Primordial Magnetic Field Generation in theories of gravity with non-minimal coupling between curvature and matter

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2021

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MASTERS THESIS

Primordial Magnetic Field Generation in theories of gravity with non-minimal coupling between curvature and matter

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A thesis submitted in fulfilment of the requirements for the degree of MSc. Astronomy and Astrophysics

at the

Faculdade de Ciências da Universidade do Porto
Departamento de Física e Astronomia

February 13, 2022

" The most incomprehensible thing about the world is that is comprehensible "

assigned to Albert Einstein (1879-1955)

Acknowledgements

I would like to thank my supervisor, Prof. Dr. Filipe Mena, and co-supervisor, Prof. Dr. Orfeu Bertolami, for all the help and support provided throughout this year. Thanks for all the challenging and interesting discussions.

I also thank all my family and friends for their ongoing patience and kindness.

I thank all those who contribute preponderantly to the evolution of science and knowledge of the universe.

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Abstract

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Primordial Magnetic Field Generation in theories of gravity with non-minimal coupling between curvature and matter

by Maria Margarida LIMA

The existence of a magnetic field in the universe is unmistakable. They are observed at almost all scales of the universe, from stars to galaxy clusters.

The origin of these fields remains enigmatic. Some scientists believe that the magnetic field seed may have emerged in a primordial phase of the universe, namely during inflation.

In this work, the scale factor is analyzed along the different evolutionary phases of the universe, with emphasis on the inflationary period, according to a theory of non-minimal coupling between curvature and matter.

The behavior of the magnetic field is then studied in this theory, which leaves open the possibility of the existence of an amplification factor for the magnetic field after inflation.

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Resumo

Faculdade de Ciências da Universidade do Porto
Departamento de Física e Astronomia

Mestrado em Astronomia e Astrofísica

Produção de Campo Magnético Primordial em teorias da gravidade com acoplamento não mínimo entre curvatura e matéria

por Maria Margarida LIMA

A existência de campo magnético no universo é inequívoca. São observados a quase todas as escalas do universo, desde estrelas aos aglomerados de galáxias.

A origem desses campos permanece enigmática. Alguns cientistas acreditam que a semente de campo magnético terá surgido numa fase primordial do universo, nomeadamente durante a inflação.

Neste trabalho é analisado o fator de escala ao longo das diferentes fases evolutivas do universo, com ênfase no período inflacionário, no âmbito de uma teoria de acoplamento não mínimo entre a curvatura e a matéria.

Foi depois estudado o comportamento do campo magnético segundo esta teoria, que deixa em aberto a possibilidade da existência de um fator de amplificação do campo magnético após a inflação.

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Chapter 1

Introduction

Magnetic fields play an important role in the dynamics of the astrophysical objects. For instance, the galactic magnetic field affects the dynamics of the compact stars and, consequently, the star formation process [1]. Although these fields are well understood, their cosmological origin represents an interesting open problem.

Our galaxy, like many others, has a coherent magnetic field of $B \sim 10^{-6}G$. Many astrophysicists believe that a plausible explanation for these observed galactic magnetic fields is the dynamo effect [2–4]. According to this effect, the differential rotation of galaxies exponentially enhances the magnetic field. This mechanism is only a means to amplify it, so a seed magnetic field needs to be created. Considering that the dynamo effect operated during the entire life of the universe ($\sim$ 10 Gyears), then it is possible to amplify a seed field by a e^{30} factor [5]. Thus, this process allows to obtain the magnetic field strength observed today in the universe, with a seed field of approximately $B \sim 3 \times 10^{-19}G$ [5].

Other lines of thought admit that the magnetic field in galaxies emerge due the compression of a primordial magnetic field during the collapse of the protogalactic cloud. In this case a stronger seed magnetic field is needed, at least $B \sim 10^{-9}$ [5].

A third possibility concerns the creation of seed magnetic fields during inflation, under certain conditions. The inflationary scenario [6] has become the current cosmological paradigm for the early universe. The existence of a primordial period with accelerated expansion solves the initial problems in standard Big Bang cosmology, such as the horizon and flatness problems. In fact, the ability of this model is broader. Through the quantum fluctuations produced in De Sitter space, inflation ensures that the electromagnetic field is excited, increasing the magnetic flux. Furthermore, using a mechanism similar to superadiabatic amplification, long wavelength ($\lambda \gtrsim H^{-1}$, where H is Hubble constant) modes are increased during the inflation and reheating.

Since during the inflationary period, the total energy density of the spacetime is constant, the magnetic field decreases with $a^{-2}(t)$, where $a(t)$ is the scale factor of the Friedmann-Robertson-Walker metric. But in order to create seed magnetic fields during inflation, the conformal symmetry of eletromagnetism must be broken [5] [7][8][9].

In this work we will investigate whether a theory of gravity with non-minimal coupling between curvature and matter, combined with inflationary models, can generate sizeable seed magnetic fields.

This thesis is divided in four chapters. In Chapter 2 we will indicate some important cosmological facts. In Chapter 3, we discuss the theories of non-minimal coupling between curvature and matter, studying the consequences of a cubic coupling in the different evolutionary phases of the universe, with an emphasis on inflation. In chapter 4, the influence of cubic non-minimal coupling is debated, deducing the modified Maxwell's equations and studying the behavior of the magnetic field during the evolutionary phases of the universe. In the last chapter some conclusions are presented.

Chapter 2

Main cosmological facts

The current evolution paradigm of the universe is based on the Friedmann-Robertson-Walker cosmological model, also called the Hot Big Bang model.

In this chapter we will present the fundamental features of this theory, based on the Robertson-Walker metric, the Friedmann equation and the evolutionary phases in this model.

During this thesis we will consider that the speed of light and the permeability of free space take the value 1, $c = \mu_0 = 1$.

2.1 The metric

The metric that describes an homogeneous and isotropic space-time is the Robertson-Walker metric, which can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) \right), \tag{2.1}$$

where $a(t)$ is the cosmic scale factor, $k = +1, -1, 0$ if the universe is closed, open or flat, respectively, $t \in \mathbb{R}_0^+$, $r \in \mathbb{R}_0^+$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$.

In this work, we will consider a spatially flat universe ($k = 0$) and the Robertson-Walker metric in cartesian coordinates and in conformal time. That is,

$$ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2 \right) \tag{2.2}$$

and

$$ds^2 = a^2(\tau)\Big(- d\tau^2 + dx^2 + dy^2 + dz^2\Big), \tag{2.3}$$

where $dt = ad\tau$.

So the metric tensor $g_{\mu\nu}$ can be written in the following matrix representation"

$$[g_{\mu\nu}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{bmatrix} \tag{2.4}$$

or, in the same way,

$$[g_{\mu\nu}] = a^2(\tau) \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.5}$$

In the cartesian metric, the non-zero Levi-Civita connection coeficients are

$$\Gamma^0_{ii} = \dot{a}a, \;\; \Gamma^i_{0i} = \frac{\dot{a}}{a}, \tag{2.6}$$

where $i \in \{1,2,3\}$ and 0, 1, 2 and 3 represent the t, x, y and z coordinates, respectively, and the dot denotes differentiation with respect to t.

The non-vanishing components of the Ricci tensor are

$$R_{00} = -3\Big(\frac{\ddot{a}}{a}\Big); \tag{2.7}$$

$$R_{ii} = \ddot{a}a + 2\dot{a}^2. \tag{2.8}$$

So the curvature scalar, for the cartesian metric, takes the form

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + 2\Big(\frac{\dot{a}}{a}\Big)^2\right), \tag{2.9}$$

where Einstein's summation convention is used.

In the conformal time metric, the curvature scalar is given by

$$R = 6\left(\frac{a''}{a^3} + 2\Big(\frac{a'}{a^2}\Big)^2\right), \tag{2.10}$$

where the prime denotes differentiation with respect to τ.

2.2 The Friedmann equations

The action functional for General Relativity over the spacetime takes the form

$$S = \int_M \sqrt{-g}\left(\frac{1}{16\pi G}R + \mathcal{L}\right)d^4x, \tag{2.11}$$

where G is the Newton's gravitation constant, g is the determinant of the metric, $\mathcal{L}$ is the Lagrangian density of the matter fields and M is the manifold.

So, varying the previous action with respect to the metric we obtain the Einstein's equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.12}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein's tensor and $T_{\mu\nu}$ is the energy-momentum tensor that can be defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\mathcal{L}\sqrt{-g})}{\delta g^{\mu\nu}}. \tag{2.13}$$

We can see that the energy-momentum tensor is conserved. This means that

$$\nabla_\nu T^{\mu\nu} = 0. \tag{2.14}$$

A perfect fluid has the energy-momentum tensor given by

$$T^{\mu\nu} = (p+\rho)u^\mu u^\nu + p g^{\mu\nu}, \tag{2.15}$$

where p and ρ are pressure and density of the matter, respectively, and u^μ is the fluid's 4-velocity orthogonal to the surfaces of constant curvature.

The relation (2.14) with the index $\mu = 0$ gives us that

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{2.16}$$

The 00 Einstein's equation for the previous energy-momentum tensor and the metric (2.2), yields Friedmann's equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \tag{2.17}$$

Using the 11 component of Einstein's equations and Friedmann's equation leads to the Raychaudhuri equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{2.18}$$

2.3 The evolutionary phase

The Lagrangian density of the inflaton, ϕ, takes the form

$$\mathcal{L}_\phi = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi), \tag{2.19}$$

where $V(\phi)$ is the potential of the inflaton.

Varying the action (2.11) with the previous Lagrangian, it is possible to deduce the inflaton equation, given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \tag{2.20}$$

where

$$H = \frac{\dot{a}}{a}. \tag{2.21}$$

Thus, the inflaton's evolution depends on the features of its potential and the evolution of the universe.

Now let's see the evolution of the energy density in the universe.

If we consider that

$$p = w\rho, \tag{2.22}$$

where w is a state parameter, and using equation (2.16), it is possible to write the temporal evolution equation of the density as:

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1 + w). \tag{2.23}$$

Solving this equation, we get

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}, \tag{2.24}$$

where $a_0 = a(t_0)$ is the value of the scale factor at present and ρ_0 is the value of the density at the present.

Then we are able to replace this relation in Friedmann's equation (2.17) to obtain the evolution of the scale factor. If $w = -1$, it is easy to see that the solution is an exponential. On the other hand, if $w \neq -1$, the solution is

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}. \tag{2.25}$$

The state parameter, w, defines the fluid equation of state. In cosmology, the most relevant fluids are dust or non-relativistic matter, radiation and the cosmological constant. Dust describes a fluid of non-relativistic particles, where pressure is negligible and $w = 0$. Radiation describes a relativistic fluid of "hot" particles and is represented by $w = 1/3$. Finally, the cosmological constant corresponds to a fluid with negative pressure, $w = -1$, in order to counteract the gravitational attracting effect of the matter.

Taking this into account, we can divide the evolution of the energy density into three distinct eras:

- De Sitter era, where the cosmological constant is dominant and the scale factor evolves as:
$$a(t) = a_0 e^{H_0(t-t_0)}, \tag{2.26}$$
where $H_0 = H(t_0)$ and $a(t_0) = a_0$ with the index 0 representing the present.

- Radiation era, where radiation dominates all other fluids and the scale factor evolves as:
$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/2}. \tag{2.27}$$

- Matter era, where dust dominates all other fluids and the scale factor evolves as:
$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}. \tag{2.28}$$

2.4 Open questions

General Relativity was proposed by Albert Einstein in 1915 and came to revolutionize modern physics. Several observations have come to demonstrate its validity. However, there is also evidence that this theory is incomplete.

For the limit of small scales, that is high energies, there is no complete consistency between quantum theories and General Relativity. Despite the efforts made, a way to unify both theories has not yet been conceived.

On the other hand, at the limit of large scales, that is small energies, problems such as dark energy and dark matter arise. Observational data reveal the need to include in the model the presence of these enigmatic components, which have not yet been observed. It has been estimated that, assuming its existence, 68.3% of the universe is constituted by dark energy and 26.8% by dark matter [10]. So only 4.9% of the matter in the universe is observed matter.

In 1930, some authors revealed a difference between the velocity dispersion of galaxies in clusters and the predicted velocity dispersion based on visible matter [11].

In 1970, two studies were published that revealed the presence of dark matter around spiral galaxies showing a flatness of the rotation curves [12] [13]. In Figure 2.1 it is possible to observe the rotation curve of the galaxy NGC 3198 [14].

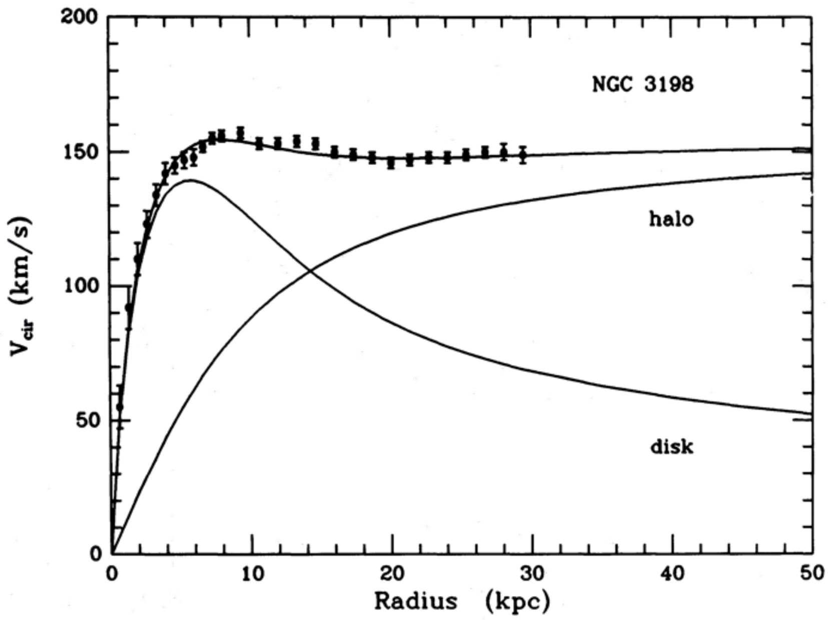

FIGURE 2.1: Rotation curve of the galaxy NGC 3198. Fit of exponential disk with maximum mass and halo to observed rotation curve (dots with error bars) [14].

In addition to several observations about the effect of dark matter, the discovery that the universe is in an accelerated expansion phase led to the conjecture of a significant ammount of dark energy. In 1998, two studies concluded this accelerated expansion through the observation of Type Ia Supernovaes [15] [16]. The simplest mechanism for including the dark energy effect was to insert the cosmological constant into Einstein's equations.

However, the existence and origin of dark energy and dark matter remains an enigma, as it is presented in the context of General Relativity.

Alternatively, it is speculated that some alternative theories to General Relativity may not need to include dark matter and dark energy to explain the rotation curve of galaxies [17].

Chapter 3

The non-minimal coupling model

The General Relativity theory has a great mathematical beauty. Nonetheless, there is also evidence to show that the theory is not complete. For this reason many alternative theories have been suggested. Some physicists suggest higher dimensional theories, such as String Theory. Others opt to add a new field content, as Quintessence.

Another possibility is to consider higher-order theories. An example of this is the $f(R)$ theory. This theory consists in providing more versatility than General Relativity, replacing the Ricci scalar by an arbitrary function $f(R)$ in the Hibert-Einstein action (2.11).

Based on this idea, some physicists considered the so-called non-minimal coupling theory, that we will explain during this chapter.

3.1 The non-minimal matter-curvature coupling theories

General Relativity rests on a principle of minimal coupling between curvature and matter. This implies a covariant conservation of the energy-momentum tensor, as seen in (2.14). However, it is possible to generalize this theory by dropping this conservation condition. This provides an extra force that can mimic the dark matter effect on galaxies [18]. Theoretically, this can be achieved by an extension of $f(R)$ theories, using a non-minimal coupling between curvature and matter.

The action functional for these non-minimal coupling models is [18]

$$S = \int \sqrt{-g}\left(\frac{1}{16\pi G}f_1(R) + f_2(R)\mathcal{L}\right)d^4x, \tag{3.1}$$

where $f_1(R)$ and $f_2(R)$ are sufficiently smooth arbitrary functions of the curvature scalar R.

Varying the action with respect to the metric, we obtain the field equations [see Appendix A]:

$$\begin{aligned}\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}\Big)G_{\mu\nu} &= 8\pi G f_2(R) T_{\mu\nu} \\ &+\Delta_{\mu\nu}\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}\Big) + \frac{1}{2}g_{\mu\nu}\Big(f_1(R) - \Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}\Big)R\Big).\end{aligned} \tag{3.2}$$

where $F_1(R) = \frac{df_1(R)}{dR}$, $F_2(R) = \frac{df_2(R)}{dR}$, $\Delta_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$ and $\Box = \nabla_\mu\nabla^\mu$. General Relativity is recovered for $f_1(R) = R$ and $f_2(R) = 1$, and (3.2) reduces to Einstein's equations (2.12).

The trace of the equations (3.2) is given by

$$\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}\Big)R - 2f_1(R) = 8\pi G f_2(R) T - 3\Box\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}\Big), \tag{3.3}$$

where $T = g^{\mu\nu}T_{\mu\nu}$.

Applying the contracted covariant derivative in equations (3.2) and using the contracted Bianchi identities, we obtain the relation

$$\nabla_\mu T^{\mu\nu} = \frac{F_2(R)}{f_2(R)}(\mathcal{L}g^{\mu\nu} - T^{\mu\nu})\nabla_\mu R. \tag{3.4}$$

It is then possible to see that in general the energy-momentum tensor is not covariantly conserved. It is easily observed that, considering the General Relativity limit where $f_2(R) = 1$, the previous equation becomes (2.14).

In this model, although the energy-momentum tensor remains the same with respect to General Relativity, we obtain different gravitational equations, depending on the chosen Lagrangian density. From equation (3.4), results an extra force appears in the geodesic equations of a perfect fluid [18], as can be seen in the following equation:

$$f^\mu = \frac{1}{\rho + p}\left(\frac{F_2(R)}{f_2(R)}(\mathcal{L} - p)\nabla_\nu R + \nabla_\nu p\right)h^{\mu\nu}, \tag{3.5}$$

where $h^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projection operator.

3.2 Inflation in non-minimal coupling theories

The first model of an expanding universe was derived from Einstein's equations by Alexander Friedmann, in 1922, and by George Lemaître, in 1927. In 1929, Edwin Hubble discovered that nearby galaxies were moving away at a rate known as the Hubble constant. It was observed that the further away the galaxies were, the faster they moved away, which suggested that the universe is expanding. It naturally led to the inference that, in the past, the universe would have been smaller, denser and hotter. This gave rise to the Big Bang model, implying the existence of an extremely hot and dense space that has continued to expand to this day.

The Big Bang model was successful in predicting the abundance of light elements (Big Bang Nucleosynthesis) and the existence of an universal background radiation. In fact, the Arno Penzias and Robert Wilson discovery, in 1964, of Cosmic Microwaves Background Radiation, showed that there is an uniform temperature, about 2.7K, that permeates the entire universe. This, together with other observations, such as the homogeneity and isotropy at large scales, led to the possible conclusion is that, in the past, all regions of space were in causal contact.

Later, other issues arose, such as the possible existence of magnetic monopoles, the flatness and the horizon problems. The solution came with the inflationary scenario [6] [19] [20], which implies that the universe had undergone an accelerated expansion at the beginning. It also provided a mechanism for the origin of large-scale observable structures, due to the quantum fluctuations of the inflaton field, ϕ.

Most of inflationary models are based on General Relativity, as seen in Chapter 2. However, this does not answer several questions, one of them being the origin of the primordial magnetic field, a problem which will be considered in the context of the theory of non-minimal coupling between curvature and matter [21].

Let's consider, as before, a homogeneous and isotropic universe, described by the Robertson-Walker metric (2.2).

The field equation of the inflaton in the context of non-minimal coupling theory is given by [see Appendix B] [21]:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = -\frac{F_2(R)}{f_2(R)}\dot{R}\dot{\phi}. \tag{3.6}$$

Comparing the previous equation with equation (2.20) we can see that, the non-minimal coupling between curvature and matter induces a friction term in the inflaton equation of the inflaton field.

Using the relation (2.13) with the previous Lagrangian we are able to define the energy-momentum tensor:

$$\begin{aligned} T_{\mu\nu} &= \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} && (3.7)\\ &= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\Big(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi)\Big), && (3.8)\end{aligned}$$

from which we get

$$\begin{aligned} T_{00} &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{(\nabla\phi)^2}{a^2} + V(\phi) && (3.9)\\ T_{ij} &= \partial_i\partial_j\phi - a^2\gamma_{ij}\Big(-\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{(\nabla\phi)^2}{a^2} + V(\phi)\Big) && (3.10)\end{aligned}$$

where $(\nabla\phi)^2 = \delta^{ij}\partial_i\phi\partial_j\phi$.

Comparing with the perfect fluid energy-momentum tensor (2.15), we can write the density, ρ_ϕ, and pressure, p_ϕ, as:

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{(\nabla\phi)^2}{a^2} + V(\phi), && (3.11)\\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}\frac{(\nabla\phi)^2}{a^2} - V(\phi). && (3.12)\end{aligned}$$

During inflation, the potential energy dominates leading to an accelerated expansion. So, if there is a region of the universe where the field is homogeneous and rolls down its potencial slowly, we get an accelerated expansion that dilutes any energy gradient.

This inflationary period will last as long as the kinetic energy is negligible, that is, during the phase where the potential is flat. When the potential becomes steeper and the field moves faster, inflation will end. After that, the inflaton potential energy will be transferred into the radiation and ordinary matter.

Therefore, as the universe expands, spatial variations will become less important. So it is physically reasonable to neglect spatial partial derivatives as they become negligible. Thus, we can simplify the expressions for density and pressure as

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{3.13}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = \mathcal{L}_\phi. \tag{3.14}$$

If we consider the time component of the non-conservation equation of the energy-momentum tensor (3.4), it is possible to find the relation:

$$\dot{\rho} + 3H(\rho + p) = -\frac{F_2(R)}{f_2(R)}(\rho + p)\dot{R}. \tag{3.15}$$

Similarly to the General Relativity case, we can write the equation of the temporal evolution of matter density, considering $p = w\rho$, as

$$\frac{\dot{\rho}}{\rho} = -(1+w)\left(3\frac{\dot{a}}{a} + \frac{F_2(R)\dot{R}}{f_2(R)}\right). \tag{3.16}$$

If $w = -1$, the previous equation reduces to

$$\rho = \rho_0. \tag{3.17}$$

We note that, in the De Sitter era, the matter density does not change even for the non-minimal coupling.

On other hand, if $w \neq -1$ we obtain

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} \left(\frac{f_2(R_0)}{f_2(R)}\right)^{1+w}, \tag{3.18}$$

where R_0 is the curvature value at the present.

In the case where we have non-relativistic matter, $w = 0$, we obtain that

$$\rho = \rho_0 \frac{f_2(R_0)}{f_2(R)} \left(\frac{a_0}{a}\right)^3. \tag{3.19}$$

In the case of radiation, this expression is also found in [22], by replacing $w = \frac{1}{3}$. We get

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^4 \left(\frac{f_2(R_0)}{f_2(R)}\right)^{\frac{4}{3}}. \tag{3.20}$$

Friedmann's equation is fundamental to the description of the expansion of the universe. Thus it would be important to deduce the generalization of this equation for the non-minimal coupling theory.

The time-time component equation of (3.2) [see Appendix C] corresponds to the modified Friedmann equation that takes the form:

$$\begin{aligned} H^2 \quad &= \quad \frac{1}{6\Big(F_1(R)+16\pi G F_2(R)\mathcal{L}_\phi\Big)} \left(16\pi G \rho_\phi f_2(R) - 6H \frac{\partial \Big(F_1(R)+16\pi G F_2(R)\mathcal{L}_\phi\Big)}{\partial t} \right. \\ & \left. -f_1(R) + \Big(F_1(R)+16\pi G F_2(R)\mathcal{L}_\phi\Big) R \right). \end{aligned} \tag{3.21}$$

In the same way, we can write the modified Raychaudhuri equation, using the ii component of (3.2) [see Appendix C]. It takes the form:

$$\begin{aligned} &\Big(F_1(R)+16\pi G F_2(R)\mathcal{L}_\phi\Big)\left(\frac{\ddot{a}}{a}+H^2\right) = -8\pi G p_\phi f_2(R) - \frac{\partial^2 \Big(F_1(R)+16\pi G F_2(R)\mathcal{L}_\phi\Big)}{\partial t^2} \\ &-3H\frac{\partial \Big(F_1(R)+16\pi G F_2(R)\mathcal{L}_\phi\Big)}{\partial t} - \frac{1}{2}\left(f_1(R) - \Big(F_1(R)+16\pi G F_2(R)\mathcal{L}_\phi\Big) R\right). \end{aligned} \tag{3.22}$$

It is easy to see that replacing $f_1(R) = R$ and $f_2(R) = 1$ in the previous equations leads to equations (2.17) and (2.18).

For inflation to occur, the inflaton field must mimic a cosmological constant. In that case, the kinetic energy is negligible when compared to the potential energy and $p_\phi \simeq -\rho_\phi$. Then, using relations (3.13) and (3.14), the following conditions, known as slow-roll conditions, are required:

1. The kinetic energy of the inflaton is much smaller than the potential energy. So,

$$\frac{1}{2}\dot{\phi} \ll V(\phi); \tag{3.23}$$

2. The acceleration of the field is small so that the inflaton velocity does not increase fast. So,

$$\ddot{\phi} \ll 3H\dot{\phi}. \tag{3.24}$$

The slow-roll conditions give us the guarantee that the inflaton's motion is sufficiently damped to allow for the accelerated expansion of the universe.

Taking into account that the slow-roll conditions give us $\mathcal{L}_\phi = p_\phi = -\rho_\phi$, the 00 and ii compoments of equation (3.2) can be written as

$$f_2(R)\rho_\phi = 3FH^2 + 3H\dot{F} + \frac{1}{2}\Big(8\pi G f_1(R) - RF\Big) \tag{3.25}$$

and

$$f_2(R)p_\phi = -3FH^2 - 3H\dot{F} - \frac{1}{2}\Big(8\pi G f_1(R) - RF\Big) - 2F\dot{H} - \ddot{F} \tag{3.26}$$

where $F = \Big(8\pi G F_1(R) - 2F_2(R)\rho_\phi\Big)$.

From these relations and from the slow-roll approximation, it follows that the temporal derivatives of the curvature scalar R and the matter density ρ vanish.

Similarly to [21], let's consider that

$$f_1(R) = R. \tag{3.27}$$

In this way we are isolating the effects of the non-minimal coupling between matter and curvature in onder to understand its impact.

Thus, it is possible to simplify the modified Friedmann's equation (3.21) in the slow-roll regime to get the expression:

$$H^2 = \left(\frac{8\pi G f_2(R)}{1 + 16\pi G \rho_\phi F_2(R)}\right)\frac{\rho_\phi}{3}. \tag{3.28}$$

Observing the above equation, it is easy to verify that the modified Friedmann equation depends on the energy density of the inflaton and on the non-minimal coupling function considered.

Likewise, we can simplify the modified Raychaudhuri equation (3.22), obtaining

$$2\frac{\ddot{a}}{a} = \frac{8\pi G\rho_\phi f_2(R)}{1+8\pi G\rho_\phi F_2(R)}\left(1-\frac{1}{3}\left(\frac{1+32\pi G\rho_\phi F_2(R)}{1+16\pi G\rho_\phi F_2(R)}\right)\right) = 2H^2. \tag{3.29}$$

It is simple to see that the limit of General Relativity, where $f_2(R) = 1$, is verified in previous equations and that there is consistency in all considered approximations.

This model proves to be versatile, being possible to consider very general non-minimal couplings between curvature and matter.

3.3 Cubic model

In the physics literature, the non-minimal coupling is written as a linear combination of the powers of the curvaturar scalar, that is

$$f_2(R) = \sum_{n=-\infty}^{+\infty} a_n\left(\frac{R}{R_0}\right)^n = \sum_{n=-\infty}^{+\infty} \Upsilon_n R^n, \tag{3.30}$$

where a_n and R_0 are constants that change with the evolutionary phase of the universe considered and Υ_n will be seen as a coupling constant that depends on the characteristics of the phase considered.

In this work we are going to consider a coupling function, that is relevant for the early universe high curvature regime, namely:

$$f_2(R) = 1 + \xi R^3, \tag{3.31}$$

where $\xi \geq 0$ parameterizes the deviation from General Relativity. Due to the characteristics of the inflationary period, this coupling constant will necessarily be small. With this representation we want to propose a new point of view on the theory of cubic non-minimal coupling proven to be relevant in [21].

Given that

$$R = 6\left(\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right) = 6\left(\frac{\ddot{a}}{a}+H^2\right), \tag{3.32}$$

considering the slow-roll regime and using equation (3.29), we get

$$R \simeq 12H^2. \tag{3.33}$$

Replacing the previous result and the non-minimal coupling function (3.31) in the modified Friedmann equation (3.28) we obtain:

$$H^2 = \frac{1}{12}\frac{\left(36\xi^2(8\pi G\rho)^3 + \sqrt{6\xi^3(8\pi G\rho)^3\left(216\xi(8\pi G\rho)^3+1\right)}\right)^{\frac{2}{3}} - 8\pi G 6^{\frac{1}{3}}\xi\rho}{8\pi G 6^{\frac{2}{3}}\xi\rho\left(36\xi^2(8\pi G\rho)^3 + \sqrt{6\xi^3(8\pi G\rho)^3\left(216\xi(8\pi G\rho)^3+1\right)}\right)^{\frac{1}{3}}}. \tag{3.34}$$

The modified Friedmann equation written in this new way allows to obtain a Taylor expansion around $\xi = 0$ thus accessing the strength of the deviation from the usual Friedmann's equation:

$$H^2 = 8\pi G\frac{\rho}{3} - 32(8\pi G)^4\frac{\rho^4}{3}\xi + \mathcal{O}(\xi^2), \tag{3.35}$$

where it is easy to see the General Relativity limite ($\xi = 0$).

We are now able to solve the modified Friedmann equation for the different phases of the universe. We will use the equations (3.17), (3.19) and (3.20) in the modified Friedmann equation.

In the De Sitter phase, it is possible to exactly solve (3.34), since the density is constant. So, from (3.17), the scale factor takes the form:

$$a(t) = a_0 e^{f(\xi)(t-t_0)}, \tag{3.36}$$

where we consider the positive solution and $f(\xi)$ is given by:

$$f(\xi) = \sqrt{\frac{1}{12}\frac{\left(36\xi^2(8\pi G\rho_0)^3 + \sqrt{6\xi^3(8\pi G\rho_0)^3\left(216\xi(8\pi G\rho_0)^3+1\right)}\right)^{\frac{2}{3}} - 8\pi G 6^{\frac{1}{3}}\xi\rho_0}{8\pi G 6^{\frac{2}{3}}\xi\rho_0\left(36\xi^2(8\pi G\rho_0)^3 + \sqrt{6\xi^3(8\pi G\rho_0)^3\left(216\xi(8\pi G\rho_0)^3+1\right)}\right)^{\frac{1}{3}}}}. \tag{3.37}$$

If we expand the previous equation in Taylor series, around $\xi = 0$, it is possible to see that

$$f(\xi) = H_0 - 1296 H_0^7\xi + \mathcal{O}(\xi^2). \tag{3.38}$$

Then $f(0) = H_0$, where H_0 is the same in equation (2.26).

Graphically, it is possible to verify that $f(\xi)$ is always less than H_0, see Figure 3.1. It is also possible to verify that the derivative of $f(\xi)$ in order to ξ takes a negative value for any $\xi > 0$.

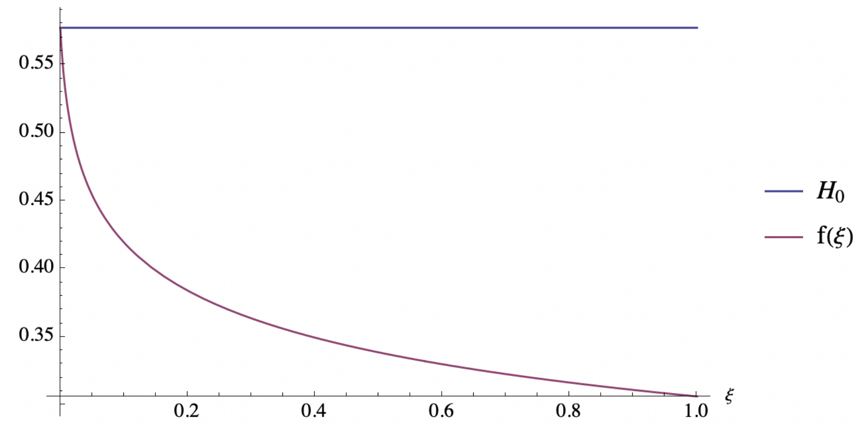

FIGURE 3.1: Comparison between H_0 and $f(\xi)$. In this representation it is considered that $8\pi G\rho_0 = 1$.

It is possible to see in the following Figure 3.2 the graphical representation of the scale factor for different values of ξ.

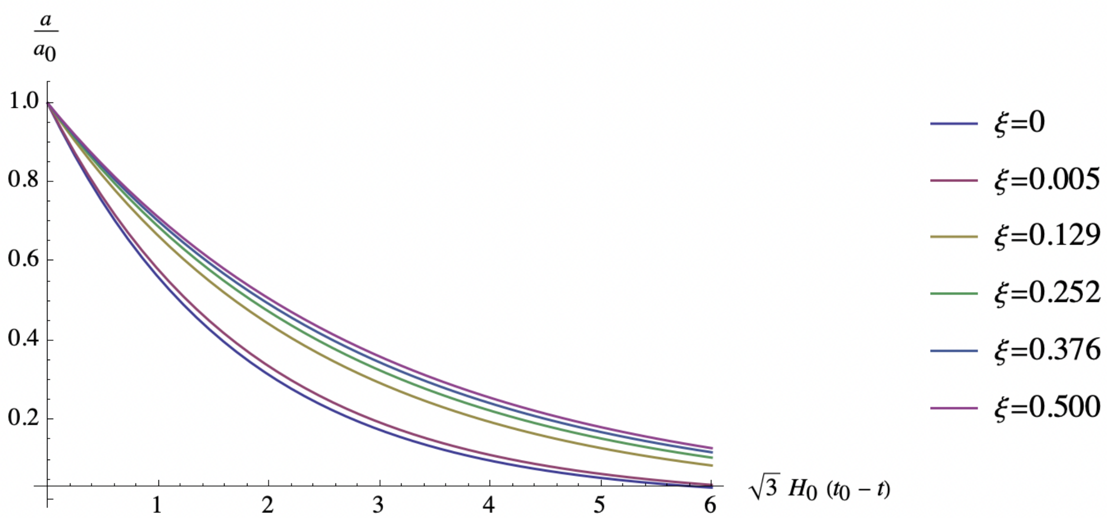

FIGURE 3.2: Behavior of the scale factor in the De Sitter phase, in the case of non-minimal cubic coupling for values of ξ between 0 and 0.500. In this representation it is considered that $8\pi G\rho_0 = 1$.

The blue curve represents the General Relativity solution. We can see that the scale factor is affected by the non-minimal coupling constant, ξ.

Note that $t \rightarrow -\infty$ points in the direction of the beginning of the universe, so the scale factor goes to zero, for all ξ. When $t \rightarrow t_0$, we are nearby the present time, so the scale factor takes the present value for all ξ.

We can see in Figure 3.2 that as we increase the coupling constant ξ, the scale factor is flattening. So we expect that the end time of inflation will vary with the coupling constant considered. In particular we expect that inflation act longer as ξ increases.

Slow-roll inflation, where the energy of the inflation potential field dominates, provides an exponentially accelerated expanding universe. This phase extends between 50 and 60 e-folds, until the potential steepens and slow-roll conditions are invalid.

After that, the kinetic energy of the inflaton becomes of the same order of magnitude as the potential energy, or maybe more. Then the field rapidly evolves to the minimum of the potential where, due to scalar field fluctuations around the minimum, allows the universe to warm up. This phase is called reheating [23][24] and will be relevant later on.

Therefore, we will consider that inflation acts for approximately 60 e-folds. So it is necessary that the scale factor increase at least e^{60}.

We will denote by a_i and a_f the scale factor at the start and end of inflation, respectively.

So, we have

$$\frac{a_f}{a_i} = e^{f(\xi)\Delta t} \simeq e^{60}, \tag{3.39}$$

where $\Delta t = t_f - t_i$ is the temporal duration of inflation.

It is easily observed that $\Delta t(\xi) \simeq \frac{60}{f(\xi)}$. Considering the approach to General Relativity $\Delta t_{GR} \simeq \frac{60}{H_0}$.

Using the result observed in Figure 3.1, it is possible to conclude that

$$\Delta t(\xi) \geq \Delta t_{GR}. \tag{3.40}$$

So it is easily demonstrated that inflation acts longer in the case of the non-minimal coupling theory, as we were expecting. Since $f(\xi)$ decreases with ξ, then stronger coupling constant implies longer inflation time.

It is easy to conclude that a larger coupling constant ξ implies greater changes in the inflation dynamics compared to General Relativity.

What we want to understand in the following chapter is whether these changes will have consequences in the behavior of the magnetic field.

Just to understand how this cubic non-minimal coupling model affects the remaining phases, due to the complexity of the differential equation, we obtain numerical solutions for the phases dominated by dust and by radiation.

In the radiation era the behavior is represented in Figure 3.3 and in the dust era the behavior is represented in Figure 3.4.

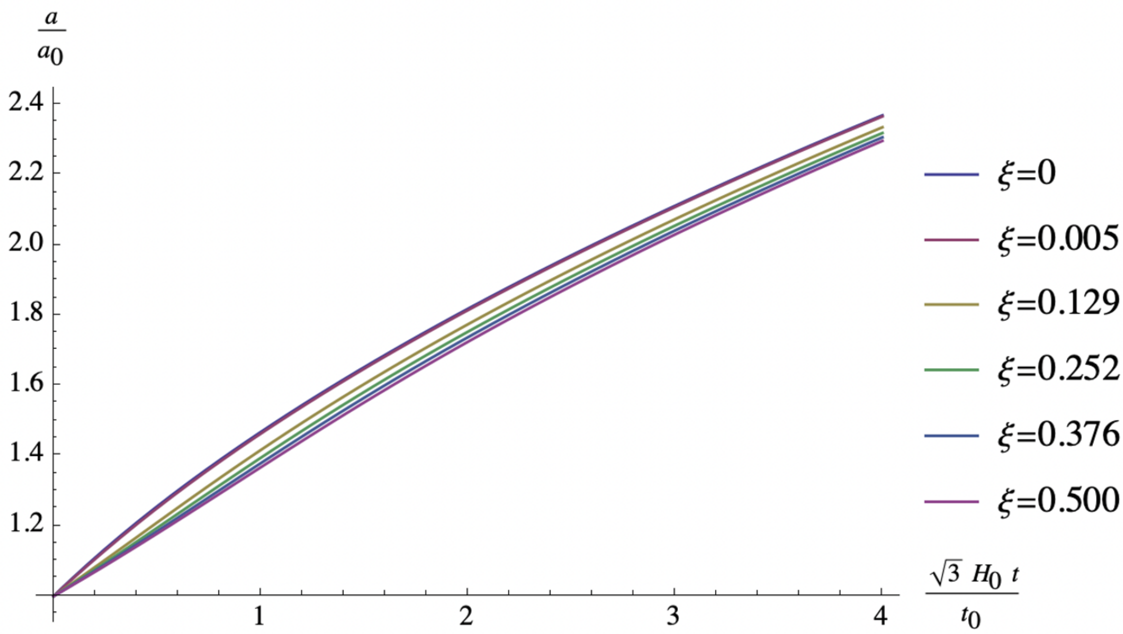

FIGURE 3.3: Behavior of the scale factor in the radiation phase, in the case of non-minimal cubic coupling for values of ξ between 0 and 0.500. In this representation it is considered that $8\pi G\rho_0 = 1$.

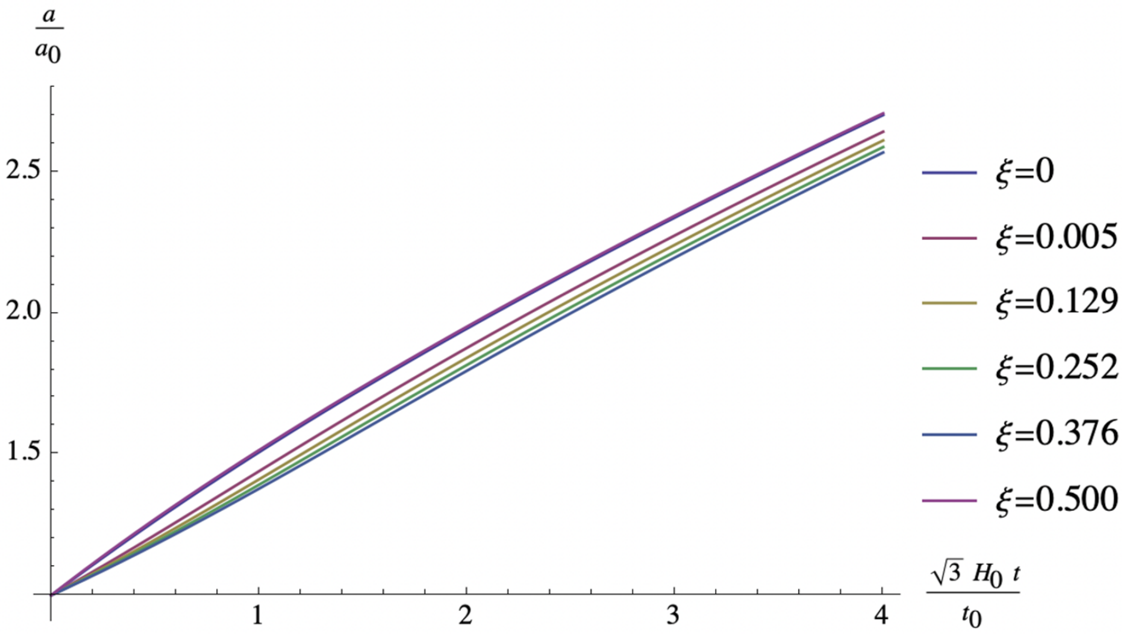

FIGURE 3.4: Behavior of the scale factor in the dust phase, in the case of non-minimal cubic coupling for values of ξ between 0 and 0.500. In this representation it is considered that $8\pi G\rho_0 = 1$.

In the previous figures it is possible to verify that the non-minimal coupling between curvature and matter does not significantly affect the remaining phases of the evolution of the universe, namely the radiation and dust phases.

Chapter 4

Primordial Magnetic field

In the 1940s the galactic magnetic field was theoretically proposed [25] [26] and detected [27][28]. Since then, the galactic and extragalactic magnetic fields have been studied extensively. However, questions about its origin remain unresolved, making it one of the most fascinating challenges of modern astrophysics.

Magnetic fields are detected on a wide variety of astrophysical scales. Observations of nearby galaxies reveal magnetic fields with an intensity of 10-30 μG [29].

One of the most discussed models explaining magnetic fields is the dynamo mechanism. With this effect, the fields are continuously generated through the combined action of differential rotation and turbulence. However, this mechanism requires a seed magnetic field to start with.

One possibility is to assume a primordial origin of the magnetic field, before the first galaxies were formed. In this chapter we will analyze primordial magnetic fields originated by electromagnetic fluctuations during the inflation period.

This idea has been first considered by Turner and Widrow in the context of General Relativity [5] and we will now analyze the behavior of a primordial magnetic field generated by inflation in the context of the non-minimal coupled matter-curvature described in the previous chapter.

This chapter is divided in two sections. In the first section we will deduce Maxwell's equations in the context of the non-minimal coupling theory. In the second we will discuss the consequences of this non-minimal coupling on the behavior of the magnetic field during different evolutionary phases of the universe, with special emphasis on the inflationary period.

4.1 Maxwell equations in the non-minimal coupling model

In this section, we deduce the modified Maxwell equations in context of the non-minimal coupling theory. For simplicity, we will use the metric (2.3), written in conformal time.

Let's consider the Electromagnetic Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.1}$$

where $F^{\mu\nu}$ is the Faraday tensor.

It is possible to write

$$F_{\mu\nu} = a^2 \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}, \tag{4.2}$$

where E_i and B_i are the electric and magnetic fields, respectively, with $i = x, y, z$.

Using the relation (2.13), it is possible to find the energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} &= -2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} - \frac{2}{\sqrt{-g}}\mathcal{L}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} && (4.3) \\ &= F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} && (4.4) \end{aligned}$$

So, we can explicitly write the Faraday tensor as

$$T_{\mu\nu} = a^2(\tau) \begin{bmatrix} \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) & -(\mathbf{E}\times\mathbf{B})_x & -(\mathbf{E}\times\mathbf{B})_y & -(\mathbf{E}\times\mathbf{B})_z \\ -(\mathbf{E}\times\mathbf{B})_x & \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) - E_x^2 - B_x^2 & -E_xE_y - B_xB_y & -E_xE_z - B_xB_z \\ -(\mathbf{E}\times\mathbf{B})_y & -E_xE_y - B_xB_y & \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) - E_y^2 - B_y^2 & -E_yE_z - B_yB_z \\ -(\mathbf{E}\times\mathbf{B})_z & -E_xE_z - B_xB_z & -E_yE_z - B_yB_z & \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) - E_z^2 - B_z^2 \end{bmatrix}, \tag{4.5}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field vectors, respectively.

In order to obtain the Maxwell equations, we can use the action (3.1) with the Lagrangian density (4.1) and vary with respect to the 4-potencial, $A_\mu = (\Phi, \mathbf{A})$, where Φ is the electric potential and $\mathbf{A}$ is the magnetic potential:

$$\delta S = \int \sqrt{-g} f_2(R) \delta \mathcal{L} d^4x = -\frac{1}{4} \int \sqrt{-g} f_2(R) \delta(F_{\mu\nu}F^{\mu\nu}) d^4x. \tag{4.6}$$

We know that $F^{\mu\nu}\delta F_{\mu\nu} = \delta F^{\mu\nu} F_{\mu\nu}$, so we can write that

$$\begin{aligned} \delta S &= -\frac{1}{2} \int \sqrt{-g} f_2(R) F^{\mu\nu} \delta F_{\mu\nu} d^4x & (4.7) \\ &= -\frac{1}{2} \int \sqrt{-g} f_2(R) F^{\mu\nu} \Big(\nabla_\mu(\delta A_\nu) - \nabla_\nu(\delta A_\mu) \Big) d^4x. & (4.8) \end{aligned}$$

Rearranging the indices in the previous equation and recalling that the Faraday tensor is antisymmetric, it is possible to rewrite δS as:

$$\delta S = -\int \sqrt{-g} f_2(R) F^{\mu\nu} \nabla_\mu(\delta A_\nu) dx^4. \tag{4.9}$$

Applying an integration by parts, the properties of tensor derivatives and simplifying, we obtain:

$$\delta S = \int \nabla_\mu(\sqrt{-g} f_2(R) F^{\mu\nu}) \delta A_\nu d^4x. \tag{4.10}$$

and thus half of Maxwell's equations:

$$\nabla_\mu \Big(\sqrt{-g} f_2(R) F^{\mu\nu} \Big) = 0. \tag{4.11}$$

Writing equations (4.11) explicitly, we get the following four equations:

$$\frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = 0; \tag{4.12}$$

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau} \Big(a^2 f_2(R) E_x \Big) + \frac{\partial}{\partial z} B_y + \frac{\partial}{\partial y} B_z = 0; \tag{4.13}$$

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau}\left(a^2 f_2(R) E_y\right) + \frac{\partial}{\partial x} B_z + \frac{\partial}{\partial z} B_x = 0; \tag{4.14}$$

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau}\left(a^2 f_2(R) E_z\right) + \frac{\partial}{\partial y} B_x + \frac{\partial}{\partial x} B_y = 0. \tag{4.15}$$

The homogeneous Maxwell equations are obtained by the variation of the action with a Lagrangian density $\mathcal{L} = -\frac{1}{2} F_{\mu\nu} F^{*\mu\nu}$, where $F^{*\mu\nu} = \frac{1}{2} F_{\alpha\beta} \epsilon^{\alpha\beta\mu\nu}$ and $\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita tensor. Thus

$$\nabla_\alpha \left(\sqrt{-g} f_2(R) F_{\mu\nu} \epsilon^{\alpha\beta\mu\nu}\right) = 0. \tag{4.16}$$

As before, we can write equations (4.16) explicitly, obtaining the following four equations:

$$\frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y + \frac{\partial}{\partial z} B_z = 0; \tag{4.17}$$

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau}\left(a^2 f_2(R) B_x\right) + \frac{\partial}{\partial y} E_z + \frac{\partial}{\partial z} E_y = 0; \tag{4.18}$$

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau}\left(a^2 f_2(R) B_y\right) + \frac{\partial}{\partial z} E_x + \frac{\partial}{\partial x} E_z = 0; \tag{4.19}$$

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau}\left(a^2 f_2(R) B_z\right) + \frac{\partial}{\partial x} E_y + \frac{\partial}{\partial y} E_x = 0. \tag{4.20}$$

So we can gather the four Maxwell equations as

$$\nabla \cdot \mathbf{B} = 0; \tag{4.21}$$

$$\nabla \cdot \mathbf{E} = 0; \tag{4.22}$$

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau}\left(a^2 f_2(R) \mathbf{B}\right) + \nabla \times \mathbf{E} = 0; \tag{4.23}$$

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau}\left(a^2 f_2(R) \mathbf{E}\right) - \nabla \times \mathbf{B} = 0. \tag{4.24}$$

It is easy to verify that for $f_2(R) = 1$, in the minimal coupling regime, we recover the usual Maxwell equations.

Applying the curl to equation (4.24) to get:

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau}\left(a^2 f_2(R) \nabla \times \mathbf{E}\right) - \nabla(\nabla \cdot \mathbf{B}) + \nabla^2 \mathbf{B} = 0. \tag{4.25}$$

Using equation (4.21) in previous equation we obtain

$$\frac{1}{a^2 f_2(R)} \frac{\partial}{\partial \tau} \left(a^2 f_2(R) \nabla \times \mathbf{E} \right) + \nabla^2 \mathbf{B} = 0. \tag{4.26}$$

Through substitution of equation (4.23) in the previous equation, we get

$$\frac{1}{a^2 f_2(R)} \frac{\partial^2}{\partial \tau^2} \left(a^2 f_2(R) \mathbf{B} \right) - \nabla^2 \mathbf{B} = 0. \tag{4.27}$$

In order to solve equation (4.27) we expand it in terms of the Fourier components of **B**.

Defining

$$\mathbf{F}_k(\tau) = a^2 f_2(R) \int e^{-\mathbf{k}.\mathbf{x}} \mathbf{B} d^3 x \tag{4.28}$$

and replacing in the equation (4.27), we obtain

$$\frac{d^2}{d\tau^2} \mathbf{F}_k(\tau) + k^2 \mathbf{F}_k(\tau) = 0, \tag{4.29}$$

where $\mathbf{F}_k$ is a measure of the magnetic flux associated with the comoving scale $\lambda \sim k^{-1}$.

The solution to the previous ordinary differential equation for each mode k is

$$\mathbf{F}_k(\tau) = c_1 e^{ik\tau} + c_2 e^{-ik\tau}, \tag{4.30}$$

where c_1 and c_2 are constants.

So we can write the magnetic field as

$$\mathbf{B} = \frac{1}{a^2 f_2(R)} \int \mathbf{F}_k(\tau) e^{ikx} dk. \tag{4.31}$$

Since the integral is bounded, it is possible to state that

$$B \propto \frac{1}{a^2 f_2(R)}, \tag{4.32}$$

where B is the intensity of magnetic field.

If we consider the minimal coupling limit, $f_2(R) = 1$, we see that magnetic field is diluted as the scale factor increases. This leads to a problem in General Relativity, as during the inflationary period the scale factor increases exponentially and the magnetic field is dramatically diluted.

4.2 Magnetic field during inflation

In this section we will discuss the results obtained for the magnetic field during inflation, with this new point of view of the non-minimal coupling theory. We want to compare our results with the results obtained for General Relativity. We will also study the consequences of our model after reheating.

What we want to understand is, to what extent will a non-minimal coupling modify the dynamical evolution of B during inflation. As we have an additional dependence on the inverse of $f_2(R)$, one might hope that the magnetic field seeds will not be completly diluted after inflation.

The dilution factor of the magnetic field during inflation is

$$\frac{B_f}{B_i} \simeq \left(\frac{a_i}{a_f}\right)^2 \frac{f_2(R_i)}{f_2(R_f)}, \tag{4.33}$$

and due to (3.39) we have

$$\frac{B_f}{B_i} \simeq 10^{-53} \frac{f_2(R_i)}{f_2(R_f)}. \tag{4.34}$$

Let's consider the cubic non-minimal coupling function for reheating

$$f_{2_{RH}}(R) = \varsigma R^3, \tag{4.35}$$

where ς is a coupling constant that depends on the characteristics of the reheating [22].

It is known that in this phase $H_{RH} \simeq \frac{\pi}{\sqrt{90}}\sqrt{8\pi G}T_{RH}^2$, where T_{RH} is the reheating temperature.

So, by (3.33), it is possible to write

$$R_{RH} \simeq \frac{2\pi^2}{15} 8\pi G T_{RH}^4. \tag{4.36}$$

Therefore, we can write

$$\frac{B_{RH}}{B_i} \simeq 10^{-53} \left(\frac{H_i}{H_{RH}}\right)^6, \tag{4.37}$$

where H_i is the expansion rate at inflation and $H_i \simeq \frac{1}{3}\sqrt{8\pi G}\Delta^2$, where we assume typically that $\Delta \simeq 10^{-3} M_{pl}$, with M_{pl} the Planck mass.

So, it is possible to write

$$\frac{B_{RH}}{B_i} \simeq 10^{-53} \left(\frac{\Delta}{T_{RH}} \right)^{12}. \tag{4.38}$$

If we assume that the reheating temperature is between the values $10^{-9} M_{pl} \lesssim T_{RH} \lesssim 10^{-4} M_{pl}$, we can estimate that

$$10^{-41} \lesssim \frac{B_{RH}}{B_i} \lesssim 10^{9}. \tag{4.39}$$

Although a more detailed study of the quantum fluctuations remains to be done, we can observe a possible amplification of the primordial magnetic field in the context of the non-minimal coupling theory.

Chapter 5

Conclusion

The origin of the cosmic magnetic field remains an open question. The dynamo effect can justify the field strength observed today, however in all these mechanisms there is the need an of initial seed magnetic field, with a mimimum intensity of $B \sim 10^{-19} G$ [5].

One class of models attributes the generation of initial magnetic fields to a primordial origin, in the beginning of the universe during inflation. It is then assumed that the magnetic fields result from the amplification of perturbations in the primordial magnetic field.

However, according to General Relativity, due to the exponential behavior of the scale factor during inflation, the magnetic field strength dilutes during this period, since $B \propto a^{-2}$. This provides a magnetic field dilution factor of 10^{-58} during the inflationary period.

In order to study the possibility of minimizing this dilution effect, it was analized in this work the influence of the theory of non-minimal cubic coupling between curvature and matter in the magnetic field behavior. This alternative theory to General Relativity has shown its versatility in previous studies [21][17].

Chapter 2 reviews some cosmological facts. We introduce the characteristics of the metric considered, briefly deduce the Friedmann equations and present the scale factor in the different phases of the universe's evolution. Open issues were also discussed.

In Chapter 3 the theory of non-minimal coupling between curvature and matter is introduced. Here it is found the field equations and the modified Friedmann equation.

It was obtained the general expression for the density of matter, considering the different types of matter in the universe: $\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} \left(\frac{f_2(R_0)}{f_2(R)}\right)^{1+w}$, with $w \neq -1$. In the case of the cosmological constant, the density of matter remains constant, not depending on the non-minimal coupling function.

Also in this chapter, it has been specified the cubic non-minimal coupling. This coupling was presented as a deviation from General Relativity, in order to understand the influence of this coupling in the scale factor and the evolution of the universe. A graphical analysis of the behavior of the scale factor during the different evolutionary phases of the universe was presented.

In the last chapter the Maxwell's equations were deduced in the context of non-minimal coupling theory. Thus, it was possible to conclude that the behavior of the magnetic field is different for this theory, namely $B \propto \frac{1}{a^2 f_2(R)}$.

It was then shown that the ratio between the magnetic field in reheating (B_{RH}) and the magnetic field at the beginning of inflation (B_i) is given by $\frac{B_{RH}}{B_i} \simeq 10^{-53} \left(\frac{H_i}{H_{RH}}\right)^6$.

In order to obtain some orders of magnitude, it was considered that the initial magnetic field is due to thermal quantum fluctuations at an inflation scale $\Delta \sim 10^{-3} M_{Pl}$, and the reheating temperature in the range $10^{-9} M_{pl} \lesssim T_{RH} \lesssim 10^{-4} M_{pl}$. So we conclude that the ratio of magnetic field intensity can take values of $10^{-41} \lesssim \frac{B_{RH}}{B_i} \lesssim 10^9$.

This study supports the idea that non-minimal coupling theories might have an impact on the problem of primordial magnetic fields. A more detailed study of the magnetic field fluctuations in the context of this theory will be presented in a future study.

Appendix A

The field equations in non-minimal matter-curvature coupling theories

It is possible to deduce field equations through the function variation of the action (3.1). So, we can write that

$$\delta S = \int \delta(\sqrt{-g})\left(\frac{1}{16\pi G}f_1(R) + f_2(R)\mathcal{L}\right) + \sqrt{-g}\left(\frac{1}{16\pi G}\delta(f_1(R)) + \delta(f_2(R))\mathcal{L} + f_2(R)\delta\mathcal{L}\right)d^4x. \tag{A.1}$$

Using relation (2.13) and $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$, we obtain:

$$\begin{aligned}\delta S &= \int \sqrt{-g}\Bigg[-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\left(\frac{1}{16\pi G}f_1(R) + f_2(R)\mathcal{L}\right) \\ &\quad + \left(\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\delta R + \frac{f_2(R)}{2}\left(\mathcal{L}g_{\mu\nu} - T_{\mu\nu}\right)\delta g^{\mu\nu}\right)\Bigg]d^4x.\end{aligned} \tag{A.2}$$

Additionally, $R = g^{\mu\nu}R_{\mu\nu}$, so $\delta R = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}$. If we replace this relation in the previous equation, we can separate it in two integrals:

$$\begin{aligned}\delta S = \int \sqrt{-g}\Bigg[-\frac{1}{2}g_{\mu\nu}\left(\frac{1}{16\pi G}f_1(R) + f_2(R)\mathcal{L}\right) + \frac{f_2(R)}{2}\left(\mathcal{L}g_{\mu\nu} - T_{\mu\nu}\right) \\ + R_{\mu\nu}\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\Bigg]\delta g^{\mu\nu}d^4x + \int \sqrt{-g}\left[\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\right]g^{\mu\nu}\delta R_{\mu\nu}d^4x.\end{aligned} \tag{A.3}$$

Using the Palatini's lemma, $\delta R_{\mu\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})$, $\nabla_\lambda g^{\mu\nu} = 0$ and $\nabla_\lambda\sqrt{-g} = 0$, we can simplify the second integral L:

$$\begin{aligned} L &= \int \sqrt{-g}\left[\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\right] g^{\mu\nu}\delta R_{\mu\nu} d^4x && \text{(A.4)} \\ &= \int \left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\nabla_\lambda(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu})d^4x && \text{(A.5)} \\ &\quad - \int \left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\nabla_\nu(\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu})d^4x. \end{aligned}$$

Thus, using the properties of the tensor density and the divergence we get:

$$\begin{aligned} L = &- \int \nabla_\lambda\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}d^4x && \text{(A.6)} \\ &+ \int \nabla_\nu\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\lambda\mu}d^4x. \end{aligned}$$

Using the Christofel symbols variation $\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\Big(\nabla_\mu(\delta g_{\sigma\nu}) + \nabla_\nu(\delta g_{\mu\sigma}) - \nabla_\sigma(\delta g_{\mu\nu})\Big)$ and a similar simplification then:

$$\begin{aligned} L &= \int \sqrt{-g}\Bigg[-\nabla_\nu\nabla_\mu\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right) + \nabla^\lambda\nabla_\lambda\Bigg(\frac{1}{16\pi G}F_1(R) && \text{(A.7)} \\ &+F_2(R)\mathcal{L}\Bigg)g_{\mu\nu}\Bigg]\delta g^{\mu\nu}d^4x = -\int\sqrt{-g}\Delta_{\mu\nu}\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\delta g^{\mu\nu}d^4x. \end{aligned}$$

Hence (A.3) can be written as:

$$\begin{aligned} \delta S &= \int \sqrt{-g}\Bigg[-\frac{1}{2}g_{\mu\nu}\left(\frac{1}{16\pi G}f_1(R) + f_2(R)\mathcal{L}\right) + \frac{f_2(R)}{2}\Big(\mathcal{L}g_{\mu\nu} - T_{\mu\nu}\Big) && \text{(A.8)} \\ &+R_{\mu\nu}\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right) - \Delta_{\mu\nu}\left(\frac{1}{16\pi G}F_1(R) + F_2(R)\mathcal{L}\right)\Bigg]\delta g^{\mu\nu}d^4x, \end{aligned}$$

from which follow the field equations

$$\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}\Big)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f_1(R) = 8\pi G f_2(R)T_{\mu\nu} + \Delta_{\mu\nu}\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}\Big), \tag{A.9}$$

where $\Delta_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$ and $\Box = \nabla_\mu\nabla^\mu$.

Or in terms of the Einstein tensor:

$$\left(F_1(R) + 16\pi G F_2(R)\mathcal{L}\right) G_{\mu\nu} = 8\pi G f_2(R) T_{\mu\nu} + \Delta_{\mu\nu}\left(F_1(R) + 16\pi G F_2(R)\mathcal{L}\right) \tag{A.10}$$
$$+\frac{1}{2} g_{\mu\nu}\left(f_1(R) - \left(F_1(R) + 16\pi G F_2(R)\mathcal{L}\right) R\right).$$

Note that when $f_1(R) = R$ and $f_2(R) = 1$, we recover Einstein's equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{A.11}$$

Appendix B

The field equation of the inflaton field in non-minimal matter-curvature coupling theories

We consider the action associated to the Lagrangian density of the inflaton field, taking into account the non-minimal matter-curvature coupling model as

$$S_\phi = \int_M \sqrt{-g}\left(\frac{1}{16\pi G}f_1(R) + f_2(R)\mathcal{L}_\phi\right)d^4x, \tag{B.1}$$

where $\mathcal{L}_\phi$ is given by equation (2.19).

We obtain here the field equation for the inflaton. We can write that

$$\delta S_\phi = \int \sqrt{-g} f_2(R)\delta(\mathcal{L}_\phi)d^4x. \tag{B.2}$$

Since $\delta(\mathcal{L}_\phi) = -\partial_\mu\phi g^{\mu\nu}\partial_\nu(\delta\phi) - \frac{dV(\phi)}{d\phi}\delta\phi$, then we get that

$$\delta S_\phi = -\int \sqrt{-g} f_2(R)\partial_\mu\phi g^{\mu\nu}\partial_\nu(\delta\phi)d^4x - \int \sqrt{-g} f_2(R)\frac{dV(\phi)}{d\phi}\delta\phi d^4x. \tag{B.3}$$

Integrating by parts the first integral it is possible to write

$$\delta S_\phi = \int \sqrt{-g}\left(\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g} f_2(R)\partial_\mu\phi g^{\mu\nu}\right) - f_2(R)\frac{dV(\phi)}{d\phi}\right)\delta\phi d^4x, \tag{B.4}$$

and from the condition $\delta S_\phi = 0$:

$$\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g} f_2(R)\partial_\mu\phi g^{\mu\nu}\right) - f_2(R)\frac{dV(\phi)}{d\phi} = 0. \tag{B.5}$$

In the Robertson-Walker metric $\sqrt{-g} = a^3$ and we obtain:

$$f_2(R)\ddot{\phi} + 3Hf_2(R)\dot{\phi} - \frac{f_2(R)}{a^2}\nabla^2\phi + f_2(R)\frac{dV(\phi)}{\phi} = -F_2(R)\dot{R}\dot{\phi}, \tag{B.6}$$

where $\nabla^2 = \gamma_{ij}\partial_i\partial_j$ and i and j are spatial coordinates.

Finally, invoking homogeneity and isotropy, we obtain the wanted equation:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = -\frac{F_2(R)}{f_2(R)}\dot{R}\dot{\phi}. \tag{B.7}$$

Appendix C

The modified Friedmann equations

The time-time component of equation (3.2) is

$$\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)G_{00}=8\pi Gf_2(R)\rho_\phi+\Delta_{00}\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big) \tag{C.1}$$
$$-\frac{1}{2}\Bigg(f_1(R)-\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)R\Bigg),$$

where $G_{00}=3H^2$.

It is possible to see that

$$g_{\mu\nu}\Box\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)=g_{\mu\nu}\nabla_\alpha\nabla^\alpha\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)$$
$$=g_{\mu\nu}g^{\beta\alpha}\Bigg(\partial_\alpha\partial_\beta\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)-\Gamma^\gamma_{\alpha\beta}\partial_\gamma\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)\Bigg). \tag{C.2}$$

So we can simplify (C.1) as:

$$\begin{aligned}\Delta_{00}\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big) &= \frac{\partial^2\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)}{\partial t^2}-g_{00}\Box\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)\\ &= -3H\frac{\partial\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)}{\partial t}.\end{aligned} \tag{C.3}$$

Replacing the previous relation in equation (C.1) we obtain the modified Friedamnn equation (3.21).

The ii component of (3.2) is given by:

$$\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)G_{11}=8\pi Gf_2(R)T_{11}+\Delta_{11}\Big(F_1(R)+16\pi GF_2(R)\mathcal{L}_\phi\Big)$$
$$+\frac{1}{2}a^2\Bigg(f_1(R)-\Big(F_1(R)+16\pi GF_2(R),\mathcal{L}_\phi\Big)R\Bigg), \tag{C.4}$$

where $G_{11} = -\ddot{a}a - \dot{a}^2$. Using the relation (C.2), it is possible to see that

$$\begin{aligned}\Delta_{11}\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}_\phi\Big) &= -g_{11}\Box\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}_\phi\Big) \\ &= a^2\frac{\partial^2\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}_\phi\Big)}{\partial t^2} + 3\dot{a}a\frac{\partial\Big(F_1(R) + 16\pi G F_2(R)\mathcal{L}_\phi\Big)}{\partial t}.\end{aligned} \tag{C.5}$$

Replacing the previous equation in equation (C.4) leads to equation (3.22).

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