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APPLICATION OF INTEGRAL TRANSFORMS COMPOSITION METHOD (ITCM) TO OBTAINING TRANSMUTATIONS VIA INTEGRAL TRANSFORMS WITH BESSEL FUNCTIONS IN KERNELS

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Abstract. In this paper we apply the Integral Transforms Composition Method (ITCM) in order to derive compositions of integral transforms with Bessel functions in kernels, and obtain norm estimates and other properties of such composition transforms. Exactly, we consider transmutations which are compositions of classical Hankel and Y integral transforms. Norms estimates in L_2 for these integral transforms with Bessel functions in kernels and their compositions are obtained. Also boundedness conditions for such transforms in weighted Lebesgue classes are proved. Classical integral transforms are used in this method as basic blocks. The ITCM and transmutations obtained by this method are applied to deriving connection formulas for solutions of singular differential equations.

Keywords: Hankel transform, Y transform, Integral Transforms Composition Method (ITCM), transmutations, Slater theorem, Bessel functions, Meijer G–function, hypergeometric functions, Mellin transform.

1. Introduction

1.1. **Integral Transforms Composition Method (ITCM) in transmutation theory.** In the paper we study applications of integral transforms composition method (ITCM) [1, 2, 3, 4, 5] for obtaining transmutations via integral transforms

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and to study integral transforms with Bessel functions in kernels. We consider four different composition of the Hankel transform and integral transform with Bessel function of the second kind in a kernel. For compositions of these transforms boundedness in L_2 and weighted Lebesgue space with exact norms are proved.

Let us note that we can construct wide range of transmutation operators by ITCM. Wherein known integral transforms as Fourier, sine and cosine–Fourier, Hankel, Mellin, Laplace and others are used as basic blocks. The ITCM and transmutations obtaining by this method are applied to deriving connection formulas for solutions to singular differential equations and non–singular ones. In [3] we considered how to apply ITCM to well–known classes of singular differential equations with Bessel operators, such as classical and generalized Euler–Poisson–Darboux equation and the generalized radiation problem of A. Weinstein. These results can be applied to more general linear partial differential equations with Bessel operators, such as multivariate Bessel–type equations, GASPT (Generalized Axially Symmetric Potential Theory) equations of A.Weinstein, Bessel–type generalized wave equations like Euler–Poisson–Darboux ones, ultra B–hyperbolic equations and others. The next step of applications of the method consists of defining new types fractional integrals and their multidimensional analogs, obviously possessing the semigroup property as consequences of ITCM representations. Such approach to inverting potential type and convolution operators as approximative inverse operators method in fact is also a variant of ITCM. So with many results and examples the main conclusion of this paper may be illustrated by the next conclusion :

the integral transforms composition method (ITCM) of constructing transmutations is important and effective tool to obtain different results for integral transforms and their compositions.

Also we mention that some formulas on compositions of Hankel transforms were announced in the preprint [7] without any connections with transmutations and ITCM method.

The formal algorithm of ITCM is the next. Let us take as input a pair of arbitrary operators A, B, and also connecting with them generalized Fourier transforms F_A, F_B, which are invertible and act by the formulas

$$F_A A = g(t) F_A, \qquad F_B B = g(t) F_B, \tag{1}$$

where t is a dual variable, g is an arbitrary function with suitable properties. It is often convenient to choose $g(t) = -t^2$ or $g(t) = -t^\alpha$, $\alpha \in \mathbb{R}$.

Then the essence of ITCM is to obtain formally a pair of transmutation operators P and S as the method output by the next formulas:

$$S = F_B^{-1} \frac{1}{w(t)} F_A, \qquad P = F_A^{-1} w(t) F_B \tag{2}$$

with arbitrary function $w(t)$. When P and S are transmutation operators intertwining A and B:

$$SA = BS, \qquad PB = AP. \tag{3}$$

A formal checking of (3) can be obtained by direct substitution. The main difficulty is the calculation of compositions (2) in an explicit integral form, as well as the choice of domains of operators P and S.

Let us list the main advantages of Integral Transform Composition Method (ITCM).

- Simplicity — many classes of transmutations are obtained by explicit formulas from elementary basic blocks, which are classical integral transforms.
- ITCM gives by a unified approach all previously explicitly known classes of transmutations.
- ITCM gives by a unified approach many new classes of transmutations for different operators.
- ITCM gives a unified approach to obtain both direct and inverse transmutations in the same composition form.
- ITCM directly leads to estimates of norms of direct and inverse transmutations using known norm estimates for classical integral transforms on different functional spaces.
- ITCM directly leads to connection formulas for solutions to perturbed and unperturbed differential equations.

An obstacle for applying ITCM is the next one: we know acting of classical integral transforms usually on standard spaces like L_2, L_p, C^k, variable exponent Lebesgue spaces and so on. But for application of transmutations to differential equations we usually need some more conditions hold, say at zero or at infinity. For these problems we may first construct a transmutation by ITCM and then expand it to the needed functional classes.

Let us stress that formulas of the type (2) of course are not new for integral transforms and its applications to differential equations. But ITCM is new when applied to transmutation theory! In other fields of integral transforms and connected differential equations theory compositions (2) for the choice of classical Fourier transform leads to famous pseudo-differential operators with symbol function $w(t)$. For the choice of the classical Fourier transform and the function $w(t) = (\pm it)^{-s}$ we obtain fractional integrals on the whole real axis, for $w(t) = |x|^{-s}$ we obtain M. Riesz potential, for $w(t) = (1+t^2)^{-s}$ in formulas (2) we obtain Bessel potential. For $w(t) = (1 \pm it)^{-s}$ we get modified Bessel potentials [2]. For this paper important are applications to compositions of integral transforms which are based on Mellin transform technique, cf. [8, 9, 18, 19, 6, 7, 1, 2, 3, 4, 5].

The next choice for ITCM algorithm,

$$A = B = B_\nu, \quad F_A = F_B = H_\nu, \quad g(t) = -t^2, \quad w(t) = \frac{2^\nu \Gamma(\nu+1)}{t^\nu}\, J_\nu(t), \qquad t \in \mathbb{R} \tag{4}$$

leads to generalized translation operators of Delsart, for this case we have to choose in ITCM algorithm defined by (1)–(2) the above values (4) in which $B_\nu = D^2 + \frac{2\nu+1}{t} D$ is the Bessel operator, H_ν is the Hankel transform (6), J_ν is the Bessel function of the first kind. In the same manner other families of operators commuting with a given one may be obtained by ITCM for the choice $A = B, F_A = F_B$ with arbitrary functions $g(t), w(t)$ (generalized translation commutes with the Bessel operator).

Direct applications of ITCM to multidimensional differential operators are obvious, in this case $t = (t_1, ..., t_n) \in \mathbb{R}^n$ is a vector and $g(t), w(t)$ are vector functions in (1)–(2). Unfortunately for this case we know and may derive some new explicit transmutations just for simple special cases. But among them are well-known and interesting classes of potentials. In case of using ITCM by (1)–(2) with Fourier transform and when $w(t)$ is a positive definite quadratic form we come to elliptic Riesz potentials [2]; when $w(t)$ is an indefinite quadratic form we come to hyperbolic Riesz potentials [2]; with $w(t) = (t_2^2 + ... + t_n^2 - it_1)^{-\alpha/2}$ we come to parabolic potentials. In the case of using ITCM by (1)–(2) with Hankel transform and when $w(t)$ is quadratic form we come to elliptic Riesz B–potentials or hyperbolic Riesz B-potentials [2]. For all above mentioned potentials we need to use distribution theory and consider for ITCM convolutions of distributions, for inversion of such potentials we need some cutting and approximation procedures. For this class of problems it is appropriate to use Schwartz or/and Lizorkin spaces for probe functions and dual spaces for distributions.

So we may conclude that the method we consider in the paper for obtaining transmutations — ITCM is effective, it is connected to many known methods and problems, it gives all known classes of explicit transmutations and works as a tool to construct new classes of transmutations. Application of ITCM needs the next three steps.

Step 1. For a given pair of operators A, B and connected generalized Fourier transforms F_A, F_B define and calculate a pair of transmutations P, S by basic formulas (1)–(2).

Step 2. Derive exact conditions and find classes of functions for which transmutations obtained by step 1 satisfy proper intertwining properties.

Step 3. Apply now correctly defined transmutations by steps 1 and 2 on proper classes of functions to deriving connection formulas for solutions of differential equations.

1.2. **Definitions.** *The Bessel function of the first kind*, denoted by $J_\alpha(x)$, is a solution of Bessel's differential equation that are finite at the origin $x = 0$. The Bessel function $J_\alpha(x)$ can be defined by the series

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

For non-integer α the functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent. If α is integer the following relationship is valid:

$$J_{-\alpha}(x) = (-1)^\alpha J_\alpha(x).$$

The Bessel function of the second kind, denoted by $Y_\alpha(x)$, for non-integer α is related to $J_\alpha(x)$ by the formula:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)},$$

In the case of integer order n, the function $Y_\alpha(x)$ is defined by taking the limit as a non-integer α tends to n:

$$Y_n(x) = \lim_{\alpha\to n} Y_\alpha(x).$$

Functions $Y_\alpha(x)$ are also called Neumann functions and denoted by $N_\alpha(x)$. The linear combination of the Bessel functions of the first and second kinds represents a complete solution of the Bessel equation:

$$y(x) = C_1 J_\alpha(x) + C_2 Y_\alpha(x).$$

A general definition of the Meijer G-function is given by the following line integral in the complex plane ([13] § 5.3.1):
(5)

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q\end{matrix}\;\middle|\; z\right)=\frac{1}{2\pi i}\int\limits_L \frac{\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\prod_{j=n+1}^p\Gamma(a_j-s)}\,z^s\,ds.$$

Here L in the integral represents the path to be followed while integrating. For this path three choices are possible (see [13] § 5.3.1).

Let S is the space of rapidly decreasing functions on $(0,\infty)$:

$$S=\left\{f\in C^\infty(0,\infty):\sup_{x\in(0,\infty)}\left|x^\alpha D^\beta f(x)\right|<\infty\quad\forall\alpha,\beta\in Z_+\right\}.$$

We will study properties of compositions of transforms with Bessel functions in the kernel:

$$H_\nu[f](x)=\int\limits_0^\infty (xt)^{\frac12}J_\nu(xt)f(t)dt,\qquad \nu\geq-\frac12, \tag{6}$$

$$\mathcal{Y}_\nu[f](x)=\int\limits_0^\infty (xt)^{\frac12}Y_\nu(xt)f(t)dt,\qquad \nu\geq-\frac12, \tag{7}$$

where J_ν is the Bessel function of the first kind of order ν and Y_ν is the Bessel functions of the second kind of order ν.

Transforms H_ν and $\mathcal{Y}_\nu$ are well studied in [20, 21, 17]. In these papers, their boundedness in $\mathcal{L}_{p,\mu}$–spaces have been completely examined. The $\mathcal{L}_{p,\mu}$–space is the space of function (or, more precisely, equivalence classes) such that their Lebesgue integral $\int\limits_0^\infty |x^\mu f(x)|^p\frac{dx}{x}$ is finite. The $\mathcal{L}_{p,\mu}$–norm is defined by

$$||f||_{p,\mu}=\left(|x^\mu f(x)|^p\frac{dx}{x}\right)^{\frac12}. \tag{8}$$

Here $1\leq p<\infty$ and μ is any number.

We will use the Mellin transform of a function $f:\mathbb{R}_+\to\mathbb{C}$ defined by formula

$$f^*(s)=\mathcal{M}[f](s)=\int\limits_0^\infty x^{s-1}f(x)dx,$$

where $s=\sigma+i\tau\in\mathbb{C}$, provided that the integral exists.

As space of originals we choose the space P_a^b, $-\infty<a<b<\infty$ which is the linear space of $\mathbb{R}_+\to\mathbb{C}$ functions such that $x^{s-1}f(x)\in L_1(\mathbb{R}_+)$ for every $s\in\{p\in\mathbb{C}:a\leq \operatorname{Re}p\leq b\}$.

If additionally $f^*(c+i\tau)\in L_1(\mathbb{R})$ with respect to τ then complex inversion formula holds:

$$\mathcal{M}^{-1}[\varphi](x)=f(x)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}x^{-s}\varphi(s)\,ds.$$

In fact, the gamma-function $\Gamma(z)$ corresponds to the Mellin transform of the negative exponential function:

$$\Gamma(z) = \mathcal{M}[e^{-x}](z).$$

If an operator A is acting in the images of Mellin transform as a multiplication to a function:

$$\mathcal{M}[Af](s) = m_A\, \mathcal{M}[f](s), \tag{9}$$

then we name m_A a *multiplier of operator* A.

The Mellin convolution $(f * g)_M(y)$ of two functions f and g is given by

$$\int\limits_0^\infty f(x) g\left(\frac{y}{x}\right) \frac{dx}{x}.$$

We have

$$\mathcal{M}\left[\int\limits_0^\infty K\left(\frac{x}{t}\right) f(t) \frac{dt}{t}\right](s) = \mathcal{M}[K](s)\, \mathcal{M}[f](s) \tag{10}$$

so the $\mathcal{M}[K](s)$ is the multiplier for the Mellin convolution (see [18]).

A *unitary operator* is a surjective bounded operator on a Hilbert space preserving the inner product. Unitary operators are usually taken as operating on a Hilbert space, but the same notion serves to define the concept of isomorphism between Hilbert spaces.

2. Applications of integral transforms composition method (ITCM) to integral transforms with Bessel functions in kernels

Now let us consider an application of integral transforms composition method (ITCM) to integral transforms with Bessel functions in kernel. Namely, we study the next operators defined as compositions of Hankel and Y transforms

$$T_1 = H_\mu\, Y_\nu, \qquad T_2 = Y_\mu\, Y_\nu, \qquad T_3 = Y_\mu\, H_\nu, \qquad T_4 = H_\mu\, H_\nu. \tag{11}$$

By integral transforms composition method (ITCM) operators T_1, T_2, T_3, T_4 are transmutations with common transmutation property

$$T\, L_\nu = L_\mu\, T, \tag{12}$$

where operator

$$L_\nu = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{\nu^2}{x^2}$$

is a modified with power change Bessel operator

$$B_\nu = \frac{d^2}{dx^2} + \frac{2\nu+1}{x}\frac{d}{dx}.$$

We apply ITCM as it is known that standard properties are valid

$$H_\nu L_\nu = -x^2\, H_\nu, \qquad Y_\nu L_\nu = -x^2\, Y_\nu$$

on proper functions. So we may classify transmutations (11) as shift operators by the parameter ν in the differential operator L_ν. And these operators are accomplishing connection formulas for pairs of differential equations with different parameters in L_ν.

Our main problem is to evaluate transmutations (11) as integral operators with special function kernels.

2.1. **Integral transforms and Mellin transform.** The next lemma that allows to reduce the question of the boundedness of some operator in L_2 to the study of its multiplier is well known (see [15, 16, 23]). We give this lemma here.

Lemma 1. *1) Let for operator A the equality (9) is true. Then the operator A can be exended as a bounded operator on L_2 if and only if*

$$\sup_{\xi\in\mathbb{R}}\left|m_A\left(i\xi+\frac{1}{2}\right)\right| = M_1 < \infty. \tag{13}$$

Thus we have $||A||_{L_2} = M_1$.

2) In order that the inverse operator A^{-1} allows an extension to bounded operator on L_2 it is necessary and sufficient that

$$\inf_{\xi\in\mathbb{R}}\left|m_A\left(i\xi+\frac{1}{2}\right)\right| = m_1 < \infty. \tag{14}$$

Thus we have $||A^{-1}||_{L_2} = \frac{1}{m_1}$.

3) Let operators A and A^{-1} are defined and bounded in L_2. Then in order to A and A^{-1} be unitary it is necessary and sufficient that an equality

$$\left|m_A\left(i\xi+\frac{1}{2}\right)\right| = 1 \tag{15}$$

is true for almost all $\xi\in\mathbb{R}$.

Here, we give some known more results from [20, 21, 17] which we will use.

Lemma 2. *1) Integral transform H_ν bijectively maps the space $\mathcal{L}_{2,\mu}$ into itself when $\frac{1}{2}\le\mu<\nu+\frac{3}{2}$.*

2) Let $f\in\mathcal{L}_{2,\mu}$, $\frac{1}{2}\le\mu<\frac{3}{2}-\nu$ then $\mathrm{Re}\, s=\mu$ we have

$$\mathcal{M}[H_\nu f](s) = m_\nu(s)\,\mathcal{M}[f](1-s), \tag{16}$$

where

$$m_\nu(s) = 2^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{2s+2\nu+1}{4}\right)}{\Gamma\left(\frac{2s-2\nu+3}{4}\right)}. \tag{17}$$

Lemma 3. *1) Integral transform $\mathcal{Y}_\nu$ bijectively maps the space $\mathcal{L}_{2,\mu}$ into itself when $\frac{1}{2}\le\mu<\frac{3}{2}-|\nu|$ except for the case $\mu=\frac{1}{2}-\nu$. For $\mu=\frac{1}{2}-\nu$ we have $\mathcal{Y}_\nu[\mathcal{L}_{2,\frac{1}{2}-\nu}] = H_\nu[\mathcal{L}_{2,\frac{1}{2}-\nu}]$. Since, $\mathcal{Y}_\nu = C_{1\nu}H_\nu + C_{2\nu}H_{-\nu}$ then for $-\frac{1}{2}<\nu<0$ we have $H_\nu[\mathcal{L}_{2,\frac{1}{2}-\nu}] = H_{-\nu}[\mathcal{L}_{2,\frac{1}{2}-\nu}] = \mathcal{L}_{2,\frac{1}{2}-\nu}$ and $\mathcal{Y}_\nu[\mathcal{L}_{2,\frac{1}{2}-\nu}]\subset\mathcal{L}_{2,\frac{1}{2}-\nu}$. (For $\mathcal{Y}_0$ it was proved in [21].)*

2) Let $f\in\mathcal{L}_{2,\mu}$, $\frac{1}{2}\le\mu<\nu+\frac{3}{2}$ then $\mathrm{Re}\, s=\mu$ we have

$$\mathcal{M}[\mathcal{Y}_\nu f](s) = -m_\nu(s)\mathrm{ctg}\left(s+\frac{1}{2}-|\nu|\right)\frac{\pi}{2}\,\mathcal{M}[f](1-s), \tag{18}$$

where $m_\nu(s) = 2^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{2s+2\nu+1}{4}\right)}{\Gamma\left(\frac{2s-2\nu+3}{4}\right)}$.

2.2. **Composition of H_ν and $\mathcal{Y}_\nu$ transforms.** Now let study composition of operators H_ν and $\mathcal{Y}_\nu$.

Theorem 1. *Let $f \in \mathcal{L}_{2,\mu}$, $\frac{1}{2} \le \mu < \frac{3}{2} - \nu$. Then*

1) Operator $H_\nu \mathcal{Y}_\nu$ acts in Mellin images according to (9) with the multiplier

$$m(s) = -\operatorname{tg}\left(\frac{2s+2\nu-1}{4}\right)\pi, \qquad \operatorname{Re} s = \mu. \tag{19}$$

2) The formula

$$H_\nu[\mathcal{Y}_\nu f](x) = \frac{2}{\pi}\int\limits_0^\infty \frac{tf(t)}{t^2-x^2}\left(\frac{x}{t}\right)^{\nu+\frac{1}{2}} dt \tag{20}$$

is valid.

Доказательство. 1) According to [21, 17], if $f \in \mathcal{L}_{2,\mu}$, then $\mathcal{Y}_\nu f \in \mathcal{L}_{2,\mu}$ for $\frac{1}{2} \le \mu < \frac{1}{2} - |\nu|$ and we can apply formula (18) to $\mathcal{Y}_\nu f$:

$$\mathcal{M}[H_\nu \mathcal{Y}_\nu f](s) = m_\nu(s)\mathcal{M}[\mathcal{Y}_\nu f](1-s).$$

Then using (16) we get

$$\mathcal{M}[H_\nu \mathcal{Y}_\nu f](s) = -m_\nu(s)\, m_\nu(1-s)\operatorname{ctg}\left(\left(s+\frac{1}{2}-\nu\right)\frac{\pi}{2}\right)\mathcal{M}[f](s).$$

To prove (19) it remains to substitute

$$m_\nu(s) = 2^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{2s+2\nu+1}{4}\right)}{\Gamma\left(\frac{2s-2\nu+3}{4}\right)}, \qquad m_\nu(1-s) = 2^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{3-2s+2\nu}{4}\right)}{\Gamma\left(\frac{5-2s-2\nu}{4}\right)}$$

(see (17)) and to apply formulas

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi},$$

and

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)}.$$

The formula (19) can be obtained directly also. We should denote $\mathcal{Y}_\nu[f](s)=g(s)$ and change the variables in (6) by $t = \frac{1}{y}$. We obtain

$$H_\nu[g](x) = \int\limits_0^\infty \mathfrak{K}_\nu\left(\frac{x}{y}\right)\frac{1}{y}\, g\left(\frac{1}{y}\right)\frac{dy}{y}, \tag{21}$$

where $\mathfrak{K}_\nu(x) = x^{\frac{1}{2}} J_\nu(x)$.

Then using formula (10) we get

$$\mathcal{M}H_\nu[g](s) = \mathcal{M}[\mathfrak{K}_\nu](s)\,\mathcal{M}\left[\frac{1}{y}\, g\left(\frac{1}{y}\right)\right] = \mathcal{M}[\mathfrak{K}_\nu](s)\,\mathcal{M}[g(y)](1-s).$$

We used formulas 3 and 4 from [14], p. 268 for obtaining the last expression. It is clear that

$$g(y) = \mathcal{Y}_\nu[f](y) = \int\limits_0^\infty \mathfrak{Y}_\nu\left(\frac{y}{p}\right)\frac{1}{p}\, f\left(\frac{1}{p}\right)\frac{dp}{p},$$

where $\mathfrak{Y}_\nu(x) = x^{\frac{1}{2}}\, Y_\nu(x)$. Then

$$\mathcal{M}[g](1-s) = \mathcal{M}[\mathcal{K}_\nu](1-s), \qquad \mathcal{M}\left[\frac{1}{p}\, f\left(\frac{1}{p}\right)\right](1-s) = \mathcal{M}[\mathcal{K}_\nu](1-s)\, \mathcal{M}[f](s).$$

So

$$\mathcal{M}[H_\nu\, \mathcal{Y}_\nu f](s) = \mathcal{M}[K_1](s)\, \mathcal{M}[K_2](1-s)\, \mathcal{M}[f](s).$$

Substitution of values $\mathcal{M}[K_1](s)$ and $\mathcal{M}[K_2](1-s)$ from [14] into this expression gives (19).

2) Denoting $m(s) = \mathcal{M}[K](s)$ we can write

$$\mathcal{M}[H_\nu\, \mathcal{Y}_\nu f](s) = \mathcal{M}[K](s)\, \mathcal{M}[f](s).$$

Using formula 18 from [14], p. 302 we get

$$K(x) = \frac{2}{\pi}\, \frac{x^{\nu+\frac{1}{2}}}{1-x^2}.$$

Applying (10) we obtain (20).

□

It is easy to see that for $\nu = \mp\frac{1}{2}$ the operator $H_\nu\, \mathcal{Y}_\nu$ is equal to Hilbert transform on semi–axes:

$$H_{\frac{1}{2}}\, [\mathcal{Y}_{\frac{1}{2}}](x) = \frac{2}{\pi} \int\limits_0^\infty \frac{x f(t)}{t^2 - x^2}\, dt.$$

It is easy to explain this fact. We know that (see formulas 14 and 15 in [13], p. 90)

$$J_{\frac{1}{2}}(x) = Y_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin(x),$$

$$Y_{\frac{1}{2}}(x) = -J_{-\frac{1}{2}}(x) = -\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos(x).$$

Then for $\nu = \frac{1}{2}$ operators H_ν and $\mathcal{Y}_\nu$ are cosine- and sine-Fourier transforms. Accordingly, for $\nu = -\frac{1}{2}$ operators H_ν and $\mathcal{Y}_\nu$ are sine- and cosine-Fourier transforms. Superposition of such transform operators is a Hilbert transform on semi–axes, cf. [7].

Theorem 2. *Let $f \in \mathcal{L}_{2,\mu}$, $\frac{1}{2} \leq \mu < \frac{3}{2} - |\nu|$. Then*

1) operator $\mathcal{Y}_\nu H_\nu$ acts in Mellin images by the formula (9) *with multiplier*

$$m(s) = -\mathrm{ctg}\left[\left(\frac{2s - 2\nu + 1}{4}\right) n\right], \qquad \mathrm{Re}\, s = \mu, \tag{22}$$

2) the formula

$$\mathcal{Y}_\nu [H_\nu f](x) = \frac{2}{\pi} \int\limits_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}-\nu} \frac{t f(t)}{x^2 - t^2}\, dt \tag{23}$$

is valid.

The proof of this theorem repeats the proof of the theorem 1.

From (20) it is clear that $\mathcal{Y}_\nu H_\nu = -H_{-\nu} \mathcal{Y}_{-\nu}$ and we can consider only one of these compositions.

3. Properties of H_ν, $\mathcal{Y}_\nu$ and their compositions

3.1. **Norms in $L_2(0,\infty)$ of H_ν, $\mathcal{Y}_\nu$ and their compositions.** Next we consider norms of H_ν, $\mathcal{Y}_\nu$ and their compositions in $L_2(0,\infty)$. For estimation of norms of H_ν and $\mathcal{Y}_\nu$ we use (18) and (16) which we write in the form:

$$\mathcal{M}[H_\nu f](s) = m_\nu(s)\,\mathcal{M}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right](s), \tag{24}$$

$$\mathcal{M}[\mathcal{Y}_\nu f](s) = -m_\nu(s)\,\cos\left(s+\frac{1}{2}-\nu\right)\frac{\pi}{2}\,\mathcal{M}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right](s). \tag{25}$$

To obtain (24) and (25) we used formulas 3 and 4 from [14], p. 268.

It is obvious that for operators H_ν and $\mathcal{Y}_\nu$ representation (9) does not hold, hence we can't apply the lemma 1 directly. We introduce auxiliary operators $\hat{H}_\nu$ and $\hat{\mathcal{Y}}_\nu$ according to the formulas

$$\hat{H}_\nu[f](x) = \int\limits_0^\infty J_\nu\left(\frac{x}{t}\right)\left(\frac{x}{t}\right)^{\frac{1}{2}} f(t)\frac{dt}{t}, \tag{26}$$

$$\hat{\mathcal{Y}}_\nu[f](x) = \int\limits_0^\infty Y_\nu\left(\frac{x}{t}\right)\left(\frac{x}{t}\right)^{\frac{1}{2}} f(t)\frac{dt}{t}. \tag{27}$$

obviously (see (21)), that for any function $f \in L_2$ we have

$$H_\nu f = \hat{H}_\nu \hat{f}, \qquad \mathcal{Y}_\nu f = \hat{\mathcal{Y}}_\nu \hat{f},$$

where

$$\hat{f}(x) = \frac{1}{x}\, f\left(\frac{1}{x}\right).$$

Wherein

$$\mathcal{M}[H_\nu f](s) = \mathcal{M}[\hat{H}_\nu \hat{f}](s) = m_\nu(s)\mathcal{M}[\hat{f}](s),$$

$$\mathcal{M}[\mathcal{Y}_\nu f](s) = \mathcal{M}[\hat{\mathcal{Y}}_\nu \hat{f}](s) = m_\nu(s)\mathrm{ctg}\left(s+\frac{1}{2}-\nu\right)\frac{\pi}{2}\mathcal{M}[\hat{f}](s).$$

Thus for (26) and (27) the representation (9) holds with multipliers of operators H_ν and $\mathcal{Y}_\nu$, accordingly. Let show that

$$\left\|\frac{1}{x}\, f\left(\frac{1}{x}\right)\right\|_{L_2} = \|f\|_{L_2}.$$

Indeed

$$\int\limits_0^\infty \left|\frac{1}{x}\, f\left(\frac{1}{x}\right)\right|^2 dx = -\int\limits_\infty^0 |tf(t)|^2\frac{dt}{t^2} = \int\limits_0^\infty |f(t)|^2 dt.$$

We have

$$||H_\nu f||_{L_2} = \sup_{f\in L_2}\frac{||H_\nu f||_{L_2}}{\|f\|_{L_2}} = \sup_{f\in L_2}\frac{||\hat{H}_\nu \hat{f}||_{L_2}}{\|\hat{f}\|_{L_2}} = ||\hat{H}_\nu||_{L_2}.$$

Similarly we obtain $||\mathcal{Y}_\nu||_{L_2} = ||\hat{\mathcal{Y}}_\nu||_{L_2}$. Now we can use lemma 1 to proof next two theorems.

Theorem 3. *For $\nu > -1$ operator H_ν is unitary in L_2.*

Доказательство. The space L_2 is obtained from $\mathcal{L}_{2,\mu}$ when $\mu=\frac{1}{2}$, therefore the multiplier of operator H_ν for $\nu>\frac{1}{2}$ is defined on the line $\mathrm{Re}\, s=\frac{1}{2}$. Let write its values on this line:

$$\left|m\left(i\xi+\frac{1}{2}\right)\right|=\left|\frac{\Gamma\left(\frac{1+\nu-i\xi}{2}\right)}{\Gamma\left(\frac{1-\nu-i\xi}{2}\right)}\right|.$$

Considering that $|z|=|\bar{z}|$ and $\bar{\Gamma}(z)=\Gamma(\bar{z})$ we obtain

$$\left|m\left(i\xi+\frac{1}{2}\right)\right|=1$$

for any $\xi\in\mathbb{R}$. That means unitarity of $\hat{H}_\nu$ and, consequently, H_ν by lemma 1. $\square$

Remark 1. *It is easy to see that for complex ν the operator H_ν is not unitary in L_2. Indeed, let $\nu=\lambda+i\mu$, $\mu\neq 0$ then*

$$\left|m\left(i\xi+\frac{1}{2}\right)\right|=\left|\frac{\Gamma\left(\frac{\lambda+1+i(\mu+\xi)}{2}\right)}{\Gamma\left(\frac{\lambda+1+i(\mu-\xi)}{2}\right)}\right|.$$

It is obvious that the modules of imaginary parts of the arguments of gamma-functions are equal here if and only if $\xi=0$ and, consequently, equality (15) is not true for almost all $\xi\in\mathbb{R}$.

Theorem 4. *For $\nu\in(-1,1)$ the operator $\mathcal{Y}_\nu$ is bounded in L_2 and*

$$||\mathcal{Y}_\nu||_{L_2}=\begin{cases}1, & -\frac{1}{2}\leq\nu\leq\frac{1}{2};\\ \left|\mathrm{tg}\left(\frac{n\nu}{n}\right)\right|, & \nu\in\left(-1,-\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right).\end{cases}$$

Доказательство. Let write multiplier of operator $\mathcal{Y}_\nu$ on the line $\mathrm{Re}\, s=\frac{1}{2}$:

$$\left|m\left(i\xi+\frac{1}{2}\right)\right|=\left|\frac{\Gamma\left(\frac{1+\nu+i\xi}{2}\right)\Gamma\left(\frac{1-\nu+i\xi}{2}\right)}{\Gamma\left(\frac{1-\nu+i\xi}{2}\right)\Gamma\left(\frac{\nu-i\xi}{2}\right)}\right|=$$

$$\left|\mathrm{tg}\left(\frac{i\xi-\nu}{2}\right)n\right|.$$

Here we use formulas $|\Gamma(z)|=|\Gamma(\bar{z})|$,

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\frac{\pi}{\cos(\pi z)},$$

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$$

and

$$|\Gamma(z)|=|\Gamma(\bar{z})|.$$

Since for $k\in\mathbb{Z}$

$$\lim_{z\to\frac{\pi}{2}+\pi k}\mathrm{tg}(z)=\infty$$

we require that $\nu\neq 2k+1$. For $-1<\nu<1$ this requirement is obviously satisfied. Because the (see [12])

$$\lim_{\xi\to\infty}\left|m\left(i\xi+\frac{1}{2}\right)\right|=1,$$

we have that $\mathcal{Y}_\nu$ is bounded in L_2. Using that

$$\operatorname{tg}(z) = (-i)\frac{e^{2iz}-1}{e^{2iz}+1}$$

we obtain

$$\left|\operatorname{tg}\left(\frac{i\xi-\nu}{2}\right)\pi\right|^2 = f_\nu(\xi),$$

where

$$f_\nu(\xi) = \frac{e^{-2i\xi} - 2e^{-n\xi}\cos(\nu\pi) + 1}{e^{-2i\xi} + 2e^{-n\xi}\cos(\nu\pi) + 1}.$$

We have

$$\sup_{\xi\in\mathbb{R}} f_\nu(\xi) = f_\nu(0) = \left|\operatorname{tg}\left(\frac{\nu}{2}\pi\right)\right|^2$$

for $\nu \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ and

$$\sup_{\xi\in\mathbb{R}} f_\nu(\xi) = f_\nu(\infty) = 1$$

for $-\frac{1}{2} \le \nu \le \frac{1}{2}$. Therefore, noticing that $||\mathcal{Y}_\nu||_{L_2} = ||\hat{\mathcal{Y}}_\nu||_{L_2}$ we get statement of the theorem, by the lemma 1 □

Theorem 5. *Operators $H_\nu\mathcal{Y}_\nu$ and $\mathcal{Y}_\nu H_\nu$ are bounded in L_2 for $-1 < \nu < 1$. The formula for norms*

$$||H_\nu\mathcal{Y}_\nu||_{L_2} = ||\mathcal{Y}_\nu H_\nu||_{L_2} = \begin{cases} 1, & -\frac{1}{2} \le \nu \le \frac{1}{2}; \\ \left|\operatorname{tg}\left(\frac{n\nu}{n}\right)\right|, & \nu \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$$

is valid.

Доказательство. The multiplier of $\mathcal{Y}_\nu H_\nu$ on a line $\operatorname{Re} s = \frac{1}{2}$ is

$$m\left(i\xi + \frac{1}{2}\right) = -\operatorname{ctg}\left(\frac{i\xi - \nu + 1}{2}\right)\pi = \operatorname{tg}\left(\frac{i\xi-\nu}{2}\right)\pi.$$

We have

$$\sup_{\xi\in\mathbb{R}}\left|\operatorname{tg}\left(\frac{i\xi-\nu}{2}\pi\right)\right| = ||\mathcal{Y}_\nu H_\nu||_{L_2}.$$

It remains to note that $||\mathcal{Y}_\nu H_\nu||_{L_2}$ is an even function of ν and is defined on symmetrical interval. Finally, since $H_\nu\mathcal{Y}_\nu = -\mathcal{Y}_{-\nu}H_{-\nu}$ we have

$$||H_\nu\mathcal{Y}_\nu||_{L_2} = ||\mathcal{Y}_\nu H_\nu||_{L_2}.$$

□

3.2. **Boundedness of H_ν, $\mathcal{Y}_\nu$ and their compositions in weighted spaces.** Let consider weighted Lebegue spaces $\mathcal{L}_{2,\mu}$. Since the boundedness of operator A acting from $\mathcal{L}_{2,\mu}$ to $\mathcal{L}_{2,\mu}$ is equal to the boundedness of operator $B = x^{\mu-\frac{1}{2}}Ax^{\frac{1}{2}-\mu}$ acting from L_2 to L_2, all statements of lemma 1 are preserved for the weighted case with changing $i\xi + \frac{1}{2}$ to $i\xi + \mu$. However, as was noticed earlier we can't apply lemma 1 directly to operators H_ν and $\mathcal{Y}_\nu$ that is why we will use $\hat{H}_\nu$ and $\hat{\mathcal{Y}}_\nu$ again. It is easy to see that

$$\left\|\frac{1}{x}f\left(\frac{1}{x}\right)\right\|_{\mathcal{L}_{2,\mu}} = \|f\|_{\mathcal{L}_{2,\mu}}$$

and

$$||H_\nu||_{\mathcal{L}_{2,\mu}\to\mathcal{L}_{2,1-\mu}} =$$

$$= \sup_{f\in\mathcal{L}_{2,\mu}} \frac{||H_\nu f||_{\mathcal{L}_{2,1-\mu}}}{\|f\|_{\mathcal{L}_{2,\mu}}} = \sup_{f\in\mathcal{L}_{2,\mu}} \frac{||\hat{H}_\nu \hat{f}||_{\mathcal{L}_{2,1-\mu}}}{\|\hat{f}\|_{\mathcal{L}_{2,1-\mu}}} = ||\hat{H}_\nu||_{\mathcal{L}_{2,1-\mu}}.$$

Similarly we find that

$$||\mathcal{Y}_\nu||_{\mathcal{L}_{2,\mu}\to\mathcal{L}_{2,1-\mu}} = ||\hat{\mathcal{Y}}_\nu||_{\mathcal{L}_{2,1-\mu}}.$$

It follows that if in lemma 1 we change $i\xi+\frac{1}{2}$ to $i\xi+1-\mu$ it is possible to use this lemma for operators H_ν and $\mathcal{Y}_\nu$ in spaces $\mathcal{L}_{2,\mu}$.

First we prove the auxiliary lemma.

Lemma 4. *Let $\alpha,\beta\in\mathbb{R}$ and $\alpha\neq\beta$. Then the next relation*

$$\sup_{\xi\in\mathbb{R}}\left|\frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)}\right| =$$

$$=\begin{cases} \infty, & \alpha>\beta;\\ \infty, & \alpha=-n,\ \beta\neq -m,\ n,m\in\mathbb{N}_0;\\ q,\ \textit{where}\ \left|\frac{\Gamma(\alpha)}{\Gamma(\beta)}\right|\leq q<\infty & \textit{in other cases}\end{cases}$$

is valid.

Доказательство. 1. Lel $\alpha>\beta$. In virtue of asymptotic formula (see [12], p. 62, formula 4)

$$\frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)}\sim (i\xi)^{\alpha-\beta}, \qquad |\xi|\to\infty$$

we obtain

$$\lim_{|\xi|\to\infty}\left|\frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)}\right| = \begin{cases}\infty, & \alpha>\beta;\\ 0, & \alpha<\beta\end{cases} \tag{28}$$

and the first statement in curly brackets is proved.

2. Let $\alpha=-n$, $\beta\neq-m$, $n,m\in\mathbb{N}_0$. In this case the numerator of the considered relation has a pole at $\xi=0$ at the same time the denominator at this point is finite.

3. Let $|\alpha|\leq\beta$, $\alpha\neq-n$, $n\in\mathbb{N}_0$. Using formula (see [25])

$$\left|\frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)}\right| = \left|\frac{\Gamma(\alpha)}{\Gamma(\beta)}\right| \prod_{s=0}^{\infty}\frac{1+\left(\frac{\xi}{(\beta+s)}\right)^2}{1+\left(\frac{\xi}{(\alpha+s)}\right)^2}$$

we obtain that $\frac{1+\left(\frac{\xi}{(\beta+s)}\right)^2}{1+\left(\frac{\xi}{(\alpha+s)}\right)^2}<1$ for $(\alpha-\beta)(\alpha+\beta+s)\leq 0$. Therefore for $\alpha<\beta$ this is true for any s is $\alpha+\beta\geq 0$. These two conditions are equal to $|\alpha|\leq\beta$ since by condition of lemma $\alpha\neq\beta$. So we have that for $|\alpha|\leq\beta$ supremum is at $\xi=0$.

4. Let $\alpha<\beta$, $\alpha+\beta<0$. From (28) follows that $\sup\limits_{\xi\in\mathbb{R}}\left|\frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)}\right|$ is finite for $\alpha\neq-n$, $n\in\mathbb{N}_0$. If $\alpha=-n$, $\beta=-m$, $n,m\in\mathbb{N}_0$, then using formula 11 from [12], p. 61 we get

$$\left|\frac{\Gamma(-n)}{\Gamma(-m)}\right| = \frac{m!}{n!}$$

and thus the lemma is completely proved.

$\square$

Theorem 6. *The operator H_ν acting from $\mathcal{L}_{2,\mu}$ to $\mathcal{L}_{2,1-\mu}$ is unbounded under any of the next conditions:*

1. $\mu-\nu=2n+\frac{3}{2}$, $n\in\mathbb{N}_0$.

2. $\mu<\frac{1}{2}$.

In other cases it is bounded and for its norm the formula

$$||H_\nu||_{\mathcal{L}_{2,\mu}\to\mathcal{L}_{2,1-\mu}}=2^{\frac{1}{2}}\left|\frac{\Gamma\left(\frac{2\nu-2\mu+3}{4}\right)}{\Gamma\left(\frac{2\nu+2\mu+3}{4}\right)}\right|$$

is valid.

Доказательство. We can write the module of the multiplier of H_ν on the line $\operatorname{Re}s=1-\mu$:

$$|m(i\xi+1-\mu)|=2^{\frac{1}{2}-\mu}\left|\frac{\Gamma\left(\frac{2\nu+i2\xi-2\mu+3}{4}\right)}{\Gamma\left(\frac{2\nu-i2\xi+2\mu+1}{4}\right)}\right|=$$

$$=2^{\frac{1}{2}-\mu}\left|\frac{\Gamma\left(\frac{2\nu+i2\xi-2\mu+3}{4}\right)}{\Gamma\left(\frac{2\nu+i2\xi+2\mu+1}{4}\right)}\right|.$$

Now setting $\alpha=\frac{2\nu-2\mu+3}{4}$, $\beta=\frac{2\nu+2\mu+3}{4}$ and applying lemma 4 we get statement of the theorem. □

Theorem 7. *The operator $\mathcal{Y}_\nu$ acting from $\mathcal{L}_{2,\mu}$ to $\mathcal{L}_{2,1-\mu}$ is unbounded under any of the next conditions*

1. $\mu<\frac{1}{2}$.

2. $\mu\pm\nu=2n+\frac{3}{2}$, $n\in\mathbb{N}_0$.

In other cases it is bounded. If at least one of the conditions

3. $\mu\geq 1$ *and* $\nu\geq-\frac{1}{2}$ *and* $\mu+\nu\leq 2$

4. $\mu\leq 2$ *and* $\nu\frac{1}{2}$ *and* $\mu+\nu\geq 1$ *is satisfied then for $\mathcal{Y}_\nu$ norm the formula*

$$||\mathcal{Y}_\nu||_{\mathcal{L}_{2,\mu}\to\mathcal{L}_{2,1-\mu}}=2^{\frac{1}{2}-\mu}\left|\frac{\Gamma\left(\frac{2\nu-2\mu+3}{4}\right)\Gamma\left(\frac{-2\nu-2\mu+3}{4}\right)}{\Gamma\left(\frac{-2\nu+2\mu+5}{4}\right)\Gamma\left(\frac{2\nu+2\mu-1}{4}\right)}\right|$$

is valid.

Доказательство. Let consider the multiplier module of $\mathcal{Y}_\nu$ on the line $\operatorname{Re}s=1-\mu$:

$$|m(i\xi+1-\mu)|=2^{\frac{1}{2}-\mu}\left|\frac{\Gamma\left(\frac{i2\xi+2\nu-2\mu+3}{4}\right)\Gamma\left(\frac{i2\xi-2\nu-2\mu+3}{4}\right)}{\Gamma\left(\frac{i2\xi-2\nu-2\mu+5}{4}\right)\Gamma\left(\frac{i2\xi+2\nu+2\mu-1}{4}\right)}\right|.$$

Now setting $\alpha=\frac{2\nu-2\mu+3}{4}$, $\beta=\frac{-2\nu-2\mu+5}{4}$, $\theta=\frac{-2\nu-2\mu+3}{4}$, $\gamma=\frac{2\nu+2\mu-1}{4}$. Noticing that if each functions $f(x)$ and $g(x)$ has maximum at x_0 then function $p(x)$=$f(x)g(x)$ has maximum at the same point. Now applying lemma 4 to relations

$$\left|\frac{\Gamma\left(\frac{\xi}{2}+\alpha\right)}{\Gamma\left(\frac{\xi}{2}+\beta\right)}\right|,\ \left|\frac{\Gamma\left(\frac{\xi}{2}+\theta\right)}{\Gamma\left(\frac{\xi}{2}+\gamma\right)}\right|,\ \left|\frac{\Gamma\left(\frac{\xi}{2}+\alpha\right)}{\Gamma\left(\frac{\xi}{2}+\gamma\right)}\right|,\ \left|\frac{\Gamma\left(\frac{\xi}{2}+\theta\right)}{\Gamma\left(\frac{\xi}{2}+\beta\right)}\right|$$

we get statement of the theorem. □

Turning to the consideration of composition of H_ν and $\mathcal{Y}_\nu$ we should notice that the representation (9) is valid for them and we can apply lemma 1 directly.

Theorem 8. *The operator $H_\nu\mathcal{Y}_\nu$ is unbounded under in $\mathcal{L}_{2,\mu}$ if $\mu\pm\nu=2n+\frac{3}{2}$, $n\in\mathbb{Z}$. In other cases it is bounded and for its norm the formula*

$$||H_\nu\mathcal{Y}_\nu||_{\mathcal{L}_{2,\mu}}=$$
$$=\begin{cases} 1, & \mu+\nu\in(2n,2n+1),\ n\in\mathbb{Z};\\ \left|\mathrm{tg}\left(\frac{2\mu+2\nu-1}{4}\right)n\right|, & \mu+\nu\in\left(2n-1,2n-\frac{1}{2}\right)\cup\left(2n-\frac{1}{2},2n\right),\ n\in\mathbb{Z}\end{cases}$$

is valid. For $\mu+\nu=n$, $n\in\mathbb{Z}$ and only for them operator $H_\nu\mathcal{Y}_\nu$ is unatary in $\mathcal{L}_{2,\mu}$.

Доказательство. By the formula (19) for $s=i\xi+\mu$ we can write out the multiplier module of $H_\nu\mathcal{Y}_\nu$

$$|m(i\xi+\mu)|=\left|\mathrm{tg}\left(\frac{i2\xi+2\mu+2\nu-1}{4}\right)n\right|.$$

Putting $\frac{2\mu+2\nu-1}{4}=-\frac{\gamma}{2}$ we get function $f_\gamma(\xi)$ which we studied in proof of theorem 5 From the properties of this function and from lemma 1 we obtain statement of the theorem. □

Следствие 1. *Since $\mathcal{Y}_\nu H_\nu=-H_{-\nu}\mathcal{Y}_{-\nu}$ we have analogous theorem for $\mathcal{Y}_\nu H_\nu$ with replacement ν to $-\nu$.*

3.3. **Composition $H_\nu H_\gamma$ integral transforms.** Similarly to what was done in the proof of theorem 6 by the formula (18) for $f\in\mathcal{L}_{2\mu}$ we obtain

$$\mathcal{M}[H_\nu H_\gamma f](s)=m_\nu(s)m_\gamma(1-s)\mathcal{M}[f](s)=m(s)\mathcal{M}[f](s),$$

$$\frac{1}{2}\leq\mu<\min\left(\frac{3}{2}+\nu,\frac{3}{2}+\gamma\right). \tag{29}$$

Here

$$m(s)=\frac{\Gamma\left(\frac{2s+2\nu+1}{4}\right)\Gamma\left(\frac{-2s+2\gamma+3}{4}\right)}{\Gamma\left(\frac{-2s+2\nu+3}{4}\right)\Gamma\left(\frac{2s+2\gamma+1}{4}\right)},\qquad \mathrm{Re}\,s=\mu.$$

It is obvious that $\left|m\left(i\xi+\frac{1}{2}\right)\right|=1$ for all $\xi\in\mathbb{R}$. So the operator $H_\nu H_\gamma$ can't be presented as Mellin convolution with a kernel from L_2. One way or another equality must be fulfilled (see [27], p.53)

$$||K||_{L_2}=\frac{1}{2\pi}\int\limits_0^\infty\left|m\left(i\xi+\frac{1}{2}\right)\right|^2d\xi.$$

We are looking for $H_\nu H_\gamma$ in the form suggested by the lemma from [11], p.78.

Lemma 5. *To each bounded linear operator T in $L_2(0,\infty)$ corresponds a function $P(x,t)$ belonging to $L_2(0,\infty)$ by t for each $x\in(0,\infty)$ and having a property: for all $f(t)\in L_2(0,\infty)$ almost everywhere on axe x*

$$T[f](x)=\frac{d}{dx}\int\limits_0^\infty P(x,t)f(t)dt.$$

Therefore, let

$$H_\nu[H_\gamma f](x)=\frac{d}{dx}\int\limits_0^\infty K\left(\frac{x}{y}\right)f(y)dy=\frac{d}{dx}\int\limits_0^\infty K\left(\frac{x}{y}\right)[yf(y)]\frac{dy}{y}, \tag{30}$$

then $\mathcal{M}[H_\nu H_\gamma f](s) = \mathcal{M}\left[\frac{d}{dx}K*(yf)\right](s) = (1-s)\mathcal{M}[K*(yf)](1-s) = (1-s)\mathcal{M}[K](1-s)M[f](s)$. Comparing with (29) we obtain

$$m(s) = (1-s)\mathcal{M}[K](1-s),$$

whence

$$\mathcal{M}[K](s) = \frac{1}{2}\frac{\Gamma\left(\frac{2s+2\nu+3}{4}\right)\Gamma\left(\frac{-2s+2\mu+1}{4}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{-2s+2\nu+1}{4}\right)\Gamma\left(\frac{2s+2\mu+3}{4}\right)\Gamma\left(\frac{s}{2}+1\right)}.$$

Applying formulas (9), p. 728 from [22] and 5, p. 268 from [25] we get

$$K(x) = -G^{2,1}_{3,3}\left(\begin{matrix}\frac{s-2\gamma}{4}, 1, \frac{s+2\gamma}{4}\\ \frac{s+2\nu}{4}, 0, \frac{s-2\nu}{4}\end{matrix}\,\middle|\, x^2\right),$$

where $G^{2,1}_{3,3}$ is the Meijer G-function (5).

The substitution in (30) gives the final expression for $H_\nu H_\gamma$:

$$H_\nu[H_\gamma f](x) = -\frac{d}{dx}\int\limits_0^\infty G^{2,1}_{3,3}\left(\begin{matrix}\frac{s-2\gamma}{4}, 1, \frac{s+2\gamma}{4}\\ \frac{s+2\nu}{4}, 0, \frac{s-2\nu}{4}\end{matrix}\,\middle|\, \left(\frac{x}{y}\right)^2\right) f(y)dy.$$

Remark 2. *Similarly, one can obtain the formula for $\mathcal{Y}_\nu\mathcal{Y}_\gamma$ and $H_\nu\mathcal{Y}_\gamma$ but they are cumbersome.*

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