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Ball Comparison between Three Sixth Order Methods for Banach Space Valued Operators

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Abstract: Three methods of sixth order convergence are tackled for approximating the solution of an equation defined on the finitely dimensional Euclidean space. This convergence requires the existence of derivatives of, at least, order seven. However, only derivatives of order one are involved in such methods. Moreover, we have no estimates on the error distances, conclusions about the uniqueness of the solution in any domain, and the convergence domain is not sufficiently large. Hence, these methods have limited usage. This paper introduces a new technique on a general Banach space setting based only the first derivative and Lipschitz type conditions that allow the study of the convergence. In addition, we find usable error distances as well as uniqueness of the solution. A comparison between the convergence balls of three methods, not possible to drive with the previous approaches, is also given. The technique is possible to use with methods available in literature improving, consequently, their applicability. Several numerical examples compare these methods and illustrate the convergence criteria.

Keywords: banach space; local convergence; system of nonlinear equations; iterative methods

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1. Introduction

Let $F : \Omega \subset \mathbb{X} \to \mathbb{Y}$ be Fréchet differentiable operator, $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces and $\Omega \subset \mathbb{X}$ be open, convex, and non-void. To solve $F(x) = 0$, we study the local convergence of the following three step methods defined for $\sigma = 0, 1, 2, \dots$ as

$$\begin{aligned} y_\sigma &= x_\sigma - \frac{2}{3}F'(x_\sigma)^{-1}F(x_\sigma) \\ z_\sigma &= x_\sigma - \frac{1}{2}\Big[I + 2F'(x_\sigma)\Big(3F'(y_\sigma) - F'(x_\sigma)\Big)^{-1}\Big]F'(x_\sigma)^{-1}F(x_\sigma) \\ x_{\sigma+1} &= z_\sigma - 2\Big(3F'(y_\sigma) - F'(x_\sigma)^{-1}\Big)F(z_\sigma), \end{aligned} \tag{1}$$

$$\begin{aligned} y_\sigma &= x_\sigma - \frac{2}{3}F'(x_\sigma)^{-1}F(x_\sigma) \\ z_\sigma &= x_\sigma - \frac{1}{2}\Big[I + 2F'(x_\sigma)\Big(3F'(y_\sigma) - F'(x_\sigma)\Big)^{-1}\Big]F'(x_\sigma)^{-1}F(x_\sigma) \\ x_{\sigma+1} &= z_\sigma - \frac{1}{4}\Big[I + 2F'(x_\sigma)\Big(3F'(y_\sigma) - F'(x_\sigma)\Big)^{-1}\Big]^2F'(x_\sigma)^{-1}F(z_\sigma), \end{aligned} \tag{2}$$

and

$$\begin{aligned} y_\sigma &= x_\sigma - F'(x_\sigma)^{-1}F(x_\sigma) \\ z_\sigma &= y_\sigma + \frac{1}{3}\Big[F'(x_\sigma)^{-1} + 2\Big(F'(x_\sigma) - 3F'(y_\sigma)\Big)^{-1}\Big]F(x_\sigma) \\ x_{\sigma+1} &= z_\sigma + \frac{1}{3}\Big[4\Big(F'(x_\sigma) - 3F'(y_\sigma)\Big)^{-1} - F'(x_\sigma)^{-1}\Big]F(z_\sigma). \end{aligned} \tag{3}$$

The application of $F(x) = 0$ is mentioned in the standard books [1–4]. The definition of the Fréchet derivative can be found for example in [5]. These methods use two operators, two Fréchet derivative evaluations, and two linear operator inversions. The sixth convergence order of methods was given in Cordero et al. [6], Soleymani et al. [7], and Esmaeili and Ahmadi [8], respectively. The conclusions were obtained for the special case when $\mathbb{X} = \mathbb{Y} = \mathbb{R}^i$, using Taylor series with hypotheses up to the seventh derivative even though it does not appear in the methods. Thus, these hypotheses restrict the applicability of the methods. Let us consider a motivational example. We assume the following function F on $\mathbb{X} = \mathbb{Y} = \mathbb{R}$ and $\mathbb{D} = [-\frac{1}{2}, \frac{3}{2}]$ such as:

$$F(\kappa) = \begin{cases} \kappa^3 \ln \kappa^2 + \kappa^5 - \kappa^4, & \kappa \neq 0 \\ 0, & \kappa = 0 \end{cases}, \tag{4}$$

which leads to

$$F'(\kappa) = 3\kappa^2 \ln \kappa^2 + 5\kappa^4 - 4\kappa^3 + 2\kappa^2,$$

$$F''(\kappa) = 6\kappa \ln \kappa^2 + 20\kappa^3 - 12\kappa^2 + 10\kappa,$$

$$F'''(\kappa) = 6 \ln \kappa^2 + 60\kappa^2 - 12\kappa + 22.$$

We note that $F'''(\kappa)$ is not bounded in $\mathbb{D}$. Therefore, results requiring the existence of $F''(\kappa)$ or higher cannot be applied for studying the convergence of Equations (1)–(3). Moreover, no computable error bounds $\|x_\sigma - x_*\|$, where x_* solves the equation $F(x) = 0$, or any information regarding the uniqueness of the solution are provided using Lipschitz-type functions. Similar types of problems can be found in [9–15]. Furthermore, the convergence criteria can not be compared, since they are based on different hypotheses. We address all these problems by using only the first derivative. Moreover, we rely on the computational order of convergence (COC) or approximated computational order of convergence $(ACOC)$ [16–18] to determine the c-order (Computational order of convergence) not requiring derivatives of order higher than one. The new technique uses the same set of conditions for the three methods. Furthermore, it can also be used to extend the applicability of other methods along the same lines.

Local convergence results are important because they demonstrate the degree of difficulty in choosing initial points within the so-called convergence ball that is in the region from which we can pick the initial points ensuring the convergence of the iterative method. In general, the convergence ball is small and, furthermore, decreases when the convergence order of the method increases. Therefore, it is very important to extend the radius of the convergence ball, but without imposing additional hypotheses that may limit the applicability of the method.

This is the main motivation for this paper that accomplishes this objective under weaker hypotheses than previous methods. It must be noted that the number of required iterations to achieve a certain error tolerance is a distinct issue. This information is also provided, as well as the uniqueness of the solution that are not clearly addressed in previous works. In fact, when applying the previous methods, we do not have sufficient information for establishing an educated guess about the convergence ball from where the initial choice point must be picked. Therefore, with those methods, the initial point may, or may not, result in convergence toward the results.

The rest of the paper includes the following sections. Section 2 analyzes the local convergence of the proposed technique. Section 3 discusses several numerical experiments. Section 4 presents the concluding results.

2. Local Convergence

Let us introduce some real functions and parameters to be used later as follows in the local convergence analysis.

Suppose that equation

$$w_0(\zeta) = 1 \tag{5}$$

has a minimal positive solution ρ_0, where $w_0 : I \to I$ is continuous, increasing, with $w_0(0) = 0$, and $I = [0, \infty)$. Consider functions $w : I_0 \to I,\ v : I_0 \to I$ to be continuous, increasing, with $w(0) = 0$, and $I_0 = [0, \rho_0)$.

Suppose that

$$\frac{v(0)}{3} - 1 < 0. \tag{6}$$

Define functions g_1 and h_1 on I_0 as follows:

$$\begin{aligned} g_1(\zeta) &= \frac{\int_0^1 w\Big((1-\theta)\zeta\Big)d\theta + \frac{1}{3}\int_0^1 v(\theta\zeta)d\theta}{1 - w_0(\zeta)}, \\ h_1(\zeta) &= g_1(\zeta) - 1. \end{aligned}$$

By (6) and these definitions, we have $h_1(0) = \frac{v(0)}{3} - 1 < 0$ and $h_1(\zeta) \to \infty$ as $t \to \rho_0^-$. Denote by r_1 the minimal solution of equation $h_1(\zeta) = 0$ in the interval $(0, \rho_0)$ with assured existence by the intermediate value theorem.

Suppose that the equation

$$p(\zeta) = 1 \tag{7}$$

has a minimal positive solution ρ_p, where

$$p(\zeta) = \frac{1}{2}\Big[3w_0\Big(g_1(\zeta)\zeta\Big) + w_0(\zeta)\Big].$$

Set $I_1 = [0, \rho_1)$, where $\rho_1 := \min\{\rho_0, \rho_p\}$. Define functions g_2 and h_2 on the interval I_1 by

$$\begin{aligned} g_2(\zeta) &= \frac{\int_0^1 w\Big((1-\theta)\zeta\Big)d\theta}{1 - w_0(\zeta)} + \frac{3}{4}\frac{\Big[w_0\Big(g_1(\zeta)\zeta\Big) + w_0(\zeta)\Big]\int_0^1 v(\theta\zeta)d\theta}{\Big(1 - p(\zeta)\Big)\Big(1 - w_0(\zeta)\Big)}, \\ h_2(\zeta) &= g_2(\zeta) - 1. \end{aligned}$$

We get again $h_2(0) = -1$ and $h_2(\zeta) \to \infty$ as $\zeta \to \rho_1^-$. Denote by r_2 the smallest solution of equation $h_2(\zeta) = 0$ in the interval $(0, \rho_1)$.

Suppose that equation

$$w_0\Big(g_2(\zeta)\zeta\Big) = 1 \tag{8}$$

has a minimal positive solution ρ_2.

Set $I_2 := [0, \rho)$, where $\rho = \min\{\rho_1, \rho_2\}$. Next, define functions g_3 and h_3 on the interval I_2 by

$$\begin{aligned} g_3(\zeta) &= \left[\frac{\int_0^1 w\Big((1-\theta)g_2(\zeta)\zeta\Big)d\theta}{1 - w_0\Big(g_2(\zeta)\zeta\Big)} + \frac{\Big[3w_0\Big(g_1(\zeta)\zeta\Big) + 2w_0\Big(g_2(\zeta)\zeta\Big) + w_0(\zeta)\Big]\int_0^1 v\Big(\theta g_2(\zeta)\zeta\Big)d\theta}{2\Big(1 - w_0\Big(g_2(\zeta)\zeta\Big)\Big)\Big(1 - p(\zeta)\Big)}\right] g_2(\zeta), \\ h_3(\zeta) &= g_3(\zeta) - 1. \end{aligned}$$

We obtain $h_3(0) = -1$ and $h_3(\zeta) \to \infty$ as $\zeta \to \rho^-$. Denote by r_3 the minimal solution of equation $h_3(\zeta) = 0$ in the interval $(0, \rho)$. Define a radius of convergence r by

$$r = \min\{r_j\}, \ j = 1, 2, 3. \tag{9}$$

It follows that, for all $\zeta \in I_3 := [0, r)$,

$$0 \le w_0(\zeta) < 1, \tag{10}$$

$$0 \le w_0\Big(g_2(\zeta)\zeta\Big) < 1, \tag{11}$$

$$0 \le p(\zeta) < 1, \tag{12}$$

$$0 \le g_j(\zeta) < 1. \tag{13}$$

The hypotheses $(A_i, \ i = 1, 2, \ldots 5)$ used in the local convergence analysis of all three methods are:

(A_1) $F : \Omega \subset \mathbb{X} \to \mathbb{Y}$, is Fréchet differentiable and there exists $x_* \in \Omega$ with $F(x_*) = 0$ and $F'(x_*)^{-1}\ell(\mathbb{Y}, \mathbb{X})$.

(A_2) There exists function $w_0 : I \to I$ continuous, increasing with $w_0(0) = 0$ such that for each $x \in \Omega$

$$\Big\| F'(x_*)^{-1}\Big(F'(x) - F'(x_*)\Big)\Big\| \le w_0(\|x - x_*\|).$$

Set $\Omega_0 = \Omega \cap U(x_*, \rho_0)$.

(A_3) There exist functions $w : I_0 \to I$ and $v :: I_0 \to I$ continuous and increasing with $w(0) = 0$, such that for each $x, y \in \Omega_0$

$$\Big\| F'(x_*)^{-1}\Big(F'(x) - F'(y)\Big)\Big\| \le w(\|x - x_*\|)\|x - y\|$$

and

$$\Big\| F'(x_*)^{-1}F'(x)\Big\| \le v(\|x - x_*\|).$$

(A_4) The ball $\bar{U}(x_*, r) \subset \Omega$, ρ_0, ρ_p, and ρ_2 are defined in previous expressions.

(A_5) There exists $r_* \ge r$ such that

$$\int_0^1 w_0(\theta r_*)d\theta < 1.$$

Set $\Omega_1 = \Omega \cap U^*(x_*, r_*)$.

Next, we provide the local convergence analysis of method (1) using the hypotheses (A) and the aforementioned symbols.

Theorem 1. *Suppose that the hypotheses (A) hold. Then, starting from any $x_0 \in U(x_*, r) - \{x_*\}$, the sequence $\{x_\sigma\}$ generated by method (1) is well defined, which remains in $U(x_*, r)$ for each $\sigma = 0, 1, 2, 3, \ldots$ and $\lim_{\sigma \to \infty} x_\sigma = x_*$. Moreover, the following error estimates are available:*

$$\|y_\sigma - x_*\| \le g_1(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \le \|x_\sigma - x_*\| < r, \tag{14}$$

$$\|z_\sigma - x_*\| \le g_2(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \le \|x_\sigma - x_*\|, \tag{15}$$

$$\|x_{\sigma+1} - x_*\| \le g_3(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \le \|x_\sigma - x_*\|, \tag{16}$$

where the functions g_j are given previously and the radius r is defined by (9). Furthermore, x_* is the only solution of equation $F(x) = 0$ in the set Ω_1 given below (A_5).

Proof. Inequations (14)–(16) are shown by using mathematical induction. Using (9) and (10), A_1, and A_2, we have for all $x \in U(x_*, r)$

$$\left\| F'(x)^{-1}\Big(F'(x) - F'(x_*)\Big) \right\| \leq w_0(\|x - x_*\|) \leq w_0(r) < 1. \tag{17}$$

By the Banach lemma on invertible operators [5,19–21], expression (17), $F'(x)^{-1} \in \ell(\mathbb{Y}, \mathbb{X})$, and

$$\left\| F'(x)^{-1}F'(x) \right\| \leq \frac{1}{w_0(\|x - x_*\|)}. \tag{18}$$

Then, y_0 is well defined by the first substep of method (1). By A_1 and A_3, we can write

$$F(x) = F(x) - F(x_*) = \int_0^1 F'\Big(x_* + \theta(x_0 - x_*)\Big) d\theta (x_0 - x_*)$$

and so, by the second hypothesis in (A_3), we have

$$\begin{aligned} \left\| F'(x)^{-1}F'(x) \right\| &= \left\| F'(x_*)^{-1} \int_0^1 F'\Big(x_* + \theta(x_0 - x_*)\Big) d\theta (x_0 - x_*) \right\| \\ &\int_0^1 v\Big(\theta\|x_0 - x_*\|\Big) d\theta \|x_0 - x_*\|. \end{aligned} \tag{19}$$

In view of method (1) (for $\sigma = 0$), expressions (9) and (13) (for $j = 1$), hypothesis (A_3), expression (18) (for $x = x_0$) and (19), we obtain

$$\begin{aligned} \|y_0 - x_*\| &= \left\| \Big(x_0 - x_* - F'(x_0)^{-1}F(x_0)\Big) + \frac{1}{3}F'(x_0)^{-1}F(x_0) \right\| \\ &\leq \left\| \Big(x_0 - x_* - F'(x_0)^{-1}F(x_0)\Big) \right\| + \frac{1}{3}\left\| F'(x_0)^{-1}F(x_0) \right\| \\ &\leq \left\| F'(x_0)^{-1}F(x_*) \right\| \left\| \int_0^1 F'(x_*)^{-1}\Big[F'\Big(x_* + \theta(x_0 - x_*)\Big) - F'(x_0)\Big] d\theta (x_0 - x_*) \right\| \\ &+ \frac{1}{3}\left\| F'(x_0)^{-1}F(x_*) \right\| \left\| F'(x_0)^{-1}F(x_0) \right\| \\ &\leq \frac{\Big[\int_0^1 w\Big((1-\theta)\|x_0 - x_*\|\Big) d\theta + \frac{1}{3}\int_0^1 v\Big(\theta\|x_0 - x_*\|\Big) d\theta\Big] \|x_0 - x_*\|}{w_0(\|x_0 - x_*\|)} \\ &= g_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned} \tag{20}$$

so that $y_0 \in U(x_*, r)$ and (14) hold for $\sigma = 0$.

By expressions (9), (11) and (20), we have

$$\begin{aligned} &\left\| \Big(2F'(x_*)\Big)^{-1}\Big[3F'(y_0) - F'(x_0) - 3F'(x_*) + F'(x_*)\Big] \right\| \\ &\leq \frac{1}{2}\Big[3\left\| F'(x_*)^{-1}\Big(F'(y_0) - F'(x_*)\Big) \right\| + \left\| F'(x_*)^{-1}\Big(F'(x_0) - F'(x_*)\Big) \right\|\Big] \\ &\leq \frac{1}{2}\Big[3w_0(\|y_0 - x_*\|) + w_0(\|x_0 - x_*\|)\Big] \\ &\leq p(\|x_0 - x_*\|) \leq p(r) < 1, \end{aligned}$$

so that

$$\Big(3F'(y_0) - F'(x_0)\Big)^{-1} \in \ell(\mathbb{Y}, \mathbb{X}),$$

and

$$\left\| \left(3F'(y_0) - F'(x_0)\right)^{-1} F'(x_*) \right\| \leq \frac{1}{2(1 - p(\|x_0 - x_*\|))}.$$

Then, z_0 is well defined by the second substep of method (1) for $\sigma = 0$. Next, by the second substep of method (1) for $\sigma = 0$, we can write

$$\begin{aligned}
z_0 - x_* &= \left(x_0 - x_* - F'(x_0)^{-1}F(x_0)\right) + F'(x_0)^{-1}F(x_0) - \frac{1}{2}F'(x_0)^{-1}F(x_0) \\
&\quad - F'(x_0)\left(3F'(y_0) - F'(x_0)\right)^{-1} F'(x_0)^{-1}F(x_0) \\
&= \left(x_0 - x_* - F'(x_0)^{-1}F(x_0)\right) + \left[\frac{1}{2}I - F'(x_0)\left(3F'(y_0) - F'(x_0)\right)^{-1}\right] F'(x_0)^{-1}F(x_0) \\
&= \left(x_0 - x_* - F'(x_0)^{-1}F(x_0)\right) + \left[\frac{3F'(y_0) - F'(x_0)}{2} - F'(x_0)\right] \left(3F'(y_0) - F'(x_0)\right)^{-1} F'(x_0)^{-1}F(x_0) \\
&= \left(x_0 - x_* - F'(x_0)^{-1}F(x_0)\right) + \frac{3}{2}\left(F'(y_0) - F'(x_0)\right)\left(3F'(y_0) - F'(x_0)\right)^{-1} F'(x_0)^{-1}F(x_0).
\end{aligned} \tag{21}$$

Hence, by expressions (9), (13) (for $j = 2$) and (19)–(21), we obtain

$$\begin{aligned}
\|z_0 - x_*\| &= \left[\frac{\int_0^1 w\left((1-\theta)\|x_0 - x_*\|\right) d\theta}{1 - w_0(\|x_0 - x_*\|)} \right. \\
&\quad \left. + \frac{3\left(w_0(\|y_0 - x_*\|) + w_0(\|x_0 - x_*\|)\right) \int_0^1 v\left(\theta\|x_0 - x_*\|\right) d\theta}{4\left(1 - w_0(\|x_0 - x_*\|)\right)\left(1 - p(\|x_0 - x_*\|)\right)} \right] \|x_0 - x_*\| \\
&= g_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r.
\end{aligned} \tag{22}$$

Thus, $z_0 \in U(x_*, r)$ and expression (15) hold for $\sigma = 0$.

In view of method (1) for $\sigma = 0$, x_1 is well defined $\left(F'(z_0)^{-1} \in \ell(\mathbb{Y}, \mathbb{X})\right.$ by (18) for $\left.x = z_0\right)$. Then, we can write

$$x_1 - x_* = \left(z_0 - x_* - F'(z_0)^{-1}F(z_0)\right) + \left[F'(z_0)^{-1} - 2\left(3F'(y_0) - F'(x_0)\right)^{-1}\right] F(z_0),$$

which further yields

$$\begin{aligned}
\|x_1 - x_*\| &= \left[\frac{\int_0^1 w\left(\theta\|z_0 - x_*\|\right) d\theta}{1 - w_0(\|z_0 - x_*\|)} \right. \\
&\quad \left. + \frac{\left[2\left(w_0(\|y_0 - x_*\|) + w_0(\|z_0 - x_*\|)\right) + w_0(\|y_0 - x_*\|) + w_0(\|x_0 - x_*\|)\right] \int_0^1 v\left(\theta\|z_0 - x_*\|\right) d\theta}{2\left(1 - w_0(\|z_0 - x_*\|)\right)\left(1 - p(\|x_0 - x_*\|)\right)} \right] \|z_0 - x_*\| \\
&= g_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r,
\end{aligned} \tag{23}$$

so that $x_1 \in U(x_*, r)$ and expression (16) hold for $\sigma = 0$. Thus far, we have shown that estimates (14)–(16) hold for $\sigma = 0$. If we simply replace x_0, y_0, z_0 and x_1 by x_m, y_m, z_m and x_{m+1}, $(m = 1, 2, 3, ..., \sigma - 1)$, respectively, in the preceding computations, then we obtain

$$\|y_{m+1} - x_*\| \leq g_1(\|x_{m+1} - x_*\|)\|x_{m+1} - x_*\| \leq \|x_{m+1} - x_*\| < r$$

$$\|z_{m+1} - x_*\| \leq g_2(\|x_{m+1} - x_*\|)\|x_{m+1} - x_*\| \leq \|x_{m+1} - x_*\| < r$$

and

$$\|x_{m+2} - x_*\| \leq g_3(\|x_{m+1} - x_*\|)\|x_{m+1} - x_*\| < r.$$

By the above estimations

$$\|x_{m+1} - x_*\| \leq c\|x_m - x_*\| < r, \quad c = g_3(\|x_0 - x_*\|) \in [0,1),$$

we deduce that $\lim_{m\to\infty} x_m = x_*$, with $x_{m+1} \in U(x_*, r)$. Consider $y_* \in \Omega_1$ with $F(y_*) = 0$ and set

$$S = \int_0^1 F'\Big(x_* + \theta(y_* - x_*)\Big)d\theta.$$

By (A_2) and (A_5), we obtain

$$\begin{aligned}\Big\|F'(x_*)^{-1}\Big(S - F'(x_*)\Big)\Big\| &\leq \int_0^1 w_0(\theta\|y_* - x_*\|)d\theta \\ &\leq \int_0^1 w_0(\theta r_*)d\theta < 1,\end{aligned}$$

so that $S^{-1} \in \ell(\mathbb{Y}, \mathbb{X})$. Then, $x_* = y_*$ follows from the identity $0 = F(y_*) - F(x_*) = S(y_* - x_*)$. □

Secondly, for the method (2), the conclusion of Theorem 1 holds, but r is defined by

$$r^{(2)} = \min\{r_1, r_2, r_3^{(2)}\}, \tag{24}$$

so that $r_3^{(2)}$ is the minimal positive solution of equation $h_3^2(\zeta) = 0$, which $h_3^2(\zeta) = g_3^2(\zeta) - 1$ and

$$g_3^{(2)}(\zeta) = \left[1 + \frac{1}{4}q(\zeta)\left(\frac{\int_0^1 v\Big(\theta g_2(\zeta)\zeta\Big)d\theta}{1 - w_0(\zeta)}\right)\right] g_2(\zeta),$$

where

$$q(\zeta) = \frac{3w_0\big(g_1(\zeta)\zeta\big) + w_0(\zeta) + 4}{2\Big(1 - p(\zeta)\Big)}.$$

Notice also that g_1, h_1, g_2, h_2, r_1, and r_2 are the same as in Theorem 1. Functions $g_3^{(2)},\ h_3^{(2)}$, and q appear due to the estimates

$$\begin{aligned}&\Big\|I + 2F'(x_\sigma)\Big(3F'(y_\sigma) - F'(x_\sigma)\Big)^{-1}\Big\| \\ &= \Big\|\Big[3\Big(F'(y_\sigma) - F'(x_\sigma)\Big) + 2F'(x_\sigma)\Big]\Big[3F'(y_\sigma) - F'(x_\sigma)\Big]^{-1}\Big\| \\ &= \Big\|\Big[3\Big(F'(y_\sigma) - F'(x_\sigma)\Big) + \Big(F'(x_\sigma) - F'(x_*)\Big) + 4F'(x_*)\Big]\Big[3F'(y_\sigma) - F'(x_\sigma)\Big]^{-1}\Big\| \\ &\leq \frac{3w_0(\|y_\sigma - x_*\| + w_0(\|x_\sigma - x_*\|) + 4}{2\Big(1 - p(\|x_\sigma - x_*\|)\Big)} \leq q(\|x_\sigma - x_*\|)\end{aligned}$$

and

$$\begin{aligned}\|x_{\sigma+1} - x_*\| &\leq \|z_\sigma - x_*\| + \frac{1}{4}\Big\|\Big[I + 2F'(x_\sigma)\Big(3F'(y_\sigma) - F'(x_\sigma)\Big)^{-1}\Big]F'(x_\sigma)^{-1}F(z_\sigma)\Big\| \\ &\leq \left[1 + \frac{1}{4}q(\|x_\sigma - x_*\|)\frac{\int_0^1 v\Big(\theta\|z_\sigma - x_*\|\Big)d\theta}{1 - w_0(\|x_\sigma - x_*\|)}\right]\|z_\sigma - x_*\| \\ &\leq g_3^{(2)}(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\| < r^{(2)}.\end{aligned}$$

Hence, we arrive at the following theorem.

Theorem 2. *Suppose that the conditions* (A) *hold, but with* r^2 *and* $g_3^{(2)}$ *replaced by* r *and* g_3*, respectively. Then, the same conclusions hold for method* (2)*, but with* (16) *replaced by*

$$\|x_{\sigma+1} - x_*\| \leq g_3^{(2)}(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\|. \tag{25}$$

Finally, for the local convergence of method (3), we introduce the functions

$$\begin{aligned}
g_2^{(3)}(\zeta) &= g_1(\zeta) + \frac{\Big(w_0(\zeta) + w_0\Big(g_1(\zeta)\zeta\Big)\Big)\int_0^1 v(\theta\zeta)d\theta}{2\Big(2 - w_0(\zeta)\Big)\Big(1 - p(\zeta)\Big)},\\
h_2^{(3)}(\zeta) &= g_2^{(3)}(\zeta) - 1,\\
g_3^{(3)}(\zeta) &= \left[1 + \frac{3\Big(w_0(\zeta) + w_0\Big(g_1(\zeta)\zeta\Big)\Big)\int_0^1 v\Big(\theta g_2^{(3)}(\zeta)\zeta\Big)d\theta}{2\Big(2 - w_0(\zeta)\Big)\Big(1 - p(\zeta)\Big)}\right] g_2^{(3)}(\zeta),\\
h_3^{(3)}(\zeta) &= g_3^{(3)}(\zeta) - 1.
\end{aligned}$$

Let us denote by $r_2^{(3)}$ and $r_3^{(3)}$ the minimal positive solutions of equations $h_2^{(3)}(\zeta) = 0$ and $h_3^{(3)}(\zeta) = 0$, respectively. Set

$$r^{(3)} = \min\{r_1, r_2^{(3)}, r_3^{(3)}\}. \tag{26}$$

These functions are defined due to the estimates

$$\begin{aligned}
\|z_\sigma - x_*\| &= \left\| y_\sigma - x_* + \frac{1}{3}F'(x_\sigma)^{-1}\left[\Big(F'(x_\sigma) - 3F'(y_\sigma) + 2F'(x_\sigma)\Big)\Big(F'(x_\sigma) - 3F'(y_\sigma)\Big)^{-1}F(x_\sigma)\right]\right\|\\
&= \left\|(y_\sigma - x_*) + F'(x_\sigma)^{-1}\Big(F'(x_\sigma) - F'(y_\sigma)\Big)\Big(F'(x_\sigma) - 3F'(y_\sigma)\Big)^{-1}F(x_\sigma)\right\|\\
&\leq \left[g_1(\|x_\sigma - x_*\|) + \frac{\Big(w_0(\|x_\sigma - x_*\|) + w_0(\|y_\sigma - x_*\|)\Big)\int_0^1 v(\theta\|x_\sigma - x_*\|)d\theta}{2\Big(1 - w_0(\|x_\sigma - x_*\|)\Big)\Big(1 - p(\|x_\sigma - x_*\|)\Big)}\right]\|x_\sigma - x_*\|\\
&\leq g_2^{(3)}(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_{\sigma+1} - x_*\| &= \left\| z_\sigma - x_* + \Big(F'(x_\sigma) - 3F'(y_\sigma)\Big)^{-1}\left[4F'(x_\sigma) - \Big(F'(x_\sigma) - 3F'(y_\sigma)\Big)\right]F'(x_\sigma)^{-1}F(z_\sigma)\right\|\\
&= \left\|(z_\sigma - x_*) + 3\Big(F'(x_\sigma) - 3F'(y_\sigma)\Big)^{-1}\Big(F'(x_\sigma) - F'(y_\sigma)\Big)F'(x_\sigma)^{-1}F(z_\sigma)\right\|\\
&\leq \left[1 + \frac{3\Big(w_0(\|x_\sigma - x_*\|) + w_0(\|y_\sigma - x_*\|)\Big)\int_0^1 v(\theta\|z_\sigma - x_*\|)d\theta}{2\Big(1 - w_0(\|x_\sigma - x_*\|)\Big)\Big(1 - p(\|x_\sigma - x_*\|)\Big)}\right]\|x_\sigma - x_*\|\\
&\leq g_3^{(3)}(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\|.
\end{aligned}$$

Theorem 3. *Let us consider hypotheses* (A)*, but with* $g_2^{(3)}, g_3^{(3)}$*, and* r^3 *replacing by* g_2, g_3*, and* r*, respectively. Then, the conclusions of Theorem 1 hold for method* (3)*, but with* (15) *and* (16) *replaced by*

$$\|z_\sigma - x_*\| = g_2^{(3)}(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\| \tag{27}$$

and

$$\|x_{\sigma+1} - x_*\| = g_3^{(3)}(\|x_\sigma - x_*\|)\|x_\sigma - x_*\| \leq \|x_\sigma - x_*\|, \tag{28}$$

respectively.

3. Numerical Examples

The theoretical results developed in the previous sections are illustrated numerically in this section. We denote the methods (1)–(3) by (CM), (SM), and (EA), respectively. We consider two real life problems and two standard nonlinear problems that are illustrated in Examples 1–4. The results are listed in Tables 1, 2, 3 (values of ψ_i and φ_i (in radians) for Example 3), 4, and 5. Additionally, we obtain the COC approximated by means of

$$\xi = \frac{\ln \frac{\|x_{\sigma+1} - x_*\|}{|x_\sigma - x_*\|}}{\ln \frac{\|x_\sigma - x_*\|}{\|x_{\sigma-1} - x_*\|}}, \quad \text{for } \sigma = 1, 2, \ldots \tag{29}$$

or $ACOC$ [18] by:

$$\xi^* = \frac{\ln \frac{\|x_{\sigma+1} - x_\sigma\|}{\|x_\sigma - x_{\sigma-1}\|}}{\ln \frac{\|x_\sigma - x_{\sigma-1}\|}{\|x_{\sigma-1} - x_{\sigma-2}\|}}, \quad \text{for } \sigma = 2, 3, \ldots \tag{30}$$

We adopt $\epsilon = 10^{-100}$ as the error tolerance and the terminating criteria to solve nonlinear system or scalar equations are: (i) $\|x_{\sigma+1} - x_\sigma\| < \epsilon$, and (ii) $\|F(x_\sigma)\| < \epsilon$.

The computations are performed with the package *Mathematica* 9 with multiple precision arithmetics.

Example 1. *Following the example presented in the Introduction, for $x_* = 1$, we can set*

$$w_0(t) = w(t) = 96.662907t \text{ and } v(t) = 2.$$

In Table 1, we present radii for example (1).

Table 1. Radii for Example (1).

Cases	r_1	r_2	r_3	$r_3^{(2)}$	$r_2^{(3)}$	$r_3^{(3)}$	r	$r^{(2)}$	$r^{(3)}$	x_0	σ	ξ
CM	0.0022989	0.0017993	0.0015625	-	-	-	0.0015625	-	-	1.001	3	6.0000
SM	0.0022989	0.0017993	-	0.001022	-	-	-	0.001022-		1.0009	3	6.0000
EA	0.0022989	-	-	-	0.0013240	0.00081403	-	-	0.00081403	1.0008	3	6.0000

(On the basis of obtained results, we conclude that method CM has a larger radius of convergence.)

Example 2. *Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}^3$ and $\Omega = \bar{\mathbb{S}}(0, 1)$. Assume F on Ω with $v = (x, y, z)^T$ as*

$$F(u) = F(u_1, u_2, u_3) = \left(e^{u_1} - 1, \frac{e-1}{2}u_2^2 + u_2, u_3\right)^T, \tag{31}$$

where $u = (u_1, u_2, u_3)^T$. Define the Fréchet-derivative as

$$F'(u) = \begin{bmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, for $x_* = (0, 0, 0)^T$ and $F'(x_*) = F'(x_*)^{-1} = diag\{1, 1, 1\}$, we have

$$w_0(t) = (e-1)t, \quad w(t) = e^{\frac{1}{e-1}}t \text{ and } v(t) = e^{\frac{1}{e-1}}.$$

We obtain the convergence radii depicted in Table 2.

Table 2. Radii for Example 2.

Cases	r_1	r_2	r_3	$r_3^{(2)}$	$r_2^{(3)}$	$r_3^{(3)}$	r	$r^{(2)}$	$r^{(3)}$	x_0	σ	ξ
CM	0.15441	0.11011	0.096467	-	-	-	0.096467	-	-	(0.094,0.094,0.094)	3	6.0000
SM	0.15441	0.11011	-	0.065471	-	-	-	0.065471	-	(0.063,0.063,0.063)	3	6.0000
EA	0.15441	-	-	-	0.092584	0.059581	-	-	0.059581	(0.054,0.054,0.054)	3	6.0000

(Among the three methods, the larger radius of convergence belong to the method CM.)

Example 3. *The kinematic synthesis problem for steering [22,23] is given as*

$$[E_i(x_2\sin(\psi_i)-x_3)-F_i(x_2\sin(\varphi_i)-x_3)]^2+[F_i(x_2\cos(\varphi_i)+1)-F_i(x_2\cos(\psi_i)-1)]^2$$
$$-[x_1(x_2\sin(\psi_i)-x_3)(x_2\cos(\varphi_i)+1)-x_1(x_2\cos(\psi_i)-x_3)(x_2\sin(\varphi_i)-x_3)]^2=0,\ \textit{for}\ i=1,2,3,$$

where

$$E_i=-x_3x_2(\sin(\varphi_i)-\sin(\varphi_0))-x_1(x_2\sin(\varphi_i)-x_3)+x_2(\cos(\varphi_i)-\cos(\varphi_0)),\ i=1,2,3$$

and

$$F_i=-x_3x_2\sin(\psi_i)+(-x_2)\cos(\psi_i)+(x_3-x_1)x_2\sin(\psi_0)+x_2\cos(\psi_0)+x_1x_3,\ i=1,2,3.$$

In Table 3, we present the values of ψ_i and φ_i (in radians).

Table 3. Values of ψ_i and φ_i (in radians) for Example 3.

i	ψ_i	φ_i
0	1.3954170041747090114	1.7461756494150842271
1	1.7444828545735749268	2.0364691127919609051
2	2.0656234369405315689	2.2390977868265978920
3	2.4600678478912500533	2.4600678409809344550

The approximated solution is for $\Omega = \bar{\mathbb{S}}(x_*, 1)$

$$x_* = (0.9051567\ldots,\ 0.6977417\ldots,\ 0.6508335\ldots)^T.$$

Then, we get

$$w_0(t) = w(t) = 3t \ \text{ and } \ v(t) = 2.$$

We provide the radii of convergence for Example 3 in Table 4.

Table 4. Radii for Example 3.

Cases	r_1	r_2	r_3	$r_3^{(2)}$	$r_2^{(3)}$	$r_3^{(3)}$	r	$r^{(2)}$	$r^{(3)}$	x_0	σ	ξ
CM	0.074074	0.057977	0.050345	-	-	-	0.050345	-	-	(0.945,0.737,0.690)	3	6.1328
SM	0.074074	0.057977	-	0.032936	-	-	-	0.032936	-	(0.933,0.726,0.678)	3	6.1377
EA	0.074074	-	-	-	0.042662	0.026229	-	-	0.026229	(0.929,0.722,0.674)	3	4.8142

Example 4. *Let us consider that* $\mathbb{X} = \mathbb{Y} = C[0,\ 1]$, $\Omega = \bar{\mathbb{S}}(0,\ 1)$ *and introduce the space of maps continuous in* $[0,\ 1]$ *having the max norm. We consider the following function* φ *on* $\mathbb{A}$*:*

$$\Psi(\phi)(x) = \Psi(x) - \int_0^1 x\tau\phi(\tau)^3 d\tau. \tag{32}$$

which further yields:

$$\Psi'(\phi(\mu))(x) = \mu(x) - 3\int_0^1 x\tau\phi(\tau)^2\mu(\tau)d\tau, \textit{ for } \mu \in \Omega.$$

We have $x_* = 0$ *and*

$$w_0(t) = \frac{3}{2},\ w(t) = 3t \text{ and } v(t) = 2.$$

We list the radii of convergence for example (4) *in Table* 5*.*

Table 5. Radii of convergence for Example 4.

Cases	r_1	r_2	r_3	$r_3^{(2)}$	$r_2^{(3)}$	$r_3^{(3)}$	r	$r^{(2)}$	$r^{(3)}$
CM	0.111111	0.105542	0.0922709	-	-	-	0.0922709	-	-
SM	0.111111	0.105542	-	0.0594758	-	-	-	0.0594758	
EA	0.111111	-	-	-	0.0718454	0.0465723	-	-	0.0465723

(CM has a larger radius of convergence as compared to other two methods.)

4. Conclusions

We have introduced a new technique capable of proving convergence relying on hypotheses only on the first derivative (used in these methods) in contrast to earlier studies using hypotheses up to the seven derivatives and the Taylor series. Moreover, the new technique provides usable error analysis for operators valued on Banach space. In order to recover the convergence order, but, without using Taylor series, we rely on the *COC* and *ACOC* that require only the first order derivative. Four numerical examples compare the radii of the convergence balls for these methods, showing that our results can be used in cases not possible before. The technique can also be used to extend the usage of other iterative methods using inverses in an analogous procedure.

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