Hodge-to-de Rham degeneration for stacks

Dmitry Kubrak, Artem Prikhodko

Abstract

We introduce a notion of a Hodge-proper stack and apply the strategy of Deligne-Illusie to prove the Hodge-to-de Rham degeneration in this setting. In order to reduce the statement in characteristic 0 to characteristic p, we need to find a good integral model of a stack (namely, a Hodge-proper spreading), which, unlike in the case of proper schemes, need not to exist in general. To address this problem we investigate the property of spreadability in more detail by generalizing standard spreading out results for schemes to higher Artin stacks and showing that all proper and some global quotient stacks are Hodge-properly spreadable. As a corollary we deduce a (non-canonical) Hodge decomposition of the equivariant cohomology for certain classes of varieties with an algebraic group action.

Contents

0 Introduction **2**
- 0.1 Deligne-Illusie method for schemes . . . 2
- 0.2 Generalization to stacks . . . 3
- 0.3 Relation to previous work and further directions . . . 6
- 0.4 Plan of the paper . . . 7

1 Degeneration of the Hodge-to-de Rham spectral sequence **8**
- 1.1 Hodge and de Rham cohomology . . . 8
- 1.2 Hodge-proper stacks . . . 10
- 1.3 Hodge-to-de Rham degeneration in positive characteristic . . . 11
- 1.4 Degeneration in characteristic zero . . . 18
- 1.5 Equivariant Hodge decomposition . . . 19

2 Spreadings **20**
- 2.1 Spreadable classes . . . 20
- 2.2 Cohomologically proper stacks . . . 23
- 2.3 Examples of Hodge-properly spreadable stacks . . . 26
  - 2.3.1 Proper stacks . . . 27
  - 2.3.2 Classifying stacks . . . 28
  - 2.3.3 Non-proper schemes . . . 30
  - 2.3.4 Higher examples . . . 32

3 Hodge-proper spreadability of quotient stacks **32**
- 3.1 Global quotients by reductive groups . . . 32
- 3.2 Global quotients coming from BB-complete $\mathbb{G}_m$-actions . . . 34
  - 3.2.1 Varieties with a $\mathbb{G}_m$-action and Bialynicki-Birula stratification . . . 34
  - 3.2.2 BB-complete quotients by $\mathbb{G}_m$ . . . 35
  - 3.2.3 Quotients by G that are BB-complete with respect to a subgroup . . . 38
- 3.3 Θ-stratified stacks and the relation to the work of Teleman . . . 40

A Computation of $H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ over $\mathbb{Z}$ **43**

References **47**

0 Introduction

0.1 Deligne-Illusie method for schemes

Let X be a smooth scheme over $\mathbb{C}$ and let $X(\mathbb{C})$ be the topological space of its complex points. Grothendieck has shown that there is a formula for the singular cohomology of $X(\mathbb{C})$ in purely algebraic terms, namely

$$H^n_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{C}) \simeq H^n_{\mathrm{dR}}(X/\mathbb{C}),$$

where the de Rham cohomology $H^n_{\mathrm{dR}}(X/\mathbb{C})$ is defined as the n-th hypercohomology of the algebraic de Rham complex of X. If, moreover, X is projective, using Hodge theory one obtains the Hodge decomposition

$$H^n_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(X,\Omega^p_X).$$

Unfortunately, it is only possible to get such a decomposition utilizing some transcendental methods (like Hodge theory). However, for X proper, using just algebraic geometry we still obtain a functorial filtration $F^\bullet H^n_{\mathrm{dR}}(X/\mathbb{C})$ whose associated graded is given by the sum above. Namely, the de Rham complex has a natural cellular (also called “stupid”) filtration $\Omega^{\geq p}_{X,\mathrm{dR}}$ given by subcomplexes

$$\Omega^{\geq p}_{X,\mathrm{dR}} \coloneqq \ \ldots \longrightarrow 0 \longrightarrow \Omega^p_X \xrightarrow{d} \Omega^{p+1}_X \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{\dim X}_X \ .$$

This filtration induces a filtration on the complex of global sections $R\Gamma_{\mathrm{dR}}(X/\mathbb{C}) \coloneqq R\Gamma(X,\Omega^\bullet_{X,\mathrm{dR}})$ whose associated graded pieces are $R\Gamma(X,\Omega^p_X[-p])$. As a consequence one gets the so-called *Hodge-to-de Rham spectral sequence*

$$E_1^{p,q} = H^q(X,\Omega^p_X) \ \Rightarrow \ H^{p+q}_{\mathrm{dR}}(X/\mathbb{C}).$$

As was shown by Deligne and Illusie [DI87] there is a purely algebraic proof of the degeneration of the spectral sequence above, thus the induced filtration $F^\bullet H^n_{\mathrm{dR}}(X/\mathbb{C})$ on the de Rham cohomology has the associated graded

$$\mathrm{gr}^\bullet_F H^n_{\mathrm{dR}}(X/\mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}(X), \quad \text{where} \quad H^{p,q}(X) := H^q(X,\Omega^p_X).$$

The strategy of Deligne-Illusie is to reduce the statement in characteristic 0 to an analogous question in big enough positive characteristic. Let k be a perfect field of characteristic p and let Y be a smooth scheme over k. Then we have:

Theorem 0.1.1 (Cartier)**.** *Let $Y^{(1)}$ denote the Frobenius twist of Y and let $\varphi_Y\colon Y \to Y^{(1)}$ be the relative Frobenius morphism. Then there exists a unique isomorphism of sheaves of $\mathcal{O}_{Y^{(1)}}$-algebras on $Y^{(1)}_{\mathrm{Zar}}$*

$$C_Y^{-1}\colon \bigoplus_i \Omega^i_{Y^{(1)}} \to \bigoplus_i \mathcal{H}^i(\varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}}),$$

determined by the property that for any local section f of $\mathcal{O}_Y$

$$C_Y^{-1}(df) = \text{“}df^p/p\text{”} \coloneqq f^{p-1}df.$$

The map C_Y^{-1} is called the inverse Cartier isomorphism.

This way we see that the Postnikov (also called “canonical”) filtration on $\varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}}$ induces another filtration on $R\Gamma(Y,\Omega^\bullet_{Y,\mathrm{dR}}) \simeq R\Gamma(Y^{(1)},\varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}})$ whose associated graded pieces are $R\Gamma(Y^{(1)},\Omega^p_{Y^{(1)}}[-p])$. Taking the spectral sequence induced by this filtration we obtain the so-called *conjugate spectral sequence*

$$E_2^{p,q} = H^p(Y^{(1)},\Omega^q_{Y^{(1)}}) \Rightarrow H^{p+q}_{\mathrm{dR}}(Y/k).$$

Note that for any spectral sequence the E_∞-page is always a subfactor of the E_r-page ($r \geq 0$), hence $\dim_k E_\infty^{*,*} \leq \dim_k E_r^{*,*}$. If all vector spaces $E_*^{*,*}$ are finite-dimensional, equality holds if and only if all differentials starting from the r-th page vanish. It follows that for Y proper, the conjugate spectral sequence degenerates if and only if $\dim_k H^n_{\mathrm{dR}}(Y) = \sum_{p+q=n} \dim_k H^{p,q}(Y^{(1)})$. Since $\dim_k H^{p,q}(Y^{(1)}) = \dim_k H^{p,q}(Y)$ this happens if and only if the Hodge-to-de Rham spectral sequence degenerates as well.

The differentials in the conjugate spectral sequence are induced by the connecting homomorphisms for the Postnikov filtration on $\varphi_{Y_*}\Omega^\bullet_{Y,\mathrm{dR}}$. In particular, if $\varphi_{Y_*}\Omega^\bullet_{Y,\mathrm{dR}}$ is formal (i.e. quasi-isomorphic to the sum of its cohomology), then the conjugate spectral sequence degenerates. While in general this is hard to guarantee, the formality of the truncation $\tau^{\leq p-1}\varphi_{Y_*}\Omega^\bullet_{Y,\mathrm{dR}}$ turns out to be equivalent to the existence of a lift to the second Witt vectors $W_2(k)$:

Theorem 0.1.2 (Deligne-Illusie)**.** *A smooth scheme Y over k admits a lift to $W_2(k)$ if and only if there exists an equivalence in the derived category of $\mathcal{O}_{Y^{(1)}}$-modules*

$$\bigoplus_{i=0}^{p-1} \Omega^i_{Y^{(1)}}[-i] \xrightarrow{\sim} \tau^{\leq p-1} \varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}}$$

inducing the inverse Cartier isomorphism C_Y^{-1} on $\mathcal{H}^*$. In particular, if Y admits a lift to $W_2(k)$ and $\dim Y < p$, then the complex $\varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}}$ is formal, and hence the Hodge-to-de Rham spectral sequence degenerates at the first page.

The proof of the degeneration in characteristic 0 is then accomplished by choosing a smooth proper model (the so-called *spreading*) X_R of X over some finitely generated $\mathbb{Z}$-subalgebra R of F. Enlarging R if needed, one can assume that the R-modules $H^q(X_R, \Omega^p_{X_R})$ and $H^n_{\mathrm{dR}}(X_R/R)$ are free of finite rank and that R is smooth over $\mathbb{Z}$. By smoothness of R any homomorphism from R to a perfect filed of positive characteristic lifts to the second Witt vectors (see Lemma 1.4.4). Picking a perfectization of closed point of $\operatorname{Spec} R$ of residue characteristic $p > \dim X$ one reduces Hodge-to-de Rham degeneration to Theorem 0.1.2.

0.2 Generalization to stacks

In this work we apply the strategy of Deligne-Illusie in the case of Artin stacks. For a smooth proper Deligne-Mumford stack one can proceed with the original arguments (see e.g. [Sat12, Corollary 1.7]), but they do not seem to work for a general smooth Artin stack (see Remark 0.2.4). Instead we use another approach relying on quasi-syntomic descent for the derived de Rham cohomology.

As in the case of schemes, to establish Hodge-to-de Rham degeneration, we need to impose some properness assumptions. However, the standard notion of a proper stack is too restrictive for our purposes. For example, the quotient stack $[X/G]$ of a proper scheme X by an action of a linear algebraic group G is proper if and only if the stabilizers of all points of X are finite group schemes. On the other hand, as we will see in Section 3.1, the Hodge-to-de Rham spectral sequence for $[X/G]$ with reductive G always degenerates.

This suggests that we should look for a more general notion of properness:

Definition 0.2.1. Let R be a Noetherian ring. A smooth Artin stack $\mathcal{X}$ over R is called *Hodge-proper* if $H^q(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/R})$ is a finitely generated R-module for all p and q, where $\mathbb{L}_{\mathcal{X}/R}$ is the cotangent complex of $\mathcal{X}$ over R.

The complex $R\Gamma(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/R})$ is a natural analogue of $R\Gamma(X, \Omega^p_X)$ and, similarly to the scheme case, the de Rham cohomology complex $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ has a natural (Hodge) filtration whose associated graded pieces are $R\Gamma(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/R}[-p])$; see Section 1.1 for more details. In this way one obtains a spectral sequence

$$E_1^{p,q} = H^q(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/R}) \;\Rightarrow\; H^{p+q}_{\mathrm{dR}}(\mathcal{X}/R).$$

In the case $R = F$ is a field this spectral sequence degenerates if and only if

$$\dim_F H^n_{\mathrm{dR}}(\mathcal{X}/F) = \sum_{p+q=n} \dim_F H^q(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/F}). \tag{1}$$

Remark 0.2.2. By smooth descent for the cotangent complex, $R\Gamma(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/F})$ produces the same answer as the definition of the Hodge cohomology via the lisse-étale site of $\mathcal{X}$ (see Proposition 1.1.4).

We will now explain the strategy of our proof of the equality (1) above. The first step is to extend Theorem 0.1.2 to the setting of stacks:

Theorem (1.3.23)**.** *Let $\mathcal{Y}$ be a smooth quasi-compact quasi-separated Artin stack over a perfect field k of characteristic p admitting a smooth lift to the ring of the second Witt vectors $W_2(k)$. Then there is a canonical equivalence*

$$R\Gamma(\mathcal{Y}, \tau^{\leq p-1}\Omega^\bullet_{\mathcal{Y},\mathrm{dR}}) \simeq R\Gamma\left(\mathcal{Y}^{(1)}, \bigoplus_{i=0}^{p-1} \wedge^i\mathbb{L}_{\mathcal{Y}^{(1)}/k}[-i]\right).$$

In particular for $n \leq p-1$ we have $H^n_{\mathrm{dR}}(\mathcal{Y}/k) \simeq H^n_{\mathrm{H}}(\mathcal{Y}^{(1)}/k)$.

Remark 0.2.3. Note that Theorem 1.3.23 gives only a partial generalization of Theorem 0.1.2. Even though the statement indeed follows from the analogous splitting of sheaves (see Theorem 1.3.21 below) there is no analogue of the "if and only if" statement of the Deligne-Illusie theorem. One reason for this is that the original approach of Deligne-Illusie is poorly suited for general Artin stacks (see Remark 0.2.4); so instead we use the (slightly enhanced) proof of the splitting due to Fontaine-Messing [FM87, Section II].

Since the de Rham cohomology for Artin stacks are defined as the right Kan extension from smooth affine schemes (Definition 1.1.3) one can more or less formally deduce the theorem above from the following very functorial form of Deligne-Illusie splitting for affine schemes:

Theorem (1.3.21)**.** *Let $\mathrm{Aff}^{\mathrm{sm}}_{/W_2(k)}$ be the category of smooth affine schemes over $W_2(k)$. Then there is a natural k-linear equivalence of functors*

$$\bigoplus_{i=0}^{p-1}\Omega^i_{-}\colon B\mapsto \bigoplus_{i=0}^{p-1}\Omega^i_{(B^{(1)}/p)/k}[-i] \quad \text{and} \quad \tau^{\le p-1}\Omega^\bullet_{-,\mathrm{dR}}\colon B\mapsto \tau^{\le p-1}\Omega^\bullet_{(B/p)/k,\mathrm{dR}}$$

from $\mathrm{Aff}^{\mathrm{sm,op}}_{/W_2(k)}$ to the ∞-category $D(\mathrm{Mod}_k)$ which induces the Cartier isomorphism on the level of the individual cohomology functors.

The splitting in Theorem 0.1.2 is already functorial with respect to liftings to $W_2(k)$, but only on the level of the underlying homotopy category and not the ∞-category of complexes $D(\mathrm{Mod}_k)$ itself. To get this higher functoriality we follow [FM87, Section II] using a more convenient language of [BMS19].

The idea is to extend the de Rham (and the crystalline) cohomology functor to a larger category of quasisyntomic algebras (Definition 1.3.1). This category, endowed with the quasisyntomic topology, has a basis consisting of quasi-regular semiperfectoid $W_n(k)$-algebras (Definition 1.3.3), on which the values of $R\Gamma_{\mathrm{dR}}$ and $R\Gamma_{\mathrm{crys}}$ (or rather their derived versions $R\Gamma_{\mathbb{L}\mathrm{dR}}$ and $R\Gamma_{\mathbb{L}\mathrm{crys}}$) become ordinary rings. Additionally, the Frobenius morphism, the Hodge filtration and the conjugate filtration can be described explicitly. This way, using quasi-syntomic descent, the question reduces to a certain computation in commutative algebra.

More concretely, for a quasi-regular semiperfect k-algebra S one can prove that $R\Gamma_{\mathbb{L}\mathrm{crys}}(S/W_n(k))\simeq \mathbb{A}_{\mathrm{crys}}(S)/p^n$, where $\mathbb{A}_{\mathrm{crys}}(S)$ is the divided power envelope of the kernel of the natural surjection $W((S)^\flat)\twoheadrightarrow S$ (see Construction 1.3.15). Under this identification the Hodge filtration on $R\Gamma_{\mathrm{crys}}(-/k)\simeq R\Gamma_{\mathrm{dR}}(-/k)$ corresponds to the filtration by the divided powers of the pd-ideal $I\triangleleft \mathbb{A}_{\mathrm{crys}}(S)/p$. The conjugate filtration $\mathrm{Fil}^{\mathrm{conj}}_*$ admits an explicit description as well (see Definition 1.3.17). Given a lifting $\widetilde{S}$ of S to $W_2(k)$ there is a natural morphism $\theta\colon \mathbb{A}_{\mathrm{crys}}(S)/p^2\to\widetilde{S}$. The image of $K\coloneqq\ker\theta$ under the first divided Frobenius map φ_1 then provides a splitting of $\mathrm{Fil}^{\mathrm{conj}}_1$ into $\mathrm{Fil}^{\mathrm{conj}}_0\oplus \mathrm{Fil}^{\mathrm{conj}}_1/\mathrm{Fil}^{\mathrm{conj}}_0\simeq S^\flat/I\oplus I/I^2$ (Proposition 1.3.22). By multiplicativity this extends to the splitting of $\mathrm{Fil}^{\mathrm{conj}}_{p-1}$ whose descent to smooth schemes gives Theorem 1.3.21.

Remark 0.2.4. The original approach of Deligne-Illusie (at least applied literally) does not seem to work for a general Artin stack; the key result of [DI87] is the equivalence of two gerbes on the étale site of $Y^{(1)}/k$ for a smooth k-scheme Y: the one of splittings of $\tau^{\le 1}\varphi_{Y*}\Omega^\bullet_{X,\mathrm{dR}}$ in $\mathrm{QCoh}(Y^{(1)})$ and the one of liftings of $Y^{(1)}$ to $W_2(k)$. A general smooth Artin stack $\mathcal{Y}$ can be covered by an affine scheme only *smooth locally*, so one needs to replace the étale site of $\mathcal{Y}$ by the smooth one. But both the space of splittings of $\tau^{\le 1}\varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}}$ and the space of liftings to $W_2(k)$ are not even presheaves there. Nevertheless, it would be still interesting to have an explicit description of the space of liftings to $W_2(k)$ for an arbitrary smooth n-Artin stack $\mathcal{Y}$. We do not discuss this question here.

Spreadings. Let now $\mathcal{X}$ be a smooth Hodge-proper stack over a field F of characteristic 0. If there exists $\mathbb{Z}$-subalgebra $R\subset F$, which is finitely generated over $\mathbb{Z}$[1], and a Hodge-proper stack $\mathcal{X}_R$ over R such that $\mathcal{X}_R\otimes_R F\simeq\mathcal{X}$ (a *Hodge-proper spreading of* $\mathcal{X}$), then one can deduce the equality (1) for the n-th cohomology from Theorem 1.3.23 by taking a suitable closed point $\mathrm{Spec}\,k\hookrightarrow\mathrm{Spec}\,R$ of characteristic $p>n$ and considering the fiber $\mathcal{X}_k$. This way we obtain

Theorem (1.4.3)**.** *Let $\mathcal{X}$ be a smooth Hodge-properly spreadable Artin stack over a field F of characteristic zero. Then the Hodge-to-de Rham spectral sequence for $\mathcal{X}$ degenerates at the first page. In particular for each $n\ge 0$ there exists a (non canonical) isomorphism*

$$H^n_{\mathrm{dR}}(\mathcal{X})\simeq\bigoplus_{p+q=n}H^{p,q}(\mathcal{X}).$$

[1] More generally, in Definition 1.4.1 we also allow subrings $R\subset F$ that are localizations of finitely generated $\mathbb{Z}$-algebras under the assumption that the image of $\mathrm{Spec}\,R$ in $\mathrm{Spec}\,\mathbb{Z}$ is open, but it is fine to assume that R is finitely generated throughout the introduction.

We must warn the reader that smooth Hodge-properly spreadable stacks do not enjoy many of the nice properties that smooth proper schemes have, in particular the natural mixed Hodge structure on the singular n-th cohomology is not necessarily pure (see Remark 2.3.18). The main motivation for the definition is that it is the most general class of stacks for which the Deligne-Illusie method can be applied. This, however, does not exceed all examples of the Hodge-to-de Rham degeneration (see Remark 2.3.15).

In order to address the question of the existence of a Hodge-proper spreading we first extend the standard spreading out results for finitely presentable schemes to the case of Artin stacks:

Theorem (2.1.13 and 2.3.2)**.** *Let $\{S_i\}$ be a filtered diagram of affine schemes with limit S. Fix a class of morphism $\mathcal{P}$ = proper, smooth, flat, surjective, or any other class of morphisms that satisfies the conditions of Definition 2.1.9. For an affine scheme T, let $\mathcal{S}\mathrm{tk}^{n\text{-Art},\mathrm{fp},\mathcal{P}}_{/T}$ denote the category of finitely presentable n-Artin stacks over T and morphisms in $\mathcal{P}$ between them. Then the natural functor*

$$\varinjlim_{i} \mathcal{S}\mathrm{tk}^{n\text{-Art},\mathrm{fp},\mathcal{P}}_{/S_i} \longrightarrow \mathcal{S}\mathrm{tk}^{n\text{-Art},\mathrm{fp},\mathcal{P}}_{/S}$$

(induced by base-change) is an equivalence.

As a corollary we deduce that any smooth n-Artin stack $\mathcal{X}$ over F admits a smooth spreading $\mathcal{X}_R$ over some finitely generated $\mathbb{Z}$-algebra $R \subset F$ and that any two such spreadings become equivalent after enlarging R. Since all smooth proper stacks are Hodge-proper (see Proposition 2.2.12), we immediately deduce the Hodge-to-de Rham degeneration in this case. Note that this includes smooth proper Deligne-Mumford stacks as a special case.

However, Hodge-proper spreadings need not to exist in general: one can show that the classifying stack BG is Hodge-proper for any finite-type group scheme G over F (see Proposition 2.3.6) but it is not necessarily Hodge-properly spreadable. Indeed, the classifying stack $B\mathbb{G}_a$ of the additive group has nontrivial Hodge cohomology but is de Rham contractible (i.e. has the de Rham cohomology of a point), so the Hodge-to-de Rham spectral sequence is clearly nondegenerate. By Theorem 1.4.3 it follows that it is not Hodge-properly spreadable and this forces the Hodge cohomology of $B\mathbb{G}_{a,\mathbb{Z}}$ to have infinitely generated p-torsion for a dense set of primes p, which one can also see from the explicit description (see Example 2.3.7 or Proposition A.2). This illustrates the general phenomenon: the non-degeneracy of the Hodge-to-de Rham spectral sequence in characteristic 0 is always reflected arithmetically, namely the Hodge cohomology of any spreading *has to be* infinitely generated over the base.

In the main case of our interest, namely the quotient stacks $\mathcal{X} = [X/G]$, we exhibit some sufficient conditions for Hodge-proper spreadability purely in terms of the geometry of X, G and the action $G \curvearrowright X$. In this case the spreadability is not easy to show, especially if we can't spread G to a linearly reductive group (which is only possible if G is a torus or an extension of a finite group by one). Nevertheless, using certain cohomological finiteness results from [FvdK10] we prove

Theorem (3.1.4)**.** *Let F be an algebraically closed field of characteristic 0. Let X be a smooth scheme and let Y be a finite-type quasi-separated scheme over F, both endowed with an action of a reductive group G. Assume that*

1. *There is a proper G-equivariant map $\pi\colon X \to Y$.*
2. *The G-action on Y is locally linear (Definition 3.1.1).*
3. *The categorical quotient $Y/\!/G$ is proper.*

Then the quotient stack $[X/G]$ is Hodge-properly spreadable[2]*.*

Theorem 3.1.4 applies to some natural examples of smooth schemes X with a G-action, in particular, equivariant "proper-over-affine" varieties (see Example 3.1.6) and the GIT quotients, whose coarse moduli space is proper (see Example 3.1.7).

We also prove a variant of Theorem 3.1.4 where we drop the reductivity assumption on G but impose an additional Bialynicki-Birula (BB)-completeness assumption on the action when restricted to a subgroup $h\colon \mathbb{G}_m \to G$. Moreover, the extra structure given by the map π is replaced by the internal condition on the properness of $h(\mathbb{G}_m)$-fixed points; see Theorem 3.2.12 for details.

Using the results of Halpern-Leistner (specifically, [HL20]) on Θ-stratifications we also show that a smooth stack $\mathcal{X}$, which is endowed with a Θ-stratification such that all strata (including the semistable locus) are cohomologically properly spreadable, is also cohomologically properly spreadable (see Corollary 3.3.5). This gives rise to new examples of Hodge-properly spreadable stacks where old ones appear as individual Θ-strata. In particular, this

[2] In fact we prove a stronger statement, namely that $[X/G]$ is cohomologically properly spreadable, see Definitions 2.2.2 and 2.3.1.

way, using Theorem 3.1.4 above, one can show that global quotients of KN-complete varieties are Hodge-properly spreadable; see Example 3.3.6.

As an application, for any Hodge-properly spreadable quotient stack $[X/G]$, we deduce an equivariant Hodge-to-de Rham degeneration:

Corollary (1.5.2)**.** *Let X be a smooth scheme over $\mathbb{C}$ endowed with an action of an algebraic group G such that the quotient stack $[X/G]$ is Hodge-properly spreadable. Then there is a (non-canonical) decomposition*

$$H^n_{G(\mathbb{C})}(X(\mathbb{C}),\mathbb{C}) \simeq \bigoplus_{p+q=n} H^q([X/G], \wedge^p \mathbb{L}_{[X/G]/\mathbb{C}}).$$

Finally, it turns out that Theorem 1.4.3 can be applied even in the case of some non-proper schemes, as we discuss in some detail in Section 2.3.3.

Remark 0.2.5. Even though all stacks in our main applications are classical (i.e. 1-Artin), the machinery developed in this work to prove Hodge-to-de Rham degeneration applies automatically to higher Artin stacks, so we did not put any artificial restrictions on the level of representablity of stacks considered in the paper. An example of a genuinely higher stack to which our method applies can be found in Section 2.3.4.

0.3 Relation to previous work and further directions

Our definition of Hodge-proper stacks is partially motivated by the work [HLP19] by Halpern-Leistner and Preygel, where several generalized notions of properness for stacks are studied. In Questions 1.3.2 and 1.3.3 of loc.cit. authors ask if any formally proper stack (Definition 1.1.3 of loc.cit.) admits a formally proper spreading and if the Hodge-to-de Rham spectral sequence degenerates for a formally proper stack over a field of characteristic 0. It follows from our work that the first statement implies the second; however, the method of Section 2.1 does not help to show the existence of a formally proper spreading. In fact, for the degeneration, only the existence of a Hodge-proper spreading would suffice, but this still seems pretty hard to show (see Question 3.3.9 in the end of our paper).

The splitting of the $(p-1)$-st truncation of the de Rham complex for a smooth tame 1-Artin stack over a perfect field k of characteristic p was established (among other things) in [Sat12]. The key observation in [Sat12] is that a smooth tame stack admits a smooth lift together with a lift of Frobenius étale-locally on its coarse moduli space, which enables to follow the original argument of Deligne-Illusie. Our proof is different and works for an arbitrary smooth n-Artin stack.

Even though the main examples of Hodge-spreadable stacks we construct in Section 2.3 are classical Artin stacks, Theorem 1.3.23 and Theorem 1.4.3 work equally well for higher ones. Thus we keep this level of generality throughout the paper. The spreading results of Section 2.1 in the case of classical Artin stacks are also covered in [Ryd15, Appendix B] and [LMB00, Chapter 4]. The use of [Pri15, Section 4] gives a clear way to extend these results to the setting of higher stacks, which we record in Section 2.1.

It is worth to mention that there is still no example of a smooth liftable scheme X in characteristic p whose Hodge-to-de Rham spectral sequence does not degenerate (recall that the Deligne-Illusie method gives such a degeneration only for $i + j < p$). Motivated by the recent examples of non-degeneration for the HKR-filtration constructed in [ABM19] one could first look for such a counterexample in the world of stacks. The de Rham cohomology of various classifying stacks were considered recently in great detail in [Tot18]; however, in all examples the Hodge-to-de Rham spectral sequence did degenerate.

The equivariant Hodge-to-de Rham degeneration for a reductive group G acting on a scheme X, under the Kempf-Ness-completeness assumption was treated (among other things) in [Tel00] by completely different methods. We reprove his result in a (slightly) more general setting (Example 3.3.6) using Θ-stratifications and the associated semiorthogonal decompositions constructed in [HL20]. The same strategy applies to any smooth Θ-stratified stack with cohomologically properly spreadable centra of the strata and the semistable locus (Corollary 3.3.5). [HL18, Section 4] could provide more examples of stacks that are Hodge-properly spreadable.

Another approach to the equivariant Hodge theory was introduced in [HLP15]. There the authors deduce (among other things) the noncommutative Hodge-to-de Rham degeneration for the category of perfect complexes $\mathrm{QCoh}([X/G])^{\mathrm{perf}}$ for a KN-complete X and for some purely non-commutative examples (like the categories of matrix factorizations), by exploiting methods of non-commutative geometry. Note that the result of Kaledin (see [Kal08] and [Kal17]) does not apply in this situation, since the DG-category $\mathrm{QCoh}([X/G])^{\mathrm{perf}}$ is usually not smooth. It is natural to ask whether the commutative degeneration implies the noncommutative one in this case. This is not immediately clear, since the relation between the Hochschild/periodic cyclic homology of the category of perfect complexes and the Hodge/de Rham cohomology for Artin stacks is more subtle than in the case of schemes.

0.4 Plan of the paper

Section 1 is devoted to a proof of the degeneration of the Hodge-to-de Rham spectral sequence for Hodge-properly spreadable stacks. In Subsections 1.1 and 1.2 we review Hodge and de Rham cohomology of stacks, define Hodge-proper stacks and prove some technical lemmas about them. In Section 1.3 we prove (a truncated version of) the Hodge-to-de Rham degeneration in positive characteristic for Hodge-proper stacks that admit a lift to $W_2(k)$. Then, in Section 1.4 we prove the Hodge-to-de Rham degeneration in characteristic 0 for stacks that are Hodge-properly spreadable. As a corollary, in Section 1.5, in the case of a quotient stack, we also deduce a (non-canonical) Hodge decomposition for the corresponding equivariant singular cohomology.

In Section 2 we study the spreadability of Hodge proper stacks. In Subsection 2.1 we extend the standard spreading out results for finitely presented schemes and their morphisms to the case of Artin stacks (see Theorem 2.1.13). In 2.2 we introduce a more convenient class of cohomologically proper stacks which includes all Hodge-proper ones. In Section 2.3 we give first examples of spreadable Hodge-proper stacks: in Section 2.3.1 we cover the case of smooth proper stacks, in Section 2.3.2 we discuss for which algebraic groups G the classifying stack BG is Hodge-properly spreadable. Then, in Section 2.3.3 we discuss the case schemes.

In Section 3 we concentrate on the spreadability of quotient stacks. In Section 3.1 we discuss the case of global quotients by reductive groups whose coarse moduli space is proper. In Section 3.2.2 we prove Hodge-proper spreadability of $[X/\mathbb{G}_m]$ under the condition that the associated Bialynicki-Birula stratification is full and $X^{\mathbb{G}_m}$ is proper; then, in Section 3.2 we use this to prove spreadability for a more general class of global quotients, including quotients by some non-reductive groups. Finally, in Section 3.3 we show that finite Θ-stratifications spread out; then using the results of [HL20] we show that if all Θ-strata (or rather their centra) together with the semistable locus have cohomologically proper spreadings, then so does the original Θ-stratified smooth stack $\mathcal{X}$. In Example 3.3.6 we show how to establish the cohomologically proper spreadability of Kempf-Ness (KN-)complete quotient stacks using this method.

Acknowledgments. We are grateful to Bhargav Bhatt for his interest in our work and helpful advice concerning the Hodge-to-de Rham degeneration in characteristic p.[3] We are also grateful to Daniel Halpern-Leistner for sharing the draft of [HL20] and answering our questions about [HL15]. We would like to thank Davesh Maulik, Sasha Petrov, Vadim Vologodsky, and Roman Travkin for many helpful discussions. We also want to thank Borys Kadets and Chris Brav for carefully reading drafts of this text and providing many useful remarks and suggestions. Finally, we would like to thank anonymous referees for many helpful advises, various comments and questions concerning different sections of the paper.

The second author was partially supported by Laboratory of Mirror Symmetry NRU HSE, RF Government grant, ag. №14.641.31.0001 and by the Simons Foundation.

Notations and conventions.

1. We will freely use the language of higher categories, modeled e.g. by quasi-categories of [Lur09]. If not explicitly stated otherwise all categories are assumed to be $(\infty, 1)$ and all (co-)limits are homotopy ones. The $(\infty, 1)$-category of Kan complexes will be denoted by $\mathcal{S}$ and we will call it *the category of spaces*. By $\operatorname{Lan}_i F$ and $\operatorname{Ran}_i F$ we will denote left and right Kan extensions of a functor F along i (see e.g. [Lur09, Definition 4.3.2.2] for more details).

2. For a commutative ring R by $D(\mathrm{Mod}_R)$ we will denote the canonical $(\infty, 1)$-enhancement of the triangulated unbounded derived category of the abelian category of R-modules Mod_R. All tensor product, pullback and push-forward functors are implicitly derived.

3. In this work by Artin stacks we always mean (higher) Artin stacks in the sense of [TV08, Section 1.3.3] or [GR17, Chapter 2.4]: these are sheaves in étale topology admitting a smooth $(n-1)$-representable atlas for some $n \geq 0$ (an inductively defined notion, see *loc.cit.* for more details). We stress that we work with non-derived Artin stacks, i.e. they are defined on the category of *ordinary* commutative rings. When we need to emphasize a precise dependence on n (usually in inductive arguments) we say that $\mathcal{X}$ is an n-Artin stack. We denote the ∞-category of n-Artin stacks over a base scheme S by $\mathcal{S}\mathrm{tk}_{/S}^{n\text{-Art}}$. We also freely use the notion of quasi-compact quasi-separated morphism between Artin stacks from [GR17, Chapter 2, Section 4.1.9].

4. For a stack $\mathcal{X}$ we will denote by $\mathrm{QCoh}(\mathcal{X})$ the category of *quasi-coherent sheaves on* $\mathcal{X}$ defined as the limit $\lim_{\operatorname{Spec} A \to \mathcal{X}} D(\mathrm{Mod}_A)$ over all affine schemes $\operatorname{Spec} A$ mapping to $\mathcal{X}$ (see [GR17, Chapter 3.1] for more details).

[3] In fact his suggestion that one should be able to prove the Hodge-to-de Rham degeneration for some "cohomologically proper" stacks via the Deligne-Illusie method basically started this project.

Note that $\mathrm{QCoh}(\mathcal{X})$ admits a natural t-structure such that $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^{\leq 0}$ if and only if $x^*(\mathcal{F}) \in D(\mathrm{Mod}_A)^{\leq 0}$ for any A-point $x \in \mathcal{X}(A)$. Moreover, by [GR17, Chapter 3, Corollary 1.5.7] if $\mathcal{X}$ is Artin stack, then $\mathrm{QCoh}(\mathcal{X})$ is left- and right-complete (i.e. Postnikov's and Whitehead's towers converge) and the truncation functors commute with filtered colimits.

5. For an affine group scheme G over a ring R, given a representation M (i.e. a comodule over the corresponding Hopf algebra $R[G]$) we denote by $R\Gamma(G, M) \in D(\mathrm{Mod}_A)$ the *rational cohomology* complex of G, namely the derived functor of G-invariants $M \mapsto M^G$. By flat descent, for G flat over R, the abelian category $\mathrm{Rep}(G) := \mathrm{Rep}_G(\mathrm{Vect}_F)^{\heartsuit}$ is identified with $\mathrm{QCoh}(BG)^{\heartsuit}$ and $R\Gamma(G, M) \simeq R\Gamma(BG, M)$.

1 Degeneration of the Hodge-to-de Rham spectral sequence

1.1 Hodge and de Rham cohomology

In this section we set up the Hodge-to-de Rham spectral sequence for n-Artin stacks and prove some technical results needed in subsequent sections of the paper. For the rest of this section fix a base ring R. We refer the reader to [TV08] for an introduction to the theory of Artin stacks and cotangent complexes.

Definition 1.1.1 (Hodge cohomology)**.** Let $\mathcal{X}$ be an Artin stack over R. Define *Hodge cohomology* $R\Gamma_{\mathrm{H}}(\mathcal{X}/R)$ *of* $\mathcal{X}$ to be

$$R\Gamma_{\mathrm{H}}(\mathcal{X}/R) := \bigoplus_{p\geq 0} R\Gamma\left(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R}[-p]\right),$$

where $\mathbb{L}_{\mathcal{X}/R}$ is the cotangent complex of $\mathcal{X}$ over R and $\wedge^p\mathbb{L}_{\mathcal{X}/R}$ is its p-th derived exterior power (see [Ill71, Chapitre I.4] or [BM19, Section 3]). For a fixed $n \in \mathbb{Z}$ we will also denote

$$H^n_{\mathrm{H}}(\mathcal{X}/R) := H^n R\Gamma_{\mathrm{H}}(\mathcal{X}/R) \simeq \bigoplus_{p+q=n} H^{p,q}(\mathcal{X}/R), \quad \text{where} \quad H^{p,q}(\mathcal{X}/R) := H^q\left(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R}\right).$$

Notation 1.1.2. Let $S := \operatorname{Spec} A$ be an affine smooth R-scheme. The algebraic de Rham complex of S over R

$$A \xrightarrow{d} \Omega^1_{A/R} \xrightarrow{d} \Omega^2_{A/R} \xrightarrow{d} \dots$$

will be denoted by $\Omega^\bullet_{S/R,\mathrm{dR}}$. We define $R\Gamma_{\mathrm{dR}}(S/R) := \Omega^\bullet_{S/R,\mathrm{dR}} \in D(\mathrm{Mod}_R)$.

Definition 1.1.3 (de Rham cohomology)**.** Let $\mathcal{X}$ be a smooth quasi-compact quasi-separated Artin stack over R. Define the *(Hodge-completed) de Rham cohomology* $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ *of* $\mathcal{X}$ to be

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) := \lim_{S\in \mathrm{Aff}^{\mathrm{sm,op}}_{/\mathcal{X}}} R\Gamma_{\mathrm{dR}}(S/R),$$

where $\mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$ is the full subcategory of stacks over $\mathcal{X}$ consisting of affine R-schemes that are smooth over $\mathcal{X}$. We will also denote $H^n R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ by $H^n_{\mathrm{dR}}(\mathcal{X}/R)$.

In fact the Hodge cohomology complex admits a description similar to our definition of the de Rham cohomology:

Proposition 1.1.4. *For any $p \in \mathbb{Z}_{\geq 0}$ the natural map*

$$R\Gamma(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/R}) \to \lim_{S\in \mathrm{Aff}^{\mathrm{sm,op}}_{/\mathcal{X}}} R\Gamma(S, \wedge^p\mathbb{L}_{S/R}) \tag{2}$$

is an equivalence.

Proof. By Proposition 1.1.5 below the left hand side satisfies smooth descent. It follows that both sides of (2) satisfy smooth descent. Since n-Artin stacks are by definition iterated smooth quotients of schemes, by induction on n we reduce to the case when $\mathcal{X}$ is a smooth affine scheme, where the assertion of the proposition is true, since $\mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$ has a final object given by $\mathcal{X}$. ∎

Proposition 1.1.5 (Flat descent for the cotangent complex)**.** *Let $p\colon \mathcal{U} \to \mathcal{X}$ be a surjective quasi-compact quasi-separated flat morphism between Artin stacks and denote by $p_\bullet\colon \mathcal{U}_\bullet \to \mathcal{X}$ the corresponding Čech nerve. Then the natural map*

$$\wedge^d\mathbb{L}_{\mathcal{X}/R} \longrightarrow \operatorname{Tot} p_{\bullet *}(\wedge^d\mathbb{L}_{\mathcal{U}_\bullet/R})$$

is an equivalence for each $d \in \mathbb{Z}_{\geq 0}$.

Proof. The proof is essentially due to Bhatt (see [Bha12a, Corollary 2.7, Remark 2.8] or [BMS19, Section 3]). For every $n \in \mathbb{Z}_{\geq 0}$ we have a co-fiber sequence

$$p_n^*\mathbb{L}_{\mathcal{X}/R} \longrightarrow \mathbb{L}_{\mathcal{U}_n/R} \longrightarrow \mathbb{L}_{\mathcal{U}_n/\mathcal{X}}$$

in $\mathrm{QCoh}(\mathcal{U}_n)$. It follows that $\wedge^d\mathbb{L}_{\mathcal{U}_n/R}$ admits a d-step filtration with associated graded pieces $\wedge^i p_n^*\mathbb{L}_{\mathcal{X}/R} \otimes \wedge^{d-i}\mathbb{L}_{\mathcal{U}_n/\mathcal{X}}$. Note that by flat descent for QCoh (see e.g. [Lur18, Corollary D.6.3.4])

$$\mathrm{Tot}\, p_{\bullet *}\, \mathrm{gr}^0 \left(\wedge^d\mathbb{L}_{\mathcal{U}_\bullet/R}\right) = \mathrm{Tot}\, p_{\bullet *}p_\bullet^* \wedge^d \mathbb{L}_{\mathcal{X}/R} \simeq \wedge^d\mathbb{L}_{\mathcal{X}/R}.$$

Hence it is enough to prove that

$$\mathrm{Tot}\, p_{\bullet *}\, \mathrm{gr}^i \left(\wedge^d\mathbb{L}_{\mathcal{U}_\bullet/R}\right) = \mathrm{Tot}\, p_{\bullet *}(p_\bullet^* \wedge^i \mathbb{L}_{\mathcal{X}/R} \otimes \wedge^{d-i}\mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}}) \simeq 0$$

for $i > 0$. Moreover, since the map p is faithfully flat, it is enough to show that the pullback

$$p^*\, \mathrm{Tot}\, p_{\bullet *}\, \mathrm{gr}^i \left(\wedge^d\mathbb{L}_{\mathcal{U}_\bullet/R}\right) \simeq p^*\, \mathrm{Tot}(\wedge^i\mathbb{L}_{\mathcal{X}/R} \otimes p_{\bullet *} \wedge^{d-i} \mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}})$$

of the totalization above is null-homotopic.

Note that by the qcqs assumption $p^*p_{n*} \simeq q_{n*}\mathrm{pr}_n^*$, where $q_\bullet\colon \mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U}$ is the pullback of the Čech nerve $p_\bullet\colon \mathcal{U}_\bullet \to \mathcal{X}$ on $\mathcal{U}$ along p and $\mathrm{pr}_n\colon \mathcal{U}_n \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U}_n$ are natural projections. It follows by base change for the relative cotangent complex that

$$p^*p_{\bullet *} \wedge^{d-i} \mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}} \simeq q_{\bullet *} \wedge^{d-i} \mathbb{L}_{\mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U}/\mathcal{U}}.$$

But since $\mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U}$ is a split simplicial object, the same holds for

$$p^*(\wedge^i\mathbb{L}_{\mathcal{X}/R} \otimes p_{\bullet *} \wedge^{d-i} \mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}}) \simeq p^*(\wedge^i\mathbb{L}_{\mathcal{X}/R}) \otimes q_{\bullet *} \wedge^{d-i} \mathbb{L}_{\mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U}/\mathcal{U}},$$

since the class of split simplicial objects is stable under any functor. It follows $\mathrm{Tot}\, p^*p_{\bullet *}\, \mathrm{gr}^i \left(\wedge^d\mathbb{L}_{\mathcal{U}_\bullet/R}\right) \simeq 0$. Finally, since by flat descent $\mathrm{QCoh}(\mathcal{X})$ is comonadic over $\mathrm{QCoh}(\mathcal{U})$, the pullback functor p^* preserves totalizations of p^*-split cosimplicial objects, hence

$$p^*\, \mathrm{Tot}\, p_{\bullet *}\, \mathrm{gr}^i \left(\wedge^d\mathbb{L}_{\mathcal{U}_\bullet/R}\right) \simeq \mathrm{Tot}\, p^*p_{\bullet *}\, \mathrm{gr}^i \left(\wedge^d\mathbb{L}_{\mathcal{U}_\bullet/R}\right) \simeq 0. \quad \square$$

Corollary 1.1.6. *Let $\mathcal{X}$ be a smooth quasi-compact quasi-separated Artin stack over R. Then*

1. *There exists a complete (decreasing)* Hodge filtration *$F^\bullet R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ such that $\mathrm{gr}\, F^\bullet R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \simeq R\Gamma_{\mathrm{H}}(\mathcal{X}/R)$.*
2. *There exists a strongly convergent spectral sequence $E_1^{p,q} = H^q(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/R}) \Rightarrow H^{p+q}_{\mathrm{dR}}(\mathcal{X}/R)$.*

Proof. Note that since $\mathcal{X}$ is smooth, all schemes $S \in \mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$ are smooth. Since the Hodge filtration on $R\Gamma_{\mathrm{dR}}(S/R)$ is complete, the same holds for $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$, since complete filtered complexes are closed under limits. Moreover, by construction we have

$$\mathrm{gr}\, F^\bullet R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \simeq \lim_{S \in \mathrm{Aff}^{\mathrm{sm,op}}_{/\mathcal{X}}} \mathrm{gr}\, F^\bullet R\Gamma_{\mathrm{dR}}(S/R) \simeq \lim_{S \in \mathrm{Aff}^{\mathrm{sm,op}}_{/\mathcal{X}}} R\Gamma_{\mathrm{H}}(S/R) \simeq R\Gamma_{\mathrm{H}}(\mathcal{X}/R),$$

where the last equivalence follows from the previous proposition. This filtration induces a spectral sequence with $E_1^{p,q}$ as stated. To prove it is strongly convergent, note that by smoothness of $\mathcal{X}$, for each n the induced filtration on $H^n_{\mathrm{dR}}(\mathcal{X}/R)$ is finite. $\square$

The following simple observation will be quite useful in what follows:

Remark 1.1.7. Let $\mathcal{X}$ be a smooth Artin stack over R. Then the cotangent complex $\mathbb{L}_{\mathcal{X}/R}$ (and its exterior powers) is concentrated in nonnegative cohomological degrees (with respect to the natural t-structure on $\mathrm{QCoh}(\mathcal{X})$). Since the global section functor $R\Gamma$ is left t-exact, it follows that the natural map $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \to R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)/F^pR\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ induces an isomorphism on $H^{<p}$.

Finally, we will need the following

Proposition 1.1.8 (Base-change)**.** *Let $\mathcal{X}$ be a smooth quasi-compact quasi-separated Artin stack over R and let $R \to R'$ be a ring homomorphism of finite Tor-amplitude. Then for $\mathcal{X}' \coloneqq \mathcal{X} \otimes_R R'$ the natural map $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \otimes_R R' \to R\Gamma_{\mathrm{dR}}(\mathcal{X}'/R')$ is a filtered equivalence. In particular, for each $p \in \mathbb{Z}_{\geq 0}$ the natural map $R\Gamma(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/R}) \otimes_R R' \to R\Gamma(\mathcal{X}', \wedge^p\mathbb{L}_{\mathcal{X}'/R'})$ is an equivalence.*

Proof. By the smoothness assumption on $\mathcal{X}$ the fiber product $\mathcal{X} \otimes_R R'$ coincides with the derived fiber product. It follows by [TV08, Lemma 1.4.1.16 (2)] that $\mathbb{L}_{\mathcal{X}/R} \otimes_R R' \simeq \mathbb{L}_{\mathcal{X}'/R'}$. By the base change for QCoh (see [GR17, Chapter 3., Proposition 2.2.2 (b)]) we deduce that the natural map $R\Gamma_{\mathrm{H}}(\mathcal{X}/R) \otimes_R R' \to R\Gamma_{\mathrm{H}}(\mathcal{X}'/R')$ is an equivalence.

Next, note that the condition on the morphism $R \to R'$ guarantees that the natural map $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \otimes_R R' \to \lim\limits_{\longleftarrow p}((R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)/F^p R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)) \otimes_R R')$ is an equivalence. Since both sides are complete with respect to the Hodge filtration, and, since by the above the induced map on the associated graded pieces

$$R\Gamma_{\mathrm{H}}(\mathcal{X}/R) \otimes_R R' \simeq \operatorname{gr} F^\bullet R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \otimes_R R' \to \operatorname{gr} F^\bullet R\Gamma_{\mathrm{dR}}(\mathcal{X}'/R') \simeq R\Gamma_{\mathrm{H}}(\mathcal{X}'/R')$$

is an equivalence, we deduce that the base-change map for de Rham cohomology is an equivalence as well. □

1.2 Hodge-proper stacks

For the rest of this subsection fix a Noetherian base ring R. In this section we will introduce a reasonable (at least from the point of view of Hodge-to-de Rham degeneration) generalization of the notion of properness for stacks.

Definition 1.2.1. A complex of R-modules M is called *bounded below coherent*[4] if it is cohomologically bounded below and for any $i \in \mathbb{Z}$ the cohomology module $H^i(M)$ is finitely generated over R. We will denote the full subcategory of $D(\mathrm{Mod}_R)$ consisting of bounded below coherent R-modules by $\mathrm{Coh}^+(R)$.

Remark 1.2.2. We use the term *nearly* coherent for objects of $\mathrm{Coh}^+(R)$ to distinguish them from coherent complexes, which in our convention are necessarily bounded (both from above and below).

We have the following basic properties of $\mathrm{Coh}^+(R)$:

Proposition 1.2.3. *Let R be a Noetherian ring. Then:*

1. *The category $\mathrm{Coh}^+(R)$ is closed under finite (co-)limits and retracts. In particular $\mathrm{Coh}^+(R)$ is a stable subcategory of $D(\mathrm{Mod}_R)$.*
2. *For each $n \in \mathbb{Z}$ the category $\mathrm{Coh}^{\geq n}(R) := \mathrm{Coh}^+(R) \cap D(\mathrm{Mod}_R)^{\geq n}$ is closed under totalizations.*

Proof. 1. This follows from the fact that for a Noetherian R the abelian category of finitely generated R-modules is closed under (co)kernels, extensions and direct summands.

2. Let $M^\bullet$ be a co-simplicial object of $\mathrm{Coh}^{\geq n}(R)$. By shifting if necessary, we can assume that $n = 0$. Since coconnective modules are closed under limits, $\mathrm{Tot}(M^\bullet) \in D(\mathrm{Mod}_R)^{\geq 0}$; hence it is enough to prove that $H^i \,\mathrm{Tot}(M^\bullet)$ is finitely generated R-module for all $i \in \mathbb{Z}_{\geq 0}$. Since all M^i are coconnective, the natural map $\mathrm{Tot}(M^\bullet) \to \mathrm{Tot}^{\leq k}(M^\bullet)$ induces an isomorphism on $H^{\leq k}$. But since $\mathrm{Tot}^{\leq k}$ is a finite limit, each $H^i\,\mathrm{Tot}^{\leq k}(M^\bullet)$ is a finitely generated R-module. □

Remark 1.2.4. Recall that the category of perfect R-modules $D(\mathrm{Mod}_R)^{\mathrm{perf}}$ is defined as the smallest full subcategory of $D(\mathrm{Mod}_R)$ containing R and closed under finite (co-)limits and direct summands. Since $R \in \mathrm{Coh}^+(R)$ it follows from Proposition 1.2.3, that $D(\mathrm{Mod}_R)^{\mathrm{perf}} \subseteq \mathrm{Coh}^+(R)$.

After this technical digression we are ready to introduce the notion of a Hodge-proper stack:

Definition 1.2.5 (Hodge-proper stacks)**.** A smooth quasi-compact quasi-separated Artin stack $\mathcal{X}$ over R is called *Hodge-proper* if for every $p \in \mathbb{Z}_{\geq 0}$ the complex $R\Gamma(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R})$ is bounded below coherent.

For us the most important implication of Hodge-properness is that the de Rham cohomology is bounded below coherent:

Proposition 1.2.6. *Let $\mathcal{X}$ be a smooth Hodge-proper Artin stack over R. Then $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ is bounded below coherent complex of R-modules.*

Proof. By smoothness $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ is bounded below by 0, hence it is enough to prove that for each $n \in \mathbb{Z}_{\geq 0}$ the cohomology module $H^n_{\mathrm{dR}}(X/R)$ is finitely generated over R. By Remark 1.1.7 the natural map $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \to R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)/F^{n+1}R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ induces an isomorphism on $H^{\leq n}$. We conclude, since $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)/F^{n+1}R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$, being a finite extension of bounded below coherent complexes $R\Gamma(\mathcal{X}, \wedge^i \mathbb{L}_{\mathcal{X}/R}[-i])$, $0 \leq i \leq n$, is bounded below coherent. □

[4] In the previous version of this text we called such complexes *almost coherent*. We decided to change the notation to avoid possible clashes with almost mathematics.

1.3 Hodge-to-de Rham degeneration in positive characteristic

Let $\mathcal{Y}$ be a Hodge-proper Artin stack over a perfect field k of characteristic p admitting a smooth lift to the ring of the second Witt vectors $W_2(k)$. In this section we will prove that the Hodge-to-de Rham spectral sequence $H^j(\mathcal{Y}, \wedge^i \mathbb{L}_{\mathcal{Y}/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(\mathcal{Y}/k)$ degenerates at the first page for $i+j < p$. Our strategy is to interpret both Hodge and de Rham cohomology in terms of crystalline cohomology and then, following Fontaine-Messing [FM87] (and Bhatt-Morrow-Scholze [BMS19]), use (quasi-)syntomic descent for the crystalline cohomology to get a very functorial form of the Deligne-Illusie splitting.

We denote by $\sigma\colon k \xrightarrow{x\mapsto x^p} k$ the absolute Frobenius morphism of k. We denote by the same letter σ the induced automorphisms $W(k) \to W(k)$ and $W_n(k) \to W_n(k)$ for any $n \in \mathbb{N}$. For a $W(k)$-algebra (e.g. a $W_s(k)$-algebra for some s) A we denote by $A^{(1)} := A \otimes_{W(k),\sigma} W(k)$ its Frobenius twist and by $A^{(-1)} := A \otimes_{W(k),\sigma^{-1}} W(k)$ its Frobenius untwist. For each $n \in \mathbb{Z}$ we have the relative Frobenius map $\varphi_A\colon A^{(n)} \to A^{(n-1)}$.

Definition 1.3.1 ([BMS19, Definition 4.10])**.** A morphism $A \to B$ of $W_n(k)$-algebras is called *quasisyntomic* if it is flat and $\mathbb{L}_{B/A}$ has cohomological Tor amplitude $[-1, 0]$. A morphism $A \to B$ is *a quasisyntomic cover* if it is quasisyntomic and faithfully flat. We will denote by QSyn_n the site consisting of quasisyntomic $W_n(k)$-algebras with the topology generated by quasisyntomic covers.

Remark 1.3.2. It probably worth clarifying how our definition of quasisyntomic site compares to the one in [BMS19, Section 4]. Namely, QSyn_n is just the *small* quasisyntomic site of $W_n(k)$ in the terminology of [BMS19]. Indeed since all algebras in QSyn_n are killed by p^n, the notions of p-complete (faithful) flatness and quasisyntomicity coincide with the classical ones. The rest of the properties can be easily seen to agree as well.

The notion of a quasisyntomic morphism is a generalization of more classical notion of a *syntomic* morphism: a flat map $A \to B$ that is locally a complete intersection in a smooth one. Syntomic morphisms include smooth morphisms, and, in the case A is a regular k-algebra, the relative Frobenius morphism $\varphi\colon A^{(1)} \to A$. The advantage of quasisyntomic morphisms is that they also include some natural non-finite-type maps, most importantly the direct limit perfection $A \to A_{\mathrm{perf}} := \varinjlim_{\varphi, n\geq 0} A^{(-n)}$ and its tensor powers $A \to A_{\mathrm{perf}} \otimes_A \ldots \otimes_A A_{\mathrm{perf}}$ for a smooth k-algebra A. Using standard properties of the cotangent complex it is not hard to show that quasisyntomic morphisms are stable under composition and pushouts along arbitrary morphisms of algebras (and same for quasisyntomic covers). We refer to Section 4 of [BMS19] for more details.

Recall that an $\mathbb{F}_p$-algebra S is called semiperfect if $\varphi\colon S \to S$ is surjective.

Definition 1.3.3. A k-algebra S is called *quasiregular semiperfect* if S is quasisyntomic and the relative Frobenius homomorphism $\varphi\colon S^{(1)} \to S$ is surjective. We call a $W_n(k)$-algebra $\widetilde{S}$ *quasiregular semiperfectoid* if it is flat over $W_n(k)$ and $\widetilde{S}/p$ is quasiregular semiperfect. We will denote by $\mathrm{QRSPerf}_n$ the site consisting of quasiregular semiperfectoid $W_n(k)$-algebras with the topology generated by quasisyntomic covers.

Remark 1.3.4. Note that if $n > 1$ our definition of a quasiregular semiperfectoid algebra over $W_n(k)$ does not agree with [BMS19, Definition 4.10] since $W_n(k)$ itself is not semiperfectoid. Nevertheless, since we assume that all our objects are flat over $W_n(k)$, all the necessary arguments go through essentially without any change by reducing modulo p.

For any k-algebra S, $H^0(\mathbb{L}_{S/k})$ is identified with the Kähler differentials $\Omega^1_{S/k}$. Since $d(x^p) = 0$, we get that $H^0(\mathbb{L}_{S/k}) = 0$ for S semiperfect, and that $\mathbb{L}_{S/k}$ is concentrated in a single cohomological degree -1 for S quasiregular semiperfect. The same is true for $\mathbb{L}_{\widetilde{S}/W_n(k)}$ for a quasiregular semiperfectoid $W_n(k)$-algebra $\widetilde{S}$. Moreover, any flat map $\widetilde{S}_1 \to \widetilde{S}_2$ between quasiregular semiperfectoids over $W_n(k)$ is quasisyntomic. This gives a map of sites $\mathrm{QRSPerf}_n \to \mathrm{QSyn}_n$.

In fact quasiregular semiperfectoid algebras form a basis of topology in QSyn_n. This leads to an equivalence between the corresponding categories of sheaves:

Proposition 1.3.5. *The restriction along the natural embedding $u\colon \mathrm{QRSPerf}_n \to \mathrm{QSyn}_n$ induces an equivalence*

$$\mathcal{S}\mathrm{hv}(\mathrm{QSyn}_n, \mathcal{C}) \xrightarrow[\sim]{u^{-1}} \mathcal{S}\mathrm{hv}(\mathrm{QRSPerf}_n, \mathcal{C})$$

of the categories of sheaves with values in any presentable ∞-category $\mathcal{C}$.

Proof. Following the proof of [BMS19, Proposition 4.31] it is enough to show, that first, any quasisyntomic algebra A has a quasisyntomic cover $A \to S$ by a semiperfectoid, and second, that all terms $S^{\otimes_A i}$ in the corresponding Čech object are automatically semiperfectoid. The cover S can be constructed as follows: we take the surjection $W_n(k)[x_a]_{a\in A} \twoheadrightarrow A$ from the free polynomial algebra on A and put $S \coloneqq A \otimes_{W_n(k)[x_a]} W_n(k)[x_a^{1/p^\infty}]$. The map $W_n(k)[x_a] \to W_n(k)[x_a^{1/p^\infty}]$ is quasisyntomic and faithfully flat, thus so is $A \to S$. Also $W_n(k)[x_a^{1/p^\infty}] \to S$ is a surjection, $k[x_a^{1/p^\infty}]$ is perfect, thus S/p is semiperfect. We get that $S \in \mathrm{QRSPerf}_n$. The statement about $S^{\otimes_A i}$ then follows from the analogous one modulo p (see e.g. [BMS19, Lemma 4.30]). □

Remark 1.3.6. For a sheaf $\mathcal{F}$ on $\mathrm{QRSPerf}_n$ we will denote its image under the inverse equivalence in Proposition 1.3.5 by $\mathcal{F}$ as well.

Example 1.3.7. Let B be a smooth algebra over $W_n(k)$. By smoothness, Zariski-locally on $\operatorname{Spec} B$, there exists an étale map $P \to B$ from the polynomial algebra $P = P_d \coloneqq W_n(k)[x_1, \dots, x_d]$ for some d. Let $P_{\mathrm{perf}} = W_n(k)[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]$ and let $B_{\mathrm{perf}} \coloneqq B \otimes_P P_{\mathrm{perf}}$; it is a quasiregular semiperfectoid $W_n(k)$-algebra[5] and the natural map $B \to B_{\mathrm{perf}}$ is a quasisyntomic cover. Moreover all terms $(B_{\mathrm{perf}} \otimes_B \ldots \otimes_B B_{\mathrm{perf}})_n$ in the corresponding Cech object are also quasiregular semiperfectoids. Given any sheaf $\mathcal{F}$ on $\mathrm{QRSPerf}_n$ its value on $B \in \mathrm{QSyn}_n$ (via Proposition 1.3.5) can be computed as "the unfolding":

$$R\Gamma_{\mathrm{QSyn}_n}(B, \mathcal{F}) \xrightarrow{\sim} \mathrm{Tot}\Big(\, \mathcal{F}(B_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}}) \rightrightarrows \cdots \Big).$$

For a ring R let $\mathrm{Poly}_R \subset \mathrm{CAlg}_{R/}$ denote the full subcategory of finitely generated *polynomial R-algebras*. Recall that one of the ways to define the cotangent complex $\mathbb{L}_{A/R}$ for an R-algebra A is to consider the left Kan extension of the functor $B \mapsto \Omega^1_{B/R}$ from the category of polynomial R-algebras, namely

$$\mathbb{L}_{A/R} \simeq \operatorname*{colim}_{\mathrm{Poly}_R/A} \Omega^1_{B/R}.$$

One can extend the de Rham and crystalline cohomology functors in a similar way:

Construction 1.3.8. Let k be a perfect field.

- The *derived de Rham cohomology* functor

$$R\Gamma_{\mathbb{L}\mathrm{dR}}(-/W_n(k))\colon \mathrm{CAlg}_{W_n(k)/} \to D(\mathrm{Mod}_{W_n(k)})$$

  is defined as the left Kan extension of the functor $B \mapsto \Omega^\bullet_{B/W_n(k),\mathrm{dR}}$ on $\mathrm{Poly}_{W_n(k)}$.

- The *derived crystalline cohomology* functor

$$R\Gamma_{\mathbb{L}\mathrm{crys}}(-/W(k))\colon \mathrm{CAlg}_{W_n(k)/} \to D(\mathrm{Mod}_{W(k)})$$

  is defined as the (derived) p-adic completion of the left Kan extension of the functor $B \mapsto R\Gamma_{\mathrm{crys}}((B/p)/W(k))$ on $\mathrm{Poly}_{W_n(k)}$.

Remark 1.3.9. For a more thorough treatment of the derived de Rham and crystalline cohomology functors we refer the reader to [Ill72] and [Bha12a] where these notions were originally considered and applied.

Remark 1.3.10. For any $W_n(k)$-algebra B the complexes $R\Gamma_{\mathbb{L}\mathrm{crys}}(B/W(k)) \otimes_{W(k)} W_n(k)$ and $R\Gamma_{\mathbb{L}\mathrm{dR}}(B/W_n(k))$ are naturally equivalent. Indeed, by construction both functors commute with geometric realizations, hence it is enough to prove the statement for B being a smooth $W_n(k)$-algebra. In this case this is a basic result in the crystalline cohomology theory, see e.g. [BO78, Corollary 7.4].

Similarly, we can extend the functor $B \mapsto \tau^{\le m}\Omega^\bullet_{B/W_n(k),\mathrm{dR}}$ to get a filtered object $(R\Gamma_{\mathbb{L}\mathrm{dR}}(-/W_n(k))\,, F^*_{\mathrm{H}})$ (Hodge filtration). If $n=1$ the functor $B \mapsto \Omega^{\le m}_{B/W_n(k),\mathrm{dR}}$ extends to $(R\Gamma_{\mathbb{L}\mathrm{dR}}(-/k), \mathrm{Fil}_*^{\mathrm{conj}})$ (conjugate filtration). The conjugate filtration on the derived de Rham cohomology is exhaustive since it is exhaustive on de Rham cohomology of polynomial algebras and since colimits commute. For $R\Gamma_{\mathbb{L}\mathrm{dR}}(-/k)$ the Cartier isomorphism identifies the corresponding associated graded with $\bigoplus_{r\ge 0} \wedge^r \mathbb{L}_{B^{(1)}/k}[-r]$. Next lemma shows that the derived de Rham cohomology on the quasi-syntomic site satisfies flat descent:

[5] In fact it is even *quasismooth perfectoid*, since $\mathbb{L}_{B/W_n(k)} \simeq 0$ and the relative Frobenius for B_{perf}/p is an isomorphism.

Lemma 1.3.11. *Let $A \to B$ be a faithfully flat homomorphism of k-algebras and let $B^\bullet$ be the corresponding Čech co-simplicial object. Assume that $\mathbb{L}_{B/k}$ and $\mathbb{L}_{B/A}$ have cohomological* Tor*-amplitude* $[-1;0]$. *Then the natural map*

$$R\Gamma_{\mathbb{L}\mathrm{dR}}(A/k) \to \mathrm{Tot}\, R\Gamma_{\mathbb{L}\mathrm{dR}}(B^\bullet/k)$$

is an equivalence.

Proof. Note that since $A \to B$ is faithfully flat the base change for cotangent complex and transitivity triangles imply that $\mathbb{L}_{B^i/k}$ has Tor-amplitude in degrees $[-1;0]$ for all $i \geq 0$, where B^i is the i-th term in the corresponding cosimplicial Čech object. Consequently, $\wedge^n \mathbb{L}_{B^{i(1)}/k}[-n]$ is 0-coconnective for any n and i. It then follows by flat descent for cotangent complex (see [BMS19, Theorem 3.1]) that the natural map

$$F^n_{\mathrm{conj}} R\Gamma_{\mathbb{L}\mathrm{dR}}(A/k) \to \mathrm{Tot}\, R\Gamma_{\mathbb{L}\mathrm{dR}}(B^\bullet/k)$$

is n-coconnective and hence induces an equivalence after passing to the colimit by n on the left hand side. □

Corollary 1.3.12. *For every $n \geq 1$ the presheaves on* QSyn_n

$$A \mapsto R\Gamma_{\mathbb{L}\mathrm{dR}}(A/W_n(k)) \quad \textit{and} \quad A \mapsto R\Gamma_{\mathbb{L}\mathrm{crys}}((A/p)/W(k))$$

are sheaves.

Proof. By Remark 1.3.10 since $W_n(k)$ is of finite Tor-amplitude over $W(k)$ it is enough to prove the assertion for $R\Gamma_{\mathbb{L}\mathrm{crys}}((-/p)/W(k))$. But since $R\Gamma_{\mathbb{L}\mathrm{crys}}((-/p)/W(k))$ is derived p-complete by construction and since k is a perfect $W(k)$-module (and thus $- \otimes_{W(k)} k$ commutes with limits) it is enough to prove that $R\Gamma_{\mathbb{L}\mathrm{crys}}((-/p)/W(k)) \otimes_{W(k)} k \simeq R\Gamma_{\mathbb{L}\mathrm{dR}}(-/k)$ is a sheaf. This is a content of the previous lemma. □

Remark 1.3.13. Note that if R were a $\mathbb{Q}$-algebra, the derived de Rham cohomology would be just equal to R [Bha12b, Remark 2.6]. Indeed, by the $\mathbb{A}^1$-invariance of the de Rham cohomology in characteristic 0, the de Rham cohomology functor restricted to Poly_R is constant with value R, hence so is its left Kan extension. In particular, the derived de Rham cohomology of a smooth R-algebra is usually not equivalent to the classical de Rham cohomology. To improve the situation one usually works with the Hodge-completed version of the derived de Rham cohomology instead.

However, in positive characteristic the non-completed derived de Rham cohomology is much better behaved. In particular it coincides with the classical de Rham cohomology for smooth $W_n(k)$-algebras. Here, the key observation, which is due to Bhatt (see [Bha12b]), is to use the conjugate filtration. Namely, to show that the natural morphism $R\Gamma_{\mathbb{L}\mathrm{dR}}(B) \to R\Gamma_{\mathrm{dR}}(B)$ is an equivalence for a smooth $W_n(k)$-algebra B, it is enough to show this modulo p, and then (since both sides are complete), that the induced map on the associated graded of the conjugate filtration is an equivalence. For B smooth, this reduces to natural isomorphisms $\Lambda^i \mathbb{L}_{B_k^{(1)}/k} \xrightarrow{\sim} \Omega^i_{B_k^{(1)}/k}$.

Remark 1.3.14. Since the absolute Frobenius $\sigma\colon k \to k$ is an automorphism, the cotangent complex $\mathbb{L}_{k/\mathbb{F}_p}$ (and all its wedge powers) vanishes. It follows that $R\Gamma_{\mathrm{dR}}(k/\mathbb{F}_p) \simeq k$. Given any k-algebra B we have a natural morphism of E_∞-algebras $R\Gamma_{\mathrm{dR}}(k/\mathbb{F}_p) \to R\Gamma_{\mathrm{dR}}(B/\mathbb{F}_p)$. This endows $R\Gamma_{\mathrm{dR}}(B/\mathbb{F}_p)$ with a natural k-linear structure. Similarly, for any k-algebra A the complex $R\Gamma_{\mathbb{L}\mathrm{crys}}(A/\mathbb{Z}_p)$ has a natural $W(k)$-linear structure. Moreover, the natural morphism

$$R\Gamma_{\mathbb{L}\mathrm{crys}}(B/\mathbb{Z}_p) \to R\Gamma_{\mathbb{L}\mathrm{crys}}(B/W(k)) \tag{3}$$

is $W(k)$-linear. We claim that (3) is an equivalence. Since both sides are p-adically complete it is enough to show that it is an equivalence mod p, where we get an analogous map, but for the derived de Rham cohomology of the reduction B/p. On the associated graded of the conjugate filtration $\mathrm{Fil}^{\mathrm{conj}}_*$ the induced map is an equivalence, since in the transitivity triangle

$$\mathbb{L}_{k/\mathbb{F}_p} \otimes_k B \to \mathbb{L}_{B/\mathbb{F}_p} \to \mathbb{L}_{B/k}$$

the term $\mathbb{L}_{k/\mathbb{F}_p}$ is equivalent to 0. Thus (3) is an equivalence.

Recall that the cotangent complex $\mathbb{L}_{\widetilde{S}/W_n(k)}$ of $\widetilde{S} \in \mathrm{QRSPerf}_n$ has cohomological Tor-amplitude consentrated in -1, thus $\bigoplus_r \wedge^r \mathbb{L}_{\widetilde{S}/W_n(k)}[-r]$ is supported in cohomological degree 0. The same holds for $R\Gamma_{\mathbb{L}\mathrm{dR}}(\widetilde{S}/W_n(k))$; in other words, it is a classical commutative ring. It has a description in terms of one of the Fontaine's period rings A_{crys}:

Construction 1.3.15. Let S be a quasiregular semiperfect k-algebra and let $S^\flat$ be the inverse limit perfection $S^\flat := \varprojlim_{\varphi, n\geq 0} S^{(n)}$. We have a natural map $S^\flat \to S$ which is surjective. The ring $\mathbb{A}_{\mathrm{crys}}(S)$ is defined as the p-adic completion of the divided power envelope of the kernel of the natural composite surjection $\theta_{1,S}\colon W(S^\flat) \twoheadrightarrow S^\flat \twoheadrightarrow S$ (where the divided power structure agrees with the one on the ideal $(p) \subset W(k)$). Note that $\mathbb{A}_{\mathrm{crys}}(S)/p$ is identified with the PD-completion $D_I^{PD}(S^\flat)$ along the ideal $I \subset S^\flat$ defined as the kernel of the natural map $S^\flat \twoheadrightarrow S$.

Theorem 8.14(3) of [BMS19] (together with Remark 1.3.14) identifies $R\Gamma_{\mathbb{L}\mathrm{crys}}(S/W(k))$ with $\mathbb{A}_{\mathrm{crys}}(S)$. The ring $\mathbb{A}_{\mathrm{crys}}(S)$ comes with a natural ring morphism $\varphi\colon \mathbb{A}_{\mathrm{crys}}(S)^{(1)} \to \mathbb{A}_{\mathrm{crys}}(S)$ induced by the relative Frobenius $\varphi\colon S^{(1)} \to S$. It is identified with the natural Frobenius $\varphi\colon R\Gamma_{\mathbb{L}\mathrm{crys}}(S/W(k))^{(1)} \to R\Gamma_{\mathbb{L}\mathrm{crys}}(S/W(k))$ on the crystalline cohomology. For each n we define a presheaf of rings $\mathbb{A}_{\mathrm{crys}}$ on $\mathrm{QRSPerf}_n$ by sending $\widetilde{S} \in \mathrm{QRSPerf}_n$ to $\mathbb{A}_{\mathrm{crys}}(\widetilde{S}/p)$. By the above identification it is in fact a sheaf. Note that by the universal property of the PD-envelope there is a natural map[6] $\theta_{n,\widetilde{S}} : \mathbb{A}_{\mathrm{crys}}(\widetilde{S}/p) \to \widetilde{S}$ which factors through $\mathbb{A}_{\mathrm{crys}}(\widetilde{S}/p)/p^n$.

The following two filtrations on $\mathbb{A}_{\mathrm{crys}}/p$ correspond to the Hodge and the conjugate filtrations:

Definition 1.3.16. Let S be a quasiregular semiperfect k-algebra and let I be the ideal of the natural projection $S^\flat \twoheadrightarrow S$. The descending *Hodge filtration* on $\mathbb{A}_{\mathrm{crys}}(S)/p \simeq D_I^{PD}(S^\flat)$ is defined as the filtration by the divided powers of I: $\mathbb{A}_{\mathrm{crys}}(S)/p \simeq I^0 \supset I^{[1]} = I \supset I^{[2]} \supset I^{[3]} \supset \cdots$. This filtration is functorial in S and thus defines a filtration by presheaves $\mathbb{I}^0 \supset \mathbb{I}^{[1]} = \mathbb{I} \supset \mathbb{I}^{[2]} \supset \mathbb{I}^{[3]} \supset \cdots$ on the sheaf $\mathbb{A}_{\mathrm{crys}}/(p)$ on $\mathrm{QRSPerf}_n$ for any n. Via Proposition 8.12 of [BMS19] it is identified with the Hodge filtration on $R\Gamma_{\mathbb{L}\mathrm{dR}}(S/k) \simeq \mathbb{A}_{\mathrm{crys}}(S)/p$ and thus is in fact a filtration by sheaves.

Definition 1.3.17. The ascending *conjugate filtration* $\mathrm{Fil}_*^{\mathrm{conj}}$ on $\mathbb{A}_{\mathrm{crys}}(S)/p \simeq D_{S^\flat}^{PD}(I)$ is defined by taking F_r^{conj} to be the $S^\flat$-submodule generated by the elements of the form $s_1^{[l_1]}s_2^{[l_2]}\dots s_m^{[l_m]}$ with $s_i \in I$ and $\sum_{i=1}^m l_i < (r+1)p$. This construction is functorial in S and determines an (ascending) filtration $\mathrm{Fil}_*^{\mathrm{conj}}$ on the sheaf $\mathbb{A}_{\mathrm{crys}}/p$ on $\mathrm{QRSPerf}_n$ for any n. By Proposition 8.12 of [BMS19] it is identified with the conjugate filtration on $R\Gamma_{\mathbb{L}\mathrm{dR}}(S/k)$ and thus is also a filtration by sheaves. Note that both filtrations are multiplicative and the conjugate filtration is exhaustive.

The following is an analogue of the inverse Cartier isomorphism (see Theorem 0.1.1) between $(\mathbb{A}_{\mathrm{crys}}/p, \mathbb{I}^{[*]})$ and $(\mathbb{A}_{\mathrm{crys}}/p, \mathrm{Fil}_*^{\mathrm{conj}})$:

Proposition 1.3.18 ([BMS19], Propositions 8.11 and 8.12)**.** *Let S be a semiperfect k-algebra. There is a well-defined surjective homomorphism of $W(k)$-algebras $\kappa_*\colon \Gamma_S^*(I/I^2)^{(1)} \to \mathrm{gr}_*^{\mathrm{conj}}(\mathbb{A}_{\mathrm{crys}}(S)/p)$[7]. If S is quasiregular, κ_* is an isomorphism.*

Proof. The map is defined as follows: for $s_i \in I$

$$\kappa_{k_1+\dots k_m} : s_1^{[k_1]}\cdots s_m^{[k_m]} \mapsto \prod_{i=1}^m \left(\frac{(pk_i)!}{p^{k_i}k_i!}\right) s_1^{[pk_1]}\cdots s_m^{[pk_m]}.$$

We have $(s_1s_2)^{[pk]} = p!(s_1^k)^{[p]}s_2^{[pk]} = 0$ and $(s_1s_2)^{[l]} \in \mathrm{Fil}_0^{\mathrm{conj}}$ for any $l<p$. This shows that for $s \in I^2$, $s^{[l]} \in \mathrm{Fil}_0^{\mathrm{conj}}$ for all l and so the map is well-defined. Elements $\{s_1^{[pk_1]}\cdots s_m^{[pk_m]}\}_{k_1+\cdots k_m<r+1}$ in fact generate $\mathrm{Fil}_r^{\mathrm{conj}}$ over $S^\flat$. Since the integer $\prod_{i=1}^m \left(\frac{(pk_i)!}{p^{k_i}k_i!}\right)$ is a p-adic unit the map κ_* is surjective. The fact that κ_* is an isomorphism for S quasiregular semiperfect is a part of Proposition 8.12 of [BMS19]. ☐

Remark 1.3.19. In particular we get an isomorphism $\kappa_*\colon \Gamma_S^*(\mathbb{I}/\mathbb{I}^2)^{(1)} \xrightarrow{\sim} \mathrm{gr}_*^{\mathrm{conj}}(\mathbb{A}_{\mathrm{crys}}/p)$ of sheaves of algebras on $\mathrm{QRSPerf_n}$.

Now we descend everything back to the quasisyntomic site QSyn_n. We record what the sheaves defined above give when computed on a smooth $W_n(k)$-algebra B.

Proposition 1.3.20. *Let B be a smooth $W_n(k)$-algebra considered as an object of QSyn_n. Then:*

1. *For any $0 \leq s \leq n$ there is a natural equivalence of E_∞-algebras $R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{A}_{\mathrm{crys}}/p^s) \simeq \Omega^\bullet_{(B/p^s)/W_s(k),\mathrm{dR}}$.*

[6] Here we endow $(p) \subset \widetilde{S}$ with the standard PD-structure, given by $p^{[k]} := p^k/k!$.

[7] Where Γ^* denotes the free commutative divided power algebra.

2. For any $r \in \mathbb{Z}_{\geq 0}$ there is a natural equivalence $R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{I}^{[r]}) \simeq \Omega^{\geq r}_{(B/p)/k,\mathrm{dR}}$, where $\Omega^{\geq r}_{B/p,\mathrm{dR}/k}$ is the r-th term of the Hodge filtration.

   In particular, $R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{I}^{[r]}/\mathbb{I}^{[r+1]}) \simeq \Omega^r_{(B/p)/k}[-r]$.

3. For any $r \in \mathbb{Z}_{\geq 0}$ there is a natural equivalence $R\Gamma_{\mathrm{QSyn}_n}(B, \mathrm{Fil}^{\mathrm{conj}}_r) \simeq \tau^{\leq r}\Omega^\bullet_{(B/p)/k,\mathrm{dR}}$.

4. The natural map $\Gamma^r_S(\mathbb{I}/\mathbb{I}^2) \to \mathbb{I}^{[r]}/\mathbb{I}^{[r+1]}$ given by multiplication induces an equivalence

$$R\Gamma_{\mathrm{QSyn}_n}(B, \Gamma^r_S(\mathbb{I}/\mathbb{I}^2)) \simeq R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{I}^{[r]}/\mathbb{I}^{[r+1]})$$

   for any $r \geq 0$.

5. The isomorphism $\kappa_*\colon \Gamma^*_S(\mathbb{I}/\mathbb{I}^2)^{(1)} \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_*(\mathbb{A}_{\mathrm{crys}}/p)$ from Proposition 1.3.18 induces the inverse Cartier isomorphism

$$\bigoplus_{r=0}^{\infty} \Omega^r_{(B^{(1)}/p)/k} \xrightarrow[\sim]{C^{-1}} \bigoplus_{r=0}^{\infty} H^r\left(\Omega^\bullet_{(B/p)/k,\mathrm{dR}}\right)$$

   via the above equivalences.

Proof. Parts $1, 2, 3$ follow from Proposition 8.12 and Theorem 8.14(3) of [BMS19] and flat descent for the derived crystalline cohomology (using Remarks 1.3.13, 1.3.10 and 1.3.14).

4. We use the notations of Example 1.3.7. We have

$$R\Gamma_{\mathrm{QSyn}_n}(B, \mathcal{F}) \xrightarrow{\sim} \mathrm{Tot}\Big(\ \mathcal{F}(B_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}}) \Rrightarrow \mathcal{F}(B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}}) \cdots \Big)$$

for any quasisyntomic sheaf $\mathcal{F}$. Moreover all terms $(B_{\mathrm{perf}} \otimes_B \ldots \otimes_B B_{\mathrm{perf}})_n$ are in fact regular semiperfect, meaning $I \subset S^\flat$ is generated by a regular sequence. Thus for them $\Gamma^r_S(I/I^2) \xrightarrow{\sim} I^{[r]}/I^{[r+1]}$ and so $R\Gamma_{\mathrm{QSyn}_n}(B, \Gamma^r_S(\mathbb{I}/\mathbb{I}^2)) \simeq R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{I}^{[r]}/\mathbb{I}^{[r+1]})$.

5. Note that the map depends only on the reduction of B modulo p, thus it is enough to consider the case $B \in \mathrm{QSyn}_1$ is of characteristic p. The inverse Cartier isomorphism C^{-1} is uniquely defined by the property that it is multiplicative, $C^{-1}(f) = f^p$ and $C^{-1}(df) = f^{p-1}df$ for any $f \in B$. The map κ_* is multiplicative, κ_0 is by definition given by Frobenius, so it remains to check the third assertion. By functoriality (considering the map $k[x] \xrightarrow{x \mapsto f} B$) it is enough to check this in the case $B = k[x]$ and $f = x$. While originally we had the proof using the relation between the Cartier isomorphism and the Bockstein operator, we will present a different proof that was kindly suggested to us by one of the referees.

One can use the explicit formula for the crystalline cohomology via the Čech-Alexander complex. Namely, for any smooth k-algebra B the homotopy groups of the cosimplicial algebra $C^\bullet_{\mathrm{crys}}(B) \coloneqq D_{\mathrm{Ker}(B^{\otimes^\bullet_k} \xrightarrow{m} B)} B^{\otimes^\bullet_k}$ compute the de Rham cohomology of B; here $B^{\otimes^\bullet_k} \xrightarrow{m} B$ is the multiplication map and $D_{\mathrm{Ker}(B^{\otimes^\bullet_k} \xrightarrow{m} B)} B^{\otimes^\bullet_k}$ is the PD-envelope corresponding to its kernel. By [BdJ11, The proof of Theorem 2.12] the totalization of the bicomplex

$$\begin{array}{ccccccc}
B & \longrightarrow & \Omega^1_B & \longrightarrow & \Omega^2_B & \longrightarrow & \ldots \\
\downarrow & & \downarrow{\scriptstyle d_{\text{Čech}}} & & \downarrow & & \\
D(2) & \xrightarrow{d_{\mathrm{dR}}} & \Omega^1_{D(2)} & \longrightarrow & \Omega^2_{D(2)} & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
D(3) & \longrightarrow & \Omega^1_{D(3)} & \longrightarrow & \Omega^2_{D(3)} & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\ldots & & \ldots & & \ldots & &
\end{array}$$

with $D(i) \coloneqq D_{\mathrm{Ker}(B^{\otimes^i_k} \xrightarrow{m} B)} B^{\otimes^i_k}$ is quasiisomorphic to both the first row and column (via the embeddings of the latter) this way establishing the comparison quasiisomorphism of $C^\bullet_{\mathrm{crys}}(B)$ and $\Omega^\bullet_{B,\mathrm{dR}}$. For $B = k[x]$ we have $D(2) = D_{(x_1 - x_2)}k[x_1, x_2]$, $d_{\text{Čech}}(x^{p-1}dx) = x_1^{p-1}dx_1 - x_2^{p-1}dx_2$ and we leave it to the reader to check that this also equals to $d_{\mathrm{dR}}(a)$ with $a := (p-1)!((x_1-x_2)^{[p]} + \sum_{i=1}^{p-1}(-1)^i x_1^{[p-1-i]}x_2^{[i]}) \in D(2)$. Thus under this comparison the class $[x^{p-1}dx] \in H^1_{\mathrm{dR}}(k[x])$ goes to $[a]$ in $H^1(C^\bullet_{\mathrm{crys}}(B))$. Note also that by an analogous but simpler computation $[dx] \in H^1_{\mathrm{dR}}(k[x])$ goes to $[x_1-x_2] \in H^1(C^\bullet_{\mathrm{crys}}(B))$.

As we saw in part 4, the cosimplicial algebra $(\mathbb{A}_{\mathrm{crys}}/p)(B_{\mathrm{perf}}^{\otimes_B \bullet}) \simeq D_{\mathrm{Ker}(B_{\mathrm{perf}}^{\otimes_k^\bullet}\to B_{\mathrm{perf}}^{\otimes_B^\bullet})} B_{\mathrm{perf}}^{\otimes_k^\bullet}$ appearing as the Čech object associated to the quasisyntomic cover $B \to B_{\mathrm{perf}}$ also computes the de Rham cohomology of B. We have a natural map of cosimplicial algebras $C^\bullet_{\mathrm{crys}}(B) \to (\mathbb{A}_{\mathrm{crys}}/p)(B_{\mathrm{perf}}^{\otimes_B \bullet})$ induced termwise by $B^{\otimes_k^\bullet} \to B_{\mathrm{perf}}^{\otimes_k^\bullet}$. The definitions (1.3.16 and 1.3.17) of the conjugate and Hodge filtrations make sense for any PD-envelope and so extend to the cosimplicial algebra $C^\bullet_{\mathrm{crys}}(B)$ as well. The map κ (see the proof of 1.3.18) extends naturally as well, the map $C^\bullet_{\mathrm{crys}}(B) \to (\mathbb{A}_{\mathrm{crys}}/p)(B_{\mathrm{perf}}^{\otimes_B \bullet})$ preserves the filtrations, commutes with κ and in fact is a filtered quasi-isomorphism[8]. Returning to the case $B = k[x]$ we see that, first, $dx \in \Omega^1_{\mathbb{A}^1_k} \simeq H^1_{\mathrm{QSyn}}(B, \mathbb{I}/\mathbb{I}^2)$ corresponds to $x_1 - x_2 \in D(2) \subset D_{\mathrm{Ker}(B_{\mathrm{perf}}^{\otimes_k^2}\to B_{\mathrm{perf}}^{\otimes_B^2})} B_{\mathrm{perf}}^{\otimes_k^2}$ under the comparison, and, second, that the class $[\kappa_1(dx)]$ in $H^1_{\mathrm{QSyn}}(B, \mathrm{gr}_1^{\mathrm{conj}})$ is given by the class of the element $(p-1)!(x_1-x_2)^{[p]} \in D(2) \subset D_{\mathrm{Ker}(B_{\mathrm{perf}}^{\otimes_k^2}\to B_{\mathrm{perf}}^{\otimes_B^2})} B_{\mathrm{perf}}^{\otimes_k^2}$ modulo $\mathrm{Fil}_0^{\mathrm{conj}}$. It remains to note that this element differs from a by $\sum_{i=1}^{p-1}(-1)^i x_1^{[p-1-i]}x_2^{[i]}$ which lies in $\mathrm{Fil}_0^{\mathrm{conj}}$. ∎

Next we prove the following enhancement of the classical Deligne-Illusie splitting:

Theorem 1.3.21. *Let $\mathrm{Aff}^{\mathrm{sm}}_{/W_2(k)}$ be the category of smooth affine schemes over $W_2(k)$. Then there is a natural k-linear equivalence of functors*

$$\bigoplus_{i=0}^{p-1}\Omega^i_{-^{(1)}} \colon B \mapsto \bigoplus_{i=0}^{p-1}\Omega^i_{(B^{(1)}/p)/k}[-i] \quad \text{and} \quad \tau^{\le p-1}\Omega^\bullet_{-,\mathrm{dR}} \colon B \mapsto \tau^{\le p-1}\Omega^\bullet_{(B/p)/k,\mathrm{dR}}$$

from $\mathrm{Aff}^{\mathrm{sm,op}}_{/W_2(k)}$ to $D(\mathrm{Mod}_k)$ which induces the Cartier isomorphism on the level of the individual cohomology functors.

By Proposition 1.3.5 and Proposition 1.3.20 to deduce the statement of the theorem it is enough to prove the following:

Proposition 1.3.22. *There is a natural isomorphism $f \colon \bigoplus_{r=0}^{p-1}\Gamma^r_S(\mathbb{I}/\mathbb{I}^2) \simeq \mathrm{Fil}^{\mathrm{conj}}_{p-1}$ of sheaves of abelian groups on $\mathrm{QRSPerf}_2$ such that it agrees with $\kappa_{\le p-1} \colon \Gamma_S^{\le p-1}(\mathbb{I}/\mathbb{I}^2) \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_{\le p-1}(\mathbb{A}_{\mathrm{crys}}/p)$ after passing to the associated graded.*

Proof. Given $\widetilde{S} \in \mathrm{QRSPerf}_2$ we denote by S the reduction of $\widetilde{S}$ modulo p. As before we denote the kernel of the natural map $S^\flat \to S$ by I. Note that $\Gamma^i_S(I/I^2) \simeq \mathrm{Sym}^i_S(I/I^2)$ for $i \le p-1$ and so, extending the map by multiplicativity, it is enough to construct a splitting $f \colon S^\flat/I \oplus I/I^2 \xrightarrow{\sim} \mathrm{Fil}_1^{\mathrm{conj}}$. Recall that we have a natural endomorphism $\varphi \colon \mathbb{A}_{\mathrm{crys}}(S) \to \mathbb{A}_{\mathrm{crys}}(S)$. We consider the Nygaard filtration (see Definition 8.9 of [BMS19])

$$\mathcal{N}^{\ge i}\mathbb{A}_{\mathrm{crys}}(S) := \{x \in \mathbb{A}_{\mathrm{crys}}(S) \mid \varphi(x) \in p^i \mathbb{A}_{\mathrm{crys}}(S)\}.$$

In fact we will be interested only in the first two of its associated graded terms. We will construct f by using the divided Frobenii, defined as follows. By Theorem 8.15(1) of [BMS19] $\mathbb{A}_{\mathrm{crys}}(S)$ is p-torsion free and so for each $i \ge 0$ one has a well defined map

$$\varphi_i := \varphi/p^i \colon \mathcal{N}^i\mathbb{A}_{\mathrm{crys}}(S) \to \mathbb{A}_{\mathrm{crys}}(S)/p$$

from the i-th graded piece $\mathcal{N}^i\mathbb{A}_{\mathrm{crys}}(S) := \mathcal{N}^{\ge i}\mathbb{A}_{\mathrm{crys}}(S)/\mathcal{N}^{\ge i+1}\mathbb{A}_{\mathrm{crys}}(S)$ of the Nygaard filtration.

It is clear that $p\cdot\mathbb{A}_{\mathrm{crys}}(S) \subset \mathcal{N}^{\ge 1}\mathbb{A}_{\mathrm{crys}}(S)$; moreover, by Theorem 8.14(4) of [BMS19], $\mathcal{N}^{\ge 1}\mathbb{A}_{\mathrm{crys}}(S) \mod p\cdot\mathbb{A}_{\mathrm{crys}}$ is given by $I \subset \mathbb{A}_{\mathrm{crys}}(S)/p$. Thus $\mathcal{N}^0 := \mathcal{N}^{\ge 0}/\mathcal{N}^{\ge 1} \simeq S^\flat/I$ and φ_0 induces an isomorphism $S^\flat/I \xrightarrow{\sim} \mathrm{Fil}_0^{\mathrm{conj}}$ (since $\kappa_0 = \varphi$, this follows from Proposition 1.3.18). We then also have a map $\varphi_1 \colon \mathcal{N}^1\mathbb{A}_{\mathrm{crys}}(S) \to \mathbb{A}_{\mathrm{crys}}(S)/p$, which, by Theorem 8.14(2) of [BMS19], is an isomorphism onto $\mathrm{Fil}_1^{\mathrm{conj}}$. Multiplication by p induces a natural map $\mathcal{N}^0\mathbb{A}_{\mathrm{crys}}(S) \to \mathcal{N}^1\mathbb{A}_{\mathrm{crys}}(S)$ which after composing with φ_1 is identified with the embedding $\mathrm{Fil}_0^{\mathrm{conj}} \subset \mathrm{Fil}_1^{\mathrm{conj}}$. In fact,

[8] Indeed, both complexes can be considered as Čech-Alexander complexes for a slightly unusual "quasisyntomic" version of the char p crystalline site: namely, we consider triples (U, T, δ) with $U \to \mathrm{Spec}\, B$ a quasisyntomic morphism and T being a char p PD-thickening of U. The complexes $C^\bullet_{\mathrm{crys}}(B)$ and $(\mathbb{A}_{\mathrm{crys}}/p)(B_{\mathrm{perf}}^{\otimes_B \bullet})$ can be interpreted as the Čech-Alexander complexes corresponding to two different covers, namely $\mathrm{Spec}\, B \to *$ and $\mathrm{Spec}\, B_{\mathrm{perf}} \to *$. The map above is then induced by the map of coverings $\mathrm{Spec}\, B_{\mathrm{perf}} \to \mathrm{Spec}\, B$ and being a map Čech-Alexander complexes (for the structure sheaf) is automatically a quasiisomorphism. From this interpretation it is also clear that it respects the Hodge and conjugate filtrations, as well as the map κ.

by flatness of $\mathbb{A}_{\mathrm{crys}}(S)$, we have $\mathcal{N}^{\geq 1}\mathbb{A}_{\mathrm{crys}}(S) \cap p \cdot \mathbb{A}_{\mathrm{crys}}(S) \simeq \mathcal{N}^{\geq 0}\mathbb{A}_{\mathrm{crys}}(S)$ and so $\mathrm{Fil}_0^{\mathrm{conj}}$ (under the isomorphism given by φ_1) is identified exactly with the subspace of those elements in $\mathcal{N}^1\mathbb{A}_{\mathrm{crys}}(S) \simeq \mathrm{Fil}_1^{\mathrm{conj}}$ that lift to elements of $\mathcal{N}^{\geq 1}\mathbb{A}_{\mathrm{crys}}(S)$ divisible by p.

We now use the lifting $\widetilde{S}$ of S to construct the splitting of $\mathrm{Fil}_1^{\mathrm{conj}}$. Recall that we have a map $\theta_{2,\widetilde{S}} \colon \mathbb{A}_{\mathrm{crys}}(S)/p^2 \twoheadrightarrow \widetilde{S}$ and let $K \coloneqq \ker \theta_{2,\widetilde{S}}$. Since both $\widetilde{S}$ and $\mathbb{A}_{\mathrm{crys}}(S)/p^2$ are flat over $W_2(k)$, we get that K is also flat over $W_2(k)$ and that $K/pK \simeq I$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \mathbb{A}_{\mathrm{crys}}(S)/p^2 & \longrightarrow & \widetilde{S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & \mathbb{A}_{\mathrm{crys}}(S)/p & \longrightarrow & S & \longrightarrow & 0. \end{array}$$

The splitting is then given by applying φ_1 to K. Namely, since $\varphi(I) = 0 \in \mathbb{A}_{\mathrm{crys}}(S)/p$ it follows that $\varphi(K) \subset p \cdot \mathbb{A}_{\mathrm{crys}}(S)/p^2$ and $K \subset \mathcal{N}^{\geq 1}\mathbb{A}_{\mathrm{crys}}(S) \mod p^2\mathbb{A}_{\mathrm{crys}}(S)$. The natural projection from K to $\mathcal{N}^1\mathbb{A}_{\mathrm{crys}}(S)$ contains $p\cdot K + \mathcal{N}^{\geq 2}\mathbb{A}_{\mathrm{crys}}(S) \mod p^2\mathbb{A}_{\mathrm{crys}}(S)$ in its kernel. Since $K/pK = I$ and the image of $\mathcal{N}^{\geq 2}\mathbb{A}_{\mathrm{crys}}$ modulo p is given by I^2 (e.g. by Theorem 8.14(4) of [BMS19]), we get that φ_1 (applied to K) gives a well-defined map $f \colon I/I^2 \to \mathrm{Fil}_1^{\mathrm{conj}}$. Moreover $K \cap (p \cdot \mathbb{A}_{\mathrm{crys}}(S)/p^2) \subset p \cdot K$, since K is flat over $W_2(k)$, and so the image of f does not intersect with $\mathrm{Fil}_0^{\mathrm{conj}}$.

It remains to check that the constructed $f \colon I/I^2 \to \mathrm{Fil}_1^{\mathrm{conj}}$ coincides with κ_1 after the projection to $\mathrm{Fil}_1^{\mathrm{conj}}/\mathrm{Fil}_0^{\mathrm{conj}}$. Given $s \in I$ let $\widetilde{s} = [s] + p \cdot s' \in K \subset \mathbb{A}_{\mathrm{crys}}/p^2$ be a lifting of s to an element of K. Then

$$\varphi(\widetilde{s}) = \varphi([s]) + p \cdot \varphi(s') = (p-1)! \cdot p \cdot [s]^{[p]} + p \cdot \varphi(s') \;\Rightarrow\; f(s) = (p-1)! \cdot s^{[p]} + \varphi(s').$$

By the discussion above (see also Theorem 8.14(2) in [BMS19]) $\varphi(s') \in \mathrm{Fil}_0^{\mathrm{conj}}$ and $f(s) = (p-1)! \cdot s^{[p]}$ modulo $\mathrm{Fil}_0^{\mathrm{conj}}$.

Since the above splitting is clearly functorial in $\widetilde{S}$ we get the statement of the proposition. ☐

As a corollary we deduce

Theorem 1.3.23. *Let $\mathcal{Y}$ be a smooth Artin stack over a perfect field k of characteristic p admitting a smooth lift to the ring of the second Witt vectors $W_2(k)$. Then there is a canonical equivalence*

$$R\Gamma(\mathcal{Y}, \tau^{\leq p-1}\Omega^\bullet_{\mathcal{Y},\mathrm{dR}}) \simeq R\Gamma\left(\mathcal{Y}^{(1)}, \bigoplus_{i=0}^{p-1} \wedge^i \mathbb{L}_{\mathcal{Y}^{(1)}/k}[-i]\right).$$

In particular for $n \leq p-1$ we have $H^n_{\mathrm{dR}}(\mathcal{Y}/k) \simeq H^n_{\mathrm{H}}(\mathcal{Y}^{(1)}/k)$.

Proof. Let $\pi \colon \mathcal{S}\mathrm{tk}^{n\text{-Art,sm}}_{/W_2(k)} \to \mathcal{S}\mathrm{tk}^{n\text{-Art,sm}}_{k}$ be the reduction functor, $\widetilde{\mathcal{Y}} \mapsto \widetilde{\mathcal{Y}} \otimes_{W_2(k)} k$. By Theorem 1.3.21 it is enough to prove that the natural map (existing by the universal property of the right Kan extensions)

$$R\Gamma_{\mathrm{dR}}(-/k) \circ \pi \to \mathrm{Ran}_{i_2}(R\Gamma_{\mathrm{dR}}(-/k) \circ \pi_{|\,\mathrm{Aff}^{\mathrm{sm}}_{/W_2(k)}}) \tag{4}$$

(where i_2 denotes the inclusion functor $\mathrm{Aff}^{\mathrm{sm}}_{/W_2(k)} \hookrightarrow \mathcal{S}\mathrm{tk}^{n\text{-Art,sm}}_{/W_2(k)}$) is an equivalence. Since both sides of (4) satisfy smooth descent, by induction on n we reduce the statement to the case of smooth affine schemes over $W_2(k)$, where (4) is evidently an equivalence. ☐

Corollary 1.3.24. *Let $\mathcal{Y}$ be a smooth Hodge-proper stack over a perfect field k of characteristic p admitting a smooth lift to $W_2(k)$. Then the Hodge-to-de Rham spectral sequence $H^j(\mathcal{Y}, \wedge^i\mathbb{L}_{\mathcal{Y}/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(\mathcal{Y}/k)$ degenerates at the first page for $i+j<p$.*

Proof. This follows from Theorem 1.3.23 and the equality of dimensions $\dim_k H^n_{\mathrm{H}}(\mathcal{Y}) = \dim_k H^n_{\mathrm{H}}(\mathcal{Y}^{(1)})$. ☐

1.4 Degeneration in characteristic zero

To reduce the statement in characteristic 0 to results of the previous section we introduce the following notion:

Definition 1.4.1. A smooth Hodge-proper Artin stack $\mathcal{X}$ over a field F of characteristic 0 is called *Hodge-properly spreadable* if there exists a $\mathbb{Z}$-subalgebra $R \subset F$ and an Artin stack $\mathcal{X}_R$ over $\operatorname{Spec} R$ such that

- R is a localization of a smooth $\mathbb{Z}$-algebra such that the image of $\operatorname{Spec} R$ in $\operatorname{Spec} \mathbb{Z}$ is open.
- $\mathcal{X}_R$ is smooth over R and $\mathcal{X} \otimes_R F \coloneqq \mathcal{X}_R \times_{\operatorname{Spec} R} \operatorname{Spec} F \simeq \mathcal{X}$.
- $\mathcal{X}_R$ is Hodge-proper over R, namely $R\Gamma(\mathcal{X}_R, \wedge^p \mathbb{L}_{\mathcal{X}_R/R})$ is bounded below coherent over R for any $p \geq 0$.

Remark 1.4.2. We note that any field F of characteristic 0 is a union of all such subrings $R \subset F$ (in fact even a union of those that are smooth over $\mathbb{Z}$). As we will see in Section 2.3.3 allowing some infinite localizations of smooth algebras makes some difference when constructing examples. The condition on openness of the image is added to guarantee that the diagram of all such $R \subset F$ is filtered and that for any such R the image of $\operatorname{Spec} R$ in $\operatorname{Spec} \mathbb{Z}$ is infinite. To see the first point: indeed, having $Q_1 = R_1[S_1^{-1}]$, $Q_2 = R_2[S_2^{-1}]$, $Q_1, Q_2 \subset F$ being localizations of smooth $\mathbb{Z}$-algebras R_1, R_2 for some subsets $S_i \subset R_i$ as in Definition 1.4.1, for any finite localization $(Q_1 \cdot Q_2)[1/f] \subset F$ the image of $\operatorname{Spec}(R_1 \cdot R_2)[1/f]$ in $\operatorname{Spec} \mathbb{Z}$ is still open. Then, picking f such that $(R_1 \cdot R_2)[1/f]$ is again smooth over $\mathbb{Z}$ we get a subring $(Q_1 \cdot Q_2)[1/f] \subset F$ that contains both Q_1, Q_2 and fits in Definition 1.4.1.

We defer a thorough discussion of spreadability of stacks till the next section. We only stress here again, that (unlike in the case of proper schemes) Hodge-proper spreadings do not exist in general (see Section 2.3.2).

Now we will deduce the promised Hodge-to-de Rham degeneration in characteristic 0:

Theorem 1.4.3. *Let $\mathcal{X}$ be a smooth Hodge-properly spreadable Artin stack over a field F of characteristic zero. Then the Hodge-to-de Rham spectral sequence for $\mathcal{X}$ degenerates at the first page. In particular for each $n \geq 0$ there exists a (non-canonical) isomorphism*

$$H^n_{\mathrm{dR}}(\mathcal{X}/F) \simeq \bigoplus_{p+q=n} H^{p,q}(\mathcal{X}/F).$$

Proof. For the rest of the proof fix $n \in \mathbb{Z}_{\geq 0}$. By Hodge-properness of $\mathcal{X}$ it is enough to prove

$$\dim_F H^n_{\mathrm{dR}}(\mathcal{X}/F) = \dim_F H^n_{\mathrm{H}}(\mathcal{X}/F).$$

Let R and $\mathcal{X}_R$ be as in Definition 1.4.1. Note that by the assumption on $\mathcal{X}_R$ and Proposition 1.2.6 both $H^n_{\mathrm{dR}}(\mathcal{X}_R/R)$ and $H^n_{\mathrm{H}}(\mathcal{X}_R/R)$ are finitely generated R-modules. Localizing R if necessary, we can assume that R is connected of some Krull dimension d, and that the i-th cohomology groups $H^i_{\mathrm{dR}}(\mathcal{X}_R/R)$ and $H^i_{\mathrm{H}}(\mathcal{X}_R/R)$ are free R-modules of finite rank for $i = n, n+1, \ldots, n+d$.[9] Note that for any point $s\colon \operatorname{Spec} k \to \operatorname{Spec} R$ the map $R \to k$ can be factored as a composition of a flat map $R \to R_s$ (where R_s is a local ring of s) and a map $R_s \to k$ of finite Tor-amplitude (by regularity assumption k is perfect as an R_s-module). Hence by Proposition 1.1.8 we have

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}_k/k) \simeq R\Gamma_{\mathrm{dR}}(\mathcal{X}_R/R) \otimes_R k \qquad \text{and} \qquad R\Gamma_{\mathrm{H}}(\mathcal{X}_k/k) \simeq R\Gamma_{\mathrm{H}}(\mathcal{X}_R/R) \otimes_R k,$$

where $\mathcal{X}_k \coloneqq \mathcal{X}_R \otimes_R k$. Since the $n, (n+1), \ldots, (n+d)$-th cohomology groups are free as R-modules and since the Tor-amplitude of k over R is bounded by d, we get $H^n_{\mathrm{dR}}(\mathcal{X}_k/k) \simeq H^n_{\mathrm{dR}}(\mathcal{X}_R/R) \otimes_R k$, so

$$\dim_F H^n_{\mathrm{dR}}(\mathcal{X}/F) = \operatorname{rank}_R H^n_{\mathrm{dR}}(\mathcal{X}_R/R) = \dim_k H^n_{\mathrm{dR}}(\mathcal{X}_k/k)$$

and analogously for the Hodge cohomology. In particular, to prove that $\dim_F H^n_{\mathrm{dR}}(\mathcal{X}/F) = \dim_F H^n_{\mathrm{H}}(\mathcal{X}/F)$ it is enough to show that $\dim_k H^n_{\mathrm{dR}}(\mathcal{X}_k/k) = \dim_k H^n_{\mathrm{H}}(\mathcal{X}_k/k)$ for some point $s\colon \operatorname{Spec} k \to \operatorname{Spec} R$.

To do so, note that by the infiniteness of the image of $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$ and Lemma 1.4.4 below, there exists a closed point $s\colon \operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$ of characteristic greater than n, such that the map $R \to k \to k^{\mathrm{perf}}$ factors through the ring of the second Witt vectors $W_2(k^{\mathrm{perf}})$. Since the base change $\mathcal{X}_{W_2(k^{\mathrm{perf}})} \coloneqq \mathcal{X}_R \times_R W_2(k^{\mathrm{perf}})$ is smooth and Hodge-proper over $W_2(k^{\mathrm{perf}})$, by Theorem 1.3.23 we have

$$\dim_{k^{\mathrm{perf}}} H^n_{\mathrm{dR}}(\mathcal{X}_{k^{\mathrm{perf}}}/k^{\mathrm{perf}}) = \dim_{k^{\mathrm{perf}}} H^n_{\mathrm{H}}(\mathcal{X}_{k^{\mathrm{perf}}}/k^{\mathrm{perf}}).$$

[9] Note that there does not necessarily exist a localization $R[s^{-1}]$ such that *for all* i the $R[s^{-1}]$-modules $H^i_{\mathrm{dR}}(\mathcal{X}_R/R)[s^{-1}]$ (or $H^i_{\mathrm{H}}(\mathcal{X}_R/R)[s^{-1}]$) are free, since there are infinitely many of them.

Finally, by base change (applied to $k \to k^{\mathrm{perf}}$) we get

$$\dim_k H^n_{\mathrm{dR}}(\mathcal{X}_k/k) = \dim_k H^n_{\mathrm{H}}(\mathcal{X}_k/k)$$

as desired. ∎

Lemma 1.4.4. *Let R be a localization of a smooth $\mathbb{Z}$-algebra. Then for any field k of positive characteristic and a map $R \to k$ the composite map $R \to k \hookrightarrow k^{\mathrm{perf}}$ factors through the ring $W_2(k^{\mathrm{perf}})$ of the second Witt vectors.*

Proof. By assumption on R the cotangent complex $\mathbb{L}_{R/\mathbb{Z}} \simeq \Omega_{R/\mathbb{Z}}[0]$ is concentrated in degree zero and is a locally free (in particular flat) R-module. By the basic deformation theory the obstruction to lift a map $R \to k^{\mathrm{perf}}$ to $R \to W_2(k^{\mathrm{perf}}) \to k^{\mathrm{perf}}$ lies in $\mathrm{Ext}^1_{k^{\mathrm{perf}}}(\mathbb{L}_{R/\mathbb{Z}} \otimes_R k^{\mathrm{perf}}, k^{\mathrm{perf}})$. But the latter group vanishes, since by flatness of $\mathbb{L}_{R/\mathbb{Z}}$, the restriction $\mathbb{L}_{R/\mathbb{Z}} \otimes_R k^{\mathrm{perf}}$ is a complex of k^{perf}-vector spaces concentrated in degree 0. ∎

1.5 Equivariant Hodge decomposition

In this section we apply Theorem 1.4.3 to obtain a (non-canonical) Hodge decomposition for the equivariant singular cohomology of an algebraic variety X with a G action, under the assumption that the corresponding quotient stack $[X/G]$ is Hodge-properly spreadable.

Let K be a homotopy type with an action of a topological group H (i.e. an $(\infty,1)$-functor $K_\bullet \colon BH \to \mathcal{S}$, where $\mathcal{S}$ denotes the $(\infty,1)$-category of spaces, see Section 0.4). Recall that the H-equivariant cohomology $C^*_H(K,\Lambda)$ of K with coefficients in a ring Λ are defined as

$$C^*_H(K,\Lambda) \coloneqq C^*(K_{hH},\Lambda),$$

where K_{hH} is the homotopy quotient of K by H (i.e. a colimit of the corresponding functor $K_\bullet$, or, more classically, $(K \times EH)/H$).

If X is a smooth algebraic variety over a field $F \subseteq \mathbb{C}$ equipped with an action of an algebraic group G, then the de Rham cohomology of $[X/G]$ gives a model for the $G(\mathbb{C})$-equivariant singular cohomology of $X(\mathbb{C})$:

Proposition 1.5.1. *Let X and G be as above. Then there is a canonical equivalence*

$$C^*_{G(\mathbb{C})}(X(\mathbb{C}),\mathbb{C}) \simeq R\Gamma_{\mathrm{dR}}([X/G]/F) \otimes_F \mathbb{C}.$$

Proof. By definition we have

$$\Big| \cdots \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} G \times G \times X \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} G \times X \rightrightarrows X \Big| \simeq [X/G],$$

$$\Big| \cdots \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} G(\mathbb{C}) \times G(\mathbb{C}) \times X(\mathbb{C}) \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} G(\mathbb{C}) \times X(\mathbb{C}) \rightrightarrows X(\mathbb{C}) \Big| \simeq X(\mathbb{C})_{hG(\mathbb{C})}.$$

Since the functor of cochains $C^*(-,\mathbb{C})$ sends colimits of homotopy types to limits of complexes and by smooth descent for $R\Gamma_{\mathrm{dR}}(-/F) \otimes_F \mathbb{C}$, the result follows from the analogous comparison between algebraic de Rham and Betti cohomology for ordinary smooth schemes $X \times G^n$. ∎

Corollary 1.5.2 (Equivariant Hodge decomposition)**.** *Let X be a smooth scheme over $\mathbb{C}$ with an action of an algebraic group G. Assume that $[X/G]$ is Hodge-properly spreadable (e.g. X and G satisfy the conditions of Theorem 3.1.4 or 3.2.12). Then for all $n \in \mathbb{Z}_{\geq 0}$ there exists an isomorphism*

$$H^n_{G(\mathbb{C})}(X(\mathbb{C}),\mathbb{C}) \simeq \bigoplus_{p+q=n} H^q([X/G], \wedge^p \mathbb{L}_{[X/G]/\mathbb{C}}).$$

Example 1.5.3. Let $X = \operatorname{Spec}\mathbb{C}$. Then $\wedge^n \mathbb{L}_{BG} \simeq \mathrm{Sym}^n(\mathfrak{g}^\vee)[-n]$ where $\mathfrak{g}$ is the Lie algebra of G endowed with the adjoint action of G. This way we get a standard isomorphism

$$H^n_{G(\mathbb{C})}(\mathrm{pt},\mathbb{C}) \simeq \begin{cases} \mathrm{Sym}^k(\mathfrak{g}^\vee)^G & \text{if } n = 2k, \\ 0 & \text{if } n = 2k+1. \end{cases}$$

In particular,

$$H^\bullet_{G(\mathbb{C})}(\mathrm{pt},\mathbb{C}) \simeq \mathrm{Sym}(\mathfrak{g}^\vee)^G, \quad \text{where} \quad \deg(\mathfrak{g}^\vee) = 2.$$

Example 1.5.4. As another example one can take a conical resolution $\pi\colon X \to \operatorname{Spec} A$ (see the second example of 3.2.16). Following Example 3.2.16, the quotient stack $[X/\mathbb{G}_m]$ is Hodge-properly spreadable and we get a decomposition for $H^\bullet_{\mathbb{C}^\times}(X(\mathbb{C}),\mathbb{C})$ as in Corollary 1.5.2. Note also that in this case $H^1_{\mathbb{C}^\times}(X(\mathbb{C}),\mathbb{C}) \simeq H^1(X(\mathbb{C}),\mathbb{C})$; indeed one can replace $\mathbb{C}^\times$ with S^1 and consider the Serre-Lerray spectral sequence

$$E_2^{p,q} = H^p(BS^1, H^q(X(\mathbb{C}),\mathbb{C})) \Rightarrow H^{p+q}_{S^1}(X(\mathbb{C}),\mathbb{C}).$$

We have $BS^1 \simeq \mathbb{C}P^\infty$, thus $H^1(BS^1, H^0(X(\mathbb{C}),\mathbb{C})) = 0$ and it's enough to show that $d_2^{0,1} = 0$. We leave it as an exercise to the reader to check that this is trues as soon as $X(\mathbb{C})$ is connected and $X(\mathbb{C})^{S^1} \neq 0$.

From all this we get a decomposition

$$H^1(X(\mathbb{C}),\mathbb{C}) \simeq H^0([X/\mathbb{G}_m], \mathbb{L}_{[X/\mathbb{G}_m]}) \oplus H^1([X/\mathbb{G}_m], \mathcal{O}_{[X/\mathbb{G}_m]}). \tag{5}$$

We have $\mathbb{L}_{[X/\mathbb{G}_m]} \simeq \Omega^1_X \xrightarrow{a^*} \mathcal{O}_X$ as a complex of $\mathbb{G}_m$-equivariant sheaves on X, where a^* is the map dual to the derivative of the action $\operatorname{Lie}(\mathbb{G}_m) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathbb{T}_X$ (where $\mathbb{T}_X$ denotes the tangent bundle). Then $\mathbb{H}^0(X, \Omega^1_X \xrightarrow{a^*} \mathcal{O}_X) \simeq \ker\left(H^0(X,\Omega^1_X) \xrightarrow{a^*} H^0(X,\mathcal{O}_X)\right)$, which is identified with the invariants of the Lie algebra action, which also identifies with the group invariants $H^0(X,\Omega^1_X)^{\mathbb{G}_m}$. Finally we get

$$H^0([X/\mathbb{G}_m], \mathbb{L}_{[X/\mathbb{G}_m]}) \simeq H^0(X, \Omega^1_X \xrightarrow{a^*} \mathcal{O}_X)^{\mathbb{G}_m} \simeq H^0(X,\Omega^1_X)^{\mathbb{G}_m}$$

as well. The second summand in (5) is just $H^1(X,\mathcal{O}_X)^{\mathbb{G}_m}$. Thus for any conical resolution we get a formula

$$H^1(X(\mathbb{C}),\mathbb{C}) \simeq H^0(X,\Omega^1_X)^{\mathbb{G}_m} \oplus H^1(X,\mathcal{O}_X)^{\mathbb{G}_m}.$$

This is a partial generalization of results of Section 6 in [KT16] to the case when $R^1\pi_*\mathcal{O}_X$ is not necessarily 0.

2 Spreadings

To apply Theorem 1.4.3 we need to find a good model of our stack over a finitely generated $\mathbb{Z}$-algebra, namely a Hodge-proper spreading. However, as we will see, such a spreading does not necessarily exist in general.

In Section 2.1 we first prove a general result about the existence of spreadings for some more natural classes of morphisms between Artin stacks (like smooth, flat, etc). Then some examples of Hodge-properly spreadable and nonspreadable stacks are given in Section 2.3.

2.1 Spreadable classes

Definition 2.1.1. Let $\mathcal{P}$ be a class of morphisms of schemes (e.g. $\mathcal{P}$ = smooth, flat or proper morphisms) containing all isomorphisms and closed under compositions. For a scheme S, define $\mathrm{Sch}^{\mathrm{fp},\mathcal{P}}_{/S}$ to be the (non-full) subcategory of schemes over S consisting of finitely-presentable S-schemes and morphisms between them that belong to $\mathcal{P}$.

Theorem 2.1.2 ([Gro66, Theorems 8.10.5, 11.2.6] and [Gro67, Proposition 17.7.8])**.** *Let $\{S_i\}$ be a filtered diagram of affine schemes with limit S and let $\mathcal{P}$ be one of the following classes of morphisms: isomorphisms, surjections, closed embeddings, flat, smooth or proper morphisms*[10]*. Then the natural functor*

$$\varinjlim_i \mathrm{Sch}^{\mathrm{fp},\mathcal{P}}_{/S_i} \to \mathrm{Sch}^{\mathrm{fp},\mathcal{P}}_{/S}$$

(induced by the base change $\mathrm{Sch}^{\mathrm{fp},\mathcal{P}}_{/S_i} \ni X \mapsto X \times_{S_i} S$) is an equivalence.

We will say that a scheme X is a *$\mathcal{P}$-scheme over S* ($\mathcal{P}$-scheme/S) if X is an S-scheme and the structure morphism $X \to S$ is in $\mathcal{P}$. From the theorem above one can formally deduces the following corollary (see Corollary 2.1.14 for a proof in a bit more general stacky setting):

Corollary 2.1.3. *Let $\{S_i\}_{i\in I}, S$ and $\mathcal{P}$ be as above. Then if X is a finitely presentable $\mathcal{P}$-scheme/S, then there exists $i \in I$ and a finitely presentable $\mathcal{P}$-scheme X_i over S_i, such that $X \simeq X_i \times_{S_i} S$.*

[10] The list is not even nearly complete. See [Poo17, Appendix C.1] for a much more exhaustive list of classes of morphisms and their properties with precise references.

Our goal in this section is to extend Theorem 2.1.2 to the setting of Artin stacks. First we recall how "finitely presentable" is defined in Artin setting:

Definition 2.1.4 (Finitely presentable Artin stacks)**.** A (-1)-Artin stack $\mathcal{X}$ over a base ring R is called *finitely presentable*, if $\mathcal{X} \simeq \operatorname{Spec} A$ and A is a finitely presentable R-algebra. Then, an n-Artin stack $\mathcal{X}$ over R is called finitely presentable if there exists a smooth atlas $U \twoheadrightarrow \mathcal{X}$ such that U is a finitely presentable affine scheme and $U \times_{\mathcal{X}} U$ is a finitely presentable $(n-1)$-Artin R-stack. We will denote the category of finitely presentable n-Artin stacks by $\mathcal{S}\mathrm{tk}^{n\text{-Art,fp}}$.

Our general strategy for proving results about spreadability is to inductively reduce to the case of finitely presentable schemes. For this end it will be technically convenient to use instead of iterative description of Artin stacks a representation as a geometric realization of a coskeletal hypercover by schemes:

Construction 2.1.5. Let $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object in a category $\mathcal{C}$ admitting finite limits. Define $X(-)\colon \mathrm{SSet}^{\mathrm{fin,op}} \to \mathcal{C}$ to be the right Kan extension of $X_\bullet$ along the inclusion of Δ^{op} into the opposite $\mathrm{SSet}^{\mathrm{fin,op}}$ of the category of finite simplicial sets (meaning simplicial sets with only finitely many non-degenerate simplices). More concretely, for a finite simplicial set K

$$X(K) \simeq \lim_{\Delta^n \in \Delta^{\mathrm{op}}_{/K}} X(\Delta^n).$$

In particular, we denote $M_n(X_\bullet) \coloneqq X(\partial\Delta^n)$ and call it the *n-th matching object of* $X_\bullet$.

Definition 2.1.6. Let $\mathbb{H}$ be an ∞-topos. An augmented simplicial object $X_\bullet \to X_{-1}$ is called a *hypercover of* X_{-1} if for any $n \in \mathbb{Z}_{\geq 0}$ the natural map $X_n \to M_n(X_\bullet)$ is an effective epimorphism (M_n is computed in the category $\mathbb{H}_{/X_{-1}}$). A hypercover $X_\bullet$ is called *n-coskeletal* if additionally for each $m > n$ the natural map $X_m \to M_m(X_\bullet)$ is an equivalence (equivalently $X_\bullet$ coincides with the right Kan extension of its restriction to $\Delta^{\mathrm{op}}_{\leq n}$). We refer interested reader to [BM19, Appendix] for a quick recap on hypercovers and to [Pri15, Section 2] for a discussion of hypergroupoids, which is most relevant for this section.

With this notation, n-Artin stacks can be thought of as some special $(n-1)$-coskeletal hypercovers:

Theorem 2.1.7 ([Pri15, Proposition 4.1 and Theorem 4.7])**.** *Let $\mathcal{X}$ be an n-Artin stack over S. Then there exists an $(n-1)$-coskeletal hypercover $X_\bullet$ of $\mathcal{X}$ such that all X_k are equivalent to coproducts of affine schemes and for all m, k, with $0 \leq m \leq k$, the maps $X_k \to X(\Lambda^k_m)$ are smooth surjections. Conversely, given $X_\bullet$ as above, its geometric realization $|X_\bullet|$ (in the category of stacks, i.e. sheaves of spaces in étale topology) is an n-Artin stack.*

Corollary 2.1.8. *Let $\mathcal{X}$ be a finitely presented n-Artin stack over S. Then there exists an $(n-1)$-coskeletal hypercover $X_\bullet$ of $\mathcal{X}$ such that all X_k are finitely presentable affine schemes and for all m, k, with $0 \leq m \leq k$, the maps $X_k \to X(\Lambda^k_m)$ are smooth surjections. Conversely, given $X_\bullet$ as above, its geometric realization $|X_\bullet|$ (in the category of stacks) is a finitely presentable n-Artin stack.*

Proof. Let $\mathcal{X}$ be a finitely presentable n-Artin stack. The simplicial scheme $X_\bullet$ from the theorem above is constructed inductively in [Pri15, Proposition 4.5] using only finite limits and atlases, hence all X_i can be chosen to be finitely presentable.

Conversely, if $X_\bullet$ is a simplicial affine scheme as in the statement of corollary, then by the theorem above $|X_\bullet|$ is an n-Artin stack. Moreover, the natural map $X_0 \to |X_\bullet|$ is a smooth finitely presentable atlas. To prove that $X_0 \times_{|X_\bullet|} X_0$ is finitely presented, recall that by [Pri15, Remark 2.25] there is a natural equivalence

$$X_0 \times_{|X_\bullet|} X_0 \simeq |X_0 \times_{X_\bullet} \mathrm{Dec}_+(X_\bullet)|,$$

where Dec_+ is the décalage functor, $\mathrm{Dec}_+(X_\bullet)_i \simeq X_{i+1}$. Since finitely presentable affine schemes are closed under fibered products, it follows that $X_0 \times_{X_\bullet} \mathrm{Dec}_+(X_\bullet)$ also satisfies conditions of the corollary. Since moreover, $X_0 \times_{X_\bullet} \mathrm{Dec}_+(X_\bullet)$ is $(n-2)$-coskeletal, its geometric realization $X_0 \times_{|X_\bullet|} X_0$ is finitely presentable by induction. ☐

For convenience we introduce the following notation:

Definition 2.1.9 (Spreadable class)**.** A class of morphism $\mathcal{P}$ between Artin stacks is called *spreadable* if

- $\mathcal{P}$ is closed under arbitrary base changes, compositions and contains all equivalences.

- (Locality on source and target) Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of finitely presentable Artin stacks. Then f lies in $\mathcal{P}$ if and only if there exist smooth finitely presentable affine atlases $U \twoheadrightarrow \mathcal{Y}$ and $V \twoheadrightarrow U \times_{\mathcal{Y}} \mathcal{X}$ such that the composite map $V \to U \times_{\mathcal{Y}} \mathcal{X} \to U$ is in $\mathcal{P}$.

- (Affine spreadability) Let $\{S_i\}$ be a filtered diagram of affine schemes with the limit S. Let $f\colon X \to Y$ be a morphism in $\mathcal{P}$ between affine finitely presentable S-schemes. Then for some i there exists a map $f_i\colon X_i \to Y_i$ in $\mathcal{P}$ of affine finitely presentable S_i-schemes, such that $f \simeq f_i \times_{S_i} S$.

Example 2.1.10. If $\mathcal{P}$ and $\mathcal{Q}$ is a pair of spreadable classes, then $\mathcal{P} \cap \mathcal{Q}$ and $\mathcal{P} \cup \mathcal{Q}$ are also spreadable. There exists the smallest spreadable class (consisting only of equivalences) and the largest one (consisting of all finitely presentable morphisms).

Example 2.1.11. Since surjective, smooth and flat morphisms of Artin stacks are by definition local on the source and the target for the flat topology, by Theorem 2.1.2 we get that these classes are spreadable.

Definition 2.1.12. Let $\mathcal{P}$ be a spreadable class and let S be a scheme. Let us denote by $\mathcal{S}\mathrm{tk}_{/S}^{n\text{-Art},\mathrm{fp},\mathcal{P}}$ the subcategory of the category of finitely presentable n-Artin stacks over S and morphisms from $\mathcal{P}$ between them.

We are now ready to prove the main technical result of this section (see [Ryd15], [LMB00, Chapter 4] for similar results in the context of 1-Artin stacks and [Lur18, Theorem 4.4.2.2] for the spectral version):

Theorem 2.1.13. *Let $\{S_i\}$ be a filtered diagram of affine schemes with limit S. Let $\mathcal{P}$ be a spreadable class. Then the natural functor*

$$\varinjlim_i \mathcal{S}\mathrm{tk}_{/S_i}^{n\text{-Art},\mathrm{fp},\mathcal{P}} \longrightarrow \mathcal{S}\mathrm{tk}_{/S}^{n\text{-Art},\mathrm{fp},\mathcal{P}}$$

(induced by the base-change $\mathcal{S}\mathrm{tk}_{/S_i}^{n\text{-Art},\mathrm{fp},\mathcal{P}} \ni \mathcal{X}_i \mapsto \mathcal{X}_i \times_{S_i} S$) is an equivalence.

Proof. We will prove the statement by induction on n. The base of the induction $n = -1$, i.e. the case of affine schemes, holds by the definition of a spreadable class. To make the induction step, we first prove the statement for $\mathcal{P} =$ "all (finitely presented) morphisms" (using the induction assumption for smooth surjective morphisms) and then deduce the statement for a general spreadable class $\mathcal{P}$.

Essential surjectivity for $\mathcal{P} =$ "all". Since all n-Artin stacks are $(n+1)$-truncated, the Yoneda embedding $\mathcal{S}\mathrm{tk}^{n\text{-Art}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$ factors through a full subcategory $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S}_{\le n+1}) =: \mathcal{PS}\mathrm{tk}_{\le n+1}$. Let now $\mathcal{X}$ be a finitely presented n-Artin S-stack and let $X_\bullet$ be a simplicial diagram of finitely presented affine S-schemes, so that $|X_\bullet| \simeq \mathcal{X}$ (as in Corollary 2.1.8). Since for any simplicial diagram $A_\bullet$ in any $(n+1,1)$-category the natural map $|A_\bullet|_{\le n+2} \to |A_\bullet|$ is an equivalence, we see that $\mathcal{X} \simeq |X_\bullet|_{\le n+2}$ in $\mathcal{PS}\mathrm{tk}_{\le n+1}$. But $X_{\bullet|\Delta^{\mathrm{op}}_{\le n+2}}$ is a finite diagram of finitely presented affine schemes, hence there exists S_i and a diagram $X_{\bullet|\Delta^{\mathrm{op}}_{\le n+2},S_i}$ such that $X_{\bullet|\Delta^{\mathrm{op}}_{\le n+2}} \simeq X_{\bullet|\Delta^{\mathrm{op}}_{\le n+2},S_i} \times_{S_i} S$. We set $\mathcal{X}_{S_i} \coloneqq |X_{\le n+2,S_i}|$. By applying the inductive assumption with $\mathcal{P} =$ "smooth surjective", we can assume that all maps $X_{k,S_j} \to X_{\bullet,S_j}(\Lambda^k_m)$ are smooth and surjective for some S_j; hence by Corollary 2.1.8 $\mathcal{X}_{S_j}$ is a finitely presented n-Artin spreading of $\mathcal{X}$.

Fully-faithfulness for $\mathcal{P} =$ "all". Let $\mathcal{X}_i, \mathcal{Y}_i$ be a pair of n-Artin stacks of finite presentation over S_i. We then have

$$\varinjlim_j \mathrm{Hom}_{\mathcal{PS}\mathrm{tk}_{/S_j}}(\mathcal{X}_i \times_{S_i} S_j, \mathcal{Y}_i \times_{S_i} S_j) \simeq \varinjlim_j \mathrm{Hom}_{\mathcal{PS}\mathrm{tk}_{/S_i}}(\mathcal{X}_i \times_{S_i} S_j, \mathcal{Y}_i) \simeq \varinjlim_j \mathrm{Hom}_{\mathcal{PS}\mathrm{tk}_{\le n+1/S_i}}(\mathcal{X}_i \times_{S_i} S_j, \mathcal{Y}_i), \quad (6)$$

where the second equivalence follows from the fact that filtered co-limits commute with π_*, hence preserve $(n+1)$-truncated spaces. Let now $X_\bullet \to \mathcal{X}$ be as in Corollary 2.1.8. Then

$$(6) \ldots \simeq \varinjlim_j \mathrm{Hom}_{\mathcal{PS}\mathrm{tk}_{\le n+1/S_i}}(|X_\bullet \times_{S_i} S_j|_{\le n+2}, \mathcal{Y}_i) \simeq \mathrm{Tot}_{\le n+2} \varinjlim_j \mathrm{Hom}_{\mathcal{PS}\mathrm{tk}_{\le n+1/S_i}}(X_\bullet \times_{S_i} S_j, \mathcal{Y}_i) \simeq$$
$$\simeq \mathrm{Tot}_{\le n+2} \varinjlim_j \mathrm{Hom}_{\mathcal{S}\mathrm{tk}_{/S_i}}(X_\bullet \times_{S_i} S_j, \mathcal{Y}_i),$$

where the second equivalence follows from the fact that, since $\Delta_{\le n+2}$ is a finite diagram, limits along $\Delta_{\le n+2}$ commute with filtered co-limits. Similarly, one shows that

$$\mathrm{Hom}_{\mathcal{S}\mathrm{tk}_{/S}}(\mathcal{X}_i \times_{S_i} S, \mathcal{Y}_i \times_{S_i} S) \simeq \mathrm{Tot}_{\le n+2} \mathrm{Hom}_{\mathcal{S}\mathrm{tk}_{/S_i}}(X_\bullet \times_{S_i} S, \mathcal{Y}_i).$$

Finally, since $\mathcal{Y}_i$ is finitely presentable, by [GR17, Chapter 2, Proposition 4.5.2]

$$\varinjlim_{j} \mathrm{Hom}_{\mathcal{S}\mathrm{tk}_{/S_i}}(X_\bullet \times_{S_i} S_j, \mathcal{Y}_i) \simeq \mathrm{Hom}_{\mathcal{S}\mathrm{tk}_{/S_i}}(X_\bullet \times_{S_i} S, \mathcal{Y}_i).$$

<u>*General* $\mathcal{P}$</u>. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism in a spreadable class $\mathcal{P}$ over S. It is enough to prove that there exists i and a map between finitely presentable n-Artin S_i-stacks $f_i\colon \mathcal{X}_i \to \mathcal{Y}_i$ such that $f_i \times_{S_i} S \simeq f$ and $f_i \in \mathcal{P}$. Choose affine finitely presentable atlases $U \twoheadrightarrow \mathcal{Y}$ and $V \twoheadrightarrow U \times_{\mathcal{Y}} \mathcal{X}$. The induced map $g\colon V \to U$ belongs to $\mathcal{P}$, so by the previous part and definition of spredable classes, the diagram

$$\begin{array}{ccc} V & \twoheadrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \twoheadrightarrow & \mathcal{Y} \end{array}$$

can be spread out to some S_i, such that g_{S_i} belongs to $\mathcal{P}$. It follows by the definition of spreadable class, that f_{S_i} is also in $\mathcal{P}$. $\square$

A stack $\mathcal{X}$ is called an *n-Artin $\mathcal{P}$-stack over S* if the structure morphism $\pi\colon \mathcal{X} \to S$ exhibits $\mathcal{X}$ as an n-Artin stack and π is in $\mathcal{P}$.

Corollary 2.1.14 (Existence of spreading in a predefined class). *Let $\{S_i\}_{i\in I}$ be a filtered diagram of affine schemes, $S := \varprojlim S_i$ and $\mathcal{P}$ be a spreadable class. Then if $\mathcal{X}$ is a finitely presentable n-Artin $\mathcal{P}$-stack over S, then there exists $i \in I$ and a finitely presentable n-Artin $\mathcal{P}$-stack $\mathcal{X}_i$ over S_i, such that $\mathcal{X} \simeq \mathcal{X}_i \times_{S_i} S$.*

Proof. Let $\pi\colon \mathcal{X} \to S$ be the structure morphism. By the previous theorem and the description of objects in filtered colimits of categories (see e.g. [Roz12]) there exists a finitely presented stack $\pi_j\colon \mathcal{X}_j \to S_j$ such that $\pi_j \times_{S_j} S = \pi$. A morphism in a filtered colimit of categories is a filtered co-limit of morphisms, hence

$$\mathrm{Hom}_{\mathcal{S}\mathrm{tk}^{n\text{-Art},\mathrm{fp},\mathcal{P}}_{/S}}(\mathcal{X}, S) \simeq \varinjlim_{k} \mathrm{Hom}_{\mathcal{S}\mathrm{tk}^{n\text{-Art},\mathrm{fp},\mathcal{P}}_{/S_i}}(\mathcal{X}_i \times_{S_j} S_i, S_i).$$

Since the left hand side is non-empty by assumption, the right hand side also must be nonempty for some i, i.e. there exists $i \in I$ such that $\pi_i\colon \mathcal{X}_i \to S_i$ is in $\mathcal{P}$. $\square$

2.2 Cohomologically proper stacks

In most examples for which we are able to construct a Hodge-proper spreading, the spreading in fact satisfies a stronger property, namely it is *cohomologically proper*. This property enjoys many natural properties that Hodge-properness does not: e.g. it translates along proper maps and a cohomologically proper scheme is necessarily proper. To introduce it we first need to extend Definition 1.2.1 to all locally Noetherian Artin stacks:

Definition 2.2.1. An Artin stack is called *locally Noetherian* if it admits an atlas $\coprod_i U_i$, where all U_i are Noetherian affine schemes. An Artin stack is called *Noetherian* if it is locally Noetherian and quasi-compact quasi-separated.

For a locally Noetherian Artin stack $\mathcal{X}$ we will denote by $\mathrm{Coh}(\mathcal{X})$ (resp. $\mathrm{Coh}^+(\mathcal{X})$) the full subcategory of $\mathrm{QCoh}(\mathcal{X})$ consisting of sheaves $\mathcal{F}$ such that the restriction of $\mathcal{F}$ to some (equivalently to any) locally Noetherian atlas has bounded (resp. bounded below) coherent cohomology sheaves.

Definition 2.2.2. A quasi-compact quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally Noetherian Artin stacks is called *cohomologically proper* if the induced functor $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ preserves the full subcategory of bounded below coherent sheaves. A locally Noetherian Artin stack $\mathcal{X}$ over a Noetherian ring R is called *cohomologically proper* if the structure morphism $\mathcal{X} \to \mathrm{Spec}\, R$ is cohomologically proper.

Remark 2.2.3. By the left exactness of f_* it is enough to prove that $f_*(\mathrm{Coh}(\mathcal{X})^\heartsuit) \subset \mathrm{Coh}^+(\mathcal{Y})$.

We have the following basic properties of cohomologically proper morphisms:

Proposition 2.2.4. *In the notations above we have:*

1. *The class of cohomologically proper morphism is closed under compositions.*

2. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a cohomologically proper morphism and assume that $\mathcal{X}$ is Noetherian. Then for any open quasi-compact embedding $\mathcal{U} \hookrightarrow \mathcal{Y}$ the pullback $\mathcal{U} \times_{\mathcal{Y}} \mathcal{X}$ is cohomologically proper over $\mathcal{U}$.*

3. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact quasi-separated morphism such that for some smooth cover $\pi\colon \mathcal{U} \to \mathcal{Y}$ the pull-back $f_{\mathcal{U}}\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is cohomologically proper. Then f is cohomologically proper.*

Proof. The first point is obvious. To prove the second one note that by base change it is enough to show that any coherent sheaf on $\mathcal{U} \times_{\mathcal{Y}} \mathcal{X}$ is a retract of a restriction of a coherent sheaf on $\mathcal{X}$. This is proved in Corollary 2.2.6 below. The third point follows by base change as well since it is enough to check that a sheaf belongs to Coh^+ on a smooth cover. ☐

Proposition 2.2.5. *Let $\mathcal{X}$ be a Noetherian Artin stack. Then for all $n \in \mathbb{Z}$ the category $\mathrm{QCoh}(\mathcal{X})^{\geq n}$ is compactly generated by $\mathrm{Coh}(\mathcal{X})^{\geq n}$.*

Proof. The shift functor $\mathcal{F} \mapsto \mathcal{F}[n]$ induces an equivalence $\mathrm{QCoh}(\mathcal{X})^{\geq n} \simeq \mathrm{QCoh}(\mathcal{X})^{\geq 0}$, hence without loss of generality we can assume that $n = 0$. During the proof we will freely use the fact that the truncation functors for the natural t-structure on $\mathrm{QCoh}(\mathcal{X})$ preserve filtered colimits (see e.g. [GR17, Chapter 3.3, Corollary 1.5.7]).

We first prove that $\mathrm{Coh}(\mathcal{X})^{[0;m]}$ is compact in $\mathrm{QCoh}(\mathcal{X})^{[0;m]}$ for all $m \geq 0$. Let $U_\bullet$ be an affine Noetherian smooth hypercover of $\mathcal{X}$. Since $\mathrm{QCoh}(\mathcal{X})^{[0;m]}$ is an $(m+1)$-category we then have

$$\mathrm{QCoh}(\mathcal{X})^{[0;m]} \simeq \mathrm{Tot}^{\leq m+2}\, \mathrm{QCoh}(U_\bullet)^{[0;m]}.$$

Since $\Delta_{\leq m+2}$ is a finite diagram it follows that a sheaf in $\mathrm{Coh}(\mathcal{X})^{[0;m]}$ is compact in $\mathrm{QCoh}(\mathcal{X})^{[0;m]}$, since all of its images are compact in $\mathrm{QCoh}(U_i)^{[0;m]}$. Note also that since for any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^{\leq m}$ and $\mathcal{G} \in \mathrm{QCoh}(\mathcal{X})$ we have

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{X})^{\geq 0}}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{F}, \tau^{\leq m}\mathcal{G})$$

and since truncation functor $\tau^{\leq m}$ preserves filtered colimits, it follows that $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})^{[0;m]}$ is compact in $\mathrm{QCoh}(\mathcal{X})^{\geq 0}$ as well.

Next we show that $\mathrm{QCoh}(\mathcal{X})^{\heartsuit} \simeq \mathrm{Ind}(\mathrm{Coh}(\mathcal{X})^{\heartsuit})$. The argument is a slight variation of [SP20, Tag 07TU]. By assumption on $\mathcal{X}$ there exists an affine Noetherian atlas $p\colon U \twoheadrightarrow \mathcal{X}$. Let $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^{\heartsuit}$ and write $p^*\mathcal{F} \simeq \varinjlim \mathcal{G}_\alpha$, where the diagram on the right runs over all finitely generated submodules of $p^*\mathcal{F}$. For each α define $\mathcal{F}_\alpha \in \mathrm{QCoh}(\mathcal{X})$ as a pullback

$$\begin{array}{ccc} \mathcal{F}_\alpha & \hookrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{H}^0 p_* \mathcal{G}_\alpha & \hookrightarrow & \mathcal{H}^0 p_* p^* \mathcal{F}. \end{array}$$

Using triangular identities one easily checks that the inclusion $p^*\mathcal{F}_\alpha \hookrightarrow p^*\mathcal{F}$ factors through an inclusion $\mathcal{G}_\alpha \hookrightarrow p^*\mathcal{F}$. In particular, $p^*\mathcal{F}_\alpha$, being a submodule of a finitely generated module $\mathcal{G}_\alpha$ over Noetherian ring $\Gamma(U, \mathcal{O}_U)$, is finitely generated itself. By definition it means that $\mathcal{F}_\alpha$ is coherent. Finally, since p is quasi-compact quasi-separated, the pushforward functor $\mathcal{H}^0 p_*$ preserves filtered colimits, hence the natural map $\varinjlim \mathcal{F}_\alpha \to \mathcal{F}$ is an isomorphism.

Let now $i\colon \mathrm{Ind}(\mathrm{Coh}(\mathcal{X})^{\geq 0}) \to \mathrm{QCoh}(\mathcal{X})^{\geq 0}$ be a natural functor. Note that since by the previous $\mathrm{Coh}(\mathcal{X})^{\geq 0}$ is compact in $\mathrm{QCoh}(\mathcal{X})^{\geq 0}$ this functor is fully faithful. Moreover, since i preserves colimits it admits a right adjoint R. Then to prove that i is essentially surjective it is enough to show that the fiber $\mathcal{G}$ of the co-unit $iR\mathcal{F} \to \mathcal{F}$ vanishes. But R being right adjoint preserves fibered products, hence $R\mathcal{G} \simeq \mathrm{fib}(RiR\mathcal{F} \to R\mathcal{F}) \simeq 0$. By Yoneda's lemma and adjunction $i \dashv R$ we conclude that $\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{X})^{\geq 0}}(\mathcal{H}, \mathcal{G}) \simeq *$ for all $\mathcal{H} \in \mathrm{Coh}(\mathcal{X})^{\geq 0}$. We claim that $\mathcal{G} \simeq 0$. To see this assume that $\mathcal{G} \not\simeq 0$ and let i be the smallest integer such that $\mathcal{H}^i(\mathcal{G}) \not\simeq 0$. By the previous part there exists a coherent subsheaf $\mathcal{H} \subseteq \mathcal{H}^i(\mathcal{G})$. It follows the composition $\mathcal{H}[-i] \to \mathcal{H}^i(\mathcal{G})[-i] \to \mathcal{G}$ is non-zero, a contradiction. ☐

Corollary 2.2.6. *Let $\mathcal{X}$ be a Noetherian Artin stack and let $j\colon \mathcal{U} \to \mathcal{X}$ be an open embedding. Then every coherent sheaf on $\mathcal{U}$ is a retract of a restriction of a coherent sheaf from $\mathcal{X}$.*

Proof. Note that since $\mathcal{X}$ is Noetherian, the stack $\mathcal{U}$ is also Noetherian. In particular the embedding j is quasi-compact and quasi-separated. Next, by pulling back to an atlas and using base change (which holds by qcqs assertion about j), one finds that the co-unit of adjunction $j^*j_*\mathcal{F} \to \mathcal{F}$ is an equivalence for any quasi-coherent sheaf on $\mathcal{U}$. Let now $\mathcal{F} \in \mathrm{Coh}(\mathcal{U})$. By the previous proposition $j_*\mathcal{F} \simeq \varinjlim \mathcal{G}_\alpha$ for some filtered diagram of coherent sheaves $\mathcal{G}_\alpha$. It follows that $\mathcal{F} \simeq \varinjlim j^*\mathcal{G}_\alpha$. By compactness of $\mathcal{F}$ we conclude that it is a retract of some $j^*\mathcal{G}_\alpha$. ☐

If R is regular, the cohomological properness is stronger than the Hodge-properness:

Proposition 2.2.7. *Let $\mathcal{X}$ be a smooth cohomologically proper Artin stack over a regular Noetherian ring R. Then $\mathcal{X}$ is Hodge-proper over R.*

Proof. Since $\mathcal{X}$ is smooth over a regular Noetherian ring R, the category of coherent sheaves on $\mathcal{X}$ coincides with the category of perfect complexes. So by assumption, it is enough to prove that $\wedge^i\mathbb{L}_{\mathcal{X}/R}$ is perfect for all $i \geq 0$. By smoothness, the cotangent complex $\mathbb{L}_{\mathcal{X}/R}$ is perfect and concentrated in non-negative cohomological degrees. It follows that $\mathbb{L}_{\mathcal{X}/R}$ admits a finite filtration with the associated graded pieces being negative shifts of vector bundles. Hence by induction it is enough to prove that if E is a quasi-coherent sheaf on $\mathcal{X}$ such that $\wedge^j E$ is perfect for $j \leq i$, then $\wedge^i(E[-1])$ is also perfect. But by construction (see [BM19, Theorem 3.35]) the functor $\wedge^i$ is i-excisive, so $\wedge^i(E[-1])$ is a finite limit of sheaves of the form $\wedge^i(E^{\oplus n}), n \leq i$, hence is perfect. ◻

Moreover, all proper morphisms are cohomologically proper. To show this, let's first recall the notion of a proper morphisms between higher stacks (following [PY14, Section 4]):

Definition 2.2.8. A 0-representable morphism $\mathcal{X} \to \mathcal{Y}$ is called *proper* if for any affine scheme S mapping to $\mathcal{Y}$, the pullback $\mathcal{X} \times_{\mathcal{Y}} S$ is a proper S-scheme. Next, assuming that the notion of a proper $(n-1)$-representable morphism is already defined, an n-representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is called *proper* if

- f is *separated*, i.e. the diagonal map $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is $(n-1)$-representable) is proper.
- For any affine scheme S mapping to $\mathcal{Y}$ the pullback $\mathcal{X}_S := \mathcal{X} \times_{\mathcal{Y}} S$ admits a surjective S-morphism $P \twoheadrightarrow \mathcal{X}_S$ such that P is a proper S-scheme.

Remark 2.2.9. Since the property of a morphism of schemes to be proper is flat local on the target, it is enough in the definition above to check the second condition only for some atlas of $\mathcal{Y}$.

Remark 2.2.10. A potentially more familiar definition of a (classical) proper algebraic stack $p : \mathcal{X} \to S$ is that p should be separated, finite type and universally closed. We note that such stacks over S are proper 1-Artin stacks in the definition above. Indeed, by [Ols05, Theorem 1.1] in this case there exists a proper surjective map $U \to \mathcal{X}$ from a proper scheme U.

From the standard results about proper morphisms of schemes and representable morphisms of stacks one formally deduces:

Proposition 2.2.11. *With the notations above:*

1. *Proper morphism are closed under base change.*
2. *The property of being a proper morphism is flat local on the target.*
3. *Proper morphisms are also closed under compositions.*

The fact that proper morphisms are cohomologically proper was proved in [PY14, Theorem 5.13], but in a slightly different context. Their proof essentially follows the argument of [LMB00, Theorem 15.6] in the case of classical proper stacks. For the reader's convenience we sketch the argument here:

Proposition 2.2.12. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper morphism between locally Noetherian Artin stacks. Then f is cohomologically proper.*

Sketch of the proof. The question is local on the target, hence we can assume that $\mathcal{Y} = Y$ is an affine Noetherian scheme. Moreover, by localizing further if necessary, we can assume that there exists a surjective map $\pi\colon P \twoheadrightarrow \mathcal{X}$ such that P is a proper scheme over Y. Let us also assume that $\mathcal{X}$ is n-Artin for some $n \geq 0$ and let us prove the statement by induction on n. The statement for the $n = 0$ is the fundamental result about the direct image of a coherent sheaf under a proper morphism of schemes [Gro61, Chapter III, Theorem 3.2.1].

By Remark 2.2.3 it is enough to prove that $f_*(\mathrm{Coh}^{\heartsuit}(\mathcal{X})) \subset \mathrm{Coh}^+(Y)$. Let $\mathcal{F} \in \mathrm{Coh}^{\heartsuit}(\mathcal{X})$. Since $\mathcal{X}$ is proper over a Noetherian base, it is Noetherian itself. It follows that there exists a finite filtration (by power of nil-radical of $\mathcal{O}_{\mathcal{X}}$) of $\mathcal{F}$ with the associated graded pieces coming from $\mathcal{X}^{\mathrm{red}}$. Since $\mathrm{Coh}^+(Y)$ is closed under finite extensions, it follows that we can assume that both $\mathcal{X}$ and Y are reduced.

Let us denote $\mathrm{Tot}\, \pi_{\bullet,*}(\mathcal{H}^0(\pi_\bullet^*\mathcal{F}))$ by $\mathcal{F}'$, where $\pi_\bullet\colon P_\bullet \to \mathcal{X}$ is the Čech nerve of the map $P \to \mathcal{X}$. By the higher "generic flatness" [PY14, Theorem 8.3] there exists an open dense substack $\mathcal{U}$ of $\mathcal{X}$ such that the induced map $P_{\mathcal{U}} := P \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U}$ is flat. In particular, $\pi^*_{\mathcal{U},n}(\mathcal{F}) \simeq \mathcal{H}^0(\pi^*_{\mathcal{U},n}(\mathcal{F}))$ for all $n \in \mathbb{Z}_{\geq 0}$. It follows by flat descent that the natural map $\mathcal{F} \to \mathcal{F}'$ becomes an equivalence when restricted to $\mathcal{U}$. By Noetherian induction we can assume that $f_*(\mathrm{fib}(\mathcal{F} \to \mathcal{F}'))$ lies in $\mathrm{Coh}^+(Y)$. So to prove that $\pi_*(\mathcal{F})$ lies in $\mathrm{Coh}^+(Y)$ it is enough to show that $\pi_*(\mathcal{F}') \in \mathrm{Coh}^+(Y)$. On the other hand, all elements P_n of the Čech nerve are $(n-1)$-Artin proper stacks over Y and all sheaves $\mathcal{H}^0(p_n^*(\mathcal{F}))$ are coherent. Since the global section functors $f_{n,*}\colon \mathrm{QCoh}(P_n) \to \mathrm{QCoh}(Y)$ are right t-exact, it follows by induction and Proposition 1.2.3 that the totalization

$$f_*(\mathcal{F}') \simeq \mathrm{Tot}\, f_{\bullet,*}(\mathcal{H}^0(p_n^*\mathcal{F}))$$

lies in $\mathrm{Coh}^+(Y)$. □

Corollary 2.2.13. *Let $\mathcal{X}$ be a smooth proper Artin stack over a regular Noetherian ring R. Then $\mathcal{X}$ is Hodge-proper.*

Proof. Follows immediately from the previous proposition and Proposition 2.2.7. □

Finally, we record the following observation, which allows to construct new examples of cohomologically proper stacks in inductive way.

Proposition 2.2.14. *Let $\pi_\bullet\colon \mathcal{U}_\bullet \to \mathcal{X}$ be a flat hypercover such that all $\mathcal{U}_n$ are cohomologically proper over a Noetherian base ring R. Then $\mathcal{X}$ is cohomologically proper over R.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. By shifting if necessary we can assume that $\mathcal{H}^{<0}(\mathcal{F}) \simeq 0$. By the flat descent

$$R\Gamma(\mathcal{X}, \mathcal{F}) \simeq \mathrm{Tot}\, R\Gamma(\mathcal{U}_\bullet, \pi_\bullet^*\mathcal{F}).$$

Since the global section functors $R\Gamma(\mathcal{U}_n, -)$ are right t-exact and by assumptions on $\mathcal{U}_n$ the diagram $R\Gamma(\mathcal{U}_\bullet, \pi_\bullet^*\mathcal{F})$ consists of coconective bounded below coherent complexes. By Proposition 1.2.3 the complex $R\Gamma(\mathcal{X}, \mathcal{F})$ is also bounded below coherent. □

2.3 Examples of Hodge-properly spreadable stacks

In this subsection we begin to study which Hodge-proper stacks in characteristic 0 admit a Hodge-proper spreading over some finitely generated $\mathbb{Z}$-algebra. We will make extensive use of Theorem 2.1.13 in the following situation: let F be an algebraically closed field of characteristic 0, then $\mathrm{Spec}\, F \simeq \varprojlim R$ where $R \subset F$ runs through subrings of F that are smooth over $\mathbb{Z}$. This diagram is filtered since for any two such subalgebras $R_1, R_2 \subset F$ some finite localization $(R_1 \cdot R_2)[1/f]$ of their composite in F is again smooth over $\mathbb{Z}$. In particular we have an equivalence

$$\varinjlim_{R \subset F} \mathcal{S}\mathrm{tk}_{/R}^{n\text{-Art},\mathrm{fp},\mathcal{P}} \xrightarrow{\sim} \mathcal{S}\mathrm{tk}_{/F}^{n\text{-Art},\mathrm{fp},\mathcal{P}}$$

for any spreadable class $\mathcal{P}$. Another option is to also allow those localizations of smooth $\mathbb{Z}$-algebras for which the image of $\mathrm{Spec}\, R$ in $\mathrm{Spec}\, \mathbb{Z}$ is open (as in Definition 1.4.1). By Remark 1.4.2 the diagram of all such R is again filtered and so Theorem 2.1.13 can be applied. In Section 2.3.3 we will see that allowing these localizations actually makes a difference. In what follows F will always denote an algebraically closed field of characteristic 0 and we will pick $R \subset F$ to be a smooth $\mathbb{Z}$-subalgebra (except Section 2.3.3, where it will be an infinite localization of one) of F. We also freely use the standard spreading out results for schemes (Theorem 2.1.2) and their easy consequences (like spreading out group schemes, group actions, group homomorphisms, closed subgroups, etc.) without any additional reference.

We start with the Hodge-proper spreadability for proper Artin stacks, which is deduced from the spreadability of proper morphisms (Theorem 2.1.13). This is done in Section 2.3.1. Then we discuss in great detail the question of Hodge-proper spreadability of BG in Section 2.3.2; the case of more general quotient stacks is postponed till Section 3. Finally, in Section 2.3.3 we try to grasp the scope of potential applications of Theorem 1.4.3 concentrating on the case of schemes: in fact a particular set of examples given by semiabelian surfaces.

As was mentioned, often, along with Hodge-proper spreadability, we are able to prove a somewhat stronger statement, saying that the stacks we consider admit a cohomologically proper spreading. For this it is convenient to introduce the following variant of Definition 2.2.2:

Definition 2.3.1. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Artin stacks over a field F of characteristic 0 is called *cohomologically properly spreadable* if there exists a finitely generated $\mathbb{Z}$-algebra $R \subset F$[11] and a morphism $f_R\colon \mathcal{X}_R \to \mathcal{Y}_R$ over $\mathrm{Spec}\, R$, such that

[11] Or a suitable localization of one, as in Definition 1.4.1.

- $f_R \otimes_R F := f_R \times_R F \simeq f$.
- $f_R \colon X_R \to Y_R$ is cohomologically proper (see Definition 2.2.2).

In the case $\mathcal{Y} = \operatorname{Spec} F$ we will call $\mathcal{X}$ *cohomologically properly spreadable*. By Proposition 2.2.7 any such $\mathcal{X}$ is also Hodge-properly spreadable.

2.3.1 Proper stacks

In this subsection we show that all proper stacks are cohomologically (and in particular Hodge-)properly spreadable. By Proposition 2.2.12 it is just enough to show that proper morphisms spread out.

Following the convention of Section 2.1, for an affine scheme S we denote by $\mathcal{S}\mathrm{tk}_{/S}^{n\text{-Art,fp,pr}} \subset \mathcal{S}\mathrm{tk}_{/S}^{n\text{-Art,fp}}$ the subcategory consisting of finitely presented n-Artin S-stacks and with morphisms given by proper maps (see Definition 2.2.8).

The results of Section 2.1 allow to deduce the spreadability of proper morphisms from the analogous statement for classical schemes:

Proposition 2.3.2. *Let $\{S_i\}$ be a filtered diagram of affine schemes with a limit S. Then the natural functor*

$$\varinjlim_i \mathcal{S}\mathrm{tk}_{/S_i}^{n\text{-Art,fp,pr}} \longrightarrow \mathcal{S}\mathrm{tk}_{/S}^{n\text{-Art,fp,pr}}$$

is an equivalence.

Proof. By Theorem 2.1.13 it is enough to prove that for a proper morphism $f \colon \mathcal{X} \to \mathcal{Y}$ there exists a proper morphism $f_i \colon \mathcal{X}_i \to \mathcal{Y}_i$ such that $f_i \times_{S_i} S \simeq f$. Assume that f is n-representable. We will prove the statement by induction on n.

For $n = 0$ let $U \twoheadrightarrow \mathcal{Y}$ be an affine finitely presentable atlas. Then by assumption $\mathcal{X}_U := \mathcal{X} \times_{\mathcal{Y}} U$ is a scheme proper over U. By Theorem 2.1.13 we can spread the commutative square

$$\begin{array}{ccc} \mathcal{X}_U & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ U & \twoheadrightarrow & \mathcal{Y} \end{array}$$

to some S_i. By spreadability of equivalences, smooth surjective morphisms of stacks and proper morphisms of schemes we can assume, taking base change to some S_j if necessary, that the natural map $\mathcal{X}_{U,j} \to U_j \times_{\mathcal{Y}_j} \mathcal{X}_j$ is an equivalence. We can also assume that $U_j \to \mathcal{Y}_j$ is a smooth atlas and that $\mathcal{X}_{U,j}$ is proper scheme over U_j. Since the property of a map of schemes to be proper is flat local on target, it follows that for any T mapping to $\mathcal{Y}_j$ the pullback $T \times_{\mathcal{Y}_j} \mathcal{X}_j$ is a proper T-scheme. So the map $f_j \colon \mathcal{X}_j \to \mathcal{Y}_j$ is proper.

Finally, assume that the statement for $(n-1)$-representable morphism is already proved. Let $U \twoheadrightarrow \mathcal{Y}$ be a smooth finitely presentable atlas and let $P \twoheadrightarrow \mathcal{X}_U$ be a surjection from a proper U-scheme P. Then by the induction assumption we can find a spreading $f_i \colon \mathcal{X}_i \to \mathcal{Y}_i$ such that f_i is separated. By taking base change to some S_j we can assume that P_j is proper over U_j and that the map $P_j \to \mathcal{X}_{U_j}$ is surjective. □

Corollary 2.3.3. *Let $\mathcal{X}$ be a smooth proper stack over a field F of characteristic 0. Then $\mathcal{X}$ is Hodge-proper and Hodge-properly spreadable.*

Proof. By Proposition 2.3.2 (and Corollary 2.1.14) applied to $\mathcal{X} \to \operatorname{Spec} F$ we get a smooth proper spreading $\mathcal{X}_R \to \operatorname{Spec} R$. Then $\mathcal{X}_R$ is Hodge-proper by Propositions 2.2.7 and 2.2.12. □

We will also use the following corollary:

Corollary 2.3.4. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a proper map of finitely presentable Artin stacks over a field F of characteristic 0 such that $\mathcal{Y}$ is cohomologically properly spreadable (see Definition 2.3.1). Then $\mathcal{X}$ is also cohomologically properly spreadable.*

Proof. Let $f_R \colon \mathcal{X}_R \to \mathcal{Y}_R$ be some proper spreading of f. Since any two spreadings become equivalent after some finite localization of R, we can assume that $\mathcal{Y}_R$ is cohomologically proper over R. Then we conclude by Proposition 2.2.12 and the first part of Proposition 2.2.4. □

Remark 2.3.5. More generally, a composition of two cohomologically properly spreadable morphisms is again cohomologically properly spreadable.

2.3.2 Classifying stacks

Let F be an algebraically closed field of characteristic zero. We start our investigation of Hodge-proper spredability by first understanding for which algebraic groups G over F the classifying stack BG is Hodge-proper. The answer turns out to be easy: for all finite type G. In fact BG is even cohomologically proper:

Proposition 2.3.6. *Let G be a finite type group scheme over F. Then BG is cohomologically (and, in particular, Hodge-)proper.*

Proof. In fact we will show a more precise statement, namely that for any coherent sheaf $\mathcal{F}$ on BG, $R\Gamma(BG, \mathcal{F})$ lies in $\mathrm{Coh}(F)$ if G is linear and in $\mathrm{Coh}^+(F)$ if G is general.

Note that, since we are in characteristic 0, G is smooth and thus so is the natural map $\mathrm{Spec}\, F \to BG$[12]. In particular, the natural t-structure on $\mathrm{Coh}(BG)$ coincides with the usual t-structure on $\mathrm{Coh}(F)$ after taking pullback $\mathrm{Coh}(BG) \to \mathrm{Coh}(F)$ (aka the forgetful functor in terms of representations). It is enough to show the statement for $\mathcal{F} \in \mathrm{Coh}(BG)^\heartsuit$ (Remark 2.2.3). Note that such $\mathcal{F}$ is the same thing as a finite-dimensional algebraic representation of G over F.

By Chevalley's structure theorem there is an exact sequence $1 \to L \to G \to A \to 1$ where L is a linear algebraic group and A is proper. Then for L we have another short exact sequence

$$1 \to U \to L \to H \to 1,$$

where U is the unipotent radical of L and $H \simeq L/U$ is reductive.

Let $j\colon BU \to BL$, $f\colon BL \to BH$, $i\colon BL \to BG$ and $p\colon BG \to BA$ be the corresponding maps between classifying stacks. We will prove the statement step by step, starting from the unipotent case.

Case 1. $G = U$ is unipotent. We assume $\mathcal{F} \in \mathrm{Coh}(BU)^\heartsuit$. Since the characteristic of F is 0 and U is unipotent, $R\Gamma(BU, \mathcal{F})$ can be computed as the cohomology of the Lie algebra $\mathfrak{u}$. Explicitly, this is given by the Chevalley complex:

$$0 \to \mathcal{F} \to \mathcal{F} \otimes \mathfrak{u}^* \to \mathcal{F} \otimes \wedge^2 \mathfrak{u}^* \to \ldots \mathcal{F} \otimes \wedge^{\dim U} \mathfrak{u}^* \to 0.$$

Since $\mathcal{F}$ is finite dimensional this complex is clearly perfect.

Case 2. $G = H$ is reductive. This follows from the fact that the abelian category $\mathrm{Rep}(H)$ is semi-simple (since $\mathrm{char}(F) = 0$). Namely for $\mathcal{F} \in \mathrm{Coh}(BH)^\heartsuit$, the complex $R\Gamma(BH, \mathcal{F})$ is equal to the H-invariants $\mathcal{F}^H$ (in cohomological degree 0). Since $\mathcal{F}$ is finite-dimensional we get $R\Gamma(BH, \mathcal{F}) \in \mathrm{Coh}(F)$.

Case 3. $G = A$ is proper. Let $\mathcal{F} \in \mathrm{Coh}(BA)$. We can compute $R\Gamma(BA, \mathcal{F})$ using the smooth $q\colon \mathrm{Spec}\, F \to BA$. Let $p_n\colon A^n \to BA$ be the map from the n-th term of the associated Čech simplicial object. We get a cosimplicial object

$$[n] \mapsto R\Gamma(A^n, p_n^* \mathcal{F}),$$

in Mod_F, and

$$R\Gamma(BA, \mathcal{F}) \simeq \mathrm{Tot}\, R\Gamma(A^\bullet, p_\bullet^* \mathcal{F}).$$

However, each term $R\Gamma(A^n, p_n^* \mathcal{F})$ lies in $\mathrm{Coh}(F)$ (since A^n is proper) and has cohomology only in non-negative degrees. By Proposition 1.2.3 it follows that $R\Gamma(BA, \mathcal{F})$ lies in $\mathrm{Coh}^+(F)$.

Case 4. $G = L$ is linear. We assume $\mathcal{F} \in \mathrm{Coh}(BU)^\heartsuit$ and consider $f_* \mathcal{F} \in \mathrm{QCoh}(BH)$ (for $f\colon BL \to BH$). We claim that $f_* \mathcal{F} \in \mathrm{Coh}(BH)$. It is enough to check that after taking pull-back to the smooth cover $q\colon \mathrm{Spec}\, F \to BH$. We have a fibered square

$$\begin{array}{ccc} BU & \xrightarrow{j} & BL \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\, F & \xrightarrow{q} & BH \end{array}$$

and by base change we have $q^* f_* \mathcal{F} \simeq R\Gamma(BU, j^* \mathcal{F})$. The map j is flat, so $j^* \mathcal{F}$ is coherent and thus $R\Gamma(BU, j^* \mathcal{F}) \in \mathrm{Coh}(F)$ by Case 1. It follows that $f_* \mathcal{F} \in \mathrm{Coh}(BH)$. But then $R\Gamma(BL, \mathcal{F}) \simeq R\Gamma(BH, f_* \mathcal{F})$ and we are done by Case 2. At this point we have the statement for G linear.

Case 5. G is general. The argument in Case 4 works here as well, replacing U with L and H with A. Namely $p_* \mathcal{F} \in \mathrm{Coh}(BA)$ and then by Case 3

$$R\Gamma(BL, \mathcal{F}) \simeq R\Gamma(BH, f_* \mathcal{F}). \quad \square$$

[12] Note that since the property of a morphism to be smooth can be checked flat locally on the source the structure map $BG \to \mathrm{Spec}\, F$ is always smooth even when G is not. We refer interested reader to [Toë11] or [SP20, Tag 0DLS] for more details.

Even though BG is Hodge-proper practically for any G, there are definitely some algebraic groups G for which BG is not Hodge-properly spreadable. Indeed, consider $G = \mathbb{G}_a$. If $B\mathbb{G}_a$ were Hodge-properly spreadable, then by Corollary 1.5.2 we would get a decomposition

$$H^n_{\mathrm{dR}}(B\mathbb{G}_a/F) \simeq \bigoplus_{p+q=n} H^q(B\mathbb{G}_a, \wedge^p \mathbb{L}_{B\mathbb{G}_a/F}).$$

However, this is impossible. Indeed, the left hand side vanishes for $n > 0$ by the $\mathbb{A}^1$-homotopy invariance of the de Rham cohomology in characteristic 0. On the other hand $\wedge^p \mathbb{L}_{B\mathbb{G}_a/F} \simeq \mathcal{O}_{B\mathbb{G}_a}[-p]$ and $H^i(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ is non-zero for $i = 0, 1$. In particular, the right hand side is non-zero for all n, contradiction.

Note that by Theorem 1.4.3 it follows that the Hodge cohomology of any spreading of $B\mathbb{G}_a$ *has to be* infinitely generated. This is also confirmed by an explicit computation of the cohomology of $\mathcal{O}_{B\mathbb{G}_a}$ over $\mathbb{Z}$ which the reader can find in Appendix A. We only slightly comment on this here:

Example 2.3.7. Let $\mathbb{G}_a$ be the additive group considered as an algebraic group scheme over $\mathbb{Z}$. By the computation in Appendix A one has an embedding

$$\left(\mathbb{Z}[v_1] \otimes_{\mathbb{Z}} \mathrm{Sym}^*_{\mathbb{Z}} \left(\bigoplus\nolimits_p \mathbb{F}_p v_p \oplus \mathbb{F}_p v_{p^2} \oplus \ldots\right)\right) \Big/ v_1^2 = v_2 \hookrightarrow H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a}) \ ,$$

where v_1 has cohomological degree 1 and all other v_{p^i} are of degree 2. In particular $H^2(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ has infinitely generated elementary p-torsion for any prime p. Given any spreading $\mathcal{X}$ of $(B\mathbb{G}_a)_F$ over some $R \subset F$, by Theorem 2.1.13 it becomes isomorphic to $(B\mathbb{G}_a)_R$ for some larger R. Choosing prime p in the image of $\operatorname{Spec} R$ in $\operatorname{Spec}\mathbb{Z}$, by flat base change we get that $H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ contains an infinite sum $(R/p)^{\oplus \mathbb{N}}$ and thus $\mathcal{X}$ is not Hodge-proper over R.

Given the complexity of $B\mathbb{G}_a$ from cohomological point of view, it is natural to ask for which algebraic groups G, the classifying stack BG is Hodge-properly spreadable. We provide a list of examples:

Example 2.3.8. BG is Hodge-properly (and in fact also cohomologically properly) spreadable if

- G is proper (=an extension of a finite group by an abelian variety). Then BG is a proper stack and this is covered by Corollary 2.3.3;
- G is reductive. This follows from Proposition 3.1.2 if we take $Y = \operatorname{Spec} F$;
- $G = P \subset H$ is some parabolic subgroup of some reductive group H. This is a particular case of Theorem 3.2.12. Alternatively, it follows from the previous point and Corollary 2.3.4 (using that $BP \to BH$ is proper).

Remark 2.3.9. By an argument similar to Proposition 2.3.6 it is also possible to show the spreadability of BG for an extension of an abelian variety by a parabolic subgroup of some reductive group.

The fact that BP is cohomologically properly spreadable can look a little surprising and we would like to illustrate what happens by the simplest non-trivial example, a Borel subgroup $B \subset \mathrm{SL}_2$:

Example 2.3.10. Let $G = B \subset \mathrm{SL}_2$ be the standard Borel subgroup of SL_2 over $\mathbb{Z}$, namely

$$B = \left\{ \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix} \right\} \subset \mathrm{SL}_2 \, .$$

In this example we will show that BB is a cohomologically proper spreading of BB_F.

Note that $B \simeq \mathbb{G}_a \rtimes \mathbb{G}_m$ with $\mathbb{G}_m = \operatorname{Spec}\mathbb{Z}[t, t^{-1}]$ acting on $\mathbb{G}_a = \operatorname{Spec}\mathbb{Z}[x]$ by multiplication of x by t^2. Consider the natural map $p\colon BB \to B\mathbb{G}_m$ and take $p_*(\mathcal{O}_{BB})$. We have a fiber square

$$\begin{array}{ccc} B\mathbb{G}_a & \xrightarrow{j} & BB \\ \downarrow & & \downarrow \\ \mathrm{pt} & \xrightarrow{q} & B\mathbb{G}_m. \end{array}$$

We have $j^*\mathcal{O}_{BB} \simeq \mathcal{O}_{B\mathbb{G}_a}$ and by base change the underlying complex $q^* p_* \mathcal{O}_{BB}$ is equal to $R\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$. It follows that

$$R\Gamma(BB, \mathcal{O}_{BB}) \simeq R\Gamma(B\mathbb{G}_m, p_*\mathcal{O}_{BB}) \simeq R\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})^{\mathbb{G}_m},$$

since $\mathbb{G}_m$-invariants is an exact functor. In the terms of the computation in Appendix A this corresponds to the 0-th graded component $R\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})_0 \subset R\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$, which, as we figure there, is just given by $\mathbb{Z}$. Consequently $R\Gamma(BB, \mathcal{O}_{BB}) = \mathbb{Z}$.

Summarizing, we can see that even though the cohomology of $\mathcal{O}_{B\mathbb{G}_a}$ is enormous, the $\mathbb{G}_m$-action contracts it, ultimately making the cohomology of $\mathcal{O}_{BB}$ finitely generated.

In fact more is true: namely for any Borel subgroup B of a split reductive group G over $\mathbb{Z}$ the stack BB is cohomologically proper. Indeed, one first shows that BG is cohomologically proper over $\mathbb{Z}$ by applying Theorem 3.0.1 in the case $A = R = \mathbb{Z}$ and, since the morphism $BB \to BG$ is proper, the rest follows from Proposition 2.2.12.

2.3.3 Non-proper schemes

Considering schemes, it is natural to ask whether a Hodge-properly spreadable scheme is necessarily proper. On the other extreme, one can ask whether any schematic example of the Hodge-to-de Rham degeneration is Hodge-properly spreadable. Below we consider an example of a semiabelian surface X given by an extension of an elliptic curve by $\mathbb{G}_m$; as we will see, appropriate choices of extensions give counterexamples to both statements above.

Example 2.3.11. Let E be an elliptic curve over a field k. Let K/k be a (not necessarily algebraic) field extension and let $\mathcal{L} \in \mathrm{Pic}^0(E)(K) \simeq E(K)$ be a degree 0 line bundle on E_K. Let X be the total space of the associated $\mathbb{G}_m$-torsor. The K-scheme X is clearly smooth and non-proper; moreover, by [Ser75, VII.3.16], X is in fact an algebraic group, more concretely a semiabelian surface.

Lemma 2.3.12. *X is Hodge-proper over K if and only if $\mathcal{L}^{\otimes n} \neq \mathcal{O}_X$ for all $n > 0$. If $\mathrm{char}\, K = 0$ this is also equivalent to the degeneration of the Hodge-to-de Rham spectral sequence for X.*

Proof. Note that $\Omega^1_X \simeq \mathcal{O}_X^{\oplus 2}$, since X is a group. Denote the natural projection $X \to E_K$ by π. We find

$$R\pi_*\mathcal{O}_X \simeq \pi_*\mathcal{O}_X \simeq \bigoplus_{n\in\mathbb{Z}} \mathcal{L}^n.$$

Next, since the degree of $\mathcal{L}$ is zero and $\mathcal{L} \neq \mathcal{O}_X$, we have $R\Gamma(E_K, \mathcal{L}) \simeq 0$. If $\mathcal{L}$ is non-torsion, the same holds for $\mathcal{L}^n$, for all $n \neq 0$. So

$$R\Gamma(X, \Omega^2_X) \simeq R\Gamma(X, \mathcal{O}_X) \simeq R\Gamma(E_K, \mathcal{O}_{E_K}) \simeq K \oplus K[-1], \quad R\Gamma(X, \Omega^1_X) \simeq R\Gamma(X, \mathcal{O}_X^{\oplus 2}) \simeq K^{\oplus 2} \oplus K[-1]^{\oplus 2}.$$

If, on the other hand, $\mathcal{L}$ is torsion, $\pi_*\mathcal{O}_X$ has infinitely many copies of $\mathcal{O}_{E_K}$ as direct summands and so $H^0(X, \mathcal{O}_X)$ is infinite-dimensional.

For the second assertion it is enough to consider the case $K = \mathbb{C}$, where we can compare the de Rham cohomology with the singular one. Since the degree of $\mathcal{L}$ is 0, $\mathcal{L}$ is topologically trivial, and X is homotopy equivalent to $(\mathbb{S}^1)^{\times 3}$; comparing the dimensions we see that the Hodge-to-de Rham spectral sequence degenerates at the first page. ☐

Remark 2.3.13. Note that if K is a subfield of $\overline{\mathbb{F}}_p$, then $\mathrm{Pic}^0(E)(K)$ is a torsion abelian group. Thus, in this case X is never Hodge-proper.

Now let E be an elliptic curve over $\mathbb{Q}$. Let's consider $\mathcal{L}_\mathbb{C} \in \mathrm{Pic}^0(E)(\mathbb{C})$; the corresponding semiabelian variety $X_\mathbb{C}$ is a variety over $\mathbb{C}$.

Proposition 2.3.14. *$X_\mathbb{C}$ is Hodge-properly spreadable if and only if $\mathcal{L}_\mathbb{C} \in \mathrm{Pic}^0(E)(\mathbb{C}) \setminus \mathrm{Pic}^0(E)(\overline{\mathbb{Q}})$.*

Proof. First, let $\mathcal{L}_\mathbb{C} \in E(\overline{\mathbb{Q}})$. Let $K \subset \overline{\mathbb{Q}}$ be the field of definition of $\mathcal{L}_\mathbb{C}$; then $\mathcal{L}_\mathbb{C}$ and $E_\mathbb{C}$ are defined over K and we will denote the corresponding line bundle and elliptic curve over K by $\mathcal{L}_K$ and E_K. We also denote by X_K the total space of $\mathcal{L}_K$. Let $\mathcal{O}_K \subset K$ be the ring of integers. Consider the filtered system $\{\mathcal{O}_K[1/n]\}_{n\in\mathbb{N}}$ of subrings of K; we have $K = \mathrm{colim}_n\, \mathcal{O}_K[1/n]$. By Corollary 2.1.3 X_K has a smooth spreading X_A over $A = \mathcal{O}_K[1/n]$ for n big enough and, taking a larger n, we can assume that X_A is the total space of a line bundle $\mathcal{L}_A$ over an elliptic curve E_A (with E_A and $\mathcal{L}_A$ being spreadings of E and $\mathcal{L}$). Note that all closed points of $\mathrm{Spec}\, A$ are of positive characteristic and have finite residue fields. Localizing A further we can assume that base change for Hodge and de Rham cohomology holds with respect to all closed points of A. Let $x\colon \mathrm{Spec}\, \mathbb{F}_q \hookrightarrow \mathrm{Spec}\, A$ be some closed point. By Remark 2.3.13 the reduction $\mathcal{L}_{\mathbb{F}_q}$ is torsion and thus $X_{\mathbb{F}_q}$ is not Hodge-proper. It follows that X_A is not Hodge-proper and thus, since any subring $R \subset K$ satisfying the conditions in Definition 1.4.1 is contained in $\mathcal{O}_K[1/n]$ for some n, X is not Hodge-properly spreadable. We claim that neither is $X_\mathbb{C}$; indeed, let X'_R be a Hodge-proper spreading over some localization $R \subset \mathbb{C}$ of some finitely generated $\mathbb{Z}$-algebra as in Definition 1.4.1.

Since $\mathbb{C}$ can be represented as a colimit of flat finitely generated R-algebras (and those fit in the framework of Definition 1.4.1), by the "spreading out" for schemes we can assume that $\{(X'_R)\}_{R\cdot K} \simeq \{(X_K)\}_{R\cdot K}$. Considering two systems: $\{(X_A)_{R\cdot A}\}_A$ and $\{(X'_R)_{R\cdot A}\}_A$ with X_A as above and A running over $\mathcal{O}_K[\frac{1}{n}]$ for various integers n we get that $(X_A)_{R\cdot A} \simeq (X'_R)_{R\cdot A}$ some A. Note that since the image of $\operatorname{Spec} R$ in $\operatorname{Spec}\mathbb{Z}$ is open and $R\cdot A$ is torsion free (over $\mathbb{Z}$, and thus also over A), $R\cdot A\subset \mathbb{C}$ becomes faithfully flat over $A=\mathcal{O}_K[1/n]$ if we take n big enough. Since X_A is not Hodge-proper, neither is $(X_A)_{R\cdot A}$. Replacing A we can also assume $R\cdot A$ is flat over R; indeed by "spreading out" of flat morphisms of schemes it is enough to show that $R\cdot K$ is flat over $R_{\mathbb{Q}}$. But $R\cdot K$ is a direct summand of $R_{\mathbb{Q}}\otimes_{\mathbb{Q}} K$, so this is clear. Thus by flat base change (applied to $R\cdot A$ which is now flat over R) we get that X'_R can't be Hodge-proper, which is a contradiction. Thus $X_{\mathbb{C}}$ is not Hodge-properly spreadable.

It remains to deal with the transcendent case $\mathcal{L}_{\mathbb{C}}\notin \operatorname{Pic}^0(E)(\overline{\mathbb{Q}})$. Let's consider the universal line bundle $\mathcal{P}$ on $E\times \operatorname{Pic}^0(E)$. Since $\mathcal{L}$ is not a $\overline{\mathbb{Q}}$-point, the corresponding map $\operatorname{Spec}\mathbb{C}\to \operatorname{Pic}^0(E)$ factors through the generic point $\operatorname{Spec}\mathbb{Q}(E)\subset E\simeq \operatorname{Pic}^0(E)$ and thus both $\mathcal{L}_{\mathbb{C}}$ and $X_{\mathbb{C}}$ are defined over $\mathbb{Q}(E)$. We denote the corresponding bundle and $\mathbb{Q}(E)$-scheme by $\mathcal{L}$ and X. Let $y^2=x^3+ax+b$ be an equation of (the affine part of) E. Let $B=\mathbb{Z}[1/n][x,y]/(y^2-x^3-ax-b)\subset \mathbb{Q}(E)$ where n is big enough to be divisible by the denominators of both a and b, and so that B is smooth over $\operatorname{Spec}\mathbb{Z}$. Note that E has a smooth proper model $E_{\mathbb{Z}[1/n]}$ over $\mathbb{Z}[1/n]$ given by the projective closure of $\operatorname{Spec} B$. Now let $R=B[S^{-1}]\subset \mathbb{Q}(E)$ be the localization of B with respect to the set S of elements $s\in B$ that are non-zero modulo all primes $p\in\mathbb{Z}$ provided $(p,n)=1$.[13] The reduction $R/(p)$ is equal to the field of fractions of $B/(p)$ which is nothing but $\mathbb{F}_p(E_{\mathbb{F}_p})$. Identifying $\operatorname{Pic}^0(E)$ with E, we obtain spreadings $\mathcal{L}_R$ (over E_R) and X_R of $\mathcal{L}$ and X (considered as a line bundle on $E_{\mathbb{Q}(E)}$ and a scheme over $\mathbb{Q}(E)$ correspondingly). Localizing R we can assume that X_R is a group scheme and thus it is enough to show that the cohomology of $R\Gamma(X_R,\mathcal{O}_{X_R})$ is finitely generated over R. We have

$$R\pi_*\mathcal{O}_{X_R}\simeq \pi_*\mathcal{O}_{X_R}\simeq \bigoplus_{n\in\mathbb{Z}}\mathcal{L}_R^n.$$

Each $\mathcal{L}_R^n$ is a coherent sheaf on E_R and $R\Gamma(E_R,\mathcal{L}_R^n)\in \operatorname{Mod}_R^{\mathrm{coh}}$. We claim that it is zero if $n\neq 0$; note that R is regular, thus $R\Gamma(E_R,\mathcal{L}_R^n)$ is perfect and so it is enough to check this modulo all primes $p\in\mathbb{Z}$. But by the construction the reduction $\mathcal{L}_{R/p}$ is the restriction of the universal line bundle on $E_{\mathbb{F}_p}\times_{\mathbb{F}_p}\operatorname{Pic}^0(E_{\mathbb{F}_p})$ to $E_{\mathbb{F}_p}\times_{\mathbb{F}_p}\operatorname{Spec}\mathbb{F}_p(E_{\mathbb{F}_p})$; in particular it is non-torsion. Thus $R\Gamma(E_R,\mathcal{L}_R^n)\simeq 0$, and so

$$R\Gamma(X_R,\mathcal{O}_{X_R})\simeq R\Gamma(E_R,\pi_*\mathcal{O}_{X_R})\simeq R\Gamma(E_R,\mathcal{O}_{E_R})$$

is coherent. □

Remark 2.3.15. Due to Proposition 2.3.14 and Lemma 2.3.12, a semiabelian surface $X_{\mathbb{C}}$, for the choice of a non-torsion line bundle $\mathcal{L}_{\mathbb{C}}\in \operatorname{Pic}^0(E)(\overline{\mathbb{Q}})$, gives an example of a scheme which is not Hodge-properly spreadable, but the Hodge-to-de Rham spectral sequence degenerates. On the other hand, taking $\mathcal{L}_{\mathbb{C}}\notin \operatorname{Pic}^0(E)(\overline{\mathbb{Q}})$ gives an example of a Hodge-properly spreadable scheme which is not proper.

Remark 2.3.16. Note that even in the case $\mathcal{L}_{\mathbb{C}}\notin \operatorname{Pic}^0(E)(\overline{\mathbb{Q}})$, the scheme $X_{\mathbb{C}}$ does not have a Hodge-proper spreading over a finitely generated $\mathbb{Z}$-algebra R (because then $\operatorname{Spec} R$ necessarily has points with finite residue fields over which $\mathcal{L}$ becomes torsion). Thus it really makes a difference to allow arbitrary localizations of the latter in Definition 1.4.1.

Remark 2.3.17. The ring R used in the proof of Proposition 2.3.14 is a slight generalization of a ring, that could be called "quantum integers/rationals". Recall that the "quantum integer n" polynomial $[n]_q\in\mathbb{Z}[q]$ is defined as $[n]_q\coloneqq 1+q+\ldots+q^{n-1}$; we then can consider $\mathbb{Q}_{\mathbf{q}}\coloneqq \mathbb{Z}[q][[n]_q^{-1}]_{n\in\mathbb{N}}$. The ring $\mathbb{Q}_{\mathbf{q}}$ is a principal ideal domain of Krull dimension 1 whose reduction modulo a prime p is given by

$$\mathbb{Q}_{\mathbf{q}}\otimes_{\mathbb{Z}}\mathbb{F}_p\simeq \mathbb{F}_p[q][[n]_q^{-1}]_{n\in\mathbb{N}}\simeq \mathbb{F}_p(q),$$

the field of rational functions over $\mathbb{F}_p$.

Remark 2.3.18. Topologically, $X_{\mathbb{C}}(\mathbb{C})\simeq \mathbb{C}^\times\times(\mathbb{S}^1)^2$, since $\mathcal{L}_{\mathbb{C}}$ has degree 0 and is topologically trivial. The space $H^1_{\mathrm{sing}}(X_{\mathbb{C}}(\mathbb{C}),\mathbb{C})\simeq\mathbb{C}^3$ is odd dimensional and thus the corresponding mixed Hodge structure can't be pure of weight 1. In particular this shows that the mixed Hodge structure[14] on the n-th singular cohomology of a Hodge-properly spreadable stack is not necessarily pure of weight n.

[13] More explicitly, one can see that it is enough to invert functions y^n-1 for $n\geq 1$.
[14] Appropriately defined, say via Section 8.3 of [Del74].

2.3.4 Higher examples

Here we also record some examples of Hodge-properly spreadable stacks that are not classical.

Example 2.3.19. Let G be a classical abelian algebraic group over F such that BG is cohomologically properly spreadable (e.g. by Remark 2.3.9 and the discussion above it, one can take G to be an algebraic torus, an abelian or even a semiabelian variety). Then the higher stack $K(G,n) := B^nG$ (given by the sheafification of $S \mapsto K(G(S),n)$) is also cohomologically properly spreadable. Indeed, after enlarging R any cohomologically proper spreading $K(G,1)_R$ becomes equal to $K(G_R,1)$ for some spreading G_R of G which has a group structure. Taking the $K(G_R,n)$ for such G_R gives a spreading of $K(G,n)$ which, we claim, is cohomologically proper. We will show this by induction. Consider the Čech simplicial object $\mathcal{U}_\bullet$ corresponding to the cover $\operatorname{Spec} R \to K(G_R,n)$. All of its terms $\mathcal{U}_k$ are given by products $K(G_R,n-1)^k$, the projection morphism $K(G_R,n-1)^k \to K(G_R,n-1)^{k-1}$ is cohomologically proper[15] and thus, projecting k times, we see that all $\mathcal{U}_k$ are cohomologically proper. It then follows from Proposition 2.2.14 that $K(G_R,n)$ is cohomologically proper as well.

3 Hodge-proper spreadability of quotient stacks

In this section we study in more detail the case of quotient stacks, providing several families of non-trivial examples of cohomologically proper spreadable stacks. Here the proofs of spreadability are much more involved; the following two important representation-theoretic results will be used:

Theorem 3.0.1 (Theorem 3 and Proposition 57 of [FvdK10][16])**.** *Let $G_\mathbb{Z}$ be a split reductive group over $\mathbb{Z}$ and R be a finitely generated algebra over $\mathbb{Z}$. Let A be a finitely generated R-algebra endowed with a (rational) action of G_R and let M be a finitely generated G_R-equivariant A-module. Then the algebra A^{G_R} of G_R-invariants is finitely generated over R and $H^n(G_R,M)$ is a finitely generated A^{G_R}-module for any $n \geq 0$.*

Theorem 3.0.2 (Kempf's theorem, see e.g. [Jan07, Proposition II.4.5])**.** *Let $G_\mathbb{Z}$ be a split connected reductive group over $\mathbb{Z}$ and let $B_\mathbb{Z} \subseteq G_\mathbb{Z}$ be a Borel subgroup. Let $(G/B)_\mathbb{Z}$ be the corresponding flag variety. Then $R\Gamma((G/B)_\mathbb{Z}, \mathcal{O}_{(G/B)_\mathbb{Z}}) \simeq \mathbb{Z}$.*

As we will see, these two theorems, together with the semiorthogonal decompositions of derived categories constructed by Halpern-Leistner ([HL15], [HL20]) allow to prove in a lot of cases that the stack is cohomologically properly spreadable. We stick to the notations of Section 2.3, in particular F will denote an algebraically closed field of characteristic 0 and we will choose the base R of the spreading to be a finitely generated $\mathbb{Z}$-subalgebra of F.

Here is a plan of the section. In Section 3.1 we show that a quotient by reductive group is cohomologically properly spreadable (Theorem 3.1.4), provided its coarse moduli space is proper and the action is locally linear. The key result is Proposition 3.1.2, where we use Theorem 3.0.1 to show that under the latter assumption the natural morphism $q\colon [Y/G] \to Y/\!/G$ is cohomologically properly spreadable. Particular examples then include a proper-over-affine scheme with an action of a reductive group (Example 3.1.6) and quotients coming from GIT (Example 3.1.7). In Section 3.2 we look at some other set of examples (Theorem 3.2.12), given by quotients that come from BB-complete $\mathbb{G}_m$-actions (Definition 3.2.3). Theorem 3.2.12 also allows some quotients by non-reductive groups, and Theorem 3.0.2 is used to pass from the quotient by a Borel subgroup to the quotient by the whole group (Lemma 3.2.15). In Section 3.3 we also give a recipe of reestablishing the degeneration results of [Tel00] in the KN-complete case using Theorem 1.4.3; here, however, we need to assume some results which are going to appear in the upcoming work of Halpern-Leistner [HL20].

3.1 Global quotients by reductive groups

In [Tel00] Teleman proved the Hodge-to-de Rham degeneration for the quotient of a smooth scheme X by a Kempf-Ness (KN) complete action of a reductive group. In this section we establish spreadability for certain class of global quotients, which includes the semistable (single KN-stratum) case $X = X^{\mathrm{ss}}(\mathcal{L})$ (3.1.7) and another standard KN-complete example given by equivariant "projective-over-affine" variety (3.1.6). Moreover, the "projectivity" condition in the latter is replaced by the "proper" one almost for free.

Let Y be a quasi-separated finite type scheme over F and let G be a reductive group acting on Y.

[15] Indeed, its pull-back to $\operatorname{Spec} R \to K(G_R,n-1)^{k-1}$ is $K(G_R,n-1) \to \operatorname{Spec} R$ which is cohomologically proper by the induction assumption, thus by Proposition 2.2.4 (3) so is the original map.

[16] Proposition 57 in *loc.cit.* is stated for $R = \mathbb{Z}$. However we can consider A as a $\mathbb{Z}$-algebra with an action of $\mathbb{G}_\mathbb{Z}$; indeed, since the action on R is trivial and $R[G_R] \simeq \mathbb{Z}[\mathbb{G}_\mathbb{Z}] \otimes_\mathbb{Z} R$ we have $A \otimes_R R[G_R] \simeq A \otimes_\mathbb{Z} \mathbb{Z}[G_\mathbb{Z}]$ and thus get the comultiplication $A \to A \otimes_R R[G_R] \simeq A \otimes_\mathbb{Z} \mathbb{Z}[G_\mathbb{Z}]$. Same thing works for any G_R-representation and, moreover, $R\Gamma(G_R,M) \simeq R\Gamma(G_\mathbb{Z},M)$ (e.g. because they are computed by the same standard complex). Thus Proposition 57 in *loc.cit.* also applies to any R that is finitely generated over $\mathbb{Z}$.

Definition 3.1.1. The action of G on Y is called *locally linear* if there exists a G-invariant affine cover of Y.

In this case there exists the categorical quotient $Y//G$; in other words, the quotient stack $[Y/G]$ has a coarse moduli $q\colon [Y/G] \to Y//G$ and $Y//G$ is representable by a scheme. More explicitly, if $\{U_i\}_{i\in I}$, with $U_i \coloneqq \operatorname{Spec} A_i$, is the G-invariant affine cover of X, the categorical quotient $X//G$ is glued out of $\{U_i//G\}_{i\in I}$, with $U_i//G \coloneqq \operatorname{Spec} A_i^G$, with the natural gluing maps induced by the gluing maps for $\{U_i\}_{i\in I}$. Note that if G is a torus and Y is normal, the action is automatically locally linear by the result of Sumihiro ([Sum74, Corollary 2]).

Proposition 3.1.2. *Let Y be a quasi-separated finite type scheme over F with a locally linear action of a reductive group G. Then the natural morphism $q\colon [Y/G] \to Y//G$ is cohomologically properly spreadable.*

Proof. The group G is split and has a Chevalley model $G_{\mathbb{Z}}$ over $\mathbb{Z}$; this defines a split reductive spreading of G over any $R \subset F$, namely just put $G_R \coloneqq G_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. We can also spread Y to a quasi-separated scheme Y_R over some R and assume that Y_R has a G_R-action. It is enough to show that after a suitable enlargement R the quotient stack $[Y_R/G_R]$ becomes cohomologically proper. Note that by picking a G-invariant affine cover $\{U_i\}_{i\in I}$ as above and a spreading $A_{i,R}$ of each A_i, localizing R, we can assume that the affine schemes $U_{i,R} \coloneqq \operatorname{Spec} A_{i,R}$ have a G_R-action and give a G_R-equivariant affine cover of Y_R.

Let $Y_R//G_R$ be the categorical quotient, namely the scheme obtained by gluing the spectra $\operatorname{Spec} A_{i,R}^{G_R}$ of the invariants in the same way that was described above. Note that by Theorem 3.0.1 the scheme $Y_R//G_R$ is of finite type and so it is a valid spreading of $Y//G$ (since $A_{i,R}^{G_R} \otimes_R F \simeq A_i$ for all $i \in I$). We have a natural map $q_R\colon [Y_R/G_R] \to Y_R//G_R$ given locally (on $Y_R//G_R$) by $q_{i,R}\colon [\operatorname{Spec} A_{i,R}/G_R] \to \operatorname{Spec} A_{i,R}^{G_R}$. The map q_R is a spreading of q, thus it is enough to show that q_R is cohomologically proper. This can be checked locally, so it is enough to show that for any finitely generated R-algebra A the map $q_R\colon [\operatorname{Spec} A/G_R] \to \operatorname{Spec} A^{G_R}$ is cohomologically proper.

Let $\mathcal{F} \in \operatorname{Coh}([\operatorname{Spec} A/G_R])^{\heartsuit}$ and let M be the corresponding G_R-equivariant finitely generated A-module. Since $\operatorname{Spec} A$ is affine, the module $q_{R*}\mathcal{F}$ has a simple description: it is just given by $R\Gamma(G_R, M)$ considered as a complex of modules over A^{G_R}. The complex $R\Gamma(G_R, M)$ lies in $\operatorname{Mod}_{A^{G_R}}^{\geq 0}$ and its cohomology are finitely generated by Theorem 3.0.1. Thus $R\Gamma(G_R, M) \in \operatorname{Coh}^+(A^{G_R})$ and q_R is cohomologically proper. ☐

Remark 3.1.3. In the case of BG some stronger results in a similar direction were established in [HLP15, Proposition 4.3.4]. Namely under some mild restrictions on the reductive group scheme $\mathcal{G}$ and the base scheme S the structure morphism $B\mathcal{G} \to S$ is formally proper (in the sense of [HLP15]).

Note that we did not assume that Y was smooth. This was on purpose: the actual cohomologically properly spreadable examples are given by the following theorem:

Theorem 3.1.4. *Let X be a smooth scheme and Y be a finite-type scheme over F, both endowed with an action of a reductive group G. Assume that*

1. *There is a proper G-equivariant map $\pi\colon X \to Y$.*
2. *The G-action on Y is locally linear.*
3. *The categorical quotient $Y//G$ is proper.*

Then the quotient stack $[X/G]$ is cohomologically properly spreadable.

Proof. Consider the map $q\colon [Y/G] \to Y//G$; by Proposition 3.1.2 it is cohomologically properly spreadable. The map $\pi\colon [X/G] \to [Y/G]$ is proper, thus $[X/G]$ is cohomologically properly spreadable by Corollary 2.3.4. ☐

Remark 3.1.5. More generally one can replace $[X/G]$ by any $\mathcal{X}$ with a cohomologically properly spreadable morphism $\pi\colon \mathcal{X} \to [Y/G]$.

We discuss some applications of Theorem 3.1.4:

Example 3.1.6. Let X be a smooth proper-over-affine scheme X with $\dim H^0(X, \mathcal{O}_X)^G < \infty$. By definition, this means that there is a proper G-equivariant map $\pi\colon X \to \operatorname{Spec} A$. By replacing $\operatorname{Spec} A$ with the image of π we can assume that π is surjective. Then A^G embeds in $H^0(X, \mathcal{O}_X)^G$ and thus is finite-dimensional; equivalently, $(\operatorname{Spec} A)//G$ is finite, and in particular proper. Applying 3.1.4 to $Y = \operatorname{Spec} A$ we get that $[X/G]$ is cohomologically properly spreadable. Also note that we were able to relax the "projective" assumption on the map π to the "proper" one.

Example 3.1.7. Let X have an ample G-equivariant line bundle $\mathcal{L}$, and let's assume that $X = X^{\mathrm{ss}} := X^{\mathrm{ss}}(\mathcal{L})$. Basically by definition, the action on $X^{\mathrm{ss}}(\mathcal{L})$ is locally linear (see e.g. the proof of [MFK94, Theorem 1.10]). Assume further that $\dim_F H^0(X, \mathcal{O}_X)^G < \infty$. Then the scheme

$$X/\!/G \simeq \mathrm{Proj}\Big(\bigoplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes n})^G\Big)$$

is projective over $\mathrm{Spec}\, H^0(X, \mathcal{O}_X)^G$ and hence also projective over F. Thus $[X/G]$ is cohomologically properly spreadable by Theorem 3.1.4 applied to $Y = X$.

3.2 Global quotients coming from BB-complete $\mathbb{G}_m$-actions

In this section we prove another result (Theorem 3.2.12) about the cohomologically proper spreadability of quotient stacks, which also allows quotients by groups that are not necessarily reductive. Another benefit of Theorem 3.2.12 (compared, say, to Theorem 3.1.4) is that the condition on X (and the G-action) is internal: no additional structure, such as a map to another scheme Y, is involved.

3.2.1 Varieties with a $\mathbb{G}_m$-action and Bialynicki-Birula stratification

Let X be a smooth scheme over an algebraically closed field F of characteristic 0 with an action $a\colon \mathbb{G}_m \curvearrowright X$. By [Sum74] such an action is always locally linear: X has a $\mathbb{G}_m$-invariant affine cover $\{U_i = \mathrm{Spec}\, A_i\}_{i\in I}$. A $\mathbb{G}_m$-action on a given affine scheme $\mathrm{Spec}\, A$ induces a $\mathbb{Z}$-grading A^* on A; we denote by $I^+ \subset A^*$ the ideal generated by $A^{>0}$ and by $I^{\pm} \subset A^*$ the ideal generated by $A^{>0}$ and $A^{<0}$.

Here are some examples:

Example 3.2.1.

- $\mathrm{pt} := \mathrm{Spec}\, F$ with the trivial $\mathbb{G}_m$-action; here $A = A^0 \simeq F$ and $I^+ = I^{\pm} = 0$.
- $\mathbb{A}^1 := \mathrm{Spec}\, F[x]$ endowed with the standard action by dilation, $s \mapsto ts$ for $s \in \mathbb{A}^1$; in this case $\deg x = -1$, so $I^+ = 0$ and $I^{\pm} = (x) \subset F[x]$.

There are natural $\mathbb{G}_m$-equivariant maps $\mathrm{pt} \overset{i_0}{\underset{p}{\rightleftarrows}} \mathbb{A}^1$ given by the projection and the embedding of $0 \in \mathbb{A}^1(F)$.

We also have a (non-equivariant) map $i_1\colon \mathrm{pt} \to \mathbb{A}^1$ given by the embedding of $1 \in \mathbb{A}^1(F)$.

To a smooth scheme X endowed with a $\mathbb{G}_m$-action one can associate the following objects:

- *The fixed points* $X^0 := \mathrm{Maps}(\mathrm{pt}, X)^{\mathbb{G}_m}$; its functor of points is given by $X^0(S) = \mathrm{Maps}(S, X)^{\mathbb{G}_m}$, meaning the $\mathbb{G}_m$-equivariant maps from S to X, where the action on S is trivial. There is a natural map $\iota\colon X^0 \to X$ sending a map $f \in X^0$ to its evaluation $f(\mathrm{pt})$. The map ι is a closed embedding ([Dri13, Proposition 1.2.2]); the affine cover $\{U_i\}_{i\in I}$ defines an affine cover $\{U_i^0\}_{i\in I}$ of X^0 with $U_i^0 := \mathrm{Spec}(A_i/I_i^{\pm})$ (glued along U_{ij}^0). There is a natural $\mathbb{G}_m$-action on X^0, which is trivial.
- *The attractor* $X^+ := \mathrm{Map}(\mathbb{A}^1, X)^{\mathbb{G}_m}$; here the functor of points is given by $X^+(S) = \mathrm{Maps}(S \times \mathbb{A}^1, X)^{\mathbb{G}_m}$, where the $\mathbb{G}_m$-action on $S \times \mathbb{A}^1$ is diagonal. By [Dri13, Corollary 1.4.3], this functor is indeed represented by a scheme. There are two natural $\mathbb{G}_m$-actions on $\mathrm{Map}(\mathbb{A}^1, X)$, one coming from the $\mathbb{G}_m$-action on $\mathbb{A}^1$, the other coming from the action on X; their restrictions to $\mathrm{Map}(\mathbb{A}^1, X)^{\mathbb{G}_m}$ coincide and thus define the same $\mathbb{G}_m$-action on X^+.

There are natural $\mathbb{G}_m$-equivariant maps $X^0 \overset{\sigma}{\underset{\pi}{\rightleftarrows}} X^+ \xrightarrow{j} X$ induced by i_0, i_1 and p. Namely,

- $\sigma(f) \in X^+$ is given by pre-composing $f\colon \mathrm{pt} \to X$ with $p\colon \mathbb{A}^1 \to \mathrm{pt}$; the map $\sigma\colon X^0 \to X^+$ is a closed embedding.
- $\pi(f) \in X^0$ is given by the evaluation of $f\colon \mathbb{A}^1 \to X$ at $0 \in \mathbb{A}^1$; by [BB73][17] $\pi\colon X^+ \to X^0$ is a Zariski locally trivial fibration with the fiber given by an affine space.
- $j(f) \in X \simeq \mathrm{Map}(\mathrm{pt}, X)$ is given by the evaluation of $f\colon \mathbb{A}^1 \to X$ at $1 \in \mathbb{A}^1$; the map $j\colon X^+ \to X$ is a locally closed embedding when restricted to each component of X^+.

Similarly to X^0, the affine cover $\{U_i\}_{i\in I}$ defines an affine cover $\{U_i^+\}_{i\in I}$ of X^+ with $U_i^+ := \mathrm{Spec}(A_i/I_i^+)$.

[17] More precisely, by [Dri13, Proposition 1.4.10] X^+ is smooth, and (say, by [Dri13, Section 1.4.3]) the action of $\mathbb{G}_m$ at any point $x \in X^0 \hookrightarrow X^+$ is definite (in the terminology of [BB73, Section 1]). Then one can apply [BB73, Theorem 2.5] to X^+.

Also, by [Dri13, Proposition 1.4.10] both X^0 and X^+ are smooth.

Remark 3.2.2. The construction of the maps σ, π, j is local and is described as follows in terms of a $\mathbb{G}_m$-invariant cover $\{U_i\}_{i\in I}$: $\sigma_i\colon U_i^0 \hookrightarrow U_i^+$ is induced by the projection $A_i/I_i^+ \to A_i/I_i^\pm$; $A_i/I_i^\pm$ is identified with the 0-th graded component of the negatively graded algebra A_i/I_i^+ and this way the contraction $\pi_i\colon U_i^+ \to U_i^0$ corresponds to the embedding $A_i/I_i^\pm \simeq (A_i/I_i^+)^0 \to A_i/I_i^+$. Finally $j_i\colon U_i^+ \to U_i$ is given by the projection $A_i \to A_i/I_i^+$. The maps for X, X^+ and X^0 are obtained by gluing along the analogous maps for $U_{i,j} := U_i \cap U_j$.

Let $\pi_0(X^0) = \pi_0(X^+)$ be the set of connected components of X^0 (equivalently, X^+). For a given $c \in \pi_0(X^0)$ we denote by $Z_c \subset X^0$ the corresponding connected component of X^0, and by $S_c := \pi^{-1}(S_c) \subset X^+$ the corresponding connected component of X^+; we call $\{S_c\}_{c\in\pi_0(X^0)}$ *the Bialynicki-Birula (BB) strata.*

Definition 3.2.3. The $\mathbb{G}_m$-action $a\colon \mathbb{G}_m \curvearrowright X$ is called *BB-complete* if the map $j\colon X^+ \to X$ is a surjection on the underlying topological spaces $|j|\colon |X^+| \twoheadrightarrow |X|$.

Equivalently, $j\colon X^+ \to X$ gives a *full* stratification of X with individual strata being locally closed. In this case, the limit of $a(t)\circ x$ for $t \to 0$ exists for any point $x \in X$; in particular, the Bialynicki-Birula stratification $\{S_c\} \in \pi_0(X^0)$ gives a full stratification of X. We will fix some ordering on $\pi_0(X^0)$ with the only condition that $c' > c$ if $\dim S_{c'} < \dim S_c$; in this case a stratum S_c is necessarily closed in $X_{\leq c} := X \setminus \cup_{c'>c} S_{c'}$.

Remark 3.2.4. Since both X^+ and X are of finite type and F is algebraically closed, $|j|\colon |X^+| \to |X|$ is surjective if and only if the corresponding map $j(F)\colon X^+(F) \to X(F)$ between the F-points is.

3.2.2 BB-complete quotients by $\mathbb{G}_m$

Spreading BB-stratification. Let X be a smooth scheme over F with a $\mathbb{G}_m$-action and let $\{U_i\}_{i\in I}$, $U_i \simeq \operatorname{Spec} A_i$ be a $\mathbb{G}_m$-invariant affine cover as above. The smooth schemes X^+ and X^0 are glued out of $\{U_i^+\}$ and $\{U_i^0\}$ (along U_{ij}^+ and U_{ij}^0) correspondingly.

By Corollary 2.1.3 we can spread X to a smooth R-scheme X_R endowed with an action of $\mathbb{G}_{m,R} := \operatorname{Spec} R[t,t^{-1}]$. We can also spread the cover $\{U_i\}_{i\in I}$ to a $\mathbb{G}_m$-invariant affine cover $\{U_{i,R}\}_{i\in I}$, $U_{i,R} \simeq \operatorname{Spec} A_{i,R}$ over some regular finitely generated $\mathbb{Z}$-algebra $R \subset F$. Each algebra $A_{i,R}$ is $\mathbb{Z}$-graded and we can consider closed subschemes $U_{i,R}^+ := \operatorname{Spec}(A_{i,R}/I_{i,R}^+)$ and $U_{i,R}^0 := \operatorname{Spec}(A_{i,R}/I_{i,R}^\pm)$, as well as schemes X_R^+ and X_R^0, obtained as their gluings along $U_{ij,R}^+$ and $U_{ij,R}^0$ (defined analogously). We have $(X_R^+)\times_R F \simeq X^+$ and $(X_R^0)\times_R F \simeq X^0$.

Recall Remark 3.2.2 to see how the maps σ, π, j between X, X^+, X^0 are defined in terms of the covering $\{U_i\}_{i\in I}$. Performing the same construction with $U_{i,R}$, $U_{i,R}^+$ and $U_{i,R}^0$ we obtain maps $X_R^0 \underset{\pi_R}{\overset{\sigma_R}{\rightleftarrows}} X_R^+ \xrightarrow{j_R} X_R$ which spread σ, π, j. After enlarging R further, by Corollary 2.1.3 we can assume that σ_R (resp. j_R) is a closed (resp. locally closed when restricted to each connected component) embedding. Picking an open cover $\{V_i\}_{i\in I}$ on which the affine fibration $\pi\colon X^+ \to X^0$ is trivial, $\pi^{-1}(V_i) \simeq V_i \times \mathbb{A}^d$, we can spread it out; enlarging R we can assume that $\{V_{i,R}\}_{i\in I}$ cover X_R^0, $\pi_R^{-1}(V_{i,R})$ cover X_R^+ and $\pi_R^{-1}(V_{i,R}) \simeq V_{i,R}\times_R \mathbb{A}_R^d$. Furthermore, we can assume that X_R^0, and consequently X_R^+, are smooth over R.

After enlarging R further, the set $\pi_0(X_R^0)$ can be identified with $\pi_0(X^0)$; the connected component $Z_{c,R}$ for $c \in \pi_0(X_R^0) \simeq \pi_0(X^0)$ then is a spreading of $Z_c \subset X^0$. Similarly, $S_{c,R} := \pi_R^{-1}(Z_{c,R})$ is a connected component of X_R^+ and a spreading of S_c.

If the $\mathbb{G}_m$-action on X is BB-complete, we can assume the same about the $\mathbb{G}_{m,R}$-action on X_R; indeed, enlarging R we can assume that the map $X_R^+ \to X_R$ is a surjection. In particular, we can assume that the spreading $\{S_{c,R}\}$ of the stratification $\{S_c\}$ of X gives a full stratification of X_R by locally closed smooth subschemes.

Semiorthogonal decomposition of $\operatorname{Coh}(X_R)$. We now discuss certain semiorthogonal decompositions of the category $\operatorname{Coh}(X_R)$ given by the theory of so-called magic windows developed by Halpern-Leistner in [HL15]. Let X be a smooth scheme over F with a $\mathbb{G}_m$-action and let X_R be a spreading of X as constructed above. Let $S_R := S_{c,R} \subset X_R$ be a closed stratum among $\{S_{c',R}\}$. Let $Z_R := Z_{c,R}$ for the same c; the scheme Z_R is called the *centrum* of S_R. The subschemes S_R and Z_R enjoy the following nice properties:

Proposition 3.2.5.
- *Both S_R and Z_R are smooth over R;*
- *There is a $\mathbb{G}_m$-equivariant map $\pi_R\colon S_R \simeq S_R^+ \to S_R^0 \simeq Z_R$, which is a Zariski locally trivial fibration with an affine space $\mathbb{A}_R^d$ (for some d) as a fiber;*
- *The conormal bundle $N_{S_R}^\vee X_R$ has strictly positive weights when restricted to the fixed locus $Z_R \subset S_R$.*

Proof. Since Z_R and S_R are connected components of X_R^0 and X_R^+ correspondingly, it is enough to show that the above properties hold for X_R^0 and X_R^+ in place of Z_R and S_R. The first two properties are included in our construction of the spreading. The last property can be checked locally, thus we can assume $X_R \simeq \operatorname{Spec} A_R$, $S_R \simeq \operatorname{Spec} A_R/I_R^+$ and $Z_R \simeq \operatorname{Spec} A_R/I_R^\pm$. The normal bundle $N^\vee_{X_R^+}X_R$ is then given by $I_R^+/(I_R^+)^2$ as a A_R/I_R^+-module and its restriction to X_R^0 is given by $I_R^+/(I_R^+\cdot I_R^\pm)$, considered as a $A_R/I_R^\pm$-module). Since by definition I_R^+ is generated by elements of strictly positive weight, the weights of $I_R^+/(I_R^+\cdot I_R^\pm)$ are also strictly positive. □

Remark 3.2.6. In other words, using the terminology of [HL15], $S_R \subset X_R$ is a smooth KN-stratum satisfying properties (A) and $(L+)$ (in the case $G = \mathbb{G}_{m,R}$).

Let $\mathcal{X}_R := [X_R/\mathbb{G}_{m,R}]$, $\mathcal{S}_R := [S_R/\mathbb{G}_{m,R}]$ and $\mathcal{Z}_R := [Z_R/\mathbb{G}_{m,R}]$. Also let $U_R := X_R \setminus S_R$ be the complement and let $\mathcal{U}_R := [U_R/\mathbb{G}_{m,R}]$. Let $\operatorname{Coh}(\mathcal{X}_R)$ be the (bounded derived) category of coherent sheaves on $\mathcal{X}_R$ and let $\operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R) \subset \operatorname{Coh}(\mathcal{X}_R)$ be the full subcategory of sheaves whose pull-back to the flat cover $X_R \twoheadrightarrow \mathcal{X}_R$ is set-theoretically supported on S_R. The action of $\mathbb{G}_{m,R}$ on Z_R is trivial and so $\mathcal{Z}_R \simeq Z_R \times_R B\mathbb{G}_{m,R}$. Thus the heart $\operatorname{Coh}(\mathcal{Z}_R)^\heartsuit$ of the category of coherent sheaves on $\mathcal{Z}_R$ is identified with the category of $\mathbb{Z}$-graded objects in $\operatorname{Coh}(Z_R)^\heartsuit$. For a given w we denote by $\operatorname{Coh}(\mathcal{Z}_R)_{<w} \subset \operatorname{Coh}(\mathcal{Z}_R)$ the full subcategory spanned by objects $\mathcal{F} \in \operatorname{Coh}(\mathcal{Z}_R)$ whose cohomology sheaves $\mathcal{H}^i(\mathcal{F}) \in \operatorname{Coh}(\mathcal{Z}_R)^\heartsuit$ have grading less than w. Similarly, by $\operatorname{Coh}(\mathcal{Z}_R)_{\geq w} \subset \operatorname{Coh}(\mathcal{Z}_R)$ we denote the full subcategory spanned by objects whose cohomology sheaves have grading greater or equal than w.

Let $i_R\colon \mathcal{U}_R \to \mathcal{X}_R$ and $j_R\colon \mathcal{S}_R \to \mathcal{X}_R$ be the natural embeddings. Then, given $w \in \mathbb{Z}$, by [HL15, Theorem 2.10], we have a semiorthogonal decomposition

$$\operatorname{Coh}(\mathcal{X}_R) = \langle\, \operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{<w}, \mathrm{G}_w, \operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{\geq w}\,\rangle, \tag{7}$$

where

- $\operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{\geq w} := \{\mathcal{F} \in \operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R) \mid \sigma_R^*\mathcal{F} \in \operatorname{Coh}(\mathcal{Z}_R)_{\geq w}\} \subset \operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$,
- $\operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{<w} := \{\mathcal{F} \in \operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R) \mid \sigma_R^* j_R^!\mathcal{F} \in \operatorname{Coh}(\mathcal{Z}_R)_{<w}\} \subset \operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$

and $\mathrm{G}_w \subset \operatorname{Coh}(\mathcal{X}_R)$ is a certain (full) subcategory such that the functor $i_R^* : \operatorname{Coh}(\mathcal{X}_R) \to \operatorname{Coh}(\mathcal{U}_R)$ restricted to G_w

$$i_R^*|_{\mathrm{G}_w}\colon \mathrm{G}_w \xrightarrow{\ \sim\ } \operatorname{Coh}(\mathcal{U}_R)$$

is an equivalence.

Remark 3.2.7. Note that the [HL15] assumes the base to be a field of characteristic 0. Even though in the case of $G = \mathbb{G}_m$ this assumption is not really necessary, the semiorthogonal decomposition above is also covered by the proof of Proposition 3.3.2. Indeed it is enough to show that $\mathcal{S}_R$ gives a Θ-stratum in $\mathcal{X}_R$. This follows from the intrinsic description of Θ-strata (see [HL20, Proposition 1.4.1] or also see the proof of Lemma 3.3.3): the third point of Proposition 3.2.5 exactly says that $\mathbb{L}_{\mathcal{S}_R/\mathcal{X}_R} \in \operatorname{QCoh}(\mathcal{S}_R)^{\geq 1}$ (meaning the term of the corresponding baric decomposition).

Proposition 3.2.8. *$\mathcal{X}_R$ is cohomologically proper if and only if both $\mathcal{U}_R$ and $\mathcal{S}_R$ are.*

Proof. Fix some $w \in \mathbb{Z}$.

"$\Rightarrow$" $\mathcal{S}_R$ is cohomologically proper, since for $\mathcal{F} \in \operatorname{Coh}(\mathcal{S}_R)$ we have $j_{R*}\mathcal{F} \in \operatorname{Coh}(\mathcal{X}_R)$ and $R\Gamma(\mathcal{S}_R, F) \simeq R\Gamma(\mathcal{X}_R, j_{R*}\mathcal{F})$. Since $\mathcal{X}_R$ is smooth, we have $\operatorname{Coh}(\mathcal{X}_R) \simeq \operatorname{Perf}(\mathcal{X}_R)$ and so any object of $\operatorname{Coh}(\mathcal{X}_R)$ is dualizable; in particular, for any $V_1, V_2 \in \operatorname{Coh}(\mathcal{X}_R)$ we have

$$\operatorname{Hom}_{\operatorname{Coh}(\mathcal{X}_R)}(V_1, V_2) \simeq R\Gamma(\mathcal{X}_R, \mathcal{H}om(V_1, V_2)),$$

where $\mathcal{H}om(V_1, V_2) \in \operatorname{Coh}(\mathcal{X}_R)$. Thus $\operatorname{Hom}_{\operatorname{Coh}(\mathcal{X}_R)}(V_1, V_2) \in \operatorname{Coh}^+(R)$, or, in other words, $\operatorname{Coh}(\mathcal{X}_R)$ is a *nearly proper* R-linear stable ∞-category. Now, taking $E \in \operatorname{Coh}(\mathcal{U}_R)$, we get

$$R\Gamma(\mathcal{U}_R, E) \simeq \operatorname{Hom}_{\operatorname{Coh}(\mathcal{U}_R)}(\mathcal{O}_{\mathcal{U}_R}, E) \simeq \operatorname{Hom}_{\operatorname{Coh}(\mathcal{X}_R)}((i_R^*|_{\mathrm{G}_w})^{-1}\mathcal{O}_{\mathcal{U}_R}, (i_R^*|_{\mathrm{G}_w})^{-1}E) \in \operatorname{Coh}^+(R),$$

via the equivalence $(i_R^*|_{\mathrm{G}_w})^{-1}\colon \operatorname{Coh}(\mathcal{U}_R) \xrightarrow{\sim} \mathrm{G}_w \subset \operatorname{Coh}(\mathcal{X}_R)$.

"$\Leftarrow$" It is enough to show that the category $\operatorname{Coh}(\mathcal{X}_R)$ is nearly proper. Let's first show that the subcategory $\operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R) \subset \operatorname{Coh}(\mathcal{X}_R)$ is nearly proper. Every object of $\operatorname{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ has a finite filtration with graded pieces of the form $j_{R*}\mathcal{F}$, where $\mathcal{F} \in \mathrm{Coh}(\mathcal{S}_R)$. Thus it is enough to show that $\mathrm{Hom}(j_{R*}\mathcal{F}_1, j_{R*}\mathcal{F}_2) \in \mathrm{Coh}^+(R)$ for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Coh}(\mathcal{S}_R)$. Since $\mathcal{S}_R$ and $\mathcal{X}_R$ are smooth we have

$$\mathrm{Hom}_{\mathrm{Coh}(\mathcal{S}_R)}(j_{R*}\mathcal{F}_1, j_{R*}\mathcal{F}_2) \simeq R\Gamma(\mathcal{S}_R, \mathcal{H}om(j_R^* j_{R*}\mathcal{F}_1, \mathcal{F}_2))$$

with $\mathcal{H}om(j_R^* j_{R*}\mathcal{F}_1, \mathcal{F}_2) \in \mathrm{Coh}(\mathcal{S}_R)$; then $R\Gamma(\mathcal{S}_R, \mathcal{H}om(j_R^* j_{R*}\mathcal{F}_1, \mathcal{F}_2)) \in \mathrm{Coh}^+(R)$ since $\mathcal{S}_R$ is cohomologically proper.

More generally, given $\mathcal{F} \in \mathrm{Coh}(\mathcal{S}_R)$ and $V \in \mathrm{Coh}(\mathcal{X}_R)$ we have

$$\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(V, j_{R*}\mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{Coh}(\mathcal{S}_R)}(j_R^* V, \mathcal{F}) \in \mathrm{Coh}^+(R),$$
$$\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(j_{R*}\mathcal{F}, V) \simeq \mathrm{Hom}_{\mathrm{Coh}(\mathcal{S}_R)}(\mathcal{F}, j_R^! V) \in \mathrm{Coh}^+(R),$$

where $j_R^! V \in \mathrm{Coh}(\mathcal{S}_R)$, since both $\mathcal{X}_R$ and $\mathcal{S}_R$ are smooth. It follows that $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(V, E)$ and $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(E, V)$ both lie in $\mathrm{Coh}^+(R)$ if $E \in \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ and V is any coherent sheaf on $\mathcal{X}_R$.

Now, due to the semiorthogonal decomposition in (7) any $V \in \mathrm{Coh}(\mathcal{X}_R)$ has a finite (in fact three-step) filtration with the associated graded pieces lying either in $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ or G_w. By the above discussion, hom-complex $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(E, -)$ for any object E in $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ is always bounded below coherent. The category $\mathrm{G}_w \simeq \mathrm{Coh}(\mathcal{U}_R)$ is nearly proper since $\mathcal{U}_R$ is cohomologically proper. Taking such filtrations for a given pair $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Coh}(\mathcal{X}_R)$ and using the exactness of Hom in each variable we get that $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(\mathcal{F}_1, \mathcal{F}_2) \in \mathrm{Coh}^+(R)$. ☐

The $\mathbb{G}_{m,R}$-action on X_R^0 is trivial; thus $\mathcal{X}_R^0$ is isomorphic to $X_R^0 \times_R B\mathbb{G}_{m,R}$. Let $p\colon \mathcal{X}_R^0 \to X_R^0$ be the projection.

Lemma 3.2.9. *The morphism $p \circ \pi_R\colon \mathcal{X}_R^+ \to X_R^0$ is cohomologically proper.*

Proof. The statement is local on X_R^0. Let $\{U_{i,R}^0\}_{i\in I}$, $U_{i,R}^0 = \mathrm{Spec}\, A_{i,R}^0$ be an affine cover of X_R^0 such that the affine bundle given by $\pi_R\colon X_R^+ \to X_R^0$ is trivial. Let $U_{i,R}^+ \coloneqq \pi_R^{-1}(U_{i,R}^0)$. It is enough to show that the morphism $\mathcal{U}_{i,R}^+ \coloneqq [U_{i,R}^+/\mathbb{G}_{m,R}] \to U_{i,R}^0$ induced by the composition of π_R and p is cohomologically proper. We have $U_{i,R}^+ \simeq U_{i,R}^0 \times_R \mathbb{A}_R^d$ where $\mathbb{G}_{m,R}$ acts with negative weights on $\mathbb{A}_R^d$. Let $A_{i,R}^+$ be the ring of functions on $U_{i,R}^+$; it is naturally $\mathbb{Z}$-graded and $A_{i,R}^+ \simeq A_{i,R}^0[x_1, \ldots, x_d]$ where x_i's can be chosen to be homogeneous of strictly negative degree.

The functor $(p \circ \pi_R)_*$ is t-exact, since π_R is affine and $\mathbb{G}_{m,R}$ is linearly reductive. We have an equivalence between the abelian category $\mathrm{Coh}(\mathcal{U}_{i,R}^+)^\heartsuit$ and the category of finitely generated graded $A_{i,R}^+$-modules. Via this equivalence, $(p \circ \pi_R)_*$ sends a graded module M^* to the $A_{i,R}^0$-module M^0. Since the degrees of x_i's are strictly negative, it is straightforward to see that if a $A_{i,R}^0[x_1, \ldots, x_d]$-module M is finitely generated, the $A_{i,R}^0$-module M^0 is finitely generated as well, and thus corresponds to a coherent sheaf on $U_{i,R}^0$. ☐

Proposition 3.2.10. *Let X be smooth scheme over F and with a BB-complete $\mathbb{G}_m$-action. Let X_R be a spreading as above. Then the following are equivalent:*

1. *$\mathcal{X}_R$ is cohomologically proper.*
2. *$\mathcal{X}_R^+$ is cohomologically proper.*
3. *X_R^0 is proper.*

Proof. $1 \Leftrightarrow 2$. Note that $\mathcal{X}_R^+ \simeq \cup_{c\in\pi_0(X^0)}\mathcal{S}_{c,R}$. Let's fix an ordering on $\pi_0(X^0)$ such that each $S_{c,R}$ is closed in $X_{\leq c,R} \coloneqq X_R \setminus \cup_{c'>c} S_{c',R}$. Since $\{S_{c,R}\}$ form a full finite stratification of X_R, applying Proposition 3.2.8 we are done by induction on c.

$2 \Rightarrow 3$. The morphism $\sigma_R\colon \mathcal{X}_R^0 \to \mathcal{X}_R^+$ is a closed embedding (in particular proper) and thus is cohomologically proper. It follows that $\mathcal{X}_R^0$ is cohomologically proper. On the other hand $\mathcal{X}_R^0 \simeq X_R^0 \times_R B\mathbb{G}_{m,R}$ and so X_R^0 is cohomologically proper as well. Indeed, $p_*\mathcal{O}_{\mathcal{X}_R^0} \simeq \mathcal{O}_{X_R^0}$ and given $\mathcal{F} \in \mathrm{Coh}(X_R^0)$ we have $R\Gamma(\mathcal{X}_R^0, p^*\mathcal{F}) \simeq R\Gamma(X_R^0, \mathcal{F})$ by the projection formula; since $p^*\mathcal{F} \in \mathrm{Coh}(\mathcal{X}_R^0)$ we get $R\Gamma(X_R^0, \mathcal{F}) \in \mathrm{Coh}^+(R)$. Being a cohomologically proper R-scheme, X_R^0 is forced to be proper (Corollary 3.8 of [Hal18]).

$2 \Leftarrow 3$: The structure morphism $\mathcal{X}_R^+ \to \mathrm{Spec}\, R$ factors as the composition of $\mathcal{X}_R^+ \xrightarrow{p\circ\pi_R} X_R^0$ and $X_R^0 \to \mathrm{Spec}\, R$. The first map is cohomologically proper by Lemma 3.2.9, the second — since X_R^0 is proper. ☐

Corollary 3.2.11. *Let X be a smooth scheme over F with an action of $\mathbb{G}_m$. Assume that the action is BB-complete and that the scheme of $\mathbb{G}_m$-fixed points X^0 is proper. Then X is cohomologically properly spreadable.*

Proof. Let X_R be a spreading as above, then X_R^0 is a spreading of X^0 and thus, after enlarging R, we can assume that X_R^0 is proper. Then $\mathcal{X}_R$ is cohomologically proper by Proposition 3.2.10. ☐

3.2.3 Quotients by G that are BB-complete with respect to a subgroup

In Corollary 3.2.11 we gave some sufficient conditions for the quotient stack $[X/\mathbb{G}_m]$ to be cohomologically spreadable. As we will see soon, Kempf's theorem allows to generalize this to a quotient by an arbitrary linear group G; however, a certain extra weight-positivity assumption with respect to a 1-parameter subgroup $h\colon \mathbb{G}_m \to G$ is necessary.

Let G be a linear algebraic group and let $B \subset G$ be a Borel subgroup[18]. Let $U \subset B$ be the unipotent radical of B and let $T \subset B$ be a maximal torus. We have a short exact sequence $1 \to U \to B \to T \to 1$. Let $X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ be the character and cocharacter lattices of T. One has $X_*(T) \simeq X^*(T)^\vee$. Given a T-representation V and a character $\lambda \in X^*(T)$ we denote by $V_\lambda \subset V$ the subspace of V of weight λ. The adjoint action of T on U induces an action on the Lie algebra $\mathfrak{u}$ of U and we denote by $\Phi^+ \subset X^*(T)$ the set of weights of $\mathfrak{u}$ with respect to this action.

Theorem 3.2.12. *Let X be a smooth scheme over F endowed with an action of a linear algebraic group G. Let $B \subset G$ be a Borel subgroup and let $T \subset B$ be a maximal torus. Let $\Phi^+ \subset X^*(T)$ be the set of T-weights of the Lie algebra $\mathfrak{u}$ of the unipotent radical $U \subset B$ with respect to the adjoint action of T on U.*

Assume that there is a subgroup $h\colon \mathbb{G}_m \to T$, $h \in X_*(T)$, such that

1. *$h(\Phi^+) > 0$.*
2. *The $\mathbb{G}_m$-action induced by h is BB-complete.*
3. *The $h(\mathbb{G}_m)$-fixed points X^0 are proper.*

Then the quotient stack $[X/G]$ is cohomologically properly spreadable.

Proof. Let's first assume that G is connected. Note that B is isomorphic to a semidirect product $T \ltimes U$ and can be spread out to a semidirect product $T_R \ltimes U_R$ of a split torus T_R and a unipotent group U_R over a finitely generated $\mathbb{Z}$-algebra $R \subset F$. Since T_R is split $X^*(T_R) \simeq X^*(T)$. In particular we have a cocharacter $h\colon \mathbb{G}_{m,R} \to T_R$. The subgroup $B \subset G$ can be spread out to a closed subgroup $B_R \subset G_R$ and we can assume that G_R is split. Let U_G be the unipotent radical of G. Then G/U_G is reductive and can be spread out to a split reductive group $(G/U_G)_R$. We then also have a spreading $p_R\colon G_R \to (G/U_G)_R$ of the projection $p\colon G \to G/U_G$ and the kernel $(U_G)_R := \mathrm{Ker}(p_G)$ is a spreading of U_G and thus can be assumed to be unipotent. Since U_G is a closed subgroup of B, we can assume that $(U_G)_R$ is a closed subgroup of B_R. The image of B_R under p_R is a spreading of $B/U_G \subseteq G/U_G$ and thus can be assumed to be a Borel subgroup of the split reductive group $(G/U)_R$. Note that with all these assumptions $G_R/B_R \simeq (G/U_G)_R/p_R(B_R)$.

We can also spread X with the action $a\colon G \curvearrowright X$ to a smooth scheme X_R over R with an action $a_R\colon G_R \curvearrowright X_R$. Note that by [Sum74] the restriction of the action of G on X to $\mathbb{G}_m$ (via h) is locally linear; consider a $\mathbb{G}_m$-invariant open cover $\{U_i\}_{i\in I}$ of X, $U_i = \operatorname{Spec} A_i$. We have spreadings $A_{i,R}$ of A_i with an action of $\mathbb{G}_{m,R}$; localizing R if necessary, we can assume that $U_{i,R} := \operatorname{Spec} A_{i,R}$ cover X_R and that the restriction of a_R to $\mathbb{G}_{m,R}$ via h is locally linear. Localizing R, we can assume that the $\mathbb{G}_{m,R}$-fixed points X_R^0 are proper, the action is BB-complete and X_R is such that the conditions of 3.2.10 are satisfied. Thus we have that $[X_R/h(\mathbb{G}_{m,R})]$ is cohomologically proper. It is enough to show that $[X_R/G_R]$ is cohomologically proper.

We split the argument into a sequence of lemmas:

Lemma 3.2.13. *Let X_R be as above. Then*

$$[X_R/h(\mathbb{G}_{m,R}))] \text{ is cohomologically proper over } R \quad \Rightarrow \quad [X_R/T_R] \text{ is cohomologically proper over } R.$$

Proof. Let $p\colon X_R \to [X_R/T_R]$ and $q\colon [X/h(\mathbb{G}_{m,R})] \to [X_R/T_R]$ be the natural smooth covers. Then, given $\mathcal{F} \in \mathrm{Coh}([X_R/T_R])$, we have $R\Gamma([X_R/T_R], \mathcal{F}) \simeq R\Gamma(X_R, p^*\mathcal{F})^{T_R}$. But

$$R\Gamma(X_R, p^*\mathcal{F})^{T_R} \simeq \left(R\Gamma(X_R, p^*\mathcal{F})^{h(\mathbb{G}_{m,R})}\right)^{T/h(\mathbb{G}_{m,R})} \simeq R\Gamma([X_R/h(\mathbb{G}_{m,R})], s^*\mathcal{F})^{T/h(\mathbb{G}_{m,R})};$$

since $s^*\mathcal{F} \in \mathrm{Coh}([X_R/h(\mathbb{G}_{m,R})])$ we have $R\Gamma([X_R/h(\mathbb{G}_{m,R})], s^*\mathcal{F}) \in \mathrm{Coh}^+(R)$. Recall that the coherence of a complex of R-modules is equivalent to being t-bounded and having finitely generated cohomology. The group scheme $T/h(\mathbb{G}_{m,R})$ is a torus and the functor of $T_R/h(\mathbb{G}_{m,R})$-invariants is t-exact. Thus $R\Gamma([X_R/h(\mathbb{G}_{m,R})], s^*\mathcal{F})^{T/h(\mathbb{G}_{m,R})}$ is also bounded and has finitely generated cohomology, so is coherent. □

[18] Recall that a subgroup $B \subset G$ is called Borel if it is a maximal Zariski-closed solvable subgroup of G.

Lemma 3.2.14. *Let X_R be as above. Then*

$$[X_R/h(\mathbb{G}_{m,R}))] \text{ is cohomologically proper over } R \quad \Rightarrow \quad [X_R/B_R] \text{ is cohomologically proper over } R.$$

Proof. Consider the natural smooth cover $q\colon [X_R/T_R] \to [X_R/B_R]$ induced by the embedding $T_R \to B_R$. Since $B_R \simeq T_R \ltimes U_R$, the n-th term of the Čech complex associated to q is given by

$$[X_R \overset{T_R}{\times} \underbrace{B_R \overset{T_R}{\times} B_R \overset{T_R}{\times} \cdots \overset{T_R}{\times} B_R}_{n}/T_R] \simeq [(X_R \times \underbrace{U_R \times U_R \times \cdots \times U_R}_{n})/T_R]$$ [19],

where the action of T_R on $X_R \times U_R \times U_R \times \cdots \times U_R$ is given by the product of the action on X_R and the adjoint action on each copy of U_R.

Note that the underlying scheme of U_R can be T_R-equivariantly identified with its Lie algebra $\mathfrak{u}_R$ (see II.1.7 in [Jan07]); this way functions on U_R (resp U_R^n) are identified with $\mathrm{Sym}_R(\mathfrak{u}_R^*)$ (resp. $\mathrm{Sym}_R(\mathfrak{u}_R^*)^{\otimes n}$). Since $h(\Phi^+) > 0$ we get that the $\mathbb{G}_{m,R}$-weights on non-constant homogeneous functions on U_R^n are strictly negative. It follows that $U_R^n \simeq (U_R^n)^+$ and $(U_R^n)^0 \simeq \operatorname{Spec} R$. We have $(X_R \times U_R^n)^+ \simeq X_R^+ \times U_R^n$, so the map $(X_R \times U_R^n)^+ \to X_R \times U_R^n$ is surjective and thus the $\mathbb{G}_{m,R}$-action on $X_R \times U_R^n$ is BB-complete for every n. Also, $(X_R \times U_R^n)^0 \simeq X_R^0$ is proper. Finally, since U_R^n is isomorphic to the affine space, the Bialynicki-Birula strata still satisfy the conditions in 3.2.5. Consequently, Proposition 3.2.10 applies to $X_R \times U_R^n$ for all n; we get that $[(X_R \times U_R^n)/h(\mathbb{G}_{m,R})]$ is cohomologically proper. By Lemma 3.2.13 it follows that $[(X_R \times U_R^n)/T_R]$ is cohomologically proper for all n. By Proposition 2.2.14 we get that $[X_R/B_R]$ is cohomologically proper. □

To pass from B_R to G_R we use the Kempf's theorem (3.0.2):

Lemma 3.2.15. *Let X_R be as above. Then*

$$[X_R/B_R] \text{ is cohomologically proper over } R \quad \Rightarrow \quad [X_R/G_R] \text{ is cohomologically proper over } R.$$

Proof. Let $j\colon BB_R \to BG_R$ be the natural morphism. Then by the projection formula

$$R\Gamma(BB_R, j^*\mathcal{F}) \simeq R\Gamma(BG_R, j_*j^*\mathcal{F}) \simeq R\Gamma(BG_R, \mathcal{F} \otimes j_*\mathcal{O}_{BB_R}).$$

By base change, the underlying complex of R-modules of $j_*\mathcal{O}_{BB_R}$ is equivalent to $R\Gamma(G_R/B_R, \mathcal{O}_{G_R/B_R})$. But $G_R/B_R \simeq (G/U_G)_R/(p_R(B_R))$, where $p_R(B_R) \subset (G/U_G)_R$ is a Borel subgroup and thus $R\Gamma(G_R/B_R, \mathcal{O}_{G_R/B_R}) \simeq R$ by Theorem 3.0.2. Consequently, $R\Gamma(BG_R, \mathcal{F}) \simeq R\Gamma(BB_R, j^*\mathcal{F})$ for any sheaf $\mathcal{F} \in \mathrm{QCoh}(BG_R)$. We now apply this as follows: there is a fibered square

$$\begin{array}{ccc} [X_R/B_R] & \xrightarrow{f} & [X_R/G_R] \\ \downarrow & & \downarrow \\ BB_R & \xrightarrow{j} & BG_R \end{array}$$

and, given a coherent sheaf $\mathcal{F} \in \mathrm{Coh}([Y_R/G_R])$, its pull-back $f^*\mathcal{F} \in \mathrm{Coh}([Y_R/B_R])$ is also coherent. Applying base change and the above isomorphism we get that $R\Gamma([Y_R/G_R], \mathcal{F}) \simeq R\Gamma([Y_R/B_R], f^*\mathcal{F})$; in particular, one complex is bounded below coherent if and only if the other one is. The statement of the lemma follows. □

It remains to cover the case of a disconnected G. We can write $[X/G] \simeq [[X/G^0]/\pi_0(G)]$, where G^0 is the connected component of $e \in G$ and $\pi_0(G)$ is the finite group of components. The homomorphism $p\colon G \to \pi_0(G) \simeq G/G^0$ can be spread out to $p_R\colon G_R \to \pi_0(G)_R$ where G_R is some spreading out of G and $\pi_0(G)_R$ is the constant group R-scheme associated to $\pi_0(G)$. Moreover the kernel G_R^0 of p_R is a spreading of G^0.

We have just shown that the quotient stack $[X_R/G_R^0]$ is cohomologically proper over R. We also have $[X_R/G_R] \simeq [[X_R/G_R^0]/\pi_0(G)_R]$. It follows that for any $\mathcal{F} \in \mathrm{Coh}([X/G_R])$

$$R\Gamma([X_R/G_R], \mathcal{F}) \simeq R\Gamma\left(B\pi_0(G)_R, R\Gamma([X_R/G_R^0], \mathcal{F})\right).$$

Replacing R with $R[1/|\pi_0(G)|]$ we can assume that $|\pi_0(G)|$ is invertible in R, and so the functor of $\pi_0(G)$-invariants is t-exact. Then we get $H^q([X_R/G_R], F) \simeq H^q([X_R/G_R^0], \mathcal{F})^{\pi_0(G)}$. In particular, $R\Gamma([X_R/G_R], \mathcal{F})$ is t-bounded and its cohomology are finitely generated over R, so $R\Gamma([X_R/G_R], \mathcal{F}) \in \mathrm{Coh}^+(R)$. □

[19] The isomorphism is given by the formula $[(x, b_1, \ldots, b_n)] \mapsto [(x, u_1, \ldots, u_n)]$, where $b_i = t_i \cdot u_i \in T_R \ltimes U_R$.

We end this subsection by giving some examples to which Theorem 3.2.12 does apply:

Example 3.2.16. 1. X is proper. In this case by the valuative criterion of properness every $\mathbb{G}_m$-orbit of an F-point has a limit as $t \to 0$, so the map $X^+ \to X$ is surjective on F-points. It follows that any $\mathbb{G}_m$-action on X is BB-complete. Moreover $X^0 \subset X$ is a closed subscheme and so is proper. Thus, the only condition to check is on G: namely there should exist $h \in X_*(T)$ such that $h(\Phi^+) > 0$ (since all Borel subgroups of G are conjugate to each other this does not depend on the choice of B). Here is a list of linear algebraic groups G which satisfy this:

- G reductive. Then one can take $h \in X_*(T)$ given by any dominant coweight. This case is also covered by Theorem 3.1.4;
- $G = P \subset H$ is a parabolic subgroup of a reductive group H. Any h that is dominant with respect to some Borel subgroup $B \subset P$ applies;
- More or less tautologically, any G with a 1-dimensional subtorus $\mathbb{G}_m \subset G$ such that the adjoint action of $\mathbb{G}_m$ on the the Lie algebra $\mathfrak{u}_G$ of the unipotent radical $U_G \subset G$ has strictly positive weights and such that the projection of $\mathbb{G}_m$ to G/U_G gives a regular coweight (meaning its centralizer is given by a maximal torus). Then one picks B as the preimage of a Borel subgroup of G/U_G, with respect to which the $\mathbb{G}_m$ above gives a dominant coweight, under the projection $G \twoheadrightarrow G/U_G$ and take h given by any lifting $\mathbb{G}_m \to B$. As a non-parabolic example of such G one can take any semidirect product $\mathbb{G}_m \ltimes U$ where U is unipotent and $\mathbb{G}_m$ acts on $\mathfrak{u}$ with strictly positive weights.

2. There are also natural examples that are more in the spirit of Theorem 3.1.4. Namely, let $\pi\colon X \to Y$ be a proper G-equivariant morphism where Y is not necessarily smooth. Then, given a cocharacter $h\colon \mathbb{G}_m \to G$ that satisfies $h(\Phi^+) > 0$ for some $B \subset G$, we have that if $Y^+ \twoheadrightarrow Y$ is a surjection and Y^0 is proper, X satisfies the conditions of Theorem 3.2.12. Indeed the induced map $X^0 \to Y^0$ is proper and so X^0 is proper. Also, given any point $x \in X(F)$, the image $\pi(\mathbb{G}_m \cdot x)$ of its orbit is the orbit $\mathbb{G}_m \cdot \pi(x)$. Since $Y^+ \twoheadrightarrow Y$ is a surjection, the limit $\lim_{t\to 0} t \circ \pi(x)$ exists. This gives a diagram

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\cdot \circ x} & X \\ \downarrow & \nearrow & \downarrow \\ \mathbb{A}^1 & \xrightarrow{\cdot \circ \pi(x)} & Y \end{array}$$

and, due to the valuative criterion of properness, the lifting $\mathbb{A}^1 \dashrightarrow X$, this way producing the limit of $t \circ x$ as $t \to 0$. We get that $X^+ \to X$ is a surjection on F-points and that the $\mathbb{G}_m$-action given by h is BB-complete.

This applies, in particular, to the case when $Y^0 \simeq \operatorname{Spec} F$ and $G = \mathbb{G}_m$ (where we can assume $h = \mathrm{id}\colon \mathbb{G}_m \to \mathbb{G}_m$). In this case we basically arrive at the definition of a conical resolution (see e.g. [KT16]). Namely, we have $Y \simeq Y^+ \to \operatorname{Spec} F$ is affine, so $Y \simeq \operatorname{Spec} A$; the induced $\mathbb{Z}$-grading on A is such that $A \simeq A^{\leq 0}$ and $A^0 \simeq F$[20]. The map $\pi\colon X \to \operatorname{Spec} A$ is proper, X is smooth and the $\mathbb{G}_m$-action on X agrees with the grading on A. The geometry of such X is the following: it is not proper itself, but it has a proper $\mathbb{G}_m$-equivariant map to $\operatorname{Spec} A$ so that the $\mathbb{G}_m$-action contracts it to the central fiber $\pi^{-1}(Y^0)$ which is proper over F. Note that even if X is smooth, $\pi^{-1}(Y^0)$ can be singular (for example in the case of the minimal resolution of the A_n-singularity for $n > 2$).

3.3 Θ-stratified stacks and the relation to the work of Teleman

The example of BB-complete quotients by $\mathbb{G}_m$ can be vastly generalized by the notion of a Θ-stratified stack introduced recently by Halpern-Leistner (and studied in great detail in [HL20]). All the stacks in this section are assumed to be derived and we also assume the base ring R to be Noetherian and regular. Let $\mathcal{X}$ be a derived stack over R and assume that it is locally almost of finite presentation with affine diagonal. We also let $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$. One can define two mapping stacks associated to $\mathcal{X}$: the stack of *filtered objects* $\mathrm{Filt}(\mathcal{X}) := \underline{\mathrm{Map}}_R(\Theta, \mathcal{X})$ and the stack of *graded objects* $\mathrm{Grad}(\mathcal{X}) := \underline{\mathrm{Map}}_R(B\mathbb{G}_m, \mathcal{X})$. We have a stacky version of maps defined in Section 3.2.1

$$\mathrm{Grad}(\mathcal{X}) \underset{\mathrm{ev}_0}{\overset{\sigma}{\rightleftarrows}} \mathrm{Filt}(\mathcal{X}) \xrightarrow{\mathrm{ev}_1} \mathcal{X}$$

induced by evaluations at $0\colon B\mathbb{G}_m \hookrightarrow \Theta$ and $1\colon \operatorname{Spec} R \simeq \left[\left(\mathbb{A}^1 \setminus 0\right)/\mathbb{G}_m\right] \to \Theta$, and the natural projection $\Theta \to B\mathbb{G}_m$. Note that if $\mathcal{X}$ is smooth (and thus classical) by [HL20, Corollary 1.3.2.1] the stack $\mathrm{Filt}(\mathcal{X})$ is also smooth

[20] Note the change of sign in the grading compared to [KT16]. In the case of a commutative group action there are two natural left actions on the space of functions on Y, induced either by the action of g or g^{-1} on Y. This is exactly the difference we are facing here.

and classical. A *derived* Θ*-stratum* $\mathcal{S}$ is by definition a union of connected components of $\mathrm{Filt}(\mathcal{X})$ with the condition that $\mathrm{ev}_1|_{\mathcal{S}} \coloneqq \mathcal{S} \to \mathcal{X}$ is a closed embedding. Let $\mathcal{Z} \coloneqq \sigma^{-1}(\mathcal{S}) \subset \mathrm{Grad}(\mathcal{X})$ be the centrum of $\mathcal{S}$; ev_0 restricts to a map $\mathcal{S} \to \mathcal{Z}$.

Definition 3.3.1 (A particular case of [HL20, Definition 1.10.1])**.** A finite Θ-*stratification* of $\mathcal{X}$ indexed by a totally ordered finite set I with a minimal element $0 \in I$ is given by:

1. A collection of open substacks $\mathcal{X}_{\leq \alpha} \subset \mathcal{X}$ with $\alpha \in I$ such that $\mathcal{X}_\alpha \subset \mathcal{X}_{\alpha'}$ if $\alpha < \alpha'$.
2. For each $\alpha > 0$ a Θ-stratum $\mathcal{S}_\alpha \subset \mathrm{Filt}(\mathcal{X}_\alpha)$ such that $\mathcal{X}_{\leq\alpha} \setminus (\cup_{\alpha'<\alpha} \mathcal{X}_{\leq \alpha'}) = \mathrm{ev}_1(\mathcal{S}_\alpha)$.
3. One should have $\mathcal{X} = \cup_{\alpha \in I} \mathcal{X}_\alpha$.

The minimal open stratum $\mathcal{X}^{ss} \subset \mathcal{X}$ is called the *semistable locus*.

Let $i_\alpha \colon \mathcal{S}_\alpha \to \mathcal{X}_\alpha$ be the embedding induced by ev_1. The pushforward $i_{\alpha *}$ has left $i_\alpha^* \colon \mathrm{QCoh}(\mathcal{X}_{\leq\alpha}) \to \mathrm{QCoh}(\mathcal{S}_\alpha)$ and right $i_\alpha^! \colon \mathrm{QCoh}(\mathcal{X}_{\leq\alpha}) \to \mathrm{QCoh}(\mathcal{S}_\alpha)$ adjoints. Also let $\mathrm{Coh}^-(-)$ denote the bounded above category of coherent sheaves.

Proposition 3.3.2. *Let $\mathcal{X}$ be a smooth Artin stack of finite type over R with affine diagonal endowed with a finite Θ-stratification. Assume that $\mathcal{X}^{ss}$ and the centra $\mathcal{Z}_\alpha$ are cohomologically proper over R. Then $\mathcal{X}$ is cohomologically proper over R.*

Proof. By induction on $|I|$ we can reduce to the case of a single Θ-stratum $\mathcal{S}$ with the complement given by $\mathcal{X}^{ss}$. The stratum $\mathcal{S}$ is a connected component of $\mathrm{Filt}(\mathcal{X})$ and thus is also smooth over R. Since both $\mathcal{X}$ and $\mathcal{S}$ are smooth, we get that i^* restricts to a functor between $\mathrm{Coh}(\mathcal{X}) \simeq \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$ and $\mathrm{Coh}(\mathcal{S}) \simeq \mathrm{QCoh}(\mathcal{S})^{\mathrm{perf}}$. Also by smoothness the direct image $i_*\mathcal{O}_{\mathcal{S}} \in \mathrm{QCoh}(\mathcal{X})$ is perfect. Indeed, by descent this is enough to check after taking a pull-back on a smooth cover by a smooth R-scheme of finite type where we get a regular embedding $i' \colon S' \to X'$ which is automatically a locally complete intersection in X'. After refining the cover in Zariski topology, we can assume the intersection is actually complete and resolve $i'_*\mathcal{O}_{S'}$ by the Koszul complex. This shows that the functor $i^!$ also restricts to a functor from $\mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$ to $\mathrm{QCoh}(\mathcal{S})^{\mathrm{perf}}$. Indeed, one has a formula $i^!\mathcal{F} \simeq \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{X})}(i_*\mathcal{O}_{\mathcal{S}}, \mathcal{F})$ where the latter has support on $\mathcal{S}$ and is considered as an object of $\mathrm{QCoh}(\mathcal{S})$. By smooth descent for $\underline{\mathrm{Hom}}$ ([Pre11, Lemma A.1.1]) we can reduce to the case $\mathrm{Spec}\, A \simeq S \to X \simeq \mathrm{Spec}\, B$ of smooth affine schemes over R, where the sheaf $\underline{\mathrm{Hom}}_{\mathrm{QCoh}(X)}(i_*\mathcal{O}_S, \mathcal{O}_X)$ is computed by the dual to the Koszul complex and thus is bounded and has finitely generated cohomology modules (and this way belongs to $\mathrm{Coh}(A)$).

Given all this, by a similar argument to Proposition 3.2.8 it is enough to get a suitable semiorthogonal decomposition of $\mathrm{Coh}(\mathcal{X}) \simeq \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$ in terms of $\mathrm{Coh}_{\mathcal{S}}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X}^{ss})$ and show that all Hom's in $\mathrm{Coh}(\mathcal{S})$ lie in $\mathrm{Coh}^+(R)$. Since $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{S})}(\mathcal{F}, \mathcal{G}) \simeq R\Gamma(\mathcal{S}, \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{S})}(\mathcal{F}, \mathcal{G}))$ with $\underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{S})}(\mathcal{F}, \mathcal{G}) \in \mathrm{Coh}(\mathcal{S})$ and $\mathcal{Z}$ is cohomologically proper for the latter point it is enough to show that $\mathrm{ev}_0 \colon \mathcal{S} \to \mathcal{Z}$ is cohomologically proper. By descent and base change this can be checked on a (suitable) smooth cover of $\mathcal{X}$; namely we can use [AHLH18, Lemma 6.11] to produce a smooth cover $[X/\mathbb{G}_m] \to \mathcal{X}$ where X is an affine scheme and such that the preimage of $\mathcal{S}$ is given by a union of connected components of $[X^+/\mathbb{G}_m]$ (in the terminology of Sections 3.2.1 and 3.2.2). The centrum $\mathcal{Z}$ is then given by a union of the corresponding components of $[X^0/\mathbb{G}_m]$ and the needed statement follows from Lemma 3.2.9.

It remains to deal with the semiorthogonal decomposition. By [HL20, Theorem 1.9.2] for each integer $w \in \mathbb{Z}$ we have a decomposition for the bounded above category $\mathrm{Coh}^-(\mathcal{X})$

$$\mathrm{Coh}^-(\mathcal{X}) = \langle\, \mathrm{Coh}^-_{\mathcal{S}}(\mathcal{X})_{<w}, \mathrm{G}^-_w, \mathrm{Coh}^-_{\mathcal{S}}(\mathcal{X})_{\geq w}\,\rangle$$

in terms of certain subcategories $\mathrm{Coh}_{\mathcal{S}}(\mathcal{X})_{<w}, \mathrm{Coh}_{\mathcal{S}}(\mathcal{X})_{\geq w} \subset \mathrm{Coh}_{\mathcal{S}}(\mathcal{X})$ forming a semiorthogonal decomposition of $\mathrm{Coh}_{\mathcal{S}}(\mathcal{X})$ on its own and with G^-_w being isomorphic to $\mathrm{Coh}^-(\mathcal{X}^{ss})$ via the restriction $\mathcal{F} \mapsto \mathcal{F}|_{\mathcal{X}^{ss}}$. This decomposition holds without extra assumptions on $\mathcal{X}$, however if we assume $\mathcal{X}, \mathcal{S}$ are smooth (and thus in particular the embedding $i \colon \mathcal{S} \to \mathcal{X}$ is regular) the proof of [HL20, Proposition 2.1.2] goes through without any changes, giving the analogous decomposition for $\mathrm{QCoh}^{\mathrm{perf}}$ (and thus also for Coh). ☐

Let's now assume that we have a smooth finite type stack $\mathcal{X}$ over a characteristic 0 field F endowed with a Θ-stratification. Filtering F by regular Noetherian rings $R \subset F$ as in Section 2.3 we get a smooth spreading $\mathcal{X}_R$; we can also spread the open substacks $\mathcal{X}_{\leq\alpha}$ to get open substacks $\mathcal{X}_{\leq\alpha,R} \subset \mathcal{X}_R$. Applying the following lemma inductively we can in fact assume that this gives a Θ-stratification. Before stating the lemma note that one has a natural monoid structure on Θ induced by the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. Having a stack $\mathcal{Y}$ with an action of Θ in the homotopy category one gets a baric structure on $\mathrm{QCoh}(\mathcal{Y})$ (see [HL20, Section 1.1]); in particular for each weight $w \in \mathbb{Z}$ one has a semiorthogonal decomposition $\langle \mathrm{QCoh}(\mathcal{Y})^{\geq w}, \mathrm{QCoh}(\mathcal{Y})^{<w} \rangle$. Moreover if $\mathcal{Y}$ is smooth over a regular Noetherian base ring R this also defines a decomposition for coherent sheaves: $\mathrm{Coh}(\mathcal{Y}) = \langle \mathrm{Coh}(\mathcal{Y})^{\geq w}, \mathrm{Coh}(\mathcal{Y})^{<w} \rangle$ ([HL20, Proposition 1.2.1(3)]); we will denote by $\beta^{\geq w}, \beta^{<w}$ the corresponding truncation functors. We note that any Θ-stratum $\mathcal{S} \subset \mathrm{Filt}(\mathcal{X})$ comes with a natural Θ-action.

Lemma 3.3.3. *Let $\mathcal{X}_F$ be a smooth Artin stack of finite type over F with affine diagonal and let $i_F \colon \mathcal{S}_F \hookrightarrow \mathcal{X}_F$ be a Θ-stratum. Then one has a spreading $i_R \colon \mathcal{S}_R \hookrightarrow \mathcal{X}_R$ which is a Θ-stratum as well.*

Proof. The key step here is to use the intrinsic description of Θ-strata ([HL20, Section 1.4]). Namely, over any Noetherian base a closed substack $i \colon \mathcal{S} \hookrightarrow \mathcal{X}$ with an action $a \colon \Theta \times \mathcal{S} \to \mathcal{S}$ gives a map $\varphi \colon \mathcal{S} \to \mathrm{Filt}(\mathcal{X})$ defined as a composition

$$\mathcal{S} \xrightarrow{a^*} \underline{\mathrm{Map}}(\Theta, \mathcal{S}) \xrightarrow{\circ i} \underline{\mathrm{Map}}(\Theta, \mathcal{X}).$$

By [HL20, Proposition 1.4.1] if $\mathcal{X}$ is locally of finite presentation with affine diagonal the map φ is also a closed embedding; moreover, $\varphi \colon \mathcal{S} \hookrightarrow \mathrm{Filt}(\mathcal{X})$ defines a Θ-stratum if and only if $\mathbb{L}_{\mathcal{S}/\mathcal{X}} \in \mathrm{QCoh}(\mathcal{S})^{\geq 1}$.

The stack $\mathcal{X}_F$ is smooth, thus by [HL20, Corollary 1.3.2.1] $\mathrm{Filt}(\mathcal{X}_F)$ is smooth and so $\mathcal{S}_F$ is smooth. The stack $\mathcal{S}_F$ is also a closed substack of a stack of finite type and so is of finite type over F as well. Using Theorem 2.1.13 we can spread the natural action $a_F \colon \Theta_F \times \mathcal{S}_F \to \mathcal{S}_F$ and the closed embedding $i_F \colon \mathcal{S}_F \hookrightarrow \mathcal{X}_F$ to get an action $a_R \colon \Theta_R \times \mathcal{S}_R \to \mathcal{S}_R$ and a closed embedding $i_R \colon \mathcal{S}_R \hookrightarrow \mathcal{X}_R$. Moreover we can assume $\mathcal{S}_R, \mathcal{X}_R$ are smooth and of finite type over R. By the above description $\mathcal{S}_R$ is a Θ-stratum if and only if $\mathbb{L}_{\mathcal{S}_R/\mathcal{X}_R} \in \mathrm{QCoh}(\mathcal{S}_R)^{\geq 1}$; this is equivalent to $\beta^{<1}(\mathbb{L}_{\mathcal{S}_R/\mathcal{X}_R}) \simeq 0$. Note that by smoothness $\mathbb{L}_{\mathcal{S}_R/\mathcal{X}_R}$ and thus also $\beta^{<1}(\mathbb{L}_{\mathcal{S}_R/\mathcal{X}_R})$ are coherent. Since the restriction to $\mathcal{X}_F$ of $\beta^{<1}(\mathbb{L}_{\mathcal{S}_R/\mathcal{X}_R})$ is given by $\beta^{<1}(\mathbb{L}_{\mathcal{S}_F/\mathcal{X}_F})$ which is zero, we get that $\beta^{<1}(\mathbb{L}_{\mathcal{S}_R/\mathcal{X}_R}) \simeq 0$ after a finite localization of R (indeed this is enough to check for a pull-back to a smooth cover by a scheme, where this follows from the Chevalley's consructibility theorem). ☐

Remark 3.3.4. We needed to use the intrinsic description of the Θ-strata in Lemma 3.3.3 because the stack $\mathrm{Filt}(\mathcal{X})$ is only locally finitely presentable; thus we can't directly apply Theorem 2.1.13 to spread $\mathcal{S}_F \to \mathrm{Filt}(\mathcal{X}_F)$ or compare $\mathrm{Filt}(\mathcal{X}_R)$ with some other spreading $\mathrm{Filt}(\mathcal{X})_R$.

From the discussion above we deduce the following result:

Corollary 3.3.5. *Let $\mathcal{X}$ be a smooth Artin stack of finite type with affine diagonal over F and let $\{\mathcal{X}_{\leq \alpha}, \mathcal{S}_\alpha\}$ be a finite Θ-stratification of $\mathcal{X}$. If the centra $\mathcal{Z}_\alpha$ of Θ-strata and the semistable locus $\mathcal{X}^{ss}$ are cohomologically properly spreadable then $\mathcal{X}$ is also cohomologically properly spreadable. In particular the Hodge-de Rham spectral sequence degenerates for $\mathcal{X}$.*

Proof. By induction on $|I|$ using Lemma 3.3.3 we can spread out the Θ-stratification (with the properties as in the proof of the latter). The centrum $\mathcal{Z}_{\alpha,R}$ is a closed substack of $\mathcal{S}_{\alpha,R}$, thus it is also of finite type and is a spreading of $\mathcal{Z}_\alpha$. Enlarging $R \subset F$ so that all $\mathcal{Z}_{\alpha,R}$ and $\mathcal{X}_R^{ss}$ become cohomologically proper we then use Proposition 3.3.2 to get that so is $\mathcal{X}_R$. ☐

In [HL18] various ways of constructing a Θ-stratifications on a stack are discussed in great detail. We will stop on a single example given by a KN-stratification of a global quotient stack.

Example 3.3.6. In [Tel00] Teleman showed the degeneration of the Hodge-to-de Rham spectral sequence for the quotient stacks $[X/G]$ with G reductive under the condition that the action on X is KN-complete (see Section 1 of *loc.cit.* for the definition of a KN-complete action and Section 7 for the proof of degeneration). We comment on how to deduce his results (in a slightly more general form) from Theorem 1.4.3 and Corollary 3.3.5.

A KN-stratification of a variety X with a G-action is a stratification

$$X = X^{\mathrm{ss}} \cup \bigcup_{\alpha \in I} S_\alpha$$

by locally closed G-invariant subschemes satisfying the following properties:

- For each α there should exist a one-parameter subgroup $\lambda_\alpha \colon \mathbb{G}_m \hookrightarrow G$; let $L_\alpha \subset G$ be the centralizer of $\lambda_\alpha(\mathbb{G}_m)$. The KN-strata S_α should come as follows: for each $\alpha \in I$ there should exist an open subvariety $Z_\alpha \subset X^{\lambda_\alpha(\mathbb{G}_m)}$ of the fixed locus of $\lambda_\alpha(\mathbb{G}_m)$ such that S_α is given by the G-span $G \cdot Y_\alpha$ of the corresponding attracting locus
$$Y_\alpha := \{x \in X \,|\, \lim_{t \to 0} \lambda_\alpha(t) \cdot x \in Z_\alpha\}.$$

The subvariety Z_α is called the centrum of S_α and it is endowed with the natural action of the centralizer L_α. The attracting locus Y_α is endowed with the natural action of the (automatically parabolic) subgroup $P_\alpha \subset G$ of elements $p \in G$ for which the limit of $\lambda_\alpha(t)p\lambda_\alpha(t)^{-1}$ in G as $t \to 0$ exists.

- Each KN-stratum S_α should satisfy one further property: namely, the natural action map $G \times Y_\alpha \to X$ should induce an isomorphism $S_\alpha \simeq G \overset{P_\alpha}{\times} Y_\alpha$. In the context of GIT the KN-stratification usually comes as follows: the centra $Z_\alpha \subset X^{\lambda_\alpha(\mathbb{G}_m)}$ are the semistable loci of the action of $L'_\alpha := L_\alpha/\lambda_\alpha(\mathbb{G}_m)$ on $X^{\lambda_\alpha(\mathbb{G}_m)}$; we note that in this case the L_α-action on Z_α is automatically locally linear and thus Theorem 3.1.4 applies to $\mathcal{Z}_\alpha := [Z_\alpha/L_\alpha]$. We will assume further on that we are in this setting.

A KN-stratification is called *complete* if the categorical quotients $Z_\alpha /\!/ L_\alpha$ and $X^{\mathrm{ss}} /\!/ G$ are projective. We will call it *locally linear* if the actions of L_α on Z_α and the action of G on X^{ss}) are locally.

Let $\mathcal{X} := [X/G]$, $\mathcal{X}^{\mathrm{ss}} := [X^{\mathrm{ss}}/G]$, $\mathcal{S}_i := [S_\alpha/G] \simeq [Y_i/P_i]$ and $\mathcal{Z}_\alpha := [Z_\alpha/L_i]$. Find a total ordering[21] on I such that for $X_{\le\alpha} := X \setminus \cup_{\alpha'>\alpha} S_{\alpha'}$ the embedding $i_\alpha\colon S_\alpha \to X_{\le\alpha}$ is closed. By the description of Θ-strata in a quotient stack ([HL18, Theorem 1.37]) applied to $\mathcal{X}_{\le\alpha}$ one can see that $\mathcal{S}_\alpha$ is naturally a Θ-stratum and that $\mathcal{Z}_\alpha$ is its centrum. Then, given the KN-stratification is complete and locally linear, by Theorem 3.1.4 we get that the stacks $\mathcal{Z}_\alpha$ and $\mathcal{X}^{ss}$ are cohomologically properly spreadable. Thus by Corollary 3.3.5 $\mathcal{X}$ is also cohomologically properly spreadable and the Hodge-de Rham spectral sequence degenerates for it.

Note that the same proof works if the categorical quotients $Z_\alpha /\!/ L_\alpha$ and $X^{\mathrm{ss}} /\!/ G$ are proper but not necessarily projective.

Remark 3.3.7. In [HLP15] the non-commutative Hodge-to-de Rham degeneration was proved for a slightly more general definition of a KN-complete stratification: namely one does not need to assume that the L_i-action on Z_i is locally linear, only that there exists a good quotient $q\colon [Z_i/L_i] \to Z_i /\!/ L_i$. In this case the strata are not necessarily covered by Theorem 3.1.4 (and the above strategy) but we still hope that they could be cohomologically properly spreadable. More generally, it would be interesting to answer the following question:

Question 3.3.8. Let $q\colon \mathcal{Y} \to Y$ be a good moduli space (e.g. see the Definition in Section 1.2 of [Alp13]) and assume that Y is a proper algebraic space. Is it true that $\mathcal{Y}$ is cohomologically properly spreadable?

The notion of a good moduli space does not spread out well unless the stabilizers are *nice*, i.e. extension of a finite group by a torus. Thus we think it would be very interesting to understand if the property of having a good proper moduli space in characteristic 0 implies any cohomological properness for its mixed characteristic spreadings (as it happens in the case of BG for a reductive G).

Motivated by Questions 1.3.2 and 1.3.3 of [HLP19], one can also ask the following:

Question 3.3.9. Let $\mathcal{X}$ be a formally proper stack (in the sense of Definition 1.1.3 of [HLP19]) over F. Is $\mathcal{X}$ cohomologically properly spreadable?

A Computation of $H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ over $\mathbb{Z}$

In this section we compute cohomology (with coefficients in the structure sheaf) of the classifying stack $B\mathbb{G}_a$ of the additive group $\mathbb{G}_a$ over the ring of integers $\mathbb{Z}$. Unfortunately, were not able to locate this result in the literature, so we do the computation here based on the Jantzen's computation of cohomology of $B\mathbb{G}_{a,\mathbb{F}_p}$. This result is included for completeness only and will not be used anywhere else in the paper.

We start by constructing sufficiently many elements in the first few cohomology groups of $\mathcal{O}_{B\mathbb{G}_a}$, the rest of the computation will then unravel from there. We consider the action of $\mathbb{G}_m$ on $\mathbb{G}_a$ with $t \in \mathbb{G}_m$ acting on the variable x in $\mathcal{O}(\mathbb{G}_a) \simeq \mathbb{Z}[x]$ by $t\colon x \mapsto t^2x$ (note the square in the formula). This makes $H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ into a $\mathbb{G}_m$-representation[22], thus providing an extra $\mathbb{Z}$-grading which we denote by $H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})_*$ using the lower indexing.

The cohomology $H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ is the same as the cohomology of $\mathbb{G}_a$ with coefficients in the trivial module $\mathbb{Z}$. We can compute it via the standard complex $C^\bullet(\mathbb{G}_a, \mathbb{Z}) := \mathbb{Z}[\mathbb{G}_a^\bullet]$ (see [Jan07, Section 4.14]):

$$0 \to \mathbb{Z} \xrightarrow{d_0} \mathbb{Z}[x] \xrightarrow{d_1} \mathbb{Z}[y,z] \xrightarrow{d_2} \dots.$$

[21] It is not hard to see that such a total ordering exists for any stratification. In the GIT "projective-over-affine" case it usually comes via the values of the Hilbert-Mumford potential. Examples of more general potentials which apply to other situations can be found in [HL18, Section 4].

[22] More precisely we should take the corresponding semidirect product $\mathbb{G}_a \rtimes \mathbb{G}_m$ with the projection $p\colon \mathbb{G}_a \rtimes \mathbb{G}_m \to \mathbb{G}_m$ and then consider the direct image $p_*\mathcal{O}_{B(\mathbb{G}_a \rtimes \mathbb{G}_m)} \in \mathrm{QCoh}(B\mathbb{G}_m)$; by base change its fiber over the point $\mathrm{Spec}\,\mathbb{Z} \to B\mathbb{G}_m$ is given by $R\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$.

The action of $\mathbb{G}_m$ extends to $C^\bullet(\mathbb{G}_a,\mathbb{Z})$ giving a $\mathbb{Z}$-grading which on each term $\mathbb{Z}[\mathbb{G}_a^n]\simeq \mathbb{Z}[x_1,\ldots,x_n]$ is given by the doubled degree of a polynomial. This splits $C^\bullet(\mathbb{G}_a,\mathbb{Z})$ as a direct sum of graded components $C^\bullet(\mathbb{G}_a,\mathbb{Z})_n$ with $i\geq 0$. Note that all non-zero components have even weight.

The 0-th component $C^\bullet(\mathbb{G}_a,\mathbb{Z})_0$ has $\mathbb{Z}$ in every component and is just the complex associated to the constant simplicial set $\mathbb{Z}$; thus $C^\bullet(\mathbb{G}_a,\mathbb{Z})_0\simeq \mathbb{Z}[0]$. The second graded component $C^\bullet(\mathbb{G}_a,\mathbb{Z})_2$ looks as

$$0\to 0\to \mathbb{Z}\cdot x\to \mathbb{Z}\cdot x\oplus \mathbb{Z}\cdot y\to \ldots$$

and is the complex associated to the simplicial interval Δ_1 (or rather the corresponding free abelian group) shifted by 1; thus $C^\bullet(\mathbb{G}_a,\mathbb{Z})_2\simeq \mathbb{Z}[-1]$. Let v_1 be the corresponding generator of $H^1(\mathbb{G}_a,\mathbb{Z})_2$.

For any $n\in\mathbb{N}$ consider $\Phi_n(y,z):=d_1(x^n)=(y-z)^n-y^n+z^n$ as an element of $C^2(\mathbb{G}_a,\mathbb{Z})_{2n}$. Note that x^n is the generator of $C^1(\mathbb{G}_a,\mathbb{Z})_{2n}$ and thus, first, $H^1(\mathbb{G}_a,\mathbb{Z})_{2n}=0$ unless $n=1$ and, second, $\Phi_n(y,z)$ generates the group of coboundaries $B^2(\mathbb{G}_a,\mathbb{Z})_{2n}\subset C^2(\mathbb{G}_a,\mathbb{Z})_{2n}$ over $\mathbb{Z}$. In particular $H^1(\mathbb{G}_a,\mathbb{Z})\simeq \mathbb{Z}v_1$. Note also that since $p|\binom{p^i}{k}$ if $0<k<p^i$, $\Phi_{p^i}(y,z)$ is divisible by p for any $i\geq 1$; moreover $d_2\left(\frac{\Phi_{p^i}(x,y)}{p}\right)=0$ since $d_2(\Phi_{p^i}(x,y))=0$ and all terms in the complex are free $\mathbb{Z}$-modules. Thus for any prime p and $i>0$ we get a class $v_{p_i}:=\left[\frac{\Phi_{p^i}(y,z)}{p}\right]\in H^2(\mathbb{G}_a,\mathbb{Z})_{p^i}$ such that $p\cdot v_{p_i}=0$. This way, we get a map

$$\chi\colon (\mathbb{Z}\oplus\mathbb{Z}v_1)\otimes_{\mathbb{Z}}\mathrm{Sym}^*_{\mathbb{Z}}\left(\bigoplus_p \mathbb{F}_p v_p\oplus \mathbb{F}_p v_{p^2}\oplus\ldots\right)\to H^*(\mathbb{G}_a,\mathbb{Z}) \tag{8}$$

which is an isomorphism on H^1 and an injection on H^2. In the context of Hodge-properness we see that this is already very bad: for any p the p-torsion in $H^2(B\mathbb{G}_a,\mathcal{O}_{B\mathbb{G}_a})$ is infinitely generated.

To compute $H^*(\mathbb{G}_a,\mathbb{Z})$ fully we will need a description of the cohomology of $\mathbb{G}_a$ over $\mathbb{F}_p$ (see e.g. [Jan07, Lemma 4.22 and Proposition 4.27]). The first cohomology $H^1((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p)$ is a span $\mathbb{F}_pw_1\oplus\mathbb{F}_pw_p\oplus\mathbb{F}_pw_{p^2}\oplus\ldots$ of classes $w_1,w_p,w_p^2\ldots$ with the $\mathbb{G}_m$-weight of each w_{p^i} given by $2p^i$. Moreover since by the universal coefficient formula the reduction map $H^1(\mathbb{G}_a,\mathbb{Z})/p\to H^1((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p)$ is an injection and preserves the $\mathbb{G}_m$-weights we get that the class w_1 equals to the reduction $\overline{v}_1$ of v_1 (up to a scalar) for any p. From the computation in [Jan07, second paragraph on p.60] it also follows the reductions $\overline{v}_{p^i}\in H^2((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p)_{2p^i}$ are non-zero (namely in the notations of [Jan07] $\overline{v}_{p^i}$ is equal to $\beta(x^{p^i})$ up to a sign change in the second variable) and linearly independent (since they have different $\mathbb{G}_m$-weights).

Lemma A.1. *For any prime p the p-primary part of $H^*(\mathbb{G}_a,\mathbb{Z})$ is elementary, i.e. it is killed by p.*

Proof. The statement will follow from the computation of the Bockstein differential

$$\beta_p\colon H^*((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p)\to H^{*+1}((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p).$$

Namely we will use the fact that if we have a class $[c]\in H^*(\mathbb{G}_a,\mathbb{Z})$ then its reduction $[\overline{c}]\in H^*((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p)$ is killed by β_p and that if $[\overline{c}]\in\operatorname{Im}\beta_p$ then $p\cdot[c]=0$ (in other words if the class of $[\overline{c}]$ in the cohomology with respect to Bockstein is 0 then $[c]$ is killed by p).

There are two cases. If $p=2$ there is an isomorphism

$$H^*((\mathbb{G}_a)_{\mathbb{F}_2},\mathbb{F}_2)\simeq \mathrm{Sym}^*_{\mathbb{F}_2}\left(H^1((\mathbb{G}_a)_{\mathbb{F}_2},\mathbb{F}_2)\right)\simeq \mathbb{F}_2[w_1,w_2,w_4,\ldots].$$

Consequently, all $\mathbb{G}_m$-weights in $H^2((\mathbb{G}_a)_{\mathbb{F}_2},\mathbb{F}_2)$ are given as sums 2^i+2^j for $i,j\geq 0$. Since the reduction $\overline{v}_{2^i}\in H^2((\mathbb{G}_a)_{\mathbb{F}_2},\mathbb{F}_2)$ is non-zero and its weight is 2^{i+1} the only option for it is $w_{2^{i-1}}^2$. Also, by the universal coefficient formula the class w_{2^i}, $i>0$ should come as the only non-zero element of $\mathrm{Tor}_1(H^2(\mathbb{G}_a,\mathbb{Z}),\mathbb{F}_2)$ of weight 2^{i+1}. The latter group is equal to the 2-torsion in $H^2(\mathbb{G}_a,\mathbb{Z})$ and contains v_{2^i} which has the correct weight. From the properties of the Bockstein operator it follows that $\beta_2(w_{2^i})$ is equal to $\overline{v}_{2^i}=w_{2^{i-1}}^2$ if $i>0$. Also $\beta_2(w_1)=0$ (since $w_1=\overline{v}_1$) and since β_2 is a differentiation this, together with the above, defines it uniquely[23]. Note that β_2 is $\mathbb{F}_2[w_1,w_2^2,w_4^2,\ldots]$-linear and that $\mathbb{F}_2[w_1,w_2,w_4,\ldots]$ is free over $\mathbb{F}_2[w_1,w_2^2,w_4^2,\ldots]$ with basis given by $w_{2^I}:=w_{2^{i_1}}w_{2^{i_2}}\ldots w_{2^{i_k}}$ where $I=\{i_1,\ldots,i_k\}$ runs over finite subsets of $\mathbb{Z}_{>0}$. We turn $(\mathbb{F}_2[w_1,w_2,w_4,\ldots],\beta_2)$ into a complex of $\mathbb{F}_2[w_1,w_2^2,w_4^2,\ldots]$-modules be defining another (homological) grading, namely putting $\deg_* w_{2^I}$ to be equal to $|I|$. In fact this way we can identify $(\mathbb{F}_2[w_1,w_2,w_4,\ldots]_*,\beta_2)$ with the Koszul

[23] In terms of the identification $H^*((\mathbb{G}_a)_{\mathbb{F}_2},\mathbb{F}_2)\simeq\mathbb{F}_2[w_1,w_2,w_4,\ldots]$, β_2 acts as the well-defined vector field $\sum_{i=1}^\infty w_{2^{i-1}}^2\frac{\partial}{\partial w_{2^i}}$.

complex $\mathrm{Kos}_{\mathbb{F}_2[w_1,w_2^2,w_4^2,\ldots]}(w_1^2, w_2^2, w_4^2, \ldots)_*$ for the infinite sequence $(w_1^2, w_2^2, w_4^2, \ldots)$;[24] indeed, one can map the k-th term $\mathrm{Kos}_{\mathbb{F}_2[w_1,w_2^2,w_4^2,\ldots]}(w_1^2, w_2^2, w_4^2, \ldots)_k \simeq \Lambda^k_{\mathbb{F}_2[w_1,w_2^2,w_4^2,\ldots]}(\mathbb{F}_2[w_1, w_2^2, w_4^2, \ldots]^{\oplus\infty})$ with the basis $(e_1, e_2, e_3, \ldots)$ of $\mathbb{F}_2[w_1, w_2^2, w_4^2, \ldots]^{\oplus\infty}$, associated to the elements $(w_1^2, w_2^2, w_4^2, \ldots)$, to the k-th graded component of $\mathbb{F}_2[w_1, w_2, w_4, \ldots]$ by sending $e_I \coloneqq e_{i_1} \wedge \ldots \wedge e_{i_k}$ to w_{2^I}. It is easy to see that the Koszul differential goes exactly to β_2. Since the sequence $(w_1^2, w_2^2, w_4^2, \ldots)$ is regular we get that the cohomology of $(H^2((\mathbb{G}_a)_{\mathbb{F}_2}, \mathbb{F}_2), \beta_2)$ is given by

$$\mathbb{F}_2[w_1, w_2^2, w_4^2, \ldots]/(w_1^2, w_2^2, w_4^2, \ldots) \simeq \mathbb{F}_2[w_1]/w_1^2$$

and is spanned by 1 and w_1 over $\mathbb{F}_2$. But we know that $w_1 = \overline{v}_1$ (and of course 1) is a reduction of a non-torsion class v_1 (resp. 1). Thus we get the statement.

If p is odd then

$$H^*((\mathbb{G}_a)_{\mathbb{F}_p}, \mathbb{F}_p) \simeq \Lambda^*_{\mathbb{F}_p}(w_1, w_p, w_{p^2}, \ldots) \otimes_{\mathbb{F}_p} \mathrm{Sym}^*_{\mathbb{F}_p}(\overline{v}_p, \overline{v}_{p^2}, \overline{v}_{p^3}, \ldots),$$

By a similar reasoning to the $p = 2$ case we get that $\beta_p(w_{p^i}) = \overline{v}_{p^i}$ (at least up to a scalar) for $i > 0$ and that $\beta_p(w_1) = 0$. Similarly, β_p is $\mathbb{F}_p[v_p, v_{p^2}, v_{p^3}, \ldots]$-linear and $H^*((\mathbb{G}_a)_{\mathbb{F}_p}, \mathbb{F}_p)$ is a free module over $\mathbb{F}_p[v_p, v_{p^2}, v_{p^3}, \ldots]$ with the basis given by $w_{p^I} \coloneqq w_{p^{i_1}} \wedge \ldots \wedge w_{p^{i_k}}$ where $I = \{i_1, \ldots, i_k\}$ with $i_1 < \ldots i_k$ runs over finite subsets of $\mathbb{Z}_{\geq 0}$. Defining a new (homological) grading on $(H^*((\mathbb{G}_a)_{\mathbb{F}_p}, \mathbb{F}_p)$ by putting $\deg_* w_{p^I} = |I|$ and $\deg_* v_i = 0$ we view $(H((\mathbb{G}_a)_{\mathbb{F}_p}, \mathbb{F}_p)_*, \beta_p)$ as a complex of $\mathrm{Sym}^*_{\mathbb{F}_p}[\overline{v}_p, \overline{v}_{p^2}, \overline{v}_{p^3}, \ldots]$-algebras, which in fact is identified with the product (now as dg-algebras)

$$\Lambda^*_{\mathbb{F}_p}[w_1] \otimes_{\mathbb{F}_p} \mathrm{Kos}_{\mathbb{F}_p[\overline{v}_p,\overline{v}_{p^2},\overline{v}_{p^3},\ldots]}(\overline{v}_p, \overline{v}_{p^2}, \overline{v}_{p^3}, \ldots) \simeq H^*((\mathbb{G}_a)_{\mathbb{F}_p}, \mathbb{F}_p)$$

where the differential on w_1 is 0. Indeed one can define a map by sending each generator $e_I \in \mathrm{Kos}_{|I|}$ to w_{p^I} and we leave it to the reader to check that it is an isomorphism. Since $\overline{v}_p, \overline{v}_{p^2}, \overline{v}_{p^3}, \ldots \in \mathbb{F}_p[\overline{v}_p, \overline{v}_{p^2}, \overline{v}_{p^3}, \ldots]$ is a regular sequence in the case of an odd p we also get that the cohomology of β_p is spanned by 1 and w_1 over $\mathbb{F}_p$, and they come as reductions of non-torsion classes. This finishes the proof. □

We finish the description of $H^*(\mathbb{G}_a, \mathbb{Z})$. By Lemma A.1 the p-primary part of $H^*(\mathbb{G}_a, \mathbb{Z})$ (as a non-unital algebra) is killed by p and thus can be described as $\mathrm{Im}\,\beta_p \subset H^*((\mathbb{G}_a)_{\mathbb{F}_p}, \mathbb{F}_p)$ via the reduction map. More explicitly $\mathrm{Im}\,\beta_p$ is freely generated by elements $\beta_p(w_I)$ as a module over $\mathbb{F}_p[\overline{v}_p, \overline{v}_{p^2}, \overline{v}_{p^3}, \ldots]$ in the notations of Lemma A.1. The elements $\beta_p(w_I)$ are not algebraically independent over $\mathbb{F}_p[\overline{v}_p, \overline{v}_{p^2}, \overline{v}_{p^3}, \ldots]$ and it seems hard to describe all the relations between them; but still this description is somewhat nice, since there is only finite number of $\beta_p(w_I)$ of a given cohomological degree. To finish the computation over $\mathbb{Z}$ it only remains to see what happens with the powers of v_1. Since v_1 is of cohomological degree 1, v_1^2 is 2-torsion and we saw in the course of proof of Lemma A.1 that in fact $v_1^2 = v_2$. All in all this gives the following description of $H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$:

Proposition A.2. *We have*

$$H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a}) \simeq \left(\mathbb{Z}[v_1] \oplus \left(\bigoplus_p \mathrm{Im}\,\beta_p\right)\right) \Big/ v_1^2 = v_2.$$

Also, returning to the map χ (see Equation (8)), we get a subalgebra

$$A = \left(\mathbb{Z}[v_1] \otimes_{\mathbb{Z}} \mathrm{Sym}^*_{\mathbb{Z}}\left(\bigoplus_p \mathbb{F}_p v_p \oplus \mathbb{F}_p v_{p^2} \oplus \ldots\right)\right) \Big/ v_1^2 = v_2 \subset H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a}),$$

and the algebra $H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ is generated by 1 and $\beta_p(w_I)$ (for various p and I) as an A-module. More precisely one can check that we have a direct sum decomposition

$$H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a}) \simeq A \oplus \bigoplus_{p,I} A \cdot \beta_p(w_I),$$

where for each $\beta_p(w_I)$ the submodule $A \cdot \beta_p(w_I)$ is just isomorphic to (non-derived) quotient A/p.

[24] We warn the reader that this is only an isomorphism $\mathbb{F}_2[w_1, w_2^2, w_4^2, \ldots]$-dg-modules and not dg-algebras.

Remark A.3. Quite remarkably the cohomology $H^*(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ turns out to be directly related to the cohomology of the Eilenberg-Maclane space $K(\mathbb{Z},3)$[25]: namely there is an isomorphism

$$H^n_{\mathrm{sing}}(K(\mathbb{Z},3),\mathbb{Z}) \simeq \bigoplus_{i=0}^{n} H^i(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})_{n-2i},$$

which in fact extends to the isomorphism of the graded algebras

$$\bigoplus_{n\geq 0} H^n_{\mathrm{sing}}(K(\mathbb{Z},3),\mathbb{Z}) \simeq \bigoplus_{n\geq 0} \left(\bigoplus_{i=0}^{n} H^i(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})_{n-2i} \right).$$

We comment more on this. Indeed, $K(\mathbb{Z},3) \simeq B(K(\mathbb{Z},2)) \simeq B\mathbb{CP}^\infty$. Realizing $K(\mathbb{Z},3)$ as the colimit of the simplicial diagram

$$\mathrm{colim}\left(\ldots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \rightrightarrows \mathbb{CP}^\infty \longrightarrow * \right) \xrightarrow{\sim} K(\mathbb{Z},3)$$

we get a spectral sequence

$$E_1^{n,q} = H^q_{\mathrm{sing}}((\mathbb{CP}^\infty)^n, \mathbb{Z}) \Rightarrow H^{n+q}_{\mathrm{sing}}(K(\mathbb{Z},3),\mathbb{Z}).$$

The cohomology $H^*_{\mathrm{sing}}(\mathbb{CP}^\infty,\mathbb{Z}) \simeq \mathbb{Z}[x]$, $\deg x = 2$ has a natural Hopf algebra structure with comultiplication induced by the addition $m\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. It is easy to see that $m^*(x) = x\otimes 1 + 1\otimes x$ and so the corresponding affine group scheme is $\mathbb{G}_a$; moreover, the cohomological grading corresponds exactly to the $\mathbb{G}_m$-action on $\mathbb{G}_a$ which we considered before. Via this identification and the Künneth formula, the first page $E_1^{n,q}$ of the spectral sequence above is identified the standard complex $C^\bullet(\mathbb{G}_a,\mathbb{Z})$:

$$0 \to \mathbb{Z} \xrightarrow{d_0} \mathbb{Z}[x] \xrightarrow{d_1} \mathbb{Z}[x]\otimes_{\mathbb{Z}} \mathbb{Z}[x] \xrightarrow{d_2} \ldots.$$

Thus we also know the second page, namely we have $E_2^{n,q} = H^n(\mathbb{G}_a,\mathbb{Z})_q$. Note that all odd rows are automatically zero.

We claim that the spectral sequence degenerates at the second page. Since all terms $E_2^{n,q}$ are finitely generated over $\mathbb{Z}$ and by Lemma A.1 all torsion they have is elementary, it is enough to check that all the differentials on the second page are zero modulo all primes p. Consider the analogous spectral sequence $E_{1,p}^{n,q} = H^q_{\mathrm{sing}}((\mathbb{CP}^\infty)^n, \mathbb{F}_p) \Rightarrow H^{n+q}_{\mathrm{sing}}(K(\mathbb{Z},3),\mathbb{F}_p)$ for $\mathbb{F}_p$-cohomology; similarly to the above its second page looks as $E_{2,p}^{n,q} = H^n((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p)_{2q}$ and as we have seen the reduction map $E_2^{n,q}/p \to E_{2,p}^{n,q}$ is an embedding. Thus it will be enough to show that $E_{*,p}^{*,*}$ degenerates at the second page for any p. All differentials in $E_{*,p}^{*,*}$ are generated by maps between some $\mathbb{F}_p$-cohomology of some spaces and thus commute with the action of the Steenrod algebra $\mathcal{A}_p$. By [Jan07, Proposition in 4.27] the algebra generators w_{p^i} in fact are related by the Frobenius $F_{\mathbb{G}_a}\colon \mathbb{G}_a \to \mathbb{G}_a$, namely $w_{p^i} = F^*_{\mathbb{G}_a} w_{p^{i-1}}$. From the topological point of view, if $x \in E_{1,p}^{1,2} \simeq H^2(\mathbb{CP}^\infty,\mathbb{F}_p)$ is a generator whose class in $E_{2,p}^{1,2} = H^2((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p)_2$ is equal to $w_1 \in E_{2,p}^{1,2}$, then such w_{p^i} comes as a generator $x^{p^i} \in H^{2p^i}(\mathbb{CP}^\infty,\mathbb{F}_p)$ and is expressed more functorially as $P^iP^{i-1}\ldots P^1 w_1$, where P^i denotes i-th Steenrod power operation. Also recall that $\overline{v}_{p^i} = \beta_p(w_{p^i})$. Since w_{p^i} and v_{p^i} together generate $E_{*,p}^{*,*}$ and are obtained from w_1 by applying cohomological operations it is enough to show that $d_{n,p}(w_1) = 0$ for any n. This is obvious for $n > 2$ and for $n =$ we have $d_{2,p}(w_1) = 0$ since $d_{2,p}(w_1) \in E_{2,p}^{3,1} = H^3((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p)_1 = 0$.

Even though the degeneration of the spectral sequence a priori only gives the description of a certain associated graded of $H^*_{\mathrm{sing}}(K(\mathbb{Z},3),\mathbb{Z})$, we claim that there also exists a natural isomorphism of the latter with E_2^{*+*}. Indeed, let c be the generator of $H^3(K(\mathbb{Z},3),\mathbb{Z}) \simeq \mathbb{Z}$ which goes to $v_1 \in E_2^{1,2} \simeq H^1(\mathbb{G}_a,\mathbb{Z})_2$ under the natural map and consider its reduction $\overline{c}$ which generates $H^3(K(\mathbb{Z},3),\mathbb{F}_p) \simeq \mathbb{F}_p$. Then putting $c_{p^i} := P^iP^{i-1}\ldots P^1\overline{c}$, and $d_{p^i} = \beta_p(c_{p^i})$ in the case of odd p, we get an isomorphism

$$\mathbb{F}_2[\overline{c}, c_2, c_4, \ldots] \simeq H^*(K(\mathbb{Z},3),\mathbb{F}_2)$$

and

$$\Lambda^*_{\mathbb{F}_p}(\overline{c}, c_p, c_{p^2}, \ldots) \otimes_{\mathbb{F}_p} \mathrm{Sym}^*_{\mathbb{F}_p}(d_p, d_{p^2}, d_{p^3}, \ldots) \simeq H^*(K(\mathbb{Z},3),\mathbb{F}_p)$$

for p odd. Moreover, the Bockstein β_p is acting on $H^*(K(\mathbb{Z},3),\mathbb{F}_p)$ by analogous formulas analogous to the ones we had in the course of the proof of Lemma A.1 and by the same argument it follows that the p-primary

[25]Recall that $K(\mathbb{Z},n)$ for $n \geq 1$ is the unique (up to homotopy) space such that $\pi_n(K(\mathbb{Z},n)) = \mathbb{Z}$ and $\pi_i(K(\mathbb{Z},n)) = 0$ for $i \neq n$.

part in $H^*_{\mathrm{sing}}(K(\mathbb{Z},3),\mathbb{F}_p)$ is killed by p. Sending c to v_1, c_{p^i} to w_{p^i} and d_{p^i} to v_{p^i} defines an isomorphism between $H^*(K(\mathbb{Z},3),\mathbb{F}_p)$ and $H^*((\mathbb{G}_a)_{\mathbb{F}_p},\mathbb{F}_p)$, which, moreover, respects Bocksteins on both sides. Finally, describing $H^*(K(\mathbb{Z},3),\mathbb{Z})$ in terms of $\mathrm{Im}\,\beta_p$ for various p and the class c as in Proposition A.2 this extends to the isomorphism of graded algebras

$$\bigoplus_{n\geq 0} H^n_{\mathrm{sing}}(K(\mathbb{Z},3),\mathbb{Z}) \simeq \bigoplus_{n\geq 0}\left(\bigoplus_{i=0}^{n} H^i(B\mathbb{G}_a,\mathcal{O}_{B\mathbb{G}_a})_{n-2i}\right)$$

as we claimed.

References

[Alp13] J. Alper, *Good moduli spaces for Artin stacks*, Annales de l'Institut Fourier **63** (2013), no. 6, 2349-2402.

[AHLH18] J. Alper, D. Halpern-Leistner, and J. Heinloth, *Existence of moduli spaces for algebraic stacks.* (2018), available at https://arxiv.org/abs/1812.01128.

[ABM19] B. Antieau, B. Bhatt, and A. Mathew, *Counterexamples to Hochschild–Kostant–Rosenberg in characteristic p* (2019), available at https://arxiv.org/pdf/1909.11437.pdf.

[BO78] P. Berthelot and A. Ogus, *Notes on crystalline cohomology*, Mathematical Notes, Princeton Univ. Pr., 1978.

[Bha12a] B. Bhatt, *Completions and derived de Rham cohomology* (2012), available at https://arxiv.org/abs/1207.6193.

[Bha12b] ———, *p-adic derived de Rham cohomology* (2012), available at https://arxiv.org/abs/1204.6560.

[BdJ11] B. Bhatt and A. de Jong, *Crystalline cohomology and de Rham cohomology* (2011), available at https://arxiv.org/abs/1110.5001.

[BMS19] B. Bhatt, M. Morrow, and P. Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math. de l'IHES Sci. **129** (2019).

[BB73] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Annals of mathematics (1973), 480–497.

[BM19] L. Brantner and A. Mathew, *Deformation Theory and Partition Lie Algebras* (2019), available at https://arxiv.org/abs/1904.07352.

[Del74] P. Deligne, *Théorie de Hodge: III*, Publications Mathématiques de l'IHÉS **44** (1974), 5-77.

[DI87] P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), 247-270.

[Dri13] V. Drinfeld, *On algebraic spaces with an action of $\mathbb{G}_m$* (2013), available at https://arxiv.org/abs/1308.2604.

[FM87] J. M. Fontaine and W. Messing, *p-adic periods and p-adic étale cohomology*, Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Contemporary math. (67), Amer. Math. Soc., Providence, RI, (1987).

[Fra] V. Franjou, *Cohomologie de de Rham entiere (Integral de Rham cohomology)*, available at https://arxiv.org/abs/math/0404123.

[FvdK10] V. Franjou and W. van der Kallen, *Power reductivity over an arbitrary base*, Doc.Math. Extra vol: Suslin (2010), 171-195.

[GR17] D. Gaitsgory and N. Rozenblyum, *A Study in Derived Algebraic Geometry, Volume I: Correspondences and Duality*, Mathematical Surveys and Monographs, American Mathematical Society, 2017.

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique III. Étude cohomologique des faisceaux cohérents, Première partie*, Publ. Math. de l'Inst. des Hautes Études Sci. **11** (1961).

[Gro66] ———, *Éléments de géométrie algébrique IV. Étude locale des schémas et des morphismes de schémas. Troisième partie*, Publ. Math. de l'Inst. des Hautes Études Sci. **28** (1966).

[Gro67] ———, *Éléments de géométrie algébrique IV. Étude locale des schémas et des morphismes de schémas. Quatrième partie*, Publ. Math. de l'Inst. des Hautes Études Sci. **32** (1967).

[Hal18] J. Hall, *GAGA Theorems* (2018), available at https://arxiv.org/pdf/1804.01976.

[HL15] D. Halpern-Leistner, *The derived category of a GIT quotient*, Journal of the American Mathematical Society **28 (3)** (2015), 871-912.

[HL18] ———, *On the structure of instability in moduli theory* (2018), available at https://arxiv.org/abs/1411.0627.

[HL20] ———, *Derived Θ-stratifications and the D-equivalence conjecture* (2020), available at https://arxiv.org/abs/2010.01127.

[HLP15] D. Halpern-Leistner and D. Pomerleano, *Equivariant Hodge theory and noncommutative geometry* (2015), available at https://arxiv.org/abs/1507.01924.

[HLP19] D. Halpern-Leistner and A. Preygel, *Mapping stacks and categorical notions of properness* (2019), available at https://arxiv.org/abs/1402.3204.

[Ill71] L. Illusie, *Complexe cotangent et déformations. I*, Lecture notes in mathematics, vol. 223, Springer-Verlag, 1971.

[Ill72] ———, *Complexe cotangent et déformations. II*, Lecture notes in mathematics, vol. 283, Springer-Verlag, 1972.

[Jan07] J. C. Jantzen, *Representations of Algebraic Groups*, Mathematical surveys and monographs, American Mathematical Society, 2007.

[Kal08] D. Kaledin, *Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie*, Pure Appl. Math. Quot. **4** (2008).

[Kal17] ———, *Spectral sequences for cyclic homology*, Algebra, Geometry, and Physics in the 21st Century **324** (2017).

[KT16] D. Kubrak and R. Travkin, *Resolutions with conical slices and descent for the Brauer group classes of certain central reductions of differential operators in characteristic p* (2016).

[LMB00] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Vol. 39, Springer, 2000.

[Lur09] J. Lurie, *Higher Topos Theory*, Princeton Univ. Pr. (2009).

[Lur17] ———, *Higher Algebra* (2017), available at http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[Lur18] ———, *Spectral Algebraic Geometry* (2018), available at https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[MFK94] D. Mumford, J. Fogarty, and F. C. Kirwan, *Geometric invariant theory*, 3rd enl. ed, Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge 34, Springer-Verlag, 1994.

[Ols05] M. C. Olsson, *On proper coverings of Artin stacks*, Advances in Mathematics **198** (2005), 93-106.

[Poo17] B. Poonen, *Rational Points on Varieties*, Graduate studies in mathematics, Amer. Mathematical Society, 2017.

[PY14] M. Porta and T. Y. Yu, *Higher analytic stacks and GAGA theorems*, Advances in Mathematics **302** (2014).

[Pre11] A. Preygel, *Thom-Sebastiani and Duality for Matrix Factorizations* (2011), available at https://arxiv.org/abs/1101.5834.

[Pri15] J. P. Pridham, *Presenting higher stacks as simplicial schemes*, Advances in Mathematics **238** (2015), 184-245.

[Ryd15] D. Rydh, *Noetherian approximation of algebraic spaces and stacks*, Journal of Algebra **422** (2015), 105–147.

[Roz12] N. Rozenblyum, *Filtered colimits of* ∞*-categories* (2012), available at http://www.math.harvard.edu/~gaitsgde/GL/colimits.pdf.

[Sat12] M. Satriano, *de Rham Theory for Tame Stacks and Schemes with Linearly Reductive Singularities*, Ann. de l'Inst. Fourier **62 (6)** (2012).

[Ser75] J. P. Serre, *Groupes algébriques et corps de classes*, Actualités scientifiques et industrielles, Hermann, 1975.

[SP20] The Stacks Project Authors, *Stacks Project* (2020), available at https://stacks.math.columbia.edu/.

[Sum74] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14(1)** (1974), 1-28.

[Tel00] C. Teleman, *The quantization conjecture revisited*, Ann. of Math. **152 (2)** (2000).

[Toë11] B. Toën, *Descente fidélement plate pour les n-champs d'Artin*, Compositio Mathematica **147** (2011), no. 5, 1382–1412.

[TV08] B. Toen and G. Vezzosi, *Homotopical Algebraic Geometry II: Geometric Stacks and Applications* (2008).

[Tot18] B. Totaro, *Hodge theory of classifying stacks*, Duke Math. J. **167** (2018).

Dmitry Kubrak, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, United States dmkubrak@mit.edu

Artem Prikhodko, Department of Mathematics, National Research University Higher School of Economics, Moscow; Center for Advanced Studies, Skoltech, Moscow, artem.n.prihodko@gmail.com