge q)$ must be one of the following sets: $\{p, p \vee q\}, \{q, p \vee q\}$ or $\{p \vee q\}$. On the other hand, a possible outcome of contracting A by $p \wedge q$ through a residual contraction is $\{p\}$. Therefore not every residual contraction is a partial meet contraction.

In the following corollary we state formally the above discussed interrelation among residual contractions and partial meet contractions.

Corollary 1 *The class of partial meet contractions is a strict subclass of the class of residual contractions (i.e., if an operator is a partial meet contraction, then it is also a residual contraction, but the converse does not always hold).*

4 From residual to partial meet contractions

As shown above, partial meet contraction are residual contractions but, in general, the converse does not hold. However, we will show in this section that if an additional requirement is imposed to the selection function on which residual contractions are based on, the class of residual contractions thus obtained coincides with the class of partial meet contractions.

We already established that $A \perp \alpha \subseteq A \wr \alpha$ but, in general, $A \wr \alpha \nsubseteq A \perp \alpha$. The following observation illustrates that the set of maximal elements of $A \wr \alpha$ (in terms of set inclusion) coincides with $A \perp \alpha$.

Observation 14 *Let A be a belief base. Then:*

$$A \perp \alpha = \{X \in A \wr \alpha : \textit{there is no } Y \in A \wr \alpha \textit{ such that } X \subset Y\}.$$

Having the above observation in mind it is natural to expect that residual contractions that are based on selection functions that select only maximal residuums are partial meet contractions.

A selection function will be designated by *maximal*, if it selects only maximal (in terms of set inclusion) residuums, and residual contractions that are based on maximal selection functions will be called *maximal residual contractions*.

Definition 26 A selection function γ for a set A is maximal if and only if the following condition holds:

$$\text{If } X \in \gamma(S), \text{ then there is no } Y \in S \text{ such that } X \subset Y.$$

An operator of residual contraction is maximal if and only if it is based on a maximal selection function.

The following observation states that an operator $\div$ is a maximal residual contraction operator if and only it is a partial meet contraction operator.

Theorem 2 *Let A be a belief base. An operator on A is a maximal residual contraction if and only if it is a partial meet contraction.*

It follows from the above result that partial meet contractions can be seen as residual contractions that are generated by selection functions which satisfy a certain (additional) condition, which ensures more conservative outcomes. We note that this relation between partial meet contractions and residual contractions is similar to the relation between smooth kernel contractions and (general) kernel contractions. Hence, in this sense, we may say that residual contractions are to partial meet contractions as kernel contractions are to smooth kernel contractions.[5]

5 Maps between classes of base contraction functions

In this section we study the interrelations among the classes of base contraction operators mentioned along this paper. We have shown above that partial meet contractions are residual contractions, but that the converse does not hold. We also showed that the classes of partial meet contractions and of maximal residual contractions coincide. Furthermore, at the end of Section 2.3 we saw the existing relations in terms of set inclusion between the classes mentioned in that section. We will now investigate if there is any set inclusion relation between the classes of residual contractions and (smooth) kernel and basic AGM-generated base contraction.

Now we revisit Example 1 this time to illustrate that, in general, residual contractions are not (smooth) kernel contractions.

[5] We are thankful to the reviewer who pointed out this interesting similarity.

Example 4 Let $A = \{p, q, p \vee q\}$ and Cn be purely truth-functional. Hence $A \perp\!\!\!\perp (p \wedge q) = \{\{p, q\}\}$. Therefore if $\div$ is a (smooth) kernel contraction then the possible outcomes of $A \div (p \wedge q)$ are $\{p, p \vee q\}$, $\{q, p \vee q\}$ or $\{p \vee q\}$. On the other hand $A \wr (p \wedge q) = \{\{p\}, \{p, p \vee q\}, \{q\}, \{q, p \vee q\}\}$. Thus $\{p\}$ is a possible outcome of contracting A by $p \wedge q$ by means of a residual contraction. Therefore not every residual contraction is a (smooth) kernel contraction.

The following example illustrates that, in general, (smooth) kernel contractions are not residual contractions.

Example 5 Let $A = \{p, q, p \vee q, p \rightarrow q\}$ and Cn be purely truth-functional. Hence $A \perp\!\!\!\perp q = \{\{q\}, \{p, p \rightarrow q\}, \{p \vee q, p \rightarrow q\}\}$. Therefore a possible outcome of contracting A by q by means of a (smooth) kernel contraction is $\{p \vee q\}$. On the other hand $A \wr q = \{\{p\}, \{p, p \vee q\}, \{p \rightarrow q\}\}$. Thus $\{p \vee q\}$ is not a possible outcome of contracting A by q by means of a residual contraction. Therefore not every (smooth) kernel contraction is a residual contraction.

The following example illustrates that the classes of basic AGM-generated base contractions and of residual contractions are not related by means of set inclusion.

Example 6 Consider a language that consists of p and q, and their truth-functional combinations. Let $A = \{p, q, p \vee q\}$ and Cn be purely truth-functional. Hence $Cn(A) = Cn(\{p, q\})$. It holds that $Cn(A) \perp p = \{Cn(q), Cn(p \leftrightarrow q)\}$. If $\div$ is a partial meet contraction on $Cn(A)$ then the possible outcomes of $Cn(A) \div p$ are: $Cn(q), Cn(p \leftrightarrow q)$ or $Cn(\neg p \vee q)$. Every partial meet contraction on belief sets is a basic AGM contraction ([1]). Therefore the possible outcomes of contracting A by p by means of a basic AGM-generated base contraction are: $\{q, p \vee q\}$ or $\emptyset$. On the other hand $A \wr p = \{\{q\}, \{q, p \vee q\}\}$. Therefore the possible outcomes of performing a contraction of A by p through a residual contraction are $\{q\}$ or $\{q, p \vee q\}$. Therefore not every basic AGM-generated base contraction is a residual contraction nor every residual contraction is a basic AGM-generated base contraction.

In Figure 1 we present a diagram that summarizes the logical relationships between the operators of base contraction mentioned along this paper. The relationships in terms of set inclusion among the classes of operators represented in this diagram are exactly those indicated by arrows (and their transitive closure).

6 Conclusion and Discussion

We have proposed a new operator of belief base change which generalizes the partial meet contractions. While partial meet contractions are essentially defined as an intersection of some *remainders*—maximal subsets of the original belief base which do not imply the sentence to be removed—, a residual contraction is obtained as an intersection of some more general constructs, namely

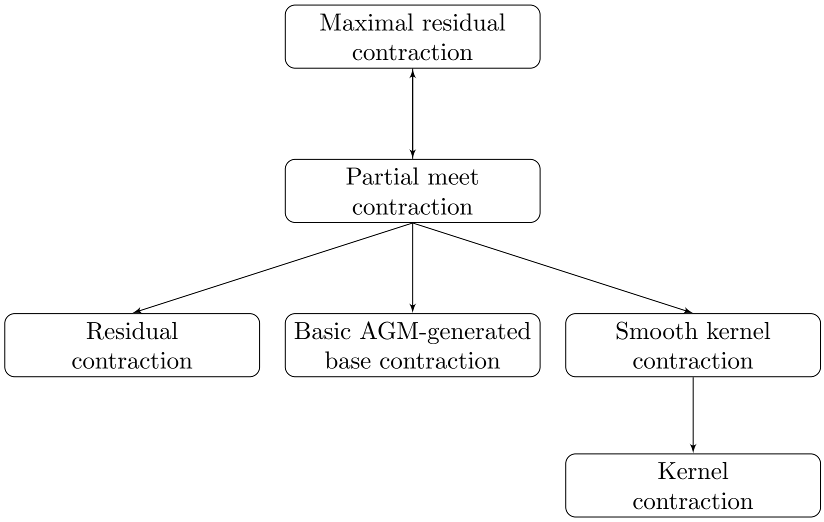

Fig. 1 Logical relationships between different operations of base contraction.

residuums, which are subsets of the original belief base A that do not imply the sentence to be removed and whose closure is maximal among the closures of the subsets of A with that property.

By using residuums, any two subsets of A which have the same consequences are treated equally in the sense that one of them can be (used in the process of obtaining) the result for the operation of contraction if and only if the other one has that same potential. This is not the case when considering remainders, since a set with exactly the same closure of a certain remainder may (itself) not be a remainder. Indeed, for instance, in the scenario described in Example 3, $A = \{p, q, p \vee q\}$ and the possible outcomes of a residual contraction of A by $p \wedge q$ are $\{p\}, \{q\}, \{p, p \vee q\}, \{q, p \vee q\}$, and $\{p \vee q\}$. On the other hand, only the latter three sets are possible outcomes of a partial meet contraction of A by $p \wedge q$. Hence, in that situation, for example, the sets $\{p\}$ and $\{p, p \vee q\}$ are not given the same importance in the context of partial meet contraction (despite the fact that those two set have exactly the same logical closure).

In [20], Levi argues that "measures of informational value ought to be carefully distinguished from measures of information". In the mentioned reference Levi defends that in a belief change process it is the loss of informational value that should be minimized rather than the loss of information. Levi's point is that not all information is valuable for the agent and, therefore, some pieces of information can be given up when moving from one belief state to another.

For the above presented discussion we may say that residual contractions address Levi's recommendations in the sense that these operators are more sensible to the informational value than to the information and, furthermore capture the notion of informational value better than partial meet contractions do. Indeed, for example the sets $\{p\}$ and $\{p, p \vee q\}$ can be seen as having the same *informational value*, although the *information* $p \vee q$ is only (explicitly) contained in the latter of them. Therefore, the fact that these two sets can be possible outcomes of one same residual contraction operation (as it is the case in the example mentioned in the second paragraph above), asserts that in a process of residual contraction it is possible to give up an (explicit) information without loosing *informational value*.

We have additionally shown that the class of residual contractions is different from all the main classes of belief base contraction operators so far presented in the literature.

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Appendix: Proofs

Lemma 1 *[19] If $A \subseteq B \subseteq Cn(A)$, then $Cn(A) = Cn(B)$.*

Lemma 2 ([2, Observation 2.2]) *Let A be a set of sentences and α a sentence. Then, $A \perp \alpha = \emptyset$ if and only if $\vdash \alpha$ (provided that the consequence operation Cn is compact).*

Lemma 3 ([19, Observation 1.39])

Let A be a set of sentences and α and β be sentences. Then the two following conditions are equivalent:

1. *$A \perp \alpha = A \perp \beta$;*
2. *For all subsets D of A: $D \vdash \alpha$ if and only if $D \vdash \beta$.*

Lemma 4 *Let A be a belief base. $A \perp \alpha = \emptyset$ if and only if $A \wr \alpha = \emptyset$.*

Proof (Right to left) Follows trivially by Observation 8.[6]
(Left to right) Let $A \perp \alpha = \emptyset$. Hence, by Lemma 2, $\vdash \alpha$, from which it follows that $A \wr \alpha = \emptyset$.

Proof (Proof of Observation 8) Let $X \in A \perp \alpha$. Hence $X \subseteq A$ and $X \nvdash \alpha$. Let $X' \subseteq A$ be such that $Cn(X) \subset Cn(X')$. We intend to prove that $X' \vdash \alpha$. From $Cn(X) \subset Cn(X')$ it follows that there exists $\delta \in X' \setminus X$. Let $Y = Cn(X') \cap A$. It holds that $X \subset X \cup \{\delta\} \subseteq Y \subseteq A$. Thus $X \subset Y \subseteq A$. Therefore $Y \vdash \alpha$ (since $X \in A \perp \alpha$). It holds that $Y \subseteq Cn(X')$. Hence $X' \vdash \alpha$.

[6] This proof uses Observation 8 whose proof is presented immediately after this lemma. However, this is not an issue because the result that is proven here is not used in the proof of that observation.

Proof (Proof of Observation 9) Let $Y \in A \wr\!\!\perp \alpha$. Hence $Y \subseteq A$ and $Y \nvdash \alpha$. By the *upper bound property* it follows that there exists X such that $Y \subseteq X \in A \perp \alpha$. From $Y \subseteq X$ it follows, by monotony of Cn, that $Cn(Y) \subseteq Cn(X)$. It holds that $X \nvdash \alpha$ and $X \subseteq A$ (since $X \in A \perp \alpha$). Hence $Cn(Y) \subset Cn(X)$ does not hold since $Y \in A \wr\!\!\perp \alpha$. Therefore $Cn(Y) = Cn(X)$.

Proof (Proof of Observation 10) Let $X \in A \perp \alpha$ and $Y \subseteq X$ be such that $Cn(X) = Cn(Y)$. From $X \in A \perp \alpha$ it holds that $X \nvdash \alpha$ and $X \subseteq A$. Hence $Y \nvdash \alpha$ and $Y \subseteq A$. It remains to show that Y satisfies condition (3) of Definition 10. Let $Z \subseteq A$ be such that $Cn(Y) \subset Cn(Z)$. Hence $Cn(X) \subset Cn(Z)$. Therefore there exists $\delta \in Z \setminus X$. Let $W = Cn(Z) \cap A$. Hence $X \subset W$. Therefore $W \vdash \alpha$ (since $X \in A \perp \alpha$). Thus $Cn(Z) \vdash \alpha$. Thus, by iteration of Cn, $Z \vdash \alpha$.

Proof (Proof of Observation 11) (2) implies (1): Let $X \in A \wr\!\!\perp \alpha$. It follows immediately from Definition 10 and (2) that $X \in A \wr\!\!\perp \beta$.
(1) implies (2): Suppose that (2) does not hold. Assume, without loss of generality, that there is some subset X of A such that $X \nvdash \alpha$ and $X \vdash \beta$. By the *residuum upper bound property* it follows that there is some set X' such that $X \subseteq X' \in A \wr\!\!\perp \alpha$. On the other hand, from $X \subseteq X'$ it follows that $X' \vdash \beta$. Hence $X' \notin A \wr\!\!\perp \beta$. This contradicts (1).

Proof (Proof of Observation 12) Let $X \in A \wr\!\!\perp \alpha$ and Y be such that $X \subseteq Y \subseteq Cn(X) \cap A$. Hence $Y \subseteq A$ and $X \subseteq Y \subseteq Cn(X)$. It follows from Lemma 1, that $Cn(X) = Cn(Y)$. Therefore, $Y \nvdash \alpha$ and for any subset Z of A if $Cn(Y) \subset Cn(Z)$, then $Z \vdash \alpha$. Therefore $Y \in A \wr\!\!\perp \alpha$.

Proof (Proof of Theorem 1) (Construction to postulates)
Let A be a belief base and $\div$ be a residual contraction operator on A. Hence there is a selection function γ for A such that for all sentences α:

1. if $A \vdash \alpha$, then $A \div \alpha = \bigcap \gamma(A \wr\!\!\perp \alpha)$ and
2. if $A \nvdash \alpha$, then $A \div \alpha = A$.

We will show that $\div$ satisfies *success, inclusion, uniformity, vacuity, failure* and *weak relevance.*
Success: Let $\nvdash \alpha$. Hence, according to Lemmas 2 and 4, $A \wr\!\!\perp \alpha \neq \emptyset$. Therefore, by definition of a selection function, $\gamma(A \wr\!\!\perp \alpha)$ is a non-empty subset of $A \wr\!\!\perp \alpha$. If $A \nvdash \alpha$, then $A \div \alpha = A$, therefore $A \div \alpha \nvdash \alpha$. If $A \vdash \alpha$, then $A \div \alpha = \bigcap \gamma(A \wr\!\!\perp \alpha)$, therefore $A \div \alpha \nvdash \alpha$ (since it holds, for any element X of $A \wr\!\!\perp \alpha$, that $X \nvdash \alpha$).
Inclusion: Let $\alpha \in \mathcal{L}$. It follows trivially by the definition of $\div$, if $A \nvdash \alpha$. Assume now that $A \vdash \alpha$. We will consider two cases:
Case 1) $\vdash \alpha$. Hence, according to Lemmas 2 and 4, $A \wr\!\!\perp \alpha = \emptyset$. Therefore, $\gamma(A \wr\!\!\perp \alpha) = \{A\}$. Hence $A \div \alpha = A$.
Case 2) $\nvdash \alpha$. Hence, according to Lemmas 2 and 4, $A \wr\!\!\perp \alpha \neq \emptyset$. Thus $\gamma(A \wr\!\!\perp \alpha)$ is a non-empty subset of $A \wr\!\!\perp \alpha$. Any element of $A \wr\!\!\perp \alpha$ is a subset of A, hence any element of $\gamma(A \wr\!\!\perp \alpha)$ is also a subset of A. Therefore, $A \div \alpha = \bigcap \gamma(A \wr\!\!\perp \alpha) \subseteq A$.
Uniformity: Let α, β be two sentences such that it holds for all subsets A' of

A that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$. It follows from Observation 11 that $A \perp \alpha = A \perp \beta$. Thus, $\bigcap \gamma(A \perp \alpha) = \bigcap \gamma(A \perp \beta)$. We will consider two cases:
Case 1) $A \nvdash \alpha$. Hence, by hypothesis, $A \nvdash \beta$. Therefore, $A \div \alpha = A = A \div \beta$.
Case 2) $A \vdash \alpha$. Hence, by hypothesis, $A \vdash \beta$. Therefore, $A \div \alpha = \bigcap \gamma(A \perp \alpha) = \bigcap \gamma(A \perp \beta) = A \div \beta$.
Vacuity: Follows trivially by the definition of $\div$.
Failure: Let $\vdash \alpha$. Hence, according to Lemmas 2 and 4, $A \perp \alpha = \emptyset$. Therefore $\gamma(A \perp \alpha) = \{A\}$. Thus $A \div \alpha = \bigcap(\gamma(A \perp \alpha)) = A$.
Weak relevance: Let β be such that $\beta \in A \setminus A \div \alpha$. Therefore, $A \div \alpha \neq A$. Hence it holds that $A \vdash \alpha$ and consequently that $A \div \alpha = \bigcap \gamma(A \perp \alpha)$. It also holds that $A \perp \alpha \neq \emptyset$. From $\beta \notin A \div \alpha$, it follows that there exists $X \in \gamma(A \perp \alpha)$ such that $\beta \notin X$. Therefore $A \div \alpha \subseteq X$. On the other hand, $X \subseteq A$ (since $X \in A \perp \alpha$) and, since $\beta \notin X$, it follows that $X \subseteq A \setminus \{\beta\}$. Furthermore, from $X \in A \perp \alpha$ it follows that $X \nvdash \alpha$ and for any subset Y of A such that $Cn(X) \subset Cn(Y)$ it holds that $Y \vdash \alpha$.

(Postulates to construction)
Let $\div$ be an operator on A that satisfies *success, inclusion, uniformity, vacuity, failure* and *weak relevance*. Let γ be such that:

(i) $\gamma(\emptyset) = \{A\}$;
(ii) If $A \perp \alpha \neq \emptyset$, then $\gamma(A \perp \alpha) = \{X \in A \perp \alpha : A \div \alpha \subseteq X\}$.

We need to show that: (1) γ is a (well-defined) function; (2) γ is a selection function; (3) for all α
(3.1) if $A \vdash \alpha$, then $A \div \alpha = \bigcap \gamma(A \perp \alpha)$ and
(3.2) if $A \nvdash \alpha$, then $A \div \alpha = A$.
(1) We must prove that for all α, β if $A \perp \alpha = A \perp \beta$ then $\gamma(A \perp \alpha) = \gamma(A \perp \beta)$.
Suppose that $A \perp \alpha = A \perp \beta$. It follows trivially if $A \perp \alpha = \emptyset$. Assume now that $A \perp \alpha \neq \emptyset$. By Observation 11 it follows that for all subsets B of A: $B \vdash \alpha$ if and only if $B \vdash \beta$. Hence, *uniformity* yields that $A \div \alpha = A \div \beta$. Hence, by definition of γ, it holds that $\gamma(A \perp \alpha) = \gamma(A \perp \beta)$.
(2) By definition $\gamma(\emptyset) = \{A\}$. Hence, in order to prove that γ is a selection function it is sufficient to show that if $A \perp \alpha \neq \emptyset$, then $\emptyset \neq \gamma(A \perp \alpha) \subseteq A \perp \alpha$. That $\gamma(A \perp \alpha) \subseteq A \perp \alpha$ follows trivially by the definition of γ. It remains to prove that $\gamma(A \perp \alpha) \neq \emptyset$. Since $A \perp \alpha \neq \emptyset$ it must be the case that $\nvdash \alpha$. Thus by *success*, $A \div \alpha \nvdash \alpha$. It follows by *inclusion* that $A \div \alpha \subseteq A$. By the *residuum upper bound property* there is some X such that $A \div \alpha \subseteq X \in A \perp \alpha$. Therefore, according to the definition of γ, $X \in \gamma(A \perp \alpha)$. Thus $\gamma(A \perp \alpha) \neq \emptyset$.
(3) Let α be an arbitrary sentence. We will prove that (3.1) and (3.2) hold.
(3.1) $A \vdash \alpha$. We will consider two cases:
Case 1: $\vdash \alpha$. Thus $A \perp \alpha = \emptyset$. By definition of γ it follows that $\gamma(A \perp \alpha) = \{A\}$. Hence $\bigcap \gamma(A \perp \alpha) = A$. On the other hand, by *failure* $A \div \alpha = A$. Therefore $\bigcap \gamma(A \perp \alpha) = A \div \alpha$.
Case 2: $\nvdash \alpha$. Thus $A \perp \alpha \neq \emptyset$. Therefore, as shown in (2), $\gamma(A \perp \alpha) \neq \emptyset$. It follows from the definition of γ that $A \div \alpha$ is a subset of every element of $\gamma(A \perp \alpha)$. Thus $A \div \alpha \subseteq \bigcap \gamma(A \perp \alpha)$. It remains to show that $\bigcap \gamma(A \perp \alpha) \subseteq A \div \alpha$. Let

$\beta \notin A \div \alpha$. We will show that $\beta \notin \bigcap \gamma(A \wr \alpha)$. This is obvious if $\beta \notin A$. Assume now that $\beta \in A$. Hence $\beta \in A \setminus A \div \alpha$. It follows by *weak relevance* that there exists $X \subseteq A \setminus \{\beta\}$ such that $A \div \alpha \subseteq X$ and $X \nvdash \alpha$ but for all $Y \subseteq A$ such that $Cn(X) \subset Cn(Y)$ it holds that $Y \vdash \alpha$. Hence $A \div \alpha \subseteq X \in A \wr \alpha$. Therefore, by definition of γ, it holds that $X \in \gamma(A \wr \alpha)$. From $\beta \notin X$ it follows that $\beta \notin \bigcap \gamma(A \wr \alpha)$.
(3.2) $A \nvdash \alpha$. Then by *vacuity* and *inclusion* it follows that $A \div \alpha = A$.

Proof (Proof of Observation 13) A be a belief base and $\div$ an operator on A that satisfies *relevance*. Let $\beta \in A \setminus A \div \alpha$. By *relevance* there exists X such that $A \div \alpha \subseteq X \subseteq A$, $X \nvdash \alpha$ but $X \cup \{\beta\} \vdash \alpha$. By the *upper bound property* there exists $X \subseteq X'$ such that $X' \in A \perp \alpha$. It must hold that $\beta \notin X'$ (otherwise it would follow that $X' \vdash \alpha$). On the other hand, by Observation 8, $X' \in A \wr \alpha$. Therefore $X' \subseteq A \setminus \{\beta\}$, $A \div \alpha \subseteq X'$, $X' \nvdash \alpha$ and for any $Y \subseteq A$ such that $Cn(X') \subset Cn(Y)$ it holds that $Y \vdash \alpha$.

Proof (Proof of Observation 14) Let $X \in A \perp \alpha$. By Observation 8 it follows that $X \in A \wr \alpha$. Assume now, by *reductio ad absurdum*, that there exists some $Y \in A \wr \alpha$ such that $X \subset Y$. From $Y \in A \wr \alpha$ it follows that $Y \subseteq A$ and $Y \nvdash \alpha$. Hence, from $Y \nvdash \alpha$ and $X \subset Y \subseteq A$, it follows (by condition (3) of Definition 2) that $X \notin A \perp \alpha$. Contradiction. Hence $A \perp \alpha \subseteq \{X \in A \wr \alpha : \nexists Y \in A \wr \_\alpha$ such that $X \subset Y\}$.
Let $X' \in \Gamma = \{X \in A \wr \alpha : \nexists Y \in A \wr \alpha$ such that $X \subset Y\}$. Hence $X' \in A \wr \alpha$ and there exists no Y' in $A \wr \alpha$ such that $X' \subset Y'$. It follows from $X' \in A \wr \alpha$ that $X' \subseteq A$ and $X' \nvdash \alpha$. Let $Z \subseteq A$ be such that $X' \subset Z$. From $X' \in \Gamma$ it follows that $Z \notin A \wr \alpha$. Assume by *reductio ad absurdum* that $Z \nvdash \alpha$. Hence, by condition (3) of Definition 10, there exists $Z' \subseteq A$ such that $Cn(Z) \subset Cn(Z')$ and $Z' \nvdash \alpha$. From $X' \subset Z$ it follows that $Cn(X') \subseteq Cn(Z) \subset Cn(Z')$ and $Z' \nvdash \alpha$. Hence $X' \notin A \wr \alpha$. Contradiction. Therefore $Z \vdash \alpha$, and we can conclude that $X' \in A \perp \alpha$. Therefore $\{X \in A \wr \alpha : \nexists Y \in A \wr \alpha$ such that $X \subset Y\} \subseteq A \perp \alpha$.

Proof (Proof of Theorem 2) Let A be a belief base. We start by showing that every maximal residual contraction on A is a partial meet contraction on A. Let $\gamma_m : \{A \wr \epsilon : \epsilon \in \mathcal{L}\} \longrightarrow \mathcal{P}(\mathcal{P}(A))$ be a maximal selection function and let $\div_{\gamma_m}$ be the maximal residual contraction based on γ_m. Now let $\gamma : \{A \perp \epsilon : \epsilon \in \mathcal{L}\} \longrightarrow \mathcal{P}(\mathcal{P}(A))$ be such that (for all $\alpha \in \mathcal{L}$):

$$\gamma(A \perp \alpha) = \gamma_m(A \wr \alpha)$$

We will show that: (1) γ is a (well-defined) function; (2) γ is a selection function; (3) For all α, $A \div_{\gamma_m} \alpha = A -_\gamma \alpha$, where $-_\gamma$ is the partial meet contraction based on γ.
(1) Let $A \perp \alpha = A \perp \beta$. Combining Lemma 3 and Observation 11 we can conclude that $A \wr \alpha = A \wr \beta$. Hence it follows from the definition of γ and from the fact that γ_m is a function, that $\gamma(A \perp \alpha) = \gamma(A \perp \beta)$.
(2) Since if $\vdash \alpha$, then $A \wr \alpha = A \perp \alpha = \emptyset$, it follows from the definition of γ

that $\gamma(\emptyset) = \gamma_m(\emptyset) = \{A\}$ (since γ_m is a selection function). It remains to show that if $A \perp \alpha \neq \emptyset$, then $\emptyset \neq \gamma(A \perp \alpha) \subseteq A \perp \alpha$. Assume that $A \perp \alpha \neq \emptyset$, then $\gamma(A \perp \alpha) = \gamma_m(A \wr\!\!\!\perp \alpha)$ and, according to Lemma 4, $A \wr\!\!\!\perp \alpha \neq \emptyset$. Since γ_m is a selection function, it holds that $\emptyset \neq \gamma_m(A \wr\!\!\!\perp \alpha) \subseteq A \wr\!\!\!\perp \alpha$. Furthermore, since γ_m is maximal, if $X \in \gamma_m(A \wr\!\!\!\perp \alpha)$ then there is no $Y \in A \wr\!\!\!\perp \alpha$ such that $X \subset Y$. Therefore it follows by Observation 14 that $\gamma_m(A \wr\!\!\!\perp \alpha) \subseteq A \perp \alpha$. Therefore $\emptyset \neq \gamma(A \perp \alpha) \subseteq A \perp \alpha$.
(3) If $A \nvdash \alpha$, then $A \div_{\gamma_m} \alpha = A = A -_{\gamma} \alpha$. Now assume that $A \vdash \alpha$. Then $A \div_{\gamma_m} \alpha = \bigcap \gamma_m(A \wr\!\!\!\perp \alpha) = \bigcap \gamma(A \perp \alpha) = A -_{\gamma} \alpha$.

Now we show that every partial meet contraction on A is a maximal residual contraction on A.
Let $\gamma : \{A \perp \epsilon : \epsilon \in \mathcal{L}\} \longrightarrow \mathcal{P}(\mathcal{P}(A))$ be a selection function and let $\div_{\gamma}$ be the partial meet contraction based on γ. Now let $\gamma_m : \{A \wr\!\!\!\perp \epsilon : \epsilon \in \mathcal{L}\} \longrightarrow \mathcal{P}(\mathcal{P}(A))$ be such that (for all $\alpha \in \mathcal{L}$):

$$\gamma_m(A \wr\!\!\!\perp \alpha) = \gamma(A \perp \alpha)$$

We will show that: (1) γ_m is a (well-defined) function; (2) γ_m is a maximal selection function; (3) For all α, $A \div_{\gamma} \alpha = A -_{\gamma_m} \alpha$, where $-_{\gamma_m}$ is the maximal residual contraction based on γ_m.
(1) Let $A \wr\!\!\!\perp \alpha = A \wr\!\!\!\perp \beta$. Combining Lemma 3 and Observation 11 we can conclude that $A \perp \alpha = A \perp \beta$. Hence it follows from the definition of γ_m and from the fact that γ is a function, that $\gamma_m(A \wr\!\!\!\perp \alpha) = \gamma_m(A \wr\!\!\!\perp \beta)$.
(2) Since if $\vdash \alpha$, then $A \wr\!\!\!\perp \alpha = A \perp \alpha = \emptyset$, it follows, from the definition of γ, that $\gamma(\emptyset) = \gamma_m(\emptyset) = \{A\}$ (since γ is a selection function). It remains to show that if $A \wr\!\!\!\perp \alpha \neq \emptyset$, then:
(a) $\emptyset \neq \gamma_m(A \wr\!\!\!\perp \alpha) \subseteq A \wr\!\!\!\perp \alpha$;
(b) If $X \in \gamma_m(A \wr\!\!\!\perp \alpha)$, then there is no $Y \in A \wr\!\!\!\perp \alpha$ such that $X \subset Y$.
Assume that $A \wr\!\!\!\perp \alpha \neq \emptyset$. Then, according to Lemma 4, $A \perp \alpha \neq \emptyset$. Therefore, since γ is a selection function, it holds that $\emptyset \neq \gamma(A \perp \alpha) \subseteq A \perp \alpha$. By Observation 8 $A \perp \alpha \subseteq A \wr\!\!\!\perp \alpha$. Hence, since $\gamma_m(A \wr\!\!\!\perp \alpha) = \gamma(A \perp \alpha)$, we can conclude that (a) holds.
Next we prove (b). Let $X \in \gamma_m(A \wr\!\!\!\perp \alpha)$. Since $\gamma_m(A \wr\!\!\!\perp \alpha) = \gamma(A \perp \alpha)$ it holds that $X \in A \perp \alpha$. Therefore, it follows from Observation 14 that there is no $Y \in A \wr\!\!\!\perp \alpha$ such that $X \subset Y$. Thus (b) holds.
(3) If $A \nvdash \alpha$, then $A \div_{\gamma} \alpha = A = A -_{\gamma_m} \alpha$. Now assume that $A \vdash \alpha$. Then $A \div_{\gamma} \alpha = \bigcap \gamma(A \perp \alpha) = \bigcap \gamma_m(A \wr\!\!\!\perp \alpha) = A -_{\gamma_m} \alpha$.

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