ABSTRACT

Title of Dissertation: Cluster Algebras and Polylogarithm Relations

Zachary Greenberg
Doctor of Philosophy, 2021

Dissertation Directed by: Professor Christian Zickert
Department of Mathematics

We seek to illuminate the connection between multiple polylogarithm relations and cluster algebras in two ways. First, we give a uniform description of the cluster modular group of affine and doubly extended cluster algebras. This will be critical for the future work of extracting polylogarithm relations from infinite type cluster algebras. Second, we introduce a differential one form, $\omega_{\mathbf{n}}$, associated to each multiple polylogarithm, which can be used to compute multiple polylogarithm relations. This form satisfies a clean recurrence relation, mirroring the inductive definition of multiple polylogarithms. We are able to use this recurrence to find several families of "small" polylogarithm relations that hold in any weight. Finally for small values of n, we extract polylogarithm relations from type A_n and D_n cluster algebras.

Cluster Algebras and Polylogarithm Relations

by

Zachary Greenberg

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2021

Advisory Committee:
Professor Christian Zickert, Chair/Advisor
Professor Jeffery Adams
Professor William Goldman
Professor Jonathan Rosenberg
Professor William Gasarch, Deans Representative

Acknowledgments

I would first like to thank my advisor, Christian for all of his guidance and support throughout this entire process. I also want to thank Dani and Haoran for all our enlightening discussions and encouragement. Finally I want to thank my family for keeping me grounded and sane.

Table of Contents

List of Figures

List of Abbreviations

$\mathbb{N}$	Natural numbers (starting at 1)
$\mathbb{Z}$	Integers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$[n]$	$\{1, \ldots, n\}$
$\mathbf{x}$	The vector $(x_1, \ldots, x_d)$
$\overleftarrow{\mathbf{x}}$	The reverse of the vector $\mathbf{x}$, $(x_d, \ldots, x_1)$
$\mathbf{x} - 1$	The vector $(x_1 - 1, \ldots, x_d - 1)$
$\binom{\mathbf{m}}{\mathbf{n}}$	The product of binomial coefficients $\prod \binom{m_i}{n_i}$
$\mathbb{Z}_n$	The cyclic group with n elements
S_n	The symmetric group on n objects
$R^{\times}$	Multiplicative group of a ring R.
$\mathrm{Aut}(Q)$	The automorphism group of a quiver Q
$\mathrm{Mod}(S)$	The mapping class group of a surface S
$\mathcal{A}_Q$	The cluster algebra associated to a quiver Q
$\mathrm{Mut}(Q)$	The set of all quivers mutation equivalent to Q
$C(\mathcal{A})$	The cluster complex of cluster algebra $\mathcal{A}$.
$S_{g,b,p,n}$	The oriented surface with genus g, b boundary components, p punctures, and n marked points
$\chi(S)$	The Euler characteristic of S
A_n, D_n, E_6, E_7, E_8	Simply laced finite Dynkin diagrams
B_n, C_n, F_4, G_2	Folded finite Dynkin diagrams
$\widetilde{A_n}, \widetilde{D_n}, \widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}$	Simply laced affine Dynkin diagrams
$\widetilde{B_n}, \widetilde{C_n}, \widetilde{F_4}, \widetilde{G_2}$	Folded affine Dynkin diagrams
${B_n}^{(2)}, {F_4}^{(2)}, {G_2}^{(2)} \ldots$	Twisted affine Dynkin diagrams
$A_n^{(1,1)}, D_n^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$	Simply laced doubly extended Dynkin diagrams
$B_3^{(1,1)}, B_2^{(2,1)}, BC_1^{(4,1)}, C_3^{(2,2)}, \ldots$	Folded doubly extended Dynkin diagrams
$\mathrm{Gr}(k, n)$	The Grassmannian of k planes in n dimensional space
p_I	Plücker Coordinate
$e2x, e2y$	The exotic A-coordinates on $\mathrm{Gr}(3, 6)$
$\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$	The standard polylogarithm
X_d	The set of singularities of a depth d polylogarithm
$\hat{U}_d$	The universal abelian covering space of $\mathbb{C}^d \setminus X_d$
$\omega_{\mathbf{n}}$	One-form associated to the polylogarithm $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$

Chapter 1: Introduction

Polylogarithms are a family of functions generalizing the classic logarithm. For any n, the weight n-logarithm $\mathrm{Li}_n(z)$ can be defined inductively by $\mathrm{Li}_n(z) = \int \frac{\mathrm{Li}_{n-1}(z)}{z}\mathrm{d}z$, where $\mathrm{Li}_1(z) = -\log(1-z)$. Polylogarithms have a wide variety of applications across mathematics and physics. In particular, the scattering amplitudes associated to particle collisions are expressed in terms of polylogarithms [1]. The dilogarithm has also been used to compute volume in hyperbolic 3 space [2].

Our motiving reason for studying polylogarithms is to obtain a concrete model of motivic cohomology. This would be a "universal cohomology theory" for smooth algebraic varieties X. In [3], Goncharov constructs a family of groups $\mathcal{B}_n(X)$ using the function relations of the polylogarithms that is conjectured to be such a concrete model. There are two key issues to be overcome. The first is that the full set of polylogarithm relations for general n are unknown. The second issue is that even by weight 4, the family of polylogarithms must be generalized further to "multiple polylogarithms". In the following we attack both problems.

To understand the difficulty of computing the polylogarithm relations we look at the example of the dilogarithm, $\mathrm{Li}_2(z)$. There is a classical "five term relation"

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{1-x}{1-xy}\right) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\left(\frac{1-y}{1-xy}\right)$$
$$= \frac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)\log(1-y) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right)$$

Already in this simple case, the relation is fairly complicated. We see that every term in this relation is either a single weight 2 polylogarithm or a product of lower weight logarithms whose total weight is 2. Our first simplification is to remove the terms that are products of logarithms, which reduces the above relation to the five dilogarithm terms. This can be justified by replacing $\mathrm{Li}_2(z)$ with the Bloch-Wigner dilogarithm (Section 1.3.4) that satisfies the relation without the product terms. This inspires us to look for "relations modulo products" in higher weights as well. Even with this simplification, the arguments $x, y, \frac{1-x}{1-xy}, 1-xy, \frac{1-y}{1-xy}$ don't lend themselves to obvious generalization. In this case $\mathrm{Li}_2(z)$ also satisfies two simple relations modulo products $\mathrm{Li}_2(z) = -\mathrm{Li}_2(1-z)$ and $\mathrm{Li}_2(z) = -\mathrm{Li}_2(\frac{1}{z})$. These combine to obtain $\mathrm{Li}_2(z) = \mathrm{Li}_2(\frac{z-1}{z}) = \mathrm{Li}_2(\frac{1}{1-z})$. Applying these combined relations to terms 1,2 and 5, allows us to rewrite the 5 term relation as:

$$\mathrm{Li}_2\left(-\frac{x-1}{-x}\right) + \mathrm{Li}_2\left(-\frac{-1}{1-y}\right) + \mathrm{Li}_2\left(-\frac{x-1}{1-xy}\right) + \mathrm{Li}_2\left(-\frac{1-xy}{-1}\right) + \mathrm{Li}_2\left(-\frac{1-xy}{y(x-1)}\right)$$

Now consider the matrix $M = \begin{bmatrix} 1 & 0 & 1 & 1 & y \\ 0 & 1 & 1 & x & 1 \end{bmatrix}$. This matrix represents a point on the affine Grassmannian $\widetilde{\mathrm{Gr}}(2,5)$, by considering the rows to be the generating vectors of a 2-plane in $\mathbb{C}^5$. The "Plücker" coordinates on $\widetilde{\mathrm{Gr}(2,5)}$ are functions, $pij = \det \begin{bmatrix} M_i & M_j \end{bmatrix}$ where M_i and M_j are the columns of M. (See Section 1.1.3 for more details). In this way each argument of the five term relation is -1 times a "cross ratio" of four Plücker coordinates:

$$\mathrm{Li}_2\left(-\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}}\right)+\mathrm{Li}_2\left(-\frac{\mathrm{p}_{15}\mathrm{p}_{23}}{\mathrm{p}_{35}\mathrm{p}_{12}}\right)+\mathrm{Li}_2\left(-\frac{\mathrm{p}_{34}\mathrm{p}_{15}}{\mathrm{p}_{13}\mathrm{p}_{45}}\right)+\mathrm{Li}_2\left(-\frac{\mathrm{p}_{12}\mathrm{p}_{45}}{\mathrm{p}_{24}\mathrm{p}_{15}}\right)+\mathrm{Li}_2\left(-\frac{\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{25}\mathrm{p}_{34}}\right)$$

For every k and n, $\mathrm{Gr}(k,n)$ has the additional structure of a "cluster algebra" (Section 1.1). These five cross ratio arguments are all five X-coordinates of the cluster algebra of $\mathrm{Gr}(2,5)$. In weight 3 the trilogarithm $\mathrm{Li}_3(z)$ also has a functional relation whose arguments are -1 times X-coordinates of the $\mathrm{Gr}(3,6)$ cluster algebra. While this relation does not use every X-coordinate, the arguments are symmetric under the symmetry group of the cluster algebra, called the "cluster modular group" (Section 1.2).

Therefore to understand potential polylogarithm relations, we seek a better understanding of the cluster modular group of cluster algebras. Both $\mathrm{Gr}(2,5)$ and $\mathrm{Gr}(3,6)$ are "finite type" cluster algebras and as such have only finitely many possible X-coordinates. One of the early results of the theory of cluster algebras is that a cluster algebra is finite type if and only if it is associated with a finite type Dynkin diagram. Most Grassmannian cluster algebras are not of finite type and it is important to understand the cluster modular groups of infinite type cluster algebras.

Chapter 2 presents a uniform computation of the cluster modular group of affine and doubly extended cluster algebras. Both $\mathrm{Gr}(4,8)$ and $\mathrm{Gr}(3,9)$ are doubly extended cluster algebras and we expect the analysis in this chapter to be critical in studying these cases. This was joint work with Dani Kaufman and covered in [4]. The key idea is that every affine and doubly extended cluster algebra is also associated to a family of quivers (directed graphs), which we call $T_{\mathbf{n},\mathbf{w}}$ (Figure 1.1). Using these quivers we are able to give a uniform description of the elements of the cluster modular group. In addition, we are able to classify every cluster algebra with a $T_{\mathbf{n},\mathbf{w}}$ quiver as either affine, doubly extended, or "infinite mutation type" (Theorem 1.0.1).

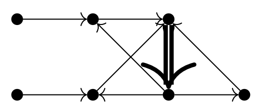


Figure 1.1: A $T_{(2,3,3),(1,1,1)}$ quiver.

Theorem 1.0.1. *For* $\mathbf{n}, \mathbf{w}$ m *dimensional vectors of integers, let*

$$\chi(T_{\mathbf{n},\mathbf{w}}) = \sum (w_i(n_i^{-1} - 1)) + 2$$

Then

1. *If* $\chi > 0$ *then* $T_{\mathbf{n},\mathbf{w}}$ *is the underlying quiver of an affine cluster algebra.*

2. *If* $\chi = 0$ *then* $T_{\mathbf{n},\mathbf{w}}$ *is the underlying quiver of a doubly extended cluster algebra.*

3. *If* $\chi < 0$ *then* $T_{\mathbf{n},\mathbf{w}}$ *is the underlying quiver of an infinite mutation type cluster algebra.*

In Chapter 3, we provide a new tool to computationally understand multiple polylogarithms. This is joint work my advisor Christian Zickert, as well as Dani

Kaufman and Haoran Li. For any vector $\mathbf{n}$ of length d, the multiple polylogarithm $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ is assigned a differential form $\omega_{\mathbf{n}}$ that lives on the universal abelian cover $\hat{U}_d$ of the domain of $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$. This generalizes the forms discovered by Zickert in [5] for the standard polylogarithms.

We show that these forms can be obtained as a further symmetrization of the "symbol modulo products" which is the classic algebraic tool used to study polylogarithm relations. The forms offer several advantages over the symbol. The first is that the forms satisfy a simple recurrence relation (Section 3.2.3):

$$\omega_{\mathbf{n}} = \sum \delta_i \omega_{\mathbf{n}} + v_{[1\ldots n]} \sum_{\mathbf{m} \prec^1 \mathbf{n}} c_{\mathbf{m}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \omega_{\mathbf{m}}$$

The second is that differential forms come with a natural chain complex with coboundary given by the differential d. As such linear combinations of forms that are closed under d can be integrated to obtain well defined functions on $\hat{U}_d$.

Using these forms we are able to establish a variety of general relations necessary to extract relations from the type A_n cluster algebras. In particular we generalize the inversion relation $\mathrm{Li}_n(z) + (-1)^n \mathrm{Li}_n(1/z) = 0$ to arbitrary depth (Section 3.3.1):

$$(-1)^d (-1)^{\sum n_i} \omega_{\mathbf{n}}(1/\mathbf{z}) = \sum_{\mathbf{m} \preceq \mathbf{n}} c_{\mathbf{m}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \sum_{\mathbf{c}} \widehat{r}_{\mathbf{c}} \omega_{\mathbf{c}\cdot\mathbf{m}}$$

We note the similarity in structure between the terms occurring in the recurrence and the inversion relation.

Finally we are able to use our understanding of the cluster algebra structure and the differential forms to compute multiple polylogarithm relations up through weight 5 coming from the A_n cluster algebras. This builds on the work of Charlton, Gangl, and Radchenko in [6] who obtained similar relations without using the cluster algebra structure. We then use the relationship between type A_n and type D_n cluster algebras to provide a method of canceling all depth 2 multiple polylogarithm terms from the relation in all known cases. We conjecture this holds for any odd weight relation (Section 3.5).

1.1 Cluster Algebras

1.1.1 Basic Definitions

In the following we focus on cluster algebras of geometric type. These are cluster algebras whose seeds are described by quivers where some nodes are considered "frozen".

Definition 1.1.1. *A **quiver** is a finite weighted directed graph without self loops or 2-cycles.*

We often think of quivers as graphical representations of skew symmetric matrices ε where there is an arrow of weight $\varepsilon_{i,j}$ from node i to node j. Note that under this interpretation a negative weight arrow from i to j is the same as a positive weight arrow from j to i. When the weight is an integer, we call the weight the number of arrows from i to j.

Definition 1.1.2. *For each node k of a quiver, **mutation** at node k produces a* *new quiver $Q' = \mu_k(Q)$ via the following process*

- *For each path $i \xrightarrow{\varepsilon_{ik}} k \xrightarrow{\varepsilon_{kj}} j$ through k add an edge of weight $\varepsilon_{ik}\varepsilon_{kj}$ from i to j. Note that if there is already an edge from i to j we add $\varepsilon_{ik}\varepsilon_{kj}$ to the weight present (ε_{ij}).*

- *Reverse every edge incident to k. So $k \xrightarrow{w} j$ becomes $j \xrightarrow{w} k$.*

See Figure 1.2 for an example mutation.

It is not hard to check that $\mu_k(\mu_k(Q)) = Q$ and so mutation is an involution. Furthermore if i isn't adjacent to j then $\mu_i\mu_j = \mu_j\mu_i$.

Figure 1.2: Mutating at node 2 transforms between the two quivers above.

Note that this rule can be encoded as a mutation of the skew symmetric matrix as follows

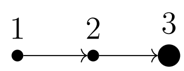
$$
\begin{aligned}
\varepsilon'_{i,j} =& -\varepsilon_{i,j} && \text{if } i = k \text{ or } j = k \\
\varepsilon'_{i,j} =& \varepsilon_{i,j} + \frac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj|}}{2} && \text{otherwise}
\end{aligned}
$$

$$
\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}
$$

Figure 1.3: The two matrices above represent the quivers in Figure 1.2. Once again mutation at node 2 transforms between the two matrices.

Remark 1.1.3. *It is possible to generalize the notion of a quiver to include weighted nodes as well as weighted edges. In this case each node is assigned a weight* $w_i > 0$*. The quiver is then represented by a* ***skew-symmetrizable*** *matrix* ε*. Such a matrix has an associated diagonal matrix* D *such that* εD^{-1} *is skew symmetric. The matrix* εD^{-1} *is the adjacency matrix of the associated quiver and the weight of node* i *is the* i^{th} *diagonal entry of* D $(D_{ii} = w_i)$*. See Figure 1.4 for an example of the correspondence.*

We use the skew symmetric matrix mutation rule to obtain the mutation rule for the skew-symmetrizable matrices. See Section 2.2 of [7] for more details.



(a) Q

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

(b) ε

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$$

(c) D

Figure 1.4: A quiver Q corresponding the skew-symmetrizable matrix ε with weight matrix D.

Definition 1.1.4. *The* ***mutation class*** *of a quiver* $\mathrm{Mut}(Q)$ *is the set of all quivers that can be obtained from* Q *via a sequence of mutations. Two quivers are* ***mutation equivalent*** *if they belong to the same mutation class.*

To define a cluster algebra (of type $\mathcal{A}$ or $\mathcal{X}$) we attach variables to the nodes of the quiver and then add rules for relations between the variables of two quivers that differ by a single mutation.

1.1.1.1 $\mathcal{A}$ Cluster Algebras

Definition 1.1.5. *Let $\mathcal{F} = \mathbb{Q}(z_1, \ldots, z_N)$ be the field of rational functions in $z_1, \ldots z_N$. A* ***seed*** *of an $\mathcal{A}$ cluster algebra is a pair $(Q, \mathbf{a})$ of a quiver Q and a list, $\mathbf{a}$, of algebraically independent elements of $\mathcal{F}$. The elements of $\mathbf{a}$ are called the* ***A-coordinates*** *of the seed. We index the A-coordinates and the nodes of the quiver with the same set, so a_i is "attached" to node i of the quiver.*

Definition 1.1.6. *Each A-coordinate of a seed is declared to be* ***unfrozen*** *or* ***frozen****. The unfrozen coordinates are also called* ***mutable*** *coordinates. As the name suggests we only allow mutation at nodes associated to mutable coordinates.*

An $\mathcal{A}$ cluster algebra will be defined by starting from an initial seed and then applying all possible mutations to it. For any mutable node, we extend the quiver mutation rule to include the A-coordinates as follows:

Definition 1.1.7. ***Mutation of a seed*** *$(Q, \mathbf{a})$ at node k produces a new seed $(Q', \mathbf{a}')$ where Q' is obtained from Q by quiver mutation and the new variables $\mathbf{a}'$ satisfy the relations $\mathbf{a}'_i = \mathbf{a}_i$ if $i \neq k$ and*

$$a_k \cdot a'_k = \prod_{i \xrightarrow{w_i} k} a_i^{w_i} + \prod_{k \xrightarrow{w_j} j} a_j^{w_j}$$

Remark 1.1.8. *These relations imply that the A-coordinates in a mutated seed, $(Q', \mathbf{a}')$ can be written as a rational function in the A-coordinates of the initial seed $(Q, \mathbf{a})$. This remains true after applying any finite sequence of mutations to initial*

Figure 1.5: The $\mathcal{A}$ mutation rule at node 2 transforms between the two quivers above.

seed. Therefore the field $\mathcal{F}$ can be taken to be $\mathbb{Q}(\mathbf{a})$ for any seed obtained from the initial seed by a finite sequence of mutations.

Definition 1.1.9. *The $\mathcal{A}$ **cluster algebra** generated by an initial seed $(Q, \mathbf{a})$ is the subalgebra of $\mathbb{Q}(\mathbf{a})$ generated by the set of all A-coordinates that appear in a seed obtained from the initial seed by a finite sequence of mutations.*

Definition 1.1.10. *The **rank** of a cluster algebra generated by a seed $(Q, \mathbf{a})$ is the number of mutable coordinates. We index $\mathbf{a}$ so the first n elements $a_1, \dots, a_n$ are mutable and the remaining m elements $a_{n+1}, \dots, a_{n+m}$ are frozen.*

See Figure 1.5 for an example of the $\mathcal{A}$ cluster algebra mutation rule.

Remark 1.1.11. *The inclusion of A-coordinates in the mutation rule, preserves the facts that mutation is an involution and mutations at nonadjacent nodes commute.*

A surprising fact about cluster algebras is that the number of seeds (and thus number of A-coordinates), only depends on the mutatable portion of the seed. In fact each cluster variable can be indexed by a length n vector called the **d-vector**. This relies on the following nontrivial property of A-coordinates, the Laurent phenomena:

Theorem 1.1.12. *Every A-coordinate in a cluster algebra can be written as a Laurent polynomial in the initial A-coordinates.*

Proof. This was shown in the original cluster algebras paper [8]. □

Definition 1.1.13. *The* ***d-vector*** *associated to an A coordinate, a is the powers of* $a_1, \ldots, a_n$ *in the denominator of the Laurent expansion of a in terms of the initial mutatable variables.*

Conjecture 1.1.14. *If a and b are two A-coordinates in a cluster algebra with the same d-vectors, then* $a = b$*.*

Proof. For finite cluster algebras this was proved in [9]. Further work on this was done in [10]. It is an open conjecture in arbitrary cluster algebras. □

1.1.1.2 $\mathcal{X}$ Cluster Algebras

The $\mathcal{X}$ cluster algebra will be defined analogously to the $\mathcal{A}$ cluster algebra, but with a different mutation rule on the coordinates.

Definition 1.1.15. *Let* $\mathcal{F} = \mathbb{Q}(z_1, \ldots, z_N)$ *be field. A* ***seed*** *of a* $\mathcal{X}$ *cluster algebra is a pair* $(Q, \mathbf{X})$ *of a quiver Q and a list,* $\mathbf{X}$*, of algebraically independent elements of* $\mathcal{F}$*. The elements of* $\mathbf{X}$ *are called* ***X-coordinates****. As in the* $\mathcal{A}$ *cluster algebra, each coordinate* X_i *is associated to node i of Q.*

Definition 1.1.16. ***Mutation of a seed*** $(Q, \mathbf{X})$ *at a node k produces a new seed* $(Q', \mathbf{X}')$ *where Q' is obtained from Q via quiver mutation and the new coordinates* $\mathbf{X}'$ *satisfy the following relations:*

$$X_i' = \mu_k(X_i) = \begin{cases} X_i^{-1} & i = k \\ X_i(1 + X_k)^w & i \xrightarrow{w} k \\ X_i(1 + X_k^{-1})^{-w} & k \xrightarrow{w} i \end{cases}$$

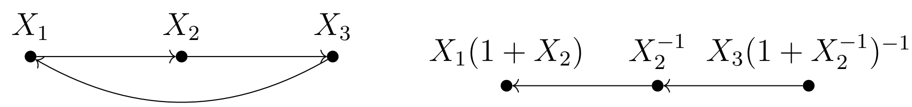


Figure 1.6: The $\mathcal{X}$ mutation at node 2 transforms between the two quivers above.

Remark 1.1.17. *As in the $\mathcal{A}$ cluster algebra, the new X-coordinates can be written as rational functions in the initial X-coordinates. This remains true after any finite sequence of mutations. Thus for any seed obtained from an initial seed $(Q, \mathbf{X})$ by finite sequence of mutations, the field $\mathcal{F}$ can be taken to be $\mathbb{Q}(\mathbf{X})$.*

Definition 1.1.18. *The $\mathcal{X}$-**cluster algebra** generated by an **initial seed** $(Q, \mathbf{X})$ is the subalgebra of $\mathbb{Q}(\mathbf{X})$ generated by the set of all X-coordinates that appear in a seed obtained from the initial seed by a finite sequence of mutations from $(Q, \mathbf{X})$.*

Remark 1.1.19. *The $\mathcal{X}$ mutation changes every X-coordinate adjacent to X_i not just X_i. See Figure 1.6 for an example.*

Let $(Q, \mathbf{a})$ be the seed of a rank n $\mathcal{A}$ cluster algebra. Using the same quiver we can define a seed $(Q, \mathbf{X})$ of a $\mathcal{X}$ cluster algebra. We have a map between the $\mathcal{A}$ and $\mathcal{X}$ cluster algebras induced via:

$$\rho_Q(X_k) = \prod_{k \xrightarrow{w} j} a_j^w \Big/ \prod_{i \xrightarrow{w} k} a_i^w$$

The image of X_k under ρ_Q is the ratio of A-coordinates out of node k to the A-coordinates coming into node k.

Claim 1.1.20. *If Q and Q' are two quivers related by a single quiver mutation μ_k*

then the following diagram commutes:

$$\begin{array}{ccc} (Q,(X_i)) & \xrightarrow{\mu_k} & (Q',(X'_i)) \\ \downarrow{\scriptstyle \rho_Q} & & \downarrow{\scriptstyle \rho_{Q'}} \\ (Q,(a_i)) & \xrightarrow{\mu_k} & (Q,(a'_i)) \end{array}$$

Proof. See [11] for the proof. □

This implies that ρ_Q respects the mutation relations and thus extends to a map ρ^* from the entire $\mathcal{X}$ cluster algebra to the $\mathcal{A}$ cluster algebra. Fock and Goncharov call the pair of the $\mathcal{A}$ cluster algebra and the $\mathcal{X}$ cluster algebra associated to the same starting quiver a **cluster ensemble**. In most cases ρ^* is injective, but isn't

Figure 1.7: In the left quiver, $\rho^* X_1 = a_2$ and $\rho^* X_3 = a_2$ even though $X_1 \neq X_3$. Adding the framing as shown on the right correctly distinguishes X_1 and X_3 as $\rho^* X_1 = a_2 a_4$ and $\rho^* X_3 = a_2 a_6$.

always. This simplest example is on the following quiver with 3 nodes (Figure 1.7). Here $\rho^*(X_1) = a_2$ and $\rho^*(X_3) = a_2$. This problem can be fixed by adding frozen vertices such that no two vertices of the quiver have the exact same set of neighbors even after arbitrary mutations. One way to guarantee this is to **frame** the quiver with one frozen node for each unfrozen node.

Definition 1.1.21. *A* ***framing*** *of a quiver Q is any quiver $\tilde{Q}$ such that the mutable portion of Q and $\tilde{Q}$ are the same. The* ***c-vectors*** *of $\tilde{Q}$ is the collection, $\{\mathbf{c}_i | 0 \leq i \leq n\}$, of m-dimensional vectors given by $\mathbf{c}_i^j = \varepsilon_{i,(j+n)}$*

Let Q be a quiver which consists of only mutable nodes. There is a canonical framing, $\hat{Q}$, obtained from Q by adding a frozen node F_i with matching weight w_i for each node N_i and a single arrow from N_i to F_i. $\hat{Q}$ is called the "ice" quiver associated with Q. The cluster algebra formed by starting with $\hat{Q}$ is called the cluster algebra with *principal coefficients*.

Remark 1.1.22. *There are two possible conventions of c-vectors, the other possibility is* $\mathbf{c}_i^j = \varepsilon_{(j+n),i)}$. *This is the convention used by Bernhard Keller's quiver mutation applet*[1]. *With the convention we chose, the matrix of c-vectors* $[\mathbf{c}_i^j]$ *associated to* $\hat{Q}$ *is the identity matrix.*

Theorem 1.1.23 ([12]). *The sets of c-vectors of quivers in* $\mathrm{Mut}(\hat{Q})$ *are in one-to-one correspondence with the clusters in the cluster algebra with principal coefficients associated with* Q.

Via this theorem, we see that by considering sets of c-vectors, one may understand whether a mutation sequence returns to a cluster with the same cluster variables without actually computing them. We only need to check that their sets of c-vectors are the same.

Definition 1.1.24. *Let* k *be a node of a quiver* Q *with frozen vertices. We call* k green *(resp. red) if the c-vector associated with* k *has all positive (resp. negative) entries.*

Remark 1.1.25. *The canonical framing* $\hat{Q}$ *is the one where every node is green.*

[1] https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/

Theorem 1.1.26 (sign coherence [13, 14])**.** *Let Q be a quiver without frozen variables. Then every quiver $R \in \mathrm{Mut}(\hat{Q})$ also has the property that every node of R is either red or green.*

Let $\check{Q}$ be the framing of Q by adding a frozen node F_i with matching weight w_i for each node N_i and a single arrow from F_i to N_i.

Theorem 1.1.27 ([15])**.** *Suppose there is $R \in \mathrm{Mut}(\hat{Q})$ satisfying that every node of R is red. Then $R \simeq \check{Q}$.*

Definition 1.1.28. *Suppose that $\check{Q} \in \mathrm{Mut}(\hat{Q})$. We call a sequence of mutations taking $\hat{Q}$ to $\check{Q}$ a* reddening sequence*.*

Remark 1.1.29. *The existence of a reddening sequence is an important property of a given quiver, and is conjectured to be related to several "niceness" properties of the cluster algebra [15].*

We will explicitly construct reddening sequences for the family of quivers introduced in Section 2.1.

Theorem 1.1.30 (Muller, [15])**.** *Let Q be a quiver with no frozen vertices and let $R \in \mathrm{Mut}(Q)$. Then $\hat{Q}$ has a reddening sequence if and only if $\hat{R}$ does.*

1.1.1.3 Poisson Structure on $\mathcal{X}$ Cluster Algebras

The X-coordinates of a cluster algebra have additional structure given by a Poisson bracket. The following definitions were given in [16]. Since every X-coordinate can be written in terms of the initial seed it suffices to define the bracket between the X-coordinates in the initial seed:

Definition 1.1.31. *The bracket of two X-coordinates x_i, y_j in the initial seed with mutation matrix ε_{ij} is given by $\{x_i, x_j\} = \varepsilon_{ij} x_i x_j$. The bracket is extended to arbitrary X-coordinates via the Leibniz rule and multi-linearity.*

Remark 1.1.32. *This bracket is preserved by mutation and so is independent of starting seed.*

Example 1.1.33. *We consider the example of an $\mathcal{X}$ mutation given in Figure 1.6. Let x_1, x_2, x_3 be the starting X coordinates on a oriented 3 cycle. Then after mutation at the node 2, the new X coordinates are $x_1(1+x_2), x_2, x_3(1+x_2^{-1})^{-1}$ on a directed path. Using the Leibniz rule and multi-linearity we compute:*

$$
\begin{aligned}
&\{x_1(1+x_2), x_3(1+x_2^{-1})^{-1}\} \\
&\qquad =\{x_1, x_3\frac{x_2}{1+x_2}\}(1+x_2) + \{1+x_2, x_3\frac{x_2}{1+x_2}\}x_1 \\
&\qquad =\{x_1, x_3\}x_2 + \{x_1, x_2\}x_3 - \{x_1, 1+x_2\}\frac{x_3x_2}{1+x_2} \\
&\qquad\qquad + \{x_2, x_3\}\frac{x_1x_2}{1+x_2} + \{x_2, x_2\}\frac{x_1x_3}{1+x_2} - \{x_2, 1+x_2\}\frac{x_1x_2x_3}{(1+x_2)^2} \\
&\qquad =-x_1x_2x_3 + x_1x_2x_3 - \frac{x_1x_2^2x_3}{1+x_2} + \frac{x_1x_2^2x_3}{1+x_2} + 0 - 0 \\
&\qquad =0
\end{aligned}
$$

This would agree with the definition of the bracket starting from the path as $x_1' = x_1(1+x_2)$ and $x_3' = x_3(1+x_2^{-1})^{-1}$ are not adjacent.

Corollary 1.1.34. *If x and y are two X coordinates that appear in a seed on non adjacent nodes, then x and y never appear on adjacent nodes in any seed.*

Proof. If x, y are not adjacent in a seed, then $\varepsilon_{ij} = 0$ when x is on node i and y is on node j. So $\{x, y\} = 0$ in the bracket starting on that seed. This implies that the bracket is zero between these two coordinates in any other seed. Thus $\varepsilon'_{i'j'} = 0$ and x and y are not adjacent. □

Definition 1.1.35. *A* ***Casimir element*** *of a cluster algebra is a function of the X-coordinates that has zero Poisson bracket with every element.*

Theorem 1.1.36. *If* $\mathbf{v}$ *is in the null space of* ε *then* $\prod x_i^{v_i}$ *is a Casimir element of the cluster algebra.*

Proof. It suffices to compute the bracket of $\prod_i x_i^{v_i}$ with x_j for each x_j in the seed with matrix ε.

$$\begin{aligned}
\{\prod_i x_i^{v_i}, x_j\} =& \sum_i x_1 \dots x_{i-1} \{x_i^{v_i}, x_j\} x_{i+1} \dots x_n \\
=& \sum_i x_1 \dots x_{i-1} v_i \{x_i, x_j\} x_{i+1} \dots x_n \\
=& \sum_i x_1 \dots x_{i-1} v_i \varepsilon_{ij} x_i x_j x_{i+1} \dots x_n \\
=& x_1 \dots x_n x_j \sum_i \varepsilon_{ij} v_i \\
=& 0
\end{aligned}$$

So $\prod_i x_i^{v_i}$ commutes with each generator and thus commutes with all the X-coordinates.

□

Corollary 1.1.37. *The cluster algebra associated to the right quiver in Figure 1.6 has a Casimir element* $x_1 x_3$.

Proof. The null space of the matrix ε associated to this quiver is generated by $(1, 0, 1)$. □

1.1.2 Dynkin Classification

Definition 1.1.38. *A cluster algebra $\mathcal{A}$ is of **finite type** if there are finitely many seeds.*

*The cluster algebra is of **finite mutation type** if there are finitely many quivers in the mutation class of Q (with potentially infinitely many coordinates)*

Theorem 1.1.39. *A cluster algebra $\mathcal{A}_Q$ is of finite type if and only if there is a quiver in the mutation class of Q whose mutable portion is an orientation of a Dynkin diagram.*

Proof. See [9] for a full proof. A key aspect of this proof is the relationship between the **almost positive roots** of the associated root system and the cluster algebra. We can take the initial quiver of the cluster algebra to be the quiver Q that is an orientation of the associated Dynkin diagram. Consider the set of simple roots $\{r_1, \ldots, r_n\}$ of the associated root system. The d-vector of each A coordinate a_i in the initial seed is $-e_i$. This directly corresponds to $-r_i$. In general the A coordinate with associated d-vector $\mathbf{v}$ corresponds to $\sum v_i r_i$. □

Remark 1.1.40. *Theorem 1.1.39 remains true even when discussing cluster algebras with weighted quivers.*

Using the Dynkin quivers, we can compute Casimir elements of the Poisson structure of finite type cluster algebras.

Remark 1.1.41. *For $n = 2k - 1$, A_n has a Casimir element that is a product of k X-coordinates.*

Proof. The Dynkin type quiver has the vector $(1, 0, -1, 0, 1, \dots)$ in the null space. So by Theorem 1.1.36 the corresponding product of k X-coordinates is a Casimir element. □

Remark 1.1.42. *The type A_n cluster algebras for n even does not have any Casimir elements of this form.*

Remark 1.1.43. *The type D_{2k} cluster algebras have a two Casimir element that are a product of k X-coordinates. These are found by freezing one of each small tail and taking the Casimir of the corresponding A_{2k-1} cluster algebra. In the Dynkin type quiver, this does not include the X-coordinate on the degree 3 vertex so the frozen tail commutes with the product.*

Remark 1.1.44. *For any n, D_n has a Casimir element given by the quotient of X_a/X_b where a and b are the short tails. When n is even this is equal to the quotient of the two A_{2k-1} Casimir elements.*

We also obtain nice classes of cluster algebras by looking at generalizations of the finite Dynkin diagrams. Cluster algebras with quivers corresponding to orientations of Affine Dynkin diagrams are called **affine cluster algebras**. These cluster algebras have infinitely many cluster variables, but can be characterized by the fact that the number of cluster variables grows at a linear rate with number of mutations. See [10] for more information between the affine root system and cluster algebra structure.

In Chapter 2 we study the **doubly extended cluster algebras**. These have quivers that are orientations of Dynkin diagrams formed by adding two nodes. For more information on the classification of doubly extended Dynkin diagrams see [17]. To see all the diagrams in this family see Figures A.7 and A.8.

1.1.3 The Cluster Structure of the Grassmannian

Our other key example of Cluster Algebras comes from the homogeneous coordinate ring of the affine cone of Grassmannian $\mathbb{C}[\widetilde{\mathrm{Gr}(k,n)}]$.

Definition 1.1.45. *The* ***Grassmannian*** *$Gr(k,n)$ is the set of k dimensional subspaces of $\mathbb{C}^n$. Recall each point in $Gr(k,n)$ can be viewed as an equivalence class of $k \times n$ matrices whose rows span the given subspace.*

There is a standard embedding of $\mathrm{Gr}(k,n)$ into projective space called the Plücker embedding.

Definition 1.1.46. *For $I \subseteq [n]$ of size k, the* ***Plücker*** *coordinate $p_I : \widetilde{Gr(k,n)} \to \mathbb{C}$ is the function that takes the determinant of the $k \times k$ submatrix using columns in I.*

Claim 1.1.47. *Taking a different basis of a subspace simultaneously changes all Plücker coordinates by the same constant. This gives the standard Plücker embedding of $Gr(k,n)$ in projective space.*

Claim 1.1.48. *For any k and n, $\widetilde{Gr(k,n)}$ has an $\mathcal{A}$ cluster algebra structure. There is an explicit initial seed where each A-coordinate corresponds to a Plücker coordinate.*

Proof. There are several recipes to obtain an initial seed of the cluster algebra structure. In [18], Scott gives a combinatorial construction of seeds that generate a cluster algebra isomorphic to the coordinate ring of $\widehat{\mathrm{Gr}(k,n)}$. In [16], Golden, et al. give a uniform description of seeds that will generate the cluster algebra structure for any k and n. The quivers in these seeds can be arranged so that all but one node is in a $k \times (n-k)$ grid with a diagonal edge through each square of the grid. See Figure 1.8 for the general shape of these "diagonal grid quivers". In this picture the blue vertices are frozen and correspond to Plücker coordinates whose index set is cyclicly adjacent. □

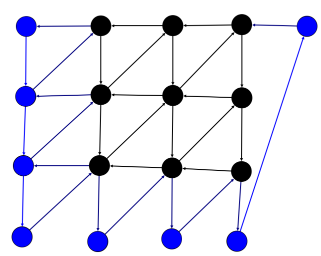


Figure 1.8: An example of the diagonal grid quiver in $\mathrm{Gr}(4,8)$.

Corollary 1.1.49. *Every Grassmannian cluster algebra has a seed whose A - coordinates are Plücker, such that the mutatable portion of the quiver is a $(k-1)\times(n-k-1)$ grid (with no diagonal edges).*

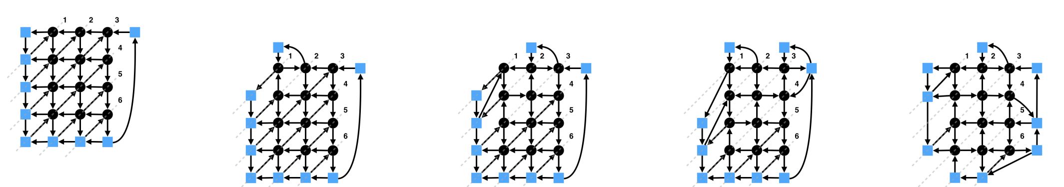
Proof. In Scott [18], he shows that mutating a node with exactly 2 arrows in and

2 arrows out transforms a Plücker coordinate into another Plücker coordinate. We call such nodes "good" for the remainder of this proof. So it suffices to specify a mutation sequence of good nodes from Goncharov's Plücker quiver (Figure 1.9a) to the grid quiver (Figure 1.9e). To do this we label the diagonals of the $k \times (n-k)$ grid parallel to the "extra" diagonal edges 1 to $n-1$. Since the edges on a diagonal aren't adjacent we can mutate at all the nodes on the diagonal in any order and achieve the same result. Call the mutation sequence for the i^{th} diagonal d_i

Mutating d_1 removes the extra edge of the first square and makes every node on the second diagonal good. In general if every node on the i^{th} diagonal is good and squares above are free of extra edges, mutating at d_i makes every node on the $(i+1)^{st}$ diagonal good. In addition, d_i removes the extra edges directly below at the cost of adding extra edges directly above. These can be removed by mutating d_{i-2}. This adds extra edges which again are removed by mutating 2 diagonals back. This can be repeated until the extra edges would be added off the grid. So let m_i be the mutation sequence $d_i d_{i-2} d_{i-4} \ldots d_1$[2].

At the start there are $n-2$ sets of extra edges to clear so the mutation sequence

[2]If i is even stop at d_2 instead of d_1.



(a) Grassmannian Diagonal Grid. (b) (c) (d) (e) Grid Quiver.

Figure 1.9: The mutation algorithm outlined in Corollary 1.1.49 transforms the quiver on the left to the one on the right.

$m_1m_2, \ldots, m_{n-2}$ takes Goncharov's quiver to a pure grid. Figure 1.9 shows the result applying m_i to a quiver from $\mathrm{Gr}(4,9)$

□

Recall that there is an isomorphism between $\mathrm{Gr}(k,n)$ and $\mathrm{Gr}(n-k,n)$ that sends a k subspace to its complementary $n-k$ dimensional subspace. This is reflected in the Plücker coordinates by sending p_I to $p_{[n] \setminus I}$ and extends to a map of cluster algebras by reversing all the arrows in the starting seed. As such we only need to study $\mathrm{Gr}(k,n)$ when $k \leq \frac{n}{2}$.

Remark 1.1.50. *For each* $1 \leq i \leq n+1$ *there are maps* $a_i : Gr(k,n+1) \to Gr(k,n)$ *given by forgetting the* i^{th} *dimension. This induces a map* $a_i^* : \mathbb{C}[Gr(k,n)] \to \mathbb{C}[Gr(k,n+1)]$ *by sending* p_I *to* $p_{f_i(I)}$ *where* $f_i(x) = \begin{cases} x & x < i \\ x+1 & x \geq i \end{cases}$. *Again this gives an inclusion of cluster algebras showing* $Gr(k,n)$ *is a subcluster algebra of* $Gr(k,n+1)$. *There is another inclusion of cluster algebras* $b_i : Gr(k,n) \to Gr(k+1,n+1)$ *obtained by conjugating* a_i *by the dual map above. This sends* p_I *to* $p_{\{i\} \cup f(I)}$.

Claim 1.1.51. *[18] The Grassmannian cluster algebra is of finite type if and only if* $(k-2)(n-k-2) < 4$ *and finite mutation type when* $(k-2)(n-k-2) \leq 4$.

In fact $\mathrm{Gr}(2,n+3)$ is type A_n, $\mathrm{Gr}(3,6)$ is type D_4, $\mathrm{Gr}(3,7)$ is type E_6 and $\mathrm{Gr}(3,8)$ is type E_8.

The only finite mutation type, but not finite type cluster algebras are $\mathrm{Gr}(3,9) = \mathrm{Gr}(6,9)$ and $\mathrm{Gr}(4,8)$.

In $\mathrm{Gr}(2,n)$ the only cluster coordinates are Plücker coordinates, but even in the other finite type cases there are "exotic" cluster coordinates. In $\mathrm{Gr}(3,6)$ there are only two exotic coordinates that Scott calls X and Y. These can be expressed as polynomials in the Plücker coordinates $X = p_{134}p_{256} - p_{156}p_{234}$ and $Y = p_{136}p_{245} - p_{126}p_{345}$.

Claim 1.1.52. *Every exotic cluster coordinate can be expressed as a polynomial in the Plücker coordinates.*

Proof. This follows from Claim 1.1.48 as the coordinate ring of the Grassmannian is generated by the Plücker coordinates. □

Remark 1.1.53. *In [18], Scott explicitly computes all of the exotic coordinates in the remaining finite cases. In particular, the only exotic coordinates in Gr$(3,7)$ are lifts of X and Y via the inclusions a_i^* (Remark 1.1.50). Additionally, in Gr$(3,8)$ there are 24 additional exotic coordinates. Scott refers to 8 as A^{ρ^i} as the polynomials in Plücker coordinates are related by applying ρ, the cyclic shift of all of the indices modulo 8. The remaining 16 have two orbits under the cyclic shift. These two orbits are additionally related by applying a dihedral flip σ (on the octagon) to the Plücker coordinates of each polynomial. As such Scott refers to these exotic coordinates as B^{ρ^i} or $B^{\sigma\rho^i}$.*

Remark 1.1.54. *When referring to exotic coordinates in Section 3.5 we use the following notation for these exotic coordinates. One goal of this new notation is to emphasize the degree of the polynomial corresponding to each exotic coordinate. For example, $X = p_{134}p_{256} - p_{156}p_{234}$ is degree 2 and so we refer to it as e2x. We* *also refer to the images of coordinates under the inclusion map by the index of the inclusion rather than the six indices of the corresponding* $Gr(3,6)$ *subalgebra. For example, we write* $e2x1$ *rather than* X^{234567}.

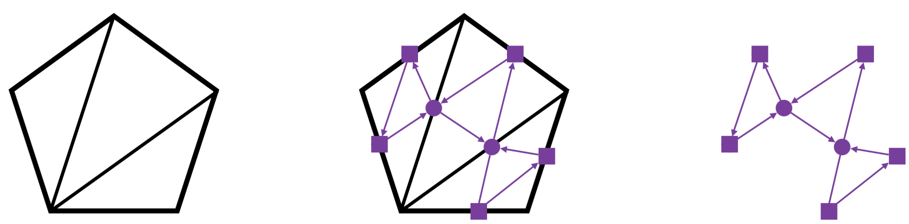
	New Notation	*Scott*
$Gr(3,6)$	$e2x$	X
	$e2y$	Y
$Gr(3,7)$	$e2x1$	X^{234567}
	$e2y1$	Y^{234567}
$Gr(3,8)$	$e2x12$	X^{345678}
	$e2y14$	Y^{235678}
	$e3A^{\rho^3}$	A^{ρ^3}
	$e3B^{\rho^4}$	B^{ρ^4}
	$e3B^{\sigma\rho^5}$	$B^{\sigma\rho^5}$

1.1.4 The Cluster Algebra of a Surface

In this section, we review cluster algebras associated to surfaces. For a complete description see [19] or Section 3 of [20].

Definition 1.1.55. *A* marked surface, $S_{g,b,p,n}$ *is an orientable surface of genus* g *with* b *boundary components,* p *punctures and* n *marked points on the boundary. We always require that each boundary component has at least one marked point. An* arc *on a marked surface* S *is a (non-contractible) isotopy class of curves between marked points or punctures on* S*. An* ideal triangulation *of a marked surface is a maximal* *collection of non-crossing arcs on S.*

Let S be a marked surface. Given an ideal triangulation Δ of S, we associate a quiver, Q_Δ to Δ, as follows: For each arc $e \in \Delta$ we add a node N_e and for each triangle $t \in \Delta$ we add a clockwise oriented cycle of arrows between the nodes associated with the arcs of t. In the situation where we have arrows between two nodes in opposite directions, we cancel them. The nodes associated to boundary edges are frozen. There are $-3\chi(S) + 2n$ total nodes and n frozen nodes (Figure 1.10).

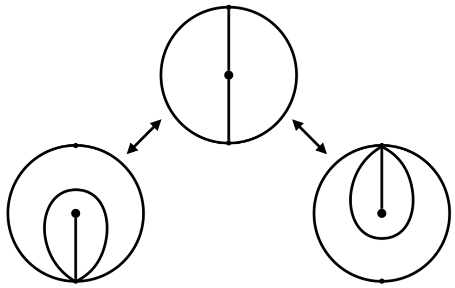


(a) (b) (c)

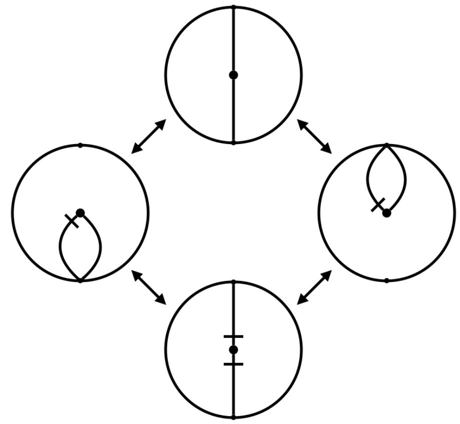
Figure 1.10: The quiver associated to a triangulation of the disk with 5 marked points. In 1.10a we see the triangulation alone. In 1.10b we place a node on each edge and attach them with a clockwise oriented cycle for each triangle. Figure 1.10c shows the resulting quiver by itself.

There is one minor complication when S has punctures. In this case it may be possible to have a "self folded" triangle in an ideal triangulation of S (Figure 1.11a). In this case, the construction mentioned above does not produce the correct quiver. However, we can always find a triangulation of S with no self folded triangles, and use this to construct a quiver associated with the triangulation.

Then mutation of nodes in Q_Δ corresponds to a "flip" or "Whitehead move" in Δ at the corresponding arc. Again, there is a caveat to this when S has punctures.



(a) Arcs in a punctured digon.



(b) The tagged arc flip graph.

Figure 1.11: Untagged vs tagged arcs in a punctured digon.

The interior arc of a self folded triangle cannot be flipped, but the corresponding node in the quiver can be mutated. This is addressed in [19] by the addition of "tagged" arcs. Essentially, we replace the outside arc of a self folded triangulation with a tagged arc as shown in Figure 1.11. There is then a rule for flipping tagged arcs which agrees with the mutation rule for quivers. With this addition, we may always flip any arc and this always agrees with mutation of corresponding quivers. We do not need the details of this in general.

Remark 1.1.56. *Since every triangulation of a surface can be reached via a series of flips, all triangulations of a surface are in the same mutation class.*

Remark 1.1.57. *A quiver associated to a surface can only have a double edge if the triangulation contains one of the two sub-triangulations in Figure 1.12.*

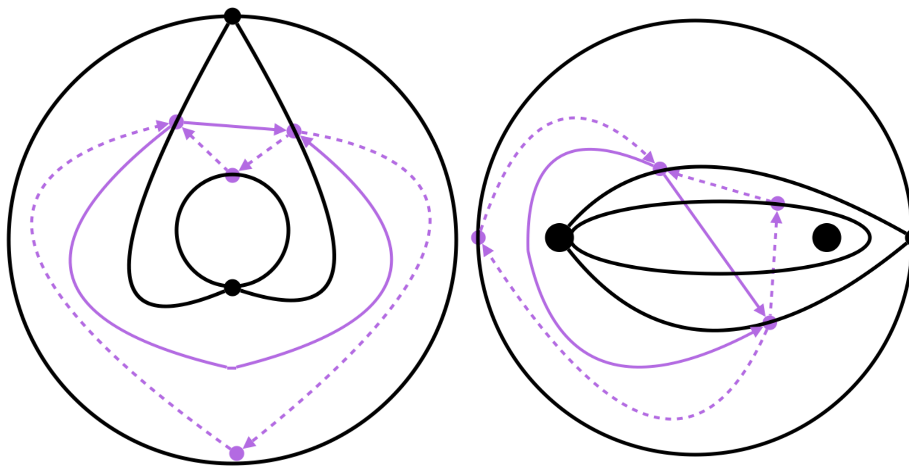


Figure 1.12: The only sub-triangulations that produce double edge quivers.

1.2 Cluster Modular Group

We will now review how to associate a group to any quiver or cluster algebra called the *cluster modular group*. The following is adapted from the discussion in my joint paper [4]. This group is essentially the automorphism group of the mutation structure of a cluster algebra associated with a given quiver. We can use our definitions of c-vectors to give a definition of this group without any reference to the cluster variables.

Let Q be a quiver without frozen vertices. By identifying the mutable nodes of Q with the integers $[n] = \{1, \dots, n\}$, we obtain a right action of $\mathbb{Z}_2^{*n}$ on quivers in the mutation class, $\mathrm{Mut}(Q)$, by mutating at each node in sequence. We refer to elements of $\mathbb{Z}_2^{*n}$ as mutation paths.

We would now like to focus on the subset of paths that return Q to an isomorphic quiver. In order to define a group structure on this subset, we need to consider pairs (P, σ) of mutation paths P and quiver isomorphisms $\sigma : Q \to P(Q)$. We write quiver isomorphisms as elements of the symmetric group S_n. The symmetric group acts on mutation paths by permuting the elements of the path and on itself by conjugation.

Given two such pairs (P, σ) and (R, τ) we can multiply by forming the composite path $P \cdot \sigma(R)$ and the composite quiver isomorphism $\sigma\tau$.

$$Q \xrightarrow{\sigma} P(Q) \xrightarrow{\sigma(\tau)} P(\sigma(R)(Q)) \tag{1.1}$$
$$Q \xrightarrow{\sigma\tau} P(\sigma(R)(Q))$$

This multiplication rule can also be obtained by viewing these pairs as elements of the semidirect product

$$\mathbb{Z}_2^{*n} \rtimes S_n. \tag{1.2}$$

This gives a group structure on the set of mutation paths which return Q to an isomorphic quiver paired with isomorphisms from the starting to ending quiver; we call this group the *quiver modular group* associated with Q denoted $\tilde{\Gamma}_Q$.

Elements of the quiver modular group act on the cluster variables of a seed $\boldsymbol{i}$ associated with Q. The path P provides a path to a new seed, and σ gives a map from the cluster variables on $\boldsymbol{i}$ to those on $P(\boldsymbol{i})$.

Definition 1.2.1. *A pair (P, σ) which acts trivially on the cluster variables of any initial seed associated with Q is called a* trivial cluster transformation. *Let T be the group of trivial cluster transformations; this is a normal subgroup of $\tilde{\Gamma}_Q$. The group $\Gamma_Q = \tilde{\Gamma}_Q / T$ is called the* cluster modular group *associated with the quiver Q.*

Equivalently, a trivial cluster transformation is an element (P, σ) of $\tilde{\Gamma}_Q$ for

(a) Q (b) Q'

Figure 1.13: A simple quiver before and after mutation.

which σ is a frozen isomorphism $\hat{Q} \to P(\hat{Q})$. In this way, we may define Γ_Q without any regard to cluster variables.

Remark 1.2.2. *Our notion of a quiver isomorphism requires that all of the arrow directions are preserved. In other definitions of the cluster modular group, such as those in [11, 21, 22], one includes arrow reversing quiver automorphisms. Our version of the cluster modular group is an index two subgroup of this more general notion.*

Example 1.2.3. *Consider the quiver Q with two nodes and a single edge between them (Figure 1.13a). Mutation at* 1 *in Q yields a quiver with the edge now going from* 2 *to* 1 *(Figure 1.13b) If we want to perform the "same" mutation in Q' that we did in Q we want to mutate at the vertex corresponding to 1 under the isomorphism $f : Q \to Q'$, which is 2. In this case there is a unique isomorphism, but in general each choice of isomorphism gives rise to a different element of the cluster modular group. It is convenient to write these isomorphisms as permutations in S_n. The element described above would be written $g = (1, (12))$. In this case g generates the cluster modular group and $g^5 = id$.*

1.2.1 The Cluster Complex

Recall that for any cluster algebra, $\mathcal{A}_Q$, there is an associated simplicial complex $C(\mathcal{A}_Q)$ called the cluster complex. This complex is defined in detail in [8, 23]. We will review the basic definitions of this complex here. First we will need the notion of compatibility of cluster variables.

Definition 1.2.4. *Two cluster variables are* compatible *if they appear in a cluster together.*

The k-dimensional simplices of $C(\mathcal{A}_Q)$ correspond to size k collections of mutually compatible cluster variables in $\mathcal{A}_Q$. In other words, the cluster complex is the "clique complex" of the compatibility rule for cluster variables. In particular each vertex corresponds to an individual cluster variable and each edge connects two cluster variables when they can be found in a cluster together. The maximal dimension simplices correspond to the clusters of $\mathcal{A}_Q$.

Remark 1.2.5. *In [11] the cluster modular group is defined to be the simplicial symmetry group of the cluster complex. This symmetry group contains the cluster modular group as described in this paper as a proper subgroup.*[3] *The distinction between these groups does not affect the main results of this thesis.*

The 1-skeleton of the dual complex of the cluster complex is called the "exchange graph" of the cluster algebra. The vertices of this graph correspond to clusters and the edges correspond to mutations between clusters.

[3]For cluster algebras of finite mutation type the only potentially missing symmetry is given by reversing all the arrows in a quiver.

1.2.2 Computing Cluster Modular Groups

We would like to have an algorithm to compute the cluster modular group. For general quivers, this can be very difficult since the mutation class can be infinite. When the quiver in question has finitely many quivers in its mutation class, there is an algorithmic construction of the cluster modular group, see Ishibashi's paper [24]. We present a simplified version of the algorithm which only computes a generating set without computing all the relations.

Definition 1.2.6. *The* directed quiver mutation graph*, G, associated to a finite mutation class cluster algebra is a multi graph with a node for each quiver isomorphism class and a directed edge for each single mutation between isomorphism classes. The* (undirected) quiver mutation graph *replaces directed two cycles corresponding to inverse mutations with a single undirected edge.*

Note, unlike the graph in [24], in our formulation the degree of each node is the rank of the cluster algebra.

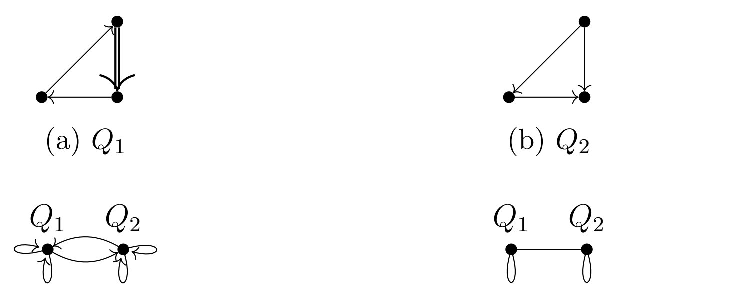
Each element (P, f) of the cluster modular group corresponds to a cycle in G by following P in G. Furthermore the set of cycles in G is finitely generated with one generator for each edge not in a fixed spanning tree of G. Since the automorphism group of each quiver is finite, this gives a finite list of generators of the cluster modular group.

In practice this method doesn't give the shortest possible list of generators of the cluster modular group. However it places an upper bound on how long the shortest path representing a generator of the cluster modular group can be. If d is the diameter of the spanning tree for G, then the maximum length of the mutation path of a generator is $2d + 1$.

Remark 1.2.7. *To check if a group surjects onto the cluster modular group it suffices to check that it reaches every quiver isomorphic to the starting quiver in distance* $2d + 1$.

Example 1.2.8. *The mutation class of an* $A_{2,1}$ *quiver has two quiver isomorphism classes* Q_1, Q_2*, shown in Figure 1.14. It is easy to compute the directed and undirected quiver mutation graphs for this quiver simply by performing each of the three mutations on each quiver isomorphism class.*

We can then compute a set of generators of the cluster modular group. There are two generators e_1, e_2 *corresponding to the two loops from* Q_1 *and* Q_2 *to themselves.*



(a) Q_1 (b) Q_2

(c) Directed mutation graph. (d) Undirected mutation graph.

Figure 1.14: The quiver mutation graphs for $A_{2,1}$.

1.2.3 Reddening Elements

If a quiver Q has a reddening sequence (Definition 1.1.28), then there is a unique element $r \in \Gamma_Q$ called the "reddening element" of Γ_Q.

Explicitly, $r = (P_r, \sigma_P)$ where P_r is any reddening sequence and $\sigma_P : Q \to P(Q)$ is the isomorphism which extends to an isomorphism $\check{Q} \to P(\hat{Q})$ by adding the identity permutation on all of the frozen vertices.

The following theorem is probably well known, but we give a proof for completeness.

Theorem 1.2.9. *The reddening element (when it exists) is in the center of* Γ_Q.

Proof. To show r is in the center we take any other group element $g = (P, f)$. Using the labeling induced by the initial framing the permutation σ_r is the identity. Then

$$g \cdot r \cdot g^{-1} = (P \cdot f(P_r) \cdot f(\sigma_r(f^{-1}(\overleftarrow{P}))), f \circ \sigma_r \circ f^{-1}) = (P \cdot f(P_r) \cdot \overleftarrow{P}, \mathrm{id}) \quad (1.3)$$

Conjugating the reddening path P_r by any other path again produces a reddening sequence (see [15]) so

$$P_r \sim P \cdot f(P_r) \cdot \overleftarrow{P} \quad (1.4)$$

and we have $r = grg^{-1}$ as needed. □

1.2.4 Surface Cluster Modular Groups

We can define a faithful action of the mapping class group, $\mathrm{Mod}(S)$, on the triangulations of S and hence identify the mapping class group as a subgroup of the cluster modular group, Γ_S, of our cluster algebra $\mathcal{S}$. We give an explicit construction of this subgroup here as a nice example of our notation. We refer to [25] section 2 for computations involving the mapping class group of selected surfaces.

Given $f \in \mathrm{Mod}(S)$ we can define $\gamma_f \in \Gamma_S$ as follows: f gives a new triangulation of S and hence by [19] there is a path of flips, P_f, taking Δ to $f(\Delta)$. Furthermore, f defines a map between the edges of Δ and $f(\Delta)$ that preserves the adjacency relations between the triangles of Δ. So Δ and $f(\Delta)$ have the same associated quivers. The path P induces a map between the nodes of $Q_{f(\Delta)}$ and $P(Q_\Delta)$ since these quivers come from the same triangulation. Let $\sigma_{f,P}$ be the isomorphism of quivers Q_Δ to $P(Q_\Delta)$ defined by the composition

$$\sigma_{f,P} \colon Q_\Delta \xrightarrow{f} Q_{f(\Delta)} \xrightarrow{P} P(Q_\Delta). \tag{1.5}$$

Thus to f we associate $\gamma_f = \{P_f, \sigma_{f,P}\}$.

It is not immediately clear that this does not depend on the choice of the path, P. Let $\{P, \sigma\}$ and $\{R, \tau\}$ be two possible representatives of γ_f. Then we have

$$\{P, \sigma\}\{R, \tau\}^{-1} = \{P, \sigma\}\{\tau^{-1}(R^{-1}), \tau^{-1}\} = \{P\sigma\tau^{-1}(R^{-1}), \sigma\tau^{-1}\}. \tag{1.6}$$

We need to show that this element is a trivial cluster transformation. First note that $\sigma\tau^{-1}$ is the quiver isomorphism from $R(Q_\Delta)$ to $P(Q_\Delta)$ coming from the fact that these both correspond to the same triangulation of S. The composite mutation path, $P\sigma\tau^{-1}(R^{-1})$, consists of following P and then following R^{-1} back to our initial cluster. This introduces a permutation on the cluster variables determined by the map $\tau\sigma^{-1} \colon P(Q_\Delta) \to R(Q_\Delta)$. Together these permutations act trivially on the cluster variables, and γ_f is well defined in the cluster modular group.

Remark 1.2.10. *For all but finitely many quivers associated with surfaces, the cluster modular group is essentially equal to the mapping class group, see [26] proposition 8.5*[4]*. For the remaining surfaces, one may check case by case that* $\mathrm{Mod}(S)$ *is always a finite index normal subgroup of* Γ.

1.2.5 Cluster Modular Group of Finite Type Cluster Algebras

From the classification of finite cluster algebras, we know every finite cluster algebra has a seed whose underlying quiver is an orientation of a finite Dynkin diagram. In fact, every orientation of the Dynkin diagram appears in the mutation class. We make a canonical choice that we call the "Dynkin quiver" where every node is either a source or a sink and there are at least as many sources as sinks. We now give a presentation the cluster modular group of each finite type based at the Dynkin quiver.

Lemma 1.2.11. *Let* Q *be a quiver where every node is a source or a sink. Let* P *be any path formed by first mutating at all the original sources and then mutating at all the original sinks. Then following* P *results in a new quiver isomorphic to* Q.

Proof. First, notice that two sources cannot be adjacent, so the mutation at two distinct sources commute. Therefore the order of the sources in P does not affect the final quiver. Since there are no directed paths through a source, the quiver

[4] The standard choice of mapping class group fixes the set of marked points on the boundary. We need to allow transformations that permute these marked points to achieve equality. We also need to include a mapping class group action that swaps the tagging at each puncture.

mutation rule is especially simple at a source: simply reverse all the arrows incident to the source. After mutating at all the sources every arrow in Q will be reversed. This makes the original sinks into sources and so we see the second half of the path returns to an isomorphic quiver. □

Definition 1.2.12. *The path P in the previous lemma is called the "sources/sinks path". It correspond to an element of the cluster modular group $g_S = (P, f)$ where the f is the "identity permutation" induces by carrying the indexing along P.*

Theorem 1.2.13. *The cluster modular group for any finite cluster algebra has order $\frac{h+2}{2}|\mathrm{Aut}(Q)|$ where h is the Coxeter number of the underlying Dynkin diagram.*

Proof. Fomin and Zelevinsky show that $\ell = \frac{h+2}{2}$ applications of g_S returns to the original quiver where h is Coxeter number of the associated root system. Furthermore they showed that every Dynkin quiver is reached during these ℓ applications and all ℓ applications are needed. So the cluster modular group is generated by g_S and $\mathrm{Aut}(Q)$.

□

Remark 1.2.14. *To identify the group exactly we must be more careful, as g_S^ℓ doesn't always return with the identity permutation. In A_{2k+1},D_{2k+1} and E_6 it turns out that $g_S^\ell = \sigma$ where σ is the order 2 generator of* $\mathrm{Aut}(Q)$*. In these two cases it is clear that σ and g_S commute and the full cluster modular group is cyclic of order 2ℓ as claimed.*

In every other case g_S^ℓ is the identity. However g_S and $\mathrm{Aut}(Q)$ *still commute as any automorphism preserves the set of sources and the set of sinks and we established P*

is independent of reordering within these sets. In these cases $\langle g_S \rangle$ *and* $\mathrm{Aut}(Q)$ *are disjoint commuting subgroups and so the overall group is* $\langle g_S \rangle \times \mathrm{Aut}(Q)$ *which has the correct order.*

Remark 1.2.15. *The previous theorem needs a slight adjustment for* A_{2k}. *In this case the Coxeter number is* $2k+1$, *so we are claiming that* g_S *has order* $\frac{2k+3}{2}$ *which is a non integer. The issue is that in this case mutating only the sources returns you to a Dynkin quiver. In this case we take* h_S, *the sources path*[5], *and we know that* $h_S^2 = g_S$. *The theorem then says that the order of* h_S *is* $2k+3$. *Furthermore the automorphism group is trivial. So in this case the cluster modular group is* $\mathbb{Z}_{2k+3} = \mathbb{Z}_{n+3}$. *Interestingly, when we compare this to the* $n = 2k+1$ *odd case we also saw the cluster modular group was* $\mathbb{Z}_{n+3}$.
Similarly the cluster modular group of $D_n = \mathbb{Z}_n \times \mathbb{Z}_2$ *regardless of if* n *is odd or even. This is because when* n *is odd* $\mathbb{Z}_{2n} \cong \mathbb{Z}_n \times \mathbb{Z}_2$

Type	Coxeter Number	Modular Group	Order of the modular group
A_n	$n+1$	$\mathbb{Z}_{n+3}$	n+3
D_4	6	$\mathbb{Z}_4 \times S_3$	24
D_n	$2n-2$	$\mathbb{Z}_n \times \mathbb{Z}_2$	$2n$
E_6	12	$\mathbb{Z}_7 \times \mathbb{Z}_2$	14
E_7	18	$\mathbb{Z}_{10}$	10
E_8	30	$\mathbb{Z}_{16}$	16

Figure 1.15: The cluster modular groups of finite simply laced cluster algebras.

See Figures 1.15, 1.16 for the modular groups in all the finite cases.

[5] Since $\mathrm{Aut}(A_{2k}) = 1$ there is only one choice of isomorphism.

Type	Coxeter Number	Modular Group	Order of the modular group
B_n	$2n$	$\mathbb{Z}_{2n+2}$	$2n+1$
C_n	$2n$	$\mathbb{Z}_{2n+2}$	$2n+1$
F_4	12	$\mathbb{Z}_7$	7
G_2	6	$\mathbb{Z}_4$	4

Figure 1.16: The cluster modular groups of finite non-simply laced cluster algebras.

1.2.6 Grassmannian Cluster Modular Groups

The Grassmannian $\mathrm{Gr}(k, n)$ has a natural action of S_n that sends the Plücker coordinate p_I to $p_{\sigma I}$. In order for this action to induce a cluster algebra action it needs to preserve the set of frozen Plücker coordinates. This restricts the group to $D_{2n} = \langle r, f | r^n = f^2 = frfr = 1 \rangle$ as the frozen coordinates have adjacent indices under the cyclic order.

Since the flip reverses the cyclic ordering, it induces an "orientation reversing" cluster automorphism, which also flips all the arrows of the quiver. As such we focus only on the cyclic group generated by r.

Claim 1.2.16. *Since every cycle in the grid quiver is even length the nodes of the grid can be two colored. As we saw with the sources/sinks path, since the nodes of each color are not adjacent the order of mutation does not affect the resulting quiver. Let P_{TC} be the mutation path given by mutating each color of node. The element of the cluster modular group corresponding to r has mutation path P_{TC} or $\overleftarrow{P_{TC}}$.*

Corollary 1.2.17. *The sources/sinks path on $Gr(2, n)$ corresponds to cyclicly shifting the indices of the Plücker coordinates modulo n.*

Remark 1.2.18. *We call the cyclic shift of indices, the rotation action on the Grassmannian.*

Proof. In $\mathrm{Gr}(2, n)$ the grid quiver is the Dynkin quiver of type A_{n-3}. The sources and sinks are the two coloring of the grid and so these two mutation paths are identical. □

Remark 1.2.19. *In $Gr(k,n)$ there is an additional symmetry called the parity map. Unlike rotation, the parity map mixes Plücker coordinates and exotic coordinates, which is critical for obtaining the full cluster modular group of $Gr(k,n)$. For this paper it suffices to know the parity map can be expressed in terms of the sources/sinks element of the cluster modular group for $Gr(3,6)$, $Gr(3,7)$ and $Gr(3,8)$.*

1.3 Polylogarithms

1.3.1 Classical

Definition 1.3.1. *Let $\mathbf{n} \in \mathbb{N}^d$ and $\mathbf{z} \in \mathbb{C}^d$ with $|z_i| < 1$.*
The multiple ***polylogarithm*** *is defined by the summation:* $\mathrm{Li}_{\mathbf{n}}(\mathbf{z}) = \sum_{0<k_1<\cdots<k_d} \frac{z_1^{k_1} \ldots z_d^{k_d}}{k_1^{n_1} \ldots k_d^{n_d}}$.

Definition 1.3.2. *The* ***weight*** *of $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ is $n = \sum n_i$ and the* ***depth*** *is d.*

When the depth is 1, we refer to $\mathrm{Li}_n(z)$ as the standard polylogarithms.

This family of functions is a natural generalization of the familiar logarithm function and in fact $\mathrm{Li}_1(z) = -\log(1-z)$. From the Taylor series definition it is simple to compute the derivatives of an arbitrary multiple polylogarithms. When $n_i > 1$, $\frac{\partial}{\partial z_i} \mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) = \frac{1}{z_i} \mathrm{Li}_{m_1,\ldots m_i-1,\ldots,m_d}(z_1,\ldots,z_d)$. When $n_i = 1$ the derivative only depends on if z_i is the first, last, or middle variable. Thus for clarity we show the derivatives for a depth 3 polylogarithm $\mathrm{Li}_{m,n,p}(x,y,z)$.

$$\begin{aligned}
\frac{\partial}{\partial x}\mathrm{Li}_{1,n,p}(x,y,z) =& \frac{1}{1-x}\mathrm{Li}_{n,p}(y,z) - \frac{1}{1-x}\mathrm{Li}_{n,p}(xy,z) - \frac{1}{x}\mathrm{Li}_{n,p}(xy,z)\\
\frac{\partial}{\partial y}\mathrm{Li}_{m,1,p}(x,y,z) =& \frac{1}{1-y}\mathrm{Li}_{m,p}(xy,z) - \frac{1}{1-y}\mathrm{Li}_{m,p}(x,yz) - \frac{1}{y}\mathrm{Li}_{m,p}(x,yz)\\
\frac{\partial}{\partial z}\mathrm{Li}_{m,n,1}(x,y,z) =& \frac{1}{1-z}\mathrm{Li}_{m,n}(x,yz)
\end{aligned} \tag{1.7}$$

1.3.2 Analytic Continuation

Definition 1.3.3. *In [27], Zhao analytically continues* $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ *to* $\mathbb{C}^d \setminus X_d$ *where* X_d *is the singularity set of a depth* d *multiple polylogarithm.*

$$X_d = \{\mathbf{z} \in \mathbb{C}^d | \prod_{i=1}^{d} z_d \cdot \prod_{1\leq i\leq j\leq d} (1 - \prod_{r=i}^{j} z_r) = 0\}$$

Definition 1.3.4. *The* ***basic liftable functions*** *in depth* d *are* z_i *for* $1 \leq i \leq d$ *and* $1 - \prod_{r=i}^{j} z_i$ *for* $1 \leq i \leq j \leq d$.

Remark 1.3.5. *The singularity set of the polylogarithm* X_d *is the zero set of the basic liftable functions.*

In order to compute the analytic continuation, Zhao writes each polylogarithm as an iterated integral. While the explicit formula is rather technical we can easily see the following:

Claim 1.3.6. *Each one-form in the iterated integral has the form* $\mathrm{d}\log f$ *where* f *is a basic liftable function.*

Proof. From the analysis of the derivative of multiple polylogarithms in Equation

1.7, we see the differential $\mathrm{d}\,\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ is a sum of terms of the form $\mathrm{d}\log f$ multiplied by a polylogarithm of lower weight whose arguments are products of adjacent coordinates. Thus inductively each smaller multiple polylogarithm can be written using products of arguments that are products in the ordinal arguments. □

Example 1.3.7. *The iterated integral for* $\mathrm{Li}_{1,1}(x,y)$ *is*

$$\int \mathrm{d}\log(1-y)\mathrm{d}\log(1-x) + \mathrm{d}\log(1-xy)\mathrm{d}\log(1-y) + \mathrm{d}\log(1-xy)\mathrm{d}\log(x) - \mathrm{d}\log(1-xy)\mathrm{d}\log(1-x)$$

Example 1.3.8. *The iterated integral for* $\mathrm{Li}_{2,1}(x,y)$ *is*

$$\begin{aligned}
&\int \mathrm{Li}_{1,1}(x,y)\mathrm{d}\log(x) + \mathrm{Li}_2(xy)\mathrm{d}\log(1-y) \\
&= \int \mathrm{d}\log(1-y)\mathrm{d}\log(1-x)\mathrm{d}\log(x) + \mathrm{d}\log(1-xy)\mathrm{d}\log(1-y)\mathrm{d}\log(x) \\
&\quad + \mathrm{d}\log(1-xy)\mathrm{d}\log(x)\mathrm{d}\log(x) - \mathrm{d}\log(1-xy)\mathrm{d}\log(1-x)\mathrm{d}\log(x) \\
&\quad + \mathrm{d}\log(1-xy)\mathrm{d}\log(x)\mathrm{d}\log(1-y) + \mathrm{d}\log(1-xy)\mathrm{d}\log(y)\mathrm{d}\log(1-y)
\end{aligned}$$

Furthermore this analytic continuation only depends on the homotopy class of path in $\mathbb{C}^d \setminus X_d$. However as $\mathbb{C}^d \setminus X_d$ isn't simply connected, we only have $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ defined as a single valued function on the universal cover of $\mathbb{C}^d \setminus X_d$. This is analogous to the situation for $\log z = \int_\gamma \frac{1}{z}\mathrm{d}z$ whose value changes by $2\pi i$ depending on how many times γ winds around $z = 0$.

To fully understand the ambiguity we build on the work of Hain (for standard polylogarithms [28]) and Zhao (multiple polylogarithms [27]) to express the multiple polylogarithms as the local system defined by a differential equation on $\mathbb{C}^d \setminus X_d$. This local system is defined by a variation matrix and a monodromy matrix that describes how the variation matrix changes around each singularity. See Figure 1.17 for the local system of the standard polylogarithms. The multiple polylogarithms have many more possible loops and so the full monodromy matrices are more complicated to enumerate.

As such we seek algebraic tools to understand multiple polylogarithms. The classic tool is called the **symbol**.

$$\begin{bmatrix} 1 & 0 & & & \\ \mathrm{Li}_1(z) & 2\pi i & & & \\ \mathrm{Li}_2(z) & * & (2\pi i)^2 & & \\ \vdots & * & * & \ddots & \\ \mathrm{Li}_n(z) & \frac{2\pi i}{n!}\mathrm{Log}^{n-1}(z) & \frac{(2\pi i)^2}{(n-1)!}\mathrm{Log}^{n-2}(z) & \dots & (2\pi i)^n \end{bmatrix}$$

(a) Variation Matrix.

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \exp\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & & \ddots & & \\ & & & 1 & 0 \end{bmatrix} \end{array}\right]$$

(b) Mondromy $z = 0$.

$$\left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \begin{matrix} -1 \\ \mathbf{0} \end{matrix} & \mathbf{Id} \end{array}\right]$$

(c) Mondromy $z = 1$.

Figure 1.17: The local system for a standard polylogarithm. This consists of a variation matrix and monodromy around the two singularities at $z = 0$ and $z = 1$.

1.3.3 Symbol

The symbol is an attempt to transfer the study of polylogarithms to an algebraic setting by assigning an element of the tensor algebra over $\mathbb{C}(X)^*$.

Definition 1.3.9. *Consider a collection of rational functions $f_{i,j}$ defined on a space X with complex coefficients. Then the **Symbol** associated to the function of the form*

$$\phi = \sum_i \int \mathrm{d}\log(f_{i,1}) \dots \mathrm{d}\log(f_{i,k})$$

is the k fold tensor $S(\phi) = \sum_i f_{i,1} \otimes \dots \otimes f_{i,k}$.

Example 1.3.10. *From the iterated integral expression for $\mathrm{Li}_{1,1}(x, y)$ in Example 1.3.7 we see the symbol of $\mathrm{Li}_{1,1}(x, y)$ is:*

$$(1-y) \otimes (1-x) + (1-xy) \otimes (1-y) + (1-xy) \otimes x - (1-xy) \otimes (1-x)$$

Example 1.3.11. *We use the iterated integral expansion of $\mathrm{Li}_{2,1}(x, y)$ in Example 1.3.8 to compute the symbol of $\mathrm{Li}_{2,1}(x, y)$:*

$$\begin{aligned}&(1-y) \otimes (1-x) \otimes x + (1-xy) \otimes (1-y) \otimes x + (1-xy) \otimes x \otimes x\\ &- (1-xy) \otimes (1-x) \otimes x + (1-xy) \otimes x \otimes (1-y) + (1-xy) \otimes y \otimes (1-y)\end{aligned}$$

Remark 1.3.12. *With this definition the symbol is only defined up to constant multiples as $\mathrm{d}\log(f_{i,j}) = \mathrm{d}\log(cf_{i,j})$ for all $c \in \mathbb{C}$. So the symbol would only live in $T^\bullet(\mathbb{C}(X)^*/\mathbb{C})$. However for multiple polylogarithms there is an algorithm [29] to* *define a unique lift of this symbol to $T^\bullet(\mathbb{C}(X)^*)$. Furthermore in this lift the functions $f_{i,j}$ are all basic liftable functions.*

Remark 1.3.13. *Since the $f_{i,j}$ are arguments to the logarithm we can treat the $f_{i,j}$ as living in a multiplicative group. As such we usually write $f_1 f_2 \otimes g$ to mean $f_1 \otimes g + f_2 \otimes g$.*

Remark 1.3.14 (Torsion)**.** *The symbol is taken to have the property that the symbol of $a_1 \otimes \ldots \otimes a_n = 0$ whenever a_i is the logarithm of a root of unity. Thus the symbol of $(2\pi i)^k q\phi$ is 0 for any k and q rational.*

Claim 1.3.15. *If $\{\phi_i\}$ is a collection of functions with symbols and $\sum c_i \phi_i = 0$ then $\sum c_i S(\phi_i) = 0$*

Proof. This follows directly from the definition. □

However computing the symbol for a multiple polylogarithm depends on knowing the iterated integral representation explicitly, and the computational complexity increases rapidly. Already for depth 3 weight 9 it can take over an hour for the algorithm in [30] to compute the symbol on an average laptop.

1.3.4 The Dilogarithm

1.3.4.1 Dilogarithm Relations

There are several known relations for the dilogarithm. The most famous is the five term relation [2]

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2(\frac{1-x}{1-xy}) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2(\frac{1-y}{1-xy}) \tag{1.8}$$

$$= \frac{\pi^2}{6} - \mathrm{Log}(x)\,\mathrm{Log}(1-x) - \mathrm{Log}(y)\,\mathrm{Log}(1-y) + \mathrm{Log}(\frac{1-x}{1-xy})\,\mathrm{Log}(\frac{1-y}{1-xy}) \tag{1.9}$$

In addition we have short relations relating $\mathrm{Li}_2(z)$ to $\mathrm{Li}_2(\frac{1}{z})$ and $\mathrm{Li}_2(1-z)$

$$\begin{aligned} \mathrm{Li}_2(\frac{1}{z}) &= -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\mathrm{Log}^2(-z) \\ \mathrm{Li}_2(1-z) &= -\mathrm{Li}_2(z) + \frac{\pi^2}{6} - \mathrm{Log}(z)\,\mathrm{Log}(1-z) \end{aligned}$$

There are two key problems to generalizing these relations to higher polylogarithms. The first is that these relations involve “product terms” of lower weight polylogarithms. To handle these product terms we generally consider relations modulo products of lower weight polylogarithms, which we refer to as “relations modulo products”. This is justified by modifying the dilogarithm by a linear combination of products of polylogarithms, so these relations are satisfied exactly. The second problem is that the arguments to the five dilogarithm terms as stated don’t satisfy a clear pattern. We will see that these arguments can be naturally interpreted as cross ratios in weight 2, and more generally correspond to $X-$coordinates in the

Grassmannian cluster algebra.

1.3.4.2 Bloch Wigner

Definition 1.3.16. *The **Bloch-Wigner dilogarithm** is a single valued real analytic function $D_2 : \mathbb{C} \setminus X_1 \to \mathbb{R}$ given by*

$$D_2(x) = \Im \operatorname{Li}_2(x) + \operatorname{Log}(|x|) \arg(1 - x)$$

This function justifies ignoring product terms as D_2 exactly satisfies the previous relations without the product terms.

1.3.4.3 Hyperbolic Volume

One nice application of the dilogarithm is computing the volume of ideal hyperbolic simplices. The key idea is that the boundary of hyperbolic 3 space can be identified with the Riemann sphere. Using hyperbolic isometries the 4 vertices of the simplex can be moved to be $\infty, 1, 0$ and z. The number z is called the cross ratio and can also be computed as $z = \operatorname{cr}(z_1, z_2, z_3, z_4) = \frac{(z_2-z_1)(z_4-z_3)}{(z_4-z_1)(z_3-z_2)}$. Notice that permutations of the vertices can at most change z to $\frac{1}{z}, 1 - z, \frac{1}{1-z}, -\frac{1-z}{z}$ or $-\frac{z}{1-z}$. The volume of the simplex is $D_2(z)$. Note that the transformations $z \mapsto \frac{1}{z}$ and $z \mapsto 1 - z$ generate all 6 possible cross ratios. Since the Bloch-Wigner dilogarithm satisfies the relations $D_2(z) = -D_2(\frac{1}{z})$ and $D_2(z) = -D_2(1-z)$ this volume function is well defined up to sign.

Furthermore choosing a consistent cross ratio gives an interpretation of the five term relation. First chose five ideal points in hyperbolic 3 space, $z_1, \ldots, z_5$. We use a hyperbolic isometry to take the points to $\infty, 0, 1, \frac{1}{x}, y$. The five cross ratios, $x_i = \mathrm{cr}(z_1, \ldots \hat{z}_i \ldots, z_d)$ are:

$$\frac{1-xy}{y(1-x)} \quad \frac{1-xy}{(1-x)} \quad 1-xy \quad 1-y \quad \frac{x-1}{x}$$

The volume of the simplex S_i given by removing point i is $D_2(x_i)$. The full volume can be dissected as $S_2 \cup S_4$ or $S_1 \cup S_3 \cup S_5$. Therefore we have

$$D_2\left(\frac{1-xy}{(1-x)}\right) + D_2\left(1-y\right) = D_2\left(\frac{1-xy}{y(1-x)}\right) + D_2(1-xy) + D_2\left(\frac{x-1}{1}\right) \quad (1.10)$$

$$-D_2\left(\frac{1-x}{1-xy}\right) - D_2\left(y\right) = D_2\left(\frac{1-y}{1-xy}\right) + D_2(1-xy) + D_2\left(x\right) \quad (1.11)$$

Note that this corresponds exactly to the five $\mathrm{Li}_2(z)$ terms of the original five term relation in Equation 1.8.

Remark 1.3.17. *In hyperbolic geometry the cross ratio is usually chosen to be* $-\mathrm{cr}(z_1, z_3, z_2, z_4)$. *Under this convention the five term relation is*

$$D_2\left(x\right) - D_2\left(y\right) + D_2\left(\frac{y}{x}\right) - D_2\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + D_2\left(\frac{1-x}{1-y}\right)$$

The cross ratio we chose aligns with the X-coordinates of the $Gr(2,5)$ *cluster algebra and so generalizes better.*

1.3.5 Stuffle Product

Let G be the free abelian group whose generators are finite strings of integers.

Definition 1.3.18. *The* ***shuffle*** *product of* $\mathbf{a} = a_1 \ldots a_m \in G$, $\mathbf{b} = b_1 \ldots b_n \in G$ $a \sqcup\!\sqcup b$ *is the sum of all possible ways to "shuffle" or interleave the elements of* $\mathbf{a}$ *and* $\mathbf{b}$. *Defined recursively we have:*

$$
\begin{aligned}
[] \sqcup\!\sqcup \mathbf{b} =& \mathbf{b} \\
\mathbf{a} \sqcup\!\sqcup [] =& \mathbf{a} \\
a_1\mathbf{a} \sqcup\!\sqcup b_1\mathbf{b} =& a_1(\mathbf{a} \sqcup\!\sqcup b_1\mathbf{b}) + b_1(a_1\mathbf{a} \sqcup\!\sqcup \mathbf{b})
\end{aligned}
$$

Definition 1.3.19. *The* ***stuffle*** *product is the shuffle product plus terms from "stuffing" entries of the two lists together by adding the entries. So inductively:*

$$
\begin{aligned}
[] \,\overline{\sqcup\!\sqcup}\, \mathbf{b} =& \mathbf{b} \\
\mathbf{a} \,\overline{\sqcup\!\sqcup}\, [] =& \mathbf{a} \\
a_1\mathbf{a} \,\overline{\sqcup\!\sqcup}\, b_1\mathbf{b} =& a_1(\mathbf{a} \,\overline{\sqcup\!\sqcup}\, b_1\mathbf{b}) + b_1(a_1\mathbf{a} \,\overline{\sqcup\!\sqcup}\, \mathbf{b}) + (a_1 + b_1)(\mathbf{a} \,\overline{\sqcup\!\sqcup}\, \mathbf{b})
\end{aligned}
$$

The product of two polylogarithms of weight n and m can be written as a sum of polylogarithms of weight $n+m$ by the stuffle product of their weight vectors. The arguments to the polylogarithm follow their weight indices and when two entries are stuffed together the corresponding arguments are multiplied. For example

$$\begin{aligned}\mathrm{Li}_{n_1,n_2}(x_1,x_2)\,\mathrm{Li}_m(y) =& \mathrm{Li}_{n_1,n_2,m}(x_1,x_2,y)+\mathrm{Li}_{n_1,m+n_2}(x_1,yx_2)\\ &+\mathrm{Li}_{n_1,m,n_2}(x_1,y,x_2)+\mathrm{Li}_{n_1+m,n_2}(x_1y,x_2)+\mathrm{Li}_{m_1n_1n_2}(y,x_1,x_2)\end{aligned}$$

1.3.6 Low Weight Relations

Previously the polylogarithm relations known for weight greater than 2 were scattered. In weight 3, Goncharov had discovered a 22 term relation consisting entirely of Li_3 terms. The following expression, modulo products is equal to $\mathrm{Li}_3(1)$:

$$\begin{aligned}&\mathrm{Li}_3(1-x+xz)+\mathrm{Li}_3\left(\frac{1-x+xz}{xz}\right)+\mathrm{Li}_3(z)+\mathrm{Li}_3\left(\frac{1-z+yz}{y(1-x+xz)}\right)-\mathrm{Li}_3\left(\frac{1-x+zx}{z}\right)+\mathrm{Li}_3\left(\frac{(1-z+yz)x}{1-x+xz}\right)-\mathrm{Li}_3\left(\frac{1-z+yz}{(1-x+xz)yz}\right)\\ &+\mathrm{Li}_3(1-y+yx)+\mathrm{Li}_3\left(\frac{1-y+yx}{yx}\right)+\mathrm{Li}_3(x)+\mathrm{Li}_3\left(\frac{1-x+zx}{z(1-y+yx)}\right)-\mathrm{Li}_3\left(\frac{1-y+xy}{x}\right)+\mathrm{Li}_3\left(\frac{(1-x+zx)y}{1-y+yx}\right)-\mathrm{Li}_3\left(\frac{1-x+zx}{(1-y+yx)zx}\right)\\ &+\mathrm{Li}_3(1-z+zy)+\mathrm{Li}_3\left(\frac{1-z+zy}{zy}\right)+\mathrm{Li}_3(y)+\mathrm{Li}_3\left(\frac{1-y+xy}{x(1-z+zy)}\right)-\mathrm{Li}_3\left(\frac{1-z+yz}{y}\right)+\mathrm{Li}_3\left(\frac{(1-y+xy)z}{1-z+zy}\right)-\mathrm{Li}_3\left(\frac{1-y+xy}{(1-z+zy)xy}\right)\\ &+\mathrm{Li}_3(-xyz)\end{aligned}$$

One can see the arguments to this relation are similar to the arguments of the five term relation, yet there is not a clear pattern.

Separately a forty term relation of Li_3 terms whose arguments come from the $\mathrm{Gr}(3,6)$ cluster algebra was discovered. However this relation doesn't use all the X-coordinates and so doesn't give a clear path to generalize to higher weights.

In weight 4 the situation is even less clear. Recent work by Gangl found a 931 term relation in $\mathcal{R}_4$ although the nature of the arguments remain mysterious [31]. There are two key issues that make generalizing to weight 4 difficult. The first is that Li_4 no longer generates all the multiple polylogarithms. By including the missing generator $\mathrm{Li}_{31}(x,y)$, Goncharov and Rudenko were able to find a relation they call

Q_4 whose arguments come from the A_4 cluster algebra. [32].

1.3.7 Bloch-Suslin Complex

In [3], Goncharov defines the "higher Bloch Complex" $\mathcal{B}_n(F)$ to be the free abelian group on $\mathbb{P}^1 F$ quotiented by $\mathcal{R}_n$ the set of weight n "polylogarithm relations". These groups fit into the chain complex

$$\mathcal{B}_n(F) \to \mathcal{B}_{n-1}(F) \otimes F^\times \to \mathcal{B}_{n-2} \otimes \bigwedge^2 F^\times \to \cdots \to \mathcal{B}_{n-2} \otimes \bigwedge^{n-2} F^\times \to \bigwedge^{n-2} F^\times.$$

It is conjectured that the cohomology of this complex rationally computes motivic cohomology. The group $\mathcal{R}_2$ is generated by 5 term relations of the Dilogarithm (Section 1.3.4.1). When $n \geq 4$ elements of $\mathcal{R}_n$ become difficult to write down. The 931 term relation found by Gangl is conjectured to be the defining relation of $\mathcal{B}_4(F)$.

Chapter 2: Cluster Modular Group and Exotic Cluster Coordinates

The following is joint work with Dani Kaufman. The following is adapted from our preprint paper [4] by moving the discussion of affine cluster algebras arising from triangulated surfaces from the Appendix into the main text.

As discussed in the introduction, particle physicists obtain polylogarithm functions in the computation of "scattering amplitudes". In particular in $\mathcal{N} = 4$ Super Yang Mills theory they obtain polylogarithms whose arguments are X-Coordinates in the $\mathrm{Gr}(4, 8)$ cluster algebra. This cluster algebra is of infinite type, in particular it is doubly extended type $E_7^{(1,1)}$. Attempts to find a finite description to find polylogarithm relations inspired the following work.

The primary results of this discussion are the following theorems:

Theorem 2.0.1. *Let $\mathbf{n}, \mathbf{w}$ be m dimensional vectors of positive integers. Let $\chi(T_{\mathbf{n},\mathbf{w}}) = \sum(w_i(n_i^{-1} - 1)) + 2$. Then we have the following:*

1. *If $\chi > 0$, then $T_{\mathbf{n},\mathbf{w}}$ provides a seed of an affine cluster algebra.*
2. *If $\chi = 0$, then $T_{\mathbf{n},\mathbf{w}}$ provides a seed of a doubly extended cluster algebra.*
3. *If $\chi < 0$, then $T_{\mathbf{n},\mathbf{w}}$ provides a seed of an infinite mutation type cluster algebra.*

Moreover almost[1] *every affine and doubly extended cluster algebra has a seed with underlying quiver isomorphic to a* $T_{\mathbf{n},\mathbf{w}}$ *for some* $\mathbf{n}, \mathbf{w}$.

Informally, the cluster modular group is the automorphism group of the mutation structure of the cluster algebra. We show that there is an abelian subgroup, Γ_τ, of the cluster modular group of cluster algebras coming from $T_{\mathbf{n},\mathbf{w}}$ quivers generated by "twists" τ_i for each "tail" $i = 1, \dots, m$ and an element γ satisfying $\tau_i^{n_i} = \gamma^{w_i}$ for all i. Let $H = \mathrm{Aut}(T_{\mathbf{n},\mathbf{w}})$ be the automorphism group of a $T_{\mathbf{n},\mathbf{w}}$ quiver. This group acts on Γ_τ by permuting twists τ_i and τ_j whenever $n_i = n_j$ and $w_i = w_j$.

Theorem 2.0.2. 1. *The cluster modular group of an affine cluster algebra is isomorphic to* $\Gamma_\tau \rtimes H$.

2. *The cluster modular group of a doubly extended cluster algebra is generated by the elements of* $\Gamma_\tau \rtimes H$ *and one new generator,* δ.

See Sections 2.1.2 and 2.3.1 for the full definitions of τ_i, γ, δ.

We conjecture the following about infinite mutation type $T_{\mathbf{n},\mathbf{w}}$ quivers, i.e. when $\chi < 0$.

Conjecture 2.0.3. *If* $\chi < 0$*, then the cluster modular group of a cluster algebra with initial seed given by a* $T_{\mathbf{n},\mathbf{w}}$ *quiver is isomorphic to* $\Gamma_\tau \rtimes H$.

We use the computation of the cluster modular group of $T_{\mathbf{n},\mathbf{w}}$ cluster algebras to construct natural finite quotients of the cluster complex.

[1] The twisted Dynkin diagrams that are Langlands dual to standard diagrams have "dual" $T_{\mathbf{n},\mathbf{w}}$ quivers. However their cluster structure is identical to their duals, so we mostly don't need to treat them. The $A_1^{(1,1)}$ and $BC_n^{(4)}$ cluster algebras are simple to treat as special cases.

In the affine case, the element γ generates a finite index subgroup of the cluster modular group. We define the *quotient cluster complex* where cells are equivalence classes up to the action of γ. The dual to the quotient complex is analogous to the generalized associahedron associated to finite type cluster algebras. We compute the basic properties of this *affine generalized associahedron* including the number of codimension 1-cells and dimension 0-cells and we conjecture that they are each homomorphic to a sphere.

We have the following theorems,

Theorem 2.0.4 (Theorem 2.2.21)**.** *The number of distinct cluster variables in an affine cluster algebra up to the action of $\langle\gamma\rangle$ is given by*

$$\sum_i (n_i - 1)n_i + \frac{n}{\chi} \tag{2.1}$$

The number of distinct clusters in an affine cluster algebra up to the action of $\langle\gamma\rangle$ is given by

$$\frac{2}{\chi}\prod_i \binom{2n_i - 1}{n_i} \tag{2.2}$$

These two equations provide the number of codimension 1-cells and dimension 0-cells of an affine generalized associahedron respectively.

In the doubly extended case, the element γ no longer generates a normal subgroup. Instead, we find that the normal closure of this element, in most cases, is a free, finite index normal subgroup of the cluster modular group. We compute the number of clusters in the quotient cluster complex by this group in Table 2.20.

We define doubly extended generalized associahedra to be the dual of this quotient complex.

We conjecture that affine and doubly extended generalized associahedra are each homeomorphic to a product of spheres.

Conjecture 2.0.5. *1. The affine generalized associahedron of an affine cluster algebra of rank $n+1$ is homeomorphic to a sphere of dimension n.*

2. The cluster complex of a doubly extended cluster algebra of rank $n+2$ is homotopy equivalent to S^{n-1}.

3. The doubly extended associahedron associated with a doubly extended cluster algebra is homeomorphic to $S^{n-1} \times S^2$ in all cases other than $E_8^{(1,1)}$ where it instead is homeomorphic to $S^7 \times S^1 \times S^1$.

2.1 Type $T_{\mathbf{n},\mathbf{w}}$ Cluster Algebras

In this section we will consider a family of quivers $T_{\mathbf{n},\mathbf{w}}$ for $\mathbf{n}, \mathbf{w}$ equal length vectors of positive integers, and their associated cluster algebras. These algebras each have a canonical subgroup of the cluster modular group with a simple description in terms of "twist mutation paths" and automorphisms of quivers. We call a cluster algebra "type $T_{\mathbf{n},\mathbf{w}}$" if it has a seed with a $T_{\mathbf{n},\mathbf{w}}$ quiver underlying it.

We then show in Sections 2.2 and 2.3 that each of the affine-type and doubly extended cluster algebras are type $T_{\mathbf{n},\mathbf{w}}$ for certain values of $\mathbf{n}$ and $\mathbf{w}$. We show that that the canonical subgroup is the cluster modular group of each affine type cluster algebra. In the doubly-extended case, we will find that this subgroup along with one extra element generates the cluster modular group. We conjecture that in all other cases, this canonical subgroup is exactly the cluster modular group.

2.1.1 $T_{\mathbf{n},\mathbf{w}}$ Quivers

Let $\mathbf{n} = (n_1, n_2, \ldots, n_m)$, $n_i > 1$ and $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ be m tuples of positive integers. We consider a weighted quiver, $T_{\mathbf{n},\mathbf{w}}$, with $n = \sum(n_i - 1) + 2$ nodes constructed in the following way: First consider the star shaped quiver $T'_{\mathbf{n},\mathbf{w}}$ with $n-1$ nodes consisting of one central node, N_1 of weight 1 and m tails of length $n_i - 1$ of weight w_i nodes $i_2, \ldots, i_{n_i}$ connected in a source-sink pattern with N_1 as a source (Figure 2.1).

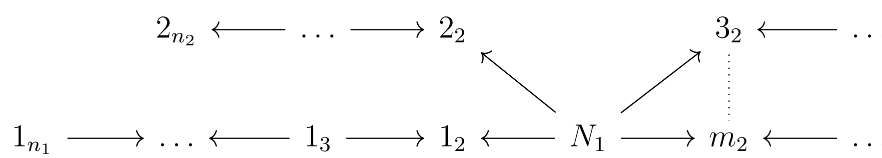


Figure 2.1: The quiver $T'_{\mathbf{n},\mathbf{w}}$.

$T_{\mathbf{n},\mathbf{w}}$ is constructed from $T'_{\mathbf{n},\mathbf{w}}$ by adding an additional weight 1 node N_∞ along with a double arrow from N_∞ to N_1 and single arrows from each of the m other neighbors of N_1 to N_∞, as shown in Figure 2.2.

When $m \leq 3$ and $w_i = 1$ for all i, we let $(p, q, r) = (n_1, n_2, n_3)$ with p, q, r possibly equal to 1 and write $T_{p,q,r}$ for $T_{\mathbf{n},\mathbf{w}}$.

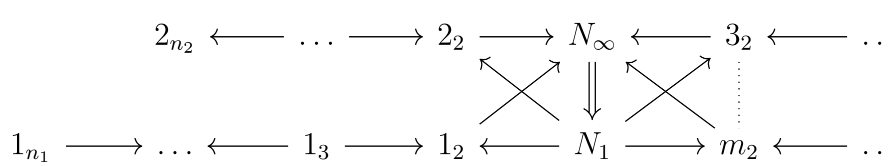


Figure 2.2: The quiver $T_{\mathbf{n},\mathbf{w}}$.

Definition 2.1.1. *The nodes i_j are called the* tail nodes *of $T_{\mathbf{n},\mathbf{w}}$. The nodes i_2 are called the* boundary *tail nodes. The i*th tail subquiver *is the quiver obtained by removing all of the tail nodes $k_j, k \neq i$.*

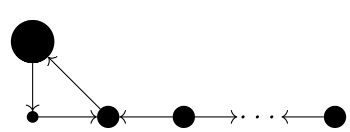
Our motivation for considering these quivers is based on the following remark:

Theorem 2.1.2. *Let $\chi(T_{\mathbf{n},\mathbf{w}}) = \sum(w_i(n_i^{-1} - 1)) + 2$. If $\chi > 0$ then $T_{\mathbf{n},\mathbf{w}}$ has a (non-twisted) affine Dynkin quiver in its mutation class and $T'_{\mathbf{n},\mathbf{w}}$ is a finite Dynkin quiver. If $\chi = 0$ then $T_{\mathbf{n},\mathbf{w}}$ is a doubly extended Dynkin quiver and $T'_{\mathbf{n},\mathbf{w}}$ is an affine Dynkin quiver.*

The first statement will be proved in Section 2.2. The second statement can be verified by checking the finitely many cases where $\chi = 0$ (Figure 2.14).

Remark 2.1.3. *χ is preserved by replacing a length n tail with weight w with w weight 1 tails of length n. This follows the idea that higher weight nodes can be analyzed by folding larger quivers.*

Remark 2.1.4. *The middle two nodes of a $T_{\mathbf{n},\mathbf{w}}$ quiver as we have described always have weight 1. The twisted affine types will have quivers which look like $T_{\mathbf{n},\mathbf{w}}$ quivers, but with weighted nodes in the middle positions. The non-BC twisted affine types are dual to ordinary affine quivers. However the type BC twisted affine quivers are special. For example, the type $BC_n^{(4)}$ quivers have the following quiver in their mutation class:*



(2.3)

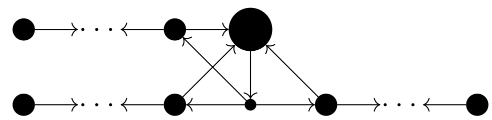


Figure 2.3: A $T_{\mathbf{n}}^{BC}$ quiver with 3 tails.

In light of this remark we will define a BC variant of $T_{\mathbf{n},\mathbf{w}}$ quiver denoted $T_{\mathbf{n}}^{BC}$ which will have the $BC_n^{(4)}$ types in their mutation class.

Definition 2.1.5. *A $T_{\mathbf{n}}^{BC}$ quiver consists of two middle nodes of weight 4 and 1 with a single arrow between them and tails of weight 2 nodes of length n_i, see Figure 2.3. We define*

$$\chi(T_{\mathbf{n}}^{BC}) = \sum_i (\frac{1}{n_i} - 1) + 1 \tag{2.4}$$

2.1.2 The Cluster Modular Group of a $T_{\mathbf{n},\mathbf{w}}$ Cluster Algebra

We will construct a subgroup, Γ_τ, of the cluster modular group of a $T_{\mathbf{n},\mathbf{w}}$ cluster algebra generated by "twist" mutation paths associated with each tail. The automorphism group $\mathrm{Aut}(T_{\mathbf{n},\mathbf{w}})$ acts on Γ_τ by permuting twists associated to tails of the same length and weight.

Definition 2.1.6. *The group $\Gamma_{T_{\mathbf{n},\mathbf{w}}} = \Gamma_\tau \rtimes \mathrm{Aut}(T_{\mathbf{n},\mathbf{w}})$ is the canonical subgroup of the cluster modular group of a $T_{\mathbf{n},\mathbf{w}}$ type cluster algebra.*

Conjecture 2.1.7. *If $\chi(T_{\mathbf{n},\mathbf{w}}) \neq 0$ then the cluster modular group of a type $T_{\mathbf{n},\mathbf{w}}$ cluster algebra is exactly $\Gamma_{T_{\mathbf{n},\mathbf{w}}}$.*

Let

$$i_{\text{odd}} = \{i_j | 3 \leq j \leq n_i, j \text{ odd}\} \text{ and } i_{\text{even}} = \{i_j | 3 \leq j \leq n_i, j \text{ even}\}. \tag{2.5}$$

Definition 2.1.8. *We have a twist $\tau_i \in \Gamma_\tau$ given by the following mutation paths depending on w_i:*

$$w_i = 1 \quad let\ \tau_i = \{i_{odd} i_{even} i_2 N_\infty N_1, (i_2 N_\infty N_1)\} \tag{2.6}$$

$$w_i = 2 \quad let\ \tau_i = \{i_{odd} i_{even} i_2 N_\infty N_1 i_2 N_1, id\} \tag{2.7}$$

$$w_i = 3 \quad let\ \tau_i = \{i_{odd} i_{even} i_2 N_\infty N_1 i_2 N_\infty i_2 N_1, id\} \tag{2.8}$$

When $w_i \geq 4$ there is no twist for tail i.

Let $\gamma = \{N_\infty, (N_1 N_\infty)\}$, which we think of as a twist of a tail of length 1.

Definition 2.1.9. Γ_τ *is the group generated by all of twists, τ_i, and γ.*

Remark 2.1.10. *Once again we see the importance of using folding to understand weighted quivers. One can verify that when $w_i = 2$, τ_i is the same as replacing tail i with two tails of the same length twisting each of them and then refolding into a tail of weight 2. The same holds for splitting into 3 tails when $w_i = 3$. However when $w_i = 4$ mutation at i_2 reverses the direction of the double edge without mutating at N_1 or N_∞ and so there is no possible equivalent twist of 4 tails. When $w_i > 4$ mutation at i_2 results in edge of weight higher than 2; this situation only happens in infinite mutation type cluster algebras which we don't consider for the remainder of the paper.*

We have the following theorem:

Theorem 2.1.11. *Γ_τ is an abelian group and the only relations are $\tau_i^{n_i} = \gamma^{w_i}$.*

Proof. In order to show that Γ_τ is abelian, we simply need to check that two twists tails of length 2 commute with each other and with γ. This is because the additional mutations which appear as the tail length increases always happen at sources. Thus they don't change the adjacency of the quiver and stay disconnected from the other tail through the entire path. Therefore all that remains is a simple computation to check commutativity for each possible combination of weights for tails of length 2.

We now focus on a single tail of length n and weight 1 and show that $\tau^n = \gamma$. It suffices to look at $T_{(n),(1)}$ since τ_i only mutates at vertices on tail i. In Section 2.2 we see that this quiver is associated to an annulus with n marked points on the interior (labeled $v_1, \ldots, v_n$ clockwise) and one marked out on the outer boundary component. Then by Lemma 2.1.12 we see that τ corresponds to rotating the interior circle by $\frac{2\pi}{n}$ radians and γ is the full Dehn twist. So τ^n is a full rotation and is equal to γ.

The previous remark completes the theorem when $w_i > 1$. □

Lemma 2.1.12. *Let S be an annulus with n marked points on the inner boundary component and 1 marked point on the outer boundary. The the twist τ (Definition 2.1.8) corresponds to rotating the inner boundary component $\frac{2\pi}{n}$ radians and γ corresponds to a full Dehn twist and thus $\gamma = \tau^n$.*

Proof. To analyze τ we break the mutation sequence into two pieces $[i_{\text{odd}} i_{\text{even}}]$, $[i_2, N_\infty, N_1]$. On the annulus, the arc associated with node i_2 begins and ends at v_1.

Thus $[i_{\text{odd}}i_{\text{even}}]$ is a "sinks then sources" sequence inside an $n-$gon. This rotates the zig-zag triangulation clockwise one tick so the outermost arc goes from v_2 clockwise around to v_1. Then treating this arc as an arc of the inner boundary component reduces puts us exactly in the situation of a $T_{(2),(1)}$ quiver.

It is then a simple computation to see that the mutation path $[i_2, N_\infty, N_1]$ returns to a quiver isomorphic to the original but with the self loop around v_2 instead of v_1. Note that N_1 is now the self loop and i_2 and N_∞ are the source and sink of the double edge respectively, justifying the permutation (i_2, N_∞, N_1). See Figure 2.4 for an example of a tail with length 4.

Therefore each application of τ moves one tick clockwise around the inner boundary component. Therefore n twists returns to v_1 having made a full clockwise twist about the inner boundary component. Furthermore, the self loop at v_1, treated as the edge of the boundary component, always separates N_1 and N_∞ from the rest of the tail.

So it suffices to analyze γ on the annulus with one marked point on each boundary component. Then it is clear applying γ is equivalent twisting once clockwise around the inner boundary component and so is equal to τ^n. □

Figure 2.4 shows the explicit action of twisting about a tail on the surface representation of the cluster algebra.

Remark 2.1.13. *Let* $\ell = \prod n_i$. *We may view* Γ_τ *as the subgroup of* $\mathbb{Z} \times \prod \mathbb{Z}_{n_i}$ *generated by the elements* $\gamma = (\ell, 0, \ldots, 0)$ *and* $\tau_i = (w_i\ell/n_i, 0, \ldots, 1, \ldots, 0)$. *Let* Γ_τ° *be the kernel of the projection* $\Gamma_\tau \to \mathbb{Z}$. *Then* $\Gamma_\tau \simeq \Gamma_\tau^\circ \rtimes \mathbb{Z}$

Remark 2.1.14. *When there are zero tails, $T_{(),()}$ is just a double edge. It is clear in this case γ^2 is the reddening element. This generalizes to the following theorem.*

Theorem 2.1.15. *The element $r \in \Gamma_\tau$ given by $r = \gamma^2 \prod_i (\tau_i \gamma^{-w_i})$ is the reddening element of $T_{\mathbf{n},\mathbf{w}}$.*

Proof. Suppose that $m = 1$. It is a simple computation to check this statement for each possible weight when $n_1 = 2$. Then, for $n_1 > 2$ we can see that the mutating at $i_{\text{even}} i_{\text{odd}}$ always mutates at a source and so is a reddening sequence for the non-boundary nodes of the tail. Since 1_3 is initially connected towards 1_2, we now have 1_2 out to 1_3. Finally, we can complete the reddening sequence by using the the $n = 2$ case.

Now consider $m > 1$. Let $r_i = \gamma^2 \tau_i \gamma^{-w_i}$ be the reddening element for the ith tail subquiver. Rewrite r as follows

$$r = \gamma^2 \prod_i (\tau_i \gamma^{-w_i}) = \big(\prod_i (r_i \gamma^{-2})\big)\gamma^2 \tag{2.9}$$

We can see that this element is reddening by noting that $r_i \gamma^{-2}$ has the effect of reddening the nodes on the tail i, while keeping the middle two nodes green. Thus for each i, the element r_i always gets applied to an all green subquiver. Therefore, the effect of the product of elements of the right hand side of equation 2.9 is to make all of the nodes other than the middle two red. Then, tacking on γ^2 makes all the nodes red. This element returns us to an isomorphic quiver with out permuting any of the frozen nodes, and is thus the reddening element.

□

Corollary 2.1.16. *When $\chi > 0$, r is a conjugation of the source-sink mutation path on the corresponding affine Dynkin diagram.*

This corollary follows since the reddening element of an affine Dynkin diagram is the source-sink mutation path. Then since $\chi > 0$ implies there is an quiver corresponding to an affine Dynkin diagram, Theorem 1.1.30 states the two reddening elements must be conjugate.

Remark 2.1.17. *In terms of the group presentation of Remark 2.1.13, the reddening sequence is given by the element $(\chi\ell, 1, 1, \ldots, 1)$.*

2.1.3 BC Type Quivers

The $T_{\mathbf{n}}^{BC}$ type quivers have an analogous abelian subgroup, $\Gamma_\tau = \langle \tau_i, \gamma | \tau_i^{n_i} = \tau_j^{n_j} = \gamma \rangle$, generated by twists of the tails. γ is the mutation path consisting of mutation at the weight 4 node and then the weight 1 node and the twist paths are the same twist paths in the $w_i = 2$ case of a regular $T_{\mathbf{n},\mathbf{w}}$ quiver.

The reddening element is given by

$$r = \gamma \prod_i (\tau_i \gamma^{-1}). \tag{2.10}$$

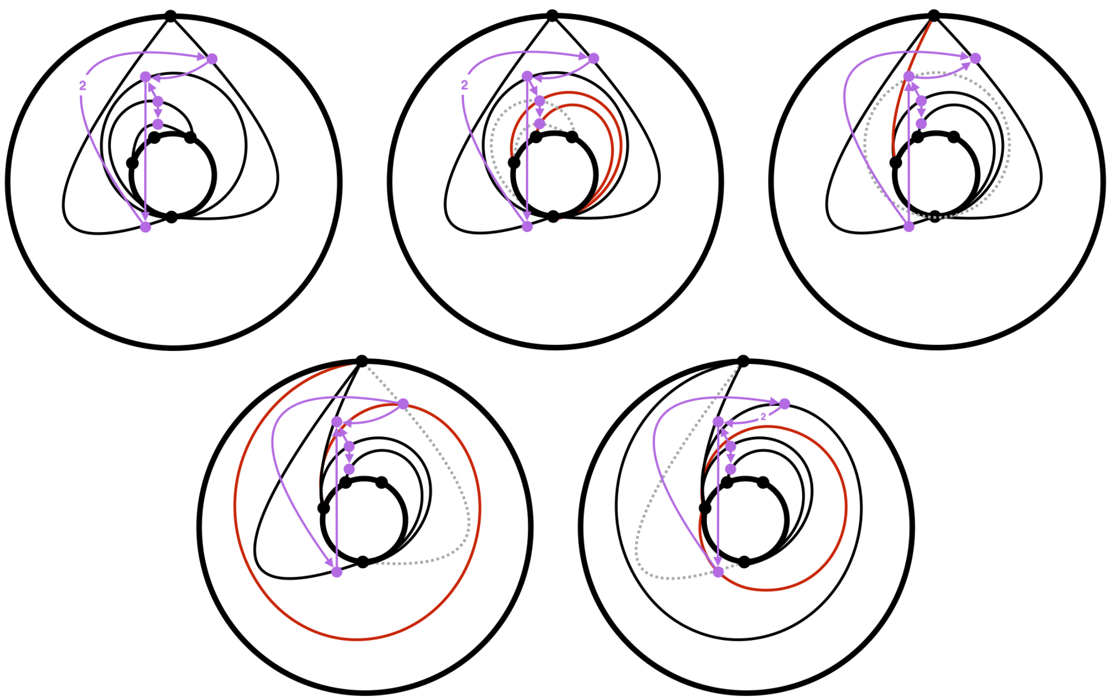


Figure 2.4: Application of single twist for a tail of length 4. The result is shown after $[i_{\text{odd}}i_{\text{even}}]$, i_2, N_∞, and then N_1. At each stage the dashed gray edges are replaced with the red edges.

2.2 Affine Cluster Algebras

Our analysis of the cluster modular group of the affine cluster algebras stems from the observation in Remark 2.1.2. Our primary goal is the following theorems:

Theorem 2.2.1. *The cluster algebra associated to the quiver $T_{\mathbf{n},\mathbf{w}}$ is of affine type if and only if $\chi > 0$. Furthermore, every affine type cluster algebra has a seed whose quiver is a $T_{\mathbf{n},\mathbf{w}}$ or $T^{BC}_{\mathbf{n}}$ with $\chi > 0$.*

Theorem 2.2.2. *The cluster modular group of a cluster algebra of affine type is $\Gamma_\tau \rtimes \mathrm{Aut}(T_{\mathbf{n},\mathbf{w}})$, where the action of the automorphism group is by permuting the twists of tails of the same weight and length.*

In order to prove Theorem 2.2.1 we need to carefully analyze the triangulations of both the annulus and the twice punctured disc.

Definition 2.2.3. *There are three classes of arcs on an annulus.* Crossing arcs *connect two marked points on different boundary components.* Boundary arcs *connect two marked points on the same boundary component. A* self loop *is a boundary arc between the same marked point that travels around the center.*

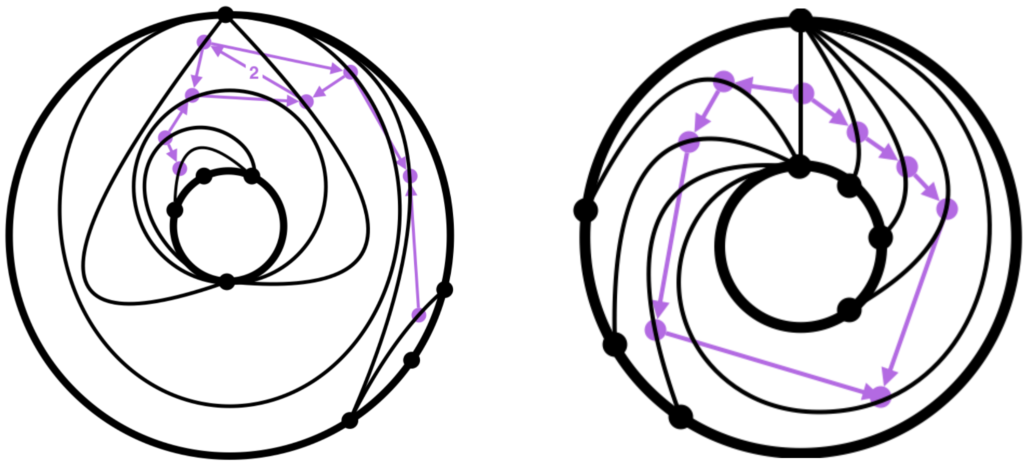
Proof of Theorem 2.2.1. First we note that we can write $T_{(n)} = T_{n,1,1}$ and $T_{(p,q)} = T_{p,q,1}$ so we can handle both of these cases together. Here we can construct a $T_{p,q,1}$ quiver from a triangulation of $S_{0,2,0,p+q}$, the annulus with p marked points on one boundary and q marked points on the other. We also construct a triangulation corresponding to an affine $A_{p,q}$ Dynkin diagram (Figure 2.12). Since any two triangulations are related by a series of flips this shows $T_{p,q,1}$ is in the same mutation class as $A_{p,q}$ as needed.

The first triangulation can be constructed by choosing a self loop on each boundary component. This divides the annulus into three regions: a p-gon, an annulus with one marked point on each boundary, and a q-gon. In the p-gon and q-gon, we then use the "zig/zag" triangulation starting from the self loop, to obtain portions of quiver that are a single line of nodes starting such that each node is a source or a sink. Finally add two distinct crossing arcs into the inner annulus completing the triangulation. See figure 2.5a for an example with $p = 4$ and $q = 4$.

The second triangulation will correspond to an orientation of the $A_{p,q}$ Dynkin diagram with a single source and sink. To construct this quiver, we first add a crossing arc between a marked point on each boundary. Next we connect the outer marked point of the initial arc to each inner marked point in a series of nested clockwise crossing arcs. Similarly attach the inner point of the initial arc to each other outer marked point in a series of nested counterclockwise crossing arcs, see figure 2.5b for an example with $p = 4$ and $q = 4$.

Similarly, $T_{(n,2,2)}$ occurs as the quiver obtained from a triangulation of twice punctured disk with n marked points on the boundary. We also construct a triangulation of the twice punctured disk that corresponds to an $\widetilde{D}_n$ Dynkin diagram. So as in the $\widetilde{A}_n$ case this shows $T_{n,2,2}$ corresponds to the type $\widetilde{D}_n$ cluster algebras.

For the first triangulation, connect the punctures with an edge and a loop from one puncture around the other (tagged arc). Then the outside of this loop is



(a) $T_{4,4}$.

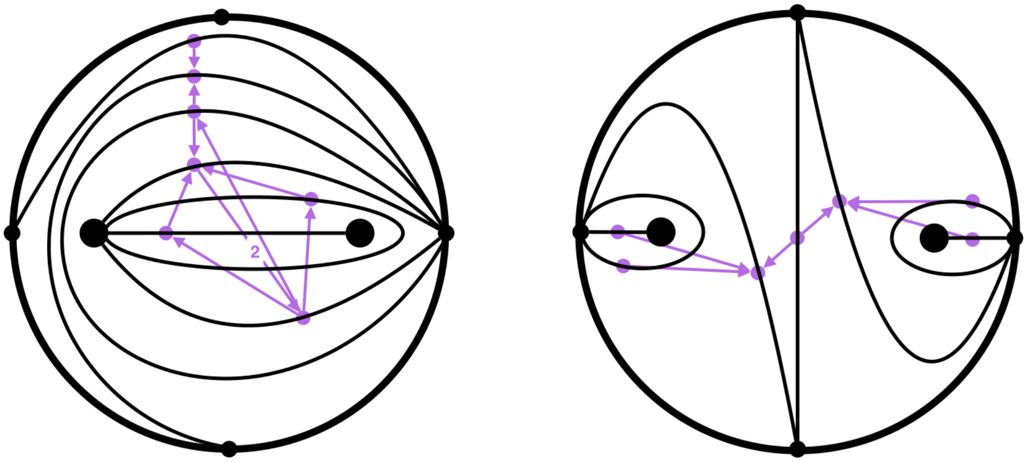
(b) $A_{4,4}$ Dynkin diagram.

Figure 2.5: Two different triangulations of an annulus with 4 marked points on each boundary component.

an annulus with one marked point on the inner "boundary" and n marked points on the outer boundary. We then complete the quiver using the construction of a $T_{n,1,1}$ quiver as described before (see figure 2.6a).

The second triangulation corresponding to a sources/sink orientation of a $\widetilde{D}_n$ Dynkin diagram. First, connect each puncture to a different boundary vertex. Then add a self loop from the boundary vertex around the corresponding puncture. Outside these self loops is a disk with n marked points that can be triangulated with a "zig/zag" starting from one self loop and ending at the other (see figure 2.6b).

For $k = 3, 4, 5$ observe that $T'_{k,3,2}$ is an E_{k+3} finite Dynkin diagram oriented so every vertex is a source or a sink. Let $g = [N_1, i_{\text{odd}}, i_{\text{even}}, i_2]$ be the mutation path corresponding to the sources/sinks move for E_{k+3}. One can verify that $g^{h/2}$ transforms $T_{k,3,2}$ into the affine Dynkin diagram for $\widetilde{E}_{k+3}$ where h is the order of g in E_{k+3} ($h = 7, 10, 16$ respectively). Note that applying g $\frac{7}{2}$ times for $T_{3,3,2}$ means



(a) $T_{4,2,2}$. (b) $\widetilde{D}_6$ Dynkin diagram.

Figure 2.6: Two different triangulations of a a twice punctured disk with 4 marked points on the boundary.

apply g 3 times, then mutate at the sources $[N_1, i_{\text{odd}}]$ one more time to achieve a sources/sinks orientation of the $\widetilde{E}_6$ diagram.

For the non simply laced cases we have explicit foldings of the simply laced cases. First consider $T_{(n,2),(1,2)}$ which we claim has type $\widetilde{B}_{n+1}$. This quiver can be obtained from the $\widetilde{D}_{n+2}$ by folding the length 2 tails of the $T_{n,2,2}$ quiver. As in the other cases doing $h/2$ applications of the underlying finite sources sink mutation transforms this quiver into the standard Dynkin type quiver for $\widetilde{B}_{n+1}$. Note this agrees with the usual Dynkin folding of $\widetilde{D}_{n+2}$ into $\widetilde{B}_{n+1}$.

The other cases are similar, $\widetilde{C}_n$ is obtained from folding the two tails $A_{n,n}$ which corresponds on the Dynkin side via $g^{h/2}$ to folding a $2n+1$ cycle in half. $\widetilde{F}_4$ is obtained from $T_{(3,2),(2,1)}$ by folding the two length three tails of $T_{3,3,2}$ ($\widetilde{E}_6$). The final affine quiver $\widetilde{G}_2$ is $T_{(2),(3)}$ obtained by folding all three tails in $T_{2,2,2}$.

Note that every possible affine Dynkin diagram (figures A.4,A.5) has appeared as one of these cases. □

We now prove theorem 2.2.2 by showing the cluster modular group is $\Gamma_\tau \rtimes \mathrm{Aut}(Q)$ in each case. It is clear that $\Gamma_\tau \rtimes \mathrm{Aut}(Q)$ is a subgroup of the cluster modular group, so it suffices to show their are no other possible cluster modular group elements.

Proof of Theorem 2.2.2 for $A_{p,q}$. Any cluster modular group element must send our original $T_{p,q}$ quiver to another $T_{p,q}$ quiver. So it suffices to construct every possible $T_{p,q}$ quiver on the annulus and show they are in the image of the proposed group.

Once again we will rely on the correspondence between seeds in the cluster algebra and triangulations of an annulus. Since this quiver has a double edge, by remark 1.1.57 the only possible construction of a $T_{p,q,1}$ quiver is the one given in the proof of theorem 2.2.1.

However there was some freedom in this construction. The first is the choice of marked point on each boundary component to add a self loop around. There are pq total possible choices for this. The other more subtle degree of freedom is the action of the mapping class group of the annulus, generated by a single Dehn twist about the center. Note the Dehn twist only changes crossing arcs which correspond to nodes N_1 and N_∞. A simple analysis shows that γ corresponds exactly to the action of the Dehn twist.

Then $\Gamma_\tau/\langle\gamma\rangle = \mathbb{Z}_p \times \mathbb{Z}_q$ has order pq. Therefore each distinct copy of $T_{p,q,1}$ up to mapping class group is the image of a distinct twist as needed. Since no other triangulation produce an isomorphic quiver we are done as long as $p \neq q$.

When $p = q$ there is an extra symmetry of the triangulation given by swapping the inner and outer boundary components. However this is exactly automorphism of $T_{p,p,1}$ that swaps each tail. This corresponds exactly to the action of $\mathrm{Aut}(T_{p,p,1})$ on Γ_τ as needed. □

Proof of Theorem 2.2.2 for $\widetilde{D}_n$. As in the $A_{p,q}$ case the only possible construction of the $T_{n,2,2}$ quiver is the one described in the proof of theorem 2.2.1. Thus we look at the ambiguity of the construction of the $T_{n,2,2}$ quiver. The obvious choices are which puncture is inside the self loop, the boundary vertex that is attached to the puncture, and the winding number of these crossing edges. There is an additional subtle choice from the tagged arc complex. In this generalization the self loop around a puncture is replaced with a singly tagged arc between the two punctures. There is then an additional way to get an isomorphic quiver by switching the tagging at a puncture. This operation at the puncture with a tagged arc simply swaps the two arcs between the punctures and thus corresponds to the extra semidirect product with $\mathbb{Z}_2$ when $n \neq 4$. However flipping the tagging at the other puncture results in a new triangulation in every case. Putting this all together gives $4n$ triangulations up to winding number. Mutation along the double edge correspond to the Dehn twist around both punctures so we can again see that $\Gamma_\tau / \langle \gamma \rangle = \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has order $4n$ and so reaches every possibility.

When $n = 4$ not every automorphism of $T_{2,2,2}$ corresponds to a symmetry of the twice punctured disk as described above, but otherwise the analysis is exactly the same.

□

Proof of Theorem 2.2.2 for $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. In [21] they compute the cluster modular group for the Dynkin type quivers as $\mathbb{Z} \times S_3$, $\mathbb{Z} \times \mathbb{Z}_2$, and $\mathbb{Z}$ for $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ respectively. In each case the $\mathbb{Z}$ is generated by the full sources/sinks move on the Dynkin quiver. This is the reddening element as is conjugate to r by theorem 2.1.15. Recall the subgroup Γ^0_τ of combinations of twists of finite order from Remark 2.1.13. The remaining finite portion of each group is given by $\Gamma^0_\tau \rtimes \mathrm{Aut}(Q)$ in each case. □

Lemma 2.2.4. *Folding the tails of the $T_{\mathbf{n},\mathbf{w}}$ quivers only changes the cluster modular group by reducing automorphism group of the quiver and identifying the generators corresponding to twists about the folded tails.*

Proof. This follows from Remark 2.1.10 that weight 2 or 3 twists are equivalent to simultaneous twists of the corresponding number of equal length tails. □

Proof of Theorem 2.2.2 for non simply laced diagrams. To prove each non simply laced affine $T_{\mathbf{n},\mathbf{w}}$ corresponded to an affine diagram, we gave an explicit folding of each simply laced $T_{\mathbf{n},\mathbf{1}}$ quiver and so the previous lemma applies. □

The association between the affine types and values of **n** and **w** is given in Figure 2.7. The following well known Lemma (included for completeness) proves that this is every possible option for $\chi > 0$.

Lemma 2.2.5. *There are finitely many families of* $(\mathbf{n}, \mathbf{w})$ *such that* $\chi > 0$.

Proof. Following Remark 2.1.3, we begin with the case where every tail has weight 1. If $\chi > 0$, we need:

$$\sum \frac{1}{n_i} > m - 2$$

The only options for $\mathbf{n}$ are (n), (p, q), $(n, 2, 2)$, $(3, 3, 2)$, $(4, 3, 2)$ and $(5, 3, 2)$. Then the higher weight tails come from folding the above cases. When $p = q$ we can fold to obtain $((p), (2))$. Similarly the two length two tails in $(n, 2, 2)$ and the length 3 tails of $(3, 3, 2)$ can be folded to obtain $((n, 2), (1, 2))$ and $((3, 2), (2, 1))$. Finally we can fold $(2, 2, 2)$ to obtain $((2), (3))$.

The BC case follows easily by direct inspection. □

Remark 2.2.6. *These cluster modular groups have already been computed [21] based at the Dynkin type quivers. However the computations based at the* $T_{\mathbf{n},\mathbf{w}}$ *quivers allows for a uniform treatment of the affine and double extended cluster algebras.*

Type	$\mathbf{n}$	$\mathbf{w}$
$A_{1,1}$	$()$	$()$
$A_{p,q}$	(p, q)	$(1, 1)$
$\widetilde{D}_n$	$(n-2, 2, 2)$	$(1, 1, 1)$
$\widetilde{E}_6$	$(3, 3, 2)$	$(1, 1, 1)$
$\widetilde{E}_7$	$(4, 3, 2)$	$(1, 1, 1)$
$\widetilde{E}_8$	$(5, 3, 2)$	$(1, 1, 1)$

Type	$\mathbf{n}$	$\mathbf{w}$
$\widetilde{C}_n$	(n)	(2)
$\widetilde{B}_n$	$(n-1, 2)$	$(1, 2)$
$\widetilde{F}_4$	$(3, 2)$	$(2, 1)$
$\widetilde{G}_2$	(2)	(3)
$BC_n^{(4)}$ (BC-Type)	(n)	-

Figure 2.7: All possible values of $T_{\mathbf{n},\mathbf{w}}$ that result in affine cluster algebras.

Affine Type	Cluster Modular Group	Quotient
$A_{p,p}$	$D_{2p} \rtimes \mathbb{Z}$	$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2$
$A_{p,q}$	$\mathbb{Z}_{gcd(p,q)} \times \mathbb{Z}$	$\mathbb{Z}_p \times \mathbb{Z}_q$
$\widetilde{D}_4$	$S_4 \times \mathbb{Z}$	$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3$
$\widetilde{D}_n$ Even	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \times \mathbb{Z}$	$\mathbb{Z}_{n-2} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
$\widetilde{D}_n$ Odd	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}$	$\mathbb{Z}_{n-2} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
$\widetilde{E}_6$	$S_3 \times \mathbb{Z}$	$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$
$\widetilde{E}_7$	$\mathbb{Z}_2 \times \mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$
$\widetilde{E}_8$	$\mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
$\widetilde{C}_n$	$\mathbb{Z}_2 \rtimes \mathbb{Z}$	$\mathbb{Z}_n$
$\widetilde{B}_n$	$\mathbb{Z}_2 \times \mathbb{Z}$	$\mathbb{Z}_{n-1} \times \mathbb{Z}_2$
$\widetilde{F}_4$	$\mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
$\widetilde{G}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
$BC_n^{(4)}$	$\mathbb{Z}$	$\mathbb{Z}_n$

Figure 2.8: Affine cluster modular groups and their quotients.

2.2.1 The Normal Subgroup Generated by γ

Our goal now is to construct a natural finite quotient of the exchange graphs and cluster complexes of each of the affine cluster algebras. We dualize the quotient cluster complexes to produce an “affine generalized associahedron”.

The subgroup of the cluster modular groups of the affine $T_{\mathbf{n},\mathbf{w}}$ quivers generated by $\gamma = \{N_\infty, (N_1 N_\infty)\}$ is a normal, finite index subgroup.

Remark 2.2.7. *In the $\widetilde{A}$ case, this subgroup can be seen to be given by the mapping class group action on the triangulations of an annulus. We therefore consider this subgroup to be an analog to the mapping class group in each of the affine cases.*

Figure 2.8 shows the cluster modular groups and quotients by the subgroup generated by γ.

We wish to understand the quotient of the exchange graph of an affine cluster algebra by the action of the group $\langle\gamma\rangle$. A possible way to accomplish this is by introducing a special framing of a $T_{\mathbf{n},\mathbf{w}}$ quiver, and compute the graph by identifying the clusters via their c-vectors as usual.

Consider the quiver $T^f_{\mathbf{n},\mathbf{w}}$ obtained from $T_{\mathbf{n},\mathbf{w}}$ by adding a frozen node for vertices $i_2, \ldots, i_{n_i}$ in each tail and one vertex associated with the double edge. In particular for each tail i add frozen nodes of weight w_i labeled $f_{i,2}, \ldots, f_{i,n_i}$ with a single arrow from i_j to $f_{i,j}$. Then add a frozen node f_1 of weight 1 along with single arrows N_1 to f_1 and f_1 to N_∞.

Conjecture 2.2.8. *Two quivers in the exchange graph of $Q = T_{\mathbf{n},\mathbf{w}}$ are in the same orbit of the action of $\langle\gamma\rangle$ if and only if the projection of those quivers in the exchange graph of $T^f_{\mathbf{n},\mathbf{w}}$ is the same.*

The "if" part of the statement follows since the framing is preserved by the action of γ. However, it is not clear that the only quivers which are identified are the ones which are in the same γ orbit.

2.2.2 Affine Associahedra

Recall the cluster complex associated to a finite cluster algebra has a dual complex called the "generalized associahedon". We cannot simply dualize an affine cluster complex immediately as there are vertices in the cluster complex with infinite degree. This is because there are cluster variables that are compatible with infinitely many other cluster variables, and so occur in infinitely many seeds. However, up to action of γ, there are only finitely many cluster variables. If we quotient the cluster complex by the action of γ, we obtain a finite cell complex.

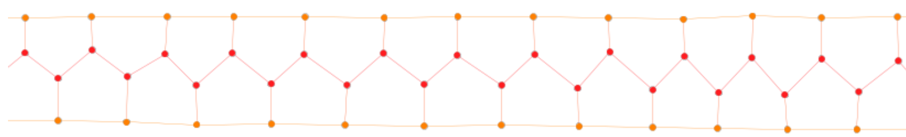
In order to construct a dual complex, we need to see that the quotient complex is a "combinatorial cell complex". Technically, the quotient by γ is not combinatorial because there are facets that contain multiple cluster variables in the same orbit. Instead we quotient by γ^3 which ensures that every maximal facet corresponds to a unique collection of distinct orbits. There is still a finite number of clusters up to γ^3 so by the work of Basak [33] this complex has a dual cell complex. We then can quotient the dual by γ to obtain the dual cell complex we originally desired.

Definition 2.2.9. *Let $C(A)$ be the cluster complex associated to the affine cluster algebra $\mathcal{A}$. The* affine associahedron *is the dual complex to $C(A)/\langle\gamma\rangle$. The 1-skeleton of an affine associahedron is the quotient exchange complex of an affine cluster algebra.*

Remark 2.2.10. *We could define an affine associahedron as a quotient of any power of γ. All of our analysis of the combinatorics of affine associahedra can be easily extended to a quotient by any other power of γ.*

Example 2.2.11. *The simplest example is the $A_{2,1}$ cluster algebra. In Figure 2.9a we see the full exchange graph extending infinitely in both directions. Below is the quotient associahedron. There are four folded 2-cells. Two correspond to the top and bottom pentagons and the remaining two correspond to the $A_{1,1}$ subalgebras. Despite the folding, this associahedron has the homology type of a sphere.*

Example 2.2.12. *A slightly more complicated example is $\widetilde{D}_4$ in Figure 2.10. Again we see the exchange graph extending infinitely in both directions. The 1-skeleton of*



(a) Exchange Graph.

(b) Associahedron.

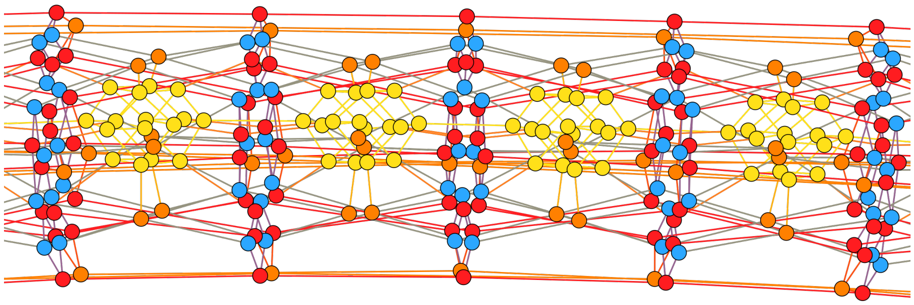
Figure 2.9: $A_{2,1}$ Exchange Graph and Associahedron.

the affine associahedron is shown. This was graph was computed using the special framing mentioned in the previous section. This computation finds the correct number of $0-$*cells in the associahedron, and thus confirms Conjecture 2.2.8 in this case. The complete counts of all subalgebras in* $\widetilde{D}_4$ *up to the action of* γ *can be found in Figure 2.11. The total counts of corank* k *subalgebras is the number of codimension* k *facets of the affine associahedron.*

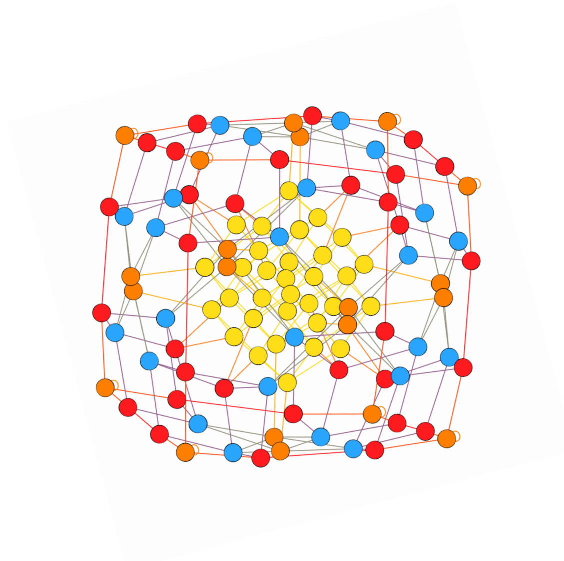
2.2.3 Counting Facets in the Affine Associahedra

Let Q be any quiver of affine mutation type of rank n, and let $\mathcal{A}$ be the cluster algebra associated to this quiver. Let $\mathbf{n} = (n_i), \mathbf{w} = (w_i)$ be the vectors defining a $T_{\mathbf{n},\mathbf{w}}$ quiver in the mutation class of Q and let $\chi(\mathcal{A}) = \sum(w_i(n_i^{-1} - 1)) + 2$.

The affine associahedron associated to $\mathcal{A}$ will have a $k-$cell for each rank k subalgebra of $\mathcal{A}$. Since a rank k subalgebra is obtained by freezing $n - k$ nodes in a quiver underlying a seed of $\mathcal{A}$, we will count codimension 1 facets by counting cluster variables up to the action of γ.



(a) Exchange Graph.



(b) 1-skeleton of the associahedron.

Figure 2.10: $\widetilde{D}_4$ exchange graph and affine associahedron.

Corank	Subalgebra Types			Total
1	$A_{2,2}$	D_4	$D_2 \times D_2$	16
	6	8	2	
2	$A_{2,1}$	A_3	$A_1 \times A_1 \times A_1$	96
	12	60	24	
3	$A_{1,1}$	A_2	$A_1 \times A_1$	244
	8	128	108	
4	A_1			270
	270			
5	A_0 (Clusters)			108
	108			

Figure 2.11: Type and number of subalgebras in the $\widetilde{D}_4$ cluster algebra up to the action of γ.

Definition 2.2.13. *Let* $R \in \mathrm{Mut}(Q)$*. We call a node* k *of* R finite *if the quiver obtained by freezing* k *is of finite mutation type. We call* k affine *if the quiver obtained by freezing* k *is of affine mutation type. We call the cluster variables associated with affine or finite nodes affine or finite respectively.*

Lemma 2.2.14. *Every node of* R *is either finite or affine.*

Proof. If Q is of type A or D then this follows by seeing that every possible arc in the associated marked surface cuts the surface into regions which are surfaces of type $\mathcal{A}$ or D. In the E case, this may be checked by brute force. The non simply laced cases then follow by folding. □

Remark 2.2.15. *The arcs which correspond to cluster variables on finite nodes in the* A *and* D *cases are exactly the arc which have non trivial intersection number with the arc that generates the Dehn twist* δ *i.e. the crossing arcs.*

Lemma 2.2.16. *The cluster variable associated with every finite node appears on a quiver which is an orientation of the associated affine Dynkin diagram. Every affine cluster variable appears on a tail node of a* $T_{\mathbf{n},\mathbf{w}}$ *quiver in the mutation class of* Q*.*

Proof. The first statement follows in the A and D cases by noticing that each arc which intersects δ can be found in a triangulation which is a source-sink orientation of the Dynkin diagram. In the E case, this follows by a slightly more sophisticated brute force calculation similar the the calculation of the previous lemma. The remaining cases again follow from the folding.

The second statement is proved in a similar way to the first. We notice that each affine cluster variable is associated with a boundary arc, and each boundary arc can be found in a triangulation corresponding to a $T_{\mathbf{n},\mathbf{w}}$ quiver. Again, in the E case this is checked by brute force. □

Remark 2.2.17. *Freezing an affine node produces an affine subalgebra of $\mathcal{A}$. Since these nodes always appear on the tail of a $T_{\mathbf{n},\mathbf{w}}$ quiver, we can see that γ is also an element of the cluster modular group of every affine subalgebra of $\mathcal{A}$. Thus, the action of γ on the cluster complex of $\mathcal{A}$ restricts to the action of γ on the cluster complex of any affine subalgebra of $\mathcal{A}$. Thus it makes sense to consider the affine associahedra of subalgebras to be facets of the affine associahedra of $\mathcal{A}$.*

Definition 2.2.18. *We write $C^k(\mathcal{A})$ resp. $C_k(\mathcal{A})$ for the sets of codimension resp. dimension k facets of the affine associahedron of $\mathcal{A}$.*

The size of $C^1(\mathcal{A})$ is equal to the number of distinct cluster variables in $\mathcal{A}$ up to the action of γ.

Theorem 2.2.19. *The number of distinct cluster variables in an affine cluster algebra up to the action of $\langle\gamma\rangle$ is given by*

$$|C^1(\mathcal{A})| = \sum_i (n_i - 1)n_i + \frac{n}{\chi(\mathcal{A})} \tag{2.11}$$

Proof. We simply need to count the number of finite and affine cluster variables up to the action of $\langle\gamma\rangle$. The action of γ is trivial on the affine cluster variables, so we simply need to count them. By Lemma 2.2.16, each affine cluster variable appears on the tail of a $T_{\mathbf{n},\mathbf{w}}$. On tail i there are $n_i - 1$ affine cluster variables, and each application of τ_i gives an entirely new collection of affine cluster variables; This may be seen by examining the $A_{p,1}$ case. Thus, in total there are $\sum_i (n_i - 1)n_i$ affine cluster variables.

To count the number of finite cluster variables up to the action of γ, we again use Lemma 2.2.16, so that we only need to count the number of cluster variables appearing on source-sink oriented Dynkin diagrams. The source-sink mutation path takes each collection of cluster variables to an entirely new collection [21]. By Theorem 2.1.15 we know that the source-sink mutation path is equivalent in the cluster modular group to r. We can calculate that the order of r in $\Gamma_Q/\langle\gamma\rangle$ is χ^{-1} using the presentation of Remark 2.1.13. Thus since there are n finite cluster variables on each source-sink oriented Dynkin quiver, there must be $\frac{n}{\chi}$ up to the action of γ. □

Remark 2.2.20. *The number of distinct cluster variables up to the action of $\langle\gamma^\ell\rangle$ is given by*

$$\sum_i (n_i - 1)n_i + \frac{\ell n}{\chi(\mathcal{A})}. \tag{2.12}$$

This is because higher powers of γ identify fewer finite cluster variables.

Lemma 2.2.21.

$$|C_k(\mathcal{A})| = \frac{1}{n-k} \sum_{\mathcal{B} \in C^1(\mathcal{A})} C_k(\mathcal{B}) \tag{2.13}$$

This follows since each dimension k facet appears $n-k$ times as a dimension k facet of distinct corank 1 subalgebras. This lemma allows us to compute the number of facets of any particular affine associahedron inductively.

Conjecture 2.2.22. *Each affine associahedron is topologically a sphere.*

This conjecture is known to be true in the type-A cases, see [34]. One may also check it case-by-case for the exceptional types.

We will now compute a uniform closed form expression for the number of vertices (number of clusters) of an affine associahedron.

Theorem 2.2.23. *The number of distinct clusters in an affine cluster algebra up to the action of $\langle\gamma\rangle$ is given by*

$$|C_0(\mathcal{A})| = \frac{2}{\chi(\mathcal{A})} \prod_i \binom{2n_i - 1}{n_i} \tag{2.14}$$

We will prove this theorem in the simply laced cases. Each of the exceptional cases can be computed inductively by Lemma 2.2.21. The non-simply laced cases have similar proofs to the one for $\widetilde{D}_n$ shown here.

First we review some facts about the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$, and the middle binomial coefficients $B_i = \binom{2i}{i}$ that will be useful in proving this counting formula. Let $C(x) = \sum_{i=0}^{\infty} C_i x^i$ and $B(x) = \sum_{i=0}^{\infty} B_i x^i$ be the generating functions for the Catalan numbers and middle binomial coefficients respectively. Then we have the following identities that hold wherever the sums converge.

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}, \quad 1 - 2xC(x) = \sqrt{1-4x} \tag{2.15}$$

$$(1-2xC(x))^{-1} = (1-4x)^{-1/2} = B(x) = \sum_{i=0}^{\infty} \binom{2i}{i} x^i \tag{2.16}$$

$$2(1-4x)^{-3/2} = \sum_{i=1}^{\infty} i \binom{2i}{i} x^{i-1} \tag{2.17}$$

It will also be helpful to define the truncated generating function $C_{\lfloor k \rfloor}(x) = \sum_{i=0}^{k-1} C_i x^i$.

We are now ready to consider the $A_{p,q}$ case. Let $A_{p,q}$ be the number of clusters in and $A_{p,q}$ cluster algebra up to γ. In this case the formula for the number of distinct clusters simplifies to:

$$A_{p,q} = \frac{pq}{2(p+q)}\binom{2p}{p}\binom{2q}{q} \tag{2.18}$$

Proof of Theorem 2.2.21 for $\widetilde{A}_n$. In Lemma 2.2.24 we establish the recurrence $A_{p,q} = 2\sum_{i=0}^{p-1} C_i A_{p-i,q} + qC_{p+q}$. Then for each q, let $A_q(x) = \sum_{i=1}^{\infty} A_{i,q} x^{i+q}$. The recurrence corresponds to the following equation of generating functions:

$$A_q(x) = 2xC(x)A_q(x) + qx\big(C(x) - C_{\lfloor q \rfloor}(x)\big) \tag{2.19}$$

Solving for $A_q(x)$ gives

$$A_q(x) = \frac{qx\big(C(x) - C_{\lfloor q \rfloor}(x)\big)}{1 - 2xC(x)} \tag{2.20}$$

In Lemma 2.2.25 we compute the powers series expansion of the right hand side is:

$$\frac{2x(C(x) - C_{\lfloor q \rfloor}(x))}{1 - 2xC(x)} = \sum_{i=1}^{\infty} \frac{i}{i+q}\binom{2i}{i}\binom{2q}{q} x^{i+q}. \tag{2.21}$$

As $A_{p,q}$ is the coefficient of x^{p+q} this means that $A_{p,q} = \frac{p}{p+q}\binom{2p}{q}\binom{2q}{q}$ as needed. $\square$

Lemma 2.2.24. $A_{p+1,q} = 2\sum_{i=0}^{p-1} C_i A_{p-i,q} + qC_{p+q}$.

Proof. We can obtain this recurrence by partitioning the set of triangulations by the triangle that contains the edge between o_1 and o_2 on the outer boundary. The third vertex of the triangle can either be on the outer or inner boundary. If the third vertex is some o the edges can either go clockwise or counterclockwise around the center. In either case it splits the annulus into a polygon with $i + 2$ sides and an annulus with $p - i$ outer marked points and q inner marked points. The triangulations of the polygon are fixed by γ and there are C_i ways to triangulate an $i + 2$ gon. So there are $2\sum_{i=0}^{p-1} C_i A_{p-i,q}$ possible triangulations where the third vertex is on the outer boundary component.

If the third vertex is on the inside there is only one possible triangle up to γ. Once this triangle is picked, it leaves a $p + q + 2$ sided polygon regardless of which of the q possible points we choose. So there are qC_{p+1} ways in this case. See Figure 2.12 for a visual of all three cases.

□

Lemma 2.2.25.

$$\frac{2x(C(x) - C_{\lfloor q \rfloor}(x))}{1 - 2xC(x)} = \sum_{i=1}^{\infty} \frac{i}{i+q} \binom{2i}{i} \binom{2q}{q} x^{i+q}. \tag{2.22}$$

Proof. In order to determine the coefficients of this power series we will examine

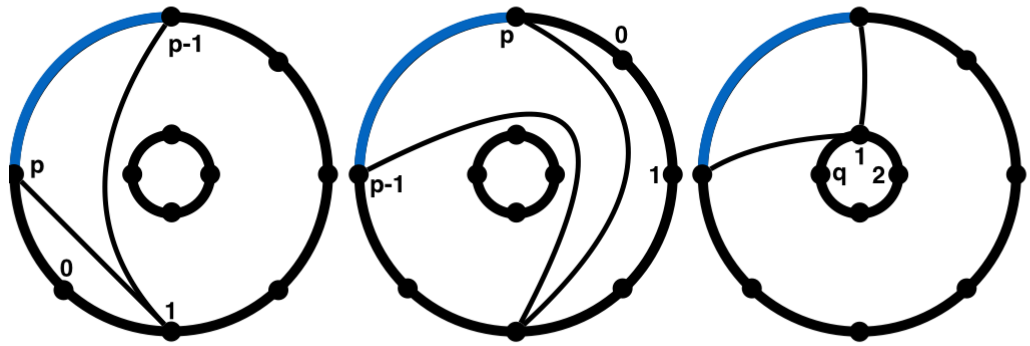


Figure 2.12: All kinds of triangles including the blue edge up to the action of the mapping class group.

the power series associated with the following integral.

$$I_q(x) = \int_0^x 2z^q(1-4z)^{-3/2}dz \tag{2.23}$$

We will evaluate I_q in two different ways. First, notice the integrand has a power series expansion given by equation 2.17. By integrating this power series we find that:

$$I_q(x) = \sum_{i=1}^{\infty} \frac{i}{i+q}\binom{2i}{i}x^{i+q} \tag{2.24}$$

Second, we use the standard calculus method of substitution to find that

$$I_q(x) = R(x)(1-4x)^{-1/2} - R(0) \tag{2.25}$$

where $R(x)$ is some polynomial of degree q.

We claim $R(x) = \binom{2q}{q}^{-1}(1-2xC_{\lfloor q \rfloor}(x))$. We verify this claim in the following two steps.

First, by comparing the two different power series representations of I_q obtained in equations 2.24 and 2.25, we may see that $R(x)(1-4x)^{-1/2}$ must have coefficient zero on x^i in its power series for $1 \leq i \leq q$. The only polynomials of degree q which we can multiply $(1-4x)^{-1/2}$ and achieve this are constant multiples of $(1-2xC(x))_{\lfloor q+1 \rfloor}$ since $1-2xC(x)$ is the inverse of $(1-4x)^{-1/2}$. Thus we have $R(x) = R(0)(1-2xC_{\lfloor q \rfloor}(x))$.

Now we may evaluate $R(0)$ by comparing the terms x^{q+1} terms of each of the power series representations. From equation 2.24, we have the $q+1$ term is $\frac{2}{1+q}x^{q+1}$. From equation 2.25, we find that the $q+1$ term is

$$R(0)\left(\binom{2q}{q} - 2\sum_{i=1}^{q} C_{i-1}\binom{2(q+1-i)}{q+1-i}\right)x^{q+1} = R(0)(2C_q)x^{q+1} \tag{2.26}$$

since $1-2xC(x)$ is the inverse power series of $\sum_{i=0}^{\infty}\binom{2i}{i}x^i$. Thus we find that

$$R(0) = \binom{2q}{q}^{-1}. \tag{2.27}$$

Finally, multiplying through by $\binom{2q}{q}$, we obtain the equation

$$\sum_{i=1}^{\infty} \frac{i}{i+q}\binom{2i}{i}\binom{2q}{q}x^{i+q} = \frac{1-2xC_{\lfloor q \rfloor}(x)}{\sqrt{1-4x}} - 1 = \frac{2x(C(x)-C_{\lfloor q \rfloor}(x))}{1-2xC(x)}. \tag{2.28}$$

□

Next we will show a similar proof for the $\widetilde{D}_n$ case. We will simply write $\widetilde{D}_n$ for the number of tagged triangulations of a twice punctured disk with $n-2$ marked points on the boundary. As before we build on the combinatorics in the finite case. Recall that $D_n = \frac{3n-2}{n}\binom{2(n-1)}{n-1}$ is the number of tagged triangulations of a once punctured disk with n marked points. For notational convenience let $D_0 = 1$. This lets us define the generating function $D(x) = \sum_{i=0}^{\infty} D_i x^i$

In this case the statement of Theorem 2.2.21 becomes:

$$\widetilde{D}_n = 9(n-2)\binom{2(n-2)}{(n-2)}, n \geq 3 \tag{2.29}$$

Proof of Theorem 2.2.21 for $\widetilde{D}_n$. In Lemma 2.2.26 we show that $\widetilde{D}_{n+1} = 2\sum_{i=0}^{n-3} C_i \widetilde{D}_{n-i} + 2\sum_{j=0}^{n} D_j D_{n-j}$. Let $\widetilde{D}(x) = \sum_{i=3}^{\infty} \widetilde{D}_i x^i$ be the generating function for $\widetilde{D}_i$. The recurrence above becomes:

$$\widetilde{D}(x) = 2xC(x)\widetilde{D}(x) + 2x(D(x)^2 - 1 - 2x) \tag{2.30}$$

Again solving for $\widetilde{D}(x)$ we find

$$\widetilde{D}(x) = \frac{2x(D(x)^2 - 1 - 2x)}{1 - 2xC(x)} \tag{2.31}$$

We can see easily that $D(x) = 3xB(x) - 2xC(x) + 1 = 3xB(x) + B^{-1}(x)$. Thus the previous equation becomes

$$\widetilde{D}(x) = \frac{2x(9x^2B^2(x) + B^{-2}(x) + 6x - 1 - 2x)}{1 - 2xC(x)} \tag{2.32}$$

and using the fact that $B^2(x) = \frac{1}{1-4x}$ (Equation 2.16) we have

$$\widetilde{D}(x) = 18x^3(1-4x)^{-3/2} = \sum_{i=3} 9(i-2)\binom{2(i-2)}{(i-2)}x^i \tag{2.33}$$

as desired. □

Lemma 2.2.26. $\widetilde{D}_{n+1} = 2\sum_{i=0}^{n-3} C_i\widetilde{D}_{n-i} + 2\sum_{j=0}^{n} D_jD_{n-j}$

Proof. As in the $\widetilde{A}_n$ case we partition the triangulations based on the triangle containing a fixed boundary edge. In this case there are six cases up to a full twist around both punctures (Figure 2.13). The first two cases correspond to triangles with third vertex on the boundary with edges going around both punctures (clockwise or counter clockwise). In either case the triangle splits the region into a i sided polygon and a twice punctured disk with $n-i$ marked points. This covers the first summation in the recurrence.

The next two cases correspond to triangle where the edges go between the punctures. If we label the $n-2$ marked points 1 to $n-1$, the triangle between the punctures going to vertex j splits the region into a punctured disk with j marked points and one with $n-j$ marked points. This covers the terms $2\sum_{j=1}^{n-1} D_jD_{n-j}$.

The final two cases are the triangles with endpoint on a puncture. Up to the full twist there is only one way to reach each puncture. There is an additional tagged triangulation in each case. In any of these cases the remaining region is a disk with n marked points. Since we took $D_0 = 1$ we can write the number of triangulations

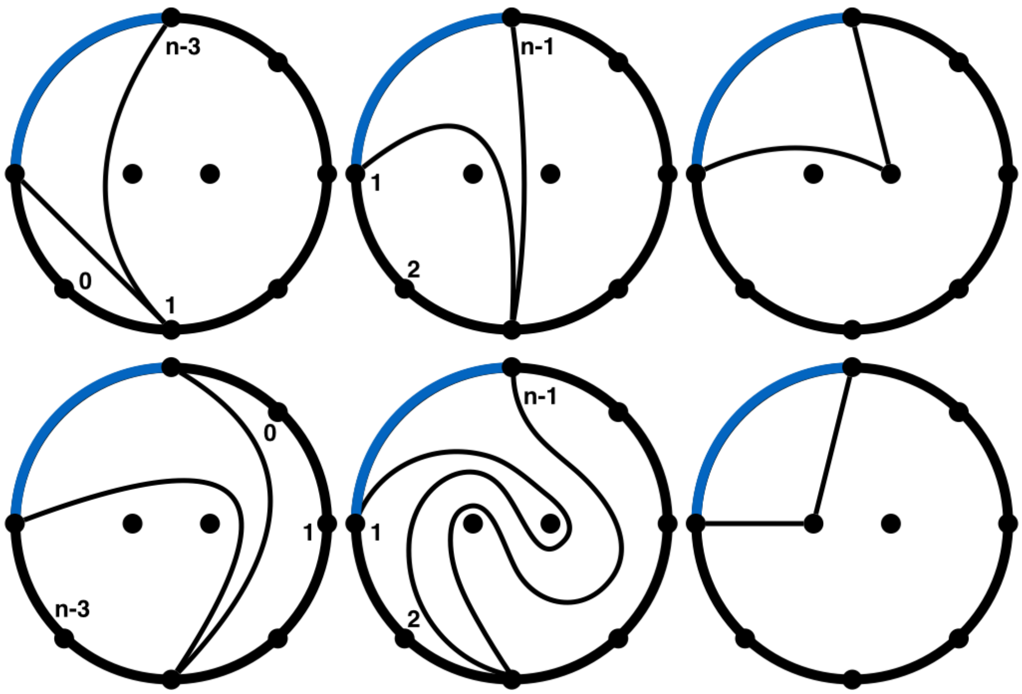


Figure 2.13: All kinds of triangles including the blue edge up to the action of the mapping class group.

in this case as D_0D_n and D_nD_0 covering the missing terms in the second summation of the recurrence.

□

2.3 Doubly Extended Cluster Algebras

In this section we consider $Q = T_{\mathbf{n},\mathbf{w}}$ to be of doubly-extended type, i.e we have $\chi = 0$. Let $\mathcal{A}$ be the cluster algebra associated to Q. There are only finitely many possibilities for $\mathbf{n}, \mathbf{w}$ with $\chi = 0$ listed in Figure 2.14. Other than $A_1^{(1,1)}$, which has to be treated separately, only $D_4^{(1,1)}$ is associated to a surface (the four punctured sphere).

Type	$\mathbf{n}$	$\mathbf{w}$	$\|N\|$	$ord(r)$	dual
$A_1^{(1,1)}$	N/A	N/A	1	1	self
$D_4^{(1,1)}$	$(2,2,2,2)$	$(1,1,1,1)$	196	2	self
$E_6^{(1,1)}$	$(3,3,3)$	$(1,1,1)$	54	3	self
$E_7^{(1,1)}$	$(4,4,2)$	$(1,1,1)$	16	4	self
$E_8^{(1,1)}$	$(6,3,2)$	$(1,1,1)$	6	6	self
$BC_1^{(4,1)}$	(2)	(4)	1	1	$BC_1^{(4,4)}$
$B_2^{(2,1)}$	$(2,2)$	$(2,2)$	4	2	self
$BC_2^{(4,2)}$	$(2,2)$	(BC-Type)	2	2	self
$B_3^{(1,1)}$	$(2,2,2)$	$(1,1,2)$	24	2	$C_3^{(2,2)}$
$F_4^{(1,1)}$	$(3,3)$	$(1,2)$	3	3	$F_4^{(2,2)}$
$F_4^{(2,1)}$	$(4,2)$	$(2,1)$	4	4	self
$G_2^{(1,1)}$	$(2,2)$	$(1,3)$	2	2	$G_2^{(3,3)}$
$G_2^{(3,1)}$	(3)	(3)	3	3	self

Figure 2.14: All possible values of $T_{\mathbf{n},\mathbf{w}}$ that result in double extended cluster algebras.

We will not consider the A or BC cases for the first part of this section, and treat them separately later. Since our $T_{\mathbf{n},\mathbf{w}}$ quivers always have weight 1 middle nodes, we will only construct quivers for the types on the left hand side of the table in Figure 2.14. The types on the right hand side are dual to types with $T_{\mathbf{n},\mathbf{w}}$ quivers.

2.3.1 Structure of the Cluster Modular Group

Let Γ be the cluster modular group of $\mathcal{A}$. Let $Q' = T'_{\mathbf{n},\mathbf{w}}$ be the underlying affine-type quiver of the doubly extended type quiver, Q. Let s be the source-sink mutation path on Q', $\chi' = \chi(Q')$ and arrange that $n_1 = \max(n_i)$ and that w_1 is minimal if there are multiple tails of the same maximal length. It is easy to verify in each case that $s^{(\chi' n_1)^{-1}}$ returns to an isomorphic quiver. Thus $\delta = (s^{(\chi' n_1)^{-1}}, \mathrm{id}) \in \Gamma$.

Theorem 2.3.1. *Γ is generated by Γ_T and δ.*

Proof. This is checked in a case by case way for each of the simply-laced doubly extended cluster modular groups. Most of these groups have been computed elsewhere. Fraser, [22], has presentations for the E_7 and E_8 cases using the Grassmannian cluster algebra structures of $\mathrm{Gr}(4,8)$ and $\mathrm{Gr}(3,9)$ respectively. We note that our notion of the cluster modular group does not include arrow reversing quiver automorphisms, so our groups are the orientation preserving subgroups of his.

Its a simple matter to check that each of Fraser's generators can be written with the above elements. For example Fraser's presentation of $\Gamma_{E_8^{(1,1)}}$ is

$$\langle \rho, P, t, : \rho^3 = P^2 = t^2, \rho^9 = 1, t\rho = \rho t, tP = Pt\rangle. \tag{2.34}$$

In our notation

$$\rho = r\delta\tau_1, \quad P = r^2\delta\tau_1\delta, \quad t = r \tag{2.35}$$

where r is the reddening element.

Fraser's presentation of the cluster modular group for $E_7^{(1,1)}$ is

$$\langle \sigma_1, \sigma_2, \sigma_3, t | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \quad \sigma_1\sigma_3 = \sigma_3\sigma_1, \tag{2.36}$$

$$\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 = (\sigma_3\sigma_2\sigma_1)^8 = 1, \quad (\sigma_3\sigma_2\sigma_1)^4 = t^2, \quad t\sigma_i = \sigma_i t\rangle. \tag{2.37}$$

In our presentation we have

$$\sigma_1 = \tau_1 \quad \sigma_2 = r\delta \quad \sigma_3 = \tau_2 \quad t = r. \tag{2.38}$$

The E_6 case is new and we have computed it using Remark 1.2.7 and Theorem 2.3.4 below. It has the following presentation:

$$\langle \tau_1, \tau_2, \tau_3, \sigma_{23}, \omega, \delta | \tau_i\tau_j = \tau_j\tau_i,\ \tau_i^3 = \tau_j^3 = \gamma,$$
$$\sigma_{23}^2 = 1,\ \omega^3 = 1,\ \sigma\omega = \omega^{-1}\sigma,\ \tau_2 = \omega\tau_1\omega^{-1},\ \tau_3 = \omega\tau_2\omega^{-1},$$
$$\tau_1\delta\tau_1 = \delta\tau_1\delta,\ (\tau_1\delta)^3 = r^2\sigma_{23}\rangle$$

where $r = \tau_1\tau_2\tau_3\gamma^{-1}$ is the reddening element. The automorphism group of $T_{3,3,3}$ is generated by σ_{23} that swaps tails 2 and 3 and ω which rotates all three tails. The non-simply laced cases follow from Remark 2.1.3. □

To best describe the relations between δ and the other generators of Γ, it will be helpful to recall some basic properties of the rank 2 Artin-Tits braid groups of type A_2, B_2 and G_2. The groups $\mathcal{B}(X_2)$ have the presentation

$$\mathcal{B}(A_2) = \{a, b | aba = bab\} \tag{2.39}$$

$$\mathcal{B}(B_2) = \{a, b | abab = baba\} \tag{2.40}$$

$$\mathcal{B}(G_2) = \{a, b | ababab = bababa\} \tag{2.41}$$

Remark 2.3.2. *$\mathcal{B}(A_2)$ is generally known as the braid group on 3 strands, $\mathcal{B}_3$. The* *center, $\mathcal{Z}$ of these groups is an infinite cyclic group generated by $z = ababab$, $z = abab$ and $z = ababab$ for $\mathcal{B}(A_2)$, $\mathcal{B}(B_2)$, $\mathcal{B}(G_2)$ respectively. We have an isomorphism*

$$\mathcal{B}(A_2)/\mathcal{Z} \simeq \mathrm{PSL}(2, \mathbb{Z}) \tag{2.42}$$

If we let $X_2(k) = A_2, B_2,$ or G_2 if $k = 1, 2,$ or 3 respectively, then the subgroup of $\mathcal{B}(A_2)$ generated by $\{a, b^k\}$ is isomorphic to $\mathcal{B}(X_2(k))$

Claim 2.3.3. *For each i we have a map $\psi_i : \mathcal{B}(X_2(n_1 w_i/n_i)) \to \Gamma$ given by $\{a, b\} \to \{\tau_i, r\delta\}$. Moreover, the image of the element z is shown in Table 2.15.*

Proof. In each case it suffices to check the images satisfy the braid relations. □

Type	$i=1$	$i=2$	$i=3$
$D_4^{(1,1)}$	id	id	id
$E_6^{(1,1)}$	$r^2\sigma_{23}$	$r^2\sigma_{13}$	$r^2\sigma_{12}$
$E_7^{(1,1)}$	r^2	r^2	$r\sigma_{12}$
$E_8^{(1,1)}$	r^2	r^4	r
$B_2^{(2,1)}$	r	r	-
$B_3^{(1,1)}$	id	id	$r\sigma_{12}$
$F_4^{(1,1)}$	r^2	r	-
$F_4^{(2,1)}$	r	r	-
$G_2^{(1,1)}$	id	r	-
$G_2^{(3,1)}$	r	-	-

Figure 2.15: Images of the central element $\psi_i(z) = c$ for the group homomorphisms of Claim 2.3.3.

Let $N = \Gamma_\tau^\circ \rtimes \mathrm{Aut}(Q)$ where Γ_τ° was defined in Remark 2.1.13 by the following exact sequence:

$$1 \to \Gamma_\tau^\circ \to \Gamma_\tau \to \mathbb{Z} \to 1 \tag{2.43}$$

Theorem 2.3.4. *The following sequence is exact:*

$$1 \to N \to \Gamma \to \mathcal{B}(X_2(w_1))/\mathcal{Z} \to 1 \tag{2.44}$$

Proof. First, it is necessary to check that N is a normal subgroup, which we may do for each of the four simply laced cases and fold to get the non simply laced cases. To see that the quotient is as described, we only need to show that the induced map

$$\mathcal{B}(X_2(w_1))/\mathcal{Z} \to \Gamma/N \tag{2.45}$$

from Claim 2.3.3 is an isomorphism. Since τ_1 has the smallest possible $\mathbb{Z}$ component in Γ_τ and $\mathrm{Aut}(Q) \subset N$, τ_1 generates $\Gamma_\tau \rtimes \mathrm{Aut}(Q)/N \simeq \mathbb{Z}$. Thus τ_1 and δ generate Γ/N. Therefore, we only need to check that the only relations come from those in the braid group modulo its center. In the simply laced cases, this follows by checking that the only relations between δ and τ_1 is $(\delta\tau_1)^3 = \mathrm{id}$.

We check this by first seeing that it is true in the $D_4^{(1,1)}$ case, since this algebra is associated with a 4-punctured sphere and δ and τ_1 correspond to elements of $\mathrm{PSL}(2,\mathbb{Z})$ as a quotient group of the mapping class group.

Then, we can check that the maps of cluster modular groups induced by folding operations of Figure 2.16 preserve this subgroup faithfully. Since folding realizes the cluster modular group of the folded algebra as a subquotient of the unfolded algebra, we must check that no extra relations are added and that δ and τ_1 appear in this subquotient.

Let $\mathcal{A} \to \mathcal{B}$ be any folding of doubly extended type cluster algebras. Let $n = n_1(\mathcal{A}), w = w_1(\mathcal{A})$ and $m = n_1(\mathcal{A}), z = w_1(\mathcal{A})$ be the length and weights of the first tail of $T_{\mathbf{n},\mathbf{w}}$ quivers representing seeds of these algebras. Let τ, η be the twist elements of the first tails and δ, ε be the extra generators in the modular groups of $\mathcal{A}$ and $\mathcal{B}$ respectively.

The double arrows corresponding to Langlands dual obviously preserve the subgroup. The solid edges, corresponding to folding the $T_{\mathbf{n},\mathbf{w}}$ quivers directly, only quotient by elements in N, which are zero in Γ/N. This follows since we fold by an automorphism of the $T_{\mathbf{n},\mathbf{w}}$ quiver which are contained in N.

Furthermore, we clearly have that $\delta = \varepsilon$ in the standard folding case. Finally, we see that if $w = z$ we have that τ directly descends to the cluster modular group of the folded algebra. Otherwise, z tails of length n are folded, and we have that η is equivalent to successive twists around each of these unfolded tails. In the quotient $\Gamma_{\mathcal{A}}/N$ we have that successive twists around z tails of the same length is equal to τ^z. Thus Remark 2.3.2 proves the theorem.

Then the only nonstandard folding (dashed arrows) we need to check are $E_7^{(1,1)} \dashrightarrow C_3^{(2,2)}$, $E_8^{(1,1)} \dashrightarrow G_2^{(3,3)}$, $E_8^{(1,1)} \dashrightarrow F_4^{(2,2)}$. See Figure 2.17 to see the folds in each case. One checks for each of these cases that these automorphisms are also contained in N. We will dualize each of these folded algebras so that we may compare their cluster modular groups using the presentation coming from $T_{\mathbf{n},\mathbf{w}}$ quivers.

We start with $E_8^{(1,1)} \dashrightarrow G_2^{(3,3)} \Leftrightarrow G_2^{(1,1)}$. We have a path of valid folds and unfolds from $D_4^{(1,1)}$ to $G_2^{(1,1)}$ so we know their are no extra relations in $G_2^{(1,1)}$. Thus it suffices to write δ^E and τ_1^E in terms of δ^G and τ_1^G. Let

$$P = (2_2 1_4 1_5 1_3 1_2 1_4 2_3 N_1 N_\infty) \tag{2.46}$$

be a path of mutations from $T_{6,3,2}$ to the triangular quiver shown in Figure 2.17a. Then

$$\delta^E = P\delta^G\tau^G(\delta^G)^{-2}\tau^G P^{-1} \tag{2.47}$$

$$\tau_1^E = P\tau^G P^{-1}. \tag{2.48}$$

By replacing P with $P' = P(\tau^G)^2$ and using braid relations, we can see that

$$\delta^E = P'\delta^G P'^{-1} \tag{2.49}$$

$$\tau_1^E = P'\tau^G P'^{-1}. \tag{2.50}$$

The next case to consider is $E_8^{(1,1)} \dashrightarrow F_4^{(2,2)} \Leftrightarrow F_4^{(1,1)}$. Once again if we can write τ_1^F and δ^F in terms of the generators τ_1^E and δ^E any extra relations in $F_4^{(1,1)}$ would descend to relations in $E_8^{(1,1)}$ which we just showed didn't have extra relations. A simple computation shows that

$$P = (1_6 3_2 1_5 1_4 1_6 1_3 1_2 N_1) \tag{2.51}$$

is a path from $T_{6,3,2}$ to the quiver shown in Figure 2.17b. Then

$$\tau_1^F = P^{-1}\tau_1^E P \tag{2.52}$$

$$\delta^F = P^{-1}(\tau_1^E)^{-1}(\delta^E)^{-3}(\tau_1^E)^{-1}(\delta^E)^{-1}(\tau_1^E)P = P^{-1}. \tag{2.53}$$

Again using braid relations we can see in the quotient that $\delta^F = P^{-1}\delta^E P$.

The final case is $E_7^{(1,1)} \dashrightarrow C_3^{(2,2)} \Leftrightarrow B_3^{(1,1)}$. Here we have a path of valid folds and unfolds $D_4^{(1,1)} \to B_3^{(1,1)}$. So all that remains is to write the generators for $E_7^{(1,1)}$, τ^E and δ^E in terms of the generators for $B_3^{(1,1)}$, δ^B and τ^B. Let

$$P = (2_4 1_4 2_3 2_2 1_3 1_2 N_1). \tag{2.54}$$

Then

$$\delta^E = P\delta^B P^{-1} \tag{2.55}$$

$$\tau_1^E = P\tau_1^B P^{-1}. \tag{2.56}$$

□

The following commutative diagram summarises the structure of the cluster modular groups of doubly extended cluster algebras in each case where $w_1 = 1$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{Z} & \hookrightarrow & \mathcal{B}_3 & \twoheadrightarrow & \mathrm{PSL}(2,\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow {\scriptstyle z \mapsto c} & & \downarrow & & \| & & \\ 1 & \longrightarrow & N & \hookrightarrow & \Gamma & \twoheadrightarrow & \mathrm{PSL}(2,\mathbb{Z}) & \longrightarrow & 1 \end{array} \tag{2.57}$$

Corollary 2.3.5. *Cluster modular groups are generated by "cluster Dehn twists" of Ishibashi, [35].*

Proof. Consider the twist generators $\tau_i \in \Gamma_\tau$. From Theorem 2.1.11, we saw that $\tau_i^{n_i} = \gamma^{w_i}$. In the surface cases γ is a Dehn twist in the surface cases, and in the exceptional cases is a cluster Dehn twist.

Furthermore, the element $\delta^{n_1} = s^{1/\chi}$ can be seen to be conjugate to γ in the following way. First by freezing nodes 1_{n_1} and N_∞ we are left with the corresponding finite type quiver. Let g be the sources sinks mutation pattern on this finite type quiver and let h be the order of this element. Then we have $\alpha = \{g^{h/2}, (1_{n_1} N_\infty)\} \in \Gamma$ and $\alpha\gamma\alpha^{-1} = \delta^{n_1}$. Thus δ is a cluster Dehn twist.

Finally, we see that the elements of $\mathrm{Aut}(Q)$ each are periodic elements akin to periodic mapping class group elements. It is possible to generate these elements in each case using cluster Dehn twists. The images of central element, c, for various maps from braid groups is always generated by the cluster Dehn twists τ_i and δ. We can see in table 2.15 that quiver automorphisms can be obtained in case from this central element. We note that in the $D_4^{(1,1)}$ case we obtain $\sigma_{12} = r(\tau_3\tau_4 r\delta)^2$, as can be seen via the folding $D_4^{(1,1)} \to B_3^{(1,1)}$

□

2.3.2 Other Cases

In the previous section, we ignored the A and BC cases. These cases are simpler, so we simply show their cluster modular groups.

$$\Gamma_{A_1^{(1,1)}} = \mathcal{B}(A_2)/\mathcal{Z} = \mathrm{PSL}(2,\mathbb{Z}) \tag{2.58}$$

$$\Gamma_{BC_1^{(4,1)}} = \Gamma_{BC_1^{(4,1)}} = \mathcal{B}(B_2)/\mathcal{Z} = \mathbb{Z} * \mathbb{Z}_2 \tag{2.59}$$

$$\Gamma_{BC_2^{(4,2)}} = \mathcal{B}(B_2)/\mathcal{Z} \times \mathbb{Z}_2 = (\mathcal{Z} * \mathbb{Z}_2) \times \mathbb{Z}_2 \tag{2.60}$$

2.3.3 Special Quotients and Counting Clusters

We will construct a special finite quotient of the cluster modular group of each of the simply laced doubly extended cluster algebras. We will use this normal subgroup to construct a finite quotient of the cluster complex and thereby construct a doubly extended generalized associahedron.

Following the ideas in the affine case, we would like to quotient Γ by $\langle\gamma\rangle$. However, $\langle\gamma\rangle$ is no longer a normal subgroup. We will now construct free normal subgroups $\mathcal{N}$, such that $\gamma^k \in \mathcal{N} \lhd \Gamma$ and $\Gamma/\mathcal{N}$ is finite group containing the normal subgroup N.

Let $n = ord(r)$ be the order of the reddening element. We can see that in the quotient $\Gamma/N = \mathrm{PSL}(2,\mathbb{Z})$, we have

$$\gamma = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \tag{2.61}$$

in each case. We denote the normal closure in Γ of the group element γ by $\mathcal{N}(\gamma)$. This is a finite index subgroup of the cluster modular group in all cases other than $E_8^{(1,1)}$ since $\mathcal{N}(\gamma)/N$ is finite index in $\mathrm{PSL}(2,\mathbb{Z})$. This group is not free in the $E_6^{(1,1)} or E_8^{(1,1)}$ cases, but $\mathcal{N}(\gamma r^2)$ and $\mathcal{N}(\gamma r^4)$ are free in these cases respectively.

Claim 2.3.6. *For $D_4^{(1,1)}$, the group $\mathcal{N}(\gamma)$ is the puncture preserving mapping class group of a four punctured sphere.*

We can verify this claim easily by seeing that γ is a Dehn twist. Thus by [25] we have that $\mathcal{N}(\gamma) \simeq F_2$. We have an exact sequence

$$1 \to \mathcal{N}(\Gamma) \to \Gamma_{D_4^{(1,1)}} \to H \to 1 \tag{2.62}$$

where H is a group of order 1152 given by an extension

$$1 \to N \to H \to S_3 \to 1. \tag{2.63}$$

Claim 2.3.7. *For $E_7^{(1,1)}$, the group $\mathcal{N}(\gamma)$ is a finite index free group. It is isomorphic to the congruence subgroup $\bar{\Gamma}(4)$ of* $\mathrm{PSL}(2,\mathbb{Z})$

We have the following diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_2 & \hookrightarrow & \mathrm{SL}(2,\mathbb{Z}) & \twoheadrightarrow & \mathrm{PSL}(2,\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & N & \hookrightarrow & \Gamma & \twoheadrightarrow & \mathrm{PSL}(2,\mathbb{Z}) & \longrightarrow & 1 \end{array} \tag{2.64}$$

The element $\gamma \in \Gamma$ is in the image of the map from $\mathrm{SL}(2,\mathbb{Z})$ and is given by the matrix $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$. The normal closure of this matrix in $\mathrm{SL}(2,\mathbb{Z})$ is the level 4 congruence subgroup $\Gamma(4)$ and is torsion free. Since γ commutes with all of N, its normal closure in Γ is isomorphic to its normal closure in $\mathrm{SL}(2,\mathbb{Z})$.

Thus we have the following diagram

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{N}(\gamma) & = & \Gamma(4) & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N & \hookrightarrow & \Gamma & \twoheadrightarrow & \mathrm{PSL}(2,\mathbb{Z}) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N & \twoheadrightarrow & \Gamma/\mathcal{N}(\gamma) & \twoheadrightarrow & S_4 & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
\cdot \qquad (2.65)
$$

Claim 2.3.8. *The normal closure $\mathcal{N}(\gamma r^2)$ is a finite index free group of the cluster modular group of $E_6^{(1,1)}$. It is isomorphic to the congruence subgroup $\bar{\Gamma}(3)$ of $\mathrm{PSL}(2,\mathbb{Z})$.*

We have the following diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{N}(\gamma r^2) & = & \Gamma(3) & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N & \hookrightarrow & \Gamma & \twoheadrightarrow & \mathrm{PSL}(2,\mathbb{Z}) & \longrightarrow & 1 \cdot \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N & \twoheadrightarrow & \Gamma/\mathcal{N}(\gamma r^2) & \twoheadrightarrow & A_4 & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{2.66}$$

Claim 2.3.9. *In the $E_8^{(1,1)}$ case, the normal closure $\mathcal{N}(\gamma r^4)$ is a free group, but is not of finite index. The groups $\mathcal{N}_k = \mathcal{N}(\gamma r^4, (r\delta)^k(\tau)^k)$ are free groups of index* $36, 108$ *and* 144 *for* $k = 1, 2, 3$.

The images of the groups $\mathcal{N}_k$ in $\mathrm{PSL}(2,\mathbb{Z})$ are normal subgroups of index $6, 18$ and 24 for $k = 1, 2, 3$. We denote these groups and their respective quotients by $F_{k,6/k}$ and $G_{k,6/k}$, see [36].

We have the following diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{N}_k & = & G_{k,6/k} & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}_6 & \hookrightarrow & \Gamma & \twoheadrightarrow & \mathrm{PSL}(2,\mathbb{Z}) & \longrightarrow & 1 \cdot \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}_6 & \twoheadrightarrow & \Gamma/\mathcal{N}_k & \twoheadrightarrow & F_{k,6/k} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{2.67}$$

While we have not explicitly described them, there are analogous finite index normal free subgroups of each of the non simply laced doubly extended cluster modular groups. These can be understood by folding the simply laced algebras. We can defined doubly extended generalized associahedra by first quotienting the cluster complexes by the action of these subgroups and then dualizing.

2.3.4 Counting Facets in Doubly Extended Generalized Associahedra

We can compute the total number of cluster variables and clusters in the quotient of doubly extended cluster complexes. The number of cluster variables is equal to the number of corank 1 subalgebras of our given algebra and is equal to the number of codimension 1 facets of the generalized associahedra.

First we will count the number of cluster variables in each coset of the action of the normal subgroup N. This count has a uniform description in each case. Since the quotient modular groups are extensions of finite groups by N, we can count the total simply by multiplying by the size of the corresponding finite group.

Theorem 2.3.10. *The number of distinct cluster variables in each coset of the action of N on the cluster complex of $\mathcal{A}$ is given by:*

$$d\frac{w_1}{n_1}(\sum(n_i-1)n_i) \tag{2.68}$$

where $d=1$ unless $\mathcal{A}$ is self dual, in which case we have $d=2$.

Proof. In each of the finitely many simply laced cases one can check that every cluster variable appears in a $T_{\mathbf{n},\mathbf{w}}$ quiver not on the double edge. This is a finite computation, as we only have to check each location in each quiver isomorphism class has a mutation path to a quiver with a double edge. For most cases this requires extensive computational aid[2]. However $D_4^{(1,1)}$ only has a 4 isomorphism classes so we can show the full computation in Figure 2.18.

Then it suffices to count how many different variables can appear on the tails of the $T_{\mathbf{n},\mathbf{w}}$ quivers up to the action of N. Recall that $N = \Gamma_\tau^\circ \rtimes \mathrm{Aut}(T_{\mathbf{n},\mathbf{w}})$. Since the action automorphism group does not generate new cluster variables, we only need to count variables in each coset of Γ_τ°.

We have the following exact sequence

$$1 \to \Gamma_\tau^\circ \to \Gamma_\tau \to \mathbb{Z} \to 1 \tag{2.69}$$

where $\gamma \in \Gamma$ maps to $n_1 \in \mathbb{Z}$. There are $n_i(n_i - 1)$ distinct cluster variables on each tail of length n_i after applying τ_i. These variables are fixed by the action of $\gamma \in \Gamma$. Finally, since Γ_τ° is index n_1/w_1 in $\Gamma_\tau/\langle\gamma\rangle$ the theorem follows.

In the cases which are each self dual, each cluster variable appears on the tail of a $T_{\mathbf{n},\mathbf{w}}$ or its dual. Thus we simply have to multiply the count by two. □

We can now count the number of clusters in each coset of the action of N on the cluster complex. Each cluster variable corresponds to a corank 1 subalgebra of the our cluster algebra. By the proof Theorem 2.3.10, these subalgebras can always be found by freezing variables on the tails of $T_{\mathbf{n},\mathbf{w}}$ quivers. Thus, every corank 1

[2]See `https://www.math.umd.edu/~zng/notes/DoubleExtendedStructureProof/` for full computational details.

subalgebra is affine type.

Let $\mathcal{A}_{i_j}$ be the affine subalgebra obtained by freezing the tail node i_j and let C_{i_j} be the number of clusters in $\mathcal{A}_{i_j}$ up to γ. The number of clusters in each affine subalgebra in each coset of N is equal $w_1 C_{i_j}/n_1$. Then the total number of clusters in each coset is

$$\frac{1}{n}\sum_i n_i \sum_{j=2}^{n_i} C_{i_j}\frac{w_1}{n_1} \tag{2.70}$$

where n is the rank of the doubly extended cluster algebra we are considering. The factor of $1/n$ appear since each cluster appears in n corank 1 subalgebras.

We can compute the number of clusters in $\Gamma/\mathcal{N}$ by multiplying the number in the coset by the size of $(\Gamma/\mathcal{N})/N$.

More generally, we can count the number of any dimension facets on a doubly extended associahedron using the formula

$$|C_k(\mathcal{A})| = \frac{1}{n-k}\sum_{\mathcal{B}\in C^1(\mathcal{A})} C_k(\mathcal{B}) \tag{2.71}$$

of Lemma 2.2.21.

Example 2.3.11. *We will compute the number of clusters in the quotient complex of type $E_7^{(1,1)}$ by $\mathcal{N}$. By freezing nodes on a tail of length 4 we can obtain subalgebras of type $\widetilde{E}_7$, $\widetilde{D}_6 \times A_1$, $A_{2,4}\times A_2$. These have sizes* 252,000*,* 5040*, and* 1400 *respectively. There is only one node to freeze on the tail of length 2 corresponding to a subalgebra of type $A_{4,4}$ which contains* 4900 *clusters up to the action of γ. So the total number* *of clusters in each coset of N is*

$$\frac{1}{9}\left(2\cdot 4\left(\frac{252,000}{4}+\frac{5040}{4}+\frac{1400}{4}\right)+2\frac{4900}{4}\right)=\frac{65730}{9}=\frac{21910}{3}. \tag{2.72}$$

It remains to multiply the size of the quotient group S_4, 24, obtaining the final count of $175,280$.

We note that doubly extended associahedra are not generally homotopy equivalent to spheres. Let $\mathcal{A}$ be a doubly extended cluster algebra of rank $n+2$. We conjecture the following:

Conjecture 2.3.12. *The exchange complex of $\mathcal{A}$ is homotopy equivalent to S^{n-1}. The doubly extended associahedron associated with $\mathcal{A}$ is homotopy equivalent to $S^{n-1}\times S^2$ in all cases other than $E_8^{(1,1)}$ where it instead is homomorphic to $S^7\times S^1\times S^1$.*

Figure 2.20 contains the results of the counting arguments for the number of clusters in the other doubly extended cases. We include the A and BC cases, which can be done individually and are somewhat degenerate. Figure 2.21 shows the total count of codimension k subalgebras obtained by inductively counting corank 1 subalgebras.

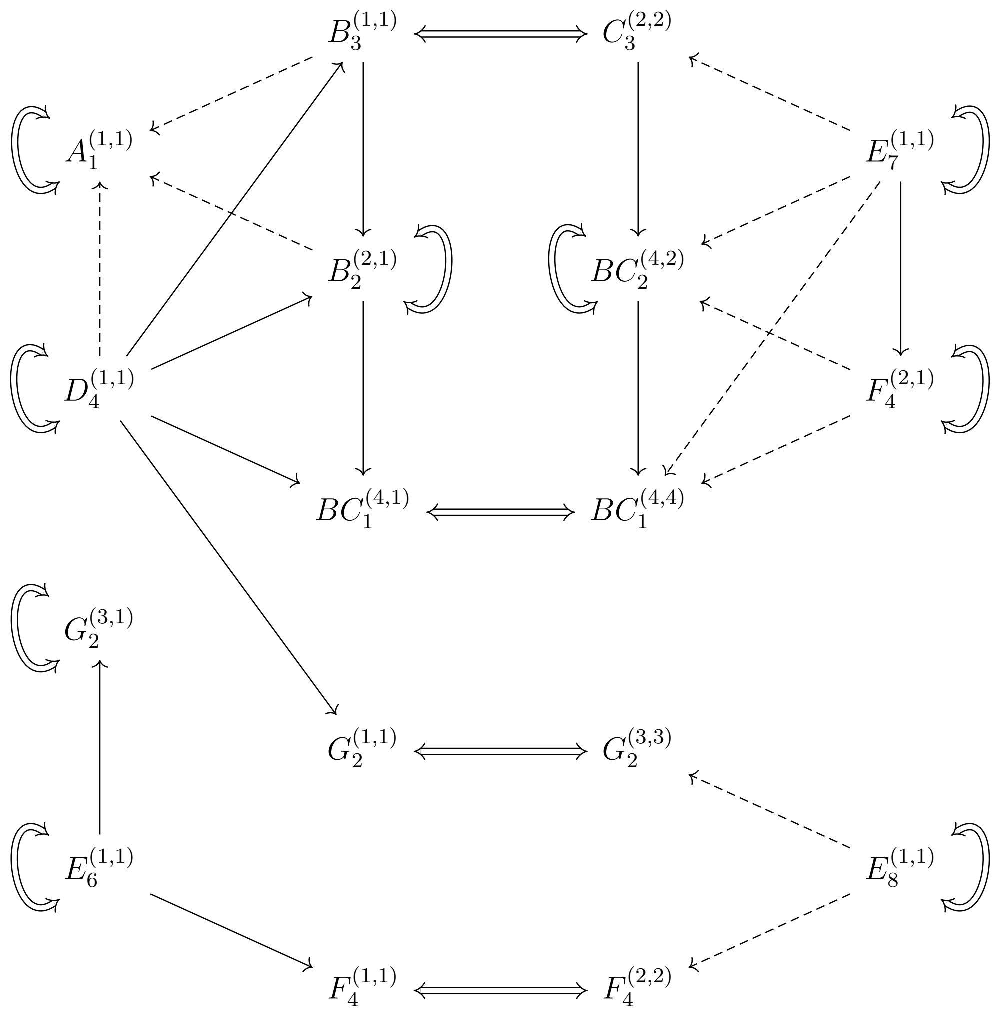
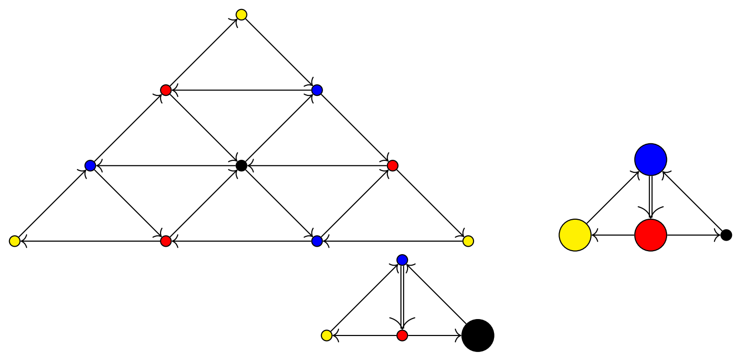
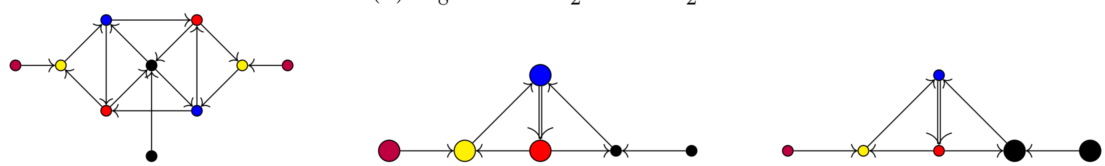


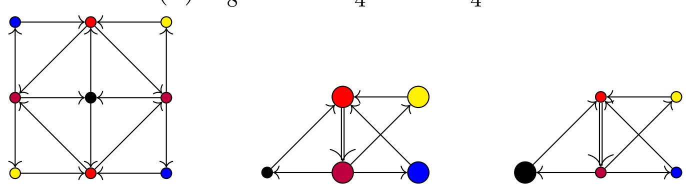
Figure 2.16: The doubly-extended family tree. The solid arrows represent folding of $T_{\mathbf{n},\mathbf{w}}$ quivers, dashed arrows are special foldings, and the double arrows represent Langland's-duality.



(a) $E_8^{(1,1)} \dashrightarrow G_2^{(3,3)} \Leftrightarrow G_2^{(1,1)}$.

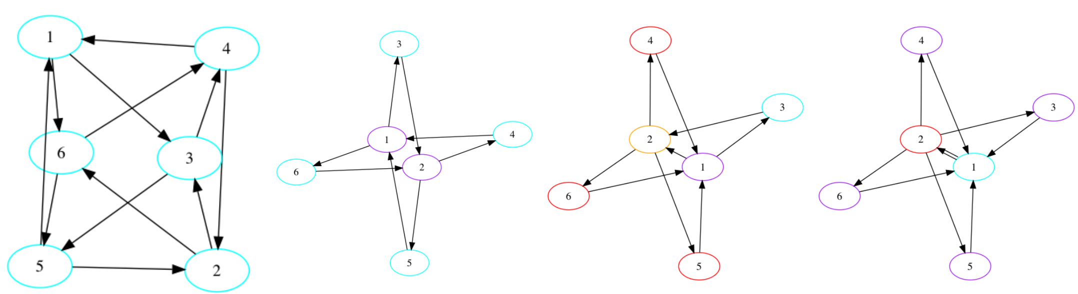


(b) $E_8^{(1,1)} \dashrightarrow F_4^{(2,2)} \Leftrightarrow F_4^{(1,1)}$.



(c) $E_7^{(1,1)} \dashrightarrow C_3^{(2,2)} \Leftrightarrow B_3^{(1,1)}$.

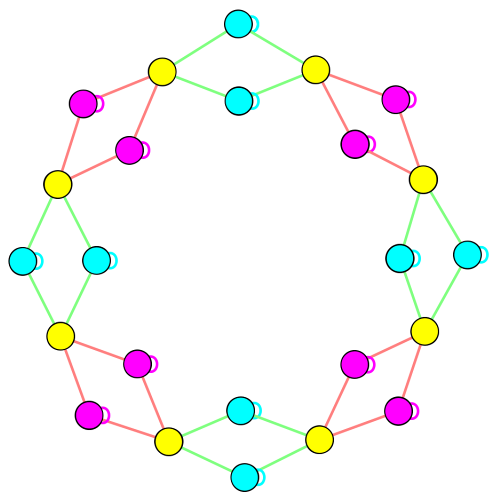
Figure 2.17: Exotic foldings of doubly extended quivers. The first folds are by the 180 degree rotational symmetry and the last is by the 3-fold rotational symmetry.



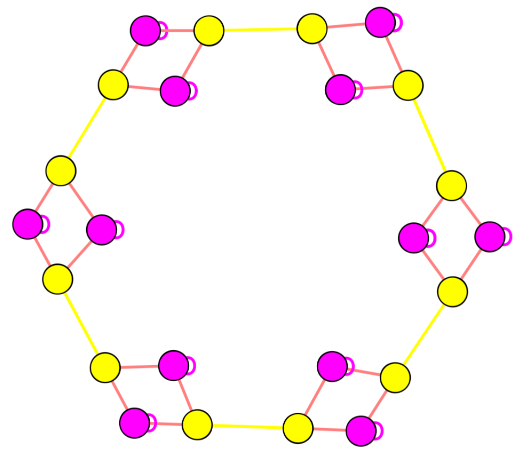
(a) Q_1. (b) Q_2. (c) Q_3. (d) Q_4.

	Q_1	Q_2	Q_3	Q_4
1	6,4,5	2,6,5,4	2,6,5,4	4,2,3,5,6
2	6,4,5	1,5,6,3	1,6,5,4	4,1,3,5,6
3	5,1,2	4,5	6,4,5	$\langle\rangle$
4	2,3,6	3,6	3	$\langle\rangle$
5	2,3,6	3,6	3	$\langle\rangle$
6	5,1,2	4,5	3	$\langle\rangle$

Figure 2.18: The four quiver isomorphism classes for $D_4^{(1,1)}$ and mutation paths so that vertex i is in a double edge quiver without mutating i.



(a) $B_2^{(2,1)}$.



(b) $G_2^{(1,1)}$.

Figure 2.19: The 1-skeleton of the doubly extended associahedra of types $B_2^{(2,1)}$ and $G_2^{(1,1)}$

Type	Number of clusters in coset of N	$\|(\Gamma/\mathcal{N})/N\|$	Number of clusters in quotient
$A_1^{(1,1)}$	1	1	1
$D_4^{(1,1)}$	72	6	432
$E_6^{(1,1)}$	1575	12	18,900
$E_7^{(1,1)}$	$\frac{21910}{3}$	24	175,280
$E_8^{(1,1)}$	34,105	6 18 24	204,630 613,890 818,520
$BC_1^{(4,1)}$	1	2	2
$B_2^{(2,1)}$	12	2	24
$BC_2^{(4,2)}$	12	2	24
$G_2^{(1,1)}$	4	6	24
$G_2^{(3,1)}$	21	3	63
$B_3^{(1,1)}$	18	6	108
$F_4^{(1,1)}$	105	12	1260
$F_4^{(2,1)}$	348	8	2784

Figure 2.20: Counting clusters in the quotient of doubly extended cluster algebras

Type	1	2	3	4	5	6	7	8	9	10
$A_1^{(1,1)}$	3	$\frac{3}{2}$	1							
$D_4^{(1,1)}$	24	192	768	1,464	1,296	432				
$E_6^{(1,1)}$	72	1,422	11,772	47,466	102,816	122,472	75,600	18,900		
$E_7^{(1,1)}$	156	4,776	53,504	288,840	857,760	1,478,400	1,474,080	788,760	175280	
$E_8^{(1,1)}$	38 114 152	1,881 5,643 7,524	28,046 84,138 112,184	196,345 589,035 785,380	763,398 2,290,194 3,053,592	177,6042 5,328,126 7,104,168	2,531,988 7,595,964 10,127,952	2,167,722 6,503,166 8,670,888	1,023,150 3,069,450 4,092,600	204,630 613,890 818,520
$BC_1^{(4,1)}$	3	3	2							
$B_2^{(2,1)}$	16	40	48	24						
$BC_2^{(4,2)}$	16	40	48	24						
$G_2^{(1,1)}$	12	36	48	24						
$G_2^{(3,1)}$	36	99	126	63						
$B_3^{(1,1)}$	18	96	244	270	108					
$F_4^{(1,1)}$	48	516	2196	4248	3780	1260				
$F_4^{(2,1)}$	112	1152	4864	9392	8352	2784				

Figure 2.21: Number of codimension k facets in the doubly extended generalized associahedra.

Chapter 3: Multiple Polylogarithm Relations

In order to study multiple polylogarithm relations we introduce a new algebraic invariant associated to a multiple polylogarithm, $\omega_{\mathbf{n}}$. This is joint work with my advisor as well as Dani Kaufman and Haoran Li. We are currently working on additional properties and characterizations of $\omega_{\mathbf{n}}$ beyond the scope of this thesis. Each multiple polylogarithm is a multi-valued function, $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})\colon \mathbb{C}^d \setminus X_d \to \mathbb{C}$. We let $\hat{U}_d \to \mathbb{C}^d \setminus X_d$ be the universal abelian cover of $\mathbb{C}^d \setminus X_d$. Then we define $\omega_{\mathbf{n}} \in \Omega^1(\hat{U}_d)$ to be a well defined one-form on the universal abelian cover.

In Section 3.2.1, we will see how these forms can be obtained from the symbol of the multiple polylogarithms via a further symmetrization operation. In Section 3.2.5, we will see these forms satisfy a simple recurrence formula that greatly speeds up the computation time of these forms. Next we describe several families of relations satisfied by $\omega_{\mathbf{n}}$. In particular, we give a closed formula for $\omega_{\mathbf{n}}(1/\mathbf{z})$ generalizing the well known standard polylogarithm relation $\mathrm{Li}_n(z) + (-1)^n \mathrm{Li}_n(1/z) = 0$ modulo products. This is critical for discussing polylogarithm relations associated to cluster algebras, as functions extracted from the cluster algebra are usually only well defined up to inversion.

In the final section we describe how to extract polylogarithm relations, which we call Q_n from the A_n cluster algebras for $n \leq 6$. Using the cluster algebra structure we are able to give evidence for the following conjecture.

Conjecture 3.0.1. *For all n odd, the signed sum $\alpha_{n+1} = \sum_{A_n \in D_n} Q_n(A_n)$ is non trivial with no depth 2 terms.*

For all n even, the corresponding sum is identically zero.

In particular α_6 is composed entirely of terms $\omega_5(x)$ or $\omega_{3,1,1}(xyz, 1/y, 1/z)$ where x, y, z are X-coordinates of D_6 and xyz is one of the two generating Casimir elements of D_6. Since $\mathrm{d}\omega_5(x) = 0$ for any x, and $\mathrm{d}\alpha_6 = 0$ this implies that the differential of all the $\omega_{3,1,1}$ terms is also 0. So this combination of terms should be integrable on the universal abelian cover and corresponds to a well defined polylogarithm function.

3.1 Universal Abelian Cover

In Section 1.3.2 we defined the "basic liftable functions" in depth d to be z_i and $1 - \prod_{r=i}^{j} z_i$ for $1 \leq i \leq j \leq d$. We saw the singularity set of a depth d polylogarithm is the zero set of the basic liftable functions. We now will define the universal abelian cover of $\mathbb{C}^d \setminus X_d$. The name "basic liftable function" will be justified as these function are lifted to the basic coordinate functions on the cover.

Definition 3.1.1. *There is a covering space of $p \colon \hat{U}_d \to \mathbb{C}^d \setminus X_d$ given by:*

$$\hat{U}_d = \{(u_i, v_{[i,j]})_{1 \leq i \leq j \leq d} | \forall i \leq j. - \prod_{r=i}^{j} e^{u_r} + e^{v_{[i,j]}} = 1\}$$

where $p(u_i, v_{[i,j]}) = (e^{u_1}, \ldots, e^{u_d})$.

Sections of the cover correspond to a choice of the branch of the logarithm around each singularity in X_d. In other words, u_i is the lift of $\mathrm{Log}(z_i)$ and $v_{i,j}$ is the lift of $\mathrm{Log}(1 - \prod_{r=i}^{j} z_j)$. Thus functions defined on $\hat{U}_d$ are well defined up to the specification of number of loops around each singularity.

Remark 3.1.2. *There is an isomorphic covering space for each choice of sign vectors* $\varepsilon_{ij} = \pm 1$ *and* $\eta_{ij} = \pm 1$ *for* $1 \leq i \leq j \leq d$*, where the relation among the u's and v's is given by* $\varepsilon_{ij} \prod_{r=i}^{j} e^{u_r} + \eta_{ij} e^{v_{[i,j]}} = 1$*. We chose* $\varepsilon_{ij} = -1$ *and* $\eta_{ij} = 1$ *so that inverting every coordinate on* $\mathbb{C}^d \setminus X_d$ *lifts to an involution of* $\hat{U}_d$*.*

Lemma 3.1.3. *The fundamental group of* $\mathbb{C}^d \setminus X_d$ *is torsion free and of rank* $n_d = 2d + \binom{d}{2}$*.*

Proof. It is simple to verify this is true for $d = 1$ where $\mathbb{C}^1 \setminus X_1$ is homeomorphic to the wedge of two circles. Thus $\pi_1(\mathbb{C}^1 \setminus X_1) = \mathbb{Z} * \mathbb{Z}$ and is torsion free of rank 2. We then prove the lemma by induction.

First, we split the singularity set X_d into three parts: the singularities that do not involve z_d, the singularities that involve only z_d, and the singularities involving both z_d and any other z_i.

$$X_d = X_{d-1} \cup \{z_d = 0, z_d = 1\} \cup \{1 - \prod_{r=i}^{d} z_r | 1 \leq i < d\}$$

Let $A = \mathbb{C}^d \setminus (X_{d-1} \cup \{z_d = 0, z_d = 1\})$ be the space without the first two sets of singularities and $B = \mathbb{C}^d \setminus (\{1 - \prod_{r=i}^{d} z_r | 1 \leq i < d\} \cup \{z_i = 0 | 1 \leq i \leq d\})$ be the total space without the final set of singularities and the set of $z_i = 0$. We then see that $A \cap B = \mathbb{C}^d \setminus X_d$.

It is then simple to compute $\pi_1(A)$ as A splits as a product $A \cong \mathbb{C}^{d-1} \setminus X_{d-1} \times \mathbb{C}^1 \setminus X_1$. So $\pi_1(A) = \pi_1(\mathbb{C}^{d-1} \setminus X_{d-1}) \times \pi_1(\mathbb{C}^1 \setminus X_1)$. Inductively $\mathbb{C}^{d-1} \setminus X_{d-1}$ is torsion free of rank $2d - 2 + \binom{d-1}{2}$ and $\pi_1(\mathbb{C}^1 \setminus X_1) = \mathbb{Z} * \mathbb{Z}$ has rank 2. So the rank of $\pi_1(A)$ is $2d - 2 + \binom{d-1}{2} + 2 = 2d + \binom{d-1}{2}$.

To analyze B we can "straighten" the product terms via the isomorphism $\mathbf{z} \mapsto (\prod_{r=1}^{d} z_i, \prod_{r=2}^{d} z_r, \dots, z_d)$. If we use $\mathbf{w}$ for the image, this map sends the singularity $1 - \prod_{r=i}^{d} z_r$ to $1 - w_i$. Similarly the singularities $z_i = 0$ as a set are sent to the set $w_i = 0$. This is necessary for the inverse $\mathbf{w} \mapsto (w_1/w_2, w_2/w_3 \dots, w_{d-1}/w_d, w_d)$ to be continuous. This shows that:

$$B \cong \mathbb{C}^d \setminus \{w_i = 1, w_i = 0 | 1 \le i \le d\} \cong \mathbb{C}^1 \setminus X_1 \times \dots \times \mathbb{C}^1 \setminus X_1$$

So $\pi_1(B) = \pi_1(\mathbb{C}^1 \setminus X_1)^d = (\mathbb{Z} * \mathbb{Z})^d$. By Van Kampen's Theorem, we have a short exact sequence:

$$0 \longrightarrow \pi_1(A \cap B) \longrightarrow \pi_1(A) * \pi_1(B) \longrightarrow \pi_1(A \cup B) \longrightarrow 0$$

Then $A \cup B = \mathbb{C} \setminus (\{z_d = 1\} \cup \{z_i = 0 | 1 < i \le d\}) \cong (S^1)^{d-1} \times \mathbb{C}^1 \setminus X_1$. So $\pi_1(A \cup B) = \mathbb{Z}^{d-1} \times (\mathbb{Z} * \mathbb{Z})$. So $\pi_1(A \cup B)$ is torsion free and of rank $d + 1$. Furthermore $\pi_1(\mathbb{C}^d \setminus X_d)$ must be torsion free as any torsion elements would map to torsion in $\pi_1(A \cup B)$.

In this case the rank of $\pi_1(A) * \pi_1(B)$ is equal to the sum of ranks of $\pi_1(\mathbb{C}^d \setminus X_d)$ and $\pi_1(A \cup B)$. So $n_{d-1} + 2 + 2d = n_d + d + 1$ and thus $n_d = n_{d-1} + d + 1$

Using our inductive hypothesis on rank this gives $n_d = 2(d-1) + \binom{d-1}{2} + d + 1 = 2d + \binom{d-1}{2} + \binom{d-1}{1} = 2d + \binom{d}{2}$ as needed. □

Corollary 3.1.4. *The first homology group of $\mathbb{C}^d \setminus X_d$ is $\mathbb{Z}^{n_d}$.*

Claim 3.1.5. *$\hat{U}_d$ is the universal abelian cover of $\mathbb{C}^d \setminus X_d$.*

Proof. By the previous corollary we know the abelianization of $\pi_1(\mathbb{C}^d \setminus X_d)$ is $H_1(\mathbb{C}^d \setminus X_d) = \mathbb{Z}^{n_d}$. We also have an obvious transitive action on the fiber of $\mathbf{z} \in \mathbb{C}^d \setminus X_d$ of $\mathbb{Z}^{n_d}$ by adding $2\pi ik$ to the appropriate coordinate. As this is the full deck transformation group of the cover $\hat{U}_d$ corresponds to the abelianization of the fundamental group as claimed. □

3.2 Differential Forms

To each multiple polylogarithm $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ we associate a differential one-form $\omega_{\mathbf{n}}$ on $\hat{U}_d$. We can think of form as the differential of a "lifted multiple polylogarithm" that is well defined on $\hat{U}_d$.

We will see these forms are related to the classical symbol of a multiple polylogarithm. However they have a major advantage in that they satisfy a simple combinatorial recurrence. Additionally differential forms have a natural coboundary map d. There is an analogous coboundary map for multiple polylogarithms defined in [37]). In low weights, we have verified the map sending $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ to $\omega_{\mathbf{n}}$ fits into a commutative diagram with these two coboundaries.

3.2.1 Relation To Symbol

Recall the symbol of a multiple polylogarithm (Definition 1.3.9) is a weight n tensor whose entries are basic liftable functions. We will define the associated form as a symmetrization of this classic construction.

Remark 3.2.1. *Let $U = \mathbb{C}^d \setminus X_d$. The symbol of a multiple polylogarithm, lives in $T^\bullet(\mathbb{C}(U)^*/\mu_\infty)$ where μ_∞ is the group all roots of unity. As such any tensor $a_1 \otimes \ldots \otimes a_n$ is 0 if any a_i is a rational multiple of $(\pi i)^k$.*

From the symbol we can define an associated one-form. This is achieved by first lifting the symbol to a tensor on $\hat{U}_d$ by sending z_i to u_i and $1 - \prod_{r=i}^{j} z_r$ to $v_{[i,j]}$. From there we can define a map to one-forms on $\hat{U}_d$ by extending the following map linearly:

$$f_1 \otimes \cdots \otimes f_n \mapsto (-1)^{n-1} \sum_{i=1}^{n} \frac{(-1)^{i-1}}{n!} \binom{n-1}{i-1} (f_1 \ldots \hat{f}_i \cdots f_n) \mathrm{d} f_i$$

Definition 3.2.2. *The map described above is the **symbol to forms** map. We can see this sends elements of $T^\bullet(\mathbb{C}(U)^*/\mu_\infty) \to \Omega^1(\hat{U}_d)$.*

The mysterious coefficients of the symbol to forms map can be explained by factoring through the product projector. The **product projector** ρ is defined recursively as follows:

$$\begin{aligned} \rho_1(a) =& a \\ \rho_n(a_1 \otimes \ldots \otimes a_n) =& \frac{n-1}{n} \left(\rho_{n-1}(a_1 \otimes \ldots \otimes a_{n-1}) \otimes a_n - \rho_{n-1}(a_2 \otimes \ldots \otimes a_n) \otimes a_1\right) \end{aligned}$$

The product projector is zero on the symbol associated to products of logarithms and thus gives a representative of the symbol of a polylogarithm "modulo products". We then define a new map ϕ as follows:

$$\phi : a_1 \otimes \ldots \otimes a_n \mapsto \frac{(-1)^{n-1}}{(n-1)!} a_2 \ldots a_n \mathrm{d}a_1$$

Theorem 3.2.3. *The symbol to forms map factors as the composition $\phi \circ \rho$.*

Proof. This is clear when $n = 1$ as a_1 is sent to $\mathrm{d}a_1$ by both maps.

For $n > 1$, we expand $\phi(\rho_n(a_1 \otimes \ldots \otimes a_n))$. Note that when $n > 1$:

$$\phi(a_1 \otimes \ldots \otimes a_n) = \frac{-1}{n-1} a_n \phi(a_1 \otimes \ldots \otimes a_n)$$

The rest of the proof follows inductively from the definition of ρ:

$$
\begin{aligned}
&\phi(\rho_n(a_1 \otimes \ldots \otimes a_n)) \\
=&\phi\left(\frac{n-1}{n}\left(\rho_{n-1}(a_1 \otimes \ldots \otimes a_{n-1}) \otimes a_n - \rho_{n-1}(a_2 \otimes \ldots \otimes a_n) \otimes a_1\right)\right) \\
=&\frac{n-1}{n}\left(\phi(\rho_{n-1}(a_1 \otimes \ldots \otimes a_{n-1}) \otimes a_n) - \phi(\rho_{n-1}(a_2 \otimes \ldots \otimes a_d) \otimes a_1)\right) \\
=&\frac{n-1}{n}\left(\frac{-1}{n-1} a_n \phi(\rho_{n-1}(a_1 \otimes \ldots \otimes a_{n-1})) - \frac{-1}{n-1} a_1 \phi(\rho_{n-1}(a_2 \otimes \ldots \otimes a_d)\right) \\
=&-\frac{1}{n}\left(a_n(-1)^{n-2} \sum_{i=1}^{n-1} \frac{(-1)^{i-1}}{(n-1)!}\binom{n-2}{i-1} a_1 \ldots \hat{a}_i \ldots a_{n-1} \mathrm{d}a_i\right. \\
&\qquad\qquad \left. -(-1)^{n-2} a_1 \sum_{i=2}^{n} \frac{(-1)^i}{(n-1)!}\binom{n-2}{i-2} a_2 \ldots \hat{a}_i \ldots a_n \mathrm{d}a_i\right) \\
=&-\frac{1}{n}\left(\frac{(-1)^{n-2}}{(n-1)!}\left(\sum_{i=1}^{n-1}\left((-1)^{i-1}\binom{n-2}{i-1} - (-1)^i\binom{n-2}{i-2}\right) a_1 \ldots \hat{a}_i \ldots a_n \mathrm{d}a_i\right)\right) \\
=&\frac{(-1)^{n-1}}{n!}\left(\sum_{i=1}^{n-1}(-1)^{i-1}\binom{n-1}{i-1} a_1 \ldots \hat{a}_i \ldots a_n\right)
\end{aligned}
$$

The final line is the image of the original symbol to product map as needed. □

Corollary 3.2.4. *If $\sum_i \mathrm{Li}_{\mathbf{n}_i}(\mathbf{z}_i)$ is a relation of polylogarithms modulo products then the corresponding sum of forms obtained by the symbol to forms map, $\sum_i \omega_{\mathbf{n}_i}(\mathbf{z}_i) = 0$.*

Proof. This is clear as the map to forms factors through the product projector where any polylogarithm relation modulo products is zero. □

3.2.2 Pullback Map Notation

Although the forms are defined on $\hat{U}_d$ it is often convenient to write the arguments as elements of $\mathbb{C}^d \setminus X_d$. This is especially useful for pulling back forms by rational maps between $f : \mathbb{C}^d \setminus X_d \to \mathbb{C}^k \setminus X_k$ where $1 - \prod_{r=i}^{j} f(x_r)$ factors into products of basic liftable functions. In this case f induces a map $\widehat{f} : \hat{U}_d \to \hat{U}_k$. We write $\omega_{\mathbf{n}}(f(x_1), \ldots, f(x_d))$ for the pullback by $\widehat{f}$, $\widehat{f}^*\omega_{\mathbf{n}} \colon \Omega^1(\hat{U}_k) \to \Omega^1(\hat{U}_d)$.

Example 3.2.5. *Consider the map $r : \mathbb{C}^2 \setminus X_2 \to \mathbb{C}^1 \setminus X_1$ by $r(x_1, x_2) = x_1x_2$. This lifts to a map $\widehat{r} : \hat{U}_2 \to \hat{U}_1$ given by $\widehat{r}(u_1, u_2, v_1, v_2, v_{12}) = (u_1 + u_2, v_{12})$. Then we write $\omega_3(xy)$ to mean $\widehat{r}^*\omega_3 = \omega_3(u_1 + u_2, v_{12})$.*

Example 3.2.6. *As a more complicated example consider $f : \mathbb{C}^2 \setminus X_2 \to \mathbb{C}^2 \setminus X_2$ by $r(x_1, x_2) = (x_1x_2, 1/x_1)$. This lifts to*

$$\widehat{r}(u_1, u_2, v_1, v_2, v_{12}) = (u_1 + u_2, -u_1, v_{12}, v_1 - u_1, v_2)$$

We write $\omega_{4,1}(x_1x_2, 1/x_1)$ instead of $\widehat{r}^*\omega_{4,1} = \omega_{4,1}(u_1 + u_2, -u_1, v_{12}, v_1 - u_1, v_2)$

3.2.3 Recurrence Relation

First we discuss some combinatorics needed to state the recurrence relation.

3.2.3.1 Compositions of an Integer

Several of the formulas in this section involve summing over the set of "compositions" of an integer:

Definition 3.2.7. *A* ***composition*** *of d is an ordered partition of positive integers $\mathbf{c} = c_1 + c_2 + \cdots + c_k$ such that $\sum c_i = d$. We write $Comp(d)$ for the set of all compositions of any length and $Comp_k(d)$ for the set of composition of length less than or equal to k. It will sometimes be helpful to allow 0 as an entry c_i in the composition. In this case we write $Comp^0(d)$ (or $Comp^0_k(d)$) for the set of compositions of d including zero of any length (or length less than or equal to k).*

Remark 3.2.8. *Compositions differ from the standard notion of a partition as $(3, 1, 2)$ and $(1, 2, 3)$ are two distinct compositions of 6.*

Definition 3.2.9. *We then define the following partial ordering on $Comp^0_k(d)$ where $\mathbf{a} \preceq \mathbf{b}$ if and only if for each i either $a_i = 0$ or $a_i > b_i$. Furthermore if an entry $a_i = 0$ then either $i = 1$ or for all j greater than i $a_j = 0$. Loosely this means that $\mathbf{a}$ can be formed from $\mathbf{b}$ by deleting entries from the end of b (and possibly the first entry) and redistributing the weight.*

Example 3.2.10. *All of the compositions that are less than $(2, 2, 2)$ under this* *ordering are*

$$(0,4,2) \quad (0,3,3) \quad (0,2,4) \quad (0,6,0) \quad (4,2,0) \quad (3,3,0) \quad (2,4,0) \quad (6,0,0)$$

Note that both $(0,0,6)$ *and* $(4,0,2)$ *are not less than* $(2,2,2)$ *in this order because the second entry is 0, but not every entry after is zero.*

Remark 3.2.11. *This partial order extends to an order on* $\bigcup_{d} Comp^0_k(d)$. *However we will use* $\prec$ *to compare vectors of the same weight and* $\prec^k$ *to compare a vector of weight* $n-k$ *with a vector of weight* n.

For $\mathbf{n} \in \mathrm{Comp}^0_k(d)$ with ℓ nonzero entries, $\omega_{\mathbf{n}}$ is understood to be the pullback form on $\hat{U}_k$ from $\hat{U}_\ell$ by ignoring the coordinates corresponding to the zero entries of $\mathbf{n}$. For example, $\omega_{0,1,2,0}(x,y,z,w) = \omega_{1,2}(y,z)$.

We will also need the following two lemmas relating to summing products of binomial coefficients of vectors under this order. The key idea in both lemmas is to use the binomial theorem to expand $(1+x+y)^p$ in both possible ways.

Definition 3.2.12. *For two vectors of integers* $\mathbf{n}$ *and* $\mathbf{m}$ *of length* d, *we define*

$$\binom{\mathbf{m}}{\mathbf{n}} = \prod_{i=1}^{d} \binom{m_i}{n_i}$$

Lemma 3.2.13. *For any vector* $\mathbf{p} \prec \mathbf{n}$ *with* $p_d = 0$,

$$\binom{\mathbf{p}-1}{\mathbf{n}-1} = \sum_{\substack{\mathbf{p}\preceq\mathbf{m}\prec\mathbf{n} \\ m_d=0, m_i>0}} \binom{\mathbf{p}-1}{\mathbf{m}-1}\binom{\mathbf{m}-1}{\mathbf{p}-1}$$

Proof. Let $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$. Then we use the binomial theorem to expand $(1 + (\mathbf{x} + \mathbf{y}))^{\mathbf{p}_{[1,d-1]}-1}$:

$$\begin{aligned}
&(1 + (\mathbf{x}+\mathbf{y}))^{\mathbf{p}_{[1,d-1]}-1} \\
&= \prod_{i=1}^{d-1} \sum_{k_i \geq 0} \binom{p_i - 1}{k_i} (x_i + y_i)^{k_i} \\
&= \prod_{i}^{d-1} \sum_{k_i, \ell_i \geq 0} \binom{p_i - 1}{k_i} \binom{k_i}{\ell_i} x_i^{\ell_i} y_i^{k_i - \ell_i} \\
&= \prod_{i}^{d-1} \sum_{m_i, n_i \geq 1} \binom{p_i - 1}{m_i - 1} \binom{m_i - 1}{n_i - 1} x_i^{n_i - 1} y_i^{m_i - n_i} \qquad [n_i - 1 = \ell_i, m_i - 1 = k_i] \\
&= \sum_{m_i, n_i \geq 1} \prod_{i=1}^{d-1} \binom{p_i - 1}{m_i - 1} \binom{m_i - 1}{n_i - 1} x_i^{n_i - 1} y_i^{m_i - n_i}
\end{aligned}$$

If we then take all $y_i = y$, each term in the sum has $\prod y_i^{m_i - n_i} = y^{\sum m_i - \sum n_i}$. In the case we are interested in $\sum m_i = \sum n_i$. So we look at the coefficient of $\mathbf{x}^{\mathbf{n}-1} y^0$. Note that $m_d = 0$, so it is fine to include it in the previous computation. Therefore the coefficient of $\mathbf{x}^{\mathbf{n}-1}$ is

$$\sum_{\substack{\mathbf{m} \\ m_i \geq 1, \sum m_i = \sum n_i}} \prod_{i=1}^{d-1} \binom{p_i - 1}{m_i - 1} \binom{m_i - 1}{n_i - 1}$$

We see that unless $m_i \geq n_i$, $\binom{m_i - 1}{n_i - 1} = 0$. Similarly if $p_i \neq 0$, then we must have $p_i \geq m_i$ to have a nonzero term. If we let $\mathbf{m} = (m_1, \ldots, m_{d-1})$ then the previous condition is exactly $\mathbf{p} \preceq \mathbf{m} \prec \mathbf{n}$ with $m_d = 0, m_i \geq 1$. We also compute that $\binom{m_d-1}{n_d-1} = (-1)^{n_d-1}$ and we consider $1 = \binom{-1}{-1} = \binom{p_d-1}{m_d-1}$. Therefore we see the coefficient of $x^{\mathbf{n}-1}$ can be written as

$$(-1)^{n_d-1} \sum_{\substack{\mathbf{p} \preceq \mathbf{m} \prec \mathbf{n} \\ m_d=0, m_i \geq 1}} \binom{\mathbf{p}-1}{\mathbf{m}-1}\binom{\mathbf{m}-1}{\mathbf{p}-1} \tag{3.1}$$

On the other hand, we can expand $((1+x_i)+y_i)^{p_i-1}$ to obtain

$$\sum_{k_i \geq 1} \prod_i \binom{p_i-1}{k_i-1}(1+x_i)^{k_i-1} y_i^{p_i-k_i}$$

Since we want the coefficient of $x_i^{n_i-1}$ we need to take $k_i \geq n_i$. Additionally we want y^0 after setting all $y_i = y$ so $\sum p_i = \sum k_i$. Furthermore $\sum p_i = \sum n_i$, so $\sum k_i = \sum n_i$. Therefore no k_i can be strictly greater than n_i. So the coefficient of $x_i^{n_i} y^0$ in this expansion is $\prod_{i=1}^{d-1} \binom{\mathbf{p}-1}{\mathbf{n}-1}$. Once again $\binom{p_d-1}{n_d-1} = (-1)^{n_d-1}$, so we can write the coefficient from the second expansion as:

$$(-1)^{n_d-1} \binom{\mathbf{p}-1}{\mathbf{n}-1} \tag{3.2}$$

Since Equations 3.1 and 3.2 are both the coefficient of $x^{\mathbf{n}-1}$, they must be equal proving the lemma. □

Lemma 3.2.14. *Let* $\mathbf{p}$ *be any vector with* $p_d = 0$ *and* $\mathbf{p} \prec^1 \mathbf{n}$. *Then we have the* *following identity*

$$\binom{\mathbf{p}-1}{\mathbf{n}-1} = \sum_{\substack{\mathbf{p}\preceq\mathbf{m}\prec^1\mathbf{n}\\ m_d=0, m_i>0}} \binom{\mathbf{p}-1}{\mathbf{m}-1}\binom{\mathbf{m}-\mathbf{1}}{\mathbf{n}-\mathbf{1}}$$

Proof. The proof of this lemma is almost identical to the last. The key difference is that $\sum m_i - \sum n_i = -1$. So we look at the coefficient of $\mathbf{x}^{\mathbf{n}-1}y^{-1}$ instead of $x^{\mathbf{n}-1}y^0$. However this doesn't affect any of the arguments and we obtain the formula we needed. □

3.2.4 Retraction Maps

Definition 3.2.15. *For any composition $\mathbf{c} = (c_1, \ldots, c_k)$ of d, the **retraction** $\widehat{r}_{\mathbf{c}}\colon \hat{U}_d \to \hat{U}_k$ is given by combining successive sets of c_i variables. Formally let $i_\ell = \sum_{i=1}^{\ell} c_i$ be the vector of partial sums of $\mathbf{c}$. Then we define $\widehat{r}$ by specifying the image in coordinate u_t and $v_{[s,t]}$:*

$$\begin{aligned} u_t =& u_{i_t} + \cdots + u_{i_{t+1}-1} \\ v_{[s,t]} =& v_{[i_s, i_{t+1}-1]} \end{aligned}$$

Remark 3.2.16. *This is a lift of the map $\mathbb{C}^d \setminus X_d \to \mathbb{C}^k \setminus X_k$ taking*

$$r_{\mathbf{c}}\colon (x_1, \ldots, x_d) \mapsto (\prod_{t=i_0+1}^{i_1} x_t, \ldots, \prod_{t=i_{k-1}+1}^{t=i_k} x_t)$$

Using the pullback notation of Section 3.2.2 we write:

$$\omega_2(x_1x_2) = \widehat{r}^*_{(2)}\omega_2$$

$$\omega_{4,1,2}(x_1x_2, x_3, x_4x_5x_6) = \widehat{r}^*_{(2,1,3)}\omega_{4,1,2}$$

We also define the "retraction action" of a composition $\mathbf{c} = (c_1, \ldots, c_k) \in \mathrm{Comp}(d)$ on a vector $\mathbf{n}$ of length d to be the length k vector formed by summing the next consecutive c_i terms of $\mathbf{n}$. Formally

$$\mathbf{c} \cdot \mathbf{n} = (\sum_{t=i_0}^{i_1} n_t, \ldots, \sum_{t=i_{k-1}+1}^{i_k} n_t)$$

Example 3.2.17. *The action of the composition* $\mathbf{c} = (2, 1, 3)$ *on* $\mathbf{n} = (3, 1, 4, 2, 5, 2)$ *yields* $(3 + 1, 4, 2 + 5 + 2) = (4, 4, 9)$.

It is often convenient to combine these two retraction actions as $\widehat{r}^*_{\mathbf{c}}\omega_{\mathbf{c}\cdot\mathbf{n}}$ to obtain forms with equal weight and shorter depth that are still defined on $\hat{U}_d$.

$$\widehat{r}^*_{\mathbf{c}}\omega_{\mathbf{c}\cdot\mathbf{n}} = \omega_{\mathbf{c}\cdot\mathbf{n}}(z_1 \ldots z_{c_1}, \ldots, z_{c_1+\cdots+c_{k-1}} \ldots z_d)$$

Example 3.2.18. *Again using* $\mathbf{c} = (2, 1, 3)$ *and* $\mathbf{n} = (3, 1, 4, 2, 5, 2)$ *the form* $\widehat{r}^*_{\mathbf{c}}\omega_{\mathbf{c}\cdot\mathbf{n}}$ *is equal to* $\omega_{4,4,9}(z_1z_2, z_3, z_4z_5z_6)$.

Note that $\mathbf{c} = \mathbf{1}_d$ leaves a form of depth d unchanged.

Remark 3.2.19. *We use 0 entries in a composition to mean removing the corresponding entry. As an example consider* $\mathbf{c} = (0, 1, 1, 2, 0)$ *and* $\mathbf{n} = (3, 1, 4, 2, 5, 2)$. *Then* $\widehat{r}^*_{\mathbf{c}}\omega_{\mathbf{c}\cdot\mathbf{n}} = \omega_{1,4,7}(x_2, x_3, x_4x_5)$.

3.2.5 Recursive Formulation

One advantage of the differential form $\omega_{\mathbf{n}}$ over the symbol is that they satisfy a clean recurrence relation. The recurrence is analogous to the derivative of the corresponding polylogarithm $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$, with additional "cross terms". There is a cross term for each vector $\mathbf{m} \prec^1 \mathbf{n}$. Let $c_{\mathbf{m}} = 1$ if $m_1 > 0$ and $c_{\mathbf{m}} = -1$ otherwise. Then the recurrence relation is:

$$
\begin{aligned}
\omega_1 &= \mathrm{d}u_1 \\
\omega_2 &= \frac{1}{2}\left(u_1\mathrm{d}v_1 - v_1\mathrm{d}u_1\right) \\
\omega_{1,1} &= \frac{1}{2}\left(v_{12}\mathrm{d}u_1 + (v_2 - v_{12})\mathrm{d}v_1 + (v_{12} - v_1)\mathrm{d}v_2 + (-u_1 + v_1 - v_2)\mathrm{d}v_{12}\right) \\
\omega_{\mathbf{n}} &= \frac{1}{\sum n_i}\left(\sum_{i=1}^{d} \delta_i\omega_{\mathbf{n}} + v_{[1,d]}\sum_{\mathbf{m}\prec^1\mathbf{n}} c_{\mathbf{m}}\binom{\mathbf{m}-1}{\mathbf{n}-1}\omega_{\mathbf{m}}(\mathbf{z})\right)
\end{aligned}
$$

Here δ_i is the derivative like operator:

$$\delta_i\omega_{\mathbf{n}} = \begin{cases} v_1\omega_{\mathbf{n}_{[2,d]}}(z_2,\ldots,z_d) - (v_1-u_1)\omega_{\mathbf{n}_{[2,d]}}(z_1z_2,z_3,\ldots,z_d) & i=1, n_1=1 \\ v_i\omega_{\mathbf{n}_{[1,i-1]}\mathbf{n}_{[i+1,d]}}(z_1,\ldots,z_{i-2},z_{i-1}z_i,z_{i+1},\ldots,z_d) & \\ \quad -(v_i-u_i)\omega_{\mathbf{n}_{[1,i-1]},\mathbf{n}_{[i+1,d]}}(z_1,\ldots,z_{i-1},z_iz_{i+1},z_{i+2},\ldots,z_d) & 1<i<d, n_i=1 \\ v_d\omega_{\mathbf{n}_{[1,d-1]}}(z_1,\ldots,z_{d-2},z_{d-1}z_d) - v_d\omega_{\mathbf{n}_{[1,d-1]}}(z_1,\ldots,z_{d-1}) & i=d \text{ and } n_d=1 \\ -u_i\omega_{n_1,\ldots,n_i-1,\ldots,n_d}(z_1,\ldots,z_d) & n_i>1 \end{cases}$$

If we compare $\delta_i\omega_{\mathbf{n}}$ to the $\partial_i \operatorname{Li}(\mathbf{z})$ we see that $\frac{1}{z_i}$ is replaced with u_i and $\frac{1}{1-z_i}$ is replaced with v_i. There is an "extra" term in δ_d when $n_i = 1$ mirroring the deleting first entry term of δ_1 when $n_1 = 1$.

Note that in depth 1 there are no smaller depth cross terms and so the recurrence simplifies to $\omega_n = -\frac{1}{n}u_1\omega_{n-1}$. A more complicated example is the depth 3 recurrence:

$$\begin{aligned}
&(p+q+r)\omega_{p,q,r} \\
&\qquad= \delta_1\omega_{p,q,r} + \delta_2\omega_{p,q,r}\omega_{p,q,r} + \delta_3\omega_{p,q,r} \\
&\qquad\quad + v_{123}(-1)^{q+r}\binom{p+(q+r-1)-1}{q+r-1}\omega_{p+q+r-1}(u_1) \\
&\qquad\quad - v_{123}(-1)^{p+r}\binom{q+(p+r-1)-1}{p+r-1}\omega_{p+q+r-1}(u_2) \\
&\qquad\quad - v_{123}\sum_{m_1+m_2=r-1}(-1)^r\binom{p+m_1-1}{m_1}\binom{q+m_2-1}{m_2}\omega_{p+m_1,q+m_2}(u_1,u_2) \\
&\qquad\quad + v_{123}\sum_{m_1+m_2=p-1}(-1)^p\binom{q+m_1-1}{m_1}\binom{r+m_2-1}{m_2}\omega_{q+m_1,r+m_2}(u_2,u_3)
\end{aligned}$$

In particular we see that $6\omega_{3,2,2} = -u_1\omega_{2,2,2} - u_2\omega_{3,1,2} - u_3\omega_{3,1,2} + v_{123}(10\omega_5(u_1) + 5\omega_5(u_2) - 3\omega_{4,2}(u_1,u_2) - 2\omega_{3,3}(u_1,u_2) - 3\omega_{4,2}(u_2,u_3) - 4\omega_{3,3}(u_2,u_3) - 3\omega_{2,4}(u_2,u_3))$

Theorem 3.2.20. *The forms $\omega_{\mathbf{n}}$ satisfy the recurrence relation.*

Proof. We can quickly check base cases as the symbol of $\mathrm{Li}_2(z)$ is $1 - z \otimes z$ which lifts to $v \otimes u$. This is sent to $\frac{1}{2}(u\mathrm{d}v - v\mathrm{d}u)$ via the symbol to forms map.

Similarly the symbol of $\mathrm{Li}_{1,1}(z_1, z_2)$ lifts to

$$-(v_{12} \otimes u_1 - v_{12} \otimes v_1 + v_{12} \otimes v_2 + v_2 \otimes v_1)$$

Under the symbol to forms map we obtain:

$$\begin{aligned}&\frac{-1}{2}\left(u_1\mathrm{d}v_{12} - v_{12}\mathrm{d}u_1 - v_1\mathrm{d}v_{12} + v_{12}\mathrm{d}v_1 + v_2\mathrm{d}v_{12} - v_{12}\mathrm{d}v_2 + v_1\mathrm{d}v_2 - v_2\mathrm{d}v_1\right)\\ &=\frac{1}{2}\left(v_{12}\mathrm{d}u_1 + (v_2 - v_{12})\mathrm{d}v_1 + (v_{12} - v_1)\mathrm{d}v_2 + (-u_1 + v_1 - v_2)\mathrm{d}v_{12}\right)\end{aligned}$$

Note that in weight 1, ω_1 corresponds directly to $\log(z)$ which has lifted symbol u, not $\mathrm{Li}_1(z)$ which has lifted symbol $-v$.

In the inductive case we must carefully examine the algorithm to assign a symbol to a multiple polylogarithm. For this we follow the algorithm of [29] to compute the symbol by extracting tensors from decorated polygons. The polygon, P, associated to $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ is a $(1 + \sum n_i)$-gon with a distinguished "root vertex". The sides of the polygon are labeled according to the conversion from $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ to "G-notation" for iterated integrals. We have that:

$$\mathrm{Li}_{\mathbf{n}}(\mathbf{z}) = (-1)^d G(\mathbf{0}_{m_d-1}, \frac{1}{z_d}, \mathbf{0}_{m_{d-1}-1}, \frac{1}{x_{d-1}x_d}, \ldots, \mathbf{0}_{m_1-1}, \frac{1}{x_1 \ldots x_d}; 1)$$

where

$$G(a_1, \ldots, a_n; x) = \int_0^x \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

If the sides of P are labeled counterclockwise from the root vertex, $G(a_1, \ldots a_n; x)$ corresponds to $P(a_n, \ldots, a_1, x)$[1]. We call side labeled x, the root side and draw a double edge to distinguish it. For a multiple polylogarithm this root side is always labeled 1. Each term in the symbol corresponds to a choice of a maximal non-crossing set of arrows from vertices to edges. This dissects the polygon into digons each of which correspond to an entry in the tensor. The ordering in the tensor product is given by first constructing the tree dual to the dissection. The root of this tree is the digon containing the root vertex and root edge. Every linear order that is compatible with partial order of the rooted tree corresponds to a term in the symbol.

For this proof we need a few key facts about this algorithm:

1. If the dual tree is not linear, then the tensors corresponding to that dissection can be written using a shuffle product. As such the product projector ρ kills any such terms.

2. If P_i is the polygon formed by deleting edge i the symbol corresponding to P,

[1] Note the reversed ordering of indices between P and G.

$S(P)$ can be written as

$$S(P) = \sum_{i=1}^{n-1} S(P_i) \otimes S(P(a_i, a_{i+1})) - \sum_{i=2}^{n-1} S(P_i) \otimes S(P(a_i, a_{i-1}))$$

Here $P(a, b)$ is the digon with root side b and non-root side a.

3. The symbol associated to the digon $P(a, b)$ is

$$S(P(a,b)) = \begin{cases} 0 & a = 0, b = 1 \\ b & a = 0, b \neq 1 \\ 1 - \frac{b}{a} & a \neq 0 \end{cases}$$

We then consider $\rho_n(S(\mathrm{Li}_{\mathbf{n}}(z))$. In the inductive case, to compute ρ we need to gather the terms of the symbol by the last entry and by the first entry. It turns out the terms gathered by the last entry correspond to the $\delta_i \omega_{\mathbf{n}}$ terms and the terms gathered by the first entry correspond to the "cross term" with the $v_{[1\ldots d]}$. The mysterious extra term of δ_d also occurs in this section when $n_d = 1$.
Explicitly we can simplify $\phi(\rho_n(S(\mathrm{Li}_{\mathbf{n}}(\mathbf{z}))))$ as follows:

$$\begin{aligned}
\phi(\rho_n(S(\mathrm{Li}_{\mathbf{n}}(\mathbf{z})))) =& \ \phi(\rho_n(S((-1)^d G(\mathbf{a};1)))) \\
=& \ (-1)^d \frac{n-1}{n} \left[\sum_{a_n} \phi(\rho_{n-1}(\sum a_1 \otimes \ldots \otimes a_{n-1}) \otimes a_n) - \sum_{a_1} \phi(\rho_{n-1}(\sum a_2 \otimes \ldots \otimes a_n) \otimes a_1) \right] \\
=& \ (-1)^d \frac{n-1}{n} \left[\sum_{a_n} \frac{-1}{n-1} a_n \phi(\rho_{n-1}(\sum a_1 \otimes \ldots \otimes a_{n-1})) - \sum_{a_1} \frac{-1}{n-1} a_1 \phi(\rho_{n-1}(\sum a_2 \otimes \ldots \otimes a_n)) \right] \\
=& \ \frac{(-1)^d}{n} \left[\sum_{a_n} -a_n \phi(\rho_{n-1}(\sum a_1 \otimes \ldots \otimes a_{n-1})) + \sum_{a_1} a_1 \phi(\rho_{n-1}(\sum a_2 \otimes \ldots \otimes a_n)) \right]
\end{aligned} \tag{3.3}$$

We now analyze the terms that appear when the symbol is gathered by last term. Fact 2 decomposes the symbol into terms of the form "the symbol of the polygon without side i" tensor "the digon associated to side i" for all non-root sides. Since the labels come from a multiple polylogarithm, any non-root side i of the polygon is labeled 0 or $\frac{1}{z_j,\ldots z_d}$ for some j. Let a_i be the label of side i.

If $a_i = 0$ then i is in a string of $m_j - 1$ zeros and $m_j > 1$. As such, deleting edge i yields the polygon corresponding to $(-1)^d \operatorname{Li}_{\mathbf{n}-e_j}(\mathbf{z})$. We also see from Fact 3 that $S(P(a_i, a_{i+1})) = 0$ unless $a_{i+1} = \frac{1}{z_{j+1}\ldots z_d}$. Similarly $S(P(a_i, a_{i-1})) = 0$ unless $a_{i-1} = \frac{1}{z_j \ldots z_d}$. So the only terms corresponding to $P((-1)^d \operatorname{Li}_{\mathbf{n}-e_j}(\mathbf{z}))$ are

$$
\begin{aligned}
&S(P((-1)^d \operatorname{Li}_{\mathbf{n}-e_j}(\mathbf{z}))) \otimes \frac{1}{z_{j+1}\ldots z_d} - S(P((-1)^d \operatorname{Li}_{\mathbf{n}-e_j}(\mathbf{z}))) \otimes \frac{1}{z_j \ldots z_d} \\
&= S(P((-1)^d \operatorname{Li}_{\mathbf{n}-e_j}(\mathbf{z}))) \otimes z_j
\end{aligned}
$$

When we lift z_j becomes u_j. At the end of Equation 3.3 this term becomes

$$
-u_j \phi(\rho_{n-1}(S(P((-1)^d \operatorname{Li}_{\mathbf{n}-e_j}(\mathbf{z})))))
$$

which inductively is equal to $-u_j(-1)^d \omega_{\mathbf{n}-e_j} = (-1)^d \delta_j \omega_{\mathbf{n}}$. Note that the original conversion has a factor of $(-1)^d$ so this distributes leaving $\delta_j \omega_{\mathbf{n}}$ as we needed.

If $a_i \neq 0$ then $a_i = \frac{1}{z_j \ldots z_d}$ for some j. If $n_j > 1$ then $a_{i+1} = 0$. So $S(P(a_i, a_{i+1})) = 1$ and this term doesn't contribute to the symbol. Otherwise $n_j = 1$ and $a_{i+1} = \frac{1}{z_{j+1} z_j \ldots z_d}$. In this case $S(P(a_i, a_{i+1})) = 1 - z_j$ which lifts to v_j. Deleting side i

corresponds to a G function whose nonzero terms skip from $\frac{1}{z_{j+1}\ldots z_d}$ to $\frac{1}{z_{j-1}z_j z_{j+1}\ldots z_d}$. When converting back to Li form the $(j-1)^{st}$ argument becomes $z_{j-1}z_j$. Therefore this situation results in $(-1)^{d-1}S(\mathrm{Li}_{n_{[1,j-1]},n_{[j+1,d]}}(z_{[1,j-2]}, z_{j-1}z_j, z_{[j+1,d]}))\otimes v_j$. If $j < d$ (and $n_j = 1$ still) we also get a contribution from $a_{i-1} = \frac{1}{z_j\ldots z_d}$ and $a_i = \frac{1}{z_{j+1}\ldots z_d}$. Here $S(P(a_i, a_{i+1})) = 1 - \frac{1}{z_j}$ which lifts to $v_j - u_j$. With the shift in index the new polygon corresponds to $(-1)^{d-1}\,\mathrm{Li}_{n_{[1,j-1]},n_{[j+1,d]}}(z_{[1,j-1]}, z_j z_{j+1}, z_{[j+2,d]})$.

So gathering all the terms with u_j or v_j for $1 \leq j < d$ results in:

$$S((-1)^{d-1}\,\mathrm{Li}_{n_{[1,j-1]},n_{[j+1,d]}}(z_{[1,j-2]}, z_{j-1}z_j, z_{[j+1,d]}))\otimes v_j \\ - S((-1)^{d-1}\,\mathrm{Li}_{n_{[1,j-1]},n_{[j+1,d]}}(z_{[1,j-1]}, z_j z_{j+1}, z_{[j+2,d]}))\otimes (v_j - u_j)$$

As in the $n_j > 1$ case we see that sending this through ϕ as in Equation 3.3 this becomes

$$-(-1)^{d-1} v_d \omega_{n_{[1,j-1]},n_{[j+1,d]}}(z_{[1,j-2]}, z_{j-1}z_j, z_{[j+1,d]}) \\ + (-1)^{d-1}(v_j - u_j)\omega_{n_{[1,j-1]},n_{[j+1,d]}}(z_{[1,j-1]}, z_j z_{j+1}, z_{[j+2,d]})$$

Distributing the $(-1)^d$ fixes the signs, so that this is $\delta_j \omega_{\mathbf{n}}$ as well. When $j = d$ we only get the term of $\delta_d \omega_{\mathbf{n}}$ that exactly matches derivative of the polylogarithm. The other term of δ_d we will see comes from the other half of the product projector.

To compute the other half of the product projector we need to gather the terms of the symbol by the first entry of the tensor. In the polygon dissection this is the entry coming from the root bigon. In Figure 3.1 we can see the 6 possible root bigons. Clearly Figures 3.1c, 3.1e, and 3.1f all correspond to nonlinear trees. As such by Fact 1 the terms coming from these cases contain a shuffle product and are 0 after the product projector.

Next we focus on Figure 3.1d. Removing the root bigon results in a new polygon where side n is now the root side. If $n_d > 1$ then this side is labeled 0 and the G function for this polygon ends in 0. Any G function whose last entry is 0 is 0 and so there are no terms in the symbol when $n_d > 1$. However if $n_d = 1$ then the new root is $\frac{1}{z_d}$. Using the change of variables $u = z_d t$ we see that $G(a_1, \ldots, a_n, \frac{1}{z_d}) = G(z_d a_1, \ldots, z_d a_n, 1)$. Since G came from $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$, the new polygon corresponds to $\mathrm{Li}_{\mathbf{n}_{[1,d-1]}}(z_{[1,d-1]})$. The symbol of the root bigon is $S(P(\frac{1}{z_d}, 1)) = 1 - z_d$ which lifts to v_d. This is the missing term from $\delta_d \omega_{\mathbf{n}}$ as needed.

The last two cases (Figures 3.1a,3.1b) together will give us all of the cross terms. In both cases the symbol of the root bigon is $1 - z_1 \ldots z_d$ which lifts to $v_{[1,d]}$ as needed. It remains to identify the G function of the polygon obtained by deleting the root bigon.

For Figure 3.1a this function is $G(\mathbf{0}, \frac{1}{z_d}, \ldots, \mathbf{0}, \frac{1}{z_2 \ldots z_d}, \mathbf{0}, 1)$ as the edge corresponding to $\frac{1}{z_1 \ldots z_d}$ is removed. This function directly converts to a degenerate polylogarithm. However by shuffle regularizing, this can be written in a non-degenerate way. In fact by Lemma 3.2.21, this is equal to $(-1)^{d-1} \sum_{\substack{\mathbf{m} \prec^1 \mathbf{n} \\ m_1 = 0, m_i > 0}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \mathrm{Li}_{\mathbf{m}}(\mathbf{z})$.

For Figure 3.1b deleting the root bigon removes the root edge. This results with the new root being $\frac{1}{z_1 \ldots z_d}$ and the entries of G reversed. We can make a change of variables to change the root to 1 resulting in every entry being multiplied by $z_1, \ldots z_d$.

Thus the G function is $G(\mathbf{0}_{m_1-1}, z_d, \mathbf{0}_{m_2-1}, z_{d-1}, \ldots, \mathbf{0}_{m_{d-1}-1}, z_1 \ldots z_{d-1}, \mathbf{0}_{m_d-1}; 1)$. Following Lemma 3.2.21 this is equal to $(-1)^{d-1} \sum_{\substack{\mathbf{m} \prec^1 \mathbf{n} \\ m_d = 0, m_i > 0}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \operatorname{Li}_{\overleftarrow{\mathbf{m}}}(1/\overleftarrow{\mathbf{z}})$. Then using the inversion theorem (Theorem 3.3.2[2]) this becomes

$$(-1)^{d-1} \sum_{\substack{\mathbf{m} \prec^1 \mathbf{n} \\ m_d = 0, m_i > 0}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \sum_{\mathbf{p} \preceq \mathbf{m}} \operatorname{Li}_{\mathbf{p}}(\mathbf{z}) = (-1)^{d-1} \sum_{\substack{\mathbf{p} \prec^1 \mathbf{n} \\ p_d = 0}} c_{\mathbf{p}} \binom{\mathbf{p}-1}{\mathbf{n}-1} \operatorname{Li}_{\mathbf{p}}(\mathbf{z})$$

The final equality is by Lemma 3.2.14. This term also picks up a negative sign as flipping the polygon reverses the condition determining the sign of the tensor. So distributing the $(-1)^d$ from Equation 3.3 leaves the terms from Figure 3.1a with a negative sign and the terms from Figure 3.1b with a positive sign. This exactly corresponds to the sign coefficient $c_{\mathbf{m}}$.

So combining these two sets of terms results in $\left(\sum_{\mathbf{m} \prec^1 \mathbf{n}} c_{\mathbf{m}} S(\operatorname{Li}_{\mathbf{m}}(\mathbf{z}))\right) \otimes v_{[1,d]}$ which is sent to the cross term via the symbol to form map as claimed.

□

Lemma 3.2.21 (Extract Trailing Zeros)**.** *For any sequence of arguments* $\mathbf{x}$*:*

$$G(\mathbf{x}, \mathbf{0}_k; y) = (-1)^k \sum_{\substack{\mathbf{z} \in \mathbf{x} \sqcup \mathbf{0}_k \\ z_{\ell+k} \neq 0}} G(\mathbf{z}; y)$$

So if $G(\mathbf{x}, 1) = (-1)^d \operatorname{Li}_{\mathbf{n}}$ *then*

$$G(\mathbf{x}, \mathbf{0}_{\mathbf{k}}; 1) = (-1)^d \sum_{\substack{\mathbf{m} \prec (k+1, \mathbf{n}) \\ m_1 = 0, m_i > 0}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \operatorname{Li}_{\mathbf{m}}$$

[2]The proof of the inversion theorem only relies on the symbol to forms map, and not on this recurrence.

Proof. The first part of the lemma follows from the algorithm outlined in Section 4.2 of [30] for extracting trailing zeros by "unshuffling powers of logarithms". In other words they consider the following sum of products:

$$\sum_{i=0}^{k-1}(-1)^{k-i}G(\mathbf{x},\mathbf{0}_i;y)G(\mathbf{0}_{k-i};y)$$

Recall that $G(\mathbf{0}_n;y) = \frac{1}{n!}\log^n(y)$ justifying the description of this a product of powers of logarithms. Furthermore any product of G functions with the same last argument can be expand as a sum of shuffles. So for each i we obtain a sum over $S_i = \mathbf{x}\mathbf{0}_i \sqcup\!\sqcup\, \mathbf{0}_{k-i}$. Let $S_i(j)$ be the set of shuffles in S_i with exactly $i+j$ trailing zeros. Then $S_i = \bigcup_{j=0}^{k-i} S_i(j)$. Clearly $S_i(j) = S_{i+1}(j-1)$ and so in the alternating series everything cancels except $S_0(0)$ and $S_{k-1}(1)$. Then $S_{k-1}(1)$ has only one term, $G(\mathbf{x},\mathbf{0}_k,y)$ which is the original function. Similarly $S_0(0)$ is the set of shuffles with no trailing zeros that we wanted. Therefore

$$\sum_{i=0}^{k-1}(-1)^{k-i}G(\mathbf{x},x_{m+1},\mathbf{0}_i;y)G(\mathbf{0}_{k-i};y) = (-1)^k\sum_{\mathbf{z}\in S_0(0)}G(\mathbf{z};y) - G(\mathbf{x},\mathbf{0}_k,y)$$

Now let $\mathbf{x}$ be the vector that makes $\mathrm{Li}_{\mathbf{n}}(\mathbf{z}) = (-1)^d G(\mathbf{x},1)$. We can further gather the terms of $S_0(0)$ by the number of zeros between the nonzero entries of $\mathbf{x}$. Let $\mathbf{m}$ be a length d vector of positive integers. For $1 \le i \le d$, in order for $m_i - 1$ to be the number of zeros before the $d+1-i^{th}$ nonzero entry of $\mathbf{x}$ we need $m_i - 1 \ge n_i - 1$ and $\sum m_i = k + \sum n_i$. In other words $(0,\mathbf{m}) \prec^1 (k+1,\mathbf{n})$ with $m_i > 0$. The number of ways to get $\mathbf{m}$ from $\mathbf{n}$ by this procedure is $\prod\binom{m_i-1}{n_i-1}$. Furthermore recall $(-1)^k =$

$\binom{-1}{k} = \binom{m_0-1}{n_0-1}$. Therefore the sum over $S_0(0)$ is $(-1)^d \sum_{\substack{\mathbf{m}\prec^1(k+1,\mathbf{n}) \\ m_1=0, m_i>0}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \mathrm{Li}_{\mathbf{m}}(\mathbf{z})$. Modulo products we then have

$$G(\mathbf{x}, \mathbf{0}_k, y) = (-1)^d \sum_{\substack{\mathbf{m}\prec^1(k+1,\mathbf{n}) \\ m_d=0, m_i>0}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \mathrm{Li}_{\mathbf{m}}(\mathbf{z})$$

□

3.2.6 Symmetrization

Definition 3.2.22. *Let s_n be the "symmetrization" operation on homogeneous polynomial one-forms that sends basis elements $p(\mathbf{x})\mathrm{d}y$ to $p(\mathbf{x})\mathrm{d}y - \frac{1}{n}(\mathrm{d}[p(\mathbf{x})y])$ where $n-1$ is the degree of p.*

It is easy to see this operation is idempotent and preserves the differential of the one-form.

Definition 3.2.23. *A one-form, ω whose coefficient polynomials that are degree n is **symmetric** if $\omega = s_n\omega$.*

We now have two proofs that $\omega_{\mathbf{n}}$ is symmetric. First we use the symbol to forms definition.

Theorem 3.2.24. *The symbol to forms map (Definition 3.2.2) results in a symmetric form.*

Proof. It is a simple computation to show the image is fixed by the symmetrization map s_n.

For each i, $s_n(f_1 \dots \hat{f}_i \dots f_n \mathrm{d} f_i) = f_1 \dots \hat{f}_i \dots f_n \mathrm{d} f_i - \frac{1}{n}\mathrm{d}[f_1, \dots f_n]$. So the subtracted term is $\frac{1}{n}\mathrm{d}[f_1, \dots, f_n]$ in each case. Combining this with the coefficients from the summation we obtain

$$\sum_{i=1}^{n}(-1)^i \binom{n-1}{i-1}\mathrm{d}[f_1 \dots f_n] = \mathrm{d}[f_1 \dots f_n]\sum_{i=1}^{n}(-1)^{i-1}\binom{n-1}{i-1}$$

It is a classic application of the binomial theorem that $0 = (1-1)^{n-1} = \sum_{i=1}^{n}(-1)^{i-1}\binom{n-1}{i-1}$ and so the extra terms of the symmetrization are zero as needed. □

The second proof uses the recurrence and the following simple lemmas:

Lemma 3.2.25. *If $\omega = \sum_i p(\mathbf{x})\mathrm{d}y_i$ is a symmetric one-form, then $z\omega$ is also symmetric.*

Proof. In order for ω to be symmetric, we have

$$0 = \frac{1}{n}\sum_i \mathrm{d}[p_i(\mathbf{x})y_i] = \frac{1}{n}\mathrm{d}\left[\sum_i p_i(\mathbf{x})y_i\right]$$

So $\sum_i p_i(\mathbf{x})y_i = C$ for some constant C. Furthermore this sum is a homogeneous degree n polynomial for $n > 1$, so to be a constant it must 0. Then

$$\begin{aligned}\frac{1}{n+1}\sum_i \mathrm{d}[zp_i(\mathbf{x})y_i] =& \frac{1}{n+1}\sum_i p_i(\mathbf{x}y_i\mathrm{d}z + z\mathrm{d}[p_i(\mathbf{x})y_i] \\ =& \frac{1}{n+1}\left(\left(\sum_i p_i(\mathbf{x}y_i)\right)\mathrm{d}z + z\left(\mathrm{d}\left[\sum_i p_i(\mathbf{x})y_i\right]\right)\right) \\ =& 0\mathrm{d}z + z(0) = 0\end{aligned}$$

□

Lemma 3.2.26. *The sum of two symmetric forms is symmetric.*

Proof. This is clear as the symmetry operator is linear. □

Theorem 3.2.27. *The one-forms defined via the recurrence relation are symmetric.*

Proof. This can be seen inductively. In the base cases clearly $\omega_1 = \mathrm{d}u$, $\omega_2 = \frac{1}{2}(u\mathrm{d}v - v\mathrm{d}u)$ and $\omega_{1,1}$ are symmetric. Then the inductive step is a sum of lower weight forms scaled by single variables. The previous two lemmas show this preserves symmetry. □

Remark 3.2.28. *For any one-form ω we have that $\mathrm{d}s_n\omega = \mathrm{d}\omega$ since the modification is by an exact form. As such symmetrization provides a method for choosing a canonical representative of the integral of a closed 2 form.*

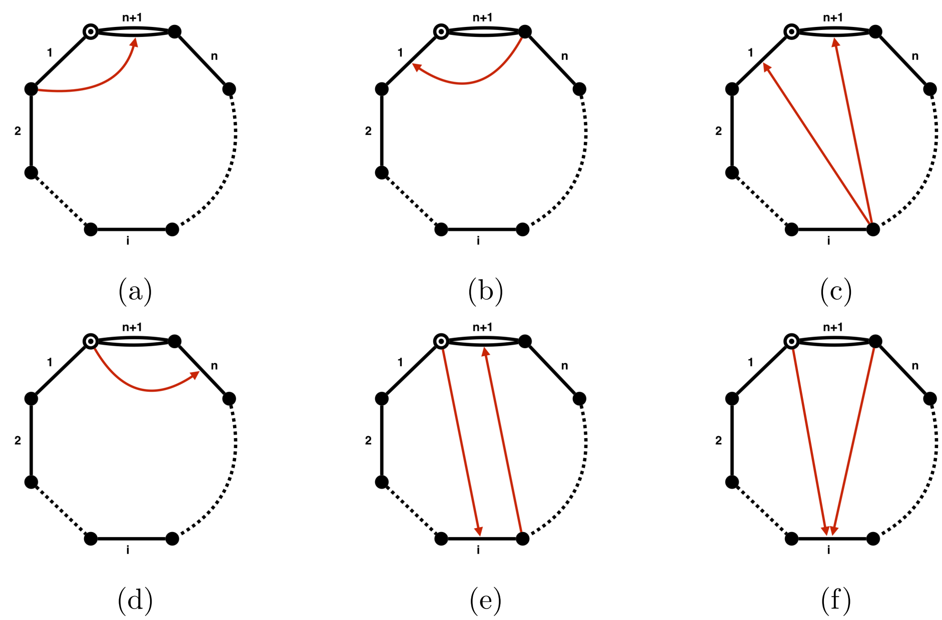


Figure 3.1: All possible root bigons.

3.3 General Relations

3.3.1 Inversion Relation

Since $z = 0 \in X_1$ the map $\tau : \mathbb{C}^1 \setminus X_1 \to \mathbb{C}^1 \setminus X_1$ sending z to $1/z$ is holomorphic and thus induces a map $\tau^* : \hat{U}_1 \to \hat{U}_1$. In coordinates this map sends $(u, v) \mapsto (-u, v - u)$. Recall that in depth 1 for any weight we have the following relation [5]:

$$\omega_n(x) + (-1)^n \omega_n(1/x) = 0$$

There is a depth d generalization of τ given by $(z_1, \ldots, z_d) \mapsto (1/z_1, \ldots, 1/z_d)$. This lifts to the map sending u_i to $-u_i$ and $v_{i,j}$ to $v_{i,j} - \sum_{r=i}^{j} u_r$. It is then natural to ask if there is a depth d relation corresponding to this involution. In fact in depth 2 we have the following relation:

Claim 3.3.1. *For all m, n the following quantity is 0:*

$$\omega_{m,n}(x, y) - (-1)^{m+n} \omega_{m,n}(1/x, 1/y) + \omega_{m+n}(xy) - (-1)^n \binom{m+n-1}{m-1} \omega_{m+n}(x) + (-1)^m \binom{m+n-1}{n-1} \omega_{m+n}(y)$$

Proof. This can be shown inductively using the recurrence in depth 2. It will also follow from Theorem 3.3.4. □

3.3.1.1 Inversion Reversing Relation

While we could state a similar "inverse" relation for any depth polylogarithm, it is most convenient to state the relation in two steps, first a relation that also reverses the arguments while inverting them and then a relation to reverse all the arguments. We use the notation $\overleftarrow{\mathbf{z}} = (z_d, \ldots, z_1)$ to indicate that a vector should be reversed.

With this idea in mind we see there is a simpler depth 2 inversion relation with one fewer term:

$$(-1)^{m+n}\omega_{n,m}(1/y,1/x)+\omega_{m,n}(x,y)-\binom{m+n-1}{n-1}\omega_{n+m}(x)+\binom{m+n-1}{m-1}\omega_{n+m}(y)$$

This can be seen to be equivalent to the previous relation by applying the stuffle relation $\omega_{m,n}(x,y) + \omega_{n,m}(y,x) + \omega_{n+m}(xy) = 0$. In general we have the following theorem:

Theorem 3.3.2 (Inversion Reversing Relation)**.** *For any vector* $\mathbf{n}$ *with* $\sum n_i > 1$ *we have*

$$-(-1)^{\sum n_i}\omega_{\overleftarrow{\mathbf{n}}}(1/\overleftarrow{\mathbf{z}}) = \sum_{\mathbf{m}\preceq\mathbf{n}} c_{\mathbf{m}} \left(\prod_i \binom{m_i-1}{n_i-1}\right)\omega_{\mathbf{m}}$$

where we recall that $\binom{-1}{k} = (-1)^k$ *and* $c_{\mathbf{m}} = \begin{cases} -1 & m_1 = 0 \\ 1 & m_1 \neq 0 \end{cases}$

Proof. This can be proved using an inductive formula of Goncharov in Section 2.6

of [38]. He defines the following generating series $B(\mathbf{z}|\mathbf{t})$:

$$B(\mathbf{z}|\mathbf{t}) = \sum_{-\infty<k_1<\cdots<k_d<\infty} \prod_i \frac{z_i^{k_i}}{k_i - t_i}$$

He then defines a multiple polylogarithm generating series:

$$\mathrm{Li}(\mathbf{z}|\mathbf{t}) = \sum \mathrm{Li}_{\mathbf{n}}(\mathbf{z})\mathbf{t}^{\mathbf{n-1}}$$

Using these two series he establishes the identity

$$\begin{aligned}
&\sum_{j=1}^{d}(-1)^j \,\mathrm{Li}(1/\overleftarrow{\mathbf{z}_{[1,j]}}| - \overleftarrow{\mathbf{t}_{[1,j]}})\, \mathrm{Li}(\mathbf{z}_{[j+1,d]}|\mathbf{t}_{[j+1,d]}) \\
&\quad + \sum_{j=1}^{d} \frac{(-1)^j}{t_j} \,\mathrm{Li}(1/\overleftarrow{\mathbf{z}_{[1,j-1]}}| - \overleftarrow{\mathbf{t}_{[1,j-1]}})\, \mathrm{Li}(\mathbf{z}_{[j+1,d]}|\mathbf{t}_{[j+1,d]}) \\
= &\sum_{j=1}^{d}(-1)^j \,\mathrm{Li}(1/\overleftarrow{\mathbf{z}_{[1,j-1]}}|t_j - \overleftarrow{\mathbf{t}_{[1,j-1]}}) B(z_1 \dots z_d|t_j)\, \mathrm{Li}(\mathbf{z}_{[j+1,d]}|\mathbf{t}_{[j+1,d]} - t_j)
\end{aligned}$$

To establish a relation of forms we can simplify the above relation modulo products. First we note that in each sum the only terms that don't contain product terms are the first and last terms.

Next we simplify $B(z_1 \dots z_d|t_j)$. There is a "classic identity" that

$$B(z|t) = -2\pi i \sum_{n\geq 0} B_n(\mathrm{Log}(z)) \frac{(2\pi i t)^{n-1}}{n!}$$

where $B_n(z)$ is the n^{th} Bernoulli polynomial. Only the constant term of $B_n(\mathrm{Log}(z))$ is nonzero mod products, so $B(z|t)$ reduces to $-\sum_{n\geq 0} B_n \frac{(2\pi i)^n}{n!} t^{n-1}$. Unless $n = 0$

each term is a rational multiple $(2\pi i)^n$. Since the symbol is defined to ignore torsion every term with $n > 0$ is 0 in the image form. This reduces to the friendlier relation:

$$\begin{aligned}\operatorname{Li}(\mathbf{z}|\mathbf{t}) + (-1)^d \operatorname{Li}(1/\overleftarrow{\mathbf{z}}| - \mathbf{t}) - \frac{1}{t_1}\operatorname{Li}(\mathbf{z}_{[2,d]}|\mathbf{t}_{[2,d]}) + \frac{1}{t_d}(-1)^d \operatorname{Li}(1/\overleftarrow{\mathbf{z}_{1,d-1}}| - \overleftarrow{\mathbf{t}_{[1,d-1]}}) \\ = \frac{-1}{t_1}\operatorname{Li}(\mathbf{z}|\mathbf{t}_{[\mathbf{2},\mathbf{d}]} - t_1) + \frac{1}{t_d}(-1)^d \operatorname{Li}(1/\overleftarrow{\mathbf{z}}|t_d - \mathbf{t}_{[1,d-1]})\end{aligned}$$

Finally we extract the coefficient of $\mathbf{t}^{\mathbf{n}-1}$. This is simple for the first terms yielding $\operatorname{Li}_{\mathbf{n}}(\mathbf{z})$. The second term, has a sign of $\prod(-1)^{n_i-1} = (-1)^d(-1)^{\sum n_i}$ from the $-\mathbf{t}$. The $(-1)^d$, from the equation, cancels out the $(-1)^d$ from $-\mathbf{t}$ to obtain exactly $(-1)^{\sum n_i}\operatorname{Li}_{\overleftarrow{\mathbf{n}}}(1/\overleftarrow{\mathbf{z}})$ from the second term. Both the third and fourth terms have no contribution, as they have t_1^{-1} or t_d^{-1} and neither n_1 or n_d can be 0.

For the right hand side, we use the binomial theorem to expand $\frac{1}{t_1}\prod_{i=2}^{d}(t_i - t_1)^{m_i-1}$ in $\frac{-1}{t_1}\operatorname{Li}(\mathbf{z}|\mathbf{t}_{[2,d]} - t_1)$ to get

$$-\sum_{k_2\cdots+k_d=n_1}(-1)^{n_1}t_1^{n_1-1}\prod_{i=2}^{d}\binom{m_i-1}{m_i-1-k_i}t_i^{m_i-1-k_i}$$

So to get $\mathbf{t}^{\mathbf{n}-1}$ we need $m_i - k_i = n_i$. In other words $\mathbf{m}$ is a vector such that $m_1 = 0$ and $m_i > n_i$ with the same weight as $\mathbf{n}$. This is $\{\mathbf{m} \prec \mathbf{n}|m_1 = 0, m_i > 0\}$. So the polylogarithms that appear as a coefficient of $\mathbf{t}^{\mathbf{n}-1}$ here as:

$$\sum_{\substack{\mathbf{m}\prec\mathbf{n} \\ m_1=0,m_i>0}}(-1)^{n_1-1}\prod\binom{m_i-1}{n_i-1}\operatorname{Li}_{\mathbf{m}}(\mathbf{z}) = \sum_{\substack{\mathbf{m}\prec\mathbf{n} \\ m_1=0,m_i\neq 0}}\binom{\mathbf{m}-1}{\mathbf{n}-1}\operatorname{Li}_{\mathbf{m}}(\mathbf{z})$$

Note that the $(-1)^{n_1-1}$ is encoded here as $\binom{-1}{n_1-1} = \binom{m_1-1}{n_1-1}$.

Similarly when we expand $\frac{1}{t_d}\prod_{i=1}^{d-1}(t_d - t_i)^{m_i-1}$ with binomial theorem, we get a $\mathbf{t}^{\mathbf{n}}$ whenever $m_i - k_i = n_i$. So this gives a sum over all $\mathbf{m} \prec \mathbf{n}$ with $m_d = 0$ and all other $m_i > 0$. We also pick up a sign $\prod_{i=1}^{d-1}(-1)^{m_i-1-k_i} = \prod_{i=1}^{d-1}(-1)^{n_i-1} = (-1)^{d-1}(-1)^{n_1+\cdots+n_{d-1}}$. As before, we want to include a $\binom{m_d-1}{n_d-1} = (-1)^{n_d-1}$ term. Multiplying in $(-1)^{n_d-1}(-1)^{n_d-1}$ leaves the summation unchanged but makes the leftover sign, $(-1)^d(-1)^{\sum n_i} = (-1)^d(-1)^{\sum m_i}$. Therefore the coefficient of $t^{\mathbf{n}-1}$ is:

$$(-1)^d \sum_{\substack{\mathbf{m}\prec\mathbf{n} \\ m_d=0, m_i>0}} \binom{\mathbf{m}-1}{\mathbf{n}-1}(-1)^{\sum m_i} \operatorname{Li}_{\overleftarrow{\mathbf{m}}}(1/\overleftarrow{\mathbf{z}})$$

Putting this all together gives us

$$\begin{aligned}
&\operatorname{Li}_{\mathbf{n}}(\mathbf{z}) + (-1)^d(-1)^{\sum n_i}\operatorname{Li}_{\overleftarrow{\mathbf{n}}}(1/\overleftarrow{\mathbf{z}}) \\
&\quad = \sum_{\substack{\mathbf{m}\prec\mathbf{n} \\ m_1=0, m_i>0}} \binom{\mathbf{m}-1}{\mathbf{n}-1}\operatorname{Li}_{\mathbf{m}}(\mathbf{z}) + (-1)^{2d}\sum_{\mathbf{m}\prec\mathbf{n}} \binom{\mathbf{m}-1}{\mathbf{n}-1}(-1)^{\sum m_i}\operatorname{Li}_{\overleftarrow{\mathbf{m}}}(1/\overleftarrow{\mathbf{z}})
\end{aligned}$$

We then can inductively apply our relation to all the terms in the final sum since $m_d = 0$, $\mathbf{m}$ has smaller depth than $\mathbf{n}$. The final summation then becomes:

$$\begin{aligned}
&\sum_{\substack{\mathbf{m}\prec\mathbf{n} \\ m_d=0, m_i>0}} \binom{\mathbf{m}-1}{\mathbf{n}-1}\left((-1)\sum_{\mathbf{p}\preceq\mathbf{m}} c_{\mathbf{p}}\binom{\mathbf{p}-1}{\mathbf{m}-1}\operatorname{Li}_{\mathbf{p}}(\mathbf{z})\right) \\
&\qquad = -\sum_{\substack{\mathbf{p}\prec\mathbf{n} \\ p_d=0}} c_{\mathbf{p}}\left(\sum_{\substack{\mathbf{p}\preceq\mathbf{m}\prec\mathbf{n} \\ m_d=0, m_i>0}} \binom{\mathbf{m}-1}{\mathbf{n}-1}\binom{\mathbf{p}-1}{\mathbf{m}-1}\right)\operatorname{Li}_{\mathbf{p}}(\mathbf{z}) \\
&\qquad = -\sum_{\substack{\mathbf{p}\prec\mathbf{n} \\ p_d=0}} c_{\mathbf{p}}\binom{\mathbf{p}-1}{\mathbf{n}-1}\operatorname{Li}_{\mathbf{p}}(\mathbf{z})
\end{aligned}$$

The final equality is by Lemma 3.2.13 showing that $\binom{\mathbf{p}-1}{\mathbf{n}-1} = \sum_{\substack{\mathbf{p}\preceq\mathbf{m}\prec\mathbf{n} \\ m_d=0, m_i>0}} \binom{\mathbf{p}-1}{\mathbf{m}-1}\binom{\mathbf{m}-1}{\mathbf{p}-1}$. Substituting this transformation into our relation gives:

$$
\begin{aligned}
&\mathrm{Li}_{\mathbf{n}}(\mathbf{z}) + (-1)^{\sum n_i} \mathrm{Li}_{\overleftarrow{\mathbf{n}}}(1/\overleftarrow{\mathbf{z}}) \\
&\quad = \sum_{\substack{\mathbf{m}\prec\mathbf{n} \\ m_1=0, m_i>0}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \mathrm{Li}_{\mathbf{m}}(\mathbf{z}) - \sum_{\substack{\mathbf{p}\prec\mathbf{n} \\ p_d=0}} c_{\mathbf{p}} \binom{\mathbf{p}-1}{\mathbf{n}-1} \mathrm{Li}_{\mathbf{p}}(\mathbf{z}) \\
&\quad = - \sum_{\substack{\mathbf{m}\prec\mathbf{n} \\ m_1=0, m_i>0}} c_{\mathbf{m}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \mathrm{Li}_{\mathbf{m}}(\mathbf{z}) - \sum_{\substack{\mathbf{p}\prec\mathbf{n} \\ p_d=0}} c_{\mathbf{p}} \binom{\mathbf{p}-1}{\mathbf{n}-1} \mathrm{Li}_{\mathbf{p}}(\mathbf{z}) \\
&\quad = - \sum_{\mathbf{m}\prec\mathbf{n}} c_{\mathbf{m}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \mathrm{Li}_{\mathbf{m}}(\mathbf{z})
\end{aligned}
$$

Finally since $\binom{\mathbf{n}-1}{\mathbf{n}-1} = 1$ and $n_1 \neq 0$ so $c_{\mathbf{n}} = 1$, we can include $\mathrm{Li}_{\mathbf{n}}(\mathbf{z})$ in the summation on the right to obtain the desired relation modulo products

$$
-(-1)^{\sum n_i} \mathrm{Li}_{\overleftarrow{\mathbf{n}}}(1/\overleftarrow{\mathbf{z}}) = \sum_{\mathbf{m}\preceq\mathbf{n}} c_{\mathbf{n}} \binom{\mathbf{m}-\mathbf{1}}{\mathbf{n}-\mathbf{1}} \mathrm{Li}_{\mathbf{m}}(\mathbf{z})
$$

Then since the forms satisfy all the relations of polylogarithms modulo products this lifts to the relation

$$
-(-1)^{\sum n_i} \omega_{\overleftarrow{\mathbf{n}}}(1/\overleftarrow{\mathbf{z}}) = \sum_{\mathbf{m}\preceq\mathbf{n}} c_{\mathbf{n}} \binom{\mathbf{m}-\mathbf{1}}{\mathbf{n}-\mathbf{1}} \omega_{\mathbf{m}}(\mathbf{z})
$$

□

3.3.1.2 Index Reversing Relations

Theorem 3.3.3 (Index Reversing Relation)**.** *For any vector* $\mathbf{n}$ *of depth* d *we have:*

$$-(-1)^d \omega_{\overleftarrow{\mathbf{n}}} = \sum_{\mathbf{c}\in Comp(d)} \widehat{r}^*_{\mathbf{c}} \omega_{\mathbf{c}\cdot\mathbf{n}}$$

Proof. This is analogous to Theorem 4.1 of [39] about reversing the arguments of iterated integrals. In this case we can find an appropriate combination of stuffle relations that results in the above formula. To simplify notation we write $(\mathbf{x})(\mathbf{y})$ to represent stuffle relation from multiplying $\mathrm{Li}_{\mathbf{m}}(\mathbf{x}) \cdot \mathrm{Li}_{\mathbf{n}}(\mathbf{y})$.

In depth 2 the stuffle relation for $(x)(y)$ is $\omega_{m,n}(x,y) + \omega_{n,m}(y,x) + \omega_{m+n}(xy) = 0$ which is exactly the relation needed.

For higher depth consider the following sum of stuffle relations

$$\sum_{k=1}^{d-1} (-1)^k (z_k, \ldots, z_1) \sum_{\mathbf{c}\in\mathrm{Comp}(d-k)} \mathbf{c} \cdot (z_{k+1}, \ldots, z_d)$$

For $\mathbf{c} \in \mathrm{Comp}(d-k)$, let $S_k(\mathbf{c})$ be the set of stuffles of $(z_k, \ldots, z_1)$ and $\mathbf{c}\cdot(z_{k+1}, \ldots, z_d) = (y_{k+1}, \ldots, y_{k+\ell})$. Following Goncharov, we partition $S_k(\mathbf{c})$ into three sets $S_k^<(\mathbf{c})$, $S_k^=(\mathbf{c})$, $S_k^>(\mathbf{c})$ based on whether z_k comes before, is stuffed with, or comes after y_{k+1}. We then observe that $S_k^>(\mathbf{c}) = \begin{cases} S_{k+1}^<(c_2, \ldots, c_\ell) & c_1 = 1 \\ S_{k+1}^=(c_1 - 1, c_2, \ldots, c_\ell) & c_1 > 1 \end{cases}$. Therefore in the alternating sum every $S_k^>(\mathbf{c})$ cancels out all of the $S_{k+1}^<$ and $S_{k+1}^=$. This leaves $S_1^<(\mathbf{c})$, $S_1^=(\mathbf{c})$ and $S_d^>(\mathbf{c})$. Each of these sets only contain a single vector. The terms that correspond to compositions of the form $1, \mathbf{c}$ come from $S_1^<(\mathbf{c})$. Similarly the terms indexed by $c_1, \mathbf{c}$ with $c_1 > 1$ correspond to $S_1^{=}(c_1 - 1, \mathbf{c})$. The fully reversed term, with the coefficient $(-1)^d$ comes from the final set $S_{d-1}^{>}((1))$. □

3.3.1.3 Inversion Relation

Theorem 3.3.4 (Inverse Relation)**.** *For any vector* $\mathbf{n}$ *with* $\sum n_i > 1$ *we have*

$$(-1)^d(-1)^{\sum n_i}\omega_{\mathbf{n}}(1/\mathbf{z}) = \sum_{\mathbf{m} \preceq \mathbf{n}} c_{\mathbf{m}} \binom{\mathbf{m}-1}{\mathbf{n}-1} \sum_{\mathbf{c}} \widehat{r}_{\mathbf{c}} \omega_{\mathbf{c}\cdot\mathbf{m}}$$

where $c_{\mathbf{m}} = \begin{cases} -1 & m_1 = 0 \\ 1 & m_1 \neq 0 \end{cases}$ *and the inner sum is over all compositions* $\mathbf{c}$ *of the number of nonzero entries of* $\mathbf{m}$*.*

Proof. This follows from taking the inversion reversing relation (Theorem 3.3.2) with $\overleftarrow{\mathbf{n}}$. Then for each term of the summation apply the index reversing relation (Theorem 3.3.3) to obtain a sum of contractions. □

3.3.2 Dynkin Reversing Relations

We now use the recurrence relation and inverse relations together to remove the ambiguity of extracting terms from an A_n cluster algebra. Recall the Dynkin quiver in type A_n is a path oriented so each node is a source or sink. In odd n there is an element of the cluster modular group, σ reversing the order of the path. Therefore any relation given by specifying coordinates from such a quiver must be invariant under σ.

We will see the arguments to the high depth polylogarithms consist of "factorizations" of the Casimir element along the Dynkin quivers. For example, for each Dynkin quiver in A_3 we label the X-coordinates at the sources x_1 and x_3. The Casimir element is then x_1/x_3. However if we apply σ this swaps x_1 and x_3 resulting in the inverse of the Casimir x_3/x_1. Thus if we use $\omega_{m,1}(\frac{x_1}{x_3}, x_3)$ we need to understand how this relates to $\omega_{m,1}(\frac{x_3}{x_1}, x_1)$.

In order to make it easier to distinguish the Casimir from its inverse we take $x_1 = x$ and $x_3 = 1/y$ so the Casimir becomes xy. We then have the following theorem:

Theorem 3.3.5 (Depth 2 Flip)**.** *For all* m, $\omega_{m,1}(xy, \frac{1}{x}) - (-1)^m \omega_{m,1}(\frac{1}{xy}, y) = 0$

Proof. This can be verified using the recurrence relation in depth 2. The base case $m = 1$ can be confirmed via simple calculation. Then for $m > 1$ the recurrence relation simplifies to

$$\omega_{m,1} = \frac{1}{m+1} \left(-u_1\omega_{m-1,1} + v_2\omega_m(xy) - v_2\omega_m(x) + (-1)^m v_{12}\omega_m(y) - v_{12}\omega_m(x)\right)$$

Since everything will be multiplied by $\frac{1}{m+1}$ we drop the fraction for the remainder of the computation. Expanding the relation we see:

$$\begin{aligned}
&\omega_{m,1}(xy,\frac{1}{x}) - (-1)^m\omega_{m,1}(\frac{1}{xy},y)\\
&\quad= -(u_1+u_2)\omega_{m-1,1}(xy,\frac{1}{x}) + (v_1-u_1)\omega_m(y) - (v_1-u_1)\omega_m(xy)\\
&\qquad + (-1)^m v_2\omega_m(\frac{1}{x}) - v_2\omega_m(xy)\\
&\qquad - (-1)^m(u_1+u_2)\omega_{m-1,1}(\frac{1}{xy},y) - (-1)^m(v_2)\omega_m(1/x) + (-1)^m v_2\omega_m(1/(xy))\\
&\qquad - (v_1-u_1)\omega_m(y) + (-1)^m(v_1-u_1)\omega_m(\frac{1}{xy})\\
&\quad= -(u_1+u_2)\left(\omega_{m-1,1}(xy,\frac{1}{x}) - (-1)^{m-1}\omega_{m-1,1}(\frac{1}{xy},y)\right)\\
&\qquad - (v_1-u_1+v_2)\left(\omega_m(xy) - (-1)^m\omega_m(\frac{1}{xy})\right)\\
&\quad= 0
\end{aligned}$$

The final equality is 0 by the inductive hypothesis and the depth 1 inversion relation.

□

Interestingly we can combine the depth 2 flip with inversion to obtain a relation for applying σ without inverting the Casimir.

Corollary 3.3.6 (Dynkin Reversal Depth 2)**.** *For all m, the following expression is trivial:*

$$\omega_{m,1}(xy,1/x) + \omega_{m,1}(xy,1/y) + m\cdot\omega_{m+1}(xy) + \omega_{m+1}(x) + \omega_{m+1}(y)$$

Proof. Apply the inverse relation to $\omega_{m,1}(1/(xy), y)$ to obtain

$$(-1)^{m+1}\omega_{m,1}(1/(xy), y) = \omega_{m,1}(xy, 1/y) + \omega_{m+1}(x) - (-1)^{m-1}\omega_{m+1}(1/y) + m \cdot \omega_{m+1}(xy)$$

Adding $\omega_{xy,1/x}$ to both sides yields:

$$\omega_{m,1}(xy, 1/x) - (-1)^{m}\omega_{m,1}(1/(xy), y) = \omega_{m,1}(xy, 1/x) + \omega_{m,1}(xy, 1/y) + \omega_{m+1}(x) + \omega_{m+1}(y) + m \cdot \omega_{m+1}(xy)$$

The left hand side is exactly the depth 2 flip relation (Theorem 3.3.5) and so we see the right hand side is 0 as needed. □

The situation in A_5 is slightly different. Here the Casimir can be written as a product of three X-coordinates $\frac{x_1 x_3}{x_2}$ and is preserved by σ which swaps x_1 and x_3. Thus we would like to relate $\omega_{m,1,1}(\frac{x_1 x_3}{x_2}, \frac{1}{x_1}, x_2)$ with $\omega_{m,1,1}(\frac{x_1 x_3}{x_2}, \frac{1}{x_3}, x_2)$. To write a general relation we take $x_1 = x$, $x_2 = 1/y$ and $x_3 = z$ and obtain the following theorem:

Theorem 3.3.7 (Dynkin Reversal Depth 3)**.** *For all m the following expression is trivial:*

$$\omega_{m,1,1}(xyz, 1/x, 1/y) + \omega_{m+1,1}(xy, 1/y) = \omega_{m,1,1}(xyz, 1/z, 1/y) + \omega_{m+1,1}(yz, 1/z)$$

Proof. This can be shown using the recurrence formula in depth 3. □

Depth 4 should be analogous to depth 2, as the Casimir in A_7 is flipped by σ. However the inversion relation in depth 4 is not only long, but requires a wider range of multiple polylogarithms. As such the depth 4 flip is not particularly useful. Nevertheless we have found the combined Dynkin reversal relation in depth 4:

Theorem 3.3.8 (Dynkin Reversal Depth 4)**.** *For all m, the following expression is trivial:*

$$+2\omega_{m,1,1,1}(xyzw, 1/x, 1/y, 1/z) + 2\omega_{m,1,1,1}(xyzw, 1/w, 1/z, 1/y)$$

$$+2\omega_{m+1,1,1}(xyz, 1/z, 1/y) + 2\omega_{m+1,1,1}(yzw, 1/y, 1/z)$$

$$+\omega_{m,1,2}(xyzw, 1/z, 1/(yz)) + \omega_{m,1,2}(xyzw, 1/w, 1/(yz))$$

$$+\omega_{m+1,2}(xyz, 1/(yz)) + \omega_{m+1,2}(yzw, 1/(yz))$$

Proof. As in the depth 3 case this can be shown inductively using the recurrence. □

Conjecture 3.3.9 (Dynkin Reversal Arbitrary Depth)**.** *There is a Dynkin reversing relation for any depth.*

3.4 Relations on A_n Cluster Algebras

We now focus on extracting the polylogarithm relations from the A_n cluster algebras. We see that the arguments in each relation are cluster X-coordinates or Casimir elements of A_{2k-1} cluster algebras. For $n \leq 5$ we present the relations so that the cluster symmetry is obvious and every coefficient is ± 1. This mirrors work in [6] to compute analogous relations using the symbol and without explicit links to the cluster algebra structure.

3.4.0.1 Subalgebra Structure

We will see that the terms of each relation come from the odd weight subalgebras of A_n. Each A_1 subalgebra has a single X-coordinate. This is trivially the Casimir element of the A_1 cluster algebra. So for each A_1 subalgebra we define

$$\mathcal{L}_n^C(A_1) = \omega_n(x)$$

Note that the depth 1 inversion relation $\omega_n(x) = -(-1)^n\omega_n(1/x)$ explains the ambiguity of choosing x or x^{-1}. As such $\mathcal{L}_n^C(A_1)$ is well defined up the "orientation" of the subalgebra.

This situation generalizes to higher weight subalgebras. While any A_3 cluster algebra has 15 distinct X-coordinates, it has a unique Casimir element (up to inverse). This Casimir element can be written as a product of X coordinates in three distinct ways corresponding, to the three seeds whose underlying quiver has the form $x_i \leftarrow \bullet \rightarrow y_i$. If we write each of these factorizations as $x_1/y_1 = x_2/y_2 = x_3/y_3$ we obtain the following quantity:

$$\mathcal{L}_n^C(A_3) = -(n-1)\omega_n(x_1/y_1) + \sum_i \omega_{n-1,1}(x_i/y_i, 1/x_i) - \omega_{n-1,1}(x_i/y_i, y_i)$$
$$+ (-1)^n\omega_n(1/x_i) + (-1)^n\omega_n(y_i)$$

Note that we have a quiver automorphism σ that switches x_i and y_i. Applying σ we obtain

$$-(n-1)\omega_n(y_1/x_1) + \sum_i \omega_{n-1,1}(y_i/x_i, 1/y_i) - \omega_{n-1,1}(y_i/x_i, x_i)$$
$$+ (-1)^n\omega_n(1/y_i) + (-1)^n\omega_n(x_i)$$

We then use the depth 2 flip (Theorem 3.3.5) on the $\omega_{n-1,1}$ terms and standard inversion on the ω_n terms to obtain

$$(n-1)(-1)^n\omega_n(x_1/y_1) + \sum_i (-1)^{n-1}\omega_{n-1,1}(x_i/y_i, 1/x_i) - (-1)^{n-1}\omega_{n-1,1}(x_i/y_i, y_i)$$
$$- \omega_n(y_i) - \omega_n(1/x_i)$$
$$= (-1)^{n-1}\Bigg(-(n-1)\omega_n(x_1/y_1) + \sum_i \omega_{n-1,1}(x_i/y_i, 1/x_i) - \omega_{n-1,1}(x_i/y_i, y_i)$$
$$(-1)^n\omega_n(y_i) + (-1)^n\omega_n(1/x_i)\Bigg)$$

As such $\mathcal{L}_n^C(A_3)$ behaves analogously to the A_1 case.

Similarly each A_5 subalgebra has a unique Casimir element C_5 that can be written as a product of 3 X coordinates. There are 12 distinct factorizations that come in 4 sets of 3 corresponding the Dynkin quiver and the two neighbors given by mutating at a single sink:

$$x_i^L \to \bullet \leftarrow x_i^M \to \bullet \leftarrow x_i^R$$

$$y_i^L \leftarrow \bullet \to y_i^M \to \bullet \leftarrow x_i^R$$

$$x_i^L \to \bullet \leftarrow z_i^M \to \bullet \leftarrow z_i^R$$

Here the full Casimir element is $C_5 = x_i^M/(x_i^L x_i^R) = y_i^M/(y_i^L x_i^R) = z_i^M/(x_i^L z_i^R)$. Furthermore, mutating at one sink preserves the A_3 type Casimir corresponding to that A_3. So we for each factorization we define

$$\begin{aligned} f(L,M,R) =& \omega_{n-2,1,1}\left(\frac{M}{LR}, L, \frac{1}{M}\right) + \omega_{n-2,1,1}\left(\frac{M}{LR}, R, \frac{1}{M}\right) \\ &+ \frac{1}{2}\left(\omega_{n-1,1}\left(\frac{M}{L}, L\right) - \omega_{n-1,1}\left(\frac{M}{L}, \frac{1}{M}\right) - (n-1)\omega_n\left(\frac{M}{L}\right) + \omega_n(L)\right) \\ &+ \frac{1}{2}\left(\omega_{n-1,1}\left(\frac{M}{R}, R\right) - \omega_{n-1,1}\left(\frac{M}{R}, \frac{1}{M}\right) - (n-1)\omega_n\left(\frac{M}{R}\right) + \omega_n(R)\right) \end{aligned}$$

We apply f to every possible factorization and include a term for the full Casimir element to obtain the following

$$\begin{aligned} \mathcal{L}_n^C(A_5) =& -2(n+1)\omega_n(C_5) \\ &+ \sum_i f(x_i^L, x_i^M, x_i^R) + \frac{1}{2}\omega_n(x_i^M) - f(y_i^L, y_i^M, x^R) - f(x_i^L, z_i^M, z_i^R) \end{aligned}$$

Remark 3.4.1. *The non-integer coefficients in $\mathcal{L}_n^C(A_5)$ are necessary to easily describe the coefficients in the relation on the A_5 cluster algebra. We can always multiply everything described in the weight 5 discussion by 2 to have integer coefficients.*

Remark 3.4.2. *Since the function $f(L, M, R)$ is defined to be symmetric under swapping L and R it is clear that $\mathcal{L}_n^C(A_5)$ is fixed by the cluster modular group of an A_5 cluster algebra.*

3.4.0.2 Relation on A_1

Recall that we define $\omega_1(u) = \mathrm{d}u$. Let x be the unique X-coordinate on an A_1 cluster algebra. Then we have:

$$\omega_1(x) + \omega_1(1/x) = \mathrm{d}x + -\mathrm{d}x = 0$$

This is the basic relation that we call $Q_1(A_1)$.

Remark 3.4.3. *In the Grassmannian we have $Q_1(Gr(2,4))$ is:*

$$\omega_1\left(\frac{p_{12}p_{34}}{p_{14}p_{23}}\right) + \omega_1\left(\frac{p_{14}p_{23}}{p_{12}p_{34}}\right)$$

3.4.0.3 Relation on A_2

We have already seen the five term relation for the dilogarithm. Using our new language the 5 term relation in weight 2 becomes:

$$Q_2(A_2) = \sum_{A_1 \subseteq A_2} \mathcal{L}_2^C(A_1)$$

For this to be unambiguous we need to consistently choose the orientation of each A_1. In general this can be done by picking a starting x and then using the cluster modular group to choose all the identical x_i on other Dynkin quivers. For A_2 we can be more explicit since each A_1 is incident to a Dynkin quiver. Here we take x from each A_1 such that x is X-coordinate of a source in the Dynkin quiver.

Remark 3.4.4. *There is a single orbit of A_1 subalgebras in A_2. Since the coefficient of every term in the orbit is the same, we can view this relation as a single term.*

Remark 3.4.5. *Using $Gr(2,5)$ as the A_2 cluster algebra we see that $Q_2(Gr(2,5))$ is*

$$\omega_2\left(\frac{p14p23}{p12p34}\right) + \omega_2\left(\frac{p12p35}{p15p23}\right) + \omega_2\left(\frac{p13p45}{p15p34}\right) + \omega_2\left(\frac{p15p24}{p12p45}\right)$$

3.4.0.4 Relation on A_3

Using our new cluster functions and knowledge of orbits to rewrite Goncharov's 22 term relation as a sum over all the A_3 and A_1 data in the cluster algebra. In $\mathrm{Gr}(2,6)$ the map from X-coordinate to ratio of A-coordinates is injective so we refer to X-coordinates by the corresponding ratio of A-coordinates. There are three orbits of X coordinates which we refer to by a representative coordinates: $\frac{\text{p14p23}}{\text{p12p34}}$, $\frac{\text{p15p23}}{\text{p12p35}}$, and $\frac{\text{p15p24}}{\text{p12p45}}$. Then the relation Q_3 is:

$$0 = \mathcal{L}_3^C(A_3) + 2\mathcal{L}_3^C\left(\frac{\text{p14p23}}{\text{p12p34}}\right) - 2\mathcal{L}_3^C\left(\frac{\text{p15p23}}{\text{p12p35}}\right) + 2\mathcal{L}_3^C(\frac{\text{p15p24}}{\text{p12p45}})$$

Note that in odd weight $\mathcal{L}_3^C(A_3)$ is unambiguous as $(-1)^{3-1} = 1$. We note under this phrasing it is obvious the relation is fixed by the cluster modular group. See Figure 3.2 to see the symmetry of the relation in the cluster complex. Each edge corresponds to an X coordinate and red edges correspond the coefficient -2, while the blue edges have coefficient 2.

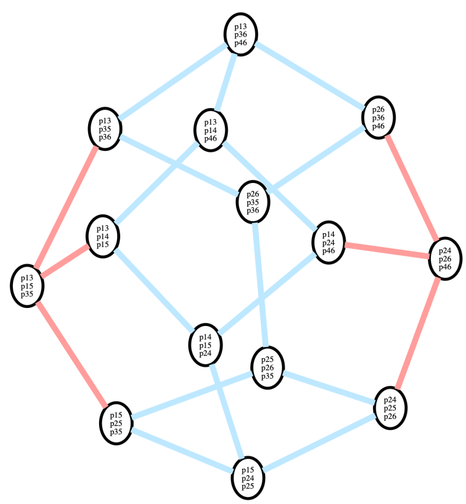


Figure 3.2: The cluster symmetry of the Q_3 relation.

3.4.0.5 Relation on A_4

Even more impressively the A_4 relation can be written a sum over all the A_1 and A_3 data where the coefficient of every term A_3 term is 1 and every A_1 term is

2.

$$0 = \sum_{A_3 \in A_4} \mathcal{L}_4^C(A_3) + \sum_{A_1 \in A_3} 2\mathcal{L}_4^C(A_1)$$

Similarly to the A_2 case, there is a choice of "orientation" for each A_3 that determines the sign of each $\mathcal{L}_4^C(A_3)$. From a cluster perspective each A_4 Dynkin quiver has two overlapping A_3 Dynkin quiver only one of which matches our standard choice of sources/sink orientation. This A_3 can be considered the source and the other A_3 is the sink. Furthermore the source A_3 has well defined outside/inside in the A_4 giving a consistent choice of initial x_1, y_1 in the A_3. This fixes a consistent orientation of the Casimir x_1/y_1.

See Appendix B.2 to see the full Q_4 relation with every term written explicitly in ratios of Plücker coordinates.

3.4.0.6 Relation on A_5

Following the pattern the relation Q_5 on A_5 consists of a signed sum over all the A_3 and A_1 subalgebras.

$$\mathcal{L}_5^C(A_5) + \sum_{A_3 \in A_5} c_{A_3}\mathcal{L}_5^C(A_3) + \sum_{A_1 \in A_5} c_{A_1}\mathcal{L}_5^C(A_1)$$

The A_5 cluster algebra has 4 orbits of A_3 subalgebras with distinct Casimir elements. We write a representative of each orbit as they appear in $\mathrm{Gr}(2,8)$.

$$\frac{\text{p12p34p56}}{\text{p16p23p45}} \quad \frac{\text{p13p46p78}}{\text{p18p34p67}} \quad \frac{\text{p13p45p67}}{\text{p17p34p56}} \quad \frac{\text{p13p46p78}}{\text{p18p34p67}}$$

The first two orbits appear with a coefficient of 1 and the second two appear with a coefficient of −1. Similarly there are 10 orbits of X-coordinates/A_1 subalgebras. Four orbits get coefficient 2:

$$\frac{\mathrm{p13p45}}{\mathrm{p15p34}} \quad \frac{\mathrm{p15p67}}{\mathrm{p17p56}} \quad \frac{\mathrm{p12p57}}{\mathrm{p17p25}} \quad \frac{\mathrm{p12p46}}{\mathrm{p16p24}}$$

The remaining orbits all have coefficient −2:

$$\frac{\mathrm{p12p56}}{\mathrm{p16p25}} \quad \frac{\mathrm{p12p45}}{\mathrm{p15p24}} \quad \frac{\mathrm{p12p47}}{\mathrm{p17p24}} \quad \frac{\mathrm{p12p34}}{\mathrm{p14p23}} \quad \frac{\mathrm{p14p56}}{\mathrm{p16p45}} \quad \frac{\mathrm{p13p57}}{\mathrm{p17p35}}$$

Remark 3.4.6. *Under the full cluster modular group, the orbit of* $\frac{p13p57}{p17p35}$ *includes its inverse. As such it technically appears with coefficient* −1*. However we can use the inversion relation in weight 5 to fix an orientation such that each term has a coefficient of 2.*

3.5 Relations on D_n Cluster Algebras

A key advantage of the cluster algebra formulation of the relations is we can easily embed relations in larger subalgebras. These subalgebra relations can overlap in interesting ways and thus can be combined to cancel out terms. Our first example recovers the 40 term relation on $D_4 = \mathrm{Gr}(3, 6)$ this way.

3.5.1 Relation on D_4

There are 12 A_3 subalgebras of D_4 corresponding to freezing each tail of the D_4 Dynkin diagram. We label the Q_3 relation corresponding to the sub-algebra obtained by freezing a, $Q_3(a)$. Furthermore these can be separated into 3 orbits under the action of the sources sinks path, g_S. Recall that g_S corresponds to the "parity map" in Gr(3, 6) (Remark 1.2.19).

Theorem 3.5.1. *Let a be an A-coordinate on the tail of a D_4 Dynkin type quiver. Then $\frac{1}{2}\sum_{i=0}^{3}(-1)^i Q_3(g_S^i a)$ consists only of ω_3 terms with coefficients ± 1. In fact this is the 40 term relation that is fixed under the full action of the D_4 cluster modular group.*

Proof. The following table (Figure 3.3) shows the X coordinates in positions 1 and 3 in each Dynkin quiver in one orbit of A_3 subalgebras.

Freeze 125		Freeze 346		Freeze 245		Freeze 136	
X_1	X_3^{-1}	X_1	X_3^{-1}	X_1	X_3^{-1}	X_1	X_3^{-1}
$\frac{p123\cdot p345}{p234\cdot p135}$	$\frac{p135\cdot p456}{p345\cdot p156}$	$\frac{p234\cdot p156}{e2x}$	$\frac{e2x}{p123\cdot p456}$	$\frac{p456\cdot p126}{p156\cdot p246}$	$\frac{p246\cdot p123}{p126\cdot p234}$	$\frac{p234\cdot p156}{e2y}$	$\frac{e2y}{p123\cdot p456}$
$\frac{p123\cdot p456}{e2x}$	$\frac{e2x}{p234\cdot p156}$	$\frac{p234\cdot p126}{p123\cdot p246}$	$\frac{p246\cdot p156}{p126\cdot p456}$	$\frac{p456\cdot p123}{e2y}$	$\frac{e2y}{p156\cdot p234}$	$\frac{p156\cdot p345}{p456\cdot p135}$	$\frac{p135\cdot p234}{p345\cdot p123}$
$\frac{p456\cdot p125}{p156\cdot p245}$	$\frac{p245\cdot p123}{p125\cdot p234}$	$\frac{p156\cdot p234}{p456\cdot p136}$	$\frac{p136\cdot p234}{p346\cdot p123}$	$\frac{p123\cdot p245}{p234\cdot p125}$	$\frac{p125\cdot p456}{p245\cdot p156}$	$\frac{p234\cdot p136}{p123\cdot p346}$	$\frac{p346\cdot p156}{p136\cdot p456}$

Figure 3.3: The arguments to $\omega_{2,1}$ terms in the relation on Gr(3, 6).

From this table we see that all the $\{-,-\}_{2,1}$ terms vanish. The second row of each column matches the first row of the column to the right but with X_1 and X_3 swapped. We saw that $\{-,-\}_{2,1}$ terms are fixed under this transformation and so cancel under the alternating signs. Similarly the third row matches two columns to the right under the transform $X_1 \leftrightarrow X_3^{-1}$. This swaps the positive and negative $\{-,-\}_{2,1}$ terms and so also these terms cancel. The Casimir term $-2\{X_1 \cdot X_3^{-1}\}_3$ is identical (or the inverse which is the same) in all 4 A_3 subalgebras and so also cancels out. The $\{X_1\}_3$ and $\{X_3^{-1}\}_3$ terms cancel from the first two rows and pick up a coefficient of 2 from the third row. The remaining 36 terms from $2\{X_2\}_3$, $-2\{\frac{1+X_3}{X_2X_3}\}$, $-2\{\frac{1+X_1}{X_1X_3}\}$ are unique in each subalgebra. Combined with the 4 uncanceled terms this gives 40 terms, each with a coefficient of ± 2 entirely in $\{-\}_3$ with X coordinates as arguments.

□

Corollary 3.5.2. *The 40 term relation is fixed (up to sign) by the full cluster modular group of D_4.*

Proof. Recall that the cluster modular group of a D_4 cluster algebra is $\mathbb{Z}_4 \times S_3$ (Figure 1.15). This can be presented from a sources sink Dynkin cluster, where the $\mathbb{Z}_4$ is generated by the sources sink element g_S and the S_3 is the symmetry group of the D_4 quiver. We defined the relation via a sum over the orbit of g_S and so g_S clearly preserves the relation up to a sign. A similar computation to what was done above shows that we obtain the same relation from the other 2 orbits of A_3 subalgebras (corresponding to freezing a different tail of the D_4). So this relation is fixed under rotating the tails. Swapping two tails is the same as switching X_1 and X_3 in an A_3 and so also preserves the 40 relation. Therefore the 40 term relation is fixed under the entire cluster modular group. □

See Figure 3.4 to see the cluster symmetry of the relation in the cluster complex. Once again we color edges blue for positive coefficients in the relation and red for negative coefficients. The black edges correspond to X-coordinates that are absent from the relation. Note in α_4 these coefficients are ± 2, but can be reduced to ± 1.

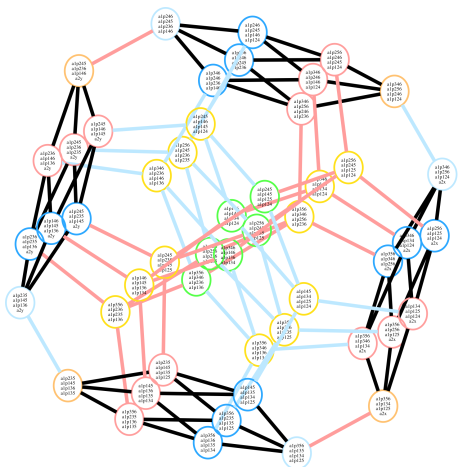


Figure 3.4: The relation on D_4 consisting entirely of $\omega_3(X)$ terms.

3.5.2 Relation on D_6

There is a similar result for A_5 relations in D_6.

Theorem 3.5.3. *Let $Q_5(a)$ be the weight 5 relation obtained from the A_5 subalgebra of D_6 where a is frozen. Let g_S be the element of cluster modular group corresponding to the sources/sinks path and σ be the element that swaps the two short tails of the Dynkin diagram. Then $\alpha_6 = \sum_{i=0}^{5} Q_5(g_S^i a) - Q_5(g_S^i \sigma a)$ has no depth 2 terms. In other words this is a relation with only ω_5 and ω_{311} terms.*

Proof. This can be shown via a long but straightforward computation. The remaining 12 ω_5 terms consist of a single orbit under the cluster modular group. There are

96 ω_{311} terms left. See Appendix B.4 for the full list of ω_{311} terms If we consider the D_6 subalgebra of Gr(3,8) given by freezing p467 and p378 we can write the X-coordinates as ratios of Plücker and exotic coordinates[3]:

$$
\begin{aligned}
2&\left(\omega_5\left(\frac{\mathrm{e2x45}}{\mathrm{p}_{128}\mathrm{p}_{367}}\right)+\omega_5\left(\frac{\mathrm{e2x45}}{\mathrm{p}_{178}\mathrm{p}_{236}}\right)+\omega_5\left(\frac{\mathrm{p}_{123}\mathrm{p}_{178}\mathrm{p}_{368}}{\mathrm{p}_{128}\mathrm{p}_{136}\mathrm{p}_{378}}\right)\right.\\
&\left.\qquad+\omega_5\left(\frac{\mathrm{p}_{136}\mathrm{p}_{234}}{\mathrm{p}_{123}\mathrm{p}_{346}}\right)+\omega_5\left(\frac{\mathrm{p}_{234}\mathrm{p}_{367}\mathrm{p}_{456}}{\mathrm{p}_{236}\mathrm{p}_{345}\mathrm{p}_{467}}\right)+\omega_5\left(\frac{\mathrm{p}_{368}\mathrm{p}_{467}}{\mathrm{p}_{346}\mathrm{p}_{678}}\right)\right)\\
-2&\left(\omega_5\left(\frac{\mathrm{e2x16p}_{178}}{\mathrm{e2x36p}_{378}}\right)+\omega_5\left(\frac{\mathrm{e2x16p}_{467}}{\mathrm{p}_{234}\mathrm{p}_{457}\mathrm{p}_{678}}\right)+\omega_5\left(\frac{\mathrm{e2x26p}_{128}}{\mathrm{p}_{123}\mathrm{p}_{178}\mathrm{p}_{458}}\right)\right.\\
&\left.\qquad+\omega_5\left(\frac{\mathrm{e2x26p}_{234}}{\mathrm{p}_{123}\mathrm{p}_{345}\mathrm{p}_{478}}\right)+\omega_5\left(\frac{\mathrm{e2x36}}{\mathrm{p}_{128}\mathrm{p}_{457}}\right)+\omega_5\left(\frac{\mathrm{p}_{458}\mathrm{p}_{467}}{\mathrm{p}_{456}\mathrm{p}_{478}}\right)\right)
\end{aligned}
$$

□

Corollary 3.5.4. *The ω_{311} terms of the D_6 relation correspond to a well defined function.*

Proof. Since $\alpha_6 = 0$, $d\alpha_6 = 0$. Furthermore $d\omega_5$ is always 0, so the combination of ω_{311} terms must also have zero differential. Therefore this combination is a closed form and has a primitive. □

Remark 3.5.5. *If we fix one tail of D_4 to be the "long tail" and compute*

$$\alpha_4 = \sum_{i=0}^{3} Q_3(\sigma^i a) - Q_3(\sigma^i \tau a)$$

we obtain 4 times the 40-term relation.

[3]See Remark 1.1.54 for an explanation of the exotic coordinate notation.

3.5.3 Relation on D_{2k+1}

Unfortunately this technique doesn't yield any new results in D_{2k+1}. Recall that the cluster modular group for D_{2k+1} is cyclic of order $2(2k+1)$ and is generated by g_S (Figure 1.15). Then $\sigma = g_S^{2k+1}$ is the automorphism of the Dynkin diagram swapping the short tails. We decompose the group as $\mathbb{Z}_{2k+1} \times \mathbb{Z}_2$ with generators $h = g_S\sigma$ and σ.

Theorem 3.5.6. *The analogous sum* $\alpha_{2k+1} = \sum_{i=0}^{2k} Q_{2k}(h^i a) - Q_{2k}(h^i \sigma a) = 0$ *for* $k = 1, 2$

Proof. It is a simple computation to take the corresponding sums of the A_2 and A_4 relations in $D_3 = A_3$ and D_5 respectively. □

Based on this information we make the following conjecture:

Conjecture 3.5.7. *For odd* n*, the relation* $\alpha_{n+1} = \sum_{i=0}^{n} Q_n(\sigma^i a) - Q_n(\sigma^i \tau a)$ *has no depth 2 forms. For even* n *the sum* $\alpha_{n+1} = 0$.

Appendix A: Dynkin Diagrams

For reference we include all the finite, affine, and doubly extended Dynkin diagrams. To align with the cluster algebras, we draw the non simply laced diagrams using "fat" nodes whose weight (Figure A.1) corresponds to the number of nodes "folded" together from the simply laced diagram. In the standard root system language these fat nodes correspond to the shorter roots of the root system. In the B, C, F cases the fat nodes are all weight 2. In the BC case there are nodes of weight 2 and 4. The G case has nodes of weight 3.

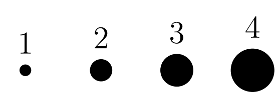
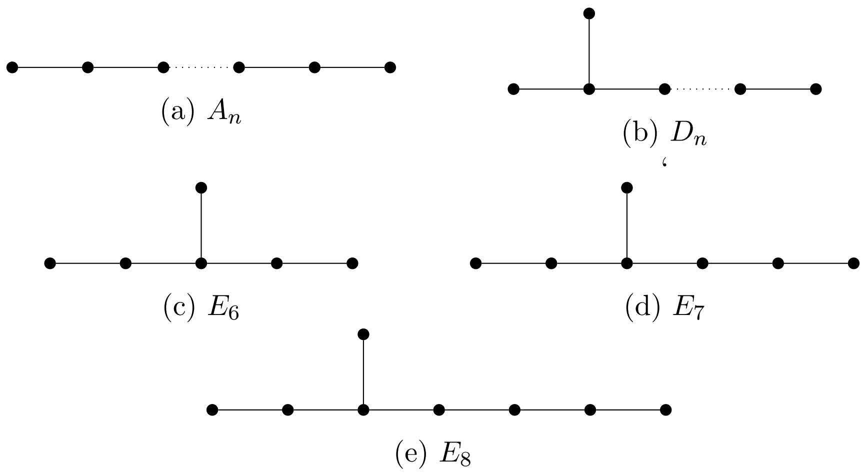
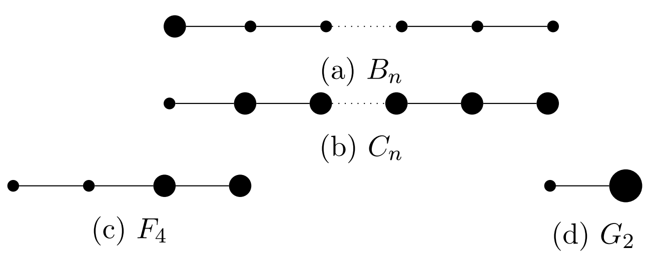


Figure A.1: Weights of nodes in Dynkin Diagrams.



Simply laced finite Dynkin diagrams.

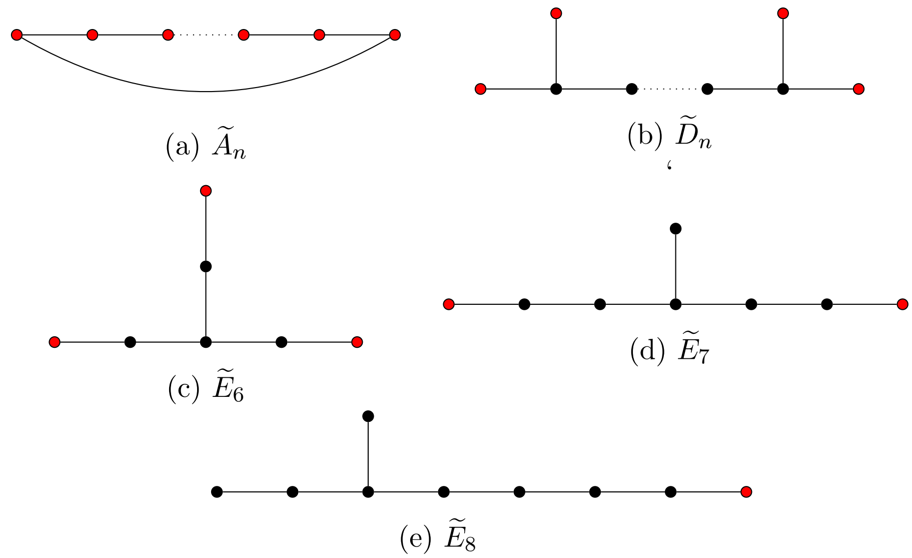
Figure A.2: Simply Laced Finite Dynkin Diagrams.



Folded finite Dynkin diagrams.

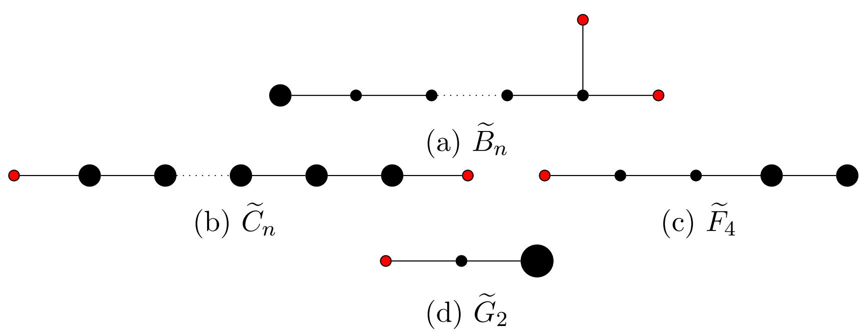
Figure A.3: Folded Finite Dynkin Diagrams.

Each affine diagram can be formed by adding a single node to the corresponding finite diagram. In figures A.4, A.5, A.6 the nodes that could be the extension are colored red.



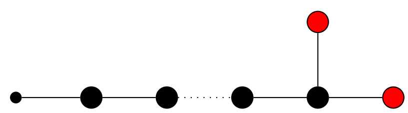
(a) $\widetilde{A}_n$ (b) $\widetilde{D}_n$ (c) $\widetilde{E}_6$ (d) $\widetilde{E}_7$ (e) $\widetilde{E}_8$

Figure A.4: Simply laced Affine Dynkin diagrams.

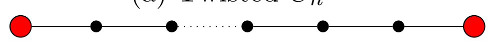


(a) $\widetilde{B}_n$ (b) $\widetilde{C}_n$ (c) $\widetilde{F}_4$ (d) $\widetilde{G}_2$

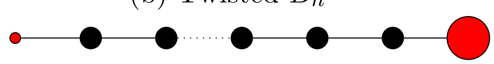
Figure A.5: Folded Affine Dynkin diagrams.



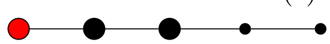
(a) Twisted $C_n^{(2)}$



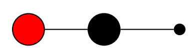
(b) Twisted $B_n^{(2)}$



(c) Twisted $BC_n^{(4)}$



(d) Twisted $F_4^{(2)}$

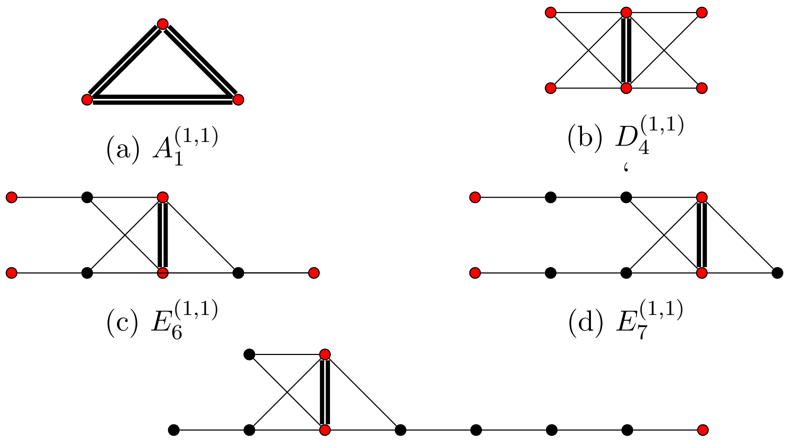


(e) Twisted $G_2^{(3)}$

Twisted Affine Dynkin diagrams.

Figure A.6: Twisted Affine Dynkin Diagrams.

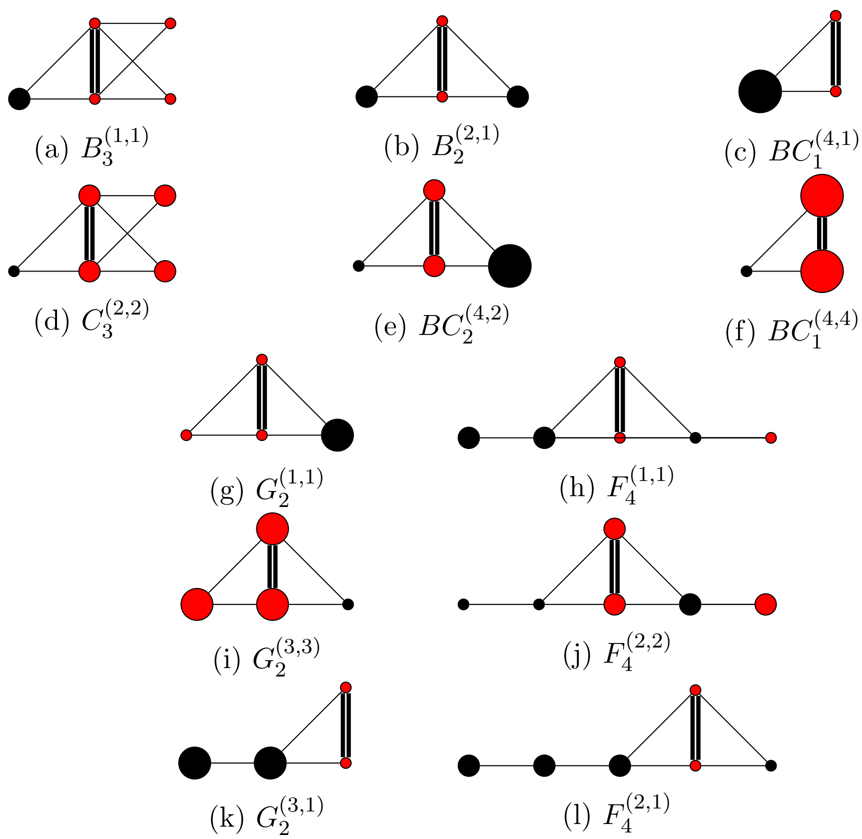
Similarly each double extended diagram can be formed by adding two nodes to a finite diagram or one node to the affine diagram. Each red node in figures A.7, A.8 is a possible extension of the corresponding affine Dynkin diagram.



(a) $A_1^{(1,1)}$ (b) $D_4^{(1,1)}$ (c) $E_6^{(1,1)}$ (d) $E_7^{(1,1)}$ (e) $E_8^{(1,1)}$

Simply laced doubly extended Dynkin diagrams.

Figure A.7: Simply Laced Doubly Extended Dynkin Diagrams.



(a) $B_3^{(1,1)}$ (b) $B_2^{(2,1)}$ (c) $BC_1^{(4,1)}$

(d) $C_3^{(2,2)}$ (e) $BC_2^{(4,2)}$ (f) $BC_1^{(4,4)}$

(g) $G_2^{(1,1)}$ (h) $F_4^{(1,1)}$

(i) $G_2^{(3,3)}$ (j) $F_4^{(2,2)}$

(k) $G_2^{(3,1)}$ (l) $F_4^{(2,1)}$

Folded doubly extended Dynkin diagrams.

Figure A.8: Folded Doubly Extended Dynkin Diagrams.

Appendix B: Full Cluster Relations

B.1 Q_3 Relation on $\mathrm{Gr}(2,6)$

$$
\begin{aligned}
&-\omega_{21}\left(\frac{p_{12}p_{34}p_{56}}{p_{16}p_{23}p_{45}},\frac{p_{14}p_{23}}{p_{12}p_{34}}\right)+\omega_{21}\left(\frac{p_{12}p_{34}p_{56}}{p_{16}p_{23}p_{45}},\frac{p_{16}p_{45}}{p_{14}p_{56}}\right)+\omega_{21}\left(\frac{p_{12}p_{34}p_{56}}{p_{16}p_{23}p_{45}},\frac{p_{23}p_{45}}{p_{25}p_{34}}\right)\\
&-\omega_{21}\left(\frac{p_{12}p_{34}p_{56}}{p_{16}p_{23}p_{45}},\frac{p_{16}p_{25}}{p_{12}p_{56}}\right)+\omega_{21}\left(\frac{p_{12}p_{34}p_{56}}{p_{16}p_{23}p_{45}},\frac{p_{16}p_{23}}{p_{12}p_{36}}\right)-\omega_{21}\left(\frac{p_{12}p_{34}p_{56}}{p_{16}p_{23}p_{45}},\frac{p_{36}p_{45}}{p_{34}p_{56}}\right)\\
&-2\omega_3\left(\frac{p_{12}p_{34}p_{56}}{p_{16}p_{23}p_{45}}\right)\\
&+2\left(\omega_3\left(\frac{p_{15}p_{24}}{p_{12}p_{45}}\right)+\omega_3\left(\frac{p_{13}p_{46}}{p_{16}p_{34}}\right)+\omega_3\left(\frac{p_{26}p_{35}}{p_{23}p_{56}}\right)\right)\\
&-2\left(\omega_3\left(\frac{p_{15}p_{23}}{p_{12}p_{35}}\right)+\omega_3\left(\frac{p_{12}p_{46}}{p_{16}p_{24}}\right)+\omega_3\left(\frac{p_{13}p_{45}}{p_{15}p_{34}}\right)\right.\\
&\qquad\left.+\omega_3\left(\frac{p_{16}p_{35}}{p_{13}p_{56}}\right)+\omega_3\left(\frac{p_{26}p_{34}}{p_{23}p_{46}}\right)+\omega_3\left(\frac{p_{24}p_{56}}{p_{26}p_{45}}\right)\right)\\
&+2\left(\omega_3\left(\frac{p_{14}p_{23}}{p_{12}p_{34}}\right)+\omega_3\left(\frac{p_{12}p_{36}}{p_{16}p_{23}}\right)+\omega_3\left(\frac{p_{16}p_{25}}{p_{12}p_{56}}\right)\right.\\
&\qquad\left.+\omega_3\left(\frac{p_{14}p_{56}}{p_{16}p_{45}}\right)+\omega_3\left(\frac{p_{25}p_{34}}{p_{23}p_{45}}\right)+\omega_3\left(\frac{p_{36}p_{45}}{p_{34}p_{56}}\right)\right)\\
&-1\left(\omega_3\left(\frac{p_{14}p_{23}}{p_{12}p_{34}}\right)+\omega_3\left(\frac{p_{16}p_{23}}{p_{12}p_{36}}\right)+\omega_3\left(\frac{p_{16}p_{25}}{p_{12}p_{56}}\right)+\omega_3\left(\frac{p_{16}p_{45}}{p_{14}p_{56}}\right)\right.\\
&\qquad\left.+\omega_3\left(\frac{p_{23}p_{45}}{p_{25}p_{34}}\right)+\omega_3\left(\frac{p_{36}p_{45}}{p_{34}p_{56}}\right)\right)
\end{aligned}
$$

B.2 Q_4 Relation on $\mathrm{Gr}(2,7)$

Recall the cluster modular group of $\mathrm{Gr}(2,7)$ is $\mathbb{Z}_7$ generated by mutating all the sources in the Dynkin quiver. In Section 1.2.6 that this corresponds to automorphism of $\mathrm{Gr}(2,7)$ give by rotating the indices of Plücker coordinates modulo 7.[1]. The sum of the orbit of following combination of multiple polylogarithm forms under this action is trivial:

$$\begin{aligned}
&2\omega_4\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}}\right)+2\omega_4\left(\frac{\mathrm{p}_{15}\mathrm{p}_{23}}{\mathrm{p}_{12}\mathrm{p}_{35}}\right)+2\omega_4\left(\frac{\mathrm{p}_{12}\mathrm{p}_{36}}{\mathrm{p}_{16}\mathrm{p}_{23}}\right)+2\omega_4\left(\frac{\mathrm{p}_{12}\mathrm{p}_{45}}{\mathrm{p}_{15}\mathrm{p}_{24}}\right)+2\omega_4\left(\frac{\mathrm{p}_{12}\mathrm{p}_{45}}{\mathrm{p}_{15}\mathrm{p}_{24}}\right)\\
&+\omega_4\left(\frac{\mathrm{p}_{14}\mathrm{p}_{23}}{\mathrm{p}_{12}\mathrm{p}_{34}}\right)+\omega_4\left(\frac{\mathrm{p}_{12}\mathrm{p}_{35}}{\mathrm{p}_{15}\mathrm{p}_{23}}\right)+\omega_4\left(\frac{\mathrm{p}_{15}\mathrm{p}_{24}}{\mathrm{p}_{12}\mathrm{p}_{45}}\right)+\omega_4\left(\frac{\mathrm{p}_{16}\mathrm{p}_{23}}{\mathrm{p}_{12}\mathrm{p}_{36}}\right)-3\omega_4\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}}\right)\\
&-\omega_{31}\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}},\frac{\mathrm{p}_{14}\mathrm{p}_{23}}{\mathrm{p}_{12}\mathrm{p}_{34}}\right)+\omega_{31}\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}},\frac{\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{25}\mathrm{p}_{34}}\right)-\omega_{31}\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}},\frac{\mathrm{p}_{36}\mathrm{p}_{45}}{\mathrm{p}_{34}\mathrm{p}_{56}}\right)\\
&\omega_{31}\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}},\frac{\mathrm{p}_{16}\mathrm{p}_{45}}{\mathrm{p}_{14}\mathrm{p}_{56}}\right)-\omega_{31}\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}},\frac{\mathrm{p}_{16}\mathrm{p}_{25}}{\mathrm{p}_{12}\mathrm{p}_{56}}\right)+\omega_{31}\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}},\frac{\mathrm{p}_{16}\mathrm{p}_{23}}{\mathrm{p}_{12}\mathrm{p}_{36}}\right)
\end{aligned}$$

B.3 Q_5 Relation on $\mathrm{Gr}(2,8)$

As in $\mathrm{Gr}(2,7)$ we recall the cluster modular group of $\mathrm{Gr}(2,8)$ is $\mathbb{Z}_8$ generated by rotating the indices of Plücker coordinates modulo 8. To simplify the expression of Q_5 we only give representatives of each orbit.

[1] $\mathrm{p}ij \mapsto \mathrm{p}(i+1)(j+1)$

The contributions of A_1 subalgebras of $\mathrm{Gr}(2,8)$ to Q_5 are

$$2\left(\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{35}}{\mathrm{p}_{15}\mathrm{p}_{23}}\right)+\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{46}}{\mathrm{p}_{16}\mathrm{p}_{24}}\right)+\omega_5\left(\frac{\mathrm{p}_{17}\mathrm{p}_{23}}{\mathrm{p}_{12}\mathrm{p}_{37}}\right)+\omega_5\left(\frac{\mathrm{p}_{13}\mathrm{p}_{47}}{\mathrm{p}_{17}\mathrm{p}_{34}}\right)\right)$$
$$-2\left(\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}}\right)+\omega_5\left(\frac{\mathrm{p}_{15}\mathrm{p}_{24}}{\mathrm{p}_{12}\mathrm{p}_{45}}\right)+\omega_5\left(\frac{\mathrm{p}_{16}\mathrm{p}_{23}}{\mathrm{p}_{12}\mathrm{p}_{36}}\right)+\omega_5\left(\frac{\mathrm{p}_{16}\mathrm{p}_{25}}{\mathrm{p}_{12}\mathrm{p}_{56}}\right)+\omega_5\left(\frac{\mathrm{p}_{16}\mathrm{p}_{34}}{\mathrm{p}_{13}\mathrm{p}_{46}}\right)\right)$$
$$-1\omega_5\left(\frac{\mathrm{p}_{17}\mathrm{p}_{35}}{\mathrm{p}_{13}\mathrm{p}_{57}}\right)$$

To write down the contributions of $\mathcal{L}_3^C(A_3)$ terms, we first gather the A_3 subalgebras into 4 orbits under $\mathbb{Z}_8$ based on their Casimir element. Despite the fact that $\mathcal{L}_3^C(A_3)$ has A_3 symmetry, if we add all the terms in a Casimir orbit we can write the result as a sum of $\mathbb{Z}_8$ orbits. The contributions of the orbit of $\frac{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}}$ are:

$$-\omega_{41}\left(\frac{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}},\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}}\right)-\omega_{41}\left(\frac{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}},\frac{\mathrm{p}_{12}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{25}}\right)+\omega_{41}\left(\frac{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}},\frac{\mathrm{p}_{12}\mathrm{p}_{36}}{\mathrm{p}_{16}\mathrm{p}_{23}}\right)$$
$$+\omega_{41}\left(\frac{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}},\frac{\mathrm{p}_{14}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{45}}\right)+\omega_{41}\left(\frac{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}},\frac{\mathrm{p}_{25}\mathrm{p}_{34}}{\mathrm{p}_{23}\mathrm{p}_{45}}\right)-\omega_{41}\left(\frac{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}},\frac{\mathrm{p}_{34}\mathrm{p}_{56}}{\mathrm{p}_{36}\mathrm{p}_{45}}\right)$$
$$+4\omega_5\left(\frac{\mathrm{p}_{16}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}}\right)+3\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}}\right)+2\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{36}}{\mathrm{p}_{16}\mathrm{p}_{23}}\right)+2\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{56}}{\mathrm{p}_{16}\mathrm{p}_{25}}\right)$$

The contributions of the orbit of $\frac{\mathrm{p}_{17}\mathrm{p}_{23}\mathrm{p}_{46}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{67}}$ are:

$$-\omega_{41}\left(\frac{\mathrm{p}_{17}\mathrm{p}_{23}\mathrm{p}_{46}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{67}},\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}}\right)+\omega_{41}\left(\frac{\mathrm{p}_{17}\mathrm{p}_{23}\mathrm{p}_{46}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{67}},\frac{\mathrm{p}_{14}\mathrm{p}_{67}}{\mathrm{p}_{17}\mathrm{p}_{46}}\right)+\omega_{41}\left(\frac{\mathrm{p}_{17}\mathrm{p}_{23}\mathrm{p}_{46}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{67}},\frac{\mathrm{p}_{26}\mathrm{p}_{34}}{\mathrm{p}_{23}\mathrm{p}_{46}}\right)$$
$$-\omega_{41}\left(\frac{\mathrm{p}_{17}\mathrm{p}_{23}\mathrm{p}_{46}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{67}},\frac{\mathrm{p}_{12}\mathrm{p}_{67}}{\mathrm{p}_{17}\mathrm{p}_{26}}\right)+\omega_{41}\left(\frac{\mathrm{p}_{17}\mathrm{p}_{23}\mathrm{p}_{46}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{67}},\frac{\mathrm{p}_{12}\mathrm{p}_{37}}{\mathrm{p}_{17}\mathrm{p}_{23}}\right)-\omega_{41}\left(\frac{\mathrm{p}_{17}\mathrm{p}_{23}\mathrm{p}_{46}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{67}},\frac{\mathrm{p}_{34}\mathrm{p}_{67}}{\mathrm{p}_{37}\mathrm{p}_{46}}\right)$$
$$+4\omega_5\left(\frac{\mathrm{p}_{17}\mathrm{p}_{24}\mathrm{p}_{56}}{\mathrm{p}_{12}\mathrm{p}_{45}\mathrm{p}_{67}}\right)+2\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{67}}{\mathrm{p}_{17}\mathrm{p}_{26}}\right)+\omega_5\left(\frac{\mathrm{p}_{14}\mathrm{p}_{67}}{\mathrm{p}_{17}\mathrm{p}_{46}}\right)$$
$$+\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}}\right)+\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{37}}{\mathrm{p}_{17}\mathrm{p}_{23}}\right)+\omega_5\left(\frac{\mathrm{p}_{26}\mathrm{p}_{34}}{\mathrm{p}_{23}\mathrm{p}_{46}}\right)$$

The contributions of the orbit of $\frac{p_{17}p_{23}p_{45}}{p_{12}p_{34}p_{57}}$ are:

$$+\omega_{41}\left(\frac{p_{17}p_{23}p_{45}}{p_{12}p_{34}p_{57}},\frac{p_{12}p_{34}}{p_{14}p_{23}}\right)-\omega_{41}\left(\frac{p_{17}p_{23}p_{45}}{p_{12}p_{34}p_{57}},\frac{p_{14}p_{57}}{p_{17}p_{45}}\right)-\omega_{41}\left(\frac{p_{17}p_{23}p_{45}}{p_{12}p_{34}p_{57}},\frac{p_{25}p_{34}}{p_{23}p_{45}}\right)$$
$$+\omega_{41}\left(\frac{p_{17}p_{23}p_{45}}{p_{12}p_{34}p_{57}},\frac{p_{12}p_{57}}{p_{17}p_{25}}\right)-\omega_{41}\left(\frac{p_{17}p_{23}p_{45}}{p_{12}p_{34}p_{57}},\frac{p_{12}p_{37}}{p_{17}p_{23}}\right)+\omega_{41}\left(\frac{p_{17}p_{23}p_{45}}{p_{12}p_{34}p_{57}},\frac{p_{34}p_{57}}{p_{37}p_{45}}\right)$$
$$-4\omega_5\left(\frac{p_{17}p_{23}p_{45}}{p_{12}p_{34}p_{57}}\right)-2\omega_5\left(\frac{p_{12}p_{34}}{p_{14}p_{23}}\right)-\omega_5\left(\frac{p_{12}p_{35}}{p_{15}p_{23}}\right)$$
$$-\omega_5\left(\frac{p_{16}p_{24}}{p_{12}p_{46}}\right)-\omega_5\left(\frac{p_{12}p_{37}}{p_{17}p_{23}}\right)-\omega_5\left(\frac{p_{12}p_{57}}{p_{17}p_{25}}\right)$$

We have to be more careful with $\frac{p_{12}p_{35}p_{67}}{p_{17}p_{23}p_{56}}$ as $\frac{p_{17}p_{23}p_{56}}{p_{12}p_{35}p_{67}}$ appears in the orbit. As we do not wish to double count any A_3 we only apply r^0, r^1, r^2, r^3 to the following:

$$-\omega_{41}\left(\frac{p_{12}p_{35}p_{67}}{p_{17}p_{23}p_{56}},\frac{p_{15}p_{23}}{p_{12}p_{35}}\right)+\omega_{41}\left(\frac{p_{12}p_{35}p_{67}}{p_{17}p_{23}p_{56}},\frac{p_{17}p_{56}}{p_{15}p_{67}}\right)+\omega_{41}\left(\frac{p_{12}p_{35}p_{67}}{p_{17}p_{23}p_{56}},\frac{p_{23}p_{56}}{p_{26}p_{35}}\right)$$
$$-\omega_{41}\left(\frac{p_{12}p_{35}p_{67}}{p_{17}p_{23}p_{56}},\frac{p_{17}p_{26}}{p_{12}p_{67}}\right)+\omega_{41}\left(\frac{p_{12}p_{35}p_{67}}{p_{17}p_{23}p_{56}},\frac{p_{17}p_{23}}{p_{12}p_{37}}\right)-4\omega_5\left(\frac{p_{12}p_{35}p_{67}}{p_{17}p_{23}p_{56}}\right)$$

However the “half orbits” of the remaining $\omega_5(x)$ terms can be assembled into 3 “full orbits”

$$-\omega_5\left(\frac{p_{15}p_{23}}{p_{12}p_{35}}\right)-\omega_5\left(\frac{p_{15}p_{24}}{p_{12}p_{45}}\right)-\omega_5\left(\frac{p_{17}p_{23}}{p_{12}p_{37}}\right)$$

Finally we include the terms from $\mathcal{L}_5^C(A_5)$. The full A_5 Casimir is fixed up to inverse by $\mathbb{Z}_8$ and so appears as

$$-6\omega_5\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{45}\mathrm{p}_{67}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}\mathrm{p}_{78}}\right)-6\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}\mathrm{p}_{78}}{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{45}\mathrm{p}_{67}}\right)$$

Once again these come in orbits under $\mathbb{Z}_8$:

$$\begin{aligned}
&+\Bigg(\omega_{311}\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{45}\mathrm{p}_{67}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}\mathrm{p}_{78}},\frac{\mathrm{p}_{12}\mathrm{p}_{38}}{\mathrm{p}_{18}\mathrm{p}_{23}},\frac{\mathrm{p}_{34}\mathrm{p}_{78}}{\mathrm{p}_{38}\mathrm{p}_{47}}\right)\\
&\qquad+\frac{1}{2}\omega_{41}\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{47}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{78}},\frac{\mathrm{p}_{34}\mathrm{p}_{78}}{\mathrm{p}_{38}\mathrm{p}_{47}}\right)-\frac{1}{2}\omega_{41}\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{47}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{78}},\frac{\mathrm{p}_{12}\mathrm{p}_{38}}{\mathrm{p}_{18}\mathrm{p}_{23}}\right)\Bigg)\\
&-\Bigg(\omega_{311}\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{45}\mathrm{p}_{67}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}\mathrm{p}_{78}},\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}},\frac{\mathrm{p}_{14}\mathrm{p}_{78}}{\mathrm{p}_{18}\mathrm{p}_{47}}\right)\\
&\qquad+\frac{1}{2}\omega_{41}\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{47}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{78}},\frac{\mathrm{p}_{14}\mathrm{p}_{78}}{\mathrm{p}_{18}\mathrm{p}_{47}}\right)-\frac{1}{2}\omega_{41}\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{47}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{78}},\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}}\right)\Bigg)\\
&-\Bigg(\omega_{311}\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{45}\mathrm{p}_{67}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{56}\mathrm{p}_{78}},\frac{\mathrm{p}_{12}\mathrm{p}_{38}}{\mathrm{p}_{18}\mathrm{p}_{23}},\frac{\mathrm{p}_{34}\mathrm{p}_{58}}{\mathrm{p}_{38}\mathrm{p}_{45}}\right)\\
&\qquad+\frac{1}{2}\omega_{41}\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{58}},\frac{\mathrm{p}_{12}\mathrm{p}_{38}}{\mathrm{p}_{18}\mathrm{p}_{23}}\right)-\frac{1}{2}\omega_{41}\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{45}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{58}},\frac{\mathrm{p}_{34}\mathrm{p}_{58}}{\mathrm{p}_{38}\mathrm{p}_{45}}\right)\Bigg)\\
&-2\omega_5\left(\frac{\mathrm{p}_{18}\mathrm{p}_{23}\mathrm{p}_{47}}{\mathrm{p}_{12}\mathrm{p}_{34}\mathrm{p}_{78}}\right)-\frac{1}{2}\omega_5\left(\frac{\mathrm{p}_{12}\mathrm{p}_{34}}{\mathrm{p}_{14}\mathrm{p}_{23}}\right)-\frac{1}{2}\omega_5\left(\frac{\mathrm{p}_{27}\mathrm{p}_{36}}{\mathrm{p}_{23}\mathrm{p}_{67}}\right)
\end{aligned}$$

B.4 α_6 Relation on D_6

We give the terms in α_6 for the D_6 subalgebra of $\mathrm{Gr}(3,8)$ given by freezing p467 and p378.

The ω_5 terms are:

$$2\left(\omega_5\left(\frac{\text{e2x45}}{\text{p}_{128}\text{p}_{367}}\right)+\omega_5\left(\frac{\text{e2x45}}{\text{p}_{178}\text{p}_{236}}\right)+\omega_5\left(\frac{\text{p}_{123}\text{p}_{178}\text{p}_{368}}{\text{p}_{128}\text{p}_{136}\text{p}_{378}}\right)\right.$$
$$\left.+\omega_5\left(\frac{\text{p}_{136}\text{p}_{234}}{\text{p}_{123}\text{p}_{346}}\right)+\omega_5\left(\frac{\text{p}_{234}\text{p}_{367}\text{p}_{456}}{\text{p}_{236}\text{p}_{345}\text{p}_{467}}\right)+\omega_5\left(\frac{\text{p}_{368}\text{p}_{467}}{\text{p}_{346}\text{p}_{678}}\right)\right)$$
$$-2\left(\omega_5\left(\frac{\text{e2x16p}_{178}}{\text{e2x36p}_{378}}\right)+\omega_5\left(\frac{\text{e2x16p}_{467}}{\text{p}_{234}\text{p}_{457}\text{p}_{678}}\right)+\omega_5\left(\frac{\text{e2x26p}_{128}}{\text{p}_{123}\text{p}_{178}\text{p}_{458}}\right)\right.$$
$$\left.+\omega_5\left(\frac{\text{e2x26p}_{234}}{\text{p}_{123}\text{p}_{345}\text{p}_{478}}\right)+\omega_5\left(\frac{\text{e2x36}}{\text{p}_{128}\text{p}_{457}}\right)+\omega_5\left(\frac{\text{p}_{458}\text{p}_{467}}{\text{p}_{456}\text{p}_{478}}\right)\right)$$

The remaining 96 ω_{311} terms in α_6 are presented below. They are grouped by the last argument, which is the middle X-Coordinates of the corresponding A_5 subalgebra.

Note that like in the α_4 case every coefficient is ± 2.

$$-2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{128}p_{137}}{p_{123}p_{178}},\frac{\mathrm{e2x28}}{p_{137}p_{456}}\right)-2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{123}p_{345}p_{467}}{\mathrm{e2x28}\,p_{234}},\frac{\mathrm{e2x28}}{p_{137}p_{456}}\right)$$

$$+2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{123}p_{178}}{p_{128}p_{137}},\frac{p_{137}p_{456}}{\mathrm{e2x28}}\right)+2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{\mathrm{e2x28}\,p_{234}}{p_{123}p_{345}p_{467}},\frac{p_{137}p_{456}}{\mathrm{e2x28}}\right)$$

$$+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{128}p_{137}}{p_{123}p_{178}},\frac{\mathrm{e2x26}\,p_{467}}{p_{137}p_{456}p_{478}}\right)+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{123}p_{345}p_{478}}{\mathrm{e2x26}\,p_{234}},\frac{\mathrm{e2x26}\,p_{467}}{p_{137}p_{456}p_{478}}\right)$$

$$-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{123}p_{178}}{p_{128}p_{137}},\frac{p_{137}p_{456}p_{478}}{\mathrm{e2x26}\,p_{467}}\right)-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{\mathrm{e2x26}\,p_{234}}{p_{123}p_{345}p_{478}},\frac{p_{137}p_{456}p_{478}}{\mathrm{e2x26}\,p_{467}}\right)$$

$$-2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{\mathrm{e2x56}}{p_{178}p_{234}},\frac{\mathrm{e2x35}\,p_{345}}{\mathrm{e2x56}\,p_{456}}\right)-2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{128}p_{467}}{\mathrm{e2x35}},\frac{\mathrm{e2x35}\,p_{345}}{\mathrm{e2x56}\,p_{456}}\right)$$

$$+2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{178}p_{234}}{\mathrm{e2x56}},\frac{\mathrm{e2x56}\,p_{456}}{\mathrm{e2x35}\,p_{345}}\right)+2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{\mathrm{e2x35}}{p_{128}p_{467}},\frac{\mathrm{e2x56}\,p_{456}}{\mathrm{e2x35}\,p_{345}}\right)$$

$$+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{\mathrm{e2x56}}{p_{178}p_{234}},\frac{p_{128}p_{345}p_{478}}{\mathrm{e2x56}\,p_{458}}\right)+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{458}p_{467}}{p_{456}p_{478}},\frac{p_{128}p_{345}p_{478}}{\mathrm{e2x56}\,p_{458}}\right)$$

$$-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{178}p_{234}}{\mathrm{e2x56}},\frac{\mathrm{e2x56}\,p_{458}}{p_{128}p_{345}p_{478}}\right)-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{456}p_{478}}{p_{458}p_{467}},\frac{\mathrm{e2x56}\,p_{458}}{p_{128}p_{345}p_{478}}\right)$$

$$+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{\mathrm{e2x45}}{p_{178}p_{236}},\frac{\mathrm{e2x35}\,p_{236}p_{345}}{\mathrm{e2x45}\,p_{234}p_{456}}\right)+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{128}p_{467}}{\mathrm{e2x35}},\frac{\mathrm{e2x35}\,p_{236}p_{345}}{\mathrm{e2x45}\,p_{234}p_{456}}\right)$$

$$-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{178}p_{236}}{\mathrm{e2x45}},\frac{\mathrm{e2x45}\,p_{234}p_{456}}{\mathrm{e2x35}\,p_{236}p_{345}}\right)-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{\mathrm{e2x35}}{p_{128}p_{467}},\frac{\mathrm{e2x45}\,p_{234}p_{456}}{\mathrm{e2x35}\,p_{236}p_{345}}\right)$$

$$+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{128}p_{367}}{\mathrm{e2x45}},\frac{\mathrm{e2x28e2x45}}{p_{123}p_{178}p_{367}p_{456}}\right)+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{123}p_{345}p_{467}}{\mathrm{e2x28}\,p_{234}},\frac{\mathrm{e2x28e2x45}}{p_{123}p_{178}p_{367}p_{456}}\right)$$

$$-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{\mathrm{e2x45}}{p_{128}p_{367}},\frac{p_{123}p_{178}p_{367}p_{456}}{\mathrm{e2x28e2x45}}\right)-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{\mathrm{e2x28}\,p_{234}}{p_{123}p_{345}p_{467}},\frac{p_{123}p_{178}p_{367}p_{456}}{\mathrm{e2x28e2x45}}\right)$$

$$+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{128}p_{378}}{p_{178}p_{238}},\frac{p_{238}p_{367}}{p_{236}p_{378}}\right)+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{236}p_{345}p_{467}}{p_{234}p_{367}p_{456}},\frac{p_{238}p_{367}}{p_{236}p_{378}}\right)$$

$$-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{178}p_{238}}{p_{128}p_{378}},\frac{p_{236}p_{378}}{p_{238}p_{367}}\right)-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{234}p_{367}p_{456}}{p_{236}p_{345}p_{467}},\frac{p_{236}p_{378}}{p_{238}p_{367}}\right)$$

$$-2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{p_{128}p_{378}}{p_{178}p_{238}},\frac{p_{238}p_{345}p_{678}}{\mathrm{e2x17}\,p_{378}}\right)-2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{\mathrm{e2x17}\,p_{467}}{p_{234}p_{456}p_{678}},\frac{p_{238}p_{345}p_{678}}{\mathrm{e2x17}\,p_{378}}\right)$$

$$+2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{178}p_{238}}{p_{128}p_{378}},\frac{\mathrm{e2x17}\,p_{378}}{p_{238}p_{345}p_{678}}\right)+2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{234}p_{456}p_{678}}{\mathrm{e2x17}\,p_{467}},\frac{\mathrm{e2x17}\,p_{378}}{p_{238}p_{345}p_{678}}\right)$$

$$+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{\mathrm{e2x26}\,p_{128}}{p_{123}p_{178}p_{458}},\frac{p_{123}p_{345}p_{458}p_{678}}{\mathrm{e2x17e2x26}}\right)+2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}},\frac{\mathrm{e2x17}\,p_{467}}{p_{234}p_{456}p_{678}},\frac{p_{123}p_{345}p_{458}p_{678}}{\mathrm{e2x17e2x26}}\right)$$

$$-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{123}p_{178}p_{458}}{\mathrm{e2x26}\,p_{128}},\frac{\mathrm{e2x17e2x26}}{p_{123}p_{345}p_{458}p_{678}}\right)-2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}},\frac{p_{234}p_{456}p_{678}}{\mathrm{e2x17}\,p_{467}},\frac{\mathrm{e2x17e2x26}}{p_{123}p_{345}p_{458}p_{678}}\right)$$

$$-2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}},\frac{p_{123}p_{178}}{p_{128}p_{137}},\frac{p_{137}p_{345}p_{678}}{\mathrm{e2x28}\,p_{378}}\right)-2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}},\frac{\mathrm{e2x28}\,p_{234}}{p_{123}p_{345}p_{467}},\frac{p_{137}p_{345}p_{678}}{\mathrm{e2x28}\,p_{378}}\right)$$

$$+2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}},\frac{p_{128}p_{137}}{p_{123}p_{178}},\frac{\mathrm{e2x28}\,p_{378}}{p_{137}p_{345}p_{678}}\right)+2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}},\frac{p_{123}p_{345}p_{467}}{\mathrm{e2x28}\,p_{234}},\frac{\mathrm{e2x28}\,p_{378}}{p_{137}p_{345}p_{678}}\right)$$

$$+2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}},\frac{p_{123}p_{178}}{p_{128}p_{137}},\frac{p_{137}p_{346}p_{678}}{p_{136}p_{378}p_{467}}\right)+2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}},\frac{p_{136}p_{234}}{p_{123}p_{346}},\frac{p_{137}p_{346}p_{678}}{p_{136}p_{378}p_{467}}\right)$$

$$-2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}},\frac{p_{128}p_{137}}{p_{123}p_{178}},\frac{p_{136}p_{378}p_{467}}{p_{137}p_{346}p_{678}}\right)-2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}},\frac{p_{123}p_{346}}{p_{136}p_{234}},\frac{p_{136}p_{378}p_{467}}{p_{137}p_{346}p_{678}}\right)$$

$$+2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}},\frac{p_{178}p_{234}}{\mathrm{e2x56}},\frac{\mathrm{e2x56}\,p_{368}}{p_{128}p_{346}p_{378}}\right)+2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}},\frac{p_{346}p_{678}}{p_{368}p_{467}},\frac{\mathrm{e2x56}\,p_{368}}{p_{128}p_{346}p_{378}}\right)$$

$$-2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}},\frac{\mathrm{e2x56}}{p_{178}p_{234}},\frac{p_{128}p_{346}p_{378}}{\mathrm{e2x56}\,p_{368}}\right)-2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}},\frac{p_{368}p_{467}}{p_{346}p_{678}},\frac{p_{128}p_{346}p_{378}}{\mathrm{e2x56}\,p_{368}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{p_{178}p_{234}}{\mathrm{e2x56}}, \frac{\mathrm{e2x56}p_{678}}{\mathrm{e2x35}p_{378}}\right) - 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{\mathrm{e2x35}}{p_{128}p_{467}}, \frac{\mathrm{e2x56}p_{678}}{\mathrm{e2x35}p_{378}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{\mathrm{e2x56}}{p_{178}p_{234}}, \frac{\mathrm{e2x35}p_{378}}{\mathrm{e2x56}p_{678}}\right) + 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{128}p_{467}}{\mathrm{e2x35}}, \frac{\mathrm{e2x35}p_{378}}{\mathrm{e2x56}p_{678}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{\mathrm{e2x16}p_{178}}{\mathrm{e2x36}p_{378}}, \frac{\mathrm{e2x36}p_{234}p_{678}}{\mathrm{e2x16e2x35}}\right) + 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{\mathrm{e2x35}}{p_{128}p_{467}}, \frac{\mathrm{e2x36}p_{234}p_{678}}{\mathrm{e2x16e2x35}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{\mathrm{e2x36}p_{378}}{\mathrm{e2x16}p_{178}}, \frac{\mathrm{e2x16e2x35}}{\mathrm{e2x36}p_{234}p_{678}}\right) - 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{128}p_{467}}{\mathrm{e2x35}}, \frac{\mathrm{e2x16e2x35}}{\mathrm{e2x36}p_{234}p_{678}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{p_{178}p_{238}}{p_{128}p_{378}}, \frac{\mathrm{e2x17}}{p_{238}p_{456}}\right) - 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{p_{234}p_{456}p_{678}}{\mathrm{e2x17}p_{467}}, \frac{\mathrm{e2x17}}{p_{238}p_{456}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{128}p_{378}}{p_{178}p_{238}}, \frac{p_{238}p_{456}}{\mathrm{e2x17}}\right) + 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{\mathrm{e2x17}p_{467}}{p_{234}p_{456}p_{678}}, \frac{p_{238}p_{456}}{\mathrm{e2x17}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{p_{178}p_{238}}{p_{128}p_{378}}, \frac{\mathrm{e2x16}}{p_{238}p_{457}}\right) + 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{p_{234}p_{457}p_{678}}{\mathrm{e2x16}p_{467}}, \frac{\mathrm{e2x16}}{p_{238}p_{457}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{128}p_{378}}{p_{178}p_{238}}, \frac{p_{238}p_{457}}{\mathrm{e2x16}}\right) - 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{\mathrm{e2x16}p_{467}}{p_{234}p_{457}p_{678}}, \frac{p_{238}p_{457}}{\mathrm{e2x16}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{p_{123}p_{178}p_{368}}{p_{128}p_{136}p_{378}}, \frac{\mathrm{e2x17}p_{136}}{p_{123}p_{368}p_{456}}\right) + 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{p_{234}p_{456}p_{678}}{\mathrm{e2x17}p_{467}}, \frac{\mathrm{e2x17}p_{136}}{p_{123}p_{368}p_{456}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{128}p_{136}p_{378}}{p_{123}p_{178}p_{368}}, \frac{p_{123}p_{368}p_{456}}{\mathrm{e2x17}p_{136}}\right) - 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{\mathrm{e2x17}p_{467}}{p_{234}p_{456}p_{678}}, \frac{p_{123}p_{368}p_{456}}{\mathrm{e2x17}p_{136}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{\mathrm{e2x36}}{p_{128}p_{457}}, \frac{p_{123}p_{178}p_{345}p_{457}p_{678}}{\mathrm{e2x28e2x36}p_{378}}\right) + 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{\mathrm{e2x28}p_{234}}{p_{123}p_{345}p_{467}}, \frac{p_{123}p_{178}p_{345}p_{457}p_{678}}{\mathrm{e2x28e2x36}p_{378}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{128}p_{457}}{\mathrm{e2x36}}, \frac{\mathrm{e2x28e2x36}p_{378}}{p_{123}p_{178}p_{345}p_{457}p_{678}}\right) - 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{123}p_{345}p_{467}}{\mathrm{e2x28}p_{234}}, \frac{\mathrm{e2x28e2x36}p_{378}}{p_{123}p_{178}p_{345}p_{457}p_{678}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}}, \frac{\mathrm{e2x26}p_{128}}{p_{123}p_{178}p_{458}}, \frac{p_{458}p_{467}}{p_{456}p_{478}}\right) - 2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}}, \frac{p_{123}p_{345}p_{478}}{\mathrm{e2x26}p_{234}}, \frac{p_{458}p_{467}}{p_{456}p_{478}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}}, \frac{\mathrm{e2x26}p_{128}}{p_{123}p_{178}p_{458}}, \frac{p_{123}p_{345}p_{478}}{\mathrm{e2x26}p_{234}}\right) - 2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}}, \frac{p_{458}p_{467}}{p_{456}p_{478}}, \frac{p_{123}p_{345}p_{478}}{\mathrm{e2x26}p_{234}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}}, \frac{p_{458}p_{467}}{p_{456}p_{478}}, \frac{\mathrm{e2x26}p_{128}}{p_{123}p_{178}p_{458}}\right) - 2\omega_{311}\left(\frac{p_{178}p_{234}p_{456}}{p_{128}p_{345}p_{467}}, \frac{p_{123}p_{345}p_{478}}{\mathrm{e2x26}p_{234}}, \frac{\mathrm{e2x26}p_{128}}{p_{123}p_{178}p_{458}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}}, \frac{p_{178}p_{236}}{\mathrm{e2x45}}, \frac{\mathrm{e2x45}}{p_{128}p_{367}}\right) + 2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}}, \frac{p_{234}p_{367}p_{456}}{p_{236}p_{345}p_{467}}, \frac{\mathrm{e2x45}}{p_{128}p_{367}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}}, \frac{p_{178}p_{236}}{\mathrm{e2x45}}, \frac{p_{234}p_{367}p_{456}}{p_{236}p_{345}p_{467}}\right) + 2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}}, \frac{\mathrm{e2x45}}{p_{128}p_{367}}, \frac{p_{234}p_{367}p_{456}}{p_{236}p_{345}p_{467}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}}, \frac{\mathrm{e2x45}}{p_{128}p_{367}}, \frac{p_{178}p_{236}}{\mathrm{e2x45}}\right) + 2\omega_{311}\left(\frac{p_{128}p_{345}p_{467}}{p_{178}p_{234}p_{456}}, \frac{p_{234}p_{367}p_{456}}{p_{236}p_{345}p_{467}}, \frac{p_{178}p_{236}}{\mathrm{e2x45}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{\mathrm{e2x16}p_{178}}{\mathrm{e2x36}p_{378}}, \frac{\mathrm{e2x36}}{p_{128}p_{457}}\right) - 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{p_{234}p_{457}p_{678}}{\mathrm{e2x16}p_{467}}, \frac{\mathrm{e2x36}}{p_{128}p_{457}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{\mathrm{e2x16}p_{178}}{\mathrm{e2x36}p_{378}}, \frac{p_{234}p_{457}p_{678}}{\mathrm{e2x16}p_{467}}\right) - 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{\mathrm{e2x36}}{p_{128}p_{457}}, \frac{p_{234}p_{457}p_{678}}{\mathrm{e2x16}p_{467}}\right)$$

$$- 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{\mathrm{e2x36}}{p_{128}p_{457}}, \frac{\mathrm{e2x16}p_{178}}{\mathrm{e2x36}p_{378}}\right) - 2\omega_{311}\left(\frac{p_{128}p_{378}p_{467}}{p_{178}p_{234}p_{678}}, \frac{p_{234}p_{457}p_{678}}{\mathrm{e2x16}p_{467}}, \frac{\mathrm{e2x16}p_{178}}{\mathrm{e2x36}p_{378}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{123}p_{346}}{p_{136}p_{234}}, \frac{p_{128}p_{136}p_{378}}{p_{123}p_{178}p_{368}}\right) + 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{368}p_{467}}{p_{346}p_{678}}, \frac{p_{128}p_{136}p_{378}}{p_{123}p_{178}p_{368}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{123}p_{346}}{p_{136}p_{234}}, \frac{p_{368}p_{467}}{p_{346}p_{678}}\right) + 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{128}p_{136}p_{378}}{p_{123}p_{178}p_{368}}, \frac{p_{368}p_{467}}{p_{346}p_{678}}\right)$$

$$+ 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{128}p_{136}p_{378}}{p_{123}p_{178}p_{368}}, \frac{p_{123}p_{346}}{p_{136}p_{234}}\right) + 2\omega_{311}\left(\frac{p_{178}p_{234}p_{678}}{p_{128}p_{378}p_{467}}, \frac{p_{368}p_{467}}{p_{346}p_{678}}, \frac{p_{123}p_{346}}{p_{136}p_{234}}\right)$$

Bibliography

[1] Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo, Alexander Goncharov, Alexander Postnikov, and Jaroslav Trnka. *Grassmannian geometry of scattering amplitudes*. Cambridge University Press, Cambridge, 2016.

[2] Don Zagier. The dilogarithm function. In *Frontiers in number theory, physics, and geometry. II*, pages 3–65. Springer, Berlin, 2007.

[3] A. B. Goncharov. Geometry of configurations, polylogarithms, and motivic cohomology. *Adv. Math.*, 114(2):197–318, 1995.

[4] Dani Kaufman and Zachary Greenberg. Cluster modular groups of affine and doubly extended cluster algebras. *arXiv:2107.10334*, 2021.

[5] Christian K. Zickert. Holomorphic polylogarithms and Bloch complexes. *arXiv:1902.03971*, 2019.

[6] Steven Charlton, Herbert Gangl, and Danylo Radchenko. Functional equations of polygonal type for multiple polylogarithms in weights 5, 6 and 7. *arXiv:2012.09840*, 2020.

[7] Robert J. Marsh. *Lecture notes on cluster algebras*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2013.

[8] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[9] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.

[10] Nathan Reading and Salvatore Stella. An affine almost positive roots model. *arXiv:1707.00340*, 2017.

[11] Vladimir V. Fock and Alexander B. Goncharov. Cluster ensembles, quantization and the dilogarithm. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(6):865–930, 2009.

[12] Tomoki Nakanishi and Andrei Zelevinsky. On tropical dualities in cluster algebras. In *Algebraic groups and quantum groups*, volume 565 of *Contemp. Math.*, pages 217–226. Amer. Math. Soc., Providence, RI, 2012.

[13] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. Quivers with potentials and their representations II: applications to cluster algebras. *J. Amer. Math. Soc.*, 23(3):749–790, 2010.

[14] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31(2):497–608, 2018.

[15] Greg Muller. The existence of a maximal green sequence is not invariant under quiver mutation. *Electron. J. Combin.*, 23(2):Paper 2.47, 23, 2016.

[16] J. K. Golden, A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich. Motivic amplitudes and cluster coordinates. 2014(1):91.

[17] Kyoji Saito. Extended affine root systems. I. Coxeter transformations. *Publ. Res. Inst. Math. Sci.*, 21(1):75–179, 1985.

[18] Joshua S. Scott. Grassmannians and cluster algebras. *Proc. London Math. Soc. (3)*, 92(2):345–380, 2006.

[19] Sergey Fomin, Michael Shapiro, and Dylan Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

[20] Lauren K. Williams. Cluster algebras: an introduction. *Bull. Amer. Math. Soc. (N.S.)*, 51(1):1–26, 2014.

[21] Ibrahim Assem, Ralf Schiffler, and Vasilisa Shramchenko. Cluster automorphisms. *Proc. Lond. Math. Soc. (3)*, 104(6):1271–1302, 2012.

[22] Chris Fraser. Braid group symmetries of Grassmannian cluster algebras. *Selecta Math. (N.S.)*, 26(2):Paper No. 17, 51, 2020.

[23] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, (103):1–211, 2006.

[24] Tsukasa Ishibashi. Presentations of cluster modular groups and generation by cluster Dehn twists. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 025, 22, 2020.

[25] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[26] Tom Bridgeland and Ivan Smith. Quadratic differentials as stability conditions. *Publ. Math. Inst. Hautes Études Sci.*, 121:155–278, 2015.

[27] Jianqiang Zhao. Multiple polylogarithms: analytic continuation, monodromy, and variations of mixed Hodge structures. pages 167–1–93, 2002.

[28] Richard M. Hain and Robert MacPherson. Higher logarithms. *Illinois Journal of Mathematics*, 34(2):392–474, 1990.

[29] Claude Duhr, Herbert Gangl, and John R. Rhodes. From polygons and symbols to polylogarithmic functions. (10):075, front matter + 77.

[30] Claude Duhr and Falko Dulat. PolyLogTools—polylogs for the masses. *J. High Energy Phys.*, (8):135, 56, 2019.

[31] Herbert Gangl. Multiple polylogarithms in weight 4. *arXiv:1609.05557*, 2016.

[32] Alexander B. Goncharov and Daniil Rudenko. Motivic correlators, cluster varieties and Zagier's conjecture on zeta(F,4). *arXiv:1803.08585*, 2018.

[33] Tathagata Basak. Combinatorial cell complexes and poincare duality. 147(1):357–387.

[34] R. C. Penner. The structure and singularities of quotient arc complexes. *J. Topol.*, 1(3):527–550, 2008.

[35] Tsukasa Ishibashi. On a Nielsen-Thurston classification theory for cluster modular groups. *Ann. Inst. Fourier (Grenoble)*, 69(2):515–560, 2019.

[36] Morris Newman. Classification of normal subgroups of the modular group. *Trans. Amer. Math. Soc.*, 126:267–277, 1967.

[37] A. B. Goncharov. Galois symmetries of fundamental groupoids and noncommutative geometry. 128(2):209–284.

[38] A. B. Goncharov. Multiple polylogarithms and mixed Tate motives.

[39] A. B. Goncharov. The dihedral Lie algebras and Galois symmetries of $\pi_1^{(l)}(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N))$. *Duke Math. J.*, 110(3):397–487, 2001.