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Higher-order q-derivatives and their applications to subclasses of multivalent Janowski type q-starlike functions

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Abstract

In the present investigation, with the help of certain higher-order q-derivatives, some new subclasses of multivalent q-starlike functions which are associated with the Janowski functions are defined. Then, certain interesting results, for example, radius problems and the results related to distortion, are derived. We also derive a sufficient condition and certain coefficient inequalities for our defined function classes. Some known consequences related to this subject are also highlighted. Finally, the well-demonstrated fact about the (p,q)-variations is also given in the concluding section.

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1 Introduction, motivation and definitions

The class of multivalent or ($\mathfrak{p}$-valent) functions in

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

with series representation

$$f(z) = z^{\mathfrak{p}} + \sum_{n=1}^{\infty} a_{n+\mathfrak{p}} z^{n+\mathfrak{p}} \quad (\mathfrak{p} \in \mathbb{N} = \{1,2,3,\ldots\}), \tag{1.1}$$

is denoted here by $\mathcal{A}(\mathfrak{p})$. We note that

$$\mathcal{A}(1) := \mathcal{A}.$$

Moreover, the class of multivalent or ($\mathfrak{p}$-valent) starlike functions is denoted by $\mathcal{S}^*(\mathfrak{p})$, which consists of functions $f \in \mathcal{A}(\mathfrak{p})$ that satisfy the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (\forall z \in \mathbb{U}).$$

It is easy to see that

$$\mathcal{S}^*(1) = \mathcal{S}^*,$$

where by $\mathcal{S}^*$ we denote the class of starlike functions in the open unit disk $\mathbb{U}$.

We next recall that the class $\mathcal{S}^*$ of starlike functions was generalized by Janowski [7] as follows.

Definition 1 ([7]) A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}^*[X, L]$ if and only if

$$\Re\left(\frac{(L-1)(\frac{zf'(z)}{f(z)}) - (X-1)}{(L+1)(\frac{zf'(z)}{f(z)}) - (X+1)}\right) \geqq 0 \quad (-1 \leqq L < X \leqq 1).$$

In order to present, we adopt the following notations and definitions. Throughout this article we assume that

$$0 < q < 1 \quad \text{and} \quad \mathfrak{p} \in \mathbb{N} = \{1, 2, 3, \dots\}.$$

Definition 2 For $0 < q < 1$, we define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C} \setminus \{0\}), \\ \sum_{k=0}^{j-1} q^k = 1 + q + q^2 + \cdots + q^{j-1} & (\lambda = j \in \mathbb{N}), \\ 0 & (\lambda = 0). \end{cases}$$

Definition 3 The q-factorial $[j]_q!$ q-factorial is defined as follows:

$$[0]_q! := 1 \quad \text{and} \quad [j]_q! = \prod_{k=1}^{j} [k]_q.$$

It is easy to see from Definition 2 and Definition 3 that

$$\lim_{q \to 1-} [j]_q = j \quad \text{and} \quad \lim_{q \to 1-} [j]_q! = j!.$$

Definition 4 For $f \in \mathcal{A}$, the q-difference (or the q-derivative) operator $\mathfrak{D}_q$ in a given subset of the set $\mathbb{C}$ of complex numbers is defined by (see [5] and [6])

$$(\mathfrak{D}_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & (z \neq 0), \\ f'(0) & (z = 0), \end{cases} \tag{1.2}$$

provided that $f'(0)$ exists.

We can easily see from (1.2) that

$$\lim_{q\to 1-}(\mathfrak{D}_q f)(z)=\lim_{q\to 1-}\frac{f(z)-f(qz)}{(1-q)z}=f'(z)$$

for a differentiable function f in a given subset of $\mathbb{C}$. Furthermore, from (1.1) and (1.2) we obtain

$$\big(\mathfrak{D}_q^{(1)}f\big)(z)=[\mathfrak{p}]_q z^{\mathfrak{p}-1}+\sum_{n=1}^{\infty}[n+\mathfrak{p}]_q a_{n+\mathfrak{p}} z^{n+\mathfrak{p}-1}, \tag{1.3}$$

$$\big(\mathfrak{D}_q^{(2)}f\big)(z)=[\mathfrak{p}]_q[\mathfrak{p}-1]_q z^{\mathfrak{p}-2}+\sum_{n=1}^{\infty}[n+\mathfrak{p}]_q[n+\mathfrak{p}-1]_q a_{n+\mathfrak{p}} z^{n+\mathfrak{p}-2}, \tag{1.4}$$

$$\begin{matrix} . & . & . \\ . & . & . \\ . & . & . \end{matrix}$$

$$\big(\mathfrak{D}_q^{(\mathfrak{p})}f\big)(z)=[\mathfrak{p}]_q!+\sum_{n=1}^{\infty}\frac{[n+\mathfrak{p}]_q!}{[n]_q!}a_{n+\mathfrak{p}} z^{n}, \tag{1.5}$$

where $(\mathfrak{D}_q^{(\mathfrak{p})}f)(z)$ denotes the q-derivative of the function $f(z)$ of order $\mathfrak{p}(\mathfrak{p}\in\mathbb{N})$.

Now, for each function f of the class $\mathcal{A}(\mathfrak{p})$, the expression in (1.1) when differentiated s times with respect to z yields

$$\big(\mathfrak{D}_q^{(s)}f\big)(z)=\frac{[\mathfrak{p}]_q!}{[\mathfrak{p}-s]_q!}z^{\mathfrak{p}-s}+\sum_{n=1}^{\infty}\frac{[n+\mathfrak{p}]_q!}{[n+\mathfrak{p}-s]_q!}a_{n+\mathfrak{p}} z^{n+\mathfrak{p}-s}.$$

The branch of q-calculus has many applications in various branches of mathematics and physics. Also, the operator $\mathfrak{D}_q$ (q-derivative operator) has some remarkable and distinctive applications, which makes it significant. Ismail *et al.* [4] were the first who made use of the q-derivative operator and starlike functions and defined a new class $\mathcal{S}^*$ of q-starlike functions. However, initially, it was Srivastava [24] who used the basic (or q-) hypergeometric functions

$${}_{\mathfrak{r}}\Phi_{\mathfrak{s}}\quad \big(\mathfrak{r},\mathfrak{s}\in\mathbb{N}_0=\{0,1,2,\dots\}=\mathbb{N}\cup\{0\}\big)$$

in geometric function theory (GFT) of complex analysis (see, for details, [24]). Very recently, Srivastava's published review article [25] gave another flavor to this subject. In his published review article [25], Srivastava highlighted the triviality of the so-called (p,q)-calculus.

The aforementioned works of Ismail *et al.* [4] and Srivastava [24, 25] motivated a number of mathematicians to give their findings. In the above-cited work by Srivastava [25], many well-known convolution and fractional q-operators were surveyed. For example, Wongsaijai and Sukantamala [35] studied certain subclasses of q-starlike functions from different viewpoints and prospectives. In particular, they studied various coefficient inequalities, inclusion properties, and sufficient conditions. Furthermore, the work of Wongsaijai and

Sukantamala [35] was systematically generalized by Srivastava *et al.* [33]. In fact, by making use of the q-calculus and the Janowski functions, Srivastava *et al.* (see [33, 34]) defined three new subclasses of q-starlike functions. Several other authors (see, for example, [16, 18, 20, 22, 29, 30]) studied and generalized the classes of q-starlike from different viewpoints and prospectives. For some more recent investigation about q-calculus and fractional calculus in geometric functions theory and in other branches of mathematics and physics, we may refer the interested reader to [1–3, 11–15, 17, 19, 21, 26, 27, 31, 32, 36, 37]. In this paper, we shall be concerned mainly with generalizations of the works presented in [8, 33], and [35].

Definition 5 (see [4]) A function $f \in \mathcal{A}$ is in the functional class $\mathcal{S}_q^*$ if

$$f(0) = f'(0) - 1 = 0 \tag{1.6}$$

and

$$\left|\frac{z}{f(z)}(\mathfrak{D}_q f)z - \frac{1}{1-q}\right| \leqq \frac{1}{1-q}. \tag{1.7}$$

By means of some higher-order q-derivatives, the following subclasses of multivalent q-starlike functions, which are associated with the Janowski functions, are defined below.

Definition 6 A multivalent function f of the class $\mathcal{A}(\mathfrak{p})$ is in the class $\mathcal{S}^*_{(q,1)}[\mathfrak{p}, \nu, s, X, L]$ if

$$\Re\left(\frac{(L-1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)}\right) \geqq 0.$$

We call $\mathcal{S}^*_{(q,1)}[\mathfrak{p}, \nu, s, X, L]$ the class of multivalent higher-order q-starlike functions of the first type involving Janowski functions.

Definition 7 A multivalent function f of the class $\mathcal{A}(\mathfrak{p})$ is in the class $\mathcal{S}^*_{(q,2)}[\mathfrak{p}, \nu, s, X, L]$ if

$$\left|\frac{(L-1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - \frac{1}{1-q}\right| < \frac{1}{1-q}.$$

We call $\mathcal{S}^*_{(q,2)}[\mathfrak{p}, \nu, s, X, L]$ the class of multivalent higher-order q-starlike functions of the second type involving Janowski functions.

Definition 8 A multivalent function f of the class $\mathcal{A}(\mathfrak{p})$ is in the class $\mathcal{S}^*_{(q,3)}[\mathfrak{p}, \nu, s, X, L]$ if

$$\left|\frac{(L-1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - 1\right| < 1.$$

We call $\mathcal{S}^*_{(q,3)}[\mathfrak{p}, \nu, s, X, L]$ the class of multivalent higher-order q-starlike functions of the third type involving Janowski functions.

Remark 1 First of all, it is easy to see that, for $(0 \leqq \alpha < 1)$,

$$\mathcal{S}^{*}_{(q,1)}\big[1,1,0,(1-2\alpha),-1\big] = \mathcal{S}^{*}_{(q,1)}(\alpha),$$
$$\mathcal{S}^{*}_{(q,2)}\big[1,1,0,(1-2\alpha),-1\big] = \mathcal{S}^{*}_{(q,2)}(\alpha),$$

and

$$\mathcal{S}^{*}_{(q,3)}\big[1,1,0,(1-2\alpha),-1\big] = \mathcal{S}^{*}_{(q,3)}(\alpha),$$

where the function classes

$$\mathcal{S}^{*}_{(q,1)}(\alpha), \qquad \mathcal{S}^{*}_{(q,2)}(\alpha), \quad \text{and} \quad \mathcal{S}^{*}_{(q,3)}(\alpha)$$

were studied by Wongsaijai and Sukantamala [35]. Secondly, we have

$$\mathcal{S}^{*}_{(q,1)}[1,1,0,X,L] = \mathcal{S}^{*}_{(q,1)}[X,L],$$
$$\mathcal{S}^{*}_{(q,2)}[1,1,0,X,L] = \mathcal{S}^{*}_{(q,2)}[X,L],$$

and

$$\mathcal{S}^{*}_{(q,3)}[1,1,0,X,L] = \mathcal{S}^{*}_{(q,3)}[X,L],$$

where the function classes

$$\mathcal{S}^{*}_{(q,1)}[X,L], \qquad \mathcal{S}^{*}_{(q,2)}[X,L], \quad \text{and} \quad \mathcal{S}^{*}_{(q,3)}[X,L],$$

were studied by Srivastava *et al.* [33]. Thirdly, for $(0 \leqq \alpha < 1)$, we have

$$\mathcal{S}^{*}_{(q,1)}\big[\mathfrak{p},\nu,s,(1-2\alpha),-1\big] = \mathcal{S}^{*}_{(q,1)}(\mathfrak{p},\nu,s,\alpha),$$
$$\mathcal{S}^{*}_{(q,2)}\big[\mathfrak{p},\nu,s,(1-2\alpha),-1\big] = \mathcal{S}^{*}_{(q,2)}(\mathfrak{p},\nu,s,\alpha),$$

and

$$\mathcal{S}^{*}_{(q,3)}\big[\mathfrak{p},\nu,s,(1-2\alpha),-1\big] = \mathcal{S}^{*}_{(q,3)}(\mathfrak{p},\nu,s,\alpha),$$

where the function classes

$$\mathcal{S}^{*}_{(q,1)}(\mathfrak{p},\nu,s,\alpha), \qquad \mathcal{S}^{*}_{(q,2)}(\mathfrak{p},\nu,s,\alpha), \quad \text{and} \quad \mathcal{S}^{*}_{(q,3)}(\mathfrak{p},\nu,s,\alpha),$$

were studied by Khan *et al.* [8].

In this paper, many properties and characteristics such as, for example, sufficient conditions, inclusion results, distortion theorems, and radius problems are investigated. We also indicate relevant connections of our results with those in a number of other related works on this subject.

2 Main results

We start here by giving an inclusion results for the classes

$$\mathcal{S}^*_{(q,1)}[\mathfrak{p},\nu,s,X,L], \qquad \mathcal{S}^*_{(q,2)}[\mathfrak{p},\nu,s,X,L], \quad \text{and} \quad \mathcal{S}^*_{(q,3)}[\mathfrak{p},\nu,s,X,L]$$

of the generalized multivalent q-starlike function classes, which involve the Janowski functions.

Theorem 1 *If* $-1 \leqq L < X \leqq 1$, *then*

$$\mathcal{S}^*_{(q,3)}[\mathfrak{p},\nu,s,X,L] \subset \mathcal{S}^*_{(q,2)}[\mathfrak{p},\nu,s,X,L] \subset \mathcal{S}^*_{(q,1)}[\mathfrak{p},\nu,s,X,L].$$

Proof Firstly, we let $f \in \mathcal{S}^*_{(q,3)}[\mathfrak{p},\nu,s,X,L]$. Then, by Definition 8, we see that

$$\left| \frac{(L-1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - 1 \right| < 1,$$

so that

$$\left| \frac{(L-1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - 1 \right| + \frac{q}{1-q} < 1 + \frac{q}{1-q}. \tag{2.1}$$

By applying the triangle inequality in equation (2.1), we get

$$\left| \frac{(L-1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}. \tag{2.2}$$

The last expression in (2.2) now implies that $f \in \mathcal{S}^*_{(q,2)}[\mathfrak{p},\nu,s,X,L]$, that is,

$$\mathcal{S}^*_{(q,3)}[\mathfrak{p},\nu,s,X,L] \subset \mathcal{S}^*_{(q,2)}[\mathfrak{p},\nu,s,X,L].$$

We next suppose $f \in \mathcal{S}^*_{(q,2)}[\mathfrak{p},\nu,s,X,L]$, so that

$$f \in \mathcal{S}^*_{(q,2)}[\mathfrak{p},\nu,s,X,L] \quad \Longleftrightarrow \quad \left| \frac{(L-1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q},$$

by Definition 7.

Since

$$\frac{1}{1-q} > \left| \frac{(L-1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - \frac{1}{1-q} \right|$$

$$= \left| \frac{1}{1-q} - \frac{(L-1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} \right|,$$

we have

$$\Re\left(\frac{(L-1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} \right) > 0 \quad (z \in \mathbb{U}).$$

This last inequality now shows that $f \in \mathcal{S}^*_{(q,1)}[\mathfrak{p}, \nu, s, X, L]$, that is,

$$\mathcal{S}^*_{(q,2)}[\mathfrak{p}, \nu, s, X, L] \subset \mathcal{S}^*_{(q,1)}[\mathfrak{p}, \nu, s, X, L],$$

which completes the proof of Theorem 1. □

Remark 2 First of all, if we put $\nu = s + 1 = \mathfrak{p} = 1$ in Theorem 1, we get the corresponding result due to Srivastava *et al.* [33]. Secondly, if we put

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad L = -1,$$

Theorem 1 gives the corresponding result that was proved by Khan *et al.* [8]. Thirdly, if we assign the following values to the parameters in Theorem 1:

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad -L = \nu = s + 1 = \mathfrak{p} = 1,$$

we get the result which was proved by Wongsaijai and Sukantamala [35].

Corollary 1 (see [35]) *For* $0 \leqq \alpha < 1$,

$$\mathcal{S}^*_{q,3}(\alpha) \subset \mathcal{S}^*_{q,2}(\alpha) \subset \mathcal{S}^*_{q,1}(\alpha).$$

Finally, below a sufficient condition for the class $\mathcal{S}^*_{(q,3)}[\mathfrak{p}, \nu, s, X, L]$ is given, which also includes the corresponding sufficient conditions for the function classes $\mathcal{S}^*_{(q,1)}[\mathfrak{p}, \nu, s, X, L]$ and $\mathcal{S}^*_{(q,2)}[\mathfrak{p}, \nu, s, X, L]$.

Theorem 2 *A function* $f \in \mathcal{A}(\mathfrak{p})$ *having form* (1.1) *is in the class* $\mathcal{S}^*_{(q,3)}[\mathfrak{p}, \nu, s, X, L]$ *if it satisfies the following coefficient inequality*:

$$\sum_{n=1}^{\infty} \bigl(2\Upsilon_{(2,n)} + |\Upsilon_{(3,n)}|\bigr)|a_{n+\mathfrak{p}}| < |\Upsilon_4| - 2\Upsilon_1, \tag{2.3}$$

where

$$\Upsilon_1 = \left(\frac{[\mathfrak{p}]_q!}{[\mathfrak{p}-s-\nu]_q!} - \frac{[\mathfrak{p}]_q!}{[\mathfrak{p}-s]_q!}\right), \tag{2.4}$$

$$\Upsilon_{(2,n)} = \left(\frac{[n+\mathfrak{p}]_q!}{[n+\mathfrak{p}-s-\nu]_q!} - \frac{[n+\mathfrak{p}]_q!}{[n+\mathfrak{p}-s]_q!}\right), \tag{2.5}$$

$$\Upsilon_{(3,n)} = \left(\frac{(L+1)[n+\mathfrak{p}]_q!}{[n+\mathfrak{p}-s-\nu]_q!} - \frac{(X-1)[n+\mathfrak{p}]_q!}{[n+\mathfrak{p}-s]_q!}\right), \tag{2.6}$$

and

$$\Upsilon_4 = \left(\frac{(L+1)[\mathfrak{p}]_q!}{[\mathfrak{p}-s-\nu]_q!} - \frac{(X+1)[\mathfrak{p}]_q!}{[\mathfrak{p}-s]_q!}\right). \tag{2.7}$$

Proof We begin the proof of Theorem 2 by assuming that the condition in (2.3) holds true. Then it is sufficient to show that

$$\left|\frac{(L-1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - 1\right| < 1.$$

We have

$$\begin{aligned}
&\left|\frac{(L-1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - 1\right| \\
&\quad = 2\left|\frac{z^\nu(\mathfrak{D}_q^{(\nu+s)}f)(z) - (\mathfrak{D}_q^{(s)}f)(z)}{(L+1)(\mathfrak{D}_q^{(\nu+s)}f)(z) - (X+1)(\mathfrak{D}_q^{(s)}f)(z)}\right| \\
&\quad = 2\left|\frac{\Upsilon_1 z^{\mathfrak{p}-s} + \sum_{n=1}^\infty \Upsilon_{(2,n)} a_{n+\mathfrak{p}} z^{n+\mathfrak{p}-s}}{\Upsilon_4 z^{\mathfrak{p}-s} + \sum_{n=1}^\infty \Upsilon_{(3,n)} a_{n+\mathfrak{p}} z^{n+\mathfrak{p}-s}}\right| \\
&\quad \leqq \frac{2\Upsilon_1 + 2\sum_{n=1}^\infty \Upsilon_{(2,n)}|a_{n+\mathfrak{p}}|}{|\Upsilon_4| - \sum_{n=1}^\infty |\Upsilon_{(3,n)}||a_{n+\mathfrak{p}}|} \quad \bigl(|z| = 1\bigr),
\end{aligned} \tag{2.8}$$

where Υ_1, $\Upsilon_{(2,n)}$, $\Upsilon_{(3,n)}$, and Υ_4 are given by (2.4), (2.5), (2.6), and (2.7) respectively. We see that the last expression in (2.8) is bounded above by 1 if

$$\sum_{n=1}^\infty \bigl(2\Upsilon_{(2,n)} + |\Upsilon_{(3,n)}|\bigr)|a_{n+\mathfrak{p}}| < |\Upsilon_4| - 2\Upsilon_1,$$

which completes the proof of Theorem 2. □

Remark 3 First of all, if we put $\nu = s + 1 = \mathfrak{p} = 1$ in Theorem 2, we get the corresponding result due to Srivastava *et al.* [33]. Secondly, if we put

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad L = -1,$$

Theorem 2 is reduced to the known result, which was stated and proved by Khan *et al.* [8].

3 Analytic functions with negative coefficients

The main aim of this section is to present some classes of multivalent q-starlike functions involving the functions with negative coefficients. A subset of the class $\mathcal{A}(\mathfrak{p})$ which contains all such functions with negative coefficient, that is,

$$f(z) = z^{\mathfrak{p}} - \sum_{n=1}^{\infty} |a_{n+\mathfrak{p}}| z^{n+\mathfrak{p}} \tag{3.1}$$

will be denoted here by $\mathcal{T}$. We also let

$$\mathcal{TS}^*_{(q,j)}[\mathfrak{p},\nu,s,X,L] := \mathcal{S}^*_{(q,j)}[\mathfrak{p},\nu,s,X,L] \cap \mathcal{T} \quad (j=1,2,3). \tag{3.2}$$

Theorem 3 *If* $-1 \leqq L < X \leqq 1$, *then*

$$\mathcal{TS}^*_{(q,1)}[\mathfrak{p},\nu,s,X,L] \equiv \mathcal{TS}^*_{(q,2)}[\mathfrak{p},\nu,s,X,L] \equiv \mathcal{TS}^*_{(q,3)}[\mathfrak{p},\nu,s,X,L].$$

Proof By the virtue of Theorem 1, it suffices to show that

$$\mathcal{TS}^*_{(q,1)}[\mathfrak{p},\nu,s,X,L] \subseteq \mathcal{TS}^*_{(q,3)}[\mathfrak{p},\nu,s,X,L].$$

Indeed, in the light of Definition 6 for a function $f \in \mathcal{TS}^*_{(q,1)}[\mathfrak{p},\nu,s,X,L]$, we have

$$\Re\left(\frac{(L-1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)}\right) \geqq 0,$$

so that

$$\Re\left(\frac{(L-1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X-1)}{(L+1)\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)} - (X+1)} - 1\right) \geqq -1.$$

After some elementary and simple calculations, we deduce that

$$\Re\left(\frac{2[(\mathfrak{D}_q^{(s)}f)(z) - z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)]}{(L+1)z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z) - (X+1)(\mathfrak{D}_q^{(s)}f)(z)}\right) \geqq -1,$$

that is,

$$-2\Re\left(\frac{\Upsilon_1 z^{\mathfrak{p}-s} - \sum_{n=1}^{\infty} \Upsilon_{(2,n)} a_{n+\mathfrak{p}} z^{n+\mathfrak{p}-s}}{\Upsilon_4 z^{\mathfrak{p}-s} - \sum_{n=1}^{\infty} \Upsilon_{(3,n)} a_{n+\mathfrak{p}} z^{n+\mathfrak{p}-s}}\right) \geqq -1.$$

If we now choose that z lies on the real axis, then

$$\frac{z^{\nu}(\mathfrak{D}_q^{(\nu+s)}f)(z)}{(\mathfrak{D}_q^{(s)}f)(z)}$$

assumes real values. In this case, upon letting $z \to 1-$ along the real line, we get

$$2\Upsilon_1 - \sum_{n=1}^{\infty} 2\Upsilon_{(2,n)}|a_{n+\mathfrak{p}}| \geqq -|\Upsilon_4| + \sum_{n=1}^{\infty} |\Upsilon_{(3,n)}||a_{n+\mathfrak{p}}|, \tag{3.3}$$

where Υ_1, $\Upsilon_{(2,n)}$, $\Upsilon_{(3,n)}$, and Υ_4 are given by (2.4), (2.5), (2.6), and (2.7) respectively. We see that the last expression in (3.3) satisfies the inequality in (2.3). Hence our proof of Theorem 3 is now completed. □

Remark 4 First of all, if we put $\nu = s + 1 = \mathfrak{p} = 1$ in Theorem 3, we get the corresponding result due to Srivastava *et al.* [33]. Secondly, if we put

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad L = -1,$$

Theorem 3 gives the corresponding result that was proved by Khan *et al.* [8]. Thirdly, if we assign the following values to the parameters in Theorem 3:

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad -L = \nu = s + 1 = \mathfrak{p} = 1,$$

we have the following known result.

Corollary 2 (see [35, Theorem 8]) *If* $0 \leqq \alpha < 1$, *then*

$$\mathcal{TS}^*_{(q,1)}(\alpha) \equiv \mathcal{TS}^*_{(q,2)}(\alpha) \equiv \mathcal{TS}^*_{(q,3)}(\alpha).$$

Corollary 3 *Let the function* f *of the form* (3.1) *be in the class* $\mathcal{TS}^*_q[j, \mathfrak{p}, \nu, s, X, L]$ $(j = 1, 2, 3)$. *Then*

$$a_{n+\mathfrak{p}} \leqq \frac{|\Upsilon_4| - 2\Upsilon_1}{(2\Upsilon_{(2,n)} + |\Upsilon_{(3,n)}|)}. \tag{3.4}$$

The following function $f_t(z)$

$$f_t(z) = z^{\mathfrak{p}} - \frac{|\Upsilon_4| - 2\Upsilon_1}{(2\Upsilon_{(2,1)} + |\Upsilon_{(3,1)}|)} z^{\mathfrak{p}+1} \tag{3.5}$$

is best possible, where Υ_1, $\Upsilon_{(2,n)}$, $\Upsilon_{(3,n)}$, *and* Υ_4 *are given by* (2.4), (2.5), (2.6), *and* (2.7), *respectively.*

By means of Theorem 3, it should be understood that Type 1, Type 2, and Type 3 of the multivalent q-starlike functions which involve the Janowski functions are similar. Therefore, for convenience, we state and prove the following distortion theorem by using the notation $\mathcal{TS}^*_q[j, \mathfrak{p}, \nu, s, X, L]$ in which it is tacitly assumed that $j = 1, 2, 3$.

Theorem 4 *If* $f \in \mathcal{TS}^*_q[j, \mathfrak{p}, \nu, s, X, L]$ $(j = 1, 2, 3)$, *then*

$$|f(z)| \geqq r^{\mathfrak{p}} - \left(\frac{|\Upsilon_4| - 2\Upsilon_1}{(2\Upsilon_{(2,1)} + |\Upsilon_{(3,1)}|)} \right) r^{\mathfrak{p}+1} \quad (n \in \mathbb{N}) \ |z^{\mathfrak{p}}| = r^{\mathfrak{p}} \ (0 < r < 1) \tag{3.6}$$

and

$$\left|f(z)\right| \leqq r^{\mathfrak{p}} + \left(\frac{|\Upsilon_4| - 2\Upsilon_1}{(2\Upsilon_{(2,1)} + |\Upsilon_{(3,1)}|)}\right) r^{\mathfrak{p}+1} \quad (n \in \mathbb{N})\ \left|z^{\mathfrak{p}}\right| = r^{\mathfrak{p}}\ (0 < r < 1). \tag{3.7}$$

The equalities in (3.6) *and* (3.7) *are attained for the function* $f(z)$ *given by* (3.5) *and where* Υ_1, $\Upsilon_{(2,n)}$, $\Upsilon_{(3,n)}$, *and* Υ_4 *are given by* (2.4), (2.5), (2.6), *and* (2.7), *respectively.*

Proof The following inequality can be easily deduced from Theorem 2:

$$(2\Upsilon_{(2,1)} + |\Upsilon_{(3,1)}|) \sum_{n=1}^{\infty} |a_{n+\mathfrak{p}}| \leqq \sum_{n=1}^{\infty} \bigl(2\Upsilon_{(2,n)} + |\Upsilon_{(3,n)}|\bigr) |a_{n+\mathfrak{p}}| < |\Upsilon_4| - 2\Upsilon_1,$$

which yields

$$\left|f(z)\right| \leqq r^{\mathfrak{p}} + \sum_{n=1}^{\infty} |a_{n+\mathfrak{p}}| r^{n+\mathfrak{p}} \leqq r^{\mathfrak{p}} + r^{\mathfrak{p}+1} \sum_{n=1}^{\infty} |a_{n+\mathfrak{p}}| \leqq r^{\mathfrak{p}} + \frac{|\Upsilon_4| - 2\Upsilon_1}{(2\Upsilon_{(2,1)} + |\Upsilon_{(3,1)}|)} r^{\mathfrak{p}+1}.$$

Similarly, we have

$$\left|f(z)\right| \geq r^{\mathfrak{p}} - \sum_{n=1}^{\infty} |a_{n+\mathfrak{p}}| r^{n+\mathfrak{p}} \geq r^{\mathfrak{p}} - r^{\mathfrak{p}+1} \sum_{n=1}^{\infty} |a_{n+\mathfrak{p}}| \geq r^{\mathfrak{p}} - \frac{|\Upsilon_4| - 2\Upsilon_1}{(2\Upsilon_{(2,1)} + |\Upsilon_{(3,1)}|)} r^{\mathfrak{p}+1},$$

where Υ_1, $\Upsilon_{(2,n)}$, $\Upsilon_{(3,n)}$, and Υ_4 are given by (2.4), (2.5), (2.6), and (2.7), respectively. The proof of Theorem 4 is now completed. □

Remark 5 First of all, if we put $v = s + 1 = \mathfrak{p} = 1$ in Theorem 1, we get the corresponding result due to Srivastava *et al.* [33]. Secondly, if we put

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad L = -1,$$

Theorem 4 will give the corresponding result that was proved by Khan *et al.* [8]. Thirdly, if we put

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad -L = v = s + 1 = \mathfrak{p} = 1$$

in Theorem 4 and let $q \longrightarrow 1-$, we have the following known result.

Corollary 4 (see [23]) *If* $f \in \mathcal{TS}^*(\alpha)$, *then*

$$r - \left(\frac{1-\alpha}{2-\alpha}\right) r^2 \leqq \left|f(z)\right| \leqq r + \left(\frac{1-\alpha}{2-\alpha}\right) r^2 \quad \bigl(|z| = r\ (0 < r < 1)\bigr).$$

The next theorem (Theorem 5) can be proven similarly as we proved Theorem 4, so we choose to omit the details involved in our proof of Theorem 5.

Theorem 5 *If* $f \in \mathcal{TS}_q^*[j, \mathfrak{p}, v, s, X, L]$ $(j = 1, 2, 3)$, *then*

$$\left|f'(z)\right| \geqq \mathfrak{p} r^{\mathfrak{p}-1} - \left(\frac{(\mathfrak{p}+1)(|\Upsilon_4| - 2\Upsilon_1)}{(2\Upsilon_{(2,1)} + |\Upsilon_{(3,1)}|)}\right) r^{\mathfrak{p}} \quad (n \in \mathbb{N})\ \left|z^{\mathfrak{p}}\right| = r^{\mathfrak{p}}\ (0 < r < 1)$$

and

$$\left|f'(z)\right| \leqq \mathfrak{p}r^{\mathfrak{p}-1} + \left(\frac{(\mathfrak{p}+1)(|\Upsilon_4| - 2\Upsilon_1)}{(2\Upsilon_{(2,1)} + |\Upsilon_{(3,1)}|)}\right) r^{\mathfrak{p}} \quad (n \in \mathbb{N}) \left|z^{\mathfrak{p}}\right| = r^{\mathfrak{p}} \ (0 < r < 1).$$

The function $f_t(z)$ given by (3.5) *is the extremal function.*

If in Theorem 5 we take

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad -L = \nu = s + 1 = \mathfrak{p} = 1,$$

and let $q \longrightarrow 1-$, we get the following known result.

Corollary 5 (see [23]) *If $f \in \mathcal{TS}^*(\alpha)$, then*

$$1 - \left(\frac{2(1-\alpha)}{2-\alpha}\right) r \leqq \left|f'(z)\right| \leqq 1 + \left(\frac{2(1-\alpha)}{2-\alpha}\right) r \quad \bigl(|z| = r \ (0 < r < 1)\bigr).$$

Finally, for the class $\mathcal{TS}_q^*[j, \mathfrak{p}, \nu, s, X, L]$ $(j = 1, 2, 3)$ of multivalent q-starlike functions with negative coefficients, the results related to the radii of close-to-convexity, starlikeness, and convexity are deduced.

Theorem 6 *Let $f \in \mathcal{TS}_q^*[j, \mathfrak{p}, \nu, s, X, L]$ $(j = 1, 2, 3)$. Then, for $|z| \leqq r_0(j, \mathfrak{p}, n, X, L, \chi)$, the function f is $\mathfrak{p}$-valent close-to-convex of order χ with $(0 \leqq \chi < \mathfrak{p})$, where*

$$r_0 = \inf_{n \geq 1} \left[\frac{(2\Upsilon_{(2,n)} + |\Upsilon_{(3,n)}|)(\mathfrak{p} - \chi)}{(|\Upsilon_4| - 2\Upsilon_1)(n + \mathfrak{p})}\right]^{\frac{1}{n}}. \tag{3.8}$$

The function $f_t(z)$ given by (3.5) *is best possible.*

Proof Using Theorem 2 in conjunction with (3.1), for $|z| < r_0$, we find that

$$\left|\frac{f'(z)}{z^{\mathfrak{p}-1}} - \mathfrak{p}\right| < \mathfrak{p} - \chi \quad \bigl(|z| \leqq r_0\bigr).$$

We have thus completed the proof of Theorem 6. □

Remark 6 If we put

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad L = -1,$$

Theorem 6 gives the corresponding result that was proved by Khan *et al.* [8].

Theorem 7 *Let $f \in \mathcal{TS}_q^*[j, \mathfrak{p}, \nu, s, X, L]$ $(j = 1, 2, 3)$. Then, for $|z| \leqq r_1(j, \mathfrak{p}, n, X, L, \chi)$, the function f is a $\mathfrak{p}$-valent starlike of order χ with $(0 \leqq \chi < \mathfrak{p})$, where*

$$r_1 = \inf_{n \geq 1} \left[\frac{(2\Upsilon_{(2,n)} + |\Upsilon_{(3,n)}|)(\mathfrak{p} - \chi)}{(|\Upsilon_4| - 2\Upsilon_1)(n + \mathfrak{p} - \chi)}\right]^{\frac{1}{n}}. \tag{3.9}$$

The result is sharp for the function $f_t(z)$ given by (3.5).

Proof Using arguments similar to those in the proof of Theorem 6, it can be seen that

$$\left|\frac{zf'(z)}{f(z)} - \mathfrak{p}\right| < \mathfrak{p} - \chi \quad (|z| \leqq r_1),$$

which evidently proves Theorem 7. □

Remark 7 If we put

$$X = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \quad \text{and} \quad L = -1,$$

Theorem 7 gives the corresponding result that was proved by Khan *et al.* [8].

Corollary 6 *Let* $f \in \mathcal{TS}_q^*[j, \mathfrak{p}, v, s, X, L]$ $(j = 1, 2, 3)$. *Then, for* $|z| \leqq r_2(j, \mathfrak{p}, n, X, L, \chi)$, *the function* f *is a* $\mathfrak{p}$*-valent convex of order* χ *with* $(0 \leqq \chi < \mathfrak{p})$ *where*

$$r_2 = \inf_{n \geq 1} \left[\frac{(2\Upsilon_{(2,n)} + |\Upsilon_{(3,n)}|)\mathfrak{p}(\mathfrak{p} - \chi)}{(|\Upsilon_4| - 2\Upsilon_1)(n + \mathfrak{p})(n + \mathfrak{p} - \chi)} \right]^{\frac{1}{n}}. \tag{3.10}$$

The result is sharp for the function $f_t(z)$ *given by* (3.5).

4 Concluding remarks and observations

The works presented in this paper are basically motivated by the well-established usage of the basic (or q-) calculus in the context of geometric function theory. Several subfamilies of multivalently (or, more precisely, $\mathfrak{p}$-valently) q-starlike functions in the open unit disk $\mathbb{U}$ have been introduced and investigated here rather systematically. We have also highlighted some known and new facts of our results in form of remarks and corollaries which significantly make the results important.

The usage of basic (or q-) series in many diverse areas of mathematics and physics makes it very important. By making use of basic (or q-) series some wonderful works have been done (see, for example, [28, pp. 350–351]; see also [9, 10, 15], and [14]). Moreover, as we described in Sect. 1, from Srivastava's observation [25] about the so-called (p, q)-calculus, we arrived at the point that indeed the result presented in this paper can be produced for the rather straightforward (p, q)-variations.

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