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ABSTRACT

We present the macroscopic dynamic description of a ferromagnetic nematic, where the nematic part and the magnetic part can move relative to each other. The relative velocity that describes such movements can be a slowly relaxing variable. Its couplings to the nematic and the magnetic degrees of freedom are particularly interesting since the symmetry properties (behavior under spatial inversion and time reversal) of the three vectorial quantities involved are all different. As a consequence, a number of new crosscouplings involving the relative velocity exist. Some of them are discussed in more detail. First, we demonstrate that transverse temperature gradients generate transverse relative velocities and, vice versa, that transverse relative velocities give rise to temperature gradients. Second, we show that a simple shear flow in the relative velocity with the preferred direction in the shear plane can lead in a stationary situation to a tilt of the magnetization.

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I. INTRODUCTION

Hydrodynamics is the prototype for a macroscopic description of the dynamics of condensed systems, e.g., fluids. The variables used are long-lived conserved variables (such as mass, momentum, and energy) and symmetry variables connected to spontaneously broken (continuous) symmetries (such as director rotations in a nematic liquid crystal, magnetization rotations in a ferromagnetic liquid, and displacements in a crystal).^{1,2} For more complex systems or more complicated situations, variables come into play that do not belong to the strictly hydrodynamic case but live long enough to take part in the dynamics of a macroscopic system. The concept of slowly relaxing variables has been pioneered by Pitaevskii³ for the relaxation of the order parameter modulus near the λ -transition in superfluid ⁴He. These slowly relaxing variables are well-known from, e.g., order parameter relaxation near second order phase transitions,^{4,5} strain relaxation in polymers,^{6,7} rotations of a nematic orientation relative to the elastomeric matrix,^{8,9} chemical reactions in binary mixtures (compare, for example, Ref. 10), and many other systems.

A rather special type of relaxational variable can occur in a two-phase system when the constituents are able to move relative to each other. The relative velocity (the difference of the individual velocities of the two phases) generally is a relaxing quantity since it is neither related to a conservation law nor to a broken symmetry.¹¹ Exceptions are superfluids, where the second (superfluid) velocity is due to the broken gauge symmetry.^{4,12,13} In the non-superfluid case, which will be dealt with here, there is always friction during relative motion and the momenta of the two constituents are not conserved. If the relative velocity is slowly relaxing, it constitutes a rather specific additional variable, odd under spatial inversion and odd under time reversal, that considerably influences the structure of the appropriate hydrodynamics. For a broader discussion of the relation between the symmetry of phases (variables) with the structure of the hydrodynamics, cf. Refs. 14 and 15.

The advantages of a two-fluid model when compared to a one fluid approach are clear: One allows for a relative motion of the constituents relative to each other, thus resulting in a more mesoscopic description, which is also applicable to shorter length

scales and/or higher frequencies. The price to be paid is also apparent: One has additional relaxing variables and, as a result, more phenomenological parameters entering the description.

Macroscopic two-fluid descriptions have already been given for nematic liquid crystals in an isotropic solvent,¹¹ for isotropic elastic media in an isotropic solvent^{11,16} with application to the cluster dynamics above the glass transition,¹⁷ for nematic liquid crystals in the presence of smectic order to describe the breakdown of flow alignment,¹⁸ for a magnetic fluid in an isotropic solvent to describe magnetorheological fluids,¹⁹ for fluids with tetrahedral order in an isotropic solvent,²⁰ and for a mixture of two isotropic elastic media.²¹ In this paper, we give a two-fluid description of ferromagnetic nematics, where the nematic part and the magnetic part can move relative to each other.

Ferromagnetic nematics have been suggested very early by Brochard and de Gennes,²² and the single-fluid Landau and hydrodynamic descriptions were given in Refs. 23–25. Experimentally, stable ferromagnetic nematics were obtained in the form of a suspension of ferromagnetic platelets in a nematic liquid crystal^{26,27} recently. In addition to a number of static properties,^{26–28} various optical and flow properties were discussed theoretically and successfully compared with experiments.^{29–31}

For the two-fluid description of ferromagnetic nematics, we restrict ourselves to the case that in equilibrium the nematic director and the magnetization (and an external magnetic field) are collinear. The material tensors involved will be given in uniaxial approximation. We first characterize the nematic and magnetic subsystems and introduce the framework to describe relative motion (Sec. II). The macroscopic dynamics includes the static part (Sec. III), the dynamic equations (Sec. IV), and the dissipative and reversible currents (Secs. IV B and IV C). A discussion of some physical implications of the two-fluid aspects is given in Sec. V, which is followed by conclusions and perspective (Sec. VI).

II. SETTING UP THE SYSTEM

We consider a mixture of a nematic liquid and a ferromagnetic one. The nematic liquid shows a preferred direction n_i according to the order parameter $Q_{ij} = (S/2)(n_i n_j - [1/3]\delta_{ij})$ with S being a measure of the strength of the orientational order.³² It is assumed to be a fast relaxing variable that is on the hydrodynamic time scale always in equilibrium. The director n_i can be used like an axial unit vector if the invariance $n_i \rightarrow -n_i$ of the appropriate equations is ensured. In the ground state (in the absence of any boundaries or external fields), the orientation of n_i^0 is arbitrary and constitutes therefore a spontaneously breaking of rotational symmetry. An external magnetic field H_i can orient the director, either parallel or perpendicular to the field, depending on the sign of the diamagnetic anisotropy,³² $\varepsilon_a = -(1/2)\chi_a(\mathbf{n} \cdot \mathbf{H})^2$.

The ferromagnetic liquid is characterized by a finite magnetization, $M_i = Mm_i$, with M being its magnitude and the axial unit vector m_i denoting its orientation. It changes sign under time reversal and also breaks rotational symmetry. In the ground state, m_i^0 is arbitrary. In the presence of a (homogeneous) magnetic field, H_i , the magnetization orients along the field, $m_i^0 \parallel H_i$, due to ferromagnetic coupling $\varepsilon_f = -M\mathbf{m} \cdot \mathbf{H}$. With the field, M_0 increases to $M_0(H)$. We will take this material function from experiment. Similar to the

nematic case, we will assume that M is fast relaxing and is always given by its (field-dependent) equilibrium value.

We will concentrate on the case, where in the ground state, the director, the magnetization, and, if present, the field are collinear. This is achieved by choosing $\chi_a > 0$ a coupling energy $\varepsilon_c = -(1/2)A_1(\mathbf{n} \cdot \mathbf{m})^2$ with $A_1 > 0$. The latter term describes the coupling between the nematic director and the magnetic particles. This strong interaction was observed in a series of experiments^{33–35} studying the Fredericks transition in ferronematics. For the experimentally accessible systems existing so far, the magnetization direction of the ferromagnetic platelets, \mathbf{m} , and the nematic director, \mathbf{n} , are parallel in the ground state without external fields. In case a ground state is observed experimentally for which the director and the magnetization direction are not parallel in equilibrium, one has to deal with a biaxial system. In this case, all property tensors will be biaxial in nature.

We assume the two subsystems to be able to move relative to each other. Thus, we need for a macroscopic dynamic description two mass densities, ρ_n and ρ_m , and two momenta, g_i^n and g_i^m . Since the nematogens and the magnetic particles are distinct entities, both densities are conserved individually. In order to simplify the analysis, we will later assume the total mass density, $\rho = \rho_n + \rho_m$, to be constant. As the remaining variable, we take the concentration of the magnetic part, $\phi = \rho_m/\rho$.

The partial momenta are not conserved since by moving around the two subsystems can exchange momentum. Only the total momentum density, $g_i = g_i^n + g_i^m$, is conserved. Therefore, it makes sense to use it as a variable. As the second variable, one takes, instead of the momenta difference, the velocity difference, $w_i = v_i^n - v_i^m$, with $v_i^n = g_i^n/\rho_n$ and $v_i^m = g_i^m/\rho_m$, which is easier to account for in experiments. The mean velocity, $v_i = g_i/\rho$, is a weighted sum of the two velocities, $v_i = (1 - \phi)v_i^n + \phi v_i^m$.

We assume that the thermal degree of freedom, the entropy density σ , is common to both subsystems, meaning that there is only one temperature.

III. STATICS

The first law of thermodynamics, a generalized energy conservation law, governs the static aspects of macroscopic dynamics. With the relevant variables discussed in Sec. II, and all others in thermodynamic equilibrium, the differential form of the first law reads (Gibbs relation)^{1,5,36}

$$d\varepsilon = Td\sigma + \mu d\rho + \Pi d\phi + v_i dg_i + h_i^w dw_i + h_i^{m'} dm_i + \Phi_{ij}^m d(\nabla_j m_i) + h_i^{n'} dn_i + \Phi_{ij}^n d(\nabla_j n_i), \quad (1)$$

with ε being the density of the total energy. The appropriate conjugate quantities are temperature (T), chemical potential (μ), relative chemical potential (Π), and mean velocity (v_i). The “molecular fields” h_i^w , $h_i^{m'}$, and $h_i^{n'}$, and Φ_{ij}^m and Φ_{ij}^n are conjugate to w_i , m_i , and n_i and to $\nabla_j m_i$ and $\nabla_j n_i$, respectively. Throughout this paper, we focus on bulk phenomena. Surface and interface effects can be rather specific and are to be considered on a case to case basis separately.

In the absence of orienting external fields or boundaries, $h_i^{m'}$ and $h_i^{n'}$ have to be zero since n_i and m_i are symmetry variables and homogeneous rotations do not cost energy. Often, the “molecular”

fields can be combined,

$$h_i^n = h_i^{n'} - \nabla_j \Phi_{ij}^n \quad \text{and} \quad h_i^m = h_i^{m'} - \nabla_j \Phi_{ij}^m. \quad (2)$$

In case the deviations of the amount of magnetization are taken into account, an extra term $h^M dM$ is added to Eq. (1).

The static behavior of our macroscopic system is conveniently described by the energy functional $\int \varepsilon dV$ that describes the energies related to deviations from the equilibrium values. It contains thermal and concentration deviations from their equilibrium values σ_0 and ϕ_0 (with density variations already suppressed), Frank-type contributions due to orientational deformations of the director and the magnetization, the kinetic energies of the subsystems, and the field-orienting contributions,

$$\begin{aligned} \varepsilon = & \frac{T}{2C_V} (\delta\sigma)^2 + \frac{1}{2\kappa_\phi} (\delta\phi)^2 + \frac{1}{\alpha_\phi} (\delta\phi)(\delta\sigma) + \frac{1}{2} K_{ijkl}^n (\nabla_j n_i)(\nabla_l m_k) \\ & + \frac{1}{2} K_{ijkl}^m (\nabla_j m_i)(\nabla_l m_k) + \tilde{K}_{ijkl} (\nabla_j n_i)(\nabla_l m_k) + \frac{1}{2\rho} \mathbf{g}^2 + \frac{1}{2} \alpha \mathbf{w}^2 \\ & + \frac{1}{2} M_0 H \delta_{ij}^\perp \delta m_i \delta m_j + \frac{\chi_a}{2} H^2 \delta_{ij}^\perp \delta n_i \delta n_j \\ & + \frac{1}{2} A_1 \delta_{ij}^\perp (\delta n_i - \delta m_i)(\delta n_j - \delta m_j) \end{aligned} \quad (3)$$

with the transverse Kronecker symbol $\delta_{ij}^\perp = \delta_{ij} - n_i n_j$. We note that the positivity of ε in Eq. (3) guarantees linear stability around the ground state assumed here. However, Eq. (3) is not suitable to describe the transition to another ground state. In this case, another method such as, for example, a Ginzburg–Landau approach is called for.²⁴

The rotational elastic material tensors read in uniaxial approximation ($* \in \{n, m\}$),

$$K_{ijkl}^* = K_1^* \delta_{ij}^\perp \delta_{kl}^\perp + K_2^* n_p n_q \varepsilon_{ijp} \varepsilon_{klq} + K_3^* n_j n_l \delta_{ik}^\perp, \quad (4)$$

$$\tilde{K}_{ijkl} = \tilde{K} \delta_{ik}^\perp (n_j m_l + n_l m_j). \quad (5)$$

Generally, all material coefficients are functions of the state variables, such as temperature, pressure, and concentration, but also of M_0 (or the external field strength). Only for a linearized description, valid for small deviations from equilibrium, are the coefficients constant.

For the kinetic energies, one finds $(1/2\rho_n)(\mathbf{g}^n)^2 + (1/2\rho_m)(\mathbf{g}^m)^2 = (1/2\rho) \mathbf{g}^2 + (\alpha/2) \mathbf{w}^2$, revealing the stiffness coefficient for relative velocities to be

$$\alpha = \phi(1 - \phi)\rho. \quad (6)$$

Since $0 < \phi < 1$, there is $0 < \alpha < \rho/2$.

From Eq. (3), the conjugate fields follow by taking variational derivatives according to the Gibbs relation, Eq. (1),

$$\delta T = \frac{T}{C_V} \delta\sigma + \frac{1}{\alpha_\phi} \delta\phi, \quad (7)$$

$$\Pi = \frac{1}{\kappa_\phi} \delta\phi + \frac{1}{\alpha_\phi} \delta\sigma + \mathbf{w} \cdot \mathbf{g} + \rho \mathbf{w}^2 (1 - 2\phi), \quad (8)$$

$$\Phi_{ij}^n = K_{ijkl}^n \nabla_l m_k + \tilde{K}_{ijkl} \nabla_l m_k, \quad (9)$$

$$\Phi_{ij}^m = K_{ijkl}^m \nabla_l m_k + \tilde{K}_{ijkl} \nabla_l m_k, \quad (10)$$

$$h_i^{n'} = \chi_a H^2 \delta_{ij}^\perp \delta n_j + A_1 \delta_{ij}^\perp (\delta n_j - \delta m_j), \quad (11)$$

$$h_i^{m'} = M_0 H \delta_{ij}^\perp \delta m_j + A_1 \delta_{ij}^\perp (\delta m_j - \delta n_j), \quad (12)$$

$$h_i^w = \alpha w_i, \quad (13)$$

where the w_i -dependence of the osmotic pressure is due to the ϕ -dependence of α , Eq. (6).

IV. DYNAMICS

A. The dynamic balance equations

The dynamical equations for the relevant variables have the form

$$\dot{\varepsilon} + \nabla_i(\varepsilon + p)v_i + \nabla_i(j_i^{\varepsilon R} + j_i^{\varepsilon D}) = 0, \quad (14)$$

$$\dot{\sigma} + \nabla_i(\sigma v_i + j_i^{\sigma R} + j_i^{\sigma D}) = 2R/T, \quad (15)$$

$$\dot{\rho} + \nabla_i(\rho v_i) = 0, \quad (16)$$

$$\dot{\phi} + v_j \nabla_j \phi + \nabla_i(\phi(1 - \phi)w_i) + \nabla_i(j_i^{\phi R} + j_i^{\phi D}) = 0, \quad (17)$$

$$\dot{\mathbf{g}} + \nabla_j(\mathbf{g}_i v_j) + \nabla_j(\sigma_{ij}^{th} + \sigma_{ij}^R + \sigma_{ij}^D) = 0, \quad (18)$$

$$\dot{w}_i + v_j \nabla_j w_i + \nabla_i(\rho^{-1}\Pi) + Z_i^R + Z_i^D = 0, \quad (19)$$

$$\dot{m}_i + v_j \nabla_j m_i + (\mathbf{m} \times \boldsymbol{\omega})_i + X_i^R + X_i^D = 0, \quad (20)$$

$$\dot{n}_i + v_j \nabla_j n_i + (\mathbf{n} \times \boldsymbol{\omega})_i + Y_i^R + Y_i^D = 0, \quad (21)$$

with the vorticity $2\boldsymbol{\omega}_i = \varepsilon_{ijk} \nabla_j v_k$ and R being the energy dissipation function.

The set of dynamic equations contains two distinct parts. First, there are the phenomenological currents with superscripts R and D denoting reversible and dissipative contributions, respectively. They will be given in Secs. IV B and IV C using general symmetry arguments. The second part consists of the transport and convective derivatives including the isotropic pressure, p , as well as σ_{ij}^{th} .

Zero entropy production requires σ_{ij}^{th} in Eq. (18) to take the form

$$\begin{aligned} \sigma_{ij}^{th} = & p \delta_{ij} - B_j H_i + \Phi_{ij}^n \nabla_l n_l + \Phi_{ij}^m \nabla_l m_l - \frac{1}{2} (n_j h_i^{n'} - n_i h_j^{n'}) \\ & - \frac{1}{2} (m_j h_i^{m'} - m_i h_j^{m'}), \end{aligned} \quad (22)$$

which can be rewritten as^{5,37}

$$2\sigma_{ij}^{th} = 2p\delta_{ij} - (B_j H_i + B_i H_j) + \Phi_{ki}^n \nabla_j n_k + \Phi_{kj}^n \nabla_i n_k + \Phi_{ki}^m \nabla_j m_k + \Phi_{kj}^m \nabla_i m_k + \nabla_k (n_j \Phi_{ik}^n - n_i \Phi_{jk}^n + m_j \Phi_{ik}^m - m_i \Phi_{jk}^m), \quad (23)$$

having used the requirement of rotational invariance of the Gibbs relation,¹ Eq. (1),

$$0 = \epsilon_{ijk} (h_j^{m'} dm_k + \Phi_{ji}^m \nabla_l m_k + \Phi_{ij}^m \nabla_k m_l + h_j^{n'} dn_k + \Phi_{ji}^n \nabla_l n_k + \Phi_{ij}^n \nabla_k n_l). \quad (24)$$

In Eq. (23), σ_{ij}^{th} is either symmetric or a divergence of an antisymmetric part. The latter can be brought into a manifestly symmetric form by some redefinitions,¹ thus demonstrating local angular momentum conservation.

We note that the reversible and irreversible phenomenological parts of the stress tensor, σ_{ij}^R and σ_{ij}^D , turn out to be manifestly symmetric [see Eqs. (42) and (35)].

In the Maxwell stress, the field $B_i = H_i + M_i$ by definition.

Putting all dynamic equations into the Gibbs relation, one finds that the non-phenomenological contributions add up to zero entropy production, as is required for reversible contributions. For the phenomenological parts of the currents, one gets (neglecting surface effects)

$$R = -j_i^{\sigma*} \nabla_i T - j_i^{\phi*} \nabla_i \Pi - \sigma_{ij}^* \nabla_j v_i + h_i^w Z_i^* + h_i^m \delta_{ik}^\perp X_k^* + h_i^n \delta_{ik}^\perp Y_k^*, \quad (25)$$

and the second law of thermodynamics requires $R \geq 0$ with the equal sign ($>$ sign) for $*$ = R ($*$ = D).

The energy conservation law Eq. (14) is redundant due to the Gibbs relation, and the continuity equation, Eq. (16), reduces to $\nabla_i v_i$ in the constant density approximation. The incompressibility condition (constant density approximation) implies that we are restricting ourselves throughout the rest of the paper to velocities that are small compared to the velocity of sound. In addition,¹¹ it also means, in the two-fluid context, that the concentration ϕ is a conserved variable, while this is, in general, no longer the case if the incompressibility condition is not implemented.

In the whole set of dynamic equations, the mean velocity v_i has been chosen as the transport/convective velocity for all variables. This ensures zero entropy production of those contributions. However, in a simplified model, one would expect that variables connected with the magnetization (with the nematic degree of freedom) are convected with the velocity v_i^m (v_i^n). This can be achieved by a special choice for the values of some reversible transport parameters, thereby preserving zero entropy production. This will be done in Sec. IV C.

B. Dissipative dynamics

The dissipative parts of the currents introduced in Eqs. (15)–(21) can be deduced from a potential, the dissipation function R . Within linear irreversible thermodynamics, R is a bilinear form of the generalized forces, which are either gradients of those conjugate quantities that are constant in equilibrium, or otherwise the conjugate quantities themselves,

$$2R = \kappa_{ij} (\nabla_i T) (\nabla_j T) + D_{ij} (\nabla_i \Pi) (\nabla_j \Pi) + 2D_{ij}^T (\nabla_i T) (\nabla_j \Pi) + v_{ijkl} A_{ij} A_{kl} + 2v_{ijkl}^c A_{ij} \nabla_l h_k^w + v_{ijkl}^w (\nabla_j h_i^w) (\nabla_l h_k^w) + \xi_{ij} h_i^w h_j^w + \frac{1}{\gamma_1} h_i^n h_j^n \delta_{ij}^\perp + b^D h_i^m h_j^m \delta_{ij}^\perp + \lambda_{ijk}^D h_i^n h_j^n A_{jk} + c_{ijk}^D h_i^m h_j^m A_{jk} + \chi_{ij}^D h_j^m h_i^n + \xi_{ij}^T h_i^w \nabla_j T + \xi_{ij}^\Pi h_i^w \nabla_j \Pi + \xi_{ijk}^{wm} (\nabla_k h_j^w) h_i^m + \xi_{ijk}^{wn} (\nabla_k h_j^w) h_i^n \quad (26)$$

with $2A_{ij} = \nabla_j v_i + \nabla_i v_j$.

In uniaxial approximation, the symmetric second rank tensors κ_{ij} , D_{ij} , D_{ij}^T , ξ_{ij} are of the standard form

$$\kappa_{ij} = \kappa_{\parallel} n_i n_j + \kappa_{\perp} \delta_{ij}^\perp, \quad (27)$$

with two coefficients each. The rank-4 tensors, viscosity v_{ijkl} and relative velocity diffusion v_{ijkl}^w , contain five parameters,^{5,38} while the mixed one v_{ijkl}^c contains six parameter due to the lack of the $v_{ijkl}^c = v_{klij}^c$ symmetry.^{11,39}

The couplings described by χ_{ij}^D , c_{ijk}^D , and λ_{ijk}^D have already been given before for ferronematics.²⁴ In the present case for which we use m_i instead of M_i with constant magnitude M_0 , the number of independent coefficients in the property tensors is overall greatly reduced, and we obtain

$$\chi_{ij}^D = \chi^D (n_p m_p \delta_{ij} - n_i m_j), \quad (28)$$

$$c_{ijk}^D = c^D (\epsilon_{ipk} n_j + \epsilon_{ipj} n_k) n_p, \quad (29)$$

$$\lambda_{ijk}^D = \lambda^D (\epsilon_{ipk} n_j + \epsilon_{ipj} n_k) m_p. \quad (30)$$

The novel tensors associated with crosscouplings of the velocity difference with temperature and concentration gradients, ξ_{ij}^T and ξ_{ij}^Π , both take the form ($*$ \in $\{T, \Pi\}$)

$$\xi_{ij}^* = \xi^* \epsilon_{ijk} m_k. \quad (31)$$

The rank-3 tensors ξ_{ijk}^{wm} and ξ_{ijk}^{wn} describe couplings of gradients of the relative velocity with h_i^m and h_i^n and read

$$\xi_{ijk}^{wm} = (\xi_1^{wm} \epsilon_{ipk} n_j + \xi_2^{wm} \epsilon_{ipj} n_k) n_p, \quad (32)$$

$$\xi_{ijk}^{wn} = (\xi_1^{wn} \epsilon_{ipk} n_j + \xi_2^{wn} \epsilon_{ipj} n_k) m_p. \quad (33)$$

We note that ξ_{ijk}^{wn} and λ_{ijk}^D are structurally quite similar, as are ξ_{ijk}^{wm} and c_{ijk}^D . However, in contrast to λ_{ijk}^D and c_{ijk}^D , ξ_{ijk}^{wn} and ξ_{ijk}^{wm} are not symmetric in j and k .

The positivity of R requires appropriate positivity conditions on the dissipative transport parameters, e.g., $\kappa_{\perp} > 0$, $\xi_{\perp} > 0$, and $\kappa_{\perp} \xi_{\perp} - (\xi_{\perp}^T)^2 > 0$.

From Eq. (26), the following dissipative currents are obtained:

$$j_i^{\sigma D} = -(\partial R) / (\partial \nabla_i T) = -\kappa_{ij} \nabla_j T - \phi(1 - \phi) d_{ij}^T \nabla_j \Pi - \xi_{ij}^T h_j^w, \quad (34)$$

$$j_i^D = -(\partial R) / (\partial A_{ij}) = -v_{ijkl} A_{kl} - c_{kij}^D h_k^m - \lambda_{kij}^D h_k^n - v_{ijkl}^c \nabla_l h_k^w, \quad (35)$$

$$\begin{aligned} Z_i^D &= (\partial R)/(\partial h_i^w) \\ &= \xi_{ij} h_j^w + \xi_{ij}^T \nabla_j T + \xi_{ij}^{\Pi} \nabla_j \Pi - \nabla_j (v_{ijkl}^w \nabla_l h_k^w + v_{klij}^c A_{lk}) \\ &\quad - \nabla_k (\xi_{jik}^w h_j^m) - \nabla_k (\xi_{jik}^w h_j^n), \end{aligned} \quad (36)$$

$$\begin{aligned} Y_i^D &= (\partial R)/(\partial h_i^n) \\ &= \frac{1}{\gamma_1} \delta_{ij}^{\perp} h_j^n + \lambda_{ijk}^D A_{jk} + \chi_{ij}^D h_j^m + \xi_{ijk}^w (\nabla_k h_j^w), \end{aligned} \quad (37)$$

$$\begin{aligned} f_i^{\phi D} &= -(\partial R)/(\partial \nabla_i \Pi) \\ &= -D_{ij} \nabla_j \Pi - \phi(1-\phi) d_{ij}^T \nabla_j T - \xi_{ji}^{\Pi} h_j^w, \end{aligned} \quad (38)$$

$$\begin{aligned} X_i^D &= (\partial R)/(\partial h_i^m) \\ &= b^D \delta_{ij}^{\perp} h_j^m + \chi_{ji}^D h_j^n + c_{ijk}^D A_{jk} + \xi_{ijk}^w (\nabla_k h_j^w), \end{aligned} \quad (39)$$

where thermodiffusion is written in the usual way with $D_{ij}^T = \phi(1-\phi)d_{ij}^T$. Since the relative velocity is not related to any broken symmetry nor to a conservation law, it always relaxes with $\alpha\xi_{\perp}$ and $\alpha\xi_{\parallel}$, the inverse transverse and longitudinal relaxation times, respectively.

C. Reversible dynamics

For the reversible dynamics, there is no potential to derive the reversible currents. Instead, one expands the currents into the forces taking into account time reversal symmetry, spatial inversion symmetry, and the $n_i \rightarrow -n_i$ invariance. We get

$$j_i^{\sigma R} = \beta_{ij} h_j^w - \kappa_{ij}^R \nabla_j T - D_{ij}^{TR} \nabla_j \Pi, \quad (40)$$

$$f_i^{\phi R} = \gamma_{ij} h_j^w - D_{ij}^R \nabla_j \Pi - D_{ij}^{TR} \nabla_j T, \quad (41)$$

$$\sigma_{ij}^R = 2\beta_2 h_i^w w_j - v_{ijkl}^R A_{kl} - c_{kij}^R h_k^m - \lambda_{kji} h_k^n, \quad (42)$$

$$\begin{aligned} Z_i^R &= -\beta_1 h_j^n \nabla_i n_j + \beta_2 w_j (\nabla_i v_j + \nabla_j v_i) + \beta_3 h_j^w (\nabla_j w_i - \nabla_i w_j) \\ &\quad + \beta_4 w_j (\nabla_j v_i - \nabla_i v_j) - \beta_5 h_j^m \nabla_i m_j - \nabla_j (\lambda_{kji}^w h_k^n) - \nabla_j (\tau_{kji}^R h_k^m) \\ &\quad + \kappa_{ij}^{wR} h_j^w + \beta_{ij} \nabla_j T + \gamma_{ij} \nabla_j \Pi, \end{aligned} \quad (43)$$

$$Y_i^R = \beta_1 h_j^w \nabla_j n_i - \lambda_{ijk} A_{jk} - \lambda_{ijk}^w \nabla_j h_k^w + b_{ij}^{nR} h_j^n + \chi^R (\mathbf{n} \times \mathbf{h}^m)_i, \quad (44)$$

$$X_i^R = \beta_5 h_j^w \nabla_j m_i - c_{ijk}^R A_{jk} - \tau_{ijk}^R \nabla_j h_k^w + b_{ij}^{mR} h_j^m + \chi^R (\mathbf{n} \times \mathbf{h}^n)_i, \quad (45)$$

containing all possible contributions linear in the forces but also some nonlinear ones ($\sim\beta_{1,2,3,4,5}$) as well as two couplings involving gradients of h_i^w . Such terms are used below to tackle the question of transport and convection velocities.

The second rank tensors β_{ij}, γ_{ij} are of the standard uniaxial form, Eq. (27), while the reversible analogs of (the dissipative) heat conduction κ_{ij}^R , diffusion D_{ij}^R , thermodiffusion D_{ij}^{TR} , relative velocity

relaxation κ_{ij}^{wR} , director b_{ij}^{nR} , and magnetization relaxation b_{ij}^{mR} are all of the form²⁴

$$\kappa_{ij}^R = \kappa^R \epsilon_{ijk} m_k. \quad (46)$$

In contrast to the analogous dissipative tensors, they are antisymmetric, e.g., $\kappa_{ji}^R = -\kappa_{ij}^R$, leading to $R = 0$, and they contain a factor $\sim m_i$ to account for the correct time reversal behavior.²³

The third rank tensors λ_{ijk} and c_{ijk}^R describe the rotation of n_i and m_i , respectively, due to the flow of the mean velocity,

$$2\lambda_{ijk} = \lambda (\delta_{ij}^{\perp} n_k + \delta_{ik}^{\perp} n_j), \quad (47)$$

$$2c_{ijk}^R = c^R (\delta_{ij}^{\perp} m_k + \delta_{ik}^{\perp} m_j), \quad (48)$$

with λ being the well-known flow alignment parameter of nematics. Analogous effects due to the flow of the relative velocity are described by the material tensors λ_{ijk}^w and τ_{ijk}^R ,

$$\lambda_{ijk}^w = \lambda_1^w \delta_{ij}^{\perp} n_k + \lambda_2^w \delta_{ik}^{\perp} n_j, \quad (49)$$

$$\tau_{ijk}^R = \tau_1^R \delta_{ij}^{\perp} m_k + \tau_2^R \delta_{ik}^{\perp} m_j, \quad (50)$$

which are not necessarily symmetric in the last two indices. We note that the sign of the coefficients in Eqs. (49) and (50) can change depending on the shape of the molecules. This is already known to be possible^{40,41} for rod-like vs disk-like molecules for the flow alignment parameter λ in usual low-molecular weight nematics. When applying Eqs. (49) and (50), a more complex behavior for flow alignment will result, in general.

The reversible analog of the (dissipative) viscosity tensor, v_{ijkl}^R , has eight components and is given in Ref. 23. It is antisymmetric in the exchange of the first pair of indices with the second one, thus guaranteeing zero entropy production, and every component of it contains a factor $\sim m_i$ to account for the correct time reversal behavior.

Finally, we discuss how we get the model, where all magnetic (nematic) degrees of freedom are transported/convected with the appropriate velocities v_i^m (v_i^n). For the conserved densities, we already have taken the individual mass currents as $\rho_m v_i^m$ and $\rho_n v_i^n$, leading to Eqs. (16) and (17). Therefore, $f_i^{\phi R}$, Eq. (41), must not contain any term that would change this, and we have to put $\gamma_{\perp} = 0 = \gamma_{\parallel}$. Similarly, we assume the entropy to be transported by the mean velocity, and therefore, we put $\beta_{ij} \equiv 0$.

In order to get n_i transported by v_i^n and m_i by v_i^m , we have to choose $\beta_1 = 1/\rho_n$ and $\beta_5 = -1/\rho_m$. In Eqs. (21) and (20), the convection terms for n_i and m_i involve the vorticity of the mean velocity. In order to use the appropriate velocities instead, one has to choose $\lambda_c^w = 1/\rho_n$ and $\tau_c^R = -1/\rho_m$. For the partial momenta, the appropriate transport velocities require $\beta_2 = 1/2 = \beta_4$ and $\beta_3 = 1/\rho_m - 1/\rho_n$.^{11,16}

V. SELECTED EFFECTS OF THE RELATIVE VELOCITY

A. Heat and concentration currents

Here, we discuss the implication of the dissipative coupling between the relative velocity and the thermal degree of freedom described by the material tensor ξ_{ij}^T , Eq. (31). It should be noted that this type of crosscoupling is possible only if at least one of the

subsystems is the magnetization or any other system described by a vorticity-like quantity. A similar analysis applies to the solutal degree of freedom involving ξ_{ij}^{Π} .

Taking the preferred direction to be the z -axis and neglecting other crosscouplings, a transverse temperature gradient triggers a dynamical change of the transverse relative velocity, Eq. (19), given by

$$\dot{w}_x = -\xi^T \nabla_y T - \xi_{\perp} \alpha w_x + \kappa^{wR} \alpha w_y, \quad (51)$$

$$\dot{w}_y = \xi^T \nabla_x T - \xi_{\perp} \alpha w_y - \kappa^{wR} \alpha w_x, \quad (52)$$

where we have also taken into account the transverse relaxation ξ_{\perp} , Eq. (36), and its reversible counterpart κ^{wR} , Eqs. (43) and (46). In the stationary limit, a finite relative velocity is generated,

$$w_x^0 = \frac{\xi^T \alpha^{-1}}{\xi_{\perp}^2 + (\kappa^{wR})^2} (-\xi_{\perp} \nabla_y T + \kappa^{wR} \nabla_x T), \quad (53)$$

$$w_y^0 = \frac{\xi^T \alpha^{-1}}{\xi_{\perp}^2 + (\kappa^{wR})^2} (\xi_{\perp} \nabla_x T + \kappa^{wR} \nabla_y T), \quad (54)$$

which is perpendicular to the preferred direction and, in the limit $\xi_{\perp} \gg |\kappa^{wR}|$, also to the temperature gradient, while the reversible κ^{wR} adds a component along the temperature gradient.

The inverse effect of the crosscoupling ξ_{ij}^T , the generation of a temperature gradient in the presence of a relative velocity (all in the plane perpendicular to the preferred direction), is found along the same lines as above. Using the entropy currents, Eqs. (34) and (40), and assuming that vertical boundaries block a constant entropy current, the temperature gradients read

$$\nabla_x T = \frac{\xi^T \alpha}{\kappa_{\perp}^2 + (\kappa^R)^2} (\kappa_{\perp} w_y + \kappa^R w_x), \quad (55)$$

$$\nabla_y T = \frac{\xi^T \alpha}{\kappa_{\perp}^2 + (\kappa^R)^2} (-\kappa_{\perp} w_x + \kappa^R w_y), \quad (56)$$

with κ_{\perp} being the transverse heat diffusion and κ^R being its reversible counterpart. Thus, in the limit $\kappa_{\perp} \gg |\kappa^R|$, the temperature gradient is perpendicular to the relative velocity, with the reversible κ^R adding a component along the relative velocity.

B. Director and magnetization rotations

We discuss the implication of the coupling between the relative velocity and rotations of the magnetization described by the dissipative material tensor ξ_{ij}^{wm} , Eq. (39), and the reversible one τ_{ijk}^R , Eq. (45). In contrast to Sec. V A, here gradients of the relative velocity are involved. A similar analysis applies to director rotations involving ξ_{ij}^{wn} and λ_{ijk}^w .

Assuming a simple shear flow in the relative velocity with the preferred direction in the shear plane,

$$\nabla_y w_z = K, \quad (57)$$

the force $\nabla_y h_z^w$ enters the (dissipative and reversible) dynamics of the magnetization. Neglecting couplings to other variables, the magnetization currents are given by

$$X_x^0 = b^D h_x^m + b^{mR} h_y^m - \xi_1^{wm} \alpha K, \quad (58)$$

$$X_y^0 = b^D h_y^m - b^{mR} h_x^m - \tau_1^R \alpha K, \quad (59)$$

where b^D is the magnetization relaxation, b^{mR} is its reversible analog, and ξ_1^{wm} and τ_1^R are the dissipative and reversible coefficients appropriate to the external force.

In the stationary limit, $X_i^0 = 0$ leads to the constant magnetic forces,

$$h_x^{m0} = \frac{\alpha K}{N} (b^D \xi_1^{wm} - b^{mR} \tau_1^R), \quad (60)$$

$$h_y^{m0} = \frac{\alpha K}{N} (b^D \tau_1^R + b^{mR} \xi_1^{wm}) \quad (61)$$

with $N \equiv (b^D)^2 + (b^{mR})^2$. These forces have to be compensated by the external magnetic field, Eq. (12), leading to a finite tilt of the magnetization,

$$m_x^0 = \frac{h_x^{m0}}{M_0 H} \quad \text{and} \quad m_y^0 = \frac{h_y^{m0}}{M_0 H}. \quad (62)$$

There is a similar tilt effect on the nematic director due to K , which can be obtained from Eqs. (60)–(62) by the replacements $\xi_1^{wm} \rightarrow \xi_1^{wn}$, $b^D \rightarrow 1/\gamma_1$, $b^{mR} \rightarrow b^{nR}$, $\tau_1^R \rightarrow \lambda_1^w$, and $M_0 H \rightarrow \chi_a H^2$. However, the director and magnetization tilt are coupled statically [via A_1 in Eqs. (11) and (12)] and dynamically [via χ_{ij}^D in Eqs. (37) and (39)], rendering the final expressions to be rather complicated.

VI. CONCLUSIONS AND PERSPECTIVE

In this paper, we have studied two-fluid effects on the macroscopic dynamic behavior of a rather recent soft matter system, ferromagnetic nematics. It turns out that the relative velocity between the two subsystems, namely, the nematic solvent and the magnetic platelets, gives rise to a number of reversible and dissipative dynamic crosscoupling terms. Several of these should be detectable experimentally. Among these effects, we have studied in some detail two groups of effects. One is the reversible and dissipative coupling between temperature and concentration gradients to the relative velocity. The other type of effect is related to the vorticity in the magnetization and its coupling to gradients in the relative velocity.

Throughout this paper, we have assumed that the nematic director, \hat{n} , and the direction of the ferromagnetic magnetizations, \hat{m} , are parallel in the ground state. We note, however, that this degeneracy can be lifted by applying electric and magnetic fields. We mention as an example the case of an electric field, which has already been studied experimentally in the static limit.²⁸ For externally applied electric or magnetic fields, the ferromagnetic nematic will become biaxial. As a consequence, all property tensors presented in the present paper will also assume a biaxial form. This can be implemented practically very easily by drawing on the experience with the macroscopic behavior of other biaxial fluid systems such as for biaxial nematics^{42,43} and biaxial ferronematics.⁴⁴

As a perspective, we would like to mention briefly that the two-fluid approach presented here can be generalized in several directions. One is ferromagnetic liquid crystalline gels and elastomers. While uniaxial ferromagnetic gels and elastomers have been reported a number of years ago already,^{45,46} ferromagnetic liquid crystalline elastomers have been synthesized only quite recently for a mainchain polymer system.⁴⁷

Another one is the incorporation of chirality concerning two-fluid aspects. Ferromagnetic cholesterics have been synthesized and characterized quite recently,⁴⁸ and the macroscopic dynamics of ferrocholesterics has been given.⁴⁹ For the approach of macroscopic dynamics, macroscopic chirality leads to the existence of a pseudoscalar quantity, the pitch q_0 , which changes sign under spatial inversion and thus makes possible in many instances a lower order in spatial gradients for crosscoupling terms.

We close by pointing out that another direction to go into for the macroscopic consequences of two-fluid effects in the domain of liquid crystals will be ferroelectric nematics, which are the analog of ferromagnetism in the electric domain.

AUTHOR DECLARATIONS

Conflict of Interest

There are no conflicts of interest to declare.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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