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ON CALIBRATED REPRESENTATIONS OF THE DEGENERATE AFFINE PERIPLECTIC BRAUER ALGEBRA

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Abstract. We initiate the representation theory of the degenerate affine periplectic Brauer algebra on n strands by constructing its finite-dimensional calibrated representations when $n = 2$. We show that any such representation that is indecomposable and does not factor through a representation of the degenerate affine Hecke algebra occurs as an extension of two semisimple representations with one-dimensional composition factors; and furthermore, we classify such representations with regular eigenvalues up to isomorphism.

1 Introduction

The degenerate affine periplectic Brauer algebra on n strands, or $s\mathbb{V}_n$ for short, belongs to a family of diagram algebras playing various roles in generalized Schur-Weyl dualities. Related algebras include the Brauer algebra, periplectic Brauer algebra, degenerate affine Hecke algebra, Nazarov–Wenzl algebra, and walled Brauer algebra. The algebra $s\mathbb{V}_n$ was first defined by Chen and Peng by generators and relations [2], where it was called "affine periplectic Brauer algebra," which is somewhat misleading as it suggests an algebra where over- and under-crossings are distinguished, and in previous work of the authors together with other collaborators as the endomorphism algebra of the object n in a certain monoidal supercategory [1], where it was called "affine VW superalgebra," which is somewhat misleading as well since there is no relation of our algebra to a German car company of tarnished reputation (also see [4, 7]). That monoidal supercategory arises from the representation theory of the periplectic Lie superalgebra, hence the word "periplectic," while "degenerate affine" indicates the close relationship of $s\mathbb{V}_n$ with the degenerate affine Hecke algebra $\mathrm{H}_n^{\mathrm{deg}}$, which is a quotient of $s\mathbb{V}_n$. We remark that there is no nondegenerate version of the algebra $s\mathbb{V}_n$. Like $\mathrm{H}_n^{\mathrm{deg}}$, the algebra $s\mathbb{V}_n$ contains a large polynomial subalgebra $\mathbb{C}[y_1, \ldots, y_n]$ which provides a point of

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leverage for its representation theory.

In the work at hand, we begin the representation theory of the algebras $s\mathbb{V}_n$ with the smallest nontrivial example of these algebras, namely $s\mathbb{V}_2$. Our goal in this paper is to explicitly construct finite-dimensional *calibrated representations*, that is, representations of $s\mathbb{V}_2$ on which the polynomial subalgebra $\mathbb{C}[y_1, y_2]$ acts diagonalizably. Our approach here is very concrete: write down matrices for the action of the generators of $s\mathbb{V}_2$, find conditions on these matrices for the representation to be indecomposable, and determine when two such indecomposable representations are isomorphic.

The representations of $s\mathbb{V}_2$ that we focus on are the ones that cannot be obtained as representations of $\mathrm{H}_2^{\mathrm{deg}}$ by setting the Temperley-Lieb type generator e equal to 0, since calibrated representations of (degenerate) affine Hecke algebras in small rank are known by work of Ram [8]. In Section 3 we give a recipe for producing "new" calibrated representations of $s\mathbb{V}_2$, i.e. ones on which e does not act by 0. Then we show that our recipe produces all such indecomposable calibrated representations; this is Theorem 11. Theorem 11 implies that an indecomposable finite-dimensional calibrated representation with nonzero action of e always occurs as an extension of two semisimple representations with 1-dimensional composition factors (Corollary 12). In Theorems 17, 20, and 21 we completely classify the indecomposable finite-dimensional calibrated representations of $s\mathbb{V}_2$ up to isomorphism on which y_1 and y_2 act with regular eigenvalues. In addition to the eigenvalues, the other classifying device turns out to be an unexpected yet natural class of matrices that we name rhizomatic, see Section 3.2.

We expect that some of the ideas in this paper will generalize to the case $n > 2$, but the algebras $s\mathbb{V}_n$ for $n > 2$ are considerably more complicated, so we also expect that more work and possibly more ideas will be needed to deal with their calibrated representations.

2 Definitions

The degenerate affine Hecke algebra $\mathrm{H}_n^{\mathrm{deg}}$ was introduced by Drinfeld [3] and Lusztig [5]. It contains $\mathbb{C}[y_1, \ldots, y_n]$ and $\mathbb{C}S_n$ as subalgebras, and together they generate $\mathrm{H}_n^{\mathrm{deg}}$. We recall its generators and relations in the case $n = 2$.

Definition 1 ([3])**.** *The degenerate affine Hecke algebra $\mathrm{H}_2^{\mathrm{deg}}$ is the $\mathbb{C}$-algebra generated by s, y_1, and y_2 with relations:*

$$s^2 = 1, \quad y_1 y_2 = y_2 y_1, \quad s y_1 = y_2 s - 1, \quad s y_2 = y_1 s + 1.$$

Multiplying both sides of the third relation by s we get the fourth relation and vice versa, but it can be convenient to use this bigger set of relations.

The following is [1, Definition 39].

Definition 2. *The degenerate affine periplectic Brauer algebra* $\mathrm{s}\mathbb{V}_2$ *is the* $\mathbb{C}$*-algebra generated by* s*,* y_1*,* y_2*, and* e *with relations:*

$$s^2 = 1, \quad y_1 y_2 = y_2 y_1, \quad s y_1 = y_2 s - 1 - e, \quad s y_2 = y_1 s + 1 - e,$$
$$e^2 = 0, \quad es = e, \quad se = -e, \quad e y_2 = e y_1 + e, \quad y_1 e = y_2 e + e.$$

Again, this is not a minimal set of generators and relations but it is convenient to use this bigger set. Notice that e is generated by y_1, y_2, and s. It follows from the relations given that $ef(y_1, y_2)e = 0$ for any polynomial $f(y_1, y_2) \in \mathbb{C}[y_1, y_2]$, see [1, Lemma 48].

We cannot hope to classify indecomposable representations of $\mathrm{H}_2^{\mathrm{deg}}$ or $\mathrm{s}\mathbb{V}_2$ in general. However, we may hope to classify a well-behaved subset of indecomposable representations: those finite-dimensional indecomposable representations on which y_1 and y_2 act diagonalizably.

Definition 3. *Let* H *be* $\mathrm{H}_2^{\mathrm{deg}}$ *or* $\mathrm{s}\mathbb{V}_2$*. A representation* V *of* H *is called calibrated if* V *has a basis with respect to which the actions of* y_1 *and* y_2 *on* V *are given by diagonal matrices.*

Notation 4. *We denote by* $M_{m\times n}(\mathbb{C})$ *the ring of* $m \times n$ *matrices with entries in* $\mathbb{C}$*. We write* $\mathbb{C}^{k,\ell}$ *for the* $\mathbb{C}$*-vector space of dimension* $k + \ell$ *whose vectors* $(\underline{a}, \underline{b})$ *are viewed as the concatenation of a vector* $\underline{a} = (a_1, \ldots, a_k)$ *of length* k *and a vector* $\underline{b} = (b_1, \ldots, b_\ell)$ *of length* ℓ*.*

3 Calibrated representations of $\mathrm{s}\mathbb{W}_2$

In this section we construct the calibrated representations of $\mathrm{s}\mathbb{V}_2$. The starting point is the obvious relationship to the degenerate affine Hecke algebra:

Lemma 5. *Let* V *be a representation of* $\mathrm{s}\mathbb{V}_2$ *on which* e *acts by* 0*. Then the action of* $\mathrm{s}\mathbb{V}_2$ *on* V *factors through* $\mathrm{H}_2^{\mathrm{deg}} \cong \mathrm{s}\mathbb{V}_2 / \langle e \rangle$*. Conversely, if* W *is a representation of* $\mathrm{H}_2^{\mathrm{deg}}$ *then we may extend* W *to a representation of* $\mathrm{s}\mathbb{V}_2$ *by declaring* e *to act by* 0*.*

Proof. Define a homomorphism of algebras $\Phi : \mathrm{s}\mathbb{V}_2 \to \mathrm{H}_2^{\mathrm{deg}}$ by sending $y_i \in \mathrm{s}\mathbb{V}_2$ to $y_i \in \mathrm{H}_2^{\mathrm{deg}}$, $i = 1, 2$, $s \in \mathrm{s}\mathbb{V}_2$ to $s \in \mathrm{H}_2^{\mathrm{deg}}$, and $e \in \mathrm{s}\mathbb{V}_2$ to 0. Then $\Phi(s) = s$ and $\Phi(y_i) = y_i$, $i = 1, 2$, satisfy the defining relations of $\mathrm{H}_2^{\mathrm{deg}}$, since these are obtained from the defining relations of $\mathrm{s}\mathbb{V}_2$ by setting $e = 0$ in each relation where it occurs. Therefore Φ is well-defined. By construction, the kernel of Φ is the two-sided ideal generated by e. Finally, Φ is surjective since $s, y_i \in \mathrm{H}_2^{\mathrm{deg}}$ generate $\mathrm{H}_2^{\mathrm{deg}}$ and are in the image of Φ. $\square$

The calibrated representations of the affine Hecke algebra are known by work of Ram [8]. As remarked by Suzuki in his study of $\mathrm{H}_n^{\mathrm{deg}}$-representations [9], Lusztig's work [5],[6] shows that the representation theory of degenerate affine Hecke algebras and affine Hecke algebras can be recovered from each other. Calibrated representations of $\mathrm{H}_2^{\mathrm{deg}}$ may therefore be considered as known. To classify calibrated representations of $\mathrm{s}\mathbb{W}_2$, we then need to construct those on which e does not act by 0. We will do this by deforming certain calibrated representations of $\mathrm{H}_2^{\mathrm{deg}}$.

Definition 6. *For any $a \in \mathbb{C}$, let V_a^+ be the one-dimensional $\mathrm{H}_2^{\mathrm{deg}}$-representation on which y_1 acts by multiplication by a and s acts by 1; let V_a^- be defined similarly except s acts by multiplication by -1.*

Using Lemma 5 and observing that $e^2 = 0$ forces e to act by 0 on any one-dimensional representation, we have:

Lemma 7. *The one-dimensional representations of $\mathrm{s}\mathbb{W}_2$ are exactly $\{V_a^+,\ V_a^- \mid a \in \mathbb{C}\}$.*

Now let $k, \ell \in \mathbb{N}$. Let S be any $k \times \ell$ matrix. Then there is a $(k+\ell)$-dimensional calibrated $\mathrm{H}_2^{\mathrm{deg}}$-representation $W_{k,\ell}(S)$ which fits into a short exact sequence:

$$0 \to (V_0^-)^{\oplus k} \to W_{k,\ell}(S) \to (V_{-1}^+)^{\oplus \ell} \to 0,$$

where y_1 acts on $W_{k,\ell}(S)$ by the diagonal matrix Y_1 with 0's in the first k diagonal entries and -1's in the last ℓ diagonal entries, and s acts by the block matrix $\widetilde{S} = \begin{pmatrix} -\mathrm{Id}_k & S \\ 0 & \mathrm{Id}_\ell \end{pmatrix}$. Using the relation $sy_1s + s = y_2$, we get that y_2 acts on $W_{k,\ell}(S)$ by the diagonal matrix Y_2 with -1's in the first k diagonal entries and 0's in the last ℓ diagonal entries.

Example 8. *Let $k = 3$ and $\ell = 2$, and let $S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \\ s_{31} & s_{32} \end{pmatrix}$. Then the actions of y_1, y_2, and s on $W_{3,2}(S)$ are given by the following matrices:*

$$Y_1 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right), \quad Y_2 = \left(\begin{array}{ccc|cc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\widetilde{S} = \left(\begin{array}{ccc|cc} -1 & 0 & 0 & s_{11} & s_{12} \\ 0 & -1 & 0 & s_{21} & s_{22} \\ 0 & 0 & -1 & s_{31} & s_{32} \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Now let $(\underline{a},\underline{b}) := (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_\ell) \in \mathbb{C}^{k,\ell}$ be any $(k+\ell)$-tuple of complex numbers. We may build a calibrated representation of $s\mathbb{V}_2$ from $W_{k,\ell}(S)$ and $(\underline{a},\underline{b})$.

Lemma 9. *Let $V_{k,\ell}(S;(\underline{a},\underline{b}))$ be a $(k+\ell)$-dimensional $\mathbb{C}$-vector space and consider the following matrices in $\mathrm{End}_{\mathbb{C}}(V_{k,\ell}(S;(\underline{a},\underline{b}))$:*

$$y_1 = Y_1 + \mathrm{diag}(\underline{a},\underline{b}), \quad y_2 = Y_2 + \mathrm{diag}(\underline{a},\underline{b}), \quad s = \widetilde{S}, \quad e = -sy_2 + y_1 s + \mathrm{Id}_{k+\ell}.$$

Then $V_{k,\ell}(S;(\underline{a},\underline{b}))$ is a calibrated representation of $s\mathbb{V}_2$ on which y_1, y_2, s, e act by the matrices with the same names.

Proof. The matrix e is a block matrix $e = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}$ where E is a $k \times \ell$ matrix with entries $e_{ij} = (a_i - b_j)s_{ij}$. It is then a straightforward computation with matrices to check that the defining relations of $s\mathbb{V}_2$ are satisfied. □

Example 10. *For $k = 3$ and $\ell = 2$ the matrices look like:*

$$y_1 = \left(\begin{array}{ccc:cc} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & b_1 - 1 & 0 \\ 0 & 0 & 0 & 0 & b_2 - 1 \end{array}\right), \quad y_2 = \left(\begin{array}{ccc:cc} a_1 - 1 & 0 & 0 & 0 & 0 \\ 0 & a_2 - 1 & 0 & 0 & 0 \\ 0 & 0 & a_3 - 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & 0 & b_2 \end{array}\right),$$

$$s = \left(\begin{array}{ccc:cc} -1 & 0 & 0 & s_{11} & s_{12} \\ 0 & -1 & 0 & s_{21} & s_{22} \\ 0 & 0 & -1 & s_{31} & s_{32} \\ \hdashline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right), \quad e = \left(\begin{array}{ccc:cc} 0 & 0 & 0 & (a_1 - b_1)s_{11} & (a_1 - b_2)s_{12} \\ 0 & 0 & 0 & (a_2 - b_1)s_{21} & (a_2 - b_2)s_{22} \\ 0 & 0 & 0 & (a_3 - b_1)s_{31} & (a_3 - b_2)s_{32} \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We can then think of the family of calibrated representations of $s\mathbb{V}_2$ constructed by this procedure as being parametrized by pairs consisting of a $\mathrm{H}_2^{\deg}$-representation $W_{k,\ell}(S)$ as above together with a vector $(\underline{a},\underline{b}) \in \mathbb{C}^{k,\ell}$; equivalently, by pairs $(S,(\underline{a},\underline{b}))$ consisting of a $k \times \ell$ matrix $S \in M_{k\times\ell}(\mathbb{C})$ and a vector $(\underline{a},\underline{b}) \in \mathbb{C}^{k,\ell}$. When we take $(\underline{a},\underline{b})$ to be the 0-vector, then e is the 0 matrix, $y_1 = Y_1$, $y_2 = Y_2$, $s = \widetilde{S}$, and so we get back the representation $W_{k,\ell}(S)$ of $\mathrm{H}_2^{\deg}$. Note that nonzero choices of $(\underline{a},\underline{b})$ may produce representations on which e acts by 0: for example, taking $(\underline{a},\underline{b}) = (a, \ldots, a, a, \ldots, a)$ for any $a \in \mathbb{C}$ forces $e = 0$. This choice of $(\underline{a},\underline{b})$ has the effect of shifting the eigenvalues by which y_1 and y_2 act by a.

3.1 The shape of calibrated representations when e does not act by 0

The next step in our classification of calibrated $s\mathbb{V}_2$-representations consists in showing that all calibrated representations on which e does not act by 0 arise via the construction just given in the preceding subsection. We will often abuse notation and give the matrices representing the generators the same names as the generators of the abstract algebra themselves.

Theorem 11. *Let V be a finite-dimensional, indecomposable calibrated representation of* $\mathrm{s}\mathbb{W}_2$ *on which e does not act by 0. Then $V = V_{k,\ell}(S;(\underline{a},\underline{b}))$ for some $k,\ell \in \mathbb{N}$, some $S \in M_{k\times\ell}(\mathbb{C})$, and some $(\underline{a},\underline{b}) \in \mathbb{C}^{k,\ell}$.*

Proof. By assumption there is a basis for V such that y_1 and y_2 act by diagonal matrices. We can choose this basis so that the matrix for $y_1 - y_2$ has the form

$$\mathrm{diag}(1,\dots,1,-1,\dots,-1,d_1,\dots,d_1,-d_1,\dots,-d_1,\dots,d_s,\dots,d_s,-d_s,\dots,-d_s),$$

where 1 occurs k times and -1 occurs ℓ times, and say d_i occurs k_i times, $-d_i$ occurs ℓ_i times. Let $e = (e_{ij})$ be the matrix of e. Using the relations $e = (y_1 - y_2)e$ and $-e = e(y_1 - y_2)$ and writing out the equations for the matrix entries e_{ij}, we see that e has all 0 entries except for in the $k \times \ell$ block $1 \le i \le k$, $k+1 \le j \le k+\ell$. Next, we look at the shape of the matrix of s. Adding the two equations mixing s and the y_i's, we have the equation $s(y_1 - y_2) + (y_1 - y_2)s = -2$. Solving this equation for the matrix entries of s, we see that s is a block matrix with blocks $\begin{pmatrix} -\mathrm{Id}_k & S \\ T & \mathrm{Id}_\ell \end{pmatrix}$ in the upper left corner; then arranged down the diagonal, further square blocks of shape $\begin{pmatrix} -\frac{1}{d_i}\mathrm{Id}_{k_i} & S_i \\ T_i & \frac{1}{d_i}\mathrm{Id}_{\ell_i} \end{pmatrix}$ and 0's everywhere else, giving us

$$s = \left(\begin{array}{cc:cc:c:cc} -\mathrm{Id}_k & S & & 0 & 0 & & 0 \\ T & \mathrm{Id}_\ell & & & & & \\ \hdashline & & -\frac{1}{d_1}\mathrm{Id}_{k_1} & S_1 & & & \\ & 0 & T_1 & \frac{1}{d_1}\mathrm{Id}_{\ell_1} & 0 & & 0 \\ \hdashline & & & & \ddots & & \\ & 0 & & 0 & \ddots & & 0 \\ \hdashline & & & & & -\frac{1}{d_s}\mathrm{Id}_{k_s} & S_s \\ & 0 & & 0 & 0 & T_s & \frac{1}{d_s}\mathrm{Id}_{\ell_s} \end{array}\right).$$

Now, considering the shapes of y_1, y_2, e, and s, we observe that they are all block matrices with

- a $(k+\ell) \times (k+\ell)$ block in the upper left corner;
- a $(\sum_i (k_i + \ell_i)) \times (\sum_i (k_i + \ell_i))$ block in the lower right corner;
- blocks made of 0's in the upper right and lower left corner.

It follows that the representation V is the direct sum $V = V_1 \oplus V_2$ where V_1 is $(k+\ell)$-dimensional and the action of y_1, y_2, e, and s on V_1 is given by the matrix block of size $(k+\ell)\times(k+\ell)$ in the upper left corner, and where e acts by 0 on V_2. Since V is an indecomposable representation on which e does not act by 0, $V = V_1$.

Write $y_1 = \mathrm{diag}(a_1,\ldots,a_k,b_1-1,\ldots,b_\ell-1)$ and $y_2 = \mathrm{diag}(a_1-1,\ldots,a_k-1,b_1,\ldots,b_\ell)$. Using the relations $es = e$, $se = -e$, and $s^2 = 1$ gives the following information about e and s:

$$s = \begin{pmatrix} -\mathrm{Id}_k & S \\ T & \mathrm{Id}_\ell \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix},$$

where $S = (s_{ij})$, $T = (t_{ji})$, $E = (e_{ij})$, $1 \le i \le k$ and $1 \le j \le \ell$, satisfy the equations

$$TS = ST = ET = TE = 0, \qquad e_{ij} = (a_i - b_j)s_{ij}, \qquad t_{ji}(b_j - a_i) = 0. \tag{3.1}$$

Suppose $T \neq 0$. We will show that V is decomposable. Since we are assuming $e \neq 0$, we also have $S \neq 0$. Let $\mathbf{v}_i$ be an eigenvector for y_1 with eigenvalue a_i and let $\mathbf{w}_j$ be an eigenvector for y_1 with eigenvalue $b_j - 1$. By assumption, $0 \subsetneq \mathrm{Im}(T) \subseteq \mathrm{Ker}(S) \subsetneq \mathbb{C}^\ell \cong \mathrm{Span}_{\mathbb{C}}(\mathbf{w}_1,\ldots,\mathbf{w}_\ell)$ and $0 \subsetneq \mathrm{Im}(S) \subseteq \mathrm{Ker}(T) \subsetneq \mathbb{C}^k \cong \mathrm{Span}_{\mathbb{C}}(\mathbf{v}_1,\ldots,\mathbf{v}_k)$. Under the isomorphism $\mathrm{Span}_{\mathbb{C}}(\mathbf{v}_1,\ldots,\mathbf{v}_k) \cong \mathbb{C}^k$ we identify $\mathbf{v}_i = (0,\ldots,0,1,0,\ldots,0,0,\ldots,0)$ with the vector $\overline{\mathbf{v}}_i := (0,\ldots,0,1,0,\ldots,0)$ where we delete the last ℓ zeros from $\mathbf{v}_i$; similarly under the isomorphism $\mathrm{Span}_{\mathbb{C}}(\mathbf{w}_1,\ldots,\mathbf{w}_\ell) \cong \mathbb{C}^\ell$ we drop the first k zeros from the vector $\mathbf{w}_\ell$ and call the resulting vector $\overline{\mathbf{w}}_\ell$. These isomorphisms are obviously equivariant for the y_1 and y_2 actions, where y_1 acts by $\mathrm{diag}(a_1,\ldots,a_k)$ on $\mathbb{C}^k$ and by $\mathrm{diag}(b_1-1,\ldots,b_\ell-1)$ on $\mathbb{C}^\ell$, and similarly with y_2.. Since $T(\overline{\mathbf{v}}_i)$ is just the i'th column of T, by Equation (3.1) it follows that for any $1 \le j \le \ell$ such that $t_{ji} \neq 0$, $b_j = a_i$ and thus $y_1\mathbf{w}_j = (a_i - 1)\mathbf{w}_j$. So $y_1 T(\overline{\mathbf{v}}_i) = y_1 \sum_{j=1}^{\ell} t_{ji}\overline{\mathbf{w}}_j = \sum_{j=1}^{\ell} t_{ji}(b_j - 1)\overline{\mathbf{w}}_j = (a_i - 1)\sum_{j=1}^{\ell} t_{ji}\overline{\mathbf{w}}_j = (a_i - 1)T(\overline{\mathbf{v}}_i)$. This shows that $\mathrm{Im}(T)$ consists of eigenvectors for y_1. If we take $\sum_{j=1}^{\ell} f_j\overline{\mathbf{w}}_j \in \mathrm{Im}(T)^\perp$, a vector space complement to $\mathrm{Im}(T)$ in $\mathbb{C}^\ell$, then $T(\overline{\mathbf{v}}_i)\cdot(y_1\sum_{j=1}^{\ell} f_j\overline{\mathbf{w}}_j) = \sum_{j=1}^{\ell} t_{ji}(b_j - 1)f_j = (a_i - 1)T(\overline{\mathbf{v}}_i)\cdot\left(\sum_{j=1}^{\ell} f_j\overline{\mathbf{w}}_j\right) = 0$. So y_1 preserves $\mathrm{Im}(T)^\perp$.

Next, we show that y_1 preserves $\mathrm{Ker}(T)$. Again, Equation (3.1) shows that $a_m = b_j = a_i$ whenever $t_{jm} \neq 0$ is in the same row as $t_{ji} \neq 0$. Take $\overline{\mathbf{u}} \in \mathrm{Ker}(T)$ and write $\overline{\mathbf{u}} = \sum_{i=1}^{k} c_i\overline{\mathbf{v}}_i$ for some $c_i \in \mathbb{C}$. Fix a row $\mathbf{t}_j$ of T. Since $t_{ji} = 0$ whenever $b_j \neq a_i$, we then have $0 = b_j\,(\mathbf{t}_j\cdot\overline{\mathbf{u}}) = b_j(\sum_{i=1}^{k} t_{ji}c_i) = \sum_{i=1}^{k} t_{ji}b_jc_i = \sum_{i=1}^{k} t_{ji}a_ic_i = \mathbf{t}_j\cdot(y_1\overline{\mathbf{u}})$, showing that y_1 preserves $\mathrm{Ker}(T)$. Then $y_1\mathrm{Im}(S) \subseteq \mathrm{Ker}(T)$ since $\mathrm{Im}(S) \subseteq \mathrm{Ker}(T)$ and $y_1\mathrm{Ker}(T) \subseteq \mathrm{Ker}(T)$. Let $\mathrm{Ker}(T)^\perp$ be a vector space complement to $\mathrm{Ker}(T)$ in $\mathbb{C}^k$. If $\sum_{i=1}^{k} d_i\overline{\mathbf{v}}_i = \overline{\mathbf{z}} \in \mathrm{Ker}(T)^\perp$ then $(y_1\overline{\mathbf{z}})\cdot\overline{\mathbf{u}} = \sum_{i=1}^{k} a_id_ic_i = \overline{\mathbf{z}}\cdot(y_1\overline{\mathbf{u}}) = 0$ so y_1 preserves $\mathrm{Ker}(T)^\perp$ as well.

All the preceding arguments apply as well to y_2 as to y_1 since $y_1 - y_2 = \begin{pmatrix} \mathrm{Id}_k & 0 \\ 0 & -\mathrm{Id}_\ell \end{pmatrix}$. Now take V_1 to be the s$\mathbb{V}_2$-subrepresentation of V generated by a vector space complement to $\mathrm{Ker}(T)$ in $\mathrm{Span}_{\mathbb{C}}(\mathbf{v}_1,\ldots,\mathbf{v}_k)$, and take V_2 to be the subrepresentation of V generated by $\mathrm{Ker}(T)$ together with a vector space complement to $\mathrm{Im}(T)$ in $\mathrm{Span}_{\mathbb{C}}(\mathbf{w}_1, \ldots, \mathbf{w}_\ell)$. By construction $s = \begin{pmatrix} -\mathrm{Id}_k & S \\ T & \mathrm{Id}_\ell \end{pmatrix}$ preserves V_1 and V_2. We checked above that y_1 and y_2 preserve V_1 and V_2. Since $e = y_1 s - s y_2 + 1$, e also preserves V_1 and V_2. Then $V_1 \neq 0$, $V_2 \neq 0$, $V_1 + V_2 = V$ and $V_1 \cap V_2 = 0$, and therefore $V \cong V_1 \oplus V_2$ is decomposable. $\square$

Theorem 11 in pictures says that if V is indecomposable and e does not act by 0, then the matrices of y_1, y_2, e, and s have the following shapes:

$$y_1 = \left(\begin{array}{ccc:cccc} a_1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & a_k & 0 & 0 & \ldots & 0 \\ \hdashline 0 & \ldots & 0 & b_1 - 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & b_2 - 1 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 & b_\ell - 1 \end{array}\right), \quad e = \left(\begin{array}{ccc:cccc} 0 & \ldots & 0 & e_{11} & e_{12} & \ldots & e_{1\ell} \\ 0 & \ldots & 0 & e_{21} & e_{22} & \ldots & e_{2\ell} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & e_{k1} & e_{k2} & \ldots & e_{k\ell} \\ \hdashline 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{array}\right),$$

$$y_2 = \left(\begin{array}{ccc:cccc} a_1 - 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & a_k - 1 & 0 & 0 & \ldots & 0 \\ \hdashline 0 & \ldots & 0 & b_1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 & b_\ell \end{array}\right),$$

$$s = \left(\begin{array}{cccc:cccc} -1 & 0 & \ldots & 0 & s_{11} & s_{12} & \ldots & s_{1\ell} \\ 0 & -1 & \ddots & \vdots & s_{21} & s_{22} & \ldots & s_{2\ell} \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & s_{k1} & s_{k2} & \ldots & s_{k\ell} \\ \hdashline 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 \end{array}\right).$$

The following corollary is immediate from the matrix descriptions of y_1, y_2, and s given by Theorem 11:

Corollary 12. *Let V be a $(k+\ell)$-dimensional indecomposable calibrated representation of $\mathrm{s}\mathbb{W}_2$ on which e does not act by 0. Then all simple composition factors of V are 1-dimensional, and V is the following extension of semisimple $\mathrm{s}\mathbb{W}_2$-modules:*

$$0 \longrightarrow \bigoplus_{i=1}^{k} V_{a_i}^{-} \longrightarrow V \longrightarrow \bigoplus_{j=1}^{\ell} V_{b_j - 1}^{+} \to 0,$$

where y_1 acts on V by $(a_1, \ldots, a_k, b_1 - 1, \ldots, b_\ell - 1) \in \mathbb{C}^{k,\ell}$. In particular, every simple calibrated representation of $\mathrm{s}\mathbb{W}_2$ is obtained from a simple calibrated representation of $\mathrm{H}_2^{\mathrm{deg}}$ by having e act by 0.

3.2 Rhizomatic matrices

We now introduce a set of matrices that we will use for determining when a calibrated representation with regular eigenvalues is indecomposable. Let $S \in M_{k\times\ell}(\mathbb{C})$ be a $k \times \ell$ matrix. Define an equivalence relation $\sim$ on the set

$$I_S := \{(i,j) \mid 1 \le i \le k,\ 1 \le j \le \ell,\ s_{ij} \neq 0\},$$

the set of indices of the nonzero matrix entries of S, by (i) considering the binary relation $\dagger$ on I_S defined by $(i,j)\dagger(m,n)$ if $i = m$ or $j = n$, which is evidently reflexive and symmetric, and then (ii) defining $\sim$ to be the transitive closure of the relation $\dagger$.

Definition 13. *Define the rhizomatic matrices to be the set of matrices $S \in M_{k\times\ell}(\mathbb{C})$ such that (i) the set I_S as above forms a single equivalence class under the relation $\sim$, and (ii) S has a nonzero entry in every row and in every column.*

Put another way, the definition says: $S \in M_{k\times\ell}(\mathbb{C})$ is rhizomatic if and only if (i) for any nonzero entries $s_{ij}, s_{mn} \in S$ it holds that $(i,j) \sim (m,n)$, and moreover, (ii) for any $1 \le i \le k$ there exists some $1 \le n \le \ell$ such that $(i,n) \in I_S$ and for any $1 \le j \le \ell$ there exists some $1 \le m \le k$ such that $(m,j) \in I_S$.

Example 14. *Any matrix all of whose entries are nonzero is rhizomatic. If $\ell \ge k$ then any $k \times \ell$ matrix S where $s_{ij} \neq 0$ whenever $j \ge i$ is rhizomatic. Any $n \times n$ diagonal matrix, and more generally any monomial matrix, is not rhizomatic.*

Example 15. *Denote a 0 entry by $\cdot$ and a nonzero entry by $\bullet$. Matrix S_1 contains two equivalence classes of nonzero entries and is not rhizomatic: one equivalence class has black entries $\bullet$, the other has blue entries $\color{blue}\bullet$. Matrix S_2 contains a single equivalence class but is not rhizomatic because it has some columns and rows that are all 0. Matrix S_3 contains a single equivalence class and is rhizomatic:*

$$S_1 = \begin{pmatrix}
\cdot & \cdot & \cdot & {\color{blue}\bullet} & {\color{blue}\bullet} & \cdot & {\color{blue}\bullet} & {\color{blue}\bullet} & \cdot & \cdot \\
\bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet \\
\cdot & {\color{blue}\bullet} & \cdot & {\color{blue}\bullet} & {\color{blue}\bullet} & {\color{blue}\bullet} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot \\
\cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet \\
\cdot & {\color{blue}\bullet} & \cdot & \cdot & {\color{blue}\bullet} & {\color{blue}\bullet} & {\color{blue}\bullet} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot
\end{pmatrix}, \qquad
S_2 = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \bullet \\
\cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot \\
\bullet & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet
\end{pmatrix},$$

$$S_3 = \begin{pmatrix}
\bullet & \cdot & \bullet & \cdot & \cdot & \cdot & \bullet & \cdot & \bullet & \cdot \\
\bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet \\
\cdot & \bullet & \cdot & \cdot & \cdot & \bullet & \cdot & \bullet & \cdot & \cdot \\
\cdot & \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \bullet & \cdot \\
\cdot & \cdot & \bullet & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \bullet \\
\cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\bullet & \cdot & \bullet & \cdot & \cdot & \cdot & \bullet & \cdot & \bullet & \cdot
\end{pmatrix}.$$

3.3 Indecomposable calibrated representations with regular eigenvalues

Definition 16. *Suppose V is a calibrated representation of $\mathrm{s}\mathbb{V}_2$ such that $y_1 - y_2$ acts on V by $\begin{pmatrix} \mathrm{Id}_k & 0 \\ 0 & -\mathrm{Id}_\ell \end{pmatrix}$ in an eigenbasis for y_1 and y_2. Set $\underline{a} = (a_1, \ldots, a_k)$ and $\underline{b} = (b_1, \ldots, b_\ell)$ and*

$$\mathbb{C}^{k,\ell,\mathrm{reg}} := \{(\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell} \mid a_i \neq a_j, b_m \neq b_n \text{ for all } 1 \leq i < j \leq k,\ 1 \leq m < n \leq \ell\}.$$

If y_1 acts on V by $\mathrm{diag}(\underline{a}, \underline{b})$ for some $(\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell,\mathrm{reg}}$ then we say that the representation V has regular eigenvalues.

Theorem 17. *Let $V = V_{k,\ell}(S; (\underline{a}, \underline{b}))$, a finite-dimensional calibrated representation of $\mathrm{s}\mathbb{V}_2$ on which e does not act by 0.*

1. *Suppose $(\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell,\mathrm{reg}}$. Then V is indecomposable if and only if S is a rhizomatic matrix.*
2. *Suppose $(\underline{a}, b) \in \mathbb{C}^{k,1}$. Then V is indecomposable if and only if $(\underline{a}, b) \in \mathbb{C}^{k,1,\mathrm{reg}}$ and all entries of S are nonzero.*
3. *Suppose $(a, \underline{b}) \in \mathbb{C}^{1,\ell}$. Then V is indecomposable if and only if $(a, \underline{b}) \in \mathbb{C}^{1,\ell,\mathrm{reg}}$ and all entries of S are nonzero.*

Proof. For part (1), suppose $(\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell,\mathrm{reg}}$. Recall that a representation V is indecomposable if and only if $\mathrm{End}(V)$ is a local ring, which is equivalent to every element of $\mathrm{End}(V)$ being either nilpotent or invertible. We determine $\mathrm{End}(V)$ as follows. Let $X \in \mathrm{End}(V)$, so by definition $X = (x_{ij})$ is a $(k+\ell) \times (k+\ell)$ matrix that commutes with the matrices for y_1, y_2, and s. (Since $e = y_1 s + 1 - s y_2$, we don't have to check commutation relations with e.) First, from $y_1 - y_2 = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ it follows that $X \in M_{k\times k}(\mathbb{C}) \times M_{\ell\times\ell}(\mathbb{C}) \subset M_{(k+\ell)\times(k+\ell)}(\mathbb{C})$ where we embed $M_{k\times k}(\mathbb{C})$ in the upper left corner and $M_{\ell\times\ell}(\mathbb{C})$ in the lower right corner of $(k+\ell) \times (k+\ell)$ matrices. Second, since $a_i \neq a_j$ for all $1 \leq i < j \leq k$, and $b_m \neq b_n$ for all $1 \leq m < n \leq \ell$, computing the matrix entries of the equation $y_1 X = X y_1$ shows that X is a diagonal matrix, and so

$$\mathrm{End}(V) \subseteq \{\mathrm{diag}(z_1, z_2, \ldots, z_k, w_1, w_2, \ldots, w_\ell) \mid z_i, w_j \in \mathbb{C}\} \cong \mathbb{C}^{k,\ell}.$$

(Computing the commutator of X with y_2 now gives no new information, since we already used y_1 and $y_1 - y_2$).

Write $X = \mathrm{diag}(z_1, z_2, \ldots, z_k, w_1, w_2, \ldots, w_\ell)$. We now determine $\mathrm{End}(V)$ as a subalgebra of diagonal matrices using the remaining equation $Xs - sX = 0$.

Computing the commutator $Xs - sX$, all entries are automatically 0 except in the upper right $k \times \ell$ corner block, where we obtain the following $k \times \ell$ entries:

$$\begin{pmatrix} s_{11}(z_1 - w_1) & s_{12}(z_1 - w_2) & \dots & s_{1\ell}(z_1 - w_\ell) \\ s_{21}(z_2 - w_1) & s_{22}(z_2 - w_2) & \dots & s_{2\ell}(z_2 - w_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ s_{k1}(z_k - w_1) & s_{k2}(z_k - w_2) & \dots & s_{k\ell}(z_k - w_\ell) \end{pmatrix}.$$

Since $Xs - sX = 0$, each of these $k\ell$ entries is equal to 0. Thus for a given pair (i, j), either $s_{ij} = 0$ or $z_i = w_j$. Taking the equivalence class of a nonzero entry s_{ij} as in Section 3.2, it follows that $z_r = w_t = z_i = w_j$ for all $s_{rt} \sim s_{ij}$, i.e. all the z_r's and w_t's are equal to each other such that r is the row or t is the column of some nonzero entry $s_{rt} \sim s_{ij}$. If s_{ij} and s_{rt} are in different equivalence classes, then there is no relation between z_i and z_r or between w_j and w_t. And finally, if some row r contains all 0 entries then we get no relation on z_r; similarly, if some column t contains all 0 entries then we get no relation on w_t. Let $n(S) \geq 1$ be the number of equivalence classes of nonzero entries of S, let Z_r be the number of rows of S that contain only 0's, and let Z_c be the number of columns that contain only 0's. We have that $\mathrm{End}(V) \cong \mathbb{C}^{n(S)+Z_r+Z_c}$, but $\mathbb{C}^{n(S)+Z_r+Z_c}$ is a local ring if and only if $n(S) + Z_r + Z_c = 1$ if and only if $n(S) = 1$ and $Z_r = Z_c = 0$ if and only if S is rhizomatic. This concludes the proof of part (1).

We turn now to part (2). One direction of the statement is simply a special case of part (1) when $\ell = 1$: if S is a $k \times 1$ matrix then S is rhizomatic if and only if all the entries of S are nonzero, thus if all entries of S are nonzero and $(\underline{a}, b) \in \mathbb{C}^{k,1,\mathrm{reg}}$ then part (1) says that V is indecomposable. For the converse direction, suppose that V is indecomposable. If some entry s_{i1} of S is 0 then we see that the actions of the generators of $s\mathbb{V}_2$ preserve the subspaces $\mathbb{C}\mathbf{v}_i$ and $\mathbb{C}\mathbf{v}_1 \oplus \ldots \oplus \mathbb{C}\mathbf{v}_{i-1} \oplus \mathbb{C}\mathbf{v}_{i+1} \oplus \ldots \oplus \mathbb{C}\mathbf{v}_{k+1}$ (where $\mathbf{v}_i$ denotes the i'th basis vector $(0, \ldots, 1, \ldots, 0)$ of $\mathbb{C}^{k+1}$ with 1 in the i'th place and 0's elsewhere); thus V splits as a direct sum of these two subrepresentations contradicting the assumption that V is indecomposable. So $s_{i1} \neq 0$ for all $i = 1, \ldots, k$. Suppose $(\underline{a}, b) \notin \mathbb{C}^{k,1,\mathrm{reg}}$, so $a_i = a_m$ for some $i \neq m$; without loss of generality we may assume $i = 1$ and $m = 2$. Then the centralizer of y_1 and y_2 contains any matrix of the form

$$X = \left(\begin{array}{cc:cccc} x_{11} & x_{12} & 0 & 0 & \dots & 0 \\ x_{21} & x_{22} & 0 & 0 & \dots & 0 \\ \hdashline 0 & 0 & x_{33} & 0 & \dots & 0 \\ 0 & 0 & 0 & x_{44} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & x_{k+\ell,k+\ell} \end{array}\right).$$

Computing the matrix entries of the equation $Xs - sX = 0$ we get the following k

entries which are all equal to 0:

$$\begin{pmatrix} (x_{11} - x_{k+1,k+1})s_{11} + x_{12}s_{21} \\ x_{21}s_{11} + (x_{22} - x_{k+1,k+1})s_{21} \\ (x_{33} - x_{k+1,k+1})s_{31} \\ \vdots \\ (x_{k,k} - x_{k+1,k+1})s_{k1} \end{pmatrix}.$$

Since all $s_{i1} \neq 0$ for $i = 1, \dots, k$, $x_{i,i} = x_{k+1,k+1}$ for all $i = 3, \dots, k$, and from the first and second lines we get that $x_{k+1,k+1}$ can be solved in terms of x_{21}, x_{22}, s_{11}, and s_{21}, and then similarly we can solve for x_{12} in terms of x_{11}, x_{21}, x_{22}, s_{11}, and s_{21} in the first equation. Then we have

$$\mathrm{End}(V) \cong \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix},$$

which is not a local ring, contradicting the assumption that V is indecomposable. Therefore all the eigenvalues a_i are distinct, and part (2) is proved.

Finally, part (3) is proved in a totally symmetric way to part (2). □

Remark 18. *In fact, if* $(\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell,\mathrm{reg}}$ *and* S *is rhizomatic, then for some* (i, j) *the entry* $e_{ij} = (a_i - b_j)s_{ij}$ *of* e *is automatically nonzero. Indeed, by way of contradiction suppose that* e *is the* 0 *matrix, but* S *is rhizomatic. Then for any* $s_{ij} \neq 0$ *there is some other* $s_{ik} \neq 0$ *in the same row or some other* $s_{hj} \neq 0$ *in the same column. In the first case,* $(a_i - b_j)s_{ij} = 0 = (a_i - b_k)s_{ik}$ *forces* $b_j = a_i$ *and* $b_k = a_i$ *and thus* $b_j = b_k$ *for some* $j \neq k$*, contradicting the assumption* $(\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell,\mathrm{reg}}$*. Similarly in the second case. Thus* $(\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell,\mathrm{reg}}$ *and* S *rhizomatic implies that* e *does not act by* 0 *on* $V_{k,\ell}(S; (\underline{a}, \underline{b}))$.

Example 19. *Let* $k = 3$ *and* $\ell = 2$ *and take* $(\underline{a}, \underline{b}) = (2i, -2i, 1, -1, 1)$. *Then* $(\underline{a}, \underline{b}) \in \mathbb{C}^{3,2,\mathrm{reg}}$. *Take* $S = \begin{pmatrix} 0 & 1 \\ -\pi & 5 \\ \frac{i\pi}{2} & 0 \end{pmatrix}$, *a rhizomatic matrix. Then* $V_{3,2}(S; (\underline{a}, \underline{b}))$ *is an indecomposable calibrated* $\mathfrak{s}\mathbb{V}_2$*-representation by Theorem 1, and* y_1, y_2, s, e *act by:*

$$y_1 = \left(\begin{array}{ccc:cc} 2i & 0 & 0 & 0 & 0 \\ 0 & -2i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad y_2 = \left(\begin{array}{ccc:cc} 2i-1 & 0 & 0 & 0 & 0 \\ 0 & -2i-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$s = \left(\begin{array}{ccc:cc} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -\pi & 5 \\ 0 & 0 & -1 & \frac{i\pi}{2} & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right), \quad e = \left(\begin{array}{ccc:cc} 0 & 0 & 0 & 0 & 1-2i \\ 0 & 0 & 0 & \pi-1 & 5+10i \\ 0 & 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

3.4 Isomorphism classes of calibrated representations with regular eigenvalues

Many of the calibrated representations we constructed in the previous section may be isomorphic to each other. In this section, we determine when two indecomposable calibrated representations with regular eigenvalues are isomorphic.

Recall the following constructions from Lie theory, which may be found in [10]. The symmetric group S_k is isomorphic to the Weyl group W of the Lie group $G = \mathrm{GL}_k(\mathbb{C})$. The Weyl group is defined by $W = N_G(T)/T$ where $T \subset G$ is a maximal torus and $N_G(T)$ is its normalizer in G. We may take T to be the diagonal matrices in $\mathrm{GL}_k(\mathbb{C})$. Then $N_G(T)$ is the group of $k \times k$ *monomial matrices* that have exactly one nonzero complex entry in every row and column, while $N_G(T)/T \cong S_k$ is the group of $k \times k$ *permutation matrices* consisting of those monomial matrices whose nonzero entries are all 1's. Let us denote by $N_k = N_G(T)$ the group of $k \times k$ monomial matrices. Similarly, let N_ℓ be the group of $\ell \times \ell$ monomial matrices.

Let $(\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell,\mathrm{reg}}$, so $\underline{a} = (a_1, \dots, a_k)$ with $a_i \neq a_j$ for all $i \neq j$, and $\underline{b} = (b_1, \dots, b_\ell)$ with $b_m \neq b_n$ for all $m \neq n$.

Theorem 20. *The group $N_k \times N_\ell$ acts naturally on the set of $(k+\ell)$-dimensional indecomposable calibrated representations with regular eigenvalues and on the space of pairs consisting of a $(k \times \ell)$ rhizomatic matrix and a vector $(\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell,\mathrm{reg}}$ which parametrize these representations:*

$$(N_k \times N_\ell) \curvearrowright \widetilde{\mathcal{V}}_{k,\ell} := \{(S, (\underline{a}, \underline{b})) \mid S \in M_{k\times\ell}(\mathbb{C}) \text{ is rhizomatic}, \ (\underline{a}, \underline{b}) \in \mathbb{C}^{k,\ell,\mathrm{reg}}\}.$$

Proof. We embed $N_k \times N_\ell$ in $\mathrm{GL}_{k+\ell}(\mathbb{C})$ in the obvious way, as block matrices:

$$N_k \times N_\ell \cong \begin{pmatrix} N_k & 0 \\ 0 & N_\ell \end{pmatrix},$$

which then act by conjugation on y_1, y_2, s, e. On the matrix S, N_k acts on the left via the left multiplication while N_ℓ acts on the right via the right (inverse) multiplication. These actions are also called left translation and right (inverse) translation, respectively. Take elements $X_1 = \sum_{i=1}^{k} \xi_i e_{i,\sigma(i)} \in N_k$ and $X_2 = \sum_{j=1}^{\ell} \phi_j e_{j,\tau(j)} \in N_\ell$ where $\xi_i, \phi_j \in \mathbb{C}^\times$, $\sigma \in S_k$, $\tau \in S_\ell$. On the matrices for y_1, y_2, s, e we get the following effect:

$$(X_1, X_2) \cdot y_1 = \mathrm{diag}(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(k)}, b_{\tau(1)} - 1, b_{\tau(2)} - 1, \dots, b_{\tau(\ell)} - 1),$$
$$(X_1, X_2) \cdot y_2 = \mathrm{diag}(a_{\sigma(1)} - 1, a_{\sigma(2)} - 1, \dots, a_{\sigma(k)} - 1, b_{\tau(1)}, b_{\tau(2)}, \dots, b_{\tau(\ell)}),$$
$$(X_1, X_2) \cdot s = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} -\mathrm{Id}_k & S \\ 0 & \mathrm{Id}_\ell \end{pmatrix} \begin{pmatrix} X_1^{-1} & 0 \\ 0 & X_2^{-1} \end{pmatrix} = \begin{pmatrix} -\mathrm{Id}_k & X_1 S X_2^{-1} \\ 0 & \mathrm{Id}_\ell \end{pmatrix},$$
$$(X_1, X_2) \cdot e = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1^{-1} & 0 \\ 0 & X_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & X_1 E X_2^{-1} \\ 0 & 0 \end{pmatrix}.$$

The action on pairs $(S, (\underline{a}, \underline{b}))$ is given explicitly by:

$$(X_1, X_2) \cdot (S, (\underline{a}, \underline{b})) = (X_1 S X_2^{-1}, (a_{\sigma(1)}, \ldots, a_{\sigma(k)}, b_{\tau(1)}, \ldots, b_{\tau(\ell)})),$$

where the effect of the action $S \to X_1 S X_2^{-1}$ is to permute the rows of S by σ and the columns by τ^{-1}, then to multiply the i'th row of the resulting matrix by ξ_i and the j'th column by ϕ_j. That is, we have $\left(X_1 S X_2^{-1}\right)_{ij} = \xi_i \phi_j s_{\sigma^{-1}(i),\tau(j)}$.

We need to check that $X_1 S X_2^{-1}$ is also a rhizomatic matrix. The minimal relations generating the relation $s_{ij} \sim s_{rp}$ in Section 3.2 are of the form $s_{ij} \sim s_{ir}$ and $s_{ij} \sim s_{pj}$, i.e. the relations given by two nonzero entries being in the same row, and two nonzero entries being in the same column. Since $\xi_i, \phi_j \neq 0$, $\left(X_1 S X_2^{-1}\right)_{ij} \neq 0$ if and only if $s_{\sigma^{-1}(i),\tau(j)} \neq 0$. Therefore $\left(X_1 S X_2^{-1}\right)_{ij} \sim \left(X_1 S X_2^{-1}\right)_{ir}$ if and only if $s_{\sigma^{-1}(i),\tau(j)} \sim s_{\sigma^{-1}(i),\tau(r)}$, and $\left(X_1 S X_2^{-1}\right)_{ij} \sim \left(X_1 S X_2^{-1}\right)_{pj}$ if and only if $s_{\sigma^{-1}(i),\tau(j)} \sim s_{\sigma^{-1}(p),\tau(j)}$. It then follows that $\left(X_1 S X_2^{-1}\right)_{ij} \sim \left(X_1 S X_2^{-1}\right)_{pr}$ if and only if $s_{\sigma^{-1}(i),\tau(j)} \sim s_{\sigma^{-1}(p),\tau(r)}$. Thus $X_1 S X_2^{-1}$ has a single equivalence class of entries since S does. Since a nonzero entry appears in every row and column of S, the same is true for $X_1 S X_2^{-1}$. Therefore $X_1 S X_2^{-1}$ is again rhizomatic, and we indeed get an action. □

Theorem 21. *Let $V_1, V_2 \in \widetilde{\mathcal{V}}_{k,\ell}$ be two indecomposable calibrated representations of dimension $k+\ell$ with regular eigenvalues. Then $V_1 \cong V_2$ as $\mathrm{s}\mathbb{V}_2$-representations if and only if V_1 and V_2 are in the same $(N_k \times N_\ell)$-orbit. Thus $\mathcal{V}_{k,\ell} := \widetilde{\mathcal{V}}_{k,\ell}/(N_k \times N_\ell)$ parametrizes the isomorphism classes of indecomposable $(k+\ell)$-dimensional calibrated $\mathrm{s}\mathbb{V}_2$-representations with regular eigenvalues.*

Proof. Let the matrices of the generators y_1, y_2, s acting on V_i, $i = 1, 2$, be given by

$$\begin{aligned} y_1^{(i)} &= \mathrm{diag}(a_1^{(i)}, \ldots, a_k^{(i)}, b_1^{(i)} - 1, \ldots, b_\ell^{(i)} - 1), \\ y_2^{(i)} &= \mathrm{diag}(a_1^{(i)} - 1, \ldots, a_k^{(i)} - 1, b_1^{(i)}, \ldots, b_\ell^{(i)}), \\ s^{(i)} &= \begin{pmatrix} -\mathrm{Id}_k & S^{(i)} \\ 0 & \mathrm{Id}_\ell \end{pmatrix}. \end{aligned}$$

By definition, $V_1 \cong V_2$ if and only if there exists $A \in \mathrm{GL}_{k+\ell}(\mathbb{C})$ such that

$$A\left(y_1^{(1)}\right)A^{-1} = y_1^{(2)}, \qquad A\left(y_2^{(1)}\right)A^{-1} = y_2^{(2)}, \qquad A\left(s^{(1)}\right)A^{-1} = s^{(2)}.$$

The first two equations imply that $A \in N_k \times N_\ell$. Thus $V_1 \cong V_2$ implies that V_1 and V_2 are in the same $(N_k \times N_\ell)$-orbit. The converse follows from the previous theorem which showed that conjugation by $N_k \times N_\ell$ respects the $\mathrm{s}\mathbb{V}_2$-action. □

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References

[1] M. Balagovic, Z. Daugherty, I. Entova-Aizenbud, I. Halacheva, J. Hennig, M.S. Im, G. Letzer, E. Norton, V. Serganova, C. Stroppel. *The affine VW supercategory*, arXiv:1801.04178. MR4073971. Zbl 07186761.

[2] C.-W. Chen and Y.-N. Peng. *Affine periplectic Brauer algebras. J. Algebra* 501 (2018), 345–372. MR3768141. Zbl 1432.17034.

[3] V. G. Drinfeld. *Degenerate affine Hecke algebras and Yangians*, Funktsional. Anal. i Prilozhen., **20**:1 (1986), 69–70; *Funct. Anal. Appl.*, 20:1 (1986), 58–60. MR0831053. Zbl 0599.20049.

[4] J. R. Kujawa and B. C. Tharp. *The marked Brauer category*, J. Lond. Math. Soc. (2) **95** (2017), no. 2, 393–413. MR3656274. Zbl 1427.17015.

[5] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Am. Math. Soc. **2**, No 3 (1989), 599–635. MR0991016. Zbl 0715.22020.

[6] G. Lusztig, *Cuspidal local systems and graded Hecke algebras*, I, Publ. Math. IHES **67** (1988), 145–202. MR0972345. Zbl 0699.22026.

[7] D. Moon, *Tensor product representations of the Lie superalgebra $\mathfrak{p}(n)$ and their centralizers*, Comm. Algebra **31** (2003), no. 5, 2095–2140. MR1976269. Zbl 1022.17005.

[8] A. Ram, *Affine Hecke algebras and generalized standard Young tableaux*, J. Algebra **260** (2003), no. 1, 367–415. MR1976700. Zbl 1076.20002.

[9] T. Suzuki, *Representations of degenerate affine Hecke algebra and $\mathfrak{gl}_n$*, Advanced Studies in Pure Mathematics **28**, 2000, *Combinatorial Methods in Representation Theory*, pp. 343–372. MR1864488. Zbl 1058.20007.

[10] E. B. Vinberg. "Tits system." *Encyclopaedia of Mathematics*, Band **9**. Ed. Michiel Hazewinkel. Springer Science and Business Media, 1993.

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