MCKAY MATRICES FOR FINITE-DIMENSIONAL HOPF ALGEBRAS

GEORGIA BENKART, REKHA BISWAL, ELLEN KIRKMAN, VAN C. NGUYEN, AND JIERU ZHU

Abstract. For a finite-dimensional Hopf algebra A, the McKay matrix M_V of an A-module V encodes the relations for tensoring the simple A-modules with V. We prove results about the eigenvalues and the right and left (generalized) eigenvectors of M_V by relating them to characters. We show how the projective McKay matrix Q_V obtained by tensoring the projective indecomposable modules of A with V is related to the McKay matrix of the dual module of V. We illustrate these results for the Drinfeld double D_n of the Taft algebra by deriving expressions for the eigenvalues and eigenvectors of M_V and Q_V in terms of several kinds of Chebyshev polynomials. For the matrix N_V that encodes the fusion rules for tensoring V with a basis of projective indecomposable D_n-modules for the image of the Cartan map, we show that the eigenvalues and eigenvectors also have such Chebyshev expressions.

1. Introduction

Assume A is a finite-dimensional associative algebra over an algebraically closed field $\Bbbk$. Let $\mathsf{S}_1, \mathsf{S}_2, \ldots, \mathsf{S}_m$ be the nonisomorphic simple (irreducible) A-modules and $\mathsf{P}_1, \mathsf{P}_2, \ldots, \mathsf{P}_m$ be their projective covers, that is, the nonisomorphic indecomposable projective modules such that for each $j = 1, \ldots, m$, the module P_j modulo its radical is isomorphic to S_j. The dimensions of these A-modules determine two column vectors in $\mathbb{Z}^m$,

$$\text{(1.0.1)} \qquad \mathbf{s} = [\mathsf{dim}(\mathsf{S}_1)\ \mathsf{dim}(\mathsf{S}_2)\ \ldots\ \mathsf{dim}(\mathsf{S}_m)]^{\mathtt{T}} \quad \text{and} \quad \mathbf{p} = [\mathsf{dim}(\mathsf{P}_1)\ \mathsf{dim}(\mathsf{P}_2)\ \ldots\ \mathsf{dim}(\mathsf{P}_m)]^{\mathtt{T}},$$

where $\mathtt{T}$ denotes "transpose." If A is viewed as a left A-module under left multiplication, then

$$\text{(1.0.2)} \qquad \mathsf{A} = \bigoplus_{j=1}^{m} \mathsf{P}_j^{\oplus \mathsf{dim}(\mathsf{S}_j)} \quad \text{and} \quad \mathbf{p}^{\mathtt{T}}\mathbf{s} = \sum_{j=1}^{m} \mathsf{dim}(\mathsf{P}_j)\ \mathsf{dim}(\mathsf{S}_j) = \mathsf{dim}(\mathsf{A}).$$

When A is semisimple, then $\mathsf{P}_j = \mathsf{S}_j$ for all j, and the second part of (1.0.2) is the familiar result $\mathsf{dim}(\mathsf{A}) = \sum_{j=1}^{m} (\mathsf{dim}(\mathsf{S}_j))^2$.

There are two Grothendieck groups, $\mathsf{G}_0(\mathsf{A})$ and $\mathsf{K}_0(\mathsf{A})$ associated to A:

- $\mathsf{G}_0(\mathsf{A})$ is the quotient of the free abelian group on the set of all isomorphism classes $[\mathsf{V}]$ of finite-dimensional A-modules V subject to the relations $[\mathsf{U}] - [\mathsf{V}] + [\mathsf{W}] = 0$ for each short exact sequence $0 \to \mathsf{U} \to \mathsf{V} \to \mathsf{W} \to 0$ of A-modules. By the Jordan-Hölder theorem, this group has a $\mathbb{Z}$-module basis consisting of the classes $[\mathsf{S}_1], [\mathsf{S}_2], ..., [\mathsf{S}_m]$, and
$$[\mathsf{A}] = \sum_{j=1}^{m} \mathsf{dim}(\mathsf{P}_j)[\mathsf{S}_j] \quad \text{in} \ \ \mathsf{G}_0(\mathsf{A}).$$
- $\mathsf{K}_0(\mathsf{A})$ is the quotient of the free abelian group on the set of all isomorphism classes $[\mathsf{V}]$ of finite-dimensional projective A-modules V, subject to the relations $[\mathsf{U}] - [\mathsf{V}] + [\mathsf{W}] = 0$ for each direct sum decomposition $\mathsf{V} = \mathsf{U} \oplus \mathsf{W}$ of A-modules. This group has a $\mathbb{Z}$-module basis

2020 *Mathematics Subject Classification.* Primary 16T05. Secondary 19A49, 33C45.

Key words and phrases. McKay matrix, Drinfeld double, character, Chebyshev polynomial.

consisting of the classes $[\mathsf{P}_1], [\mathsf{P}_2], \ldots, [\mathsf{P}_m]$ due to the Krull-Remak-Schmidt theorem, and (1.0.2) says

$$[\mathsf{A}] = \sum_{j=1}^{m} \mathsf{dim}(\mathsf{S}_j)[\mathsf{P}_j] \quad \text{in } \mathsf{K}_0(\mathsf{A}).$$

When A is a Hopf algebra, the tensor product of two A-modules is an A-module with the A-action given by the coproduct, and the dual vector space of an A-module is an A-module via the antipode. Both $\mathsf{G}_0(\mathsf{A})$ and $\mathsf{K}_0(\mathsf{A})$ have products using $\otimes$, so that $[\mathsf{U}][\mathsf{W}] = [\mathsf{U} \otimes \mathsf{W}]$, where $\mathsf{G}_0(\mathsf{A})$ is a ring with a unit element, which is the one-dimensional A-module $\Bbbk$ with action given by the counit, and $\mathsf{K}_0(\mathsf{A})$ is a ring without a unit element.

Let V be an A-module, and set $d = \mathsf{dim}(\mathsf{V})$. The *McKay matrix* M_V for tensoring with V has as its (i, j) entry $\mathsf{M}_{ij} = [\mathsf{S}_i \otimes \mathsf{V} : \mathsf{S}_j]$, the multiplicity of S_j as a composition factor of the A-module $\mathsf{S}_i \otimes \mathsf{V}$, or equivalently, the coefficient of $[\mathsf{S}_j]$ when $[\mathsf{S}_i \otimes \mathsf{V}]$ is expressed as a $\mathbb{Z}$-linear combination of the basis elements in $\mathsf{G}_0(\mathsf{A})$. Then

$$d\,\mathbf{s}_i = \mathsf{dim}(\mathsf{V})\mathsf{dim}(\mathsf{S}_i) = \mathsf{dim}(\mathsf{S}_i \otimes \mathsf{V}) = \sum_{j=1}^{m} [\mathsf{S}_i \otimes \mathsf{V} : \mathsf{S}_j]\mathsf{dim}(\mathsf{S}_j) = \sum_{j=1}^{m} \mathsf{M}_{ij}\mathsf{dim}(\mathsf{S}_j) = (\mathsf{M}_\mathsf{V}\,\mathbf{s})_i$$

shows that the ith entry of $\mathsf{M}_\mathsf{V}\,\mathbf{s}$ is d times the ith entry of $\mathbf{s}$ for $1 \leq i \leq m$, which implies that $\mathbf{s}$ is a right eigenvector of M_V for the eigenvalue $d = \mathsf{dim}(\mathsf{V})$.

One can argue as in the paper [20] by Grinberg, Huang, and Reiner that the dimension vector $\mathbf{p}^\mathsf{T} = [\mathsf{dim}(\mathsf{P}_1)\ \ \mathsf{dim}(\mathsf{P}_2)\ \ \ldots\ \ \mathsf{dim}(\mathsf{P}_m)]$ is a left eigenvector for M_V with eigenvalue d. (But note that their McKay matrix and ours are transposes of one another, so for them $\mathbf{p}^\mathsf{T}$ is a right eigenvector, and $\mathbf{s}$ is a left one.)

The matrix M_V can be viewed as the adjacency matrix of a quiver (the so-called *McKay quiver determined by* V) having nodes labeled by $1 \leq i \leq m$ that correspond to the simple modules S_i. There are $[\mathsf{S}_i \otimes \mathsf{V} : \mathsf{S}_j]$ arrows from i to j. If there is an arrow from i to j and one from j to i, they are replaced by a single undirected edge. The idea to consider such a quiver and matrix goes back to McKay's insight [30] that the quivers determined by tensoring with the G-module $\mathsf{V} = \mathbb{C}^2$ for a finite subgroup G of SU_2 exactly correspond to the affine Dynkin diagrams of types $\mathsf{A}, \mathsf{D}, \mathsf{E}$. This result, subsequently referred to as the *McKay correspondence*, has been the inspiration for much work on a host of topics in singularity theory, group theory, orbifolds, and many other subjects.

In expanding on McKay's result, Steinberg [35] showed that for the group algebra $\mathbb{C}\mathsf{G}$ of any finite group G, the columns of the character table of G give a complete set of right eigenvectors for the McKay matrix determined by tensoring with *any* finite-dimensional G-module V, and the associated eigenvalues are given by the character values $\chi_\mathsf{V}(g_j)$, as g_j ranges over a set of conjugacy class representatives of G. Group algebras over algebraically closed fields of characteristic $p > 0$ were considered in [20], where it was shown that the columns of the Brauer character table of G are right eigenvectors of M_V with corresponding eigenvalues $\chi_\mathsf{V}(g_j)$, as g_j ranges over a set of representatives for the p' conjugacy classes of G (that is, elements whose order is relatively prime to p). Each such g_j determines a left eigenvector whose coordinates are the character values of the projective indecomposable modules P_j evaluated at g_j, and whose eigenvalue is $\chi_\mathsf{V}(g_j)$.

Aside from the result of Grinberg, Huang, and Reiner in [20] mentioned earlier, very little is known about the eigenvalues and eigenvectors of the McKay matrices for *arbitrary* finite-dimensional Hopf algebras, and the purpose of this paper is to remedy that situation. The specific case of the quantum group $\mathfrak{u}_q(\mathfrak{sl}_2)$ at q an odd root of unity was examined in depth in [2]. In that work, it was shown that the McKay matrix for tensoring with $\mathsf{V} = \mathbb{C}^2$ has two-dimensional generalized eigenspaces for every eigenvalue different from $2 = \mathsf{dim}(\mathsf{V})$, and it is necessary to work with Jordan blocks of size 2×2 in that example. The main point of [2] is that McKay matrices and quivers determine interesting Markov chains, and in the particular case of $\mathfrak{u}_q(\mathfrak{sl}_2)$, the chain exhibits new phenomena due to the existence of such Jordan blocks. The examples in [2] show that the analysis of the rates of convergence of the Markov chains determined by McKay matrices is quite delicate.

Most of the work on characters of Hopf algebras has been in the case that the algebra is semisimple. For example, Witherspoon defined a notion of a character table for finite-dimensional semisimple, almost cocommutative Hopf algebras and showed that the characters provide eigenvectors for the McKay matrix for tensoring with the simple modules (see [39, proof of Thm. 3.2]). In [11], Cohen and Westreich determined Verlinde formulas for semisimple, almost cocommutative Hopf algebras, and in [9, 10], for (nonsemisimple) factorizable ribbon Hopf algebras such as the Drinfeld double of the Taft algebra considered here. Such formulas were introduced by Verlinde [38] for diagonalizing fusion relations in 2D rational conformal field theory and have played an important role in physics. They have been considered subsequently in many different contexts.

For a Hopf algebra A with nonisomorphic simple modules S_i and corresponding projective covers P_i, the Cartan matrix $\mathsf{C} = (\mathsf{C}_{ij})$ records the multiplicity $\mathsf{C}_{ij} = [\mathsf{P}_i : \mathsf{S}_j]$ of $[\mathsf{S}_j]$ when $[\mathsf{P}_i]$ is expressed in the $\mathbb{Z}$-basis of $\mathsf{G}_0(\mathsf{A})$. Let r be the rank of C and select r of the P_i so that the corresponding rows of C are linearly independent. For each simple module S_j there is an $r \times r$ matrix N^j, that contains the fusion rules for tensoring those r projective modules with S_j and writing the answer $[\mathsf{P}_i \otimes \mathsf{S}_j]$ as a linear combination of the r chosen projectives. It was shown in [9] that the matrices N^j are diagonalizable and have eigenvectors that can be expressed using the primitive idempotents e_i corresponding to the projective modules P_i.

In Section 2 of this work, we consider *arbitrary* finite-dimensional Hopf algebras A and prove general results about McKay matrices, their eigenvalues, and their (left and right) eigenvectors by using the coproduct and the characters of simple and projective modules (Theorem 2.1.2 and Proposition 2.1.4). The tensor product $\mathsf{P} \otimes \mathsf{V}$ of a projective module P with a finite-dimensional module V is projective, and so $[\mathsf{P} \otimes \mathsf{V}]$ can be written as an integral combination of the classes $[\mathsf{P}_j]$ of the projective indecomposable modules. Letting $\mathsf{Q}_\mathsf{V} = (\mathsf{Q}_{ij})$, where Q_{ij} is the multiplicity $[\mathsf{P}_i \otimes \mathsf{V} : \mathsf{P}_j]$ of $[\mathsf{P}_j]$ in $[\mathsf{P}_i \otimes \mathsf{V}]$, we obtain what we term a *projective McKay matrix.* Theorems 2.3.3 and 2.4.1 and Corollary 2.4.2 show how Q_V and its eigenvectors are related to the McKay matrix of the dual module V^*. In the special case that the Hopf algebra is semisimple, the McKay matrix M_V and the projective McKay matrix Q_V are the same, and if the module V is self-dual, then the McKay matrix is orthogonally diagonalizable (Corollary 2.3.4).

In Section 3, we illustrate the general results of Section 2 by applying them to a family of nonsemisimple Hopf algebras, namely, the Drinfeld double D_n of the Taft algebra A_n for n odd, $n \geq 3$. When n is even, the eigenvalues exhibit different patterns, and that case will not be considered here. The algebras D_n provide a convenient testing ground, as their representation theory has been developed in great detail by Chen and coauthors (see [4–8,36]). Unlike the situation for semisimple, almost cocommutative Hopf algebras, the McKay matrices for D_n fail to be diagonalizable. More specifically, we

- determine the eigenvalues, right and left eigenvectors and generalized eigenvectors of the McKay matrix M_V obtained by tensoring the simple D_n-modules with one of the two-dimensional simple D_n-modules, $\mathsf{V}(2, 0)$ (Secs. 3.3-3.9);
- express the coordinates of these vectors using Chebyshev polynomials (Secs. 3.4, 3.5, 3.8);

- relate the eigenvectors to character values of the grouplike elements and other special elements of D_n (Secs. 3.6, 3.7, 3.9) and show that the character value of a grouplike element on any simple D_n-module can be computed using Chebyshev polynomials (Thm. 3.6.1 of Sec. 3.6;
- prove that the (generalized) eigenvectors for M_V, $\mathsf{V} = \mathsf{V}(2,0)$, are (generalized) eigenvectors for the McKay matrix of *any* simple D_n-module (Sec. 3.10);
- show that the eigenvalues of the McKay matrix of any simple D_n-module can be expressed in terms of Chebyshev polynomials of the second kind (Thm. 3.11.1 of Sec. 3.11);
- find the eigenvectors and eigenvalues of the projective McKay matrix Q_V by relating them to the McKay matrix of the dual module V^* (Prop. 3.12.1 of Sec. 3.12);
- determine the structure of the complex Grothendieck algebra $\mathsf{G}_0^\mathbb{C}(\mathsf{D}_n) = \mathbb{C} \otimes_\mathbb{Z} \mathsf{G}_0(\mathsf{D}_n)$ and prove that its Jacobson radical squares to 0; (Thm. 3.13.8 of Sec. 3.13);
- construct certain idempotents in $\mathsf{G}_0^\mathbb{C}(\mathsf{D}_n)$, and show how they provide an alternate approach to producing the eigenvectors and generalized eigenvectors of M_V (Sec. 3.13);
- compute the eigenvectors and eigenvalues of the matrix N_V that encodes the fusion rules for tensoring a maximal set of independent projective covers in $\mathsf{G}_0(\mathsf{D}_n)$ with $\mathsf{V} = \mathsf{V}(2,0)$ (Sec. 3.14).

It has been said that "Chebyshev polynomials are everywhere dense in numerical analysis" (see [28, Sec. 1.1] for a discussion of this quotation). In this paper, Chebyshev polynomials (of the second, third, and fourth kind) are everywhere dense in expressing eigenvalues, eigenvectors, and generalized eigenvectors of McKay matrices and fusion rules for D_n. The characters of the simple D_n-modules evaluated on the grouplike elements of D_n also have Chebyshev polynomial expressions.

When n is odd, D_n is a ribbon Hopf algebra [25], and D_n provides an invariant of 3-manifolds [21]. In [3], we determine the unique ribbon element of D_n explicitly. We use the R-matrix and ribbon element of the quasitriangular Hopf algebra D_n to obtain an algebra homomorphism from the Temperley-Lieb algebra $\mathsf{TL}_k(-(q^{\frac{1}{2}} + q^{-\frac{1}{2}}))$ to the centralizer algebra $\mathsf{End}_{\mathsf{D}_n}(\mathsf{V}^{\otimes k})$, when V is any two-dimensional simple D_n-module. In the special case that V is the unique two-dimensional simple module that is self-dual, we show that the homomorphism is injective for all $k \geq 1$ and an isomorphism for $k \leq 2(n-1)$. This leads to a realization of $\mathsf{End}_{\mathsf{D}_n}(\mathsf{V}^{\otimes k})$ for V self-dual as a diagram algebra.

Acknowledgments. Our joint work began in connection with the workshop WINART2 at the University of Leeds in May 2019. The authors would like to extend thanks to the organizers of WINART2 and to the University of Leeds for its hospitality, and to acknowledge support provided by a University of Leeds conference grant, the London Mathematical Society Workshop Grant WS-1718-03, the U.S. National Science Foundation grant DMS-1900575, the Association for Women in Mathematics (NSF Grant DMS-1500481), and by a research fellowship from the Alfred P. Sloan Foundation. Van C. Nguyen was also supported by the Naval Academy Research Council in summer 2020. Rekha Biswal gratefully acknowledges the Max Planck Institute for the fellowship she received as a postdoctoral researcher there and for providing an excellent atmosphere for research. The authors thank the referees for their helpful comments.

2. McKay Matrices for Arbitrary Hopf Algebras

In the classical representation theory of finite groups, the columns of the character table are obtained by evaluating the characters (traces) of the simple modules on one element from each conjugacy class [18, Sec. 2.1]. It is known that these columns are right eigenvectors for any McKay matrix determined by tensoring with a finite-dimensional module of the group (see [35]). In this section, we develop an analog of this result by showing how grouplike elements and generalizations of skew primitive elements in an arbitrary Hopf algebra can be used to construct eigenvectors for McKay matrices. In the special case that the algebra is semisimple, more detailed results are possible.

Throughout Section 2, A *denotes a finite-dimensional Hopf algebra over an algebraically closed field* $\Bbbk$*. All* A*-modules are assumed to be finite-dimensional, and all tensor products are over* $\Bbbk$*. We adopt Sweedler's notation for the coproduct* Δ *applied to an element* $x \in \mathsf{A}$*,*

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}.$$

2.1. **Right eigenvectors from traces of simple modules.** We will use the coproduct of A to obtain right (generalized) eigenvectors of McKay matrices by taking the trace of certain elements of A on simple modules. To accomplish this, we apply the following well-known results on traces. Parts (a) and (b) hold for any finite-dimensional algebra A.

Lemma 2.1.1. (a) *Assume* S, T, U *are finite-dimensional modules over the algebra* A *such that* $\mathsf{U} \simeq \mathsf{S}/\mathsf{T}$*. Then for any* $x \in \mathsf{A}$*, the trace* $\mathsf{tr}_\mathsf{U}(x)$ *of* x *on* U *satisfies* $\mathsf{tr}_\mathsf{U}(x) = \mathsf{tr}_\mathsf{S}(x) + \mathsf{tr}_\mathsf{T}(x)$.

(b) *If* $\mathsf{U}_1, \dots, \mathsf{U}_s$ *are the composition factors of a finite-dimensional* A*-module* U*, where* U_i *occurs with multiplicity* c_i*, then for any* $x \in \mathsf{A}$*,* $\mathsf{tr}_\mathsf{U}(x) = c_1\mathsf{tr}_{\mathsf{U}_1}(x) + \cdots + c_s\mathsf{tr}_{\mathsf{U}_s}(x)$.

(c) (See [27, Proposition 10.21 (b)].) For any finite-dimensional A-modules U and W, and any $x \in \mathsf{A}$,

$$\mathsf{tr}_{\mathsf{U}\otimes\mathsf{W}}(x) = \sum_x \mathsf{tr}_\mathsf{U}\left(x_{(1)}\right) \mathsf{tr}_\mathsf{W}\left(x_{(2)}\right). \tag{2.1.1}$$

Let $\mathsf{S}_1, \mathsf{S}_2, \dots, \mathsf{S}_m$ be a $\mathbb{Z}$-basis of simple modules for the Grothendieck ring $\mathsf{G}_0(\mathsf{A})$ of the Hopf algebra A, and for any $x \in \mathsf{A}$ set

$$\mathsf{Tr}_\mathsf{S}(x) := [\mathsf{tr}_{\mathsf{S}_1}(x)\ \ \mathsf{tr}_{\mathsf{S}_2}(x)\ \ \dots\ \ \mathsf{tr}_{\mathsf{S}_m}(x)]^\mathtt{T}. \tag{2.1.2}$$

Observe that $\mathsf{Tr}_\mathsf{S}(1) = [\mathsf{dim}(\mathsf{S}_1)\ \ \mathsf{dim}(\mathsf{S}_2)\ \ \dots\ \ \mathsf{dim}(\mathsf{S}_m)]^\mathtt{T} = \mathbf{s}$, the vector of dimensions of the simple A-modules. We know that $\mathbf{s}$ gives a right eigenvector for any McKay matrix determined by tensoring with a finite-dimensional A-module. Next we explore some other elements of the Hopf algebra A that give such right eigenvectors.

Theorem 2.1.2. *Assume* $\mathsf{M}_\mathsf{V} = (\mathsf{M}_{ij})$*, where* $\mathsf{M}_{ij} = [\mathsf{S}_i \otimes \mathsf{V} : \mathsf{S}_j]$*, is the McKay matrix associated to tensoring with an* A*-module* V*. Then the trace of* $x \in \mathsf{A}$ *on* $\mathsf{S}_i \otimes \mathsf{V}$ *for any* $1 \le i \le m$ *is given by*

$$\sum_x \mathsf{tr}_{\mathsf{S}_i}(x_{(1)})\mathsf{tr}_\mathsf{V}(x_{(2)}) = \sum_{j=1}^m \mathsf{M}_{ij}\mathsf{tr}_{\mathsf{S}_j}(x), \ \textit{and} \tag{2.1.3}$$

$$\mathsf{M}_\mathsf{V}\,\mathsf{Tr}_\mathsf{S}(x) = \sum_x \mathsf{tr}_\mathsf{V}(x_{(2)})\,\mathsf{Tr}_\mathsf{S}(x_{(1)}). \tag{2.1.4}$$

Proof. This follows from Lemma 2.1.1 (c) and the fact that M_{ij} is the multiplicity of S_j as a composition factor of $\mathsf{S}_i \otimes \mathsf{V}$. Equation (2.1.3) gives the ith coordinate of the matrix equation in (2.1.4). □

Corollary 2.1.3. *For the McKay matrix* M_V *associated to tensoring the simple* A*-modules with* V*, the following hold:*

(a) *When $g \in \mathsf{A}$ is grouplike, then $\Delta(g) = g \otimes g$, and* (2.1.4) *says*

$$\mathsf{M}_\mathsf{V}\,\mathsf{Tr}_\mathsf{S}(g) = \mathsf{tr}_\mathsf{V}(g)\,\mathsf{Tr}_\mathsf{S}(g). \tag{2.1.5}$$

Consequently, for every grouplike element of A, $\mathsf{Tr}_\mathsf{S}(g)$ is a right eigenvector for M_V with eigenvalue $\mathsf{tr}_\mathsf{V}(g)$. When $g = 1$, this reverts to the result that the dimension vector $\mathbf{s}$ is a right eigenvector with eigenvalue $\mathsf{tr}_\mathsf{V}(1) = \mathsf{dim}(\mathsf{V})$.

(b) *When $x \in \mathsf{A}$ has the property that $\Delta(x) = x \otimes y + z \otimes x$ for some nonzero $y, z \in \mathsf{A}$, then* (2.1.4) *says*

$$\mathsf{M}_\mathsf{V}\,\mathsf{Tr}_\mathsf{S}(x) = \mathsf{tr}_\mathsf{V}(y)\,\mathsf{Tr}_\mathsf{S}(x) + \mathsf{tr}_\mathsf{V}(x)\,\mathsf{Tr}_\mathsf{S}(z). \tag{2.1.6}$$

The next result draws some useful conclusions from this equation. In part (2)(b) of Proposition 2.1.4, the phrase *generalized right eigenvector of* M_V *with eigenvalue* λ refers to a vector u such that $(\mathsf{M}_\mathsf{V} - \lambda \mathrm{I})u$ is a nonzero right eigenvector of M_V for λ. Similarly, in Proposition 2.2.2 (2)(b) below, a generalized left eigenvector for the matrix $\mathsf{Q}_\mathsf{V}^\mathsf{T}$ with eigenvalue λ is a vector w such that $w(\mathsf{Q}_\mathsf{V}^\mathsf{T} - \lambda \mathrm{I})$ is a nonzero left eigenvector for $\mathsf{Q}_\mathsf{V}^\mathsf{T}$ with eigenvalue λ.

Proposition 2.1.4. *Assume* (2.1.6) *holds for some $x \in \mathsf{A}$, and $\mathsf{Tr}_\mathsf{S}(x) \neq \mathbf{0}$.*

(1) *If $\mathsf{tr}_\mathsf{V}(x) = 0$, then $\mathsf{Tr}_\mathsf{S}(x)$ is a right eigenvector for M_V with eigenvalue $\mathsf{tr}_\mathsf{V}(y)$.*

(2) *If $\mathsf{tr}_\mathsf{V}(x) \neq 0$, and $\mathsf{Tr}_\mathsf{S}(z)$ is a right eigenvector for M_V with eigenvalue λ, then*
   - (a) *$\mathsf{tr}_\mathsf{V}(y) \neq \lambda$ implies that $(\mathsf{tr}_\mathsf{V}(y) - \lambda)\,\mathsf{Tr}_\mathsf{S}(x) + \mathsf{tr}_\mathsf{V}(x)\,\mathsf{Tr}_\mathsf{S}(z)$ is a right eigenvector for M_V with eigenvalue $\mathsf{tr}_\mathsf{V}(y)$;*
   - (b) *$\mathsf{tr}_\mathsf{V}(y) = \lambda$ and $\mathsf{Tr}_\mathsf{S}(x) \notin \Bbbk\,\mathsf{Tr}_\mathsf{S}(z)$ imply that $\mathsf{Tr}_\mathsf{S}(x)$ is a generalized right eigenvector for M_V with eigenvalue λ: $\quad \mathsf{M}_\mathsf{V}\,\mathsf{Tr}_\mathsf{S}(x) = \lambda\,\mathsf{Tr}_\mathsf{S}(x) + \mathsf{tr}_\mathsf{V}(x)\,\mathsf{Tr}_\mathsf{S}(z)$.*

Remark 2.1.5. The case $\mathsf{tr}_\mathsf{V}(y) = \lambda$ and $\mathsf{Tr}_\mathsf{S}(x) = \delta\,\mathsf{Tr}_\mathsf{S}(z)$ for some $0 \neq \delta \in \Bbbk$ cannot happen when $\mathsf{Tr}_\mathsf{S}(z)$ is assumed to be a right eigenvector for M_V with eigenvalue λ in (2), as this would imply that $\mathsf{Tr}_\mathsf{S}(x)$ is a right eigenvector for M_V with both eigenvalue λ and eigenvalue $\lambda + \delta^{-1}\mathsf{tr}_\mathsf{V}(x) \neq \lambda$.

2.2. **Left eigenvectors from traces of projective modules.** This section discusses using projective modules to produce left eigenvectors. Suppose $\mathsf{P}_1, \mathsf{P}_2, \dots, \mathsf{P}_m$ are the nonisomorphic indecomposable projective modules for the Hopf algebra A. The tensor product of any A-module V with a projective A-module is projective, hence the corresponding isomorphism class is a $\mathbb{Z}$-linear combination of the $[\mathsf{P}_j]$ in the Grothendieck group $\mathsf{K}_0(\mathsf{A})$ (see [26, Sec. 3.1]). We use $[\mathsf{P}_i \otimes \mathsf{V} : \mathsf{P}_j]$ to denote the multiplicity of $[\mathsf{P}_j]$ in $[\mathsf{P}_i \otimes \mathsf{V}]$ and define

$$\mathsf{Q}_\mathsf{V} = (\mathsf{Q}_{ij}), \quad \text{where} \quad \mathsf{Q}_{ij} = [\mathsf{P}_i \otimes \mathsf{V} : \mathsf{P}_j]. \tag{2.2.1}$$

We refer to the matrix Q_V as the *projective McKay matrix* and relate Q_V to a McKay matrix in Theorem 2.3.3 below. If $x \in \mathsf{A}$ and $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$, then

$$\sum_x \mathsf{tr}_{\mathsf{P}_i}(x_{(1)})\mathsf{tr}_\mathsf{V}(x_{(2)}) = \mathsf{tr}_{\mathsf{P}_i \otimes \mathsf{V}}(x) = \sum_{j=1}^m \mathsf{Q}_{ij}\mathsf{tr}_{\mathsf{P}_j}(x) = \sum_{j=1}^m \mathsf{tr}_{\mathsf{P}_j}(x)\left(\mathsf{Q}_\mathsf{V}^\mathsf{T}\right)_{ji}. \tag{2.2.2}$$

This is the ith component of the matrix equation

$$\mathsf{Tr}_\mathsf{P}(x)\mathsf{Q}_\mathsf{V}^\mathsf{T} = \sum_x \mathsf{tr}_\mathsf{V}(x_{(2)})\,\mathsf{Tr}_\mathsf{P}(x_{(1)}), \quad \text{where} \tag{2.2.3}$$

$$\mathsf{Tr}_\mathsf{P}(x) := \left[\mathsf{tr}_{\mathsf{P}_1}(x)\ \ \mathsf{tr}_{\mathsf{P}_2}(x)\ \ \dots\ \ \mathsf{tr}_{\mathsf{P}_m}(x)\right]. \tag{2.2.4}$$

We have the following analogs of (2.1.5) and (2.1.6):

Corollary 2.2.1. *Let $\mathsf{Q}_\mathsf{V}^\mathtt{T}$ be the McKay matrix associated to tensoring the projective indecomposable* A*-modules with* V*. Then*

(a) *When $g \in \mathsf{A}$ is grouplike,*

$$\mathsf{Tr}_\mathsf{P}(g)\mathsf{Q}_\mathsf{V}^\mathtt{T} = \mathsf{tr}_\mathsf{V}(g)\,\mathsf{Tr}_\mathsf{P}(g). \tag{2.2.5}$$

Consequently, for every grouplike element of A*, $\mathsf{Tr}_\mathsf{P}(g)$ is a left eigenvector for $\mathsf{Q}_\mathsf{V}^\mathtt{T}$ with eigenvalue $\mathsf{tr}_\mathsf{V}(g)$. When $g = 1$, the eigenvalue is $\mathsf{tr}_\mathsf{V}(1) = \mathsf{dim}(\mathsf{V})$, and the eigenvector $\mathsf{Tr}_\mathsf{P}(1)$ is just the dimension vector $\mathbf{p} = \big[\mathsf{dim}(\mathsf{P}_1)\ \mathsf{dim}(\mathsf{P}_2)\ \dots\ \mathsf{dim}(\mathsf{P}_m)\big]$.*

(b) *When $x \in \mathsf{A}$ has the property that $\Delta(x) = x \otimes y + z \otimes x$ for some nonzero $y, z \in \mathsf{A}$, then* (2.2.3) *says*

$$\mathsf{Tr}_\mathsf{P}(x)\mathsf{Q}_\mathsf{V}^\mathtt{T} = \mathsf{tr}_\mathsf{V}(y)\,\mathsf{Tr}_\mathsf{P}(x) + \mathsf{tr}_\mathsf{V}(x)\,\mathsf{Tr}_\mathsf{P}(z). \tag{2.2.6}$$

The next result is the projective version of Proposition 2.1.4.

Proposition 2.2.2. *Assume* (2.2.6) *holds for $x \in \mathsf{A}$ and $\mathsf{Tr}_\mathsf{P}(x) \neq \mathbf{0}$.*

(1) *If $\mathsf{tr}_\mathsf{V}(x) = 0$, then $\mathsf{Tr}_\mathsf{P}(x)$ is a left eigenvector for $\mathsf{Q}_\mathsf{V}^\mathtt{T}$ with eigenvalue $\mathsf{tr}_\mathsf{V}(y)$.*

(2) *If $\mathsf{tr}_\mathsf{V}(x) \neq 0$, and $\mathsf{Tr}_\mathsf{P}(z)$ is a left eigenvector for $\mathsf{Q}_\mathsf{V}^\mathtt{T}$ with eigenvalue λ, then*

   (a) *$\mathsf{tr}_\mathsf{V}(y) \neq \lambda$ implies that $(\mathsf{tr}_\mathsf{V}(y) - \lambda)\,\mathsf{Tr}_\mathsf{P}(x) + \mathsf{tr}_\mathsf{V}(x)\,\mathsf{Tr}_\mathsf{P}(z)$ is a left eigenvector for $\mathsf{Q}_\mathsf{V}^\mathtt{T}$ with eigenvalue $\mathsf{tr}_\mathsf{V}(y)$;*

   (b) *$\mathsf{tr}_\mathsf{V}(y) = \lambda$ and $\mathsf{Tr}_\mathsf{P}(x) \notin \Bbbk\,\mathsf{Tr}_\mathsf{P}(z)$ imply that $\mathsf{Tr}_\mathsf{P}(x)$ is a generalized left eigenvector for $\mathsf{Q}_\mathsf{V}^\mathtt{T}$ with eigenvalue λ: $\mathsf{Tr}_\mathsf{P}(x)\mathsf{Q}_\mathsf{V}^\mathtt{T} = \lambda\,\mathsf{Tr}_\mathsf{P}(x) + \mathsf{tr}_\mathsf{V}(x)\,\mathsf{Tr}_\mathsf{P}(z)$.*

Remark 2.2.3. As in Remark 2.1.5, the case $\mathsf{tr}_\mathsf{V}(y) = \lambda$ and $\mathsf{Tr}_\mathsf{P}(x) = \delta\,\mathsf{Tr}_\mathsf{P}(z)$ for some $0 \neq \delta \in \Bbbk$ cannot happen when $\mathsf{Tr}_\mathsf{P}(z)$ is assumed to be a left eigenvector for $\mathsf{Q}_\mathsf{V}^\mathtt{T}$ with eigenvalue λ.

2.3. **The Cartan map and Cartan matrix.** We will use the Cartan map to relate McKay matrices and projective McKay matrices for any finite-dimensional Hopf algebra A.

The two Grothendieck groups mentioned in the Introduction are related by the *Cartan map* $\mathsf{c} : \mathsf{K}_0(\mathsf{A}) \to \mathsf{G}_0(\mathsf{A})$, $[\mathsf{P}] \mapsto [\mathsf{P}]$. The *Cartan matrix* is the integral matrix $\mathsf{C} = \big(\mathsf{C}_{ij}\big)$ representing c in the bases $\{[\mathsf{P}_j] \mid 1 \leq j \leq m\}$ and $\{[\mathsf{S}_j] \mid 1 \leq j \leq m\}$ of $\mathsf{K}_0(\mathsf{A})$ and $\mathsf{G}_0(\mathsf{A})$ respectively. It has as its (i, j) entry $\mathsf{C}_{ij} = [\mathsf{P}_i : \mathsf{S}_j]$, the multiplicity of $[\mathsf{S}_j]$, when $[\mathsf{P}_i]$ is written as a $\mathbb{Z}$-combination of the classes $[\mathsf{S}_j]$, which is also the multiplicity of S_j in a composition series for P_i. When A is semisimple, C is the identity matrix, as $\mathsf{P}_j = \mathsf{S}_j$ for all j. In general, the Cartan matrix is not invertible.

Here is a tiny example exhibiting a non-invertible C: The Sweedler Hopf algebra has a basis $1, a, b, ab$, where $a^2 = 0$, $b^2 = 1$, $ba = -ab$ (it is the baby Taft algebra A_2 - compare Section 3.1 with $q = -1$). It has two one-dimensional simple modules S_α and S_ε, where $\alpha(a) = 0$, $\alpha(b) = -1$, and ε is the counit with $\varepsilon(a) = 0$ and $\varepsilon(b) = 1$. The corresponding projective covers P_α and P_ε are two-dimensional and have both simple modules as composition factors. Thus, $\mathsf{C} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. In fact, the Cartan matrix of any Taft algebra A_n defined using a primitive root of unity of order $n \geq 2$ is the $n \times n$ matrix with all entries equal to 1 (see [27, Exer. 10.2.4]).

Now suppose that $\mathsf{Q}_\mathsf{V} = \big(\mathsf{Q}_{ij}\big)$ is the projective McKay matrix for tensoring with V as in (2.2.1). Then the relation between Q_V and the McKay matrix M_V for tensoring simple modules with V can be described as follows:

Proposition 2.3.1. *For any* A*-module* V*,*

(a) $\mathsf{Q_V C} = \mathsf{C M_V}$, *where* $\mathsf{C} = \big(\mathsf{C}_{ij}\big)$ *is the Cartan matrix.*
(b) *If* v *is a right eigenvector for* $\mathsf{M_V}$ *with eigenvalue* λ, *then* $\mathsf{C}v$ *is a right eigenvector for* $\mathsf{Q_V}$ *with eigenvalue* λ.
(c) *If* w *is a left eigenvector for* $\mathsf{Q_V}$ *with eigenvalue* μ, *then* $w\mathsf{C}$ *is a left eigenvector for* $\mathsf{M_V}$ *with eigenvalue* μ.

Proof. (a) With notation as in (2.2.1), we have on one hand

$$[\mathsf{P}_i \otimes \mathsf{V}] = \sum_{t=1}^{m} \mathsf{Q}_{it}[\mathsf{P}_t] \xrightarrow{\ \mathsf{c}\ } \sum_{t=1}^{m} \mathsf{Q}_{it}[\mathsf{P}_t] = \sum_{t,\ell=1}^{m} \mathsf{Q}_{it}\mathsf{C}_{t\ell}[\mathsf{S}_\ell],$$

and on the other,

$$[\mathsf{P}_i \otimes \mathsf{V}] \xrightarrow{\ \mathsf{c}\ } [\mathsf{P}_i \otimes \mathsf{V}] = [\mathsf{P}_i][\mathsf{V}] = \sum_{t=1}^{m} \mathsf{C}_{it}[\mathsf{S}_t][\mathsf{V}] = \sum_{t,\ell=1}^{m} \mathsf{C}_{it}\mathsf{M}_{t\ell}[\mathsf{S}_\ell].$$

Thus, $\mathsf{Q_V C} = \mathsf{C M_V}$.

(b) If v is a right eigenvector for $\mathsf{M_V}$ with eigenvalue λ, then $\mathsf{Q_V C}v = \mathsf{C M_V}v = \lambda \mathsf{C}v$ so that $\mathsf{C}v$ is a right eigenvector for $\mathsf{Q_V}$ with eigenvalue λ. (c) Similarly, if w is a left eigenvector for $\mathsf{Q_V}$ with eigenvalue μ, then $w\mathsf{C}$ is a left eigenvector for $\mathsf{M_V}$ with eigenvalue μ. $\square$

For simplicity, we usually omit the brackets on the isomorphism class representatives of $\mathsf{K}_0(\mathsf{A})$ and $\mathsf{G}_0(\mathsf{A})$ in what follows unless they are needed for clarity. We will use the next result, which can be found for example in [16, Prop. 9.2.3].

Proposition 2.3.2. *Let* P_i *be the projective cover of the simple module* S_i. *Then for any* A*-module* N,

$$\mathsf{dim}_{\Bbbk}\mathsf{Hom}_{\mathsf{A}}(\mathsf{P}_i, \mathsf{N}) = [\mathsf{N} : \mathsf{S}_i],$$

the multiplicity of S_i *in a Jordan-Hölder series of* N.

For every $\mathsf{P} \in \mathsf{K}_0(\mathsf{A})$ and $\mathsf{V}, \mathsf{W} \in \mathsf{G}_0(\mathsf{A})$, the following holds (see [26, Sec. 3.1])

$$\mathsf{dim}_{\Bbbk}\mathsf{Hom}_{\mathsf{A}}(\mathsf{P} \otimes \mathsf{V}, \mathsf{W}) = \mathsf{dim}_{\Bbbk}\mathsf{Hom}_{\mathsf{A}}(\mathsf{P}, \mathsf{W} \otimes \mathsf{V}^*). \tag{2.3.1}$$

This will enable us to relate the projective McKay matrix to the McKay matrix of the dual module.

Assume $\mathsf{M_V} = \big(\mathsf{M}_{ij}\big)$ is the McKay matrix for tensoring with $\mathsf{V} \in \mathsf{G}_0(\mathsf{A})$, and $\mathsf{M}_{\mathsf{V}^*} = \big(\mathsf{M}^*_{ij}\big)$ is the McKay matrix for tensoring with the dual module V^*. Let $\mathsf{M}^{\mathtt{T}}_{\mathsf{V}^*} = \big(\mathsf{M}^*_{ji}\big)$ be the transpose of $\mathsf{M}_{\mathsf{V}^*}$, and let $\mathsf{Q_V} = \big(\mathsf{Q}_{ij}\big)$ be the projective McKay matrix for V. Then we have the following consequence of Proposition 2.3.2 and (2.3.1). Variations of this result have appeared in several different contexts such as [1, Lem. 8] and [29, Lem. 10], and it can be regarded as a special case of the tensor category result [15, Prop. 6.1.2].

Theorem 2.3.3. *Assume* V *is a module for a finite-dimensional Hopf algebra* A. *Then*

$$\mathsf{Q}_{ij} = \mathsf{dim}_{\Bbbk}\mathsf{Hom}_{\mathsf{A}}(\mathsf{P}_i \otimes \mathsf{V}, \mathsf{S}_j) = \mathsf{dim}_{\Bbbk}\mathsf{Hom}_{\mathsf{A}}(\mathsf{P}_i, \mathsf{S}_j \otimes \mathsf{V}^*) = [\mathsf{S}_j \otimes \mathsf{V}^* : \mathsf{S}_i] = \mathsf{M}^*_{ji}.$$

Therefore, $\mathsf{Q_V} = \mathsf{M}^{\mathtt{T}}_{\mathsf{V}^*}$.

Corollary 2.3.4. *For any module* V *over a finite-dimensional Hopf algebra* A,

(a) $\mathsf{M}^{\mathtt{T}}_{\mathsf{V}^*}\,\mathsf{C} = \mathsf{C}\,\mathsf{M_V}$, *where* C *is the Cartan matrix.*
(b) *If* v *is a right eigenvector of* $\mathsf{M_V}$ *with eigenvalue* λ, *then* $\mathsf{C}v$ *is a right eigenvector for* $\mathsf{M}^{\mathtt{T}}_{\mathsf{V}^*}$ *with eigenvalue* λ. *Similarly, if* w *is a left eigenvector of* $\mathsf{M}^{\mathtt{T}}_{\mathsf{V}^*}$ *with eigenvalue* μ, *then* $w\mathsf{C}$ *is a left eigenvector for* $\mathsf{M_V}$ *with eigenvalue* μ.

(c) *If* C *is invertible, then* $\mathsf{M}_{\mathsf{V}^*}^{\mathtt{T}} = \mathsf{C}\mathsf{M}_{\mathsf{V}}\mathsf{C}^{-1}$. *Therefore,* M_{V} *and* $\mathsf{M}_{\mathsf{V}^*}^{\mathtt{T}}$ *(and also* $\mathsf{M}_{\mathsf{V}^*}$*) have the same eigenvalues when* C *is invertible.*

(d) *If* A *is semisimple, the Cartan matrix is the identity matrix, and consequently,* $\mathsf{M}_{\mathsf{V}^*}^{\mathtt{T}} = \mathsf{M}_{\mathsf{V}}$. *Moreover, if* V *is self-dual, then* M_{V} *is symmetric, hence orthogonally diagonalizable.*

2.4. **Eigenvectors from projective modules.** We combine results from the previous sections to obtain left eigenvectors from traces of projective modules. Theorem 2.3.3 implies that $\mathsf{Q}_{\mathsf{V}}^{\mathtt{T}} = \mathsf{M}_{\mathsf{V}^*}$ holds for any finite-dimensional Hopf algebra A, where V^* is the dual module to V, and Q_{V} is the projective McKay matrix. As a consequence of (2.2.3), we have

Theorem 2.4.1. *For any* A*-module* V *and all* $x \in \mathsf{A}$,

(a) $\mathsf{Tr}_{\mathsf{P}}(x)\mathsf{M}_{\mathsf{V}^*} = \sum_x \mathsf{tr}_{\mathsf{V}}(x_{(2)})\,\mathsf{Tr}_{\mathsf{P}}(x_{(1)})$, *where* $\mathsf{Tr}_{\mathsf{P}}(x) = [\mathsf{tr}_{\mathsf{P}_1}(x)\ \ \mathsf{tr}_{\mathsf{P}_2}(x)\ \ \dots\ \ \mathsf{tr}_{\mathsf{P}_m}(x)]$.

(b) *If* $(\mathsf{V}^*)^* \cong \mathsf{V}$, *then* $\mathsf{Tr}_{\mathsf{P}}(x)\mathsf{M}_{\mathsf{V}} = \sum_x \mathsf{tr}_{\mathsf{V}^*}(x_{(2)})\,\mathsf{Tr}_{\mathsf{P}}(x_{(1)})$.

Corollary 2.4.2. *Under the assumption that* $(\mathsf{V}^*)^* \cong \mathsf{V}$ *for the* A*-module* V*, the following results hold (compare these to the corresponding results for simple modules* (2.1.5) *and* (2.1.6)*):*

(a) *When* $g \in \mathsf{A}$ *is grouplike,*

$$\mathsf{Tr}_{\mathsf{P}}(g)\mathsf{M}_{\mathsf{V}} = \mathsf{tr}_{\mathsf{V}^*}(g)\,\mathsf{Tr}_{\mathsf{P}}(g). \tag{2.4.1}$$

Hence, for every grouplike element of A, $\mathsf{Tr}_{\mathsf{P}}(g)$ *is a left eigenvector of* M_{V} *of eigenvalue* $\mathsf{tr}_{\mathsf{V}^*}(g)$. *The vector* $\mathsf{Tr}_{\mathsf{P}}(1)$ *is just the dimension vector* $\mathbf{p} = \big[\mathsf{dim}(\mathsf{P}_1)\ \ \mathsf{dim}(\mathsf{P}_2)\ \ \dots\ \ \mathsf{dim}(\mathsf{P}_m)\big]$, *and the eigenvalue is* $\mathsf{tr}_{\mathsf{V}^*}(1) = \mathsf{dim}(\mathsf{V}^*) = \mathsf{dim}(\mathsf{V}) = \mathsf{tr}_{\mathsf{V}}(1)$.

(b) *When* $x \in \mathsf{A}$ *has the property that* $\Delta(x) = x \otimes y + z \otimes x$ *for some nonzero* $y, z \in \mathsf{A}$, *then* (2.2.3) *says*

$$\mathsf{Tr}_{\mathsf{P}}(x)\mathsf{M}_{\mathsf{V}} = \mathsf{tr}_{\mathsf{V}^*}(y)\,\mathsf{Tr}_{\mathsf{P}}(x) + \mathsf{tr}_{\mathsf{V}^*}(x)\,\mathsf{Tr}_{\mathsf{P}}(z). \tag{2.4.2}$$

2.5. **Eigenvectors for McKay matrices from the Grothendieck algebra** $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$**.** Next we describe a way to produce left eigenvectors and generalized left eigenvectors for McKay matrices using the Grothendieck algebra $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A}) = \mathbb{C}\otimes_{\mathbb{Z}}\mathsf{G}_0(\mathsf{A})$ of any Hopf algebra A. The classes $[\mathsf{S}_1], [\mathsf{S}_2], \dots, [\mathsf{S}_m]$ of the nonisomorphic simple modules give a $\mathbb{C}$-basis for $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$.

Proposition 2.5.1. (a) *Let* V *be an* A*-module, and assume* $[\mathsf{X}] = c_1[\mathsf{S}_1]+\dots+c_m[\mathsf{S}_m] \in \mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$, $c_j \in \mathbb{C}$ *for all* j, *is an eigenvector for the right multiplication operator* $\mathsf{R}_{[\mathsf{V}]}$ *of* $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ *with eigenvalue* λ. *Then* $[c_1\ c_2\ \ \dots\ \ c_m]$ *is a left eigenvector for* M_{V} *with eigenvalue* λ.

(b) *Suppose that* $[\mathsf{Y}] = d_1[\mathsf{S}_1] + \dots + d_k[\mathsf{S}_m] \in \mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ *has the property that* $(\mathsf{R}_{[\mathsf{V}]} - \lambda\mathrm{I})^{\ell}(\mathsf{Y}) = 0$. *Then for the McKay matrix* M_{V}, *we have* $[d_1\ \ d_2\ \ \dots\ \ d_m](\mathsf{M}_{\mathsf{V}} - \lambda\mathrm{I})^{\ell} = 0$.

Proof. (a) We are assuming that $\mathsf{R}_{[\mathsf{V}]}([\mathsf{X}]) = \lambda[\mathsf{X}]$, or more specifically,

$$(c_1[\mathsf{S}_1] + \dots + c_m[\mathsf{S}_m])[\mathsf{V}] = c_1[\mathsf{S}_1][\mathsf{V}] + \dots + c_m[\mathsf{S}_m][\mathsf{V}] = \lambda(c_1[\mathsf{S}_1] + \dots + c_m[\mathsf{S}_m]).$$

Since multiplication in $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ is given by tensoring, this implies that

$$\sum_{i=1}^{m} c_i[\mathsf{S}_i \otimes \mathsf{V} : \mathsf{S}_j] = \lambda c_j$$

holds for each j, where $[\mathsf{S}_i\otimes\mathsf{V} : \mathsf{S}_j]$ is the multiplicity of $[\mathsf{S}_j]$ in $[\mathsf{S}_i\otimes\mathsf{V}]$. However, $[\mathsf{S}_i\otimes\mathsf{V} : \mathsf{S}_j] = \mathsf{M}_{ij}$, (i, j) entry of M_{V}. Therefore $\sum_{i=1}^{m} c_i\mathsf{M}_{ij} = \lambda c_j$ for all j, which says

$$[c_1\ \ c_2\ \ \dots\ \ c_m]\mathsf{M}_{\mathsf{V}} = \lambda[c_1\ \ c_2\ \ \dots\ \ c_m].$$

Part (b) follows by induction, with part (a) providing the $\ell = 1$ case. □

Remark 2.5.2. Grothendieck algebras in general do not have to be commutative, so we do need to specify the "right" multiplication above.

2.6. **Eigenvectors for McKay matrices from $\mathsf{K}_0^{\mathbb{C}}(\mathsf{A})$.** To determine additional information about the McKay matrix $\mathsf{M_V}$, we consider the operator $\mathsf{R}_{[\mathsf{V}^*]}$ of right multiplication by $[\mathsf{V}^*]$ on the finite-dimensional complex vector space $\mathsf{K}_0^{\mathbb{C}}(\mathsf{A}) = \mathbb{C} \otimes_{\mathbb{Z}} \mathsf{K}_0(\mathsf{A})$. Suppose $[\mathsf{Y}]$ is an eigenvector for $\mathsf{R}_{[\mathsf{V}^*]}$ with eigenvalue ξ. Then $[\mathsf{Y}][\mathsf{V}^*] = \xi[\mathsf{Y}]$, where $[\mathsf{Y}] = \sum_{i=1}^m y_i[\mathsf{P}_i]$, a $\mathbb{C}$-linear combination of the projective covers. The matrix $\mathsf{Q}_{\mathsf{V}^*}$ has (i,j) entry equal to the multiplicity of $[\mathsf{P}_j]$ in $[\mathsf{P}_i \otimes \mathsf{V}^*] = [\mathsf{P}_i][\mathsf{V}^*]$, which implies

$$[y_1 \ y_2 \ \dots \ y_m]\mathsf{Q}_{\mathsf{V}^*} = \xi[y_1 \ y_2 \ \dots \ y_m]. \tag{2.6.1}$$

Using the fact that $\mathsf{Q}_{\mathsf{V}^*}^{\mathtt{T}} = \mathsf{M}_{(\mathsf{V}^*)^*}$ from Theorem 2.3.3 and (2.6.1), we have

Proposition 2.6.1. *Assume that the* A*-module* V *satisfies* $(\mathsf{V}^*)^* \cong \mathsf{V}$*. Let* $[\mathsf{Y}]$ *be a left eigenvector for* $\mathsf{R}_{[\mathsf{V}^*]}$ *with eigenvalue* ξ *in* $\mathsf{K}_0^{\mathbb{C}}(\mathsf{A})$*, and let* $[y_1 \, y_2 \ \dots \ y_m]$ *be its coordinate vector relative to the basis* $\{[\mathsf{P}_i]\}_{i=1}^m$ *of* $\mathsf{K}_0^{\mathbb{C}}(\mathsf{A})$ *of projective covers. Then*

$$\mathsf{M_V}[y_1 \ y_2 \ \dots \ y_m]^{\mathtt{T}} = \xi[y_1 \, y_2 \ \dots \ y_m]^{\mathtt{T}},$$

so that $[y_1 \ y_2 \ \dots \ y_m]^{\mathtt{T}}$ *is a right eigenvector for* $\mathsf{M_V}$ *with eigenvalue* ξ*.*

Remark 2.6.2. When the antipode S of A has the property that S^2 is an inner automorphism of A, then $(\mathsf{V}^*)^* \cong \mathsf{V}$ holds for all A-modules V (see [27, Lem. 10.2(a)]). Any semisimple Hopf algebra will have $(\mathsf{V}^*)^* \cong \mathsf{V}$ for all V, as will any symmetric algebra [32]. Drinfeld doubles are always symmetric (i.e. have a nondegenerate, symmetric, associative bilinear form). In particular, the Drinfeld double D_n, which we investigate in detail in Section 3, has the property $(\mathsf{V}^*)^* \cong \mathsf{V}$ for all V, and D_n is not semisimple. Also, since the quantum group $\mathfrak{u}_q(\mathfrak{sl}_2)$ for q a root of unity has a unique simple module of each dimension, $(\mathsf{V}^*)^* \cong \mathsf{V}$ holds for all $\mathfrak{u}_q(\mathfrak{sl}_2)$-modules V, and $\mathfrak{u}_q(\mathfrak{sl}_2)$ is not semisimple.

2.7. **Eigenvectors for McKay matrices of semisimple Hopf algebras.** In this section, we assume that the Hopf algebra A is semisimple. The assumption of semisimplicity enables us to say more about the eigenvectors and characters of A.

Following [26] and [39, p. 886], we define

$$\langle \mathsf{X}, \mathsf{Y} \rangle = \mathsf{dim}_{\Bbbk}\mathsf{Hom}_{\mathsf{A}}(\mathsf{X}^*, \mathsf{Y}),$$

for any two A-modules X and Y in $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$, where X^* denotes the dual module of X. This generates a nondegenerate, symmetric associative $\mathbb{C}$-bilinear form on $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$. Therefore, $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ is a symmetric algebra with dual bases $\{\mathsf{S}_1, \mathsf{S}_2, \dots, \mathsf{S}_m\}$ and $\{\mathsf{S}_1^*, \mathsf{S}_2^*, \dots, \mathsf{S}_m^*\}$, where the S_i are representatives of the isomorphism classes of simple A-modules.

Let $\zeta_1, \zeta_2, \dots, \zeta_s$ be the $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$-characters afforded by the simple $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$-modules. Since A is semisimple, so is the Grothendieck algebra $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ (see, for example, [41] and [39]). Therefore, by [13, Sec. 9B] we know that the characters of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ are linearly independent over $\mathbb{C}$. Moreover, if $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_s$ are the primitive central idempotents of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$, then $\zeta_i(\mathcal{E}_i) = \zeta_i(1)$, and $\zeta_i(\mathcal{E}_j) = 0$ for $j \neq i$. The idempotents $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_s$ form a $\mathbb{C}$-basis for the center of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$.

By Proposition 9.17 (i) of [13], the duality of the bases $\{\mathsf{S}_j^*\}$, $\{\mathsf{S}_j\}$ relative to the symmetric bilinear form above can be used to define

$$\mathcal{D}_i = \sum_{j=1}^{m} \zeta_i(\mathsf{S}_j^*)\mathsf{S}_j = \sum_{j=1}^{m} \zeta_i(\mathsf{S}_j)\mathsf{S}_j^*, \qquad \text{for } 1 \leq i \leq s,$$

and to prove that $\zeta_i(\mathcal{D}_i) \neq 0$, Then applying [13, Prop. 9.17 (ii)], we have

Proposition 2.7.1. *Assume that the Hopf algebra* A *is semisimple, and let* $\zeta_1, \zeta_2, \ldots, \zeta_s$ *be the characters of the simple* $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$*-modules. Then for all* $1 \leq i \leq s$*, the primitive central idempotent* $\mathcal{E}_i$ *of* $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ *corresponding to the character* ζ_i *is given by*

$$\mathcal{E}_i = \zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\mathcal{D}_i, \quad \textit{where} \quad \zeta_i(\mathcal{D}_i) = \sum_{j=1}^{m} \zeta_i(\mathsf{S}_j^*)\zeta_i(\mathsf{S}_j) = \zeta_i\left(\bigoplus_{j=1}^{m} \mathsf{S}_j^* \otimes \mathsf{S}_j\right),$$

and S_j*,* $1 \leq j \leq m$*, are the simple* A*-modules.*

Remarks 2.7.2. • In [39], Witherspoon investigated $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ when A is a semisimple, almost cocommutative Hopf algebra. Under these assumptions, $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ is both semisimple and commutative, so the simple modules for $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ are one-dimensional. The expression for the primitive central idempotents coming from [39, (3.4)] is basically the same as the one given in Proposition 2.7.1; however, since $\zeta_i(1) = 1$ for all i when A is semisimple and almost cocommutative, that factor does not appear in [39].
• The primitive central idempotents $\mathcal{E}_i$ form a basis for $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ when A is semisimple and almost cocommutative, and in fact, the simple $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$-modules are exactly the $\mathbb{C}\mathcal{E}_i$. Therefore, $s = m$ in the situation considered in [39]. This is not true when A is an *arbitrary* semisimple Hopf algebra. Each $\mathcal{E}_i$ is the identity element of a matrix block of the semisimple algebra $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$, but these blocks do not have to be one-dimensional. The Hopf algebra that is (14) in Kashina's classification [24] of 16-dimensional semisimple Hopf algebras is an example of this phenomenon. The Grothendieck algebra $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ has a 2×2 matrix block in that case.

Assume A is semisimple and V is an A-module that is central in $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$. Then $\mathsf{V} = \sum_{i=1}^{s} \lambda_i \mathcal{E}_i$, which implies that $\mathcal{E}_i\mathsf{V} = \lambda_i\mathcal{E}_i$, that is, $\mathcal{E}_i$ is an eigenvector for right multiplication by V in $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$. Therefore, by Propositions 2.5.1 and 2.7.1, we have for the McKay matrix $\mathsf{M}_\mathsf{V} = (\mathsf{M}_{ij})$,

$$\begin{aligned}
\lambda_i\mathcal{E}_i = \lambda_i\zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\sum_{j=1}^{m} \zeta_i(\mathsf{S}_j^*)\mathsf{S}_j &= \mathcal{E}_i\mathsf{V} = \zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\sum_{j=1}^{m} \zeta_i(\mathsf{S}_j^*)\mathsf{S}_j\mathsf{V} \\
&= \zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\sum_{j=1}^{m} \zeta_i(\mathsf{S}_j^*)\left(\sum_{\ell=1}^{m} [\mathsf{S}_j \otimes \mathsf{V} : \mathsf{S}_\ell]\mathsf{S}_\ell\right) = \zeta_i(1)\zeta_i(\mathcal{D}_i)^{-1}\sum_{\ell=1}^{m}\left(\sum_{j=1}^{m} \zeta_i(\mathsf{S}_j^*)\mathsf{M}_{j\ell}\right)\mathsf{S}_\ell.
\end{aligned}$$

In other words, $\lambda_i\zeta_i(\mathsf{S}_\ell^*) = \sum_{j=1}^{m} \zeta_i(\mathsf{S}_j^*)\mathsf{M}_{j\ell}$. This says that $[\zeta_i(\mathsf{S}_1^*) \ \ \zeta_i(\mathsf{S}_2^*) \ \ \ldots \ \ \zeta_i(\mathsf{S}_m^*)]$ for $1 \leq i \leq s$, is a left eigenvector for M_V with eigenvalue λ_i when $\mathsf{V} = \sum_{i=1}^{s} \lambda_i\mathcal{E}_i$ is a central element of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$. Hence, we have shown the following

Proposition 2.7.3. *Assume that the Hopf algebra* A *is semisimple, and* V *is an* A*-module that is central in the Grothendieck algebra* $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$*. Let* $\zeta_1, \zeta_2, \ldots, \zeta_s$ *be the simple characters of* $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$*. Then* $[\zeta_i(\mathsf{S}_1^*) \ \ \zeta_i(\mathsf{S}_2^*) \ \ \ldots \ \ \zeta_i(\mathsf{S}_m^*)]$ *is a left eigenvector of* M_V *for* $1 \leq i \leq s$*, where the characters* ζ_i *are evaluated on the dual modules* $\mathsf{S}_1^*, \mathsf{S}_2^*, \ldots, \mathsf{S}_m^*$ *of the nonisomorphic simple* A*-modules* $\mathsf{S}_1, \mathsf{S}_2, \ldots, \mathsf{S}_m$*.*

Remark 2.7.4. When A is semisimple and almost cocommutative (as in [39]), then all the left eigenvectors for M_V for any choice of V are obtained in this fashion, since the $\mathcal{E}_i$ form a $\mathbb{C}$-basis of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$ in this case, and every A-module V can be expressed as $\mathsf{V} = \sum_{i=1}^{s=m} \lambda_i \mathcal{E}_i$ for some $\lambda_i \in \mathbb{C}$. Consequently, the following result holds.

Corollary 2.7.5. *When A is a semisimple, almost cocommutative Hopf algebra, the left eigenvectors of M_V are the same for the McKay matrix of* any *finite-dimensional A-module V, and they can be gotten by evaluating the simple characters ζ_i, $1 \le i \le m$, of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{A})$,*

$$[\zeta_i(\mathsf{S}_1^*)\ \zeta_i(\mathsf{S}_2^*)\ \ldots\ \zeta_i(\mathsf{S}_m^*)], \tag{2.7.1}$$

on the dual modules S_j^* of the nonisomorphic simple A-modules S_j, $1 \le j \le m$.

3. The Drinfeld Double of the Taft Algebra and its Modules

Throughout Section 3, we assume n is an odd integer ≥ 3, $\Bbbk$ is an algebraically closed field of characteristic 0, and q is a primitive nth root of unity in $\Bbbk$. When n is even, the Drinfeld double and its modules are defined similarly, but different behavior is exhibited, and so this case will not be considered in this work.

3.1. **Preliminaries.** The Drinfeld double D_n of the Taft algebra A_n has a presentation as the Hopf algebra over $\Bbbk$ with generators a, b, c, d that satisfy the following relations:

$$\begin{aligned} ba &= qab, & db &= qbd, \\ ca &= qac, & dc &= qcd, \\ bc &= cb, & da - qad &= 1 - bc, \\ a^n &= 0 = d^n, & b^n &= 1 = c^n. \end{aligned} \tag{3.1.1}$$

The coproduct, counit, and antipode of D_n are given by

$$\begin{aligned} \Delta(a) &= a \otimes b + 1 \otimes a, & \Delta(d) &= d \otimes c + 1 \otimes d, \\ \Delta(b) &= b \otimes b, & \Delta(c) &= c \otimes c, \\ \varepsilon(a) &= 0 = \varepsilon(d), & \varepsilon(b) &= 1 = \varepsilon(c), \\ S(a) = -ab^{-1}, \quad S(b) &= b^{-1}, & S(c) = c^{-1}, \quad S(d) &= -dc^{-1}. \end{aligned} \tag{3.1.2}$$

The Taft algebra A_n is the Hopf subalgebra generated by a and b. It follows from (3.1.2) and the fact that Δ is an algebra homomorphism that the elements $b^i c^k$ for $0 \le i, k \le n-1$ are grouplike.

3.1.1. *The simple and projective D_n-modules.* The simple D_n-modules $\mathsf{V}(\ell, r)$ are indexed by a pair (ℓ, r) where $1 \le \ell \le n$ and $r \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ (the integers modulo n). Then $\mathsf{V}(\ell, r)$ is a $\Bbbk$-vector space of dimension ℓ with basis $v_1, v_2, \ldots, v_\ell$ and with D_n-action given by

$$\begin{aligned} &a.v_j = v_{j+1},\ 1 \le j < \ell, && a.v_\ell = 0, \\ &b.v_j = q^{r+j-1} v_j, && c.v_j = q^{j-(r+\ell)} v_j, \quad 1 \le j \le \ell, \\ &d.v_j = \alpha_{j-1}(\ell) v_{j-1},\ 1 < j \le \ell, && d.v_1 = 0, \end{aligned} \tag{3.1.3}$$

where

$$\alpha_i(\ell) = \frac{(q^i - 1)\left(1 - q^{i-\ell}\right)}{q - 1} \qquad \text{for } 1 \le i \le n-1. \tag{3.1.4}$$

From Chen et al. [6, 8], the following hold:

(1) $\mathsf{V}(1, 0)$ is the trivial D_n-module with action given by the counit ε.

(2) $\mathsf{V}(\ell,r)\otimes\mathsf{V}(1,s)\cong\mathsf{V}(\ell,r+s)$.
(3) $\mathsf{V}(\ell,r)\otimes\mathsf{V}(\ell',s)$ is completely reducible if and only if $\ell+\ell'\leq n+1$. In this case, if $m=\mathsf{min}(\ell,\ell')$, then

$$\mathsf{V}(\ell,r)\otimes\mathsf{V}(\ell',s)\cong\bigoplus_{j=1}^{m}\mathsf{V}(\ell+\ell'+1-2j,r+s+j-1). \tag{3.1.5}$$

Let $\mathsf{P}(\ell,r)$ be the projective cover of the simple D_n-module $\mathsf{V}(\ell,r)$. Chen [7] has shown that any indecomposable projective left D_n-module is isomorphic to one of the modules $\mathsf{P}(\ell,r)$ for $1\leq\ell<n$ or to $\mathsf{V}(n,r)$ for some $r\in\mathbb{Z}_n$, and the module $\mathsf{P}(\ell,r)$ for $1\leq\ell<n$ has the following structure. There is a chain of submodules $\mathsf{P}(\ell,r)\supset\mathsf{soc}^2(\mathsf{P}(\ell,r))\supset\mathsf{soc}(\mathsf{P}(\ell,r))\supset(0)$ such that

(1) $\mathsf{soc}(\mathsf{P}(\ell,r))$ is the socle of $\mathsf{P}(\ell,r)$ (the sum of all the simple submodules), and $\mathsf{soc}(\mathsf{P}(\ell,r))\cong\mathsf{V}(\ell,r)$;
(2) $\mathsf{soc}^2(\mathsf{P}(\ell,r))/\mathsf{soc}(\mathsf{P}(\ell,r))\cong\mathsf{V}(n-\ell,r+\ell)\oplus\mathsf{V}(n-\ell,r+\ell)$;
(3) $\mathsf{P}(\ell,r)/\mathsf{soc}^2(\mathsf{P}(\ell,r))\cong\mathsf{V}(\ell,r)$.

Therefore, $[\mathsf{P}(\ell,r)]=2[\mathsf{V}(\ell,r)]+2[\mathsf{V}(n-\ell,r+\ell)]$ in the Grothendieck group $\mathsf{G}_0(\mathsf{D}_n)$, and the dimension of the indecomposable module $\mathsf{P}(\ell,r)$ is $2n$ for $1\leq\ell<n$. Hence, it follows that $[\mathsf{P}(\ell,r)]=[\mathsf{P}(n-\ell,r+\ell)]$ holds in $\mathsf{G}_0(\mathsf{D}_n)$ for all $1\leq\ell<n$ and all $r\in\mathbb{Z}_n$. The modules $\mathsf{V}(n,r)$ for all $r\in\mathbb{Z}_n$ are the only D_n-modules that are both simple and projective.

3.1.2. *The Cartan map for* D_n. We consider an extension of the Cartan map of D_n to a $\mathbb{C}$-linear map (also denoted c), $\mathsf{c}:\mathsf{K}_0^{\mathbb{C}}(\mathsf{D}_n)\to\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$, $[\mathsf{P}]\mapsto[\mathsf{P}]$. Then $\mathsf{c}\big([\mathsf{P}(\ell,r])\big)=[\mathsf{P}(\ell,r)]=2[\mathsf{V}(\ell,r)]+2[\mathsf{V}(n-\ell,\ell+r)]=\mathsf{c}\big([\mathsf{P}(n-\ell,\ell+r)]\big)$ for all $1\leq\ell\leq\frac{n-1}{2}$, and $r\in\mathbb{Z}_n$, and $\mathsf{c}\big([\mathsf{V}(n,r)]\big)=[\mathsf{V}(n,r)]$ for all $r\in\mathbb{Z}_n$. Therefore, the images under c of the basis elements of $\mathsf{K}_0^{\mathbb{C}}(\mathsf{D}_n)$ are $\mathbb{C}$-linearly independent elements of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$, and it follows that $\mathsf{dim}\big(\mathsf{im}(\mathsf{c})\big)=\frac{n(n+1)}{2}$. Since the elements $[\mathsf{P}(\ell,r)]-[\mathsf{P}(n-\ell,\ell+r)]$, $1\leq\ell\leq\frac{n-1}{2}$, $r\in\mathbb{Z}_n$, are linearly independent elements of the kernel of c and $\mathsf{dim}\big(\mathsf{ker}(\mathsf{c})\big)=n^2-\frac{n(n+1)}{2}=\frac{n(n-1)}{2}$, they form a basis for the kernel. To summarize, we have the following result.

Proposition 3.1.1. (a) *The elements* $2[\mathsf{V}(\ell,r)]+2[\mathsf{V}(n-\ell,\ell+r)]$ *for all* $1\leq\ell\leq\frac{n-1}{2}$, $r\in\mathbb{Z}_n$, *and* $[\mathsf{V}(n,r)],r\in\mathbb{Z}_n$, *form a basis for the image* $\mathsf{im}(\mathsf{c})$ *of the Cartan map* $\mathsf{c}:\mathsf{K}_0^{\mathbb{C}}(\mathsf{D}_n)\to\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$. *Therefore,* $\mathsf{dim}\big(\mathsf{im}(\mathsf{c})\big)=\frac{n(n+1)}{2}$.
(b) *The rank of the Cartan matrix* C *of* D_n *is* $\frac{n(n+1)}{2}$.
(c) *The elements* $[\mathsf{P}(\ell,r)]-[\mathsf{P}(n-\ell,\ell+r)]$, $1\leq\ell\leq\frac{n-1}{2}$, $r\in\mathbb{Z}_n$, *form a basis for the kernel of* c, *and* $\mathsf{dim}\big(\mathsf{ker}(\mathsf{c})\big)=\frac{n(n-1)}{2}$.

3.2. **The McKay matrix for tensoring with the D_n-module $\mathsf{V}(2,0)$.** Throughout this section, we assume that V is the two-dimensional simple D_n-module $\mathsf{V}(2,0)$ with basis $\{v_1,v_2\}$. Now it follows from (3.1.5) and [8, Prop. 3.1, Thms. 3.3 and 3.5] that for $\mathsf{V}=\mathsf{V}(2,0)$,

(1) $\mathsf{V}(1,r)\otimes\mathsf{V}=\mathsf{V}(2,r)$;
(2) $\mathsf{V}(\ell,r)\otimes\mathsf{V}=\mathsf{V}(\ell+1,r)\oplus\mathsf{V}(\ell-1,r+1)$ for $2\leq\ell<n$;
(3) $\mathsf{V}(n,r)\otimes\mathsf{V}\cong\mathsf{P}(n-1,r+1)$;
(4) $\mathsf{P}(1,r)\otimes\mathsf{V}\cong\mathsf{P}(2,r)\oplus2\mathsf{V}(n,r+1)$;
(5) $\mathsf{P}(\ell,r)\otimes\mathsf{V}\cong\mathsf{P}(\ell+1,r)\oplus\mathsf{P}(\ell-1,r+1)$ for $2\leq\ell<n-1$;
(6) $\mathsf{P}(n-1,r)\otimes\mathsf{V}\cong\mathsf{P}(n-2,r+1)\oplus2\mathsf{V}(n,r)$.

The McKay matrix for tensoring the simple D_n-modules $\mathsf{V}(\ell, r)$ with $\mathsf{V} := \mathsf{V}(2,0)$ is the $n^2 \times n^2$ matrix $\mathsf{M}_{\mathsf{V}} = \big(\mathsf{M}_{(\ell,r),(\ell',s)}\big)$, whose entry $\mathsf{M}_{(\ell,r),(\ell',s)}$ is given by the composition series multiplicity

$$\mathsf{M}_{(\ell,r),(\ell',s)} = [\mathsf{V}(\ell, r) \otimes \mathsf{V} : \mathsf{V}(\ell', s)].$$

We assume that the numbering of the rows and columns of M_{V} is first by $\ell = 1$, then by $\ell = 2$, etc., and for each ℓ the numbering is $r = 0, 1, \ldots, n-1$; that is, we are numbering by lexicographic order, and we will often simply write M for M_{V} when the choice of V is unambiguous. Using the decomposition formulas above, we observe that the McKay matrix M then can be displayed using $n \times n$ blocks, where $\mathrm{I} = \mathrm{I}_n$ is the $n \times n$ identity matrix, and Z is the $n \times n$ cyclic permutation matrix as presented below

$$(3.2.1) \qquad \mathsf{M} = \begin{pmatrix} 0 & \mathrm{I} & 0 & & \cdots & 0 & 0 \\ \mathrm{Z} & 0 & \mathrm{I} & & \cdots & 0 & 0 \\ 0 & \mathrm{Z} & 0 & \mathrm{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \mathrm{Z} & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \mathrm{I} & 0 \\ 0 & 0 & 0 & 0 \cdots & \mathrm{Z} & 0 & \mathrm{I} \\ 2\mathrm{I} & 0 & 0 & 0 & \cdots & 2\mathrm{Z} & 0 \end{pmatrix} \qquad \mathrm{Z} = \begin{pmatrix} 0 & 1 & 0 & & \cdots & 0 & 0 \\ 0 & 0 & 1 & & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

There are identity matrices on the superdiagonal of M, and the matrix Z is on the subdiagonal of M except for the last row (corresponding to the modules $\mathsf{V}(n, r)$ for $r \in \mathbb{Z}_n$), where the nonzero entries are 2I and 2Z, due to the fact that $\mathsf{V}(n, r) \otimes \mathsf{V}(2, 0) = \mathsf{P}(n-1, r+1)$, which has composition factors $\mathsf{V}(n-1, r+1)$ (twice) and $\mathsf{V}(1, r)$ (twice).

Assume $\mathsf{Y} = \mathsf{diag}\{\mathsf{X}, \mathsf{X}, \ldots, \mathsf{X}\}$, where the $n \times n$ matrix X diagonalizes Z,

$$\mathsf{X}\mathrm{Z}\mathsf{X}^{-1} = \mathrm{D} := \mathsf{diag}\{1, q, \ldots, q^{n-1}\},$$

and q is as before, a primitive nth root of unity in $\Bbbk$ for n odd and ≥ 3. Then

$$(3.2.2) \qquad \mathsf{M}' = \mathsf{Y}\mathsf{M}\mathsf{Y}^{-1} = \begin{pmatrix} 0 & \mathrm{I} & 0 & & \cdots & 0 & 0 \\ \mathrm{D} & 0 & \mathrm{I} & & \cdots & 0 & 0 \\ 0 & \mathrm{D} & 0 & \mathrm{I} & \cdots & 0 & 0 \\ ,\vdots & \vdots & \vdots & & \ddots & \mathrm{I} & 0 \\ 0 & 0 & 0 & \cdots & \mathrm{D} & 0 & \mathrm{I} \\ 2\mathrm{I} & 0 & 0 & \cdots & 0 & 2\mathrm{D} & 0 \end{pmatrix}.$$

3.3. **Characteristic polynomial and characteristic roots of M_{V}, $\mathsf{V} = \mathsf{V}(2, 0)$.** We want to determine the characteristic roots of $\mathsf{M} = \mathsf{M}_{\mathsf{V}}$, which we can do by computing the characteristic roots of M', as they are the same. The advantage to working with M' is that its entries lie in the commutative ring of diagonal matrices, and so usual matrix operations apply.

Consider the characteristic polynomial of M', which can be found by computing

$$\det\left(t\mathrm{I}_{n^2}-\mathsf{M}'\right)=\det\begin{pmatrix} t\mathrm{I} & -\mathrm{I} & 0 & & \cdots & 0 & 0\\ -\mathrm{D} & t\mathrm{I} & -\mathrm{I} & & \cdots & 0 & 0\\ 0 & -\mathrm{D} & t\mathrm{I} & -\mathrm{I} & \cdots & 0 & 0\\ \vdots & \vdots & -\mathrm{D} & \ddots & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & & \ddots & -\mathrm{I} & 0\\ 0 & 0 & 0 & 0\cdots & -\mathrm{D} & t\mathrm{I} & -\mathrm{I}\\ -2\mathrm{I} & 0 & 0 & 0 & \cdots & -2\mathrm{D} & 0 \end{pmatrix}.$$

Define polynomials $\mathcal{U}_k(t,\mathrm{D})$ recursively by

$$\mathcal{U}_0(t,\mathrm{D})=\mathrm{I},\;\; \mathcal{U}_1(t,\mathrm{D})=t\mathrm{I},\;\; \mathcal{U}_k(t,\mathrm{D})=t\mathcal{U}_{k-1}(t,\mathrm{D})-\mathrm{D}\mathcal{U}_{k-2}(t,\mathrm{D}),\;\; k\geq 2. \tag{3.3.1}$$

These polynomials are related to Chebyshev polynomials of the second kind, as we explain in the next section. In computing the determinant of $t\mathrm{I}_{n^2}-\mathsf{M}'$, we will abbreviate $\mathcal{U}_k(t,\mathrm{D})$ as $\mathcal{U}_k$. We perform row operations on $t\mathrm{I}_{n^2}-\mathsf{M}'$ using the matrices $-\mathrm{I}$ on the superdiagonal to eliminate the entries beneath them. Therefore, after using the $-\mathrm{I}$ in the first row and then the $-\mathrm{I}$ in the second row, we have

$$\begin{pmatrix} \mathcal{U}_1 & -\mathrm{I} & 0 & & \cdots & 0 & 0\\ t\mathcal{U}_1-\mathrm{D}\mathcal{U}_0 & 0 & -\mathrm{I} & & \cdots & 0 & 0\\ -\mathrm{D}\mathcal{U}_1 & 0 & t\mathrm{I} & -\mathrm{I} & \cdots & 0 & 0\\ \vdots & \vdots & -\mathrm{D} & \ddots & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & & \ddots & -\mathrm{I} & 0\\ 0 & 0 & 0 & 0\cdots & -\mathrm{D} & t\mathrm{I} & -\mathrm{I}\\ -2\mathrm{I} & 0 & 0 & 0 & \cdots & -2\mathrm{D} & t\mathrm{I} \end{pmatrix},\quad \begin{pmatrix} \mathcal{U}_1 & -\mathrm{I} & 0 & & \cdots & 0 & 0\\ \mathcal{U}_2 & 0 & -\mathrm{I} & & \cdots & 0 & 0\\ t\mathcal{U}_2-\mathrm{D}\mathcal{U}_1 & 0 & 0 & -\mathrm{I} & \cdots & 0 & 0\\ \vdots & \vdots & & \ddots & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & & \ddots & -\mathrm{I} & 0\\ 0 & 0 & 0 & 0\cdots & -\mathrm{D} & t\mathrm{I} & -\mathrm{I}\\ -2\mathrm{I} & 0 & 0 & 0 & \cdots & -2\mathrm{D} & t\mathrm{I} \end{pmatrix},$$

respectively. Continuing, we obtain

$$\begin{pmatrix} \mathcal{U}_1 & -\mathrm{I} & 0 & & \cdots & 0 & 0\\ \mathcal{U}_2 & 0 & -\mathrm{I} & & \cdots & 0 & 0\\ \mathcal{U}_3 & 0 & 0 & -\mathrm{I} & \cdots & 0 & 0\\ \vdots & \vdots & & \ddots & \ddots & 0 & 0\\ \mathcal{U}_{n-2} & \vdots & \vdots & & \ddots & -\mathrm{I} & 0\\ \mathcal{U}_{n-1} & 0 & 0 & 0\cdots & 0 & 0 & -\mathrm{I}\\ -2\mathrm{D}\mathcal{U}_{n-2}-2\mathrm{I} & 0 & 0 & 0 & \cdots & 0 & t\mathrm{I} \end{pmatrix},$$

so that after the final step, the result of using the bottommost $-\mathrm{I}$ on the superdiagonal is

$$\begin{pmatrix} \mathcal{U}_1 & -\mathrm{I} & 0 & & \cdots & 0 & 0\\ \mathcal{U}_2 & 0 & -\mathrm{I} & & \cdots & 0 & 0\\ \mathcal{U}_3 & 0 & 0 & -\mathrm{I} & \cdots & 0 & 0\\ \vdots & \vdots & & \ddots & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & & \ddots & -\mathrm{I} & 0\\ \mathcal{U}_{n-1} & 0 & 0 & 0 & \cdots & 0 & -\mathrm{I}\\ t\mathcal{U}_{n-1}-2\mathrm{D}\mathcal{U}_{n-2}-2\mathrm{I} & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Therefore, setting

$$\mathsf{p}_n(t,\mathrm{D}) := t\mathcal{U}_{n-1}(t,\mathrm{D}) - 2\mathrm{D}\mathcal{U}_{n-2}(t,\mathrm{D}) - 2\mathrm{I} = \mathcal{U}_n(t,\mathrm{D}) - \mathrm{D}\mathcal{U}_{n-2}(t,\mathrm{D}) - 2\mathrm{I}, \tag{3.3.2}$$

we have $\mathsf{det}(t\mathrm{I}_{n^2} - \mathsf{M}') = \mathsf{p}_n(t,\mathrm{D})$, where M' is as in (3.2.2), and the characteristic roots of M', hence also of M, are the roots of $\mathsf{p}_n(t,\mathrm{D})$.

Here are the first few polynomials $\mathsf{p}_n(t,\mathrm{D})$:

$$\begin{aligned}
n = 3: &\quad t^3 - 3\mathrm{D}t - 2\mathrm{I} \\
n = 5: &\quad t^5 - 5\mathrm{D}t^3 + 5\mathrm{D}^2t - 2\mathrm{I} \\
n = 7: &\quad t^7 - 7\mathrm{D}t^5 + 14\mathrm{D}^2t^3 - 7\mathrm{D}^3t - 2\mathrm{I} \\
n = 9: &\quad t^9 - 9\mathrm{D}t^7 + 27\mathrm{D}^2t^5 - 30\mathrm{D}^3t^3 + 9\mathrm{D}^4t - 2\mathrm{I} \\
n = 11: &\quad t^{11} - 11\mathrm{D}t^9 + 44\mathrm{D}^2t^7 - 77\mathrm{D}^3t^5 + 55\mathrm{D}^4t^3 - 11\mathrm{D}^5t - 2\mathrm{I} \\
n = 13: &\quad t^{13} - 13\mathrm{D}t^{11} + 65\mathrm{D}^2t^9 - 156\mathrm{D}^3t^7 + 182\mathrm{D}^4t^5 - 91\mathrm{D}^5t^3 + 13\mathrm{D}^6t - 2\mathrm{I}.
\end{aligned} \tag{3.3.3}$$

These are polynomials with coefficients that are $n \times n$ diagonal matrices. Each diagonal entry q^k of D for $k \in \mathbb{Z}_n$ determines a polynomial $\mathsf{p}_n(t,q^k)$ in q and t with coefficients in $\mathbb{Z}$. The characteristic roots of M are obtained by setting those n polynomials equal to 0. For example, when $n = 7$, the polynomials are $\mathsf{p}_n(t,q^k) = t^7 - 7q^kt^5 + 14q^{2k}t^3 - 7q^{3k}t - 2$ for $k \in \mathbb{Z}_7$, and the characteristic roots of M are the roots of those 7 polynomials.

3.4. **Right eigenvectors for M_{V}, $\mathsf{V} = \mathsf{V}(2,0)$, and Chebyshev polynomials.** The polynomials $\mathsf{p}_n(t,\mathrm{D})$ are related to Chebyshev polynomials $U_k(t)$ of the second kind, which are defined recursively by the formulas

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t) \ \text{ for } k \geq 2. \tag{3.4.1}$$

Setting $\mathcal{U}_k(t) = U_k(\frac{t}{2})$, we have

$$\mathcal{U}_0(t) = 1, \quad \mathcal{U}_1(t) = t, \quad \mathcal{U}_k(t) = t\mathcal{U}_{k-1}(t) - \mathcal{U}_{k-2}(t) \ \text{ for } k \geq 2. \tag{3.4.2}$$

There are a number of closed-form formulas for Chebyshev polynomials of the second kind. Replacing t by $\frac{t}{2}$ in one such formula (see for example, [31, 18.5.10 with $\lambda = 1$] or [14, (23), p. 185]) gives the following expression for $\mathcal{U}_k(t)$:

$$\mathcal{U}_k(t) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{k-j}{j} t^{k-2j}. \tag{3.4.3}$$

The polynomials $\mathcal{U}_k(t,\mathrm{D})$, which were defined in the previous section, satisfy a similar recursion (3.3.1), and as a result,

$$\mathcal{U}_k(t,\mathrm{D}) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{k-j}{j} t^{k-2j}\mathrm{D}^j. \tag{3.4.4}$$

The polynomial $\mathcal{U}_k(t)$ is $\mathcal{U}_k(t,\mathrm{D})$ with D and the $n \times n$ identity matrix I replaced by 1. Thus, when D and I are replaced by 1 in $\mathsf{p}_n(t,\mathrm{D})$, we obtain

$$\mathsf{p}_n(t) := \mathsf{p}_n(t,1) = t\mathcal{U}_{n-1}(t) - 2\mathcal{U}_{n-2}(t) - 2 = \mathcal{U}_n(t) - \mathcal{U}_{n-2}(t) - 2. \tag{3.4.5}$$

Assume now that $n = 2h+1$ for $h \geq 1$. We aim to show

$$\mathsf{p}_n(t) = \mathsf{p}_{2h+1}(t) = (t-2)\mathcal{W}_h^2(t), \tag{3.4.6}$$

where $\mathcal{W}_h(t) = \mathcal{U}_h(t) + \mathcal{U}_{h-1}(t)$ for all $h \geq 1$, by appealing to results on Chebyshev polynomials of the fourth kind.

The sum of two consecutive Chebyshev polynomials of the second kind is a *Chebyshev polynomial of the fourth kind*. These polynomials are defined recursively by the following formulas (see [28, Secs. 1.2.3, 1.2.4]):

$$W_0(t) = 1, \quad W_1(t) = 2t + 1, \quad W_k(t) = 2tW_{k-1}(t) - W_{k-2}(t) \text{ for } k \geq 2. \tag{3.4.7}$$

Thus, they satisfy the same recursion as the polynomials $U_k(t)$, except $W_1(t) = 2t + 1$, while $U_1(t) = 2t$. In particular, $W_k(t) = U_k(t) + U_{k-1}(t)$ for all $k \geq 1$ by [28, (1.54)].

Chebyshev polynomials have integer coefficients and complex roots. Suppose $x = e^{i\theta} \in \mathbb{C}$. Then $z := \frac{x+x^{-1}}{2} = \cos(\theta) \in \mathbb{C}$, and it follows from [28, (1.54)] that the relation

$$U_k(z) = x^k + x^{k-2} + \cdots + x^{-(k-2)} + x^{-k} = \frac{x^{k+1} - x^{-(k+1)}}{x - x^{-1}} \tag{3.4.8}$$

holds. Moreover, by [28, (1.57)],

$$W_k(z) = \frac{x^{\frac{2k+1}{2}} - x^{-\frac{2k+1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} = x^k + x^{k-1} + \cdots + x^{-(k-1)} + x^{-k} = x^{-k}\,\frac{x^{2k+1} - 1}{x - 1}. \tag{3.4.9}$$

Hence, for $\mathcal{U}_k(x+x^{-1}) = U_k(\frac{x+x^{-1}}{2})$ and $\mathcal{W}_k(x+x^{-1}) := W_k(\frac{x+x^{-1}}{2})$, we can conclude the following:

Proposition 3.4.1. *Assume $\mathcal{U}_k(t)$ is defined as in (3.4.2). Set $\mathcal{W}_0(t) = 1 = \mathcal{U}_0(t)$, and let $\mathcal{W}_k(t) = \mathcal{U}_k(t) + \mathcal{U}_{k-1}(t)$ for $k \geq 1$. Let $x = e^{i\theta} \in \mathbb{C}$ be chosen so that $t = x + x^{-1}$. Then for all $k \geq 1$, the following hold:*

(a) $\mathcal{U}_k(t) = x^k + x^{k-2} + \cdots + x^{-(k-2)} + x^{-k} = \frac{x^{k+1}-x^{-(k+1)}}{x-x^{-1}}$;
(b) $\mathcal{W}_k(t) = x^k + x^{k-1} + x^{k-2} + \cdots + x^{k-2} + x^{-(k-1)} + x^{-k} = x^{-k}\,\frac{x^{2k+1}-1}{x-1}$;
(c) $\mathsf{p}_n(t) = \mathcal{U}_n(t) - \mathcal{U}_{n-2}(t) - 2 = (t-2)\mathcal{W}_h^2(t)$ *for* $n = 2h+1$, $h \geq 1$.

Proof. Only the last equality in (c) needs to be verified, and we proceed to show that the two sides of (c) are equal by computing both by induction on h and comparing them. When $h = 1$,

$$\mathcal{W}_1^2(t) = x^2 + 2x + 3 + 2x^{-1} + x^{-2} = (x^2 + 1 + x^{-2}) + 2(x + x^{-1}) + 2 = \mathcal{U}_2(t) + 2\mathcal{U}_1(t) + 2\mathcal{U}_0(t).$$

Assuming the statement

$$\mathcal{U}_{2h}(t) + 2\mathcal{U}_{2h-1}(t) + \cdots + 2\mathcal{U}_1(t) + 2\mathcal{U}_0(t) = \mathcal{W}_h^2(t) \tag{3.4.10}$$

for $h \geq 1$, we have

$$\begin{aligned}
&\mathcal{U}_{2h+2}(t) + 2\mathcal{U}_{2h+1}(t) + 2\mathcal{U}_{2h}(t) + \cdots + 2\mathcal{U}_1(t) + 2\mathcal{U}_0(t) \\
&\quad = \Big(x^{2h+2} + x^{2h} + \cdots + x^{-2h} + x^{-2h-2}\Big) + \Big(2x^{2h+1} + 2x^{2h-1} + \cdots + 2x^{-2h+1} + 2x^{-2h-1}\Big) \\
&\qquad + \Big(x^{2h} + x^{2h-2} + \cdots + x^{-2h+2} + x^{-2h}\Big) + \mathcal{U}_{2h}(t) + 2\mathcal{U}_{2h-1}(t) + \cdots + 2\mathcal{U}_1(t) + 2\mathcal{U}_0(t) \\
&\quad = \Big(x^{2h+2} + 2x^{2h+1} + 2x^{2h} + \cdots + 2 + \cdots + 2x^{-2h} + 2x^{-2h-1} + x^{-2h-2}\Big) \\
&\qquad\qquad + \Big(x^{2h} + 2x^{2h-1} + 3x^{2h-2} + \cdots + (2h+1) + \cdots + 3x^{-2h+2} + 2x^{-2h+1} + x^{-2h}\Big) \\
&\quad = x^{2h+2} + 2x^{2h+1} + 3x^{2h} + \cdots + (2h+3) + \cdots + 3x^{-2h} + 2x^{-2h-1} + x^{-2h-2} \;=\; \mathcal{W}_{h+1}^2(t).
\end{aligned}$$

This completes the induction step and proves (3.4.10).

Now on the other hand, we claim that when $n = 2h + 1 \geq 3$,

$$\mathsf{p}_n(t) = (t-2)\Big(\mathcal{U}_{n-1}(t) + 2\mathcal{U}_{n-2}(t) + 2\mathcal{U}_{n-3}(t) + \cdots + 2\mathcal{U}_0(t)\Big). \tag{3.4.11}$$

We argue this by induction on n. The base case when $n = 3$ follows from a direct calculation. Suppose the statement is true for n. Then by (3.4.5), for $n + 1$ we have

$$\begin{aligned}
\mathsf{p}_{n+1}(t) =& t\mathcal{U}_n(t) - 2\mathcal{U}_{n-1}(t) - 2\\
=&(t-2)\mathcal{U}_n(t) + 2\mathcal{U}_n(t) - 2\mathcal{U}_{n-1}(t) - 2\\
=&(t-2)\mathcal{U}_n(t) + 2\big[t\mathcal{U}_{n-1}(t) - \mathcal{U}_{n-2}(t)\big] - 2\mathcal{U}_{n-1}(t) - 2\\
=&(t-2)\big[\mathcal{U}_n(t) + \mathcal{U}_{n-1}(t)\big] + \mathsf{p}_n(t)\\
=&(t-2)\big[\mathcal{U}_n(t) + \mathcal{U}_{n-1}(t) + \mathcal{U}_{n-1}(t) + 2\mathcal{U}_{n-2}(t) + \cdots + 2\mathcal{U}_0(t)\big]\\
=&(t-2)\big[\mathcal{U}_n(t) + 2\mathcal{U}_{n-1}(t) + 2\mathcal{U}_{n-2}(t) + \cdots + 2\mathcal{U}_0(t)\big].
\end{aligned}$$

Therefore, the assertion $\mathsf{p}_n(t) = (t-2)\mathcal{W}_h^2(t)$ follows by comparing (3.4.10) and (3.4.11). □

Applying (3.4.9) with $k = h$, we deduce

Corollary 3.4.2. *For $h \geq 1$, the roots of $\mathcal{W}_h(t)$ as a polynomial in x are all $x \neq 1$ which are roots of unity in $\mathbb{C}$ of order $2h+1$. As a polynomial in t, the roots of $\mathcal{W}_h(t)$ are all $t = x + x^{-1}$, where x is a root of unity of order $2h+1$ in $\mathbb{C}$ and $x \neq 1$.*

Remark 3.4.3. We are assuming that D_n is defined over an algebraically closed field $\Bbbk$ of characteristic 0, and q is a primitive nth root of unity in $\Bbbk$ for $n = 2h+1$, $h \geq 1$. Since the subfield of $\Bbbk$ generated by 1 and q is isomorphic to the subfield of $\mathbb{C}$ generated by 1 and x, where $x \neq 1$ is a root of unity of order $2h+1$ as in Corollary 3.4.2, we will identify x with q in what follows.

Corollary 3.4.4. *Assume $n \geq 3$, n odd. The characteristic roots of the McKay matrix M in (3.2.1) are $\lambda_{j,r} = q^r(q^j + q^{-j}) = q^r\mathcal{U}_1(q^j + q^{-j})$, where $r \in \mathbb{Z}_n$, $0 \leq j \leq \frac{n-1}{2}$, and q is a primitive nth root of unity. Each root $\lambda_{0,r} = 2q^r$ has multiplicity 1, and each root $\lambda_{j,r}$ for $j \neq 0$ has multiplicity 2.*

The next result gives an expression for right eigenvectors of $\mathsf{M} = \mathsf{M}_{\mathsf{V}}$, $\mathsf{V} = \mathsf{V}(2,0)$, in terms of the Chebyshev polynomials.

Proposition 3.4.5. *Assume $r \in \mathbb{Z}_n$ and $0 \leq j \leq \frac{n-1}{2}$, and let $\mathbf{v}_0$ be the right eigenvector of the matrix Z in (3.2.1) corresponding to the eigenvalue q^{2r} given by $\mathbf{v}_0 = \begin{bmatrix}1 & q^{2r} & \cdots & q^{(n-1)2r}\end{bmatrix}^{\mathtt{T}}$. For $1 \leq \ell \leq n-1$, set $\mathbf{v}_\ell := q^{\ell r}\mathcal{U}_\ell(q^j + q^{-j})\mathbf{v}_0$, where $\mathcal{U}_\ell$ is as in (3.4.2). Then $\mathbf{v}_{j,r} = [\mathbf{v}_0\ \mathbf{v}_1\ \ldots\ \mathbf{v}_{n-1}]^{\mathtt{T}}$ is a right eigenvector of M corresponding to the eigenvalue $\lambda_{j,r} = q^r(q^j + q^{-j})$.*

Proof. We check directly that the given vector $\mathbf{v}_{j,r}$ is a right eigenvector of M, that is, we verify

$$\mathsf{M}\mathbf{v}_{j,r} = \mathsf{M}\begin{bmatrix}\mathbf{v}_0\\ \mathbf{v}_1\\ \vdots\\ \mathbf{v}_{n-1}\end{bmatrix} = \lambda_{j,r}\begin{bmatrix}\mathbf{v}_0\\ \mathbf{v}_1\\ \vdots\\ \mathbf{v}_{n-1}\end{bmatrix} \tag{3.4.12}$$

holds by comparing both sides of (3.4.12). We assume that $\mathbf{v}_0$ is as in the statement of the proposition, and argue this forces $\mathbf{v}_\ell := q^{\ell r}\mathcal{U}_\ell(q^j + q^{-j})\mathbf{v}_0$ to hold for $1 \leq \ell \leq n-1$, where $\mathcal{U}_\ell$ is as in (3.4.2). The comparison involves checking

$$\mathbf{v}_1 \stackrel{?}{=} \lambda_{j,r}\mathbf{v}_0 = q^r(q^j + q^{-j})\mathbf{v}_0 = q^r\mathcal{U}_1(q^j + q^{-j})\mathbf{v}_0, \tag{Row 0}$$

$$\begin{aligned}\mathbf{v}_\ell &\stackrel{?}{=} \lambda_{j,r}\mathbf{v}_{\ell-1} - \mathrm{Z}\mathbf{v}_{\ell-2} && \text{(Row } 1 \le \ell < n-1)\\ &= \lambda_{j,r} q^{(\ell-1)r}\mathcal{U}_{\ell-1}(q^j+q^{-j})\mathbf{v}_0 - \mathrm{Z}q^{(\ell-2)r}\mathcal{U}_{\ell-2}(u^j+u^{-j})\mathbf{v}_0,\\ &= q^{\ell r}[(q^j+q^{-j})\mathcal{U}_{\ell-1}(q^j+q^{-j}) - \mathcal{U}_{\ell-2}(q^j+q^{-j})]\mathbf{v}_0 = q^{\ell r}\mathcal{U}_\ell(q^j+q^{-j})\mathbf{v}_0.\end{aligned}$$

For the final row, we compare $2\mathbf{v}_0+2\mathrm{Z}\mathbf{v}_{n-2}$ with $\lambda_{j,r}\mathbf{v}_{n-1}$, by showing $2\mathbf{v}_0+2\mathrm{Z}\mathbf{v}_{n-2}-\lambda_{j,r}\mathbf{v}_{n-1} = \mathbf{0}$:

$$\begin{aligned}2\mathbf{v}_0 + 2\mathrm{Z}\mathbf{v}_{n-2} - \lambda_{j,r}\mathbf{v}_{n-1} &= 2\mathbf{v}_0 + 2q^{2r}q^{(n-2)r}\mathcal{U}_{n-2}(q^j+q^{-j})\mathbf{v}_0 - \lambda_{j,r}\, q^{(n-1)r}\mathcal{U}_{n-1}(q^j+q^{-j})\mathbf{v}_0\\ &= \left(2 + 2q^{nr}\mathcal{U}_{n-2}(q^j+q^{-j}) - q^{nr}(q^j+q^{-j})\mathcal{U}_{n-1}(q^j+q^{-j})\right)\mathbf{v}_0\\ &= -\mathsf{p}_n(\lambda_{j,0})\mathbf{v}_0 = \mathbf{0},\end{aligned}$$

because $\lambda_{j,0} = q^j + q^{-j}$ is a root of $\mathsf{p}_n(t)$ by Proposition 3.4.6 (b). $\square$

Remark 3.4.6. In the expression for the right eigenvector $\mathbf{v}_{j,r}$ of M in (3.4.12) corresponding to the eigenvalue $\lambda_{j,r} = q^r(q^j+q^{-j})$, the last vector component is

$$\mathbf{v}_{n-1} = q^{(n-1)r}\mathcal{U}_{n-1}(q^j+q^{-j})\mathbf{v}_0 = q^{(n-1)r}\frac{q^{jn}-q^{-jn}}{q^j-q^{-j}}\mathbf{v}_0 = \mathbf{0} \tag{3.4.13}$$

when $j \neq 0$ by Proposition 3.4.1 (a).

3.5. **Generalized right eigenvectors for M_V.** Using the Chebyshev polynomials $\mathcal{U}_k(q^j+q^{-j})$, we now describe generalized right eigenvectors for M.

Theorem 3.5.1. *Assume $r \in \mathbb{Z}_n$ and fix a choice of $j \in \{1, 2, \ldots, \frac{n-1}{2}\}$. Let $\mathbf{v} = \begin{bmatrix}1 & q^{2r} & \cdots & q^{(n-1)2r}\end{bmatrix}^{\mathtt{T}}$ be the right eigenvector of the matrix Z corresponding to the eigenvalue q^{2r} as in Proposition 3.4.5, and set $\mathbf{x}_0 = \mathbf{v}$. For any $1 \le k \le n-1$, assume*

$$\mathbf{x}_k := q^{kr}\mathcal{U}_k\mathbf{v} + q^{(k-1)r}\sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor}(k-2s)\mathcal{U}_{k-1-2s}\mathbf{v}, \tag{3.5.1}$$

where $\mathcal{U}_k$ is shorthand for $\mathcal{U}_k(q^j+q^{-j})$. Then $\mathbf{x}_{j,r} = [\mathbf{x}_0\ \ \mathbf{x}_1\ \ \ldots\ \ \mathbf{x}_{n-2}\ \ \mathbf{x}_{n-1}]^{\mathtt{T}}$ is a generalized right eigenvector of M_V corresponding to the eigenvalue $\lambda_{j,r} = q^r(q^j+q^{-j})$, and $\mathsf{M}_\mathsf{V}\mathbf{x}_{j,r} = \lambda_{j,r}\mathbf{x}_{j,r} + \mathbf{v}_{j,r}$, where $\mathbf{v}_{j,r}$ is the right eigenvector for M_V in Proposition 3.4.5.

Proof. The proof amounts to showing that the matrix equation below holds

$$\mathsf{M}_\mathsf{V}\mathbf{x}_{j,r} = \begin{pmatrix} 0 & \mathrm{I} & 0 & & \cdots & 0 & 0\\ \mathrm{Z} & 0 & \mathrm{I} & & \cdots & 0 & 0\\ 0 & \mathrm{Z} & 0 & \mathrm{I} & \cdots & 0 & 0\\ \vdots & \vdots & \mathrm{Z} & \ddots & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & & \ddots & \mathrm{I} & 0\\ 0 & 0 & 0 & 0\cdots & \mathrm{Z} & 0 & \mathrm{I}\\ 2\mathrm{I} & 0 & 0 & 0 & \cdots & 2\mathrm{Z} & 0\end{pmatrix}\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\\ \vdots\\ \mathbf{x}_{n-3}\\ \mathbf{x}_{n-2}\\ \mathbf{x}_{n-1}\end{bmatrix} = \lambda_{j,r}\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\\ \vdots\\ \mathbf{x}_{n-3}\\ \mathbf{x}_{n-2}\\ \mathbf{x}_{n-1}\end{bmatrix} + \begin{bmatrix}\mathbf{v}\\ q^r\mathcal{U}_1\mathbf{v}\\ q^{2r}\mathcal{U}_2\mathbf{v}\\ \vdots\\ q^{(n-3)r}\mathcal{U}_{n-3}\mathbf{v}\\ q^{(n-2)r}\mathcal{U}_{n-2}\mathbf{v}\\ q^{(n-1)r}\mathcal{U}_{n-1}\mathbf{v}\end{bmatrix}, \tag{3.5.2}$$

when $\mathbf{x}_{j,r}$ has vector components given by (3.5.1).

Row 0 of $\mathsf{M}_\mathsf{V}\mathbf{x}_{j,r}$ says that $\mathbf{x}_1 = \lambda_{j,r}\mathbf{x}_0 + \mathbf{v} = q^r\mathcal{U}_1\mathbf{x}_0 + \mathbf{v}$, which is true for $\mathbf{x}_1$ in (3.5.1). Next we compute rows $k = 2, \ldots, n-1$ of $\mathsf{M}_\mathsf{V}\mathbf{x}_{j,r}$ proceeding by induction to show that $\mathbf{x}_k = \lambda_{j,r}\mathbf{x}_{k-1} - \mathrm{Z}x_{k-2} + q^{(k-1)r}\mathcal{U}_{k-1}\mathbf{v}$ must hold for $2 \le k \le n-1$. To facilitate this, we write

$$\mathbf{x}_k = q^{kr}\mathcal{U}_k\mathbf{v} + q^{(k-1)r}\sum_{s=0}^{\lfloor\frac{k-1}{2}\rfloor}(k-2s)\mathcal{U}_{k-1-2s}\mathbf{v} = q^{kr}\mathcal{U}_k\mathbf{v} + q^{(k-1)r}\Sigma(k), \quad \text{where} \tag{3.5.3}$$

$$\Sigma(k) = \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2s)\mathcal{U}_{k-1-2s}\mathbf{v}. \tag{3.5.4}$$

Now for row k of $\mathsf{M}_{\mathsf{V}}\mathbf{x}_{j,r}$ verifying that (3.5.3) holds involves using the Chebyshev recursion $\mathcal{U}_{k+1} = (q^j + q^{-j})\mathcal{U}_k - \mathcal{U}_{k-1}$ for $1 \leq k \leq n-1$ (which we write $\mathcal{U}_{k+1} = \mathcal{U}_1\mathcal{U}_k - \mathcal{U}_{k-1}$ here for the sake of brevity) and showing that

(3.5.5)

$$\begin{aligned}
\mathbf{x}_{k+1} &= \lambda_{j,r}\mathbf{x}_k - \mathrm{Z}\mathbf{x}_{k-1} + q^{kr}\mathcal{U}_k\mathbf{v} \\
&= q^{(k+1)r}\mathcal{U}_1\mathcal{U}_k\mathbf{v} + q^{kr}\mathcal{U}_1\Sigma(k) - q^{2r}\mathbf{x}_{k-1} + q^{kr}\mathcal{U}_k\mathbf{v}, \\
&= q^{(k+1)r}\left(\mathcal{U}_{k+1}\mathbf{v} + \mathcal{U}_{k-1}\mathbf{v}\right) + q^{kr}\left(\mathcal{U}_1\Sigma(k) + \mathcal{U}_k\mathbf{v}\right) - q^{2r}\left(q^{(k-1)r}\mathcal{U}_{k-1}\mathbf{v} + q^{(k-2)r}\Sigma(k-1)\right) \\
&= q^{(k+1)r}\mathcal{U}_{k+1}\mathbf{v} + q^{kr}\left(\mathcal{U}_1\Sigma(k) - \Sigma(k-1) + \mathcal{U}_k\mathbf{v}\right).
\end{aligned}$$

We see that (3.5.3) will hold for $k+1$, if we can show that

$$\Sigma(k+1) = \mathcal{U}_1\Sigma(k) - \Sigma(k-1) + \mathcal{U}_k\mathbf{v}, \qquad (\mathcal{U}_1 = q^j + q^{-j}). \tag{3.5.6}$$

Case k odd, $k = 2t+1$ for $t \geq 1$. We start from the right-hand side and use the Chebyshev recursion relation. When we encounter a term $\mathcal{U}_\ell\mathbf{v}$ with $\ell < 0$, we assume it is 0 and drop it from the equation. We use the fact that $\lfloor\frac{k-1}{2}\rfloor = t = \lfloor\frac{k}{2}\rfloor$ and $\lfloor\frac{k-2}{2}\rfloor = t-1$. Then

$$\begin{aligned}
&\mathcal{U}_1\Sigma(k) - \Sigma(k-1) + \mathcal{U}_k\mathbf{v} \\
&\qquad = \mathcal{U}_1\sum_{s=0}^{t}(k-2s)\mathcal{U}_{k-1-2s}\mathbf{v} - \sum_{s=0}^{t-1}(k-1-2s)\mathcal{U}_{k-2-2s}\mathbf{v} + \mathcal{U}_k\mathbf{v} \\
&\qquad = \sum_{s=0}^{t}(k-2s)\mathcal{U}_{k-2s}\mathbf{v} + \sum_{s=0}^{t-1}(k-2s)\mathcal{U}_{k-2-2s}\mathbf{v} - \sum_{s=0}^{t-1}(k-1-2s)\mathcal{U}_{k-2-2s}\mathbf{v} + \mathcal{U}_k\mathbf{v} \\
&\qquad = \sum_{s=0}^{t}(k-2s)\mathcal{U}_{k-2s}\mathbf{v} + \sum_{s=0}^{t-1}\mathcal{U}_{k-2-2s} + \mathcal{U}_k\mathbf{v} \\
&\qquad = \sum_{s=0}^{t}(k+1-2s)\mathcal{U}_{k-2s}\mathbf{v} - \sum_{s=0}^{t}\mathcal{U}_{k-2s}\mathbf{v} + \sum_{s=0}^{t-1}\mathcal{U}_{k-2-2s}\mathbf{v} + \mathcal{U}_k\mathbf{v} \\
&\qquad = \sum_{s=0}^{t}(k+1-2s)\mathcal{U}_{k-2s}\mathbf{v} = \sum_{s=0}^{\lfloor\frac{k}{2}\rfloor}(k+1-2s)\mathcal{U}_{k-2s}\mathbf{v} = \Sigma(k+1).
\end{aligned}$$

Case k even. Since the argument just requires minor adjustments to the one above when k is even, we omit the proof.

What remains to be done is to compute the last row of $\mathsf{M}_{\mathsf{V}}\mathbf{x}_{j,r}$, and to show that $2\mathbf{x}_0 + 2\mathrm{Z}\mathbf{x}_{n-2} = \lambda_{j,r}\mathbf{x}_{n-1} + q^{(n-1)r}\mathcal{U}_{n-1}\mathbf{v}$. We will use the relations $\mathrm{Z}\mathbf{x}_k = q^{2r}\mathbf{x}_k$ for all $0 \leq k \leq n-1$ and $\mathcal{U}_{n-1}\mathbf{v} = \mathbf{0}$ and $\mathcal{U}_n\mathbf{v} = \mathbf{v}$ which come from $\mathcal{U}_k\mathbf{v} = \frac{q^{(k+1)j} - q^{-(k+1)j}}{q^j - q^{-j}}\mathbf{v}$ when $k = n-1, n$. Since $\mathcal{U}_{n-1}\mathbf{v} = \mathbf{0}$, what we end up showing is that $2\mathbf{x}_0 + 2\mathrm{Z}\mathbf{x}_{n-2} - \lambda_{j,r}\mathbf{x}_{n-1} = \mathbf{0}$. Now the computation in (3.5.5) for $k = n-1$ with a little rearranging and with $\mathcal{U}_1\Sigma(n-1) - \Sigma(n-2) + \mathcal{U}_{n-1}\mathbf{v}$ replaced with $\Sigma(n) = \sum_{s=0}^{\frac{n-1}{2}} (n-2s)\mathcal{U}_{n-1-2s}\mathbf{v}$ implies

$$\mathrm{Z}\mathbf{x}_{n-2} - \lambda_{j,r}\mathbf{x}_{n-1} = q^{(n-1)r}\mathcal{U}_{n-1}\mathbf{v} - q^{nr}\mathcal{U}_n\mathbf{v} - q^{(n-1)r}\sum_{s=0}^{\frac{n-1}{2}}(n-2s)\mathcal{U}_{n-1-2s}\mathbf{v}$$
$$= -\mathbf{v} - q^{(n-1)r}\sum_{s=1}^{\frac{n-1}{2}}(n-2s)\mathcal{U}_{n-1-2s}\mathbf{v} \qquad (\text{using } \mathcal{U}_{n-1}\mathbf{v} = \mathbf{0} \text{ and } \mathcal{U}_n\mathbf{v} = \mathbf{v}).$$

Therefore, for the last row we have

$$2\mathbf{x}_0 + 2\mathrm{Z}\mathbf{x}_{n-2} - \lambda_{j,r}\mathbf{x}_{n-1} = 2\mathbf{v} + q^{2r}\mathbf{x}_{n-2} - \mathbf{v} - q^{(n-1)r}\sum_{s=1}^{\frac{n-1}{2}}(n-2s)\mathcal{U}_{n-1-2s}\mathbf{v}$$
$$= \mathbf{v} + q^{2r}\left(q^{(n-2)r}\mathcal{U}_{n-2}\mathbf{v} + q^{(n-3)r}\sum_{s=0}^{\frac{n-3}{2}}(n-2-2s)\mathcal{U}_{n-3-2s}\mathbf{v}\right) - q^{(n-1)r}\sum_{s=1}^{\frac{n-1}{2}}(n-2s)\mathcal{U}_{n-1-2s}\mathbf{v}$$
$$= \mathbf{v} + \mathcal{U}_{n-2}\mathbf{v} = \mathbf{v} + \mathcal{U}_{n-2}\mathbf{v} - \frac{1}{2}\mathcal{U}_1\mathcal{U}_{n-1}\mathbf{v} = -\frac{1}{2}\mathsf{p}_n(q^j + q^{-j})\mathbf{v} = \mathbf{0},$$

since $\lambda_{j,0} = q^j + q^{-j}$ is a root of $\mathsf{p}_n(t)$. □

3.6. **Right eigenvectors from grouplike elements of D_n.** In this section, we focus on the grouplike elements b^ic^k, $0 \le i,k \le n-1$, of D_n and compute the trace vectors $\mathsf{Tr}_\mathsf{S}(b^ic^k)$ explicitly. Relation (2.1.5) with $g = b^ic^k$ says $\mathsf{M}_\mathsf{V}\,\mathsf{Tr}_\mathsf{S}(b^ic^k) = \mathsf{tr}_\mathsf{V}(b^ic^k)\,\mathsf{Tr}_\mathsf{S}(b^ic^k)$, so that $\mathsf{Tr}_\mathsf{S}(b^ic^k)$ is a right eigenvector of eigenvalue $\mathsf{tr}_\mathsf{V}(b^ic^k)$ for the McKay matrix M_V, $\mathsf{V} = \mathsf{V}(2,0)$. We identify the eigenvalue $\mathsf{tr}_\mathsf{V}(b^ic^k)$ with $\lambda_{j,r} = q^r(q^j + q^{-j})$ for certain values of j and r that depend on i and k. Since there is a unique right eigenvector of M_V corresponding to the eigenvalue $\lambda_{j,r}$ up to scalar multiples by Theorem 3.5.1, by comparing the vectors $\mathsf{Tr}_\mathsf{S}(b^ic^k)$ with the vectors $\mathbf{v}_{j,r}$ in Section 3.4, we obtain an expression for the characters $\eta_{\ell,s}$ of the simple D_n-modules $\mathsf{V}(\ell,s)$ evaluated on the grouplike elements b^ic^k of D_n in terms of Chebyshev polynomials of the second kind. The trace vectors $\mathsf{Tr}_\mathsf{S}(b^ic^k) = \mathsf{Tr}_\mathsf{S}(b^{-k}c^{-i})$ for $i,k \in \mathbb{Z}_n$ are shown to give a complete set of right eigenvectors for M_V.

The simple D_n-module $\mathsf{V}(\ell,s)$, $1 \le \ell \le n$ and $s \in \mathbb{Z}_n$, has a basis $\{v_1, v_2, \ldots, v_\ell\}$ with D_n-action prescribed by (3.1.3), which implies that b^ic^k has the following character value on $\mathsf{V}(\ell,s)$:

$$\eta_{\ell,s}(b^ic^k) := \mathsf{tr}_{\mathsf{V}(\ell,s)}(b^ic^k) = \sum_{t=1}^{\ell} q^{(s+t-1)i+(t-(s+\ell))k} = q^{(s-1)i-(s+\ell)k}\sum_{t=1}^{\ell} q^{t(i+k)}. \tag{3.6.1}$$

We assume that the simple modules of D_n are ordered so that for a fixed grouplike element g,

$$\begin{aligned} \mathsf{Tr}_\mathsf{S}(g) &= [\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_{n-1}\ \mathbf{u}_n]^\mathsf{T}, \quad \text{where} \\ \mathbf{u}_\ell &= [\eta_{\ell,0}(g)\ \eta_{\ell,1}(g)\ \ldots\ \eta_{\ell,n-1}(g)] \quad \text{for } 1 \le \ell \le n. \end{aligned} \tag{3.6.2}$$

Theorem 3.6.1. *Assume $i,k \in \mathbb{Z}_n$, and $\mathsf{V} = \mathsf{V}(2,0)$. Then*

(a) $\mathsf{Tr}_\mathsf{S}(b^ic^k)$ *is a right eigenvector for* M_V *of eigenvalue* $\lambda_{j,r} = \eta_{2,0}(b^ic^k) = q^i + q^{-k}$ *if*

$$j = \pm\frac{i+k}{2} \pmod n \qquad r = \frac{i-k}{2} \pmod n, \tag{3.6.3}$$

or equivalently, if for $0 \le j \le \frac{n-1}{2}$ *and* $r \in \mathbb{Z}_n$,

$$\begin{array}{ll} i = j + r \\ k = j - r \end{array} \ (\mathsf{mod}\, n), \qquad \textit{or} \qquad \begin{array}{ll} i = -j + r \\ k = -j - r \end{array} \ (\mathsf{mod}\, n). \tag{3.6.4}$$

(b) $\mathsf{Tr}_{\mathsf{S}}(b^i c^k) = \mathbf{v}_{j,r}$, *and for all* $1 \le \ell \le n$ *and* $s \in \mathbb{Z}_n$

$$\eta_{\ell,s}(b^i c^k) = q^{(\ell+s-1)r}\mathcal{U}_{\ell-1}(q^j + q^{-j}), \tag{3.6.5}$$

where j *and* r *are as in* (3.6.3), *and* $\mathbf{v}_{j,r}$ *is as in Proposition 3.4.5.*

(c) $\mathsf{Tr}_{\mathsf{S}}(b^i c^k) = \mathsf{Tr}_{\mathsf{S}}(b^{-k} c^{-i})$ *for all* $i, k \in \mathbb{Z}_n$. *Therefore, the character* $\eta_{\ell,s}$ *of* $\mathsf{V}(\ell, s)$ *satisfies*

$$\eta_{\ell,s}(b^i c^k) = \eta_{\ell,s}(b^{-k} c^{-i}) \tag{3.6.6}$$

for all $1 \le \ell \le n$ *and* $s \in \mathbb{Z}_n$.

(d) *The vectors* $\mathsf{Tr}_{\mathsf{S}}(b^i c^k) = \mathsf{Tr}_{\mathsf{S}}(b^{-k} c^{-i})$ *for* $i, k \in \mathbb{Z}_n$ *give a complete set of right eigenvectors for the McKay matrix* M_{V} *for tensoring with the* D_n*-module* $\mathsf{V} = \mathsf{V}(2, 0)$.

Proof. We already know that $\mathsf{Tr}_{\mathsf{S}}(b^i c^k)$ is a right eigenvector of M_{V} of eigenvalue $\eta_{2,0}(b^i c^k)$ for $i, k \in \mathbb{Z}_n$, (see (1) of Corollary 2.1.3).

(a) By (3.6.1) with $\ell = 2$ and $s = 0$,

$$\eta_{2,0}(b^i c^k) = q^i + q^{-k} = q^{\frac{i-k}{2}}(q^{\frac{i+k}{2}} + q^{-\frac{i+k}{2}}) = q^r(q^j + q^{-j}) = \lambda_{j,r}, \tag{3.6.7}$$

when j and r are as in (3.6.3). Conversely, given $r \in \mathbb{Z}_n$ and $0 \le j \le \frac{n-1}{2}$, it is easy to verify that $\eta_{2,0}(b^i c^k) = \lambda_{j,r}$ for the specified values of i and k in (3.6.4).

(b) Let $\mathsf{Tr}_{\mathsf{S}}(g)$ and $\mathbf{u}_\ell$ be as in (3.6.2) for $g = b^i c^k$, and assume $2r = i - k$ and $2j = i + k\, (\mathsf{mod}\, n)$. We argue first that $\mathbf{u}_1$ is the right eigenvector $\begin{bmatrix} 1 & q^{2r} & q^{4r} & \dots & q^{2(n-1)r} \end{bmatrix}^{\mathsf{T}}$ for the cyclic $n \times n$ matrix Z with corresponding eigenvalue q^{2r}. For this, observe that by (3.6.1),

$$\eta_{1,s}(b^i c^k) = q^{(s-1)i-(s+1)k} q^{i+k} = q^{si-sk}. \tag{3.6.8}$$

Thus, $\dfrac{\eta_{1,s}(b^i c^k)}{\eta_{1,s-1}(b^i c^k)} = q^{i-k} = q^{2r}$ for all $s \in \mathbb{Z}_n$. Since $\eta_{1,0}(b^i c^k) = 1$, we have

$$\mathbf{u}_1 = \begin{bmatrix} 1 & q^{2r} & q^{4r} & \dots & q^{2(n-1)r} \end{bmatrix}^{\mathsf{T}},$$

which is an eigenvector for Z of eigenvalue q^{2r}, and $\eta_{1,s}(b^i c^k) = q^{2sr}$, for all $s \in \mathbb{Z}_n$.

Now the first vector components $\mathbf{u}_1$ and $\mathbf{v}_0$ of $\mathsf{Tr}_{\mathsf{S}}(g)$ and $\mathbf{v}_{j,r}$ are identical, and the subsequent vector components $\mathbf{u}_\ell$ of $\mathsf{Tr}_{\mathsf{S}}(g)$ must satisfy the same relations as the vector components $\mathbf{v}_{\ell-1}$ in the proof of Proposition 3.4.5 for $\ell \ge 1$. Thus, $\mathsf{Tr}_{\mathsf{S}}(b^i c^k) = \mathbf{v}_{j,r}$, where j and r are as in (3.6.3) and

$$\mathbf{u}_\ell = \mathbf{v}_{\ell-1} = q^{(\ell-1)r}\mathcal{U}_{\ell-1}(q^j + q^{-j})\mathbf{v}_0 = q^{(\ell-1)r}\mathcal{U}_{\ell-1}(q^j + q^{-j})\mathbf{u}_1. \tag{3.6.9}$$

Equating component s on both sides for $0 \le s \le n-1$ gives the assertion in (3.6.5) – the character value $\eta_{\ell,s}(b^i c^k)$ of $b^i c^k$ on $\mathsf{V}(\ell, s)$ is given by $\eta_{\ell,s}(b^i c^k) = q^{(\ell+s-1)r}\mathcal{U}_{\ell-1}(q^j + q^{-j})$, where j and r are as in (3.6.3).

(c) We have seen in the proof of (a) that $\eta_{2,0}(b^i c^k) = q^i + q^{-k}$ for any $i, k \in \mathbb{Z}_n$. Therefore, $\eta_{2,0}(b^{-k} c^{-i}) = q^{-k} + q^i = \eta_{2,0}(b^i c^k)$. Since $\mathsf{Tr}_{\mathsf{S}}(b^i c^k)$ and $\mathsf{Tr}_{\mathsf{S}}(b^{-k} c^{-i})$ are two right eigenvectors of M_{V} with the same eigenvalue, and since $\eta_{1,s}(b^i c^k) = q^{si-sk} = \eta_{1,s}(b^{-k} c^{-i})$ for all $s \in \mathbb{Z}_n$ follows from (3.6.8), we obtain as in the proof of (b) that $\mathsf{Tr}_{\mathsf{S}}(b^i c^k) = \mathsf{Tr}_{\mathsf{S}}(b^{-k} c^{-i})$. As a result, $b^i c^k$ and $b^{-k} c^{-i}$ have the same character value on any simple D_n-module $\mathsf{V}(\ell, s)$, that is $\eta_{\ell,s}(b^i c^k) = \eta_{\ell,s}(b^{-k} c^{-i})$ for all $1 \le \ell \le n$, and $s \in \mathbb{Z}_n$.

(d) We know that the McKay matrix M_V has $\frac{n(n+1)}{2}$ distinct eigenvalues $\lambda_{j,r}$, and there are $\frac{n(n+1)}{2}$ right eigenvectors $\mathsf{Tr}_\mathsf{S}(b^i c^k), i, k \in \mathbb{Z}_n$, giving all these distinct eigenvalues, since for a given pair j, r we can take $i = j + r \,(\mathsf{mod}\, n)$ and $k = j - r \,(\mathsf{mod}\, n)$ as in (a). Consequently, the vectors $\mathsf{Tr}_\mathsf{S}(b^i c^k)$ give a complete set of right eigenvectors for M_V. □

3.7. **Generalized right eigenvectors as trace vectors.** The goal of this section is to show that generalized right eigenvectors for the McKay matrix $\mathsf{M} = \mathsf{M}_\mathsf{V}$ for $\mathsf{V} = \mathsf{V}(2,0)$ can also be realized as trace vectors on simple modules, but for traces of non-grouplike elements. Generalized eigenvectors occur only for the eigenvalues $\lambda_{j,r}$ with $j \neq 0$, and the corresponding (generalized) eigenspace is two-dimensional in that case. We will use the coproduct expression for the trace in (2.1.4) and will require quantum integers $[\ell] = 1 + q + \cdots + q^{\ell-1}$, the quantum factorial $[\ell]! = [\ell][\ell-1]\cdots[1]$, and the quantum binomial coefficient

$$\begin{bmatrix} \ell \\ i \end{bmatrix} = \frac{[\ell]!}{[i]!\,[\ell-i]!}$$

for $\ell, i \in \mathbb{Z}_{\geq 0}$, $\ell \geq i$, where $[0] = [0]! = 1$ is understood.

Chen has studied a family of Hopf algebras $\mathsf{H}(p,q)$, where p and q are arbitrary scalars, and Lemma 2.7 of [4] gives an expression for coproduct for the algebra $\mathsf{H}(p,1)$. When q is an nth root of unity, $\mathsf{H}(p,q)$ modulo a certain Hopf ideal is a finite-dimensional quasi-triangular Hopf algebra $\mathsf{H}_n(p,q)$, and the algebra $\mathsf{H}_n(1,q)$ is isomorphic to D_n. Chen's coproduct formula can be modified by replacing binomial coefficients with their q analogues to give a coproduct formula for specific powers of the generators of D_n. We will use these specializations in the next result.

Lemma 3.7.1. *For $\ell \in \mathbb{Z}_{\geq 0}$,*

$$\begin{aligned}\Delta(d^\ell a^\ell) &= \sum_{t=0}^{\ell} \begin{bmatrix} \ell \\ t \end{bmatrix}^2 d^t a^t \otimes b^t c^t d^{\ell-t} a^{\ell-t} + \textit{nilpotent terms} \\ &= d^\ell a^\ell \otimes b^\ell c^\ell + [\ell]^2 d^{\ell-1} a^{\ell-1} \otimes b^{\ell-1} c^{\ell-1} da + \textit{nilpotent terms.}\end{aligned}$$

By nilpotent terms, we mean terms $x \otimes y$ such that y acts as a nilpotent endomorphism on V.

Proof. From [4, Lemma 2.7], we deduce

$$\Delta(a^\ell) = \sum_{i=0}^{\ell} \begin{bmatrix} \ell \\ i \end{bmatrix} a^i \otimes a^{\ell-i} b^i, \qquad \Delta(d^\ell) = \sum_{i=0}^{\ell} \begin{bmatrix} \ell \\ i \end{bmatrix} d^i \otimes c^i d^{\ell-i}.$$

Recall that $ba = qab$, $db = qbd$ in D_n. Therefore,

$$\begin{aligned}\Delta(d^\ell a^\ell) &= \left(\sum_{j=0}^{\ell} \begin{bmatrix} \ell \\ j \end{bmatrix} d^j \otimes c^j d^{\ell-j} \right) \left(\sum_{i=0}^{\ell} \begin{bmatrix} \ell \\ i \end{bmatrix} a^i \otimes a^{\ell-i} b^i \right) \\ &= \sum_{t=0}^{\ell} \begin{bmatrix} \ell \\ t \end{bmatrix}^2 d^t a^t \otimes c^t d^{\ell-t} a^{\ell-t} b^t + \text{nilpotent terms} \qquad (3.7.1) \\ &= \sum_{t=0}^{\ell} \begin{bmatrix} \ell \\ t \end{bmatrix}^2 d^t a^t \otimes b^t c^t d^{\ell-t} a^{\ell-t} + \text{nilpotent terms.}\end{aligned}$$

The term $b^t c^t d^{\ell-t} a^{\ell-t}$ is nilpotent on V only if $\ell - t = 0$ or $\ell - t = 1$, i.e. $t = \ell$ or $t = \ell - 1$, hence

$$\Delta(d^\ell a^\ell) = d^\ell a^\ell \otimes b^\ell c^\ell + [\ell]^2 d^{\ell-1} a^{\ell-1} \otimes b^{\ell-1} c^{\ell-1} da + \text{nilpotent terms.}$$ □

The following result gives a formulation of the generalized right eigenvectors for M_V, $\mathsf{V} = \mathsf{V}(2,0)$, as trace vectors on the simple modules. Recall that $\{\mathsf{Tr}_\mathsf{S}(b^ic^k)\}$ gives a list of right eigenvectors for M_V, with repetitions described by Theorem 3.6.1 (c).

Theorem 3.7.2. *Given $0 \leq i,k \leq n-1$ with $k \neq -i \,(\mathsf{mod}\, n)$, choose $1 \leq s \leq n-1$ such that $s = -(i+k)\,(\mathsf{mod}\, n)$. Let $\gamma_1, \dots, \gamma_s$ be defined recursively by $\gamma_s = 1$ and*

$$\gamma_\ell = \frac{[\ell+1]^2 q^{-1-\ell+s}}{[\ell][s-\ell](q-1)}\gamma_{\ell+1}$$

for $\ell = s-1, s-2, \dots, 1$. Then $\sum_{\ell=1}^s \gamma_\ell \,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell)$ is in the generalized eigenspace of $\mathsf{M} = \mathsf{M}_\mathsf{V}$, $\mathsf{V} = \mathsf{V}(2,0)$, containing $\mathsf{Tr}_\mathsf{S}(b^ic^k)$.

Proof. From the coproduct expression in (3.7.1) and Lemma 2.1.1 (c), we have

$$\mathsf{M}\,\mathsf{Tr}_\mathsf{S}(d^\ell a^\ell) = \mathsf{tr}_\mathsf{V}(b^\ell c^\ell)\,\mathsf{Tr}_\mathsf{S}(d^\ell a^\ell) + [\ell]^2 \mathsf{tr}_\mathsf{V}(b^{\ell-1}c^{\ell-1}da)\,\mathsf{Tr}_\mathsf{S}(d^{\ell-1}a^{\ell-1}).$$

And similarly

$$\mathsf{M}\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell) = \mathsf{tr}_\mathsf{V}(b^{\ell+i}c^{\ell+k})\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell) + \mathsf{tr}_\mathsf{V}(b^{\ell+i-1}c^{\ell+k-1}da)[\ell]^2\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^{\ell-1}a^{\ell-1}).$$

Now from (3.6.7) we know that $\mathsf{tr}_\mathsf{V}(b^ic^k) = q^i + q^{-k}$ for any $i,k \in \mathbb{Z}_n$. Using that fact and the action of the generators of D_n on the basis $\{v_1, v_2\}$ of V in (3.1.3), we have

$$\begin{aligned}
&\mathsf{tr}_\mathsf{V}(b^{\ell+i}c^{\ell+k}) = q^{\ell+i} + q^{-\ell-k},\\
&da.v_1 = \alpha_1(2)v_1 = (1-q^{-1})v_1, \qquad b^{\ell+i-1}c^{\ell+k-1}.v_1 = q^{-\ell-k+1}v_1,\\
&\mathsf{tr}_\mathsf{V}(b^{\ell+i-1}c^{\ell+k-1}da) = (1-q^{-1})q^{-\ell-k+1}.
\end{aligned}$$

Therefore,

$$\mathsf{M}\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell) = (q^{\ell+i} + q^{-\ell-k})\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell) + [\ell^2](1-q^{-1})q^{-\ell-k+1}\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^{\ell-1}a^{\ell-1}). \tag{3.7.2}$$

By a change of the index of summation on the second summand,

$$\begin{aligned}
&\mathsf{M}\left(\sum_{\ell=1}^s \gamma_\ell\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell)\right)\\
&\quad= \sum_{\ell=1}^s (q^{\ell+i}+q^{-\ell-k})\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell)\gamma_\ell + \sum_{\ell=0}^{s-1}[\ell+1]^2(1-q^{-1})q^{-\ell-k}\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell)\gamma_{\ell+1}\\
&\quad= (1-q^{-1})q^{-k}\,\mathsf{Tr}_\mathsf{S}(b^ic^k)\gamma_1 + (q^{s+i}+q^{-s-k})\gamma_s\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^sa^s)\\
&\qquad+ \sum_{\ell=1}^{s-1}\Big((q^{\ell+i}+q^{-\ell-k})\gamma_\ell + [\ell+1]^2(1-q^{-1})q^{-\ell-k}\gamma_{\ell+1}\Big)\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell).
\end{aligned}$$

We claim this is equal to

$$\lambda\left(\sum_{\ell=1}^s \gamma_\ell\,\mathsf{Tr}_\mathsf{S}(b^ic^kd^\ell a^\ell)\right) + \text{constant}\cdot\mathsf{Tr}_\mathsf{S}(b^ic^k),$$

for $\lambda = q^i + q^{-k} = \mathsf{tr}_\mathsf{V}(b^ic^k) = \lambda_{j,r}$, where j and r are as in (3.6.3).

First, the terms agree at $\ell = s$, where we recall that $s = -(i+k)\,(\mathsf{mod}\, n)$ so that

$$\lambda = q^{s+i} + q^{-s-k} = q^{-k} + q^i.$$

When $1 \le \ell \le s-1$, the following is true

$$(q^{\ell+i}+q^{-\ell-k})\gamma_\ell+[\ell+1]^2(1-q^{-1})q^{-\ell-k}\gamma_{\ell+1}=\lambda\gamma_\ell=\gamma_\ell(q^i+q^{-k}),$$

if and only if the following is true:

$$\begin{aligned}\gamma_\ell =&\frac{[\ell+1]^2(1-q^{-1})q^{-\ell-k}\gamma_{\ell+1}}{q^i+q^{-k}-q^{\ell+i}-q^{-\ell-k}}=\frac{[\ell+1]^2(q-1)q^{-1-\ell-k}\gamma_{\ell+1}}{q^i(1-q^\ell)-q^{-\ell-k}(1-q^\ell)}\\ =&\frac{[\ell+1]^2q^{-1-\ell-k}}{[\ell](q^{-\ell-k}-q^{-s-k})}\gamma_{\ell+1}=\frac{[\ell+1]^2q^{-1-\ell-k}}{[\ell]q^{-s-k}(q^{s-\ell}-1)}\gamma_{\ell+1}=\frac{[\ell+1]^2q^{-1-\ell+s}}{[\ell][s-\ell](q-1)}\gamma_{\ell+1}.\end{aligned}$$ □

3.8. **Left eigenvectors and generalized left eigenvectors of M_V, $\mathsf{V}=\mathsf{V}(2,0)$.** In this section, we compute left (generalized) eigenvectors of the McKay matrix $\mathsf{M}=\mathsf{M}_\mathsf{V}$ for $\mathsf{V}=\mathsf{V}(2,0)$ using modified Chebyshev polynomials $\mathcal{L}_j(t)$ that are defined recursively by

(3.8.1) $$\mathcal{L}_0(t)=2,\quad \mathcal{L}_1(t)=t,\quad \mathcal{L}_k(t)=t\mathcal{L}_{k-1}(t)-\mathcal{L}_{k-2}(t)\ \text{ for } k\ge 2.$$

Proposition 3.8.1. *Assume $x\neq 1$, and let $t=x+x^{-1}$ as in Proposition 3.4.1. Then*

(a) *For $k\ge 2$, $\mathcal{L}_k(t)=\mathcal{U}_k(t)-\mathcal{U}_{k-2}(t)$.*
(b) *For all $k\ge 0$, $\mathcal{L}_k(t)=x^k+x^{-k}$.*

Proof. (a) We have $\mathcal{L}_2(t)=t\mathcal{L}_1(t)-\mathcal{L}_0(t)=t^2-2=t^2-1-1=\mathcal{U}_2(t)-\mathcal{U}_0(t)$, and for $k>2$ the proof of (a) is an easy inductive argument starting from this base case. (b) The relation $\mathcal{L}_k(t)=x^k+x^{-k}$ clearly holds for $k=0,1$. Proposition 3.4.1 (a) says that $\mathcal{U}_k(t)=x^k+x^{k-2}+\cdots+x^{-(k-2)}+x^{-k}$ for all $k\ge 2$. Part (b) follows readily from that and part (a). □

Proposition 3.8.2. *For $r\in\mathbb{Z}_n$, let $\mathbf{w}_0\neq\mathbf{0}$ be a left eigenvector of Z corresponding to the eigenvalue q^{2r}. Assume $0\le j\le\frac{n-1}{2}$, and set $\mathbf{w}_k=q^{kr}\mathcal{L}_k(q^j+q^{-j})\mathbf{w}_0=q^{kr}(q^{kj}+q^{-kj})\mathbf{w}_0$ for $1\le k\le n-1$, where $\mathcal{L}_k$ is the modified Chebyshev polynomial defined in* (3.8.1)*. Then $\mathbf{w}_{j,r}=[\mathbf{w}_{n-1}\ \mathbf{w}_{n-2}\ \ldots\ \mathbf{w}_1\ \mathbf{w}_0]$ is a left eigenvector of M corresponding to eigenvalue $\lambda_{j,r}=q^r(q^j+q^{-j})$.*

Proof. The proof amounts to comparing the left and right sides of $\mathbf{w}_{j,r}\mathsf{M}=\lambda_{j,r}\mathbf{w}_{j,r}$ and showing that equality holds when $\mathbf{w}_k=q^{kr}\mathcal{L}_k(q^j+q^{-j})\mathbf{w}_0=q^{kr}(q^{jk}+q^{-jk})\mathbf{w}_0$, where the last "=" results from $\mathcal{L}_k(q^j+q^{-j})=q^{jk}+q^{-jk}$, which is a direct consequence of Proposition 3.8.1 (b).

We expand the left-hand side of $\mathbf{w}_{j,r}\mathsf{M}$ starting from the rightmost column and ask if the two sides of $\mathbf{w}_{j,r}\mathsf{M}=\lambda_{j,r}\mathbf{w}_{j,r}$ are equal:

$$\begin{aligned}\mathbf{w}_1 &\overset{?}{=}\lambda_{j,r}\mathbf{w}_0=q^r(q^j+q^{-j})\mathbf{w}_0=q^r\mathcal{L}_1(q^j+q^{-j})\mathbf{w}_0,\\ \mathbf{w}_2 &\overset{?}{=}\lambda_{j,r}\mathbf{w}_1-\mathbf{w}_0(2\mathsf{Z})=\mathbf{w}_0(\lambda_{j,r}^2-2\mathsf{Z})=\mathbf{w}_0(\lambda_{j,r}^2-2q^{2r})\\ &=\mathbf{w}_0q^{2r}\Big((q^j+q^{-j})^2-2\Big)=q^{2r}\mathcal{L}_2(q^j+q^{-j})\mathbf{w}_0,\\ \mathbf{w}_k &\overset{?}{=}\lambda_{j,r}\mathbf{w}_{k-1}-\mathbf{w}_{k-2}\mathsf{Z}=\lambda_{j,r}q^{(k-1)r}\mathcal{L}_{k-1}(q^j+q^{-j})\mathbf{w}_0-q^{(k-2)r}\mathcal{L}_{k-2}(q^j+q^{-j})\mathbf{w}_0\mathsf{Z}\\ &=q^{kr}\Bigg((q^j+q^{-j})\mathcal{L}_{k-1}(q^j+q^{-j})-\mathcal{L}_{k-2}(q^j+q^{-j})\Bigg)\mathbf{w}_0=q^{kr}\mathcal{L}_k(q^j+q^{-j})\mathbf{w}_0,\ (3\le k\le n-1).\end{aligned}$$

The leftmost column involves comparing $\mathbf{w}_{n-2}\mathsf{Z}+2\mathbf{w}_0$ with $\lambda_{j,r}\mathbf{w}_{n-1}$, which we do by showing that

$$\begin{aligned}\mathbf{w}_{n-2}\mathsf{Z}+2\mathbf{w}_0-\lambda_{j,r}\mathbf{w}_{n-1}&=q^{(n-2)r}\mathcal{L}_{n-2}(q^j+q^{-j})\mathbf{w}_0\mathsf{Z}+2\mathbf{w}_0-\lambda_{j,r}\,q^{(n-1)r}\mathcal{L}_{n-1}(q^j+q^{-j})\mathbf{w}_0\\ &=q^{nr}\mathcal{L}_{n-2}(q^j+q^{-j})\mathbf{w}_0+2\mathbf{w}_0-q^{nr}(q^j+q^{-j})\mathcal{L}_{n-1}(q^j+q^{-j})\mathbf{w}_0\\ &=\big(q^{j(n-2)}+q^{-j(n-2)}\big)\mathbf{w}_0+2\mathbf{w}_0-\big(q^j+q^{-j}\big)\big(q^{j(n-1)}+q^{-j(n-1)}\big)\mathbf{w}_0\end{aligned}$$

$$= (q^{j(n-2)} + q^{-j(n-2)} + 2 - q^{jn} - q^{-j(n-2)} - q^{j(n-2)} - q^{-jn})\mathbf{w}_0 = \mathbf{0}. \qquad \square$$

The next corollary relates the above eigenvector results to the dimension vectors.

Corollary 3.8.3. *The dimension vector* $\mathbf{s} = [\mathsf{dim}(\mathsf{S}_1)\ \mathsf{dim}(\mathsf{S}_2)\ \ldots\ \mathsf{dim}(\mathsf{S}_{n^2})]^{\mathsf{T}}$ *of the simple* D_n*-modules is a right eigenvector corresponding to eigenvalue* $\lambda_{0,0} = 2 = \mathsf{dim}\,(\mathsf{V}(2,0))$*. The dimension vector* $\mathbf{p}^{\mathsf{T}} = [\mathsf{dim}(\mathsf{P}_1)\ \mathsf{dim}(\mathsf{P}_2)\ \ldots\ \mathsf{dim}(\mathsf{P}_{n^2})]$ *of the projective indecomposable* D_n*-modules is a left eigenvector corresponding to the eigenvalue* $\lambda_{0,0}$*.*

Proof. Observe that $\mathbf{v}_0 = [1\,1\ \cdots\ 1\,1]^{\mathsf{T}}$ and $\mathbf{w}_0 = \mathbf{v}_0^{\mathsf{T}}$ are right and left (resp.) eigenvectors of the matrix Z corresponding to the eigenvalue 1, and $\mathcal{U}_k(2) = k+1$ and $\mathcal{L}_k(2) = 2$ for all values of $k \geq 0$. Therefore, by Propositions 3.4.5 and 3.8.2, $\mathbf{v}_k = (k+1)\mathbf{v}_0$ and $\mathbf{w}_k = 2\mathbf{w}_0$ for all $1 \leq k \leq n-1$. Consequently, $\mathbf{v}_{0,0} = \mathbf{s}$ and $\mathbf{w}_{0,0} = \frac{1}{n}\mathbf{p}^{\mathsf{T}}$, and the corresponding eigenvalue is $\lambda_{0,0} = 2 = \mathsf{dim}(\mathsf{V}(2,0))$. $\square$

Remark 3.8.4. Corollary 3.8.3 confirms the result in [20, Sec. 3] mentioned in the Introduction in the specific case of the Drinfeld double D_n of the Taft algebra and the McKay matrix for tensoring with $\mathsf{V}(2,0)$.

Proposition 3.8.5. *Let* $r \in \mathbb{Z}_n$*, and assume* $\mathbf{w} = \mathbf{w}_0$*, where* $\mathbf{w}_0\mathrm{Z} = q^{2r}\mathbf{w}_0$ *as in Proposition 3.8.2. Fix a choice of* $j \in \{1, \ldots, \frac{n-1}{2}\}$ *and set* $\mathcal{U}_k = \mathcal{U}_k(q^j + q^{-j})$ *for all* $k \geq 0$*. Assume* $\mathbf{y}_0 = \mathbf{w}$*,* $\mathbf{y}_1 = q^r\mathcal{U}_1\mathbf{w} + \mathbf{w}$*, and* $\mathbf{y}_k = q^{kr}(\mathcal{U}_k - \mathcal{U}_{k-2})\mathbf{w} + kq^{(k-1)r}\mathcal{U}_{k-1}\mathbf{w}$ *for* $2 \leq k \leq n-1$*, Then* $\mathbf{y}_{j,r} := [\mathbf{y}_{n-1}\ \mathbf{y}_{n-2}\ \ldots\ \mathbf{y}_1\ \mathbf{y}_0]$ *is a generalized left eigenvector for* M_V *corresponding to the eigenvalue* $\lambda_{j,r} = q^r(q^j + q^{-j})$*, and* $\mathbf{y}_{j,r}$ *satisfies*

(3.8.2)

$$\mathbf{y}_{j,r}\mathsf{M} = [\mathbf{y}_{n-1}\ \mathbf{y}_{n-2}\ \ldots\ \mathbf{y}_1\ \mathbf{y}_0]\begin{pmatrix} 0 & \mathrm{I} & 0 & & \cdots & 0 & 0 \\ \mathrm{Z} & 0 & \mathrm{I} & & \cdots & 0 & 0 \\ 0 & \mathrm{Z} & 0 & \mathrm{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \mathrm{Z} & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \mathrm{I} & 0 \\ 0 & 0 & 0 & 0\cdots & \mathrm{Z} & 0 & \mathrm{I} \\ 2\mathrm{I} & 0 & 0 & 0 & \cdots & 2\mathrm{Z} & 0 \end{pmatrix}$$

$$= \lambda_{j,r}\,[\mathbf{y}_{n-1}\ \mathbf{y}_{n-2}\ \ldots\ \mathbf{y}_1\ \mathbf{y}_0] + \Big[q^{(n-1)r}\mathcal{L}_{n-1}\mathbf{w}\ q^{(n-2)r}\mathcal{L}_{n-2}\mathbf{w}\ \ldots\ q^r\mathcal{L}_1\mathbf{w}\ \mathbf{w}\Big] = \lambda_{j,r}\mathbf{y}_{j,r} + \mathbf{w}_{j,r},$$

where $\mathbf{w}_{j,r}$ *is as in Proposition 3.8.2, and* $\mathcal{L}_k = \mathcal{L}_k(q^j + q^{-j})$ *for* $k \geq 1$*.*

Proof. The proof entails comparing the entries on both sides of (3.8.2) starting with column 0 (the rightmost) and proceeding to column $n-1$ (the leftmost). This involves noting that

$$\begin{aligned}\mathbf{y}_1 &= \lambda_{j,r}\mathbf{y}_0 + \mathbf{w} = q^r\mathcal{U}_1\mathbf{w} + \mathbf{w},\\ \mathbf{y}_2 &= \lambda_{j,r}\mathbf{y}_1 - 2\mathbf{y}_0\mathrm{Z} + q^r\mathcal{L}_1\mathbf{w}\\ &= q^r(q^j + q^{-j})\big(q^r\mathcal{U}_1\mathbf{w} + \mathbf{w}\big) - 2q^{2r}\mathbf{w} + q^r\mathcal{L}_1\mathbf{w} = q^{2r}\left(\mathcal{U}_2 - \mathcal{U}_0\right)\mathbf{w} + 2\mathcal{U}_1\mathbf{w},\end{aligned}$$

and then arguing inductively for $k \geq 3$ using the relation $\mathcal{L}_k = \mathcal{U}_k - \mathcal{U}_{k-2}$ to show that

$$\begin{aligned}\mathbf{y}_{k+1} &= \lambda_{j,r}\mathbf{y}_k - \mathbf{y}_{k-1}\mathrm{Z} + q^{kr}\mathcal{L}_k\mathbf{w}\\ &= q^r(q^j + q^{-j})\Big(q^{kr}\left(\mathcal{U}_k - \mathcal{U}_{k-2}\right) + kq^{(k-1)r}\mathcal{U}_{k-1}\Big)\mathbf{w}\end{aligned}$$

$$
\begin{aligned}
&\quad - q^{(k+1)r}\big(\mathcal{U}_{k-1} - \mathcal{U}_{k-3}\big)\mathbf{w} - (k-1)q^{kr}\mathcal{U}_{k-2}\mathbf{w} + q^{kr}\big(\mathcal{U}_k - \mathcal{U}_{k-2}\big)\mathbf{w}\\
&= q^{(k+1)r}\big(\mathcal{U}_{k+1} - \mathcal{U}_{k-1}\big)\mathbf{w} + kq^{kr}(q^j + q^{-j})\mathcal{U}_{k-1}\mathbf{w} - (k-1)q^{kr}\mathcal{U}_{k-2}\mathbf{w} + q^{kr}\big(\mathcal{U}_k - \mathcal{U}_{k-2}\big)\mathbf{w}\\
&= q^{(k+1)r}\big(\mathcal{U}_{k+1} - \mathcal{U}_{k-1}\big)\mathbf{w} + kq^{kr}\big(\mathcal{U}_k + \mathcal{U}_{k-2}\big)\mathbf{w} - (k-1)q^{kr}\mathcal{U}_{k-2}\mathbf{w} + q^{kr}\big(\mathcal{U}_k - \mathcal{U}_{k-2}\big)\mathbf{w}\\
&= q^{(k+1)r}\big(\mathcal{U}_{k+1} - \mathcal{U}_{k-1}\big)\mathbf{w} + (k+1)q^{kr}\mathcal{U}_k\mathbf{w}.
\end{aligned}
$$

It remains to check the leftmost column, which involves verifying that

$$\mathbf{y}_{n-2}\mathrm{Z} + 2\mathbf{y}_0 - \lambda_{j,r}\mathbf{y}_{n-1} - q^{(n-1)r}\mathcal{L}_{n-1}\mathbf{w} = \mathbf{0}.$$

Now by the right-hand side of the above calculation for $\mathbf{y}_{k+1}$ with $k = n-1$, we have

$$
\begin{aligned}
\mathbf{y}_{n-2}\mathrm{Z} &+ 2\mathbf{y}_0 - \lambda_{j,r}\mathbf{y}_{n-1} - q^{(n-1)r}\mathcal{L}_{n-1}\mathbf{w}\\
&= -q^{nr}(\mathcal{U}_n - \mathcal{U}_{n-2})\mathbf{w} - nq^{(n-1)r}\mathcal{U}_{n-1}\mathbf{w} + 2\mathbf{w}\\
&= -\mathcal{L}_n\mathbf{w} - \mathbf{0} + 2\mathbf{w} = -(q^{nj} + q^{-nj})\mathbf{w} + 2\mathbf{w} = \mathbf{0},
\end{aligned}
$$

since $\mathcal{U}_{n-1}\mathbf{w} = \frac{q^{nj}-q^{-nj}}{q^j-q^{-j}}\mathbf{w} = \mathbf{0}$ by Proposition 3.4.1 (a). □

3.9. **Left eigenvectors from grouplike elements of D_n.** Next we determine the vector $\mathsf{Tr}_\mathsf{P}(g)$ explicitly for $g = b^ic^k$, a grouplike element of D_n. In contrast to the situation for right eigenvectors, only the n vectors $\mathsf{Tr}_\mathsf{P}(b^ic^{-i})$, $i \in \mathbb{Z}_n$, are nonzero.

We assume an ordering $\mathsf{P}_1, \mathsf{P}_2, \dots, \mathsf{P}_{n^2}$ of the projective indecomposable D_n-modules $\mathsf{P}(\ell, r)$, $1 \le \ell \le n-1$, $\mathsf{V}(n,r)$, $r \in \mathbb{Z}_n$, first by ℓ and then by r. Since the dual of the simple module $\mathsf{V}(\ell, r)$ is $\mathsf{V}(\ell, 1-r-\ell)$ by [22, Thm. 4.3], we see $\big(\mathsf{V}(\ell,r)^*\big)^* \cong \mathsf{V}(\ell,r)$, so that $(\mathsf{V}^*)^* \cong \mathsf{V}$ holds for any finite-dimensional D_n-module. Using that fact, we have from (2.4.1) that for every grouplike element of D_n and every D_n-module V, $\mathsf{Tr}_\mathsf{P}(g)\mathsf{M}_\mathsf{V} = \mathsf{tr}_{\mathsf{V}^*}(g)\,\mathsf{Tr}_\mathsf{P}(g)$, where $\mathsf{Tr}_\mathsf{P}(g) = \Big[\mathsf{tr}_{\mathsf{P}_1}(g)\ \ \mathsf{tr}_{\mathsf{P}_2}(g)\ \ \dots\ \ \mathsf{tr}_{\mathsf{P}_{n^2}}(g)\Big]$. Hence for every grouplike element of D_n, $\mathsf{Tr}_\mathsf{P}(g)$ is a left eigenvector of M_V of eigenvalue $\mathsf{tr}_{\mathsf{V}^*}(g)$.

To compute the vector $\mathsf{Tr}_\mathsf{P}(g)$ for $g = b^ic^k$, we use the explicit description of the projective modules $\mathsf{P}(\ell,r)$ given in [5, Lem. 2.1], showing for $1 \le \ell < n$ that $\mathsf{P}(\ell,r)$ has a basis $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{2n}$ such that the actions of b and c are diagonal:

$$(b^ic^k).\mathbf{p}_t = \begin{cases} q^{(r+t-1)i+(t-r-\ell)k}\mathbf{p}_t & \text{for } 1 \le t \le n,\\ q^{(r+t-1+\ell)i+(t-r)k}\mathbf{p}_t & \text{for } n+1 \le t \le 2n.\end{cases}$$

Thus, for $\ell < n$,

$$\mathsf{tr}_{\mathsf{P}(\ell,r)}(b^ic^k) = q^{(r-1)i+(-r-\ell)k}\sum_{t=1}^{n} q^{t(i+k)} + q^{(r-1+\ell)i-rk}\sum_{t=n+1}^{2n} q^{t(i+k)}.$$

As shown in (3.6.1), we have for $\mathsf{V}(n,r)$,

$$\mathsf{tr}_{\mathsf{V}(n,r)}(b^ic^k) = q^{(r-1)i-rk}\sum_{t=1}^{n} q^{t(i+k)}.$$

Observe that when $i+k \ne 0 \pmod n$, the trace in these expressions is 0; consequently, $\mathsf{Tr}_\mathsf{P}(b^ic^k) = 0$, when $i+k \ne 0 \pmod n$. Hence, we may assume that $k = -i \pmod n$ and obtain

$$
\begin{aligned}
&\mathsf{tr}_{\mathsf{P}(\ell,r)}(b^ic^{-i}) = \Big(q^{(r-1)i+(r+\ell)i} + q^{(r-1+\ell)i+ri}\Big)\,n = 2nq^{(2r+\ell-1)i} \qquad \text{for } \ell < n,\\
&\mathsf{tr}_{\mathsf{V}(n,r)}(b^ic^{-i}) = q^{(r-1)i+ri}\sum_{t=1}^{n} q^{t(i+-i)} = nq^{(2r-1)i}.
\end{aligned}
\tag{3.9.1}
$$

Except for the extra factor of 2 that occurs when $\ell < n$, the second equation in (3.9.1) is the same as the first one with $\ell = n$. This motivates us to define for a fixed value of i, a vector with the following n components:

$$\mathsf{t}_{\ell,i} = \begin{cases} \begin{bmatrix} 2nq^{(\ell-1)i} & 2n^{(\ell+1)i} & 2n^{(\ell+3)i} & \dots & 2nq^{(\ell+n-3)i} \end{bmatrix} & \ell \neq n, \\ \begin{bmatrix} nq^{(n-1)i} & nq^{i} & nq^{3i} & \dots & nq^{(n-3)i} \end{bmatrix} & \ell = n. \end{cases} \tag{3.9.2}$$

From the calculations above, we can conclude that the following holds.

Proposition 3.9.1. *Assume $i \in \mathbb{Z}_n$, and let $\mathsf{t}_{\ell,i}$ be as in* (3.9.2) *for $1 \leq \ell \leq n$. Then*

$$\mathsf{Tr}_{\mathsf{P}}(b^i c^{-i}) = \begin{bmatrix} \mathsf{t}_{1,i} & \mathsf{t}_{2,i} & \dots & \mathsf{t}_{n,i} \end{bmatrix}$$

is a left eigenvector of the McKay matrix M_{V} corresponding to the eigenvalue $\mathsf{tr}_{\mathsf{V}^*}(b^i c^{-i})$ for any finite-dimensional D_n-module V.

Remark 3.9.2. In comparison to the right eigenvector situation in Theorem 3.6.1 (d), no nonzero left eigenvectors corresponding to the eigenvalues $q^r(q^j + q^{-j})$ for $j \neq 0$ arise from evaluating projective characters on grouplike elements of D_n.

Example 3.9.3. When $\mathsf{V} = \mathsf{V}(2,0)$, we have $\mathsf{V}^* = \mathsf{V}(2,-1)$, and (3.1.3) tells us that the matrix of $b^i c^{-i}$ on V^* relative to the basis $\{v_1, v_2\}$ is

$$\begin{pmatrix} q^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{i} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-i} = \begin{pmatrix} q^{-i} & 0 \\ 0 & q^{-i} \end{pmatrix}.$$

Hence, $\mathsf{tr}_{\mathsf{V}^*}(b^i c^{-i}) = 2q^{-i} = \lambda_{0,(n-1)i}$. Since these eigenvalues are distinct for $i \in \mathbb{Z}_n$, the left eigenvectors $\mathsf{Tr}_{\mathsf{P}}(b^i c^{-i})$ for M_{V} are linearly independent. When $n = 3$, the vectors $\mathsf{Tr}_{\mathsf{P}}(b^i c^{-i})$ and their corresponding eigenvalues are

$$\begin{aligned} \mathsf{Tr}_{\mathsf{P}}(1) &= [6\ \ 6\ \ 6\ \ 6\ \ 6\ \ 6\ \ 3\ \ 3\ \ 3] & \lambda_{0,0} &= 2, \\ \mathsf{Tr}_{\mathsf{P}}(bc^{-1}) &= [6\ \ 6q\ \ 6q^2\ \ 6q^2\ \ 6\ \ 6q\ \ 3q\ \ 3q^2\ \ 3] & \lambda_{0,1} = \lambda_{0,-2} &= 2q, \\ \mathsf{Tr}_{\mathsf{P}}(b^2c^{-2}) &= [6\ \ 6q^2\ \ 6q\ \ 6q\ \ 6\ \ 6q^2\ \ 3q^2\ \ 3q\ \ 3] & \lambda_{0,2} = \lambda_{0,-1} &= 2q^2. \end{aligned} \tag{3.9.3}$$

3.10. **Tensoring with $\mathsf{V}(\ell, s)$.** Let $g = [\mathsf{V}(1,1)]$ and $x = [\mathsf{V}(2,0)]$ in the Grothendieck ring $\mathsf{G}_0(\mathsf{D}_n)$. Then $g^n = 1 = [\mathsf{V}(1,0)]$ and $x^k = [\mathsf{V}(2,0)]^k = [\mathsf{V}^{\otimes k}] = [\mathsf{V}]^k$ in $\mathsf{G}_0(\mathsf{D}_n)$. The following results are consequences of Lemmas 3.1-3.3 of [40] and the tensor rules in Section 3.2:

$$[\mathsf{V}(\ell,0)] = x[\mathsf{V}(\ell-1,0)] - g[\mathsf{V}(\ell-2,0)] \quad \text{for all } 3 \leq \ell \leq n, \tag{3.10.1}$$

and since $[\mathsf{V}(\ell,s)] = g^s[\mathsf{V}(\ell,0)]$ for all $s \in \mathbb{Z}_n$, it follows that $\mathsf{G}_0(\mathsf{D}_n)$ is generated by g and x. Moreover,

$$[\mathsf{V}(\ell,0)] = \sum_{i=0}^{\lfloor \frac{\ell-1}{2} \rfloor} (-1)^i \binom{\ell-1-i}{i} g^i x^{\ell-1-2i} \quad \text{for all } 1 \leq \ell \leq n. \tag{3.10.2}$$

Thus, this defines a sequence of elements of $\mathsf{G}_0(\mathsf{D}_n)$ given by

$$\begin{aligned} &f_0(x,g) = 1 = [\mathsf{V}(1,0)], \quad f_1(x,g) = x = [\mathsf{V}(2,0)], \\ &f_{\ell-1}(x,g) \;=\; [\mathsf{V}(\ell,0)] \;=\; \sum_{i=0}^{\lfloor \frac{\ell-1}{2} \rfloor} (-1)^i \binom{\ell-1-i}{i} g^i x^{\ell-1-2i}, \qquad 2 \leq \ell \leq n, \end{aligned} \tag{3.10.3}$$

and satisfying

$$f_\ell(x,g) = x f_{\ell-1}(x,g) - g f_{\ell-2}(x,g) = \sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} (-1)^i \binom{\ell-i}{i} g^i x^{\ell-2i}, \qquad 2 \le \ell \le n.$$

In addition, if $f(x,g) := f_n(x,g) - g f_{n-2}(x,g) - 2$, then by Lemma 3.3 of [40] we have

$$f(x,g) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \frac{n}{n-i} \binom{n-i}{i} g^i x^{n-2i} \ - 2 = 0. \tag{3.10.4}$$

If R is the group algebra $\mathbb{Z}\mathsf{G}$, where G is the cyclic group generated by g, then $\mathsf{G}_0(\mathsf{D}_n) \cong R[x]/\langle f(x,g) \rangle$ is a commutative $\mathbb{Z}$-algebra of dimension n^2 with a basis given by $\{g^i x^k \mid i \in \mathbb{Z}_n, 0 \le k \le n-1\}$. Under the correspondence $1 \leftrightarrow \mathrm{I} = \mathrm{I}_n,\ t\mathrm{I} \leftrightarrow x,\ \mathrm{D} \leftrightarrow g$, we have that $\mathcal{U}_k(t,\mathrm{D}) \leftrightarrow f_k(x,g)$ for $0 \le k \le n$, since the recursion relations are the same. Moreover, $\mathsf{p}_n(t,\mathrm{D}) \leftrightarrow f(x,g)$ implies that

$$\mathsf{p}_n(t,\mathrm{D}) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \frac{n}{n-i} \binom{n-i}{i} \mathrm{D}^i t^{n-2i} \ - 2 = 0. \tag{3.10.5}$$

These considerations give the following result (compare [40, Thm. 3.4]).

Proposition 3.10.1. *The Grothendieck ring* $\mathsf{G}_0(\mathsf{D}_n) \cong \mathbb{Z}[g,x]/\langle g^n - 1, f(x,g) \rangle$.

Example 3.10.2. When $n = 11$, (3.10.4) says

$$\begin{aligned} f(x,g) &= x^{11} - \frac{11}{10}\binom{10}{1} g x^9 + \frac{11}{9}\binom{9}{2} g^2 x^7 - \frac{11}{8}\binom{8}{3} g^3 x^5 + \frac{11}{7}\binom{7}{4} g^4 x^3 - \frac{11}{6}\binom{6}{5} g^5 x - 2 \\ &= x^{11} - 11 g x^9 + 44 g^2 x^7 - 77 g^3 x^5 + 55 g^4 x^3 - 11 g^5 x - 2, \ \text{ and (3.10.5) says} \\ \mathsf{p}_n(t,\mathrm{D}) &= t^{11} - 11\mathrm{D}t^9 + 44\mathrm{D}^2 t^7 - 77\mathrm{D}^3 t^5 + 55\mathrm{D}^4 t^3 - 11\mathrm{D}^5 t - 2\mathrm{I} = 0 \ \text{ (compare (3.3.3)).} \end{aligned}$$

Now suppose that $\mathsf{M}_{(\ell,s)}$ is the McKay matrix for tensoring with $\mathsf{V}(\ell,s)$. In computing matrices here, we order the rows and columns as usual, first by ℓ and then by s, so that the order is $(1,0),(1,1),\ldots,(1,n-1),(2,1),\ldots,(2,n-1),\ldots,(n,0),\ldots,(n,n-1)$ as before. Let $\mathrm{Z}^{(s)} = \mathsf{diag}\{\mathrm{Z}^s, \mathrm{Z}^s, \ldots, \mathrm{Z}^s\}$ (n copies), where Z is the cyclic permutation matrix in (3.2.1), and let I_{n^2} be the $n^2 \times n^2$ identity matrix. Assume $\mathsf{M} = \mathsf{M}_{(2,0)}$, the McKay matrix for tensoring with $\mathsf{V} = \mathsf{V}(2,0)$. Then (3.10.2) implies that

$$\mathsf{M}_{(\ell,s)} = \sum_{i=0}^{\lfloor \frac{\ell-1}{2} \rfloor} (-1)^i \binom{\ell-1-i}{i} \mathrm{Z}^{(i+s)} \mathsf{M}^{\ell-1-2i} \quad \text{for all } 1 \le \ell \le n, s \in \mathbb{Z}_n. \tag{3.10.6}$$

Here are a few special cases:

$$\begin{gathered} \mathsf{M}_{(1,0)} = \mathrm{I}_{n^2}, \qquad \mathsf{M}_{(2,0)} = \mathsf{M}, \qquad \mathsf{M}_{(3,0)} = \mathsf{M}^2 - \mathrm{Z}^{(1)}, \\ \mathsf{M}_{(4,0)} = \mathsf{M}^3 - 2\mathrm{Z}^{(1)}\mathsf{M}, \qquad \mathsf{M}_{(5,0)} = \mathsf{M}^4 - 3\mathrm{Z}^{(1)}\mathsf{M}^2 + \mathrm{Z}^{(2)}. \end{gathered}$$

Corollary 3.10.3. *The left and right (generalized) eigenvectors are the same for all the matrices* $\mathsf{M}_{(\ell,s)}$, $1 \le \ell \le n$, $s \in \mathbb{Z}_n$.

Proof. Since M and $\mathrm{Z}^{(1)}$ commute, we can simultaneously upper-triangularize them and find a basis of common right (generalized) eigenvectors for them. Similarly, we can simultaneously lower-triangularize M and $\mathrm{Z}^{(1)}$ and find a basis of common left (generalized) eigenvectors. These vectors will be common left and right (generalized) eigenvectors for all powers of M and $\mathrm{Z}^{(1)}$, hence for all the matrices $\mathsf{M}_{(\ell,s)} = \mathrm{Z}^{(s)}\mathsf{M}_{(\ell,0)}$. □

3.11. **Eigenvalues for the McKay matrix of any simple D_n-module.** In this section, we use the results above to determine the eigenvalues of the McKay matrix $\mathsf{M}_{(\ell,s)}$ for all $1 \le \ell \le n$ and $s \in \mathbb{Z}_n$. Recall from Proposition 3.4.5 that $\mathbf{v}_{j,r} = [\mathbf{v}_0\ \mathbf{v}_1\ \ \dots\ \ \mathbf{v}_{n-1}]^{\mathsf{T}}$ is a right eigenvector for $\mathsf{M} = \mathsf{M}_{(2,0)}$ with eigenvalue $\lambda_{j,r} = q^r(q^j + q^{-j})$ when $\mathbf{v}_0$ is a right eigenvector for Z with eigenvalue q^{2r}, and $\mathbf{v}_k = q^{kr}\mathcal{U}_k(q^j + q^{-j})\mathbf{v}_0$ for $1 \le k \le n-1$. It follows that $\mathsf{Z}^{(s)}\mathbf{v}_{j,r} = q^{2sr}\mathbf{v}_{j,r}$ for all $s \in \mathbb{Z}_n$. As a consequence, we have the next result by combining (3.10.6) with (3.4.3).

Theorem 3.11.1. (a) *Assume $\mathbf{v}_{j,r} = \begin{bmatrix}\mathbf{v}_0 & \mathbf{v}_1 & \dots & \mathbf{v}_{n-1}\end{bmatrix}^{\mathsf{T}}$ is a right eigenvector for $\mathsf{M} = \mathsf{M}_{(2,0)}$ with eigenvalue $\lambda_{j,r} = q^r(q^j + q^{-j})$ as in Proposition 3.4.5. Then*

(3.11.1)
$$\begin{aligned}\mathsf{M}_{(\ell,0)}\mathbf{v}_{j,r} &= \sum_{i=0}^{\lfloor\frac{\ell-1}{2}\rfloor} (-1)^i\binom{\ell-1-i}{i} q^{2ir}q^{(\ell-1-2i)r}(q^j+q^{-j})^{\ell-1-2i}\mathbf{v}_{j,r}\\ &= q^{(\ell-1)r}\sum_{i=0}^{\lfloor\frac{\ell-1}{2}\rfloor} (-1)^i\binom{\ell-1-i}{i}(q^j+q^{-j})^{\ell-1-2i}\mathbf{v}_{j,r} = q^{(\ell-1)r}\mathcal{U}_{\ell-1}(q^j+q^{-j})\mathbf{v}_{j,r}.\end{aligned}$$

Hence, for all $1 \le \ell \le n$ and for all $s \in \mathbb{Z}_n$, $\mathbf{v}_{j,r}$ is a right eigenvector for $\mathsf{M}_{(\ell,s)} = \mathsf{Z}^{(s)}\mathsf{M}_{(\ell,0)}$ with eigenvalue

$$q^{(\ell-1+2s)r}\mathcal{U}_{\ell-1}(q^j+q^{-j}) = q^{(\ell-1+2s)r}\frac{q^{j\ell}-q^{-j\ell}}{q^j-q^{-j}} \quad \textit{when } j \neq 0,$$

and with eigenvalue $q^{(\ell-1+2s)r}\ell$ when $j = 0$.

(b) *Assume $\mathbf{w}_{j,r} = [\mathbf{w}_{n-1}\ \mathbf{w}_{n-2}\ \ \dots\ \ \mathbf{w}_1\ \mathbf{w}_0]$ is a left eigenvector for M with eigenvalue $q^r(q^j + q^{-j})$ as in Proposition 3.8.2. Then $\mathbf{w}_{j,r}$ is a left eigenvector for $\mathsf{M}_{(\ell,s)} = \mathsf{Z}^{(s)}\mathsf{M}_{(\ell,0)}$ with eigenvalue $q^{(\ell-1+2s)r}(q^j+q^{-j})$ for all $1 \le \ell \le n$ and for all $s \in \mathbb{Z}_n$.*

Proof. Relation (3.11.1) follows directly from (3.11.1) and the fact that $\mathbf{v}_{j,r}$ is a right eigenvector for $\mathsf{Z}^{(s)}$ with eigenvalue q^{2sr}. The last equality in (3.11.1) comes from Proposition 3.4.1 (a). Part (b) is similar. □

3.12. **Eigenvectors for the projective McKay matrices of D_n.** Recall from Section 2.2 that the projective McKay matrix for tensoring with a finite-dimensional module V is given by $\mathsf{Q}_{\mathsf{V}} = (\mathsf{Q}_{ij})$, where $\mathsf{Q}_{ij} = [\mathsf{P}_i \otimes \mathsf{V} : \mathsf{P}_j]$. We have shown in Theorem 2.3.3 that $\mathsf{Q}_{\mathsf{V}} = \mathsf{M}_{\mathsf{V}^*}^{\mathsf{T}}$, where $\mathsf{M}_{\mathsf{V}^*}$ is the McKay matrix for tensoring with the dual module V^*. In particular, since for the simple D_n-module $\mathsf{V}(\ell,s)$, we have $\mathsf{V}(\ell,s)^* = \mathsf{V}(\ell, 1-\ell-s)$, we can conclude the following about $\mathsf{Q}_{(\ell,s)}$ using Theorem 3.11.1.

Proposition 3.12.1. *Assume $\mathbf{v}_{j,r}$ and $\mathbf{w}_{j,r}$ are as in Theorem 3.11.1. Then for all $1 \le \ell \le n$ and for all $s \in \mathbb{Z}_n$,*

(a) *$\mathbf{v}_{j,r}^{\mathsf{T}} = [\mathbf{v}_0\ \mathbf{v}_1\ \ \dots\ \ \mathbf{v}_{n-1}]$ is a left eigenvector for $\mathsf{Q}_{(\ell,s)}$ with eigenvalue*

(3.12.1)
$$q^{(1-\ell-2s)r}\mathcal{U}_{\ell-1}(q^j+q^{-j}) = q^{(1-\ell-2s)r}\frac{q^{j\ell}-q^{-j\ell}}{q^j-q^{-j}} \quad \textit{when } j \neq 0,$$

and with eigenvalue $q^{(1-\ell-2s)r}\ell$ when $j = 0$.

(b) *$\mathbf{w}_{j,r}^{\mathsf{T}} = [\mathbf{w}_{n-1}\ \mathbf{w}_{n-2}\ \ \dots\ \ \mathbf{w}_1\ \mathbf{w}_0]^{\mathsf{T}}$ is a right eigenvector for $\mathsf{Q}_{(\ell,s)}$ with eigenvalue as in (a) for all $1 \le \ell \le n$ and for all $s \in \mathbb{Z}_n$.*

3.13. **Multiplication operators in Grothendieck algebras and idempotents.** In Section 2.7, we have described a method for constructing a left eigenvector for the McKay matrix M_V using an eigenvector for the right multiplication operator R_V in the Grothendieck algebra of an arbitrary Hopf algebra. The next result has a similar flavor but is stated in greater generality for an arbitrary finite-dimensional algebra A.

Proposition 3.13.1. *Suppose* A *is an algebra with basis* $\{b_1, b_2, \ldots, b_k\}$, $k \in \mathbb{Z}_{\geq 1}$, *such that* $b_i b_j = \sum_{t=1}^{k} a_{i,t}^{(j)} b_t$ *for all* $1 \leq i, j \leq k$. *Let* $\mathsf{M}_j = \left(a_{i,t}^{(j)}\right)$ *be the McKay matrix for multiplying by* b_j *(on the right). Assume* $\mathbf{u} = [u_1 \ u_2 \ \ldots \ u_k]$ *is a common left eigenvector for the* M_j *with* $\mathbf{u}\mathsf{M}_j = \beta_j \mathbf{u}$ *for all* j. *Then* $e_\mathbf{u} = \sum_{i=1}^{k} u_i b_i \in$ A *satisfies* $e_\mathbf{u}^2 = c_\mathbf{u} e_\mathbf{u}$, *where* $c_\mathbf{u} = \sum_{j=1}^{k} \beta_j u_j$.

Proof. It follows from the relations $\mathbf{u}\mathsf{M}_j = \beta_j \mathbf{u}$ that $\sum_{i=1}^{k} u_i a_{i,t}^{(j)} = \beta_j u_t$ for all $1 \leq j, t \leq k$. Then for $e_\mathbf{u} = \sum_{i=1}^{k} u_i b_i$, we have

$$\begin{aligned} e_\mathbf{u}^2 = \left(\sum_{i=1}^{k} u_i b_i\right)^2 &= \left(\sum_{i=1}^{k} u_i b_i\right)\left(\sum_{j=1}^{k} u_j b_j\right) = \sum_{i,j=1}^{k} u_i u_j b_i b_j \\ &= \sum_{i,j,t=1}^{k} u_i u_j a_{i,t}^{(j)} b_t = \sum_{j,t=1}^{k} u_j \left(\sum_{i=1}^{k} u_i a_{i,t}^{(j)}\right) b_t = \sum_{j,t=1}^{k} u_j \beta_j u_t b_t \\ &= \left(\sum_{j=1}^{k} \beta_j u_j\right)\left(\sum_{t=1}^{k} u_t b_t\right) = c_\mathbf{u} e_\mathbf{u}. \end{aligned}$$

□

Corollary 3.13.2. *Assume* $\mathrm{A} = \mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n) = \mathbb{C} \otimes_\mathbb{Z} \mathsf{G}_0(\mathsf{D}_n)$, *and let* $b_1, b_2, \ldots, b_{n^2}$ *be an ordering of the simple modules for the Drinfeld double* D_n, *first by* $\ell = 1, \ldots, n$, *and then for a given value of* ℓ *by* $r = 0, 1, \ldots, n-1$, *so that* $b_1, b_2, \ldots, b_{n^2}$ *is a basis for* A, *and* $b_1 = \mathsf{V}(1, 0)$ *is the unit element of* A. *Let* M_j *be the McKay matrix for tensoring with* b_j, *and let* $\mathbf{u} = [u_1 \ u_2 \ \ldots \ u_{n^2}]$ *be a nonzero common left eigenvector for matrices* M_j *(such exist by Corollary 3.10.3). Assume* $\mathbf{u}$ *has eigenvalue* β_j *relative* M_j *for all* j. *Then* $e_\mathbf{u} = \sum_{i=1}^{n^2} u_i b_i \in$ A *satisfies* $e_\mathbf{u}^2 = c_\mathbf{u} e_\mathbf{u}$, *where* $c_\mathbf{u} = \sum_{j=1}^{n^2} \beta_j u_j$, *and when* $c_\mathbf{u} = \sum_{j=1}^{n^2} \beta_j u_j \neq 0$, *then* $e_\mathbf{u} = c_\mathbf{u}^{-1} e_\mathbf{u}$ *is an idempotent in* A.

Remark 3.13.3. In Proposition 3.8.2, we have described $\frac{n(n+1)}{2}$ left eigenvectors corresponding to distinct eigenvalues of the McKay matrix M for tensoring with $\mathsf{V}(2, 0)$. They correspond to eigenvectors for the right multiplication operator R_x, $x = [\mathsf{V}(2, 0)]$, of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$ as in Section 3.10. They are common left eigenvectors for the n^2 McKay matrices that result from tensoring with *any* simple module $\mathsf{V}(\ell, s)$. Each such left eigenvector $\mathbf{u} = [u_1 \ u_2 \ \ldots \ u_{n^2}]$ with $c_\mathbf{u} \neq 0$ gives an idempotent $e_\mathbf{u} = c_\mathbf{u}^{-1} \sum_{i=1}^{n^2} u_i b_i$ in $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$. Moreover, such idempotents are distinct, $\mathbf{u} \neq \mathbf{v} \implies e_\mathbf{u} \neq e_\mathbf{v}$, because the b_i determine a basis for $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$. This suggests that we should be able to locate $\frac{n(n+1)}{2}$ linearly independent idempotents in $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$. We accomplish this in Theorem 3.13.8 (b) below. Moreover, we show in part (a) of that theorem, that there are $\frac{n(n-1)}{2}$ linearly independent elements that square to 0 and form a basis for the Jacobson radical of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$.

Our aim here is to identify a $\mathbb{C}$-basis of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$ consisting of elements that square to 0 and idempotents. Our calculations will be based on the following well-known result (see for example, [18, Sec. 2.2]). if $\mathbb{C}\mathsf{G}$ is the group algebra of a finite group and S is a simple G-module, then

$$\varepsilon_\mathsf{S} := \frac{\mathsf{dim}(\mathsf{S})}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \chi_\mathsf{S}(g^{-1})\, g$$

is a central idempotent in $\mathbb{C}\mathsf{G}$, and ε_{S} projects any finite-dimensional G-module onto the S-isotypic component. Moreover, $\varepsilon_{\mathsf{S}}\varepsilon_{\mathsf{T}} = 0$ whenever S and T are simple, nonisomorphic modules.

Recall that $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n) \cong \mathbb{C}\mathsf{G}[x]/\langle f(x,g)\rangle$, where $\mathsf{G} = \langle g\rangle$,

$$f(x,g) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \frac{n}{n-i}\binom{n-i}{i} g^i x^{n-2i} \ - 2,$$

and $g = [\mathsf{V}(1,1)]$, $x = [\mathsf{V}(2,0)]$. Therefore, it follows for $u \in \mathbb{Z}_n$ that $\mathcal{E}_u = \frac{1}{n}\sum_{v=0}^{n-1} q^{-uv} g^v$ is an idempotent in $\mathbb{C}\mathsf{G} \subset \mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$ corresponding to the character $\chi_u(g^v) = q^{uv}$ of $\mathsf{G} = \langle g\rangle$, and these idempotents are orthogonal $\mathcal{E}_u\mathcal{E}_{u'} = \delta_{u,u'}\mathcal{E}_u$. Note that

$$g\mathcal{E}_u = \frac{1}{n}\sum_{v=0}^{n-1} q^{-uv} g^{v+1} = \frac{q^u}{n}\sum_{v=0}^{n-1} q^{-u(v+1)} g^{v+1} = q^u \mathcal{E}_u, \tag{3.13.1}$$

so that $\mathbb{C}\mathsf{G} = \bigoplus_{u=0}^{n-1} \mathbb{C}\mathcal{E}_u$ is a decomposition of the group algebra $\mathbb{C}\mathsf{G}$ into simple G-modules, where $\mathbb{C}\mathcal{E}_u$ is the one-dimensional G-module with corresponding character χ_u.

As a consequence of (3.13.1), we know that

$$f(x,g)\mathcal{E}_u = f(x,q^u)\mathcal{E}_u = \mathsf{p}_n(x,q^u)\mathcal{E}_u \tag{3.13.2}$$

in the notation of Corollary 3.4.4. We can write $u = 2r$ for some r, since n is odd and 2 is invertible modulo n, and then

$$f(x,q^{2r}) = \mathsf{p}_n(x,q^{2r}) = (x - \lambda_{0,r}) \prod_{j=1}^{\frac{n-1}{2}} (x - \lambda_{j,r})^2. \tag{3.13.3}$$

Definition 3.13.4. For $r \in \mathbb{Z}_n$, let

$$\begin{aligned} \mathcal{F}_{j,r} &:= \frac{f(x,q^{2r})}{x - \lambda_{j,r}}\mathcal{E}_{2r}, \qquad 0 \le j \le \frac{n-1}{2}, \\ \mathcal{G}_{j,r} &:= \frac{f(x,q^{2r})}{(x - \lambda_{j,r})^2}\mathcal{E}_{2r}, \qquad 1 \le j \le \frac{n-1}{2}. \end{aligned} \tag{3.13.4}$$

Proposition 3.13.5. *The elements defined in* (3.13.4) *satisfy the relations*

(a) $x\mathcal{F}_{j,r} = \lambda_{j,r}\mathcal{F}_{j,r}$ *and* $g\mathcal{F}_{j,r} = q^{2r}\mathcal{F}_{j,r}$;
(b) $x\mathcal{G}_{j,r} = \lambda_{j,r}\mathcal{G}_{j,r} + \mathcal{F}_{j,r}$ *and* $g\mathcal{G}_{j,r} = q^{2r}\mathcal{G}_{j,r}$.

Proof. The fact that $g\mathcal{F}_{j,r} = q^{2r}\mathcal{F}_{j,r}$ and $g\mathcal{G}_{j,r} = q^{2r}\mathcal{G}_{j,r}$ hold is a consequence of $g\mathcal{E}_{2r} = q^{2r}\mathcal{E}_{2r}$. The relations involving x follow easily from

$$\begin{aligned} (x - \lambda_{j,r})\mathcal{F}_{j,r} &= (x - \lambda_{j,r})\frac{f(x,q^{2r})}{x - \lambda_{j,r}}\mathcal{E}_{2r} = f(x,q^{2r})\mathcal{E}_{2r} = f(x,g)\mathcal{E}_{2r} = 0, \\ (x - \lambda_{j,r})\mathcal{G}_{j,r} &= (x - \lambda_{j,r})\frac{f(x,q^{2r})}{(x - \lambda_{j,r})^2}\mathcal{E}_{2r} = \mathcal{F}_{j,r}. \end{aligned}$$

□

Proposition 3.13.6. *For the elements defined in* (3.13.4)*, the following hold:*

(a) $\mathcal{F}_{j,r}\mathcal{F}_{k,s} = \mathcal{F}_{j,r}\mathcal{G}_{k,s} = \mathcal{G}_{j,r}\mathcal{G}_{k,s} = 0$ *when* $r \neq s$;
(b) *The following products are 0 whenever* $j \neq k$,

$$\mathcal{F}_{j,r}\mathcal{F}_{k,r}, \qquad \mathcal{F}_{j,r}\mathcal{G}_{k,r}, \qquad \mathcal{G}_{j,r}\mathcal{G}_{k,r};$$

(c) $\mathcal{F}_{j,r}^2 = 0$ *when* $1 \le j \le \frac{n-1}{2}$;

(d) *The ideal* $\mathcal{J} = \mathsf{span}_{\Bbbk}\{\mathcal{F}_{j,r} \mid 1 \le j \le \frac{n-1}{2},\, r \in \mathbb{Z}_n\}$ *satisfies* $\mathcal{J}^2 = (0)$.

Proof. (a) Since $\mathcal{E}_{2r}\mathcal{E}_{2s} = 0$ whenever $r \neq s$, it is apparent that $\mathcal{F}_{j,r}\mathcal{F}_{k,s} = \mathcal{F}_{j,r}\mathcal{G}_{k,s} = \mathcal{G}_{j,r}\mathcal{G}_{k,s} = 0$ for $r \neq s$ and any choice of j and k. (b) Suppose $m_j = 1$ when $j = 0$, and $m_j = 1$ or 2 when $1 \le j \le \frac{n-1}{2}$. Then when $j \neq k$, we have

$$\begin{aligned}\frac{f(x,q^{2r})}{(x-\lambda_{j,r})^{m_j}}\mathcal{E}_{2r} \cdot \frac{f(x,q^{2r})}{(x-\lambda_{k,r})^{m_k}}\mathcal{E}_{2r} &= \frac{f(x,q^{2r})}{(x-\lambda_{j,r})^{m_j}(x-\lambda_{k,r})^{m_k}} \cdot f(x,q^{2r})\mathcal{E}_{2r}, \\ &= \frac{f(x,q^{2r})}{(x-\lambda_{j,r})^{m_j}(x-\lambda_{k,r})^{m_k}} \cdot f(x,g)\mathcal{E}_{2r} = 0.\end{aligned} \tag{3.13.5}$$

Part (b) is now clear from the calculation in (3.13.5). For part (c), observe that equation (3.13.5) holds when $k = j \neq 0$, and $m_j = m_k = 1$. Part (d) follows from (b) and (c). ☐

Proposition 3.13.7. *The elements* $\{\xi_r^{-1}\mathcal{F}_{0,r} = \xi_r^{-1}\frac{f(x,q^{2r})}{x-\lambda_{0,r}}\mathcal{E}_{2r} \mid r \in \mathbb{Z}_n\}$ *are (nonzero) orthogonal idempotents, where* $\xi_r = \prod_{j=1}^{\frac{n-1}{2}}\Big(2q^r - q^r(q^j+q^{-j})\Big)^2 \neq 0$.

Proof. Orthogonality is a consequence of Proposition 3.13.6 (a), and the remaining assertions follow from (3.13.3) and the calculation

$$\begin{aligned}\mathcal{F}_{0,r}^2 &= \frac{f(x,q^{2r})}{x-\lambda_{0,r}}\mathcal{E}_{2r} \cdot \frac{f(x,q^{2r})}{x-\lambda_{0,r}}\,\mathcal{E}_{2r} = \frac{f(x,q^{2r})}{x-\lambda_{0,r}}\mathcal{F}_{0,r} \\ &= \left(\prod_{j=1}^{\frac{n-1}{2}}(x-\lambda_{j,r})^2\right)\mathcal{F}_{0,r} = \left(\prod_{j=1}^{\frac{n-1}{2}}(\lambda_{0,r}-\lambda_{j,r})^2\right)\mathcal{F}_{0,r} \\ &= \left(\prod_{j=1}^{\frac{n-1}{2}}\big(2q^r - q^r(q^j+q^{-j})\big)^2\right)\mathcal{F}_{0,r} = \xi_r\,\mathcal{F}_{0,r},\end{aligned}$$

where $\xi_r = \left(\prod_{j=1}^{\frac{n-1}{2}}\big(2q^r - q^r(q^j+q^{-j})\big)^2\right) \neq 0$, which implies $\xi_r^{-1}\mathcal{F}_{0,r}$ is an idempotent. ☐

So far we have identified n idempotents $\xi_r^{-1}\mathcal{F}_{0,r}, r \in \mathbb{Z}_n$, in $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$ and $\frac{n(n-1)}{2}$ elements $\mathcal{F}_{j,r}$, $1 \le j \le \frac{n-1}{2}$, $r \in \mathbb{Z}_n$, that square to 0. Next we will find some more idempotents using the elements $\mathcal{G}_{j,r}$. In Sec. 3.13.2, we will examine the elements $\mathcal{F}_{j,r}$ and $\mathcal{G}_{j,r}$ explicitly for $n = 3$.

3.13.1. *Idempotents from the elements* $\mathcal{G}_{j,r}$. We begin by computing $\mathcal{G}_{j,r}\mathcal{F}_{j,r}$ and $\mathcal{G}_{j,r}^2$ for $\le j \le \frac{n-1}{2}$, $r \in \mathbb{Z}_n$. Now

$$\begin{aligned}\mathcal{G}_{j,r}\mathcal{F}_{j,r} &= \left(\frac{f(x,q^{2r})}{(x-\lambda_{j,r})^2}\right)\mathcal{F}_{j,r} = (x-\lambda_{0,r})\left(\prod_{k=1,k\neq j}^{\frac{n-1}{2}}(x-\lambda_{k,r})^2\right)\mathcal{F}_{j,r} \\ &= (\lambda_{j,r}-\lambda_{0,r})\left(\prod_{k=1,k\neq j}^{\frac{n-1}{2}}(\lambda_{j,r}-\lambda_{k,r})^2\right)\mathcal{F}_{j,r} = \vartheta_{j,r}\mathcal{F}_{j,r}, \qquad \text{where}\end{aligned}$$

$$\vartheta_{j,r} = (\lambda_{j,r}-\lambda_{0,r})\left(\prod_{k=1,k\neq j}^{\frac{n-1}{2}}(\lambda_{j,r}-\lambda_{k,r})^2\right). \tag{3.13.6}$$

This shows that $\mathcal{G}_{j,r}\mathcal{F}_{j,r}$ is a nonzero multiple $\vartheta_{j,r}$ of $\mathcal{F}_{j,r}$. We also have

$$\mathcal{G}_{j,r}^2 = \left(\frac{f(x,q^{2r})}{(x-\lambda_{j,r})^2}\mathcal{E}_{2r}\right)^2 = \frac{f(x,q^{2r})}{(x-\lambda_{j,r})^2}\mathcal{G}_{jr} = \left((x-\lambda_{0,r})\prod_{k=1,k\neq j}^{\frac{n-1}{2}}(x-\lambda_{k,r})^2\right)\mathcal{G}_{j,r}. \tag{3.13.7}$$

For $1 \le k \le \frac{n-1}{2}$, $k \neq j$,

$$\begin{aligned}(x-\lambda_{k,r})\mathcal{G}_{j,r} &= (\lambda_{j,r}-\lambda_{k,r})\mathcal{G}_{j,r} + \mathcal{F}_{j,r}\\ (x-\lambda_{k,r})^2\mathcal{G}_{j,r} &= (\lambda_{j,r}-\lambda_{k,r})(x-\lambda_{k,r})\mathcal{G}_{j,r} + (x-\lambda_{k,r})\mathcal{F}_{j,r}\\ &= (\lambda_{j,r}-\lambda_{k,r})^2\mathcal{G}_{j,r} + 2(\lambda_{j,r}-\lambda_{k,r})\mathcal{F}_{j,r}\\ (x-\lambda_{0,r})\mathcal{G}_{j,r} &= (\lambda_{j,r}-\lambda_{0,r})\mathcal{G}_{j,r} + \mathcal{F}_{j,r}.\end{aligned}$$

These computations and (3.13.7) imply that

$$\mathcal{G}_{j,r}^2 = \vartheta_{j,r}\mathcal{G}_{j,r} + \nu_{j,r}\mathcal{F}_{j,r},$$

for some scalar $\nu_{j,r}$, where $\vartheta_{j,r} \neq 0$ is as in (3.13.6). Then

$$\begin{aligned}\big(\mathcal{G}_{j,r} - \frac{\nu_{j,r}}{\vartheta_{j,r}}\mathcal{F}_{j,r}\big)^2 &= \mathcal{G}_{j,r}^2 - 2\frac{\nu_{j,r}}{\vartheta_{j,r}}\mathcal{G}_{j,r}\mathcal{F}_{j,r}\\ &= \vartheta_{j,r}\mathcal{G}_{j,r} + \nu_{j,r}\mathcal{F}_{j,r} - 2\frac{\nu_{j,r}}{\vartheta_{j,r}}\vartheta_{j,r}\mathcal{F}_{j,r}\\ &= \vartheta_{j,r}\mathcal{G}_{j,r} - \nu_{j,r}\mathcal{F}_{j,r} = \vartheta_{j,r}\left(\mathcal{G}_{j,r} - \frac{\nu_{j,r}}{\vartheta_{j,r}}\mathcal{F}_{j,r}\right),\end{aligned}$$

which shows that

$$\mathcal{G}'_{j,r} := \vartheta_{j,r}^{-1}\left(\mathcal{G}_{j,r} - \frac{\nu_{j,r}}{\vartheta_{j,r}}\mathcal{F}_{j,r}\right) \tag{3.13.8}$$

is an idempotent in $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$ for $\vartheta_{j,r}$ as in (3.13.6) and some $\nu_{j,r} \in \mathbb{C}$ for each $1 \le j \le \frac{n-1}{2}$, $r \in \mathbb{Z}_n$.

In summary, we have the following

Theorem 3.13.8. (a) *The $\frac{n(n-1)}{2}$ elements $\mathcal{F}_{j,r}$, $1 \le j \le \frac{n-1}{2}$, $r \in \mathbb{Z}_n$, determine a $\mathbb{C}$-basis for the Jacobson radical $\mathcal{J}$ of the Grothendieck algebra $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$ and $\mathcal{J}^2 = (0)$.*

(b) *The elements $\xi_{0,r}^{-1}\mathcal{F}_{0,r}$ and $\mathcal{G}'_{j,r}$ for $r \in \mathbb{Z}_n$ and $1 \le j \le \frac{n-1}{2}$ are orthogonal idempotents, and they form a basis for $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$ modulo $\mathcal{J}$.*

(c) *Suppose $\mathsf{S}_1, \mathsf{S}_2, \dots, \mathsf{S}_{n^2}$ is an ordering of the nonisomorphic simple D_n-modules, first by $\ell = 1, \dots, n$ and then by $r = 0, 1, \dots, n-1$. If $\mathcal{F}_{j,r} = c_1^{j,r}\mathsf{S}_1 + c_2^{j,r}\mathsf{S}_2 + \cdots + c_{n^2}^{j,r}\mathsf{S}_{n^2}$, then $\mathsf{f}_{j,r} := [c_1^{j,r}\ \ c_2^{j,r}\ \ \dots\ \ c_{n^2}^{j,r}]$ is a left eigenvector for the McKay matrix M_{V}, $\mathsf{V} = \mathsf{V}(2,0)$, for all $0 \le j \le \frac{n-1}{2}$ and $r \in \mathbb{Z}_n$ corresponding to the eigenvalue $\lambda_{j,r} = q^r(q^j + q^{-j})$.*

(d) *If $\mathcal{G}_{j,r} = d_1^{j,r}\mathsf{S}_1 + d_2^{j,r}\mathsf{S}_2 + \cdots + d_{n^2}^{j,r}\mathsf{S}_{n^2}$, then $\mathsf{g}_{j,r} := [d_1^{j,r}\ \ d_2^{j,r}\ \ \dots\ \ d_{n^2}^{j,r}]$ is a generalized left eigenvector for the McKay matrix M_{V} such that $\mathsf{g}_{j,r}\mathsf{M}_{\mathsf{V}} = \lambda_{j,r}\mathsf{g}_{j,r} + \mathsf{f}_{j,r}$ for all $1 \le j \le \frac{n-1}{2}$ and $r \in \mathbb{Z}_n$.*

(e) *The vectors $\mathsf{f}_{j,r}$, $0 \le j \le \frac{n-1}{2}, r \in \mathbb{Z}_n$, give a complete set of left eigenvectors, and the vectors $\mathsf{g}_{j,r}, 1 \le j \le \frac{n-1}{2}, r \in \mathbb{Z}_n$, give a complete set of generalized left eigenvectors for the McKay matrix M_{V}, $\mathsf{V} = \mathsf{V}(2,0)$, hence, for any McKay matrix $\mathsf{M}_{(\ell,s)}$ by Corollary 3.10.3.*

Proof. From Proposition 3.13.5 (a) we know that the elements $\mathcal{F}_{j,r}$ are eigenvectors for the multiplication operator R_x, $x = [\mathsf{V}(2,0)]$, of $\mathsf{G}_0^{\mathbb{C}}(\mathsf{D}_n)$, so the corresponding coordinate vectors $\mathsf{f}_{j,r}$ relative to the basis of nonisomorphic simple modules will be left eigenvectors for $\mathsf{M} = \mathsf{M}_{\mathsf{V}}$ by Proposition 2.5.1. Part (b) of Proposition 3.13.5 shows that $x\mathcal{G}_{j,r} = \lambda_{j,r}\mathcal{G}_{j,r} + \mathcal{F}_{j,r}$. Therefore, the coordinate vector $\mathsf{g}_{j,r}$ of $\mathcal{G}_{j,r}$ will be a generalized left eigenvector for M_V corresponding to the eigenvalue $\lambda_{j,r}$ by Proposition 2.5.1 (b). □

3.13.2. *Computations for* D_3. When $n = 3$, we have $f(x, q^{2r}) = x^3 - 3q^{2r}x - 2$, and

$$\frac{f(x, q^{2r})}{(x - 2q^r)} = x^2 + 2q^r x + q^{2r} = (x + q^r)^2, \quad \text{and} \quad \xi_r = \big(2q^r - q^r(q + q^{-1})\big)^2 = (3q^r)^2 = 9q^{2r}.$$

Thus, $\xi_r^{-1}\mathcal{F}_{0,r} = (9q^{2r})^{-1}\big(x^2 + 2q^r x + q^{2r}\big)\mathcal{E}_{2r} = \frac{1}{9}\big(q^r x^2 + 2q^{2r}x + 1\big)\mathcal{E}_{2r}$ is an idempotent for $r = 0, 1, 2$. Now

$$\mathcal{F}_{1,r} = \frac{f(x, q^{2r})}{x - \lambda_{1,r}}\mathcal{E}_{2r} = (x - \lambda_{0,r})(x - \lambda_{1,r})\mathcal{E}_{2r}, \tag{3.13.9}$$

so that $\mathcal{F}_{1,r}^2 = (x - \lambda_{0.r})^2(x - \lambda_{1,r})^2\mathcal{E}_{2r} = (x - \lambda_{0,r})f(x, q^{2r})\mathcal{E}_{2r} = (x - \lambda_{0,r})f(x, g)\mathcal{E}_{2r} = 0$. Consequently, the elements $\mathcal{F}_{1,r}$, $r \in \mathbb{Z}_3$, square to 0 in agreement with Proposition 3.13.6 (b).

Finally,

$$\mathcal{G}_{1,r} = \frac{f(x, q^{2r})}{(x - \lambda_{1,r})^2}\mathcal{E}_{2r} = (x - \lambda_{0,r})\mathcal{E}_{2r} \quad \text{and} \quad \mathcal{G}_{1,r}^2 = (x - \lambda_{0,r})\mathcal{G}_{1,r} = (\lambda_{1,r} - \lambda_{0,r})\mathcal{G}_{1,r} + \mathcal{F}_{1,r}.$$

This tells us that by taking $\nu_{1,r} = 1$ and $\vartheta_{1,r} = \lambda_{1,r} - \lambda_{0,r} = q^r(q + q^{-1}) - 2q^r = -3q^r$,

$$\left(\mathcal{G}_{1,r} + \frac{1}{3q^r}\mathcal{F}_{1,r}\right)^2 = -3q^r\mathcal{G}_{1,r} + \mathcal{F}_{1,r} - 2\mathcal{F}_{1,r} = -3q^r\left(\mathcal{G}_{1,r} + \frac{1}{3q^r}\mathcal{F}_{1,r}\right),$$

and therefore by setting $\mathcal{G}'_{1,r} = -\frac{1}{3q^r}\big(\mathcal{G}_{1,r} + \frac{1}{3q^r}\mathcal{F}_{1,r}\big)$, we get an idempotent for $r \in \mathbb{Z}_3$.

Writing (ℓ, r) for $\mathsf{V}(\ell, r)$ and recalling that $x = [\mathsf{V}(2, 0)]$ and $g = [\mathsf{V}(1, 1)]$ gives

$$\begin{aligned}
\xi_r^{-1}\mathcal{F}_{0,r} &= (9q^{2r})^{-1}\big(x^2 + 2q^r x + q^{2r}\big)\mathcal{E}_{2r} = \frac{1}{27}\big(q^r x^2 + 2q^{2r}x + 1\big)\big(q^{-4r}g^2 + q^{-2r}g + 1\big)\\
&= \frac{1}{27}\Big(q^r(3,0) + q^r(1,1) + 2q^{2r}(2,0) + (1,0)\Big)\Big(q^{2r}(1,2) + q^r(1,1) + (1,0)\Big)\\
&= \frac{1}{27}\Big((3,2) + q^{2r}(3,1) + q^r(3,0) + 2q^r(2,2) + 2(2,1) + 2q^{2r}(2,0) + 2q^{2r}(1,2) + 2q^r(1,1) + 2(1,0)\Big).
\end{aligned}$$

Ordering the summands from $(1, 0)$ to $(3, 2)$, ignoring the factor of $\frac{1}{27}$, and recording the coefficients, we have

$$\mathsf{f}_{0,r} = [2 \;\; 2q^r \;\; 2q^{2r} \;\; 2q^{2r} \;\; 2 \;\; 2q^r \;\; q^r \;\; q^{2r} \;\; 1].$$

Multiplying $\mathsf{f}_{0,r}$ by 3 and then setting $r = 0, 1, 2$, we obtain the left eigenvectors of M_V for $\mathsf{V} = \mathsf{V}(2, 0)$ in (3.9.3) exactly:

$$\begin{array}{rcll}
\mathsf{Tr}_\mathsf{P}(1) &=& [6 \;\; 6 \;\; 6 \;\; 6 \;\; 6 \;\; 6 \;\; 3 \;\; 3 \;\; 3] & \qquad \lambda_{0,0} = 2,\\
\mathsf{Tr}_\mathsf{P}(bc^{-1}) &=& [6 \;\; 6q \;\; 6q^2 \;\; 6q^2 \;\; 6 \;\; 6q \;\; 3q \;\; 3q^2 \;\; 3] & \qquad \lambda_{0,1} = 2q,\\
\mathsf{Tr}_\mathsf{P}(b^2c^{-2}) &=& [6 \;\; 6q^2 \;\; 6q \;\; 6q \;\; 6 \;\; 6q^2 \;\; 3q^2 \;\; 3q \;\; 3] & \qquad \lambda_{0,2} = 2q^2.
\end{array}$$

Now

$$\begin{aligned}
\mathcal{G}_{1,r} &= \frac{f(x, q^{2r})}{(x - \lambda_{1,r})^2}\mathcal{E}_{2r} = (x - \lambda_{0,r})\mathcal{E}_{2r} = (x - 2q^r)\big(\frac{1}{3}(q^{2r} + q^r g + 1)\big)\\
&= \frac{1}{3}\big(- 2q^r(1,0) - 2q^{2r}(1,1) - 2(1,2) + (2,0) + q^r(2,1) + q^{2r}(2,2)\big).
\end{aligned}$$

Therefore, $\mathsf{g}_{1,r} = \frac{1}{3}[-2q^r, -2q^{2r}, -2, 1, q^r, q^{2r}, 0, 0, 0]$, and $\lambda_{1,r} = q^r(q+q^{-1}) = -q^r$. From (3.13.9), we can deduce that $\mathsf{f}_{1,r} = \frac{1}{3}[-q^{2r}, -1, -q^r, -q^r, -q^{2r}, -1, 1, q^r, q^{2r}]$. Then

$$\begin{aligned}
\mathsf{g}_{1,r}\mathsf{M}_{\mathsf{V}} &= \frac{1}{3}[-2q^r, -2q^{2r}, -2, 1, q^r, q^{2r}, 0, 0, 0]\begin{pmatrix} 0 & \mathrm{I} & 0 \\ \mathrm{Z} & 0 & \mathrm{I} \\ 2\mathrm{I} & 2\mathrm{Z} & 0 \end{pmatrix} \\
&= \frac{1}{3}[q^{2r}, 1, q^r, -2q^r, -2q^{2r}, -2, 1, q^r, q^{2r}] \\
&= -q^r\frac{1}{3}[-2q^r, -2q^{2r}, -2, 1, q^r, q^{2r}, 0, 0, 0] + \frac{1}{3}[-q^{2r}, -1, -q^r, -q^r, -q^{2r}, -1, 1, q^r, q^{2r}] \\
&= \lambda_{1,r}\mathsf{g}_{1,r} + \mathsf{f}_{1,r},
\end{aligned}$$

so that $\mathsf{g}_{1,r}$ is a generalized left eigenvector for M_{V} with eigenvalue $\lambda_{1,r}$ for $r \in \mathbb{Z}_3$.

3.14. **Fusion rules for tensoring a maximal set of independent projective modules in $\mathsf{G}_0(\mathsf{D}_n)$ with V.** We have seen in Proposition 3.1.1 that the Cartan map c for D_n has rank $\frac{n(n+1)}{2}$, and that the modules $\mathsf{P}(\ell, r) - \mathsf{P}(n-\ell, \ell+r)$ lie in the kernel of c for $1 \le \ell \le \frac{n(n-1)}{2}$. Following [9], we let N_{V} be the matrix that records tensoring a projective module P with $\mathsf{V} = \mathsf{V}(2,0)$ and writing the answer $[\mathsf{P} \otimes \mathsf{V}]$ as a $\mathbb{Z}$-combination of isomorphism classes of projectives whose images form a $\mathbb{Z}$-basis for $\mathsf{c}\big(\mathsf{K}_0(\mathsf{D}_n)\big) \subseteq \mathsf{G}_0(\mathsf{D}_n)$. Since the Cartan map has rank $\frac{n(n+1)}{2}$, we use only the modules $\mathsf{V}(n,r), \mathsf{P}(1,r), \ldots, \mathsf{P}(\frac{n-1}{2}, r)$ in forming N_{V}. We assume that ordering and take all values of r for each type, first for $\mathsf{V}(n,r)$, then for $\mathsf{P}(1,r)$ etc.. From the tensor rules (3.2), we have that the resulting matrix N_{V} is $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ for any $n = 2h+1$ with $h \ge 1$, and

$$\mathsf{N}_{\mathsf{V}} = \begin{pmatrix}
0 & \mathrm{I} & 0 & & \cdots & 0 & 0 \\
2\mathrm{Z} & 0 & \mathrm{I} & & \cdots & 0 & 0 \\
0 & \mathrm{Z} & 0 & \mathrm{I} & \cdots & 0 & 0 \\
\vdots & \vdots & \mathrm{Z} & \ddots & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & & \ddots & \mathrm{I} & 0 \\
0 & 0 & 0 & \cdots & \mathrm{Z} & 0 & \mathrm{I} \\
0 & 0 & 0 & 0 & \cdots & \mathrm{Z} & \mathrm{Z}^{h+1}
\end{pmatrix}, \tag{3.14.1}$$

where I is the $n \times n$ identity matrix, and Z is the $n \times n$ cyclic matrix in (3.2.1). In this section, we show that the matrix N_{V} has right eigenvectors whose components involve the modified Chebyshev polynomials $\mathcal{L}_k(t)$ of Section 3.8, and left eigenvectors whose components involve the Chebyshev polynomials $\mathcal{V}_k(t)$ of the third kind [28, Sec. 1.2.3], which are defined by

$$\mathcal{V}_0(t) = 1, \ \mathcal{V}_1(t) = t-1, \ \mathcal{V}_k(t) = t\mathcal{V}_{k-1}(t) + \mathcal{V}_{k-2}(t), \quad k \ge 2. \tag{3.14.2}$$

They are expressible in terms of other Chebyshev polynomials via the relations

$$\mathcal{V}_k(t) = \mathcal{U}_k(t) - \mathcal{U}_{k-1}(t), \ \text{ and } \mathcal{L}_k(t) = \mathcal{V}_k(t) + \mathcal{V}_{k-1}(t) \text{ for all } k \ge 1. \tag{3.14.3}$$

The first identity can be found in [28, 1.17], and the second comes from $\mathcal{L}_k(t) = \mathcal{U}_k(t) - \mathcal{U}_{k-2}(t)$ and the first. More specifically, we show the following for all $n = 2h+1 \ge 3$:

Theorem 3.14.1. (a) *The matrix N_{V} in (3.14.1) has eigenvalues $\lambda_{j,r} = q^r(q^j + q^{-j})$ for $0 \le j \le \frac{n-1}{2}, r \in \mathbb{Z}_n$, (each with multiplicity one), so the matrix N_{V} is diagonalizable (as expected from [9]).*

(b) *Let* $\mathbf{v} \neq \mathbf{0}$ *satisfy* $\mathrm{Z}\mathbf{v} = q^{2r}\mathbf{v}$*, and assume* $\mathcal{L}_k$ *stands for* $\mathcal{L}_k(q^j + q^{-j})$*. Then*

$$[\mathbf{v} \;\; q^r\mathcal{L}_1\mathbf{v} \;\; \dots \;\; q^{hr}\mathcal{L}_h\mathbf{v}]^{\mathtt{T}}$$

is a right eigenvector for N_{V} *of eigenvalue* $\lambda_{j,r}$ *for* $0 \le j \le \frac{n-1}{2}, r \in \mathbb{Z}_n$.

(c) *Let* $\mathbf{w} \neq \mathbf{0}$ *satisfy* $\mathbf{w}\mathrm{Z} = q^{2r}\mathbf{w}$*, and assume* $\mathcal{V}_k$ *stands for* $\mathcal{V}_k(q^j + q^{-j})$*. Then*

$$[q^{hr}\mathcal{V}_h\mathbf{w} \;\; \dots \;\; q^r\mathcal{V}_1\mathbf{w} \;\; \mathbf{w}]$$

is a left eigenvector for N_{V} *with eigenvalue* $\lambda_{j,r}$ *for* $0 \le j \le \frac{n-1}{2}, r \in \mathbb{Z}_n$.

Proof. We will argue that (b) and (c) hold, and part (a) will follow.

(b) We compare both sides of this equation and verify they are indeed equal:

$$\begin{pmatrix} 0 & \mathrm{I} & 0 & & \cdots & 0 & 0 \\ 2\mathrm{Z} & 0 & \mathrm{I} & & \cdots & 0 & 0 \\ 0 & \mathrm{Z} & 0 & \mathrm{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \mathrm{Z} & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \mathrm{I} & 0 \\ 0 & 0 & 0 & \cdots & \mathrm{Z} & 0 & \mathrm{I} \\ 0 & 0 & 0 & 0 & \cdots & \mathrm{Z} & \mathrm{Z}^{h+1} \end{pmatrix} \begin{bmatrix} \mathbf{v} \\ q^r\mathcal{L}_1\mathbf{v} \\ \\ q^{2r}\mathcal{L}_2\mathbf{v} \\ \vdots \\ q^{(h-1)r}\mathcal{L}_{h-1}\mathbf{v} \\ q^{hr}\mathcal{L}_h\mathbf{v} \end{bmatrix} = \lambda_{j,r} \begin{bmatrix} \mathbf{v} \\ q^r\mathcal{L}_1\mathbf{v} \\ \\ q^{2r}\mathcal{L}_2\mathbf{v} \\ \vdots \\ q^{(h-1)r}\mathcal{L}_{h-1}\mathbf{v} \\ q^{hr}\mathcal{L}_h\mathbf{v} \end{bmatrix}.$$

Row 0 on the left is $q^r\mathcal{L}_1\mathbf{v} = q^r(q^j + q^{-j})\mathbf{v}$, which equals $\lambda_{j,r}\mathbf{v}$, the entry in row 0 on the right. Row 1 says $2q^{2r}\mathbf{v} + q^{2r}\mathcal{L}_2\mathbf{v} = q^{2r}\left(\mathcal{L}_0 + \mathcal{L}_2\right)\mathbf{v} = q^{2r}(q^j + q^{-j})\mathcal{L}_1\mathbf{v} = \lambda_{j,r}q^r\mathcal{L}_1\mathbf{v}$.
Now for rows $2 \le s \le h-1$, we have

$$\mathrm{Z}q^{(s-2)r}\mathcal{L}_{s-2}\mathbf{v} + q^{sr}\mathcal{L}_s\mathbf{v} = q^{sr}\left(\mathcal{L}_{s-2} + \mathcal{L}_s\right)\mathbf{v} = q^{sr}(q^j + q^{-j})\mathcal{L}_{s-1}\mathbf{v} = \lambda_{j,r}q^{(s-1)r}\mathcal{L}_{s-1}\mathbf{v}.$$

Finally, for the last row, recall that $h = \frac{n-1}{2}$ so that $h+1 = \frac{n+1}{2}$. Then on the left we have

$$\mathrm{Z}q^{(h-1)r}\mathcal{L}_{h-1}\mathbf{v} + \mathrm{Z}^{h+1}q^{hr}\mathcal{L}_h\mathbf{v} = \left(q^{(h+1)r}\mathcal{L}_{h-1} + q^{\left(2(h+1)+h\right)r}\mathcal{L}_h\right)\mathbf{v} = q^{(h+1)r}\left(\mathcal{L}_{h-1} + \mathcal{L}_h\right)\mathbf{v}.$$

On the right, the last entry is

$$\lambda_{j,r}q^{hr}\mathcal{L}_h\mathbf{v} = q^{(h+1)r}(q^j + q^{-j})\mathcal{L}_h = q^{(h+1)r}\big(\mathcal{L}_{h+1} + \mathcal{L}_{h-1}\big)\mathbf{v}.$$

So comparing the left and right sides, we see that the argument boils down to whether $\mathcal{L}_h\mathbf{v} = \mathcal{L}_{h+1}\mathbf{v}$. But since

$$\mathcal{L}_h(q^j + q^{-j}) = q^{hj} + q^{-hj} = q^{-(h+1)j} + q^{(h+1)j} = \mathcal{L}_{h+1}(q^j + q^{-j}),$$

the left and right sides are indeed equal, so (b) holds.

(c) The connection with the Chebyshev polynomials $\mathcal{U}_k(t)$ in (3.14.3) is the one that will be most helpful in proving part (c). Recall we know by Proposition 3.4.1(a) that for $t = x + x^{-1}$ and all $k \ge 1$,

$$\mathcal{U}_k(t) = x^k + x^{k-2} + \cdots + x^{-(k-2)} + x^{-k} = \frac{x^{k+1} - x^{-(k+1)}}{x - x^{-1}}.$$

Therefore,

$$\mathcal{V}_k(t) = \mathcal{U}_k(t) - \mathcal{U}_{k-1}(t) = \frac{x^{k+1} - x^{-(k+1)}}{x - x^{-1}} - \frac{x^k - x^{-k}}{x - x^{-1}}$$

for all $k \geq 2$. In particular, taking $k = h+1$ for $h = \frac{n-1}{2}$, and assuming $x^n = 1$, we obtain

$$\begin{aligned} \mathcal{V}_{h+1}(t) &= \frac{x^{h+2} - x^{-(h+2)}}{x - x^{-1}} - \frac{x^{h+1} - x^{-(h+1)}}{x - x^{-1}} \\ &= \frac{x^{-(h-1)} - x^{h-1}}{x - x^{-1}} - \frac{x^{-h} - x^{h}}{x - x^{-1}} = \frac{x^{h} - x^{-h}}{x - x^{-1}} - \frac{x^{h-1} - x^{-(h-1)}}{x - x^{-1}} = \mathcal{V}_{h-1}(t). \end{aligned} \tag{3.14.4}$$

We will use (3.14.4) and identify x with q when we argue that the following equation holds:

$$[q^{hr}\mathcal{V}_h\mathbf{w} \ \ \ldots \ \ q^r\mathcal{V}_1\mathbf{w} \ \ \mathbf{w}] \begin{pmatrix} 0 & \mathrm{I} & 0 & & \cdots & 0 & 0 \\ 2\mathrm{Z} & 0 & \mathrm{I} & & \cdots & 0 & 0 \\ 0 & \mathrm{Z} & 0 & \mathrm{I} & \cdots & 0 & 0 \\ \vdots & \vdots & \mathrm{Z} & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \mathrm{I} & 0 \\ 0 & 0 & 0 & \cdots & \mathrm{Z} & 0 & \mathrm{I} \\ 0 & 0 & 0 & 0 & \cdots & \mathrm{Z} & \mathrm{Z}^{h+1} \end{pmatrix} = \lambda_{j,r}[q^{hr}\mathcal{V}_h\mathbf{w} \ \ \ldots \ \ q^r\mathcal{V}_1\mathbf{w} \ \ \mathbf{w}].$$

Consider column h on both sides (numbering columns h to 0 from left to right). On the left we have $2q^{(h-1)r}\mathcal{V}_{h-1}\mathbf{w}\mathrm{Z} = 2q^{(h+1)r}\mathcal{V}_{h-1}\mathbf{w}$. On the right we have for column h,

$$\lambda_{j,r}q^{hr}\mathcal{V}_h\mathbf{w} = q^{(h+1)r}(q^j + q^{-j})\mathcal{V}_h\mathbf{w} = q^{(h+1)r}\left(\mathcal{V}_{h+1} + \mathcal{V}_{h-1}\right)\mathbf{w} = 2q^{(h+1)r}\mathcal{V}_{h-1}\mathbf{w}$$

by (3.14.4), so the two are equal.

Now for $s = h, \ldots, 2$, column $s-1$ on the left gives $q^{sr}\mathcal{V}_s\mathbf{w} + q^{(s-2)r}\mathcal{V}_{s-2}\mathbf{w}\mathrm{Z} = q^{sr}\left(\mathcal{V}_s + \mathcal{V}_{s-2}\right)\mathbf{w}$. The corresponding column on the right has entry

$$\lambda_{j,r}q^{(s-1)r}\mathcal{V}_{s-1}\mathbf{w} = q^{sr}(q^j + q^{-j})\mathcal{V}_{s-1} = q^{sr}\left(\mathcal{V}_s + \mathcal{V}_{s-2}\right)\mathbf{w},$$

so the two are equal. Finally, for column 0, we have on the left

$$\begin{aligned} q^r\mathcal{V}_1\mathbf{w} + \mathbf{w}\mathrm{Z}^{h+1} &= q^r\mathcal{V}_1\mathbf{w} + q^{2(h+1)r}\mathbf{w} = q^r(\mathcal{V}_1 + 1)\mathbf{w} \\ &= q^r(q^j + q^{-j})\mathbf{w} = \lambda_{j,r}\mathbf{w}, \end{aligned}$$

which is precisely the entry in column 0 on the right-hand side.

We have produced $\frac{n(n+1)}{2}$ right (and left) eigenvectors with distinct eigenvalues $\lambda_{j,r}$, for $0 \leq j \leq \frac{n-1}{2}, r \in \mathbb{Z}_n$, for the $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ matrix N_V, so the $\lambda_{j,r}$ are exactly the eigenvalues of N_V. □

3.15. **Further Questions.** In this paper, we have proven results on McKay matrices of arbitrary finite-dimensional Hopf algebras and illustrated them for the Drinfeld double D_n of the Taft algebra, but there remain many interesting open questions, even for semisimple Hopf algebras.

- When is the McKay matrix symmetric or normal, hence orthogonally diagonalizable? It is shown in [39] that the McKay matrix corresponding to any simple module is orthogonally diagonalizable when A is semisimple and almost cocommutative, and we have shown in Corollary 2.3.4 that if A is semisimple and V is self-dual, then M_V is symmetric. The McKay matrix M_V, $\mathsf{V} = \mathsf{V}(2,0)$, for the nonsemisimple Drinfeld double D_n is not symmetric, and for the algebra A that is (14) in Kashina's classification [24] of 16-dimensional semisimple Hopf algebras there is a module $\mathsf{V} \not\cong \mathsf{V}^*$ such that M_V is not symmetric.
- For which Hopf algebras do the (generalized) right eigenvectors of McKay matrices correspond to columns in something that can be regarded as a character table?

- When can all the (right) eigenvectors of the McKay matrix $\mathsf{M_V}$ be obtained from traces of grouplike elements? We have shown this is possible for D_n in Sec. 3.6. It is possible for Radford's Hopf algebra $\mathsf{A}(n, m)$ which is also not semisimple [33, Exer. 10.5.9], but fails to be true for the Kac-Palyutkin algebra which is semisimple [23]. We have seen in Sec. 3.9 that for D_n, only n of the $\frac{n(n+1)}{2}$ linearly independent left eigenvectors can be realized as trace vectors of grouplike elements on projective covers.
- Under what assumptions can the (generalized) eigenvectors of McKay matrices be related to central elements of the Hopf algebra A or cocommutative elements in A^*?
- When are the eigenvalues of the fusion matrix $\mathsf{N_V}$ obtained by tensoring a maximal independent set of indecomposable projective modules with V the same as the eigenvalues for the McKay matrix $\mathsf{M_V}$? They are for Drinfeld double D_n and $\mathsf{V} = \mathsf{V}(2, 0)$.
- What can be said about the (generalized) eigenvectors of matrices that encode the fusion relations in the more general context of tensor or fusion categories (see e.g. [19], [34])? In [17], Etingof, Nikshych, and Ostrik introduced the Frobenius-Perron dimension of a fusion category as the spectral radius of a matrix representing the fusion relations.
- When n is even, do the (generalized) eigenvectors of the McKay matrices for tensoring D_n-modules have expressions in terms of Chebyshev polynomials, and what can be said about the multiplicities of the eigenvalues?

References

[1] T. Agerholm and V. Mazorchuk, On selfadjoint functors satisfying polynomial relations, *J. Algebra* **330** (2011), 448–467.

[2] G. Benkart, P. Diaconis, M. Liebeck, and P. Tiep, Tensor product Markov chains, *J. Algebra* **561** (2020), 17–83.

[3] G. Benkart, R. Biswal, E. Kirkman, V.C. Nguyen, J. Zhu, Tensor representations for the Drinfeld double of the Taft algebra, arXiv #2012.1527.

[4] H.-X. Chen, A class of noncommutative and noncocommutative Hopf algebras: the quantum version, *Comm. Algebra* **27** (1999), no. 10, 5011–5032.

[5] H.-X. Chen, Representations of a class of Drinfeld doubles, *Comm. Algebra* **33** (2005), 2809–2825.

[6] H.-X. Chen, Irreducible representations of a class of quantum doubles, *J. Algebra* **225** (2000), 391–409.

[7] H.-X. Chen, Finite-dimensional representations of a quantum double, *J. Algebra* **251** (2002), 751–789.

[8] H.-X. Chen; H.S.E. Mohammed; H. Sun, Indecomposable decomposition of tensor products of modules over Drinfeld doubles of Taft algebras, *J. Pure Appl. Algebra* **221** (2017), no. 11, 2752–2790.

[9] M. Cohen and S. Westreich, Characters and a Verlinde-type formula for symmetric Hopf algebras, *J. Algebra* **320** (2008), no. 12, 4300–4316.

[10] M. Cohen and S. Westreich, Higman ideals and Verlinde-type formulas for Hopf algebras, in *Ring and module theory*, Trends Math., 91–114, Birkhäuser/Springer Basel AG, Basel, 2010.

[11] M. Cohen and S. Westreich, Structure constants related to symmetric Hopf algebras, *J. Algebra* **324** (2010), no.11, 3219–3240.

[12] M. Cohen and S. Westreich, Conjugacy classes, class sums and character tables for Hopf algebras, *Comm. Algebra* **39** (2011), no. 12, 4618–4633.

[13] C.W. Curtis and I. Reiner, Methods of Representation Theory. Vol. I. With applications to finite groups and orders. Reprint of the 1981 original. Wiley Classics Library. A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1990.

[14] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, Higher transcendental functions. Vol. II. Based on notes left by Harry Bateman. Reprint of the 1953 original. Robert E. Krieger Publishing Co., Inc. (1981).

[15] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, Tensor categories. Mathematical Surveys and Monographs, **205**. *American Mathematical Society, Providence, RI*, 2015.

[16] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yudovina, Introduction to representation theory. Student Mathematical Library **59**, *American Mathematical Society, Providence, RI*, 2011.

[17] P. Etingof, D. Nikshych and V. Ostrik, On fusion categories, *Ann. of Math.*, (2) **162** (2005), no. 2, 581–642.

[18] W. Fulton, J. Harris: Representation theory. A first course. Graduate Texts in Mathematics **129**. Readings in Mathematics. *Springer-Verlag, New York*, 1991.

[19] N. Geer, J. Kujawa, and B. Patureau-Mirand, Ambidextrous objects and trace functions for nonsemisimple categories, *Proc. Amer. Math. Soc.* **141** (2013), no. 9, 2963–2978.

[20] D. Grinberg, J. Huang, V. Reiner, Critical groups for Hopf algebra modules, *Math. Proc. Cambridge Philos. Soc.* **168** (2020), no. 3, 473–503.

[21] M.A. Hennings, Invariants of links and 3-manifolds obtained from Hopf algebras, *J. Lond. Math Soc.* **54** (1996), 594–624.

[22] A. Jedwab, A trace-like invariant for representations of Hopf algebras, *Comm. Algebra* **38** (2010), no. 9, 3456–3468.

[23] G.I. Kac and V.G. Palyutkin, Finite ring groups, *Trudy Moskov. Mat. Obsc.* **15** (1966), 224–261.

[24] Y. Kashina, Semisimple Hopf algebras of dimension 16, *J. Algebra* **232** (2000), 617–663.

[25] L. Kauffman, D. Radford, A necessary and sufficient condition for a finite-dimensional Drinfeld double to be a ribbon Hopf algebra, *J. Algebra* **159** (1993), no. 1, 98–114.

[26] M. Lorenz, Representations of finite-dimensional Hopf algebras, *J. Algebra* **188** (1997), 476–505.

[27] M. Lorenz, A tour of representation theory, *Graduate Studies in Mathematics* **193** *American Math. Society, Providence, RI*, 2018.

[28] J.C. Mason and D.C. Handscomb, Chebyshev Polynomials, *Chapman & Hall/CRC, Boca Raton, FL, 2003*.

[29] V. Mazorchuk and V. Miemietz, Transitive 2-representations of finitary 2-categories. *Trans. Amer. Math. Soc.* **368** (2016), no. 11, 7623–7644.

[30] J. McKay, Graphs, singularities, and finite groups. *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, pp. 183–186, Proc. Sympos. Pure Math., **37**, *Amer. Math. Soc., Providence, R.I.*, 1980.

[31] NIST Digital Library of Mathematical Functions at https://dlmf.nist.gov/.

[32] U. Oberst and H.-J. Schneider, Über Untergruppen endlicher algebraischer Gruppen, *Manuscripta Math.* **8** (1973), 217–241.

[33] D. Radford, Hopf Algebras, *Series on Knots and Everything* **49** World Scientific 2012.

[34] K. Shimizu, The monoidal center and the character algebra, *J. Pure Appl. Algebra* **221** (2017), no. 9, 2338–2371.

[35] R. Steinberg, Finite subgroups of SU2, Dynkin diagrams and affine Coxeter elements, *Pacific J. Math.* **118** (1985), no. 2, 587–598.

[36] H. Sun, H.S.E. Mohammed, W. Lin, H.-X. Chen, Green rings of Drinfeld doubles of Taft algebras, *Comm. Algebra* **48** (2020), no. 9, 3933–3947.

[37] M. Sweedler, Hopf algebras, Mathematics Lecture Note Series, *W.A. Benjamin, Inc., New York* 1969.

[38] E. Verlinde, Fusion rules and modular transformations in 2D conformal field theory, *Nuclear Phys. B* **300** (1988), no. 3, 360–376.

[39] S.J. Witherspoon, The representation ring and the centre of a Hopf algebra *Canad. J. Math.* **51** (1999), no. 4, 881–896.

[40] Y. Zhang, F. Wu, L. Liu, and H.-X. Chen, Grothendieck groups of a class of Drinfeld doubles, *Alg. Colloq.* **15**:3 (2008), 431–448.

[41] Y. Zhu, Hopf algebras of prime dimension, *Int. Math. Res. Not.* **1** (1994), 53–59.

(G. Benkart) Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, U.S.A.
Email address: benkart@math.wisc.edu

(R. Biswal) Max Planck Institute for Mathematics, Bonn, Germany
Email address: rekhabiswal27@gmail.com

(E. Kirkman) Department of Mathematics and Statistics, Wake Forest University, Winston-Salem, NC, 27109, U.S.A.
Email address: kirkman@wfu.edu

(V.C. Nguyen) Department of Mathematics, United States Naval Academy, Annapolis, MD 21402, U.S.A.
Email address: vnguyen@usna.edu

(J. Zhu) Department of Mathematics, University at Buffalo, Buffalo, NY 14260-2900, U.S.A.
Email address: jieruzhu699@gmail.com