



The real cycle class map

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The classical cycle class map for a smooth complex variety sends cycles in the Chow ring to cycles in the singular cohomology ring. We study two cycle class maps for smooth real varieties: the map from the I -cohomology ring to singular cohomology induced by the signature, and a new cycle class map defined on the Chow–Witt ring. For both maps, we establish compatibility with pullbacks, pushforwards and cup products. As a first application of these general results, we show that both cycle class maps are isomorphisms for cellular varieties.

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1. Introduction

Classically, for smooth varieties X over a field F with a complex embedding $\sigma : F \hookrightarrow \mathbb{C}$, there are cycle class maps $\mathrm{CH}^k(X) \rightarrow H^{2k}(X(\mathbb{C}), \mathbb{Z})$ from Chow groups to singular cohomology, and these are compatible with pullbacks, pushforwards and intersection products.

We consider real cycle class maps in the analogous situation for real varieties. Note that integral Chow groups alone are not the right object to look at when studying real varieties. For example, it is known that the rational motivic Eilenberg–Mac Lane spectrum maps to zero under real realization. Rather, the general philosophy due to Morel, Fasel and others is that we should study I -cohomology: sheaf cohomology with coefficients in the sheafified powers I^k of the fundamental ideal of the Witt ring (see Definitions 2.31 and 2.36 below). It is this theory

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that is to singular cohomology of real varieties what Chow groups are to singular cohomology for complex varieties.

Concretely, for varieties X over a field F with a real embedding $\sigma : F \hookrightarrow \mathbb{R}$ (or more generally, an embedding into a real closed field) there are *real cycle class maps* $H^k(X, \mathbf{I}^k) \rightarrow H^k(X(\mathbb{R}), \mathbb{Z})$. These were discussed in [Jacobson 2017], and they lift the more classical maps constructed by Borel–Haefliger and Scheiderer.

The main goal of this article is to study this real cycle class map in detail. In particular, we will show that it is compatible with all the structure we are interested in, notably pushforwards, pullbacks and ring structures. We also study the cycle class map with twisted coefficients $\mathbf{I}^k(\mathcal{L})$, whose target is singular cohomology with coefficients in local coefficient systems arising from real realization of the twisting line bundle.

One important application of the compatibility results for the real cycle class maps is that for a smooth cellular variety X over \mathbb{R} , the real cycle class map induces a ring isomorphism

$$\bigoplus_{k, \mathcal{L}} H^k(X, \mathbf{I}^k(\mathcal{L})) \rightarrow \bigoplus_{n, \mathcal{L}} H_{\text{sing}}^k(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})),$$

where $\mathbb{Z}(\mathcal{L})$ denotes coefficients in the local system associated to the line bundle \mathcal{L} . More generally, the result holds over real closed fields, using semialgebraic cohomology instead of singular cohomology.

Another important goal of the present article is to extend the real cycle class map to a *full cycle class map* on the twisted Chow–Witt groups $\widetilde{\text{CH}}^k(X, \mathcal{L})$ and establish the compatibility of the cycle class maps with pushforwards, pullbacks and intersection products in Chow–Witt theory. The target of this full cycle class map combines both the singular cohomology of $X(\mathbb{R})$ with integral coefficients (possibly twisted by a local system) and the C_2 -equivariant cohomology of $X(\mathbb{C})$, in the same way that Milnor–Witt K-theory combines the sheaves \mathbf{I}^k and \mathbf{K}_k^M , and the full cycle class map refines the recently defined C_2 -equivariant cycle class map of [Benoist and Wittenberg 2020].

The results concerning real realization and real cycle class maps discussed here were motivated by recent computations of Chow–Witt rings and \mathbf{I} -cohomology rings of classifying spaces [Hornbostel and Wendt 2019], Grassmannians [Wendt 2018] and split quadrics [Hornbostel et al. 2020]. In particular, we discuss the compatibility of real realization with the characteristic classes left open both in [Wendt 2018] and in [HW 2019]; see in particular [HW 2019, Remark 9.2].

For most of our considerations we deal only with “geometric” bidegrees, that is, we concentrate on $H^k(X, \mathbf{I}^k)$ or $\widetilde{\text{CH}}^k(X) = H^k(X, \mathbf{K}_k^{\text{MW}})$ and ignore the full bigraded theories (Milnor–Witt motivic cohomology $H^i(X, \widetilde{\mathbb{Z}}(j))$, etc.) that have recently been developed and studied by Fasel and others.

1.1. Remark. The focus of this article is on schemes over \mathbb{R} and realization to classical topological manifolds. However, several results refine to the real-étale site and the real topology; see Sections 2.D and 2.E.2, Remarks 4.2 and 5.10 and the Appendix. To start with, there are evident extensions of the results to the case of real closed fields, using the theory of semialgebraic cohomology as discussed in [Delfs 1991]. It is also possible to go beyond the real closed case for parts of the results. In full generality, one has to replace singular cohomology by real-étale cohomology as in [Scheiderer 1994]. In this setting, it is possible to define cycle class maps $H^i(X, \mathbf{I}^j) \rightarrow H_{\text{rét}}^i(X, \mathbb{Z})$ and prove all the relevant compatibilities. For a formally real field F , the kernel of the natural homomorphism

$$W(F) \rightarrow \prod_P W(F_P) = H_{\text{rét}}^0(F, \mathbb{Z})$$

is equal to the torsion subgroup of $W(F)$, where the product runs over all orderings P of the field F and F_P denotes the real closure for F with the ordering P . In particular, to determine an element up to the torsion, it suffices to consider all the signatures. Consequently, claims concerning the image of an element in real cohomology over a formally real field may then be deduced from the corresponding claims for real closed fields (whose proofs can be obtained from the case \mathbb{R} discussed in the present article). Lacking significant applications for this extension, we won't go into further details here.

1.2. Remark. At least for smooth varieties over \mathbb{R} , taking real points is expected to induce a morphism of spectral sequences from the Gersten–Witt spectral sequence defined in [Balmer and Walter 2002] to the Atiyah–Hirzebruch spectral sequence for topological KO-theory, each potentially twisted by a line bundle in an appropriate sense. On E_2 -terms in degree zero, this morphism is the (twisted) signature map studied here, and on E_∞ -terms it is expected to be (a twisted version of) the “signature map” that appears in Brumfiel’s theorem [Brumfiel 1984; Karoubi et al. 2016]. It thus seems plausible that variants of Brumfiel’s theorem could be obtained by such a comparison of spectral sequences. However, a careful implementation of such arguments requires higher signature maps, which are beyond the scope of this paper. We refer to [Jacobson 2015] for some work in this direction, and for related questions.

Structure of the paper. We provide a review of the various cohomology theories involved in Section 2. Section 3 discusses sheaf-theoretic and geometric constructions of the real cycle class maps and shows that they agree. The compatibility results with pullbacks, pushforwards and intersection products are proved in Section 4. The special case of cellular varieties is discussed in Section 5. The comparison of algebraic and topological characteristic classes is established in Section 6.

2. Preliminaries and notation

All schemes are assumed to be separated of finite type over a field F which has a real embedding $\sigma : F \hookrightarrow \mathbb{R}$, although many results obviously hold in greater generality. In particular, the conditions of being perfect, infinite and of characteristic $\neq 2$ which frequently occur in the literature on \mathbb{A}^1 -homotopy and Chow–Witt theory are automatically satisfied. Occasionally, we might refer to reduced schemes as varieties.

We write $\mathcal{A}b(X)$ for the category of sheaves of abelian groups on a topological space X . Given a continuous map $f : X \rightarrow Y$, we write f_* and f^* for the components of the canonical adjunction $\mathcal{A}b(X) \rightleftarrows \mathcal{A}b(Y)$. (Note that some authors prefer to write f^{-1} for the left adjoint and reserve the notation f^* for \mathcal{O} -modules over ringed spaces.)

2.A. Reminders on sheaf cohomology. The *sheaf cohomology* of a topological space X with coefficients in a sheaf of abelian groups \mathcal{F} on X is given by the right derived functors of the global sections functor $\Gamma : H^i(X, \mathcal{F}) := (R^i \Gamma)(\mathcal{F})$. Thus, sheaf cohomology depends on a pair (X, \mathcal{F}) consisting of a topological space X and a sheaf of abelian groups \mathcal{F} , and we refer to such a pair (X, \mathcal{F}) as *coefficient datum*. We define a *morphism of coefficient data* $(f, \phi) : (X, \mathcal{F}_X) \rightarrow (Y, \mathcal{F}_Y)$ to be a pair consisting of a morphism of spaces $f : X \rightarrow Y$ and a morphism of sheaves $\phi : f_* \mathcal{F}_X \leftarrow \mathcal{F}_Y$. There’s an obvious notion of composition of such morphisms:

$$(g, \gamma) \circ (f, \phi) = (gf, g_* \phi \circ \gamma). \quad (2.1)$$

2.2. Example. A morphism of ringed spaces $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is, in particular, a morphism of coefficient data in the above sense.

2.3. Example. For any morphism of spaces $f : X \rightarrow Y$ and arbitrary sheaves \mathcal{F}_X and \mathcal{F}_Y on X and Y , respectively, we have canonical associated morphisms of coefficient data $(X, \mathcal{F}_X) \rightarrow (Y, f_* \mathcal{F}_X)$ and $(X, f^* \mathcal{F}_Y) \rightarrow (Y, \mathcal{F}_Y)$.

2.4. Remark. In [Bredon 1997], the component ϕ of a morphism of coefficient data (f, ϕ) is referred to as an “ f -cohomomorphism”.

In order to compute sheaf cohomology with coefficients in \mathcal{F} from the definition, one needs to first find an injective resolution of \mathcal{F} . However, one gets the same result using resolutions by flasque or more generally by acyclic sheaves [Hartshorne 1977, Proposition III.1.2A], and this additional flexibility is often useful. The Godement resolution at the center of this section is one particularly well-behaved choice of flasque resolution.

2.A.1. Godement resolutions. We refer to the classical and established sources for the Godement resolution $\mathcal{F} \rightarrow \mathbb{G}\mathcal{F}$ of a sheaf \mathcal{F} : [Godement 1958; Iversen 1986;

[Bredon 1997]. Here, we are mainly interested in a good definition of cup and cross products. Note that a morphism of coefficient data $(f, \phi) : (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$ induces a morphism of resolutions (see [Bredon 1997, II.8] for a direct construction):

$$f_* \mathbb{G}\mathcal{F} \leftarrow \mathbb{G}\mathcal{G}.$$

2.5. Definition. Given a morphism of coefficient data $(f, \phi) : (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$, we define the *pullback in sheaf cohomology*

$$H^*(X, \mathcal{F}) \xleftarrow{(f, \phi)^*} H^*(Y, \mathcal{G})$$

to be the morphism induced by passing to global sections above.

2.A.2. Cohomology with supports. Suppose $i : V \hookrightarrow X$ is a closed embedding. Then the pushforward $i_* : \mathcal{A}b(V) \rightarrow \mathcal{A}b(X)$ has a right adjoint $i^!$ [SGA 4_I 1972, Exposé IV, Proposition 14.5]. Both functors i_* and $i^!$ and the global sections functor Γ on X are left exact, hence so is the composition $\Gamma i_* i^!$.

2.6. Definition. Given coefficient data (X, \mathcal{F}) and a closed embedding $i : V \hookrightarrow X$, as above, we define the *global sections functor with support on V* as the composition $\Gamma_V := \Gamma i_* i^!$, and the *cohomology with support in V* via the right derived functors of Γ_V as $H_V^s(X, \mathcal{F}) := R^s \Gamma_V \mathcal{F}$.

2.7. Lemma. Consider a continuous map $f : X \rightarrow Y$. Suppose $i_V : V \hookrightarrow X$ and $i_W : W \hookrightarrow Y$ are closed subsets such that $V \supset f^{-1}W$. Then we have a canonical natural transformations of functors $(i_W)_* i_W^! f_* \rightarrow f_* (i_V)_* i_V^!$ and, hence, a canonical natural transformation $\Gamma_W f_* \rightarrow \Gamma_V$. If $V = f^{-1}W$, then this natural transformation is an isomorphism. \square

2.8. Definition (pullback with support). Suppose we have a morphism of coefficient data $(f, \phi) : (X, \mathcal{F}) \rightarrow (Y, \mathcal{G})$, and suppose we have closed subsets $V \subset X$ and $W \subset Y$ such that $f^{-1}W \subset V$. Then the *pullback with support along (f, ϕ)* is the morphism

$$H_V^s(X, \mathcal{F}) \xleftarrow{(f, \phi)^*} H_W^s(Y, \mathcal{G})$$

induced by the composition

$$\Gamma_V \mathbb{G}\mathcal{F} \xleftarrow{\text{Lemma 2.7}} \Gamma_W f_* \mathbb{G}\mathcal{F} \leftarrow \Gamma_W \mathbb{G}\mathcal{G}.$$

The first arrow, on the right, is essentially the same morphisms as used in the definition of pullback without support (Definition 2.5).

2.9. Proposition (universality of the pullback). *In the situation of Definition 2.8, suppose we have resolutions $\mathcal{F} \rightarrow \mathcal{R}_{\mathcal{F}}$ and $\mathcal{G} \rightarrow \mathcal{R}_{\mathcal{G}}$ on X and Y , respectively,*

together with a morphism of complexes of sheaves $\hat{\phi} : f_*\mathcal{R}_{\mathcal{F}} \leftarrow \mathcal{R}_{\mathcal{G}}$ such that

$$\begin{array}{ccc} f_*\mathcal{R}_{\mathcal{F}} & \xleftarrow{\hat{\phi}} & \mathcal{R}_{\mathcal{G}} \\ \uparrow & & \uparrow \\ f_*\mathcal{F} & \xleftarrow{\phi} & \mathcal{G} \end{array}$$

commutes. Then also the following square commutes:

$$\begin{array}{ccc} H^*(\Gamma_V \mathcal{R}_{\mathcal{F}}) & \xleftarrow{\quad} & H^*(\Gamma_W \mathcal{R}_{\mathcal{G}}) \\ \downarrow & & \downarrow \\ H_V^*(X, \mathcal{F}) & \xleftarrow{(f, \phi)^*} & H_W^*(Y, \mathcal{G}) \end{array}$$

Here, the vertical arrows are the canonical maps, the upper horizontal arrow is induced by $\hat{\phi}$ and the canonical transformation of Lemma 2.7, and the lower horizontal arrow is the pullback of Definition 2.8. \square

2.10. Proposition (functoriality of pullback). *Sheaf pullback is functorial with respect to morphisms of coefficient data. That is, given two composable morphisms of coefficient data (f, ϕ) and (g, γ) and adequate choices of support,*

$$(f, \phi)^* \circ (g, \gamma)^* = ((g, \gamma) \circ (f, \phi))^*. \quad \square$$

2.11. Proposition (localization sequence). *Let X be a topological space, and let $i : V \hookrightarrow X$ be the inclusion of a closed subspace with open complement $j : U \hookrightarrow X$. With the definition of cohomology with support from Definition 2.6, we have a long exact sequence of cohomology groups as follows:*

$$\cdots \rightarrow H_V^s(X, \mathcal{F}) \rightarrow H^s(X, \mathcal{F}) \rightarrow H^s(U, j^*\mathcal{F}) \rightarrow H_V^{s+1}(X, \mathcal{F}) \rightarrow \cdots$$

The maps preserving cohomological degree are the obvious pullback maps, as defined in Definition 2.8. \square

2.12. Proposition. *Given a short exact sequence of sheaves $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ on X , we have an associated long exact sequence of cohomology groups:*

$$\cdots \rightarrow H_V^s(X, \mathcal{E}) \rightarrow H_V^s(X, \mathcal{F}) \rightarrow H_V^s(X, \mathcal{G}) \xrightarrow{\partial} H_V^{s+1}(X, \mathcal{E}) \rightarrow \cdots$$

The maps preserving cohomological degree are the obvious change-of-coefficient morphisms. Moreover, given a continuous map $f : X \rightarrow Y$ such that $0 \rightarrow f_*\mathcal{F} \rightarrow f_*\mathcal{G} \rightarrow f_*\mathcal{H} \rightarrow 0$ is still exact on Y , and given a closed subset $W \subset Y$ such that $f^{-1}W \subset V$, the pullback along f and the two long exact cohomology sequences fit together to a commutative ladder diagram. \square

Finally, we note that on *Noetherian* spaces cohomology commutes with sequential colimits. Note that all schemes considered in this article are Noetherian, as they are assumed to be of finite type over a field.

2.13. Lemma. *Let X be a Noetherian topological space, $V \subset X$ a closed subset. For any sequence of abelian sheaves $\mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow \cdots$ on X , the canonical maps $\operatorname{colim}_j H_V^s(X, \mathcal{F}_j) \rightarrow H_V^s(X, \operatorname{colim}_j \mathcal{F}_j)$ are isomorphisms. \square*

2.A.3. Godement's cup and cross products. Of course, the above pullbacks could also be constructed using injective resolutions. For the following cup product structure on sheaf cohomology, however, we know of no construction using injective resolutions.

2.14. Theorem (cup product [Godement 1958; Bredon 1997]). *Let X be a topological space or, more generally, a site with a conservative set of points. Let $V, V' \subset X$ be closed subsets. Consider a sheaf of commutative rings \mathcal{A} on X and the ring of global sections over $V \cap V'$, which we abbreviate to $\Gamma' \mathcal{A} := \Gamma_{V \cap V'} \mathcal{A}$. For \mathcal{A} -modules \mathcal{M} and \mathcal{N} , there exists a unique cup product*

$$H_V^p(X, \mathcal{M}) \otimes_{\Gamma' \mathcal{A}} H_{V'}^q(X, \mathcal{N}) \rightarrow H_{V \cap V'}^{p+q}(X, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}),$$

$$a \otimes b \mapsto a \cup b.$$

That is, there exist unique such homomorphisms that are natural in \mathcal{M} and \mathcal{N} and satisfy the following properties:

- (i) *For $p = q = 0$, $\Gamma_V \mathcal{M} \otimes_{\Gamma' \mathcal{A}} \Gamma_{V'} \mathcal{N} \rightarrow \Gamma_{V \cap V'}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$ is the canonical map.*
- (ii) *The cup product is compatible with the boundary maps ∂ in the long exact cohomology sequences associated with short exact sequences of sheaves: given an \mathcal{A} -module \mathcal{N} and a short exact sequence of left \mathcal{A} -modules $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$ that remains exact after tensoring over \mathcal{A} with \mathcal{N} , we have $\partial(a \cup b) = (\partial a) \cup b$.*
- (iii) *The cup product is graded commutative: $a \cup b = (-1)^{p \cdot q} \tau^*(b \cup a)$, where $\tau : \mathcal{N} \otimes \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{N}$ denotes the canonical isomorphism.*
- (iv) *The cup product is associative. \square*

For the comparison theorems of this paper, we will need a few further properties of the cup product in sheaf cohomology.

2.15. Proposition (naturality of the cup product). *The cup product is natural with respect to continuous maps: Given a continuous map $f : X \rightarrow Y$ and closed subsets $V, V' \subset X$ and $W, W' \subset Y$ such that $f^{-1}W \subset V$ and $f^{-1}W' \subset V'$, the following*

square commutes:

$$\begin{array}{ccc} H_W^*(Y, f_*\mathcal{M}) \otimes_{\Gamma'\mathcal{A}} H_{W'}^*(Y, f_*\mathcal{N}) & \xrightarrow{\cup} & H_{W \cap W'}^*(Y, f_*\mathcal{M} \otimes_{f_*\mathcal{A}} f_*\mathcal{N}) \\ \downarrow f^* \otimes f^* & & \downarrow f^* \\ H_V^*(X, \mathcal{M}) \otimes_{\Gamma'\mathcal{A}} H_{V'}^*(X, \mathcal{N}) & \xrightarrow{\cup} & H_{V \cap V'}^*(X, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) \quad \square \end{array}$$

We also have a cross product in sheaf cohomology. Given two topological spaces X and Y , let $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ denote the canonical projections. For sheaves of abelian groups \mathcal{M} over X and \mathcal{N} over Y , we write $\mathcal{M} \boxtimes \mathcal{N} := \pi_X^* \mathcal{M} \otimes \pi_Y^* \mathcal{N}$ for the exterior tensor product sheaf over $X \times Y$, and similarly for complexes of sheaves.

2.16. Definition (cross product). For sheaves \mathcal{M} on X and \mathcal{N} on Y , and for closed subsets $V \subset X$ and $W \subset Y$, the cross product

$$\times : H_V^p(X, \mathcal{M}) \otimes H_W^q(Y, \mathcal{N}) \rightarrow H_{V \times W}^{p+q}(X \times Y, \mathcal{M} \boxtimes \mathcal{N}) \quad (2.17)$$

is given by $a \times b := \pi_X^* a \cup \pi_Y^* b$.

Alternatively, this cross product can be obtained directly by a construction entirely analogous to the construction of the cup product, but we will not need this. If we take $X = Y$ in Definition 2.19 and write $\Delta : X \rightarrow X \times X$ for the diagonal, we clearly have $a \cup b = \Delta^*(a \times b)$ in $H^*(X, \mathcal{M} \otimes \mathcal{N})$, by the naturality of the cup product; see Proposition 2.15.

Now suppose X and Y are schemes over our base field F . Then we need to distinguish (the underlying topological space of) the scheme-theoretic product $X \times_F Y$ from the topological product $X \times Y$. Let $\pi_{F,X} : X \times_F Y \rightarrow X$ and $\pi_{F,Y} : X \times_F Y \rightarrow Y$ denote the respective scheme-theoretic projections. As these morphisms are continuous, we have an induced continuous map

$$u : X \times_F Y \rightarrow X \times Y \quad (2.18)$$

such that $\pi_X u = \pi_{X,F}$ and $\pi_Y u = \pi_{Y,F}$. Given closed subsets $V \subset X$ and $W \subset Y$, we have $V \times_F W \subset u^{-1}(V \times W)$. Now consider sheaves \mathcal{M} and \mathcal{N} on X and Y , respectively. The scheme-theoretic exterior tensor product of \mathcal{M} and \mathcal{N} is given by

$$\begin{aligned} \mathcal{M} \boxtimes_F \mathcal{N} &:= \pi_{F,X}^* \mathcal{M} \otimes \pi_{F,Y}^* \mathcal{N} \\ &\cong u^*(\pi_X^* \mathcal{M} \otimes \pi_Y^* \mathcal{N}). \end{aligned}$$

2.19. Definition (cross product for schemes). For F -schemes X and Y , closed subsets $V \subset X$ and $W \subset Y$, and sheaves \mathcal{M} and \mathcal{N} on X and Y , respectively, the *sheaf-theoretic cross product* \times_F is the composition of the usual cross product with the pullback along u :

$$H_V^p(X, \mathcal{M}) \otimes H_W^q(Y, \mathcal{N}) \xrightarrow{\times} H_{V \times W}^{p+q}(X \times Y, \mathcal{M} \boxtimes \mathcal{N}) \xrightarrow{u^*} H_{V \times_F W}^{p+q}(X \times_F Y, \mathcal{M} \boxtimes_F \mathcal{N}).$$

We clearly have the following analogues of the formulas relating the cup and the usual cross product:

$$a \times_F b = \pi_{X,F}^* a \cup \pi_{Y,F}^* b, \quad (2.20)$$

$$a \cup b = \Delta_F^*(a \times_F b) \quad \text{when } X = Y. \quad (2.21)$$

Here, $\Delta_F : X \rightarrow X \times_F X$ denotes the scheme-theoretic diagonal.

2.22. Theorem (universality of the cross product). *Suppose we have resolutions*

$$\mathcal{M} \rightarrow \mathcal{R}_{\mathcal{M}}, \mathcal{N} \rightarrow \mathcal{R}_{\mathcal{N}} \quad \text{and} \quad \mathcal{M} \boxtimes_F \mathcal{N} \rightarrow \mathcal{R}_{\boxtimes},$$

together with a homomorphism

$$\mathcal{R}_{\mathcal{M}} \boxtimes_F \mathcal{R}_{\mathcal{N}} \xrightarrow{\mu} \mathcal{R}_{\boxtimes}$$

that is compatible with the two augmentation maps from $\mathcal{M} \boxtimes_F \mathcal{N}$. Then the following diagram commutes:

$$\begin{array}{ccc} H^*(\Gamma_V \mathcal{R}_{\mathcal{M}}) \otimes H^*(\Gamma_W \mathcal{R}_{\mathcal{N}}) & \xrightarrow{\mu} & H^*(\Gamma_{V \times_F W} \mathcal{R}_{\boxtimes}) \\ \downarrow & & \downarrow \\ H_V^*(X, \mathcal{M}) \otimes H_W^*(Y, \mathcal{N}) & \xrightarrow{\times} & H_{V \times_F W}^*(X \times_F Y, \mathcal{M} \boxtimes_F \mathcal{N}) \end{array}$$

Here, the top horizontal map is induced by the pairing of complexes, while the two vertical maps are induced by the augmentations. \square

From now on, whenever dealing with schemes over F , the product $X \times Y$ and the cross product will be understood to be taken over F , even when the subscript F is suppressed.

2.A.4. Twisted coefficients. So far, we have been considering arbitrary sheaves of abelian groups on arbitrary topological spaces (or even sites) in this section. We now consider an operation on coefficient sheaves that requires slightly more structure: ringed spaces (X, \mathcal{O}_X) , and coefficient sheaves of abelian groups \mathcal{F} that come equipped with a \mathbb{G}_m -action. We always assume this action to be *linear* in the sense that sections of \mathbb{G}_m act on \mathcal{F} as automorphisms of abelian groups.

2.23. Definition. Consider a ringed space (X, \mathcal{O}_X) equipped with a line bundle \mathcal{L} , i.e., with a locally free \mathcal{O}_X -module of rank one. Given an abelian sheaf \mathcal{F} over X with a (linear) \mathbb{G}_m -action, the *twisted sheaf* $\mathcal{F}(\mathcal{L})$ is defined as $\mathcal{F} \otimes_{\mathbb{Z}[\mathbb{G}_m]} \mathbb{Z}[\mathcal{L}^\times]$, i.e., as the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathbb{Z}[\mathcal{O}_X^\times(U)]} \mathbb{Z}[\mathcal{L}(U)^\times]$. Given an equivariant morphism of sheaves with \mathbb{G}_m -action $\phi : \mathcal{F} \rightarrow \mathcal{G}$, the *twisted morphism* $\phi(\mathcal{L}) : \mathcal{F}(\mathcal{L}) \rightarrow \mathcal{G}(\mathcal{L})$ is defined as $\phi \otimes \text{id}$.

Here, $\mathbb{Z}[\mathbb{G}_m(U)]$ denotes the free abelian group generated by the invertible functions on U and $\mathbb{Z}[\mathcal{L}(U)^\times]$ denotes the free abelian group generated by the

nowhere-vanishing sections of the line bundle \mathcal{L} over U . (See [Asok and Fasel 2016, Definition 2.6], where the notation \mathcal{L}° is used to denote the complement of the zero section of the line bundle \mathcal{L} ; $\mathcal{L}^\circ(U) = \mathcal{L}(U)^\times$.)

2.24. Lemma. *Consider a morphism of ringed spaces $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$. For any sheaf \mathcal{F}_X over X with $\widehat{\mathbb{G}}_m$ -action and any line bundle \mathcal{L} over Y , we have a canonical isomorphism of sheaves of abelian groups $f_*(\mathcal{F}_X(f^*\mathcal{L})) \cong (f_*\mathcal{F}_X)(\mathcal{L})$, natural in \mathcal{F}_X and \mathcal{L} .¹ \square*

More generally, we can twist an arbitrary morphism of coefficient data. To formalize this, let us call a pair $((X, \mathcal{O}_X), \mathcal{F})$ consisting of a ringed space (X, \mathcal{O}_X) and a sheaf of abelian groups \mathcal{F} on X with a linear $\widehat{\mathbb{G}}_m$ -action a *ringed coefficient datum*. Note that \mathcal{F} is not assumed to be an \mathcal{O} -module; when we write $f^*\mathcal{F}$ in the following, we still mean the pullback of \mathcal{F} as a sheaf of abelian groups. If we forget the structure sheaf and the $\widehat{\mathbb{G}}_m$ -action, we obtain an underlying coefficient datum (X, \mathcal{F}) . A *morphism of ringed coefficient data*

$$(f, \phi) : ((X, \mathcal{O}_X), \mathcal{F}_X) \rightarrow ((Y, \mathcal{O}_Y), \mathcal{F}_Y)$$

is a pair such that $(f, \phi) : (X, \mathcal{F}_X) \rightarrow (Y, \mathcal{F}_Y)$ is morphism of coefficient data in the previous sense, and such that $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces whose structure morphism $f_*\mathcal{O}_X \leftarrow \mathcal{O}_Y$ is compatible with the given actions of \mathcal{O}_X^\times on \mathcal{F}_X and \mathcal{O}_Y^\times on \mathcal{F}_Y .

2.25. Example. Suppose \mathcal{F} is an abelian sheaf with linear $\widehat{\mathbb{G}}_m$ -action on the big Zariski site Sm/F . Then any morphism of smooth F -schemes $f : X \rightarrow Y$ can be viewed as a morphism of ringed coefficient data $((X, \mathcal{O}_X), \mathcal{F}|_X) \rightarrow ((Y, \mathcal{O}_Y), \mathcal{F}|_Y)$.

2.26. Definition. Consider a morphism $(f, \phi) : ((X, \mathcal{O}_X), \mathcal{F}_X) \rightarrow ((Y, \mathcal{O}_Y), \mathcal{F}_Y)$ of ringed coefficient data. Given a line bundle \mathcal{L} over Y , the *twisted morphism of ringed coefficient data*

$$(f, \phi)(\mathcal{L}) := (f, \phi_{\mathcal{L}}) : ((X, \mathcal{O}_X), \mathcal{F}_X(f^*\mathcal{L})) \rightarrow ((Y, \mathcal{O}_Y), \mathcal{F}_Y(\mathcal{L}))$$

has the same underlying morphism of ringed spaces f , while the structure morphism $\phi_{\mathcal{L}}$ is defined as the composition of the canonical isomorphism of Lemma 2.24 with the morphism $\phi(\mathcal{L})$ of Definition 2.23:

$$f_*(\mathcal{F}_X(f^*\mathcal{L})) \cong (f_*\mathcal{F}_X)(\mathcal{L}) \xleftarrow[\phi(\mathcal{L})]{} \mathcal{F}_Y(\mathcal{L}).$$

2.27. Lemma. *Twisting is compatible with composition: given two composable morphisms of ringed coefficient data*

$$((X, \mathcal{O}_X), \mathcal{F}_X) \xrightarrow{(f, \phi)} ((Y, \mathcal{O}_Y), \mathcal{F}_Y) \xrightarrow{(g, \gamma)} ((Z, \mathcal{O}_Z), \mathcal{F}_Z)$$

¹In the context of line bundles, $f^*\mathcal{L}$ denotes the pullback of \mathcal{L} as an \mathcal{O}_X -module, so that $f^*\mathcal{L}$ is again a line bundle.

and a line bundle \mathcal{L} over Z , $(g, \gamma)(\mathcal{L}) \circ (f, \phi)(g^*\mathcal{L}) = ((g, \gamma)(f, \phi))(\mathcal{L})$, up to the canonical identification $f^*g^*\mathcal{L} \cong (gf)^*\mathcal{L}$ on X . \square

We will mostly use this lemma in the following form:

2.28. Corollary. *Suppose we are given a commutative square of morphisms of ringed coefficient data:*

$$\begin{array}{ccc} ((X, \mathcal{O}_X), \mathcal{F}_X) & \xrightarrow{(g, \gamma)} & ((X', \mathcal{O}_{X'}), \mathcal{F}_{X'}) \\ \downarrow (f, \phi) & & \downarrow (f', \phi') \\ ((Y, \mathcal{O}_Y), \mathcal{F}_Y) & \xrightarrow{(h, \eta)} & ((Y', \mathcal{O}_{Y'}), \mathcal{F}_{Y'}) \end{array}$$

Then for any line bundle \mathcal{L} on Y' , the following square still commutes, where the isomorphism in the top left corner is induced by the canonical identification $f^*h^*\mathcal{L} \cong g^*f'^*\mathcal{L}$:

$$\begin{array}{ccc} ((X, \mathcal{O}_X), \mathcal{F}_X(g^*f'^*\mathcal{L})) & \xrightarrow{(g, \gamma)} & ((X', \mathcal{O}_{X'}), \mathcal{F}_{X'}(f'^*\mathcal{L})) \\ \parallel \cong & & \downarrow (f', \phi^*) \\ ((X, \mathcal{O}_X), \mathcal{F}_X(f^*h^*\mathcal{L})) & & \\ \downarrow (f, \phi) & & \downarrow \\ ((Y, \mathcal{O}_Y), \mathcal{F}_Y(h^*\mathcal{L})) & \xrightarrow{(h, \eta)} & ((Y', \mathcal{O}_{Y'}), \mathcal{F}_{Y'}(\mathcal{L})) \end{array} \quad \square$$

2.29. Example. Let F be a field, let $f : X \rightarrow Y$ be a morphism of smooth F -schemes, and let \mathcal{L} be a line bundle over Y . Any equivariant morphism of abelian sheaves with \mathbb{G}_m -action $\phi : \mathcal{F} \rightarrow \mathcal{G}$ on the big Zariski site Sm_F induces well-defined morphisms $\phi_Y(\mathcal{L}) : \mathcal{F}|_Y(\mathcal{L}) \rightarrow \mathcal{G}|_Y(\mathcal{L})$ and $\phi_X(f^*\mathcal{L}) : \mathcal{F}|_X(f^*\mathcal{L}) \rightarrow \mathcal{G}|_X(f^*\mathcal{L})$ of abelian sheaves such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}|_Y(\mathcal{L}) & \xrightarrow{\phi_Y(\mathcal{L})} & \mathcal{G}|_Y(\mathcal{L}) \\ \downarrow & & \downarrow \\ f_*(\mathcal{F}|_X(f^*\mathcal{L})) & \xrightarrow{f_*\phi_X(f^*\mathcal{L})} & f_*(\mathcal{G}|_X(f^*\mathcal{L})) \end{array}$$

Here the vertical morphisms are the ones induced from the sheaf restriction morphisms as in Example 2.25.

2.B. Reminders on Witt groups and fundamental ideals. Given a scheme with a line bundle (X, \mathcal{L}) , let $\text{VB}(X)$ denote the category of vector bundles over X and let $\#_{\mathcal{L}} : \text{VB}(X)^{\text{op}} \rightarrow \text{VB}(X)$ denote the functor sending \mathcal{F} to $\mathcal{F}^{\vee \mathcal{L}} := \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$.

2.30. Definition. The Witt group $W(X, \mathcal{L})$ of the pair (X, \mathcal{L}) is the Witt group $W(X, \mathcal{L}) := W(\text{VB}(X), \#_{\mathcal{L}})$ of the exact category with duality $(\text{VB}(X), \#_{\mathcal{L}})$, equipped with the usual double-dual identification.

To simplify notation, we often omit \mathcal{L} when $\mathcal{L} = \mathcal{O}$. Note that $W(X, \mathcal{L})$ is a $W(X)$ -module by left multiplication/tensoring with symmetric forms.

There is a well-defined ring homomorphism

$$\mathrm{rk} : W(X) \rightarrow H_{\text{ét}}^0(X, \mathbb{Z}/2\mathbb{Z})$$

which assigns to an isometry class of a symmetric bilinear form the rank of its underlying vector bundle modulo two. The kernel of the rank homomorphism is denoted by $I(X)$ and is called the fundamental ideal. Let $I^j(X)$ denote the j -th power of this fundamental ideal, and let $I^j(X, \mathcal{L}) := I^j(X) \cdot W(X, \mathcal{L})$ denote the $W(X)$ -submodule of $W(X, \mathcal{L})$ generated by $I^j(X)$.

2.31. Definition. We write $\mathbf{W}_{\mathcal{L}}$ to denote the Zariski sheafification of the presheaf $(U \subseteq X) \mapsto W(U, \mathcal{L})$ on the small Zariski site of X , and $\mathbf{I}_{\mathcal{L}}^j$ to denote the Zariski sheafification of the presheaf $(U \subseteq X) \mapsto I^j(U, \mathcal{L})$.

Again, $\mathcal{L} = \mathcal{O}$ will usually be suppressed in the notation. The \mathbf{I}^k -cohomology of a scheme X is by definition the sheaf cohomology of X with coefficients in \mathbf{I}^k , or with coefficients in the twisted variants $\mathbf{I}_{\mathcal{L}}^k$.

We can also twist the sheaves \mathbf{I}^j using the construction of Definition 2.23. Indeed, for any scheme U , we have a morphism $\mathbb{G}_m(U) \rightarrow W(U)$ sending an invertible function to the associated symmetric form of rank one. This defines a morphism of sheaves of rings $\mathbb{G}_m \rightarrow \mathbf{W}$ and hence equips each of the sheaves \mathbf{I}^j with a canonical linear \mathbb{G}_m -action. Thus, for any line bundle \mathcal{L} over U , we have twisted sheaves $\mathbf{I}^j(\mathcal{L})$. Luckily, twists in this sense agree with twists in the sense considered above:

2.32. Proposition. *For any smooth variety X over a field and any line bundle \mathcal{L} over X there is an isomorphism $\mathbf{I}^j(\mathcal{L}) \cong \mathbf{I}_{\mathcal{L}}^j$ of Zariski sheaves which is compatible with the linear \mathbb{G}_m -action on both sides. \square*

In the following, we sometimes denote the sheaf \mathbf{W} on X by \mathbf{W}_X for clarity. The functoriality of the Witt sheaves furnishes us with structure morphisms $\phi : f_* \mathbf{W}_X \leftarrow \mathbf{W}_Y$ for morphisms of schemes $f : X \rightarrow Y$ over F . In particular, any such morphism f induces a morphism of ringed coefficient data

$$(f, \phi) : ((X, \mathcal{O}_X), \mathbf{W}_X) \rightarrow ((Y, \mathcal{O}_Y), \mathbf{W}_Y).$$

More generally, we have associated morphisms of coefficient data for twists of the fundamental ideal sheaves as follows:

2.33. Definition. Consider a morphism of schemes $f : X \rightarrow Y$ over F . Given an exponent j and a line bundle \mathcal{L} over Y , the *induced morphism of coefficient data*

$$((X, \mathcal{O}_X), \mathbf{I}^j(f^* \mathcal{L})) \rightarrow ((Y, \mathcal{O}_Y), \mathbf{I}^j(\mathcal{L}))$$

is obtained by restricting (f, ϕ) to the sheaves \mathbf{I}^j and by twisting via the construction of Definition 2.26.

2.34. Remark (big site). When we work on the big Zariski site Sm_F of smooth schemes over F , \mathbf{W} may occasionally denote the sheafification of the presheaf $(U \mapsto W(U))$ on this big site. The restriction of this sheaf to a scheme X agrees with the sheaf \mathbf{W}_X considered above, i.e., with the sheafification of the restricted Witt presheaf, so no confusion seems likely.

2.35. Definition (Mahé). Let U be a scheme. The global signature is the ring homomorphism

$$\begin{aligned} \text{sign}^U : W(U) &\rightarrow C(U_r, \mathbb{Z}) = H^0(U_r, \mathbb{Z}), \\ [\phi] &\mapsto (\text{sign}([\phi]) : (x, P) \mapsto \text{sign}_P(i_x^* \phi)), \end{aligned}$$

where $\text{sign}_P : W(\kappa(x)) \rightarrow \mathbb{Z}$ sends a form $[\varphi]$ to its signature with respect to the ordering P on the residue field $\kappa(x)$ and U_r is the real spectrum of U discussed in Section 2.D.

If U is a scheme over \mathbb{R} , we may replace sign_P by the classical signature for the closed point x , and we denote the resulting variant $W(U) \rightarrow C(U(\mathbb{R}), \mathbb{Z})$ by sign^U as well.

2.C. Reminders on \mathbf{I} -cohomology.

2.36. Definition. The \mathbf{I}^k -cohomology of a scheme X equipped with a line bundle \mathcal{L} is the Zariski sheaf cohomology of X with coefficients in the sheaf of abelian groups $\mathbf{I}^k(\mathcal{L})$ defined in Definition 2.31/Proposition 2.32. We will denote it as $H_{\text{Zar}}^i(X, \mathbf{I}^k(\mathcal{L}))$, or simply as $H^i(X, \mathbf{I}^k(\mathcal{L}))$. In general statements that hold for all exponents k , we simply say \mathbf{I} -cohomology in place of \mathbf{I}^k -cohomology.

In principle, \mathbf{I}^k -cohomology can be computed using an injective resolution of \mathbf{I}^k . However, apart from the existence of pullback maps along morphisms of schemes, most of the fundamental properties of \mathbf{I} -cohomology are difficult to establish using injective resolutions. Instead, Fasel [2008], in his foundational text, uses explicit Gersten resolutions to establish the fundamental properties. In Section 9.3 of that work, pushforwards for proper morphisms and pullbacks for flat morphisms are constructed. The latter extend to arbitrary maps between smooth schemes; see Definition 2.5 and Proposition 2.45 below for more details. We have a long exact localization sequence [Fasel 2008, Théorème 9.3.4 and Remarque 9.3.5], at least if the closed complement is smooth. We also have homotopy invariance for vector bundles over regular schemes; see [Fasel 2008, Théorème 11.2.9]. Moreover, the \mathbf{I} -cohomology groups form a graded ring [Fasel 2007].

For the comparison results in later sections, in particular those regarding the ring structure, the sheaf-theoretic constructions are more suitable than the constructions

using Gersten complexes. We therefore discuss and compare both. We begin by briefly recalling the approach using Gersten complexes in Section 2.C.1, noting in Theorem 2.38 that the I^k -cohomology groups defined in this way indeed coincide with the sheaf cohomology groups with coefficients in I^k for smooth X . In Section 2.C.3, we verify, following [Asok and Fasel 2016], that pullbacks constructed using Gersten complexes coincide with the sheaf-theoretic pullbacks of Section 2.A.1. In Section 2.C.4, we verify that, likewise, products constructed using Gersten complexes coincide with the products constructed via Godement resolutions in Section 2.A.3. Lastly, in Section 2.C.5, we recall the construction of Thom classes and pushforwards.

Everything in this section applies verbatim to J -cohomology, where $J^k \cong K_k^{\text{MW}}$, with one notable exception concerning the commutativity of the ring structure (see Remark 2.53). In fact, all arguments and almost all references below apply to Chow–Witt groups or their bigraded extension.

2.C.1. Gersten resolutions. We recall the basic construction of Gersten resolutions from [Balmer and Walter 2002]; see also [Gille 2007b] and [Fasel 2008]. Other constructions of the Gersten complex for W and GW are known to agree with this one: the construction in [Morel 2012] agrees with the one of Schmid, and the latter one agrees with the ones of Fasel et. al. discussed in [Fasel 2008, Section 7].

Let X be a regular scheme of dimension n . Let \mathcal{L} be a line bundle. We denote by $(D^b(X), \vee_{\mathcal{L}}, \varpi_{\mathcal{L}})$ the derived category with duality, where $\vee_{\mathcal{L}} : D^b(X)^{\text{op}} \rightarrow D^b(X)$ is given by $V \mapsto \text{Hom}_{\mathcal{O}_X}(V, \mathcal{L})$ and $\varpi_{\mathcal{L}}$ is the double-dual identification. Let $D^{(i)} \subset D^b(X)$ be the triangulated subcategory of objects whose homology is supported in codimension $\geq i$, and let D^i denote the triangulated category $D^{(i)}/D^{(i+1)}$. The augmented Gersten–Witt complex on X has the form

$$0 \rightarrow W(X, \mathcal{L}) \xrightarrow{d} W(D^0, \mathcal{L}) \xrightarrow{d} W^1(D^1, \mathcal{L}) \xrightarrow{d} \dots \xrightarrow{d} W^n(D^n, \mathcal{L}) \rightarrow 0,$$

where W^i denotes Balmer’s derived Witt groups. The terms of the complex can be identified via isomorphisms

$$\Sigma : W^i(D^i, \mathcal{L}) \cong \bigoplus_{x \in X^{(i)}} W(k(x), \omega_x^{\mathcal{L}}) \tag{2.37}$$

obtained from localizations and dévissage, where $\omega_x^{\mathcal{L}} := \text{Ext}_{\mathcal{O}_x}^i(k(x), \mathcal{L}_x)$ is a one-dimensional vector space over $k(x)$ for each point $x \in X^{(i)}$.

Denote by $I^{i,j}(X, \mathcal{L}) \subset W^i(D^i, \mathcal{L})$ the subgroup corresponding to the sum of the j -th powers of fundamental ideals in the Witt groups $W(k(x), \omega_x^{\mathcal{L}})$ under the isomorphism Σ , and by $I_{\mathcal{L}}^{i,j}$ the sheafification of $U \mapsto I^{i,j}(U, \mathcal{L}|_U)$ on X . Note that $I_{\mathcal{L}}^{i,j}$ is a flasque sheaf. The Gersten–Witt complex is compatible with the fundamental ideal in [Gille 2007b] or more precisely [Fasel 2008, Théorème 9.2.4]

and [Fasel 2013]; i.e., the differential d in the Gersten–Witt complex satisfies $d(I^m(k(x), \omega_x^{\mathcal{L}})) \subset I^{m-1}(k(y), \omega_y^{\mathcal{L}})$ for any $m \in \mathbb{Z}$, $x \in X^{(i)}$ and $y \in X^{(i+1)}$. We thus obtain, by restriction, a complex of abelian groups,

$$C^\bullet(X, \mathbf{I}^j(\mathcal{L})) : I^{0,j}(X, \mathcal{L}) \rightarrow I^{1,j-1}(X, \mathcal{L}) \rightarrow \dots \rightarrow I^{n,j-n}(X, \mathcal{L}) \rightarrow 0,$$

and a complex of flasque sheaves on X ,

$$C^\bullet(-, \mathbf{I}^j(\mathcal{L})) : \mathbf{I}_{\mathcal{L}}^{0,j} \rightarrow \mathbf{I}_{\mathcal{L}}^{1,j-1} \rightarrow \dots \rightarrow \mathbf{I}_{\mathcal{L}}^{n,j-n} \rightarrow 0,$$

which we also refer to as Gersten–Witt complexes.

2.38. Theorem. *Let X be a smooth F -scheme of dimension n . Then, for any i and any j , we have canonical isomorphisms*

$$H^i(C^\bullet(X, \mathbf{I}^j(\mathcal{L}))) \cong H_{\text{Zar}}^i(X, \mathbf{I}^j(\mathcal{L})) \cong H_{\text{Nis}}^i(X, \mathbf{I}^j(\mathcal{L})).$$

We therefore simply write $H^i(X, \mathbf{I}^j(\mathcal{L}))$ for the above cohomology groups from now onwards.

Proof. For the first isomorphism, the augmented Gersten complex provides a flasque resolution of the sheaf $\mathbf{I}_{\mathcal{L}}^j$, i.e., the complex

$$0 \rightarrow \mathbf{I}_{\mathcal{L}}^j \rightarrow \mathbf{I}_{\mathcal{L}}^{0,j} \rightarrow \mathbf{I}_{\mathcal{L}}^{1,j-1} \rightarrow \dots \rightarrow \mathbf{I}_{\mathcal{L}}^{n,j-n} \rightarrow 0$$

is exact. This is shown for $j = 0$, i.e., for the sheaf \mathbf{W} , in [Balmer et al. 2002, Lemma 4.2 and Theorem 6.1]; for the general case of fundamental ideals and twisted line bundles see [Fasel 2008, Proof of Theorem 9.2.4] and [Gille 2007b, Corollary 7.2]. So the first isomorphism is the canonical isomorphism coming from the uniqueness of sheaf cohomology. The second isomorphism is [Morel 2012, Corollary 5.43], using that the sheaves \mathbf{I}^j satisfy the conditions of Morel’s framework by [Morel 2012, Example 3.34 and Lemma 3.35]. \square

2.C.2. Supports. For a closed subset $i : Z \hookrightarrow X$ of a smooth F -scheme X , we have a definition of \mathbf{I} -cohomology with support in Z as the derived functors of $\Gamma_Z = \Gamma(Z, -) \circ i^!$; see Definition 2.6. This can also be expressed in terms of Gersten complexes, as follows.

2.39. Proposition. *Let $j : U \hookrightarrow X$ denote the open complement of $i : Z \hookrightarrow X$. Consider the natural transformation $\Gamma(X, -) \rightarrow \Gamma(U, -) \circ j^*$. Following [Fasel 2008, Définition 9.3.3], define the Gersten complex with support on Z as the kernel of this natural transformation applied to the Gersten complex on X :*

$$C^\bullet(X, \mathbf{I}^j(\mathcal{L}))_Z := \ker(C^\bullet(X, \mathbf{I}^j(\mathcal{L})) \rightarrow C^\bullet(U, \mathbf{I}^j(\mathcal{L}))).$$

There is a natural isomorphism $H_Z^s(X, \mathbf{I}^j(\mathcal{L})) \cong H^s(C^\bullet(X, \mathbf{I}^j(\mathcal{L}))_Z)$.

Proof. We need to derive the functor $\Gamma_Z = \Gamma(Z, -) \circ i^!$. We can compute $(R^s\Gamma_Z)(I^j)$ by evaluating Γ_Z on a flasque resolution. Choosing this resolution to be the flasque resolution $C^\bullet(-, I^j(\mathcal{L}))$ of $I^j(\mathcal{L})$, we can identify Γ_Z with the functor $\ker(\Gamma(X, -) \rightarrow \Gamma(U, -))$. \square

In the above situation, we have a short exact sequence of Gersten complexes $0 \rightarrow C^\bullet(X, I^j(\mathcal{L}))_Z \rightarrow C^\bullet(X, I^j(\mathcal{L})) \rightarrow C^\bullet(U, I^j(\mathcal{L})) \rightarrow 0$. The associated long exact cohomology sequence is the localization sequence of [Fasel 2008, Théorème 9.3.4], and coincides via the isomorphism above with the localization sequence in Proposition 2.11.

2.40. Remark. The boundary map in this localization sequence is induced by a morphism of complexes $\partial : C^\bullet(U, I^j(\mathcal{L})) \rightarrow C^{\bullet+1}(X, I^j(\mathcal{L}))_Z$, at least up to a sign. To see this, one notes that, in each degree, $C^i(U, I^j(\mathcal{L}))$ is a direct summand of $C^i(X, I^j(\mathcal{L}))$. The differential on $C^i(X, I^j(\mathcal{L}))$ thus has a component that maps the direct summand $C^i(U, I^j(\mathcal{L}))$ to the direct summand $C^{i+1}(X, I^j(\mathcal{L}))_Z$. See [Asok and Fasel 2016, Lemma 2.10] for more details.

2.41. Theorem (dévissage). *Suppose $i : Z \hookrightarrow X$ is a closed embedding of codimension r of regular schemes. Let ω_Z denote the determinant line bundle of the normal bundle of this embedding, and let \mathcal{L} be an arbitrary line bundle over X . We have an isomorphism of complexes $C^{\bullet-r}(Z, I^{j-r}(i^*\mathcal{L} \otimes \omega_Z)) \cong C^\bullet(X, I^j(\mathcal{L}))_Z$ inducing isomorphisms in cohomology*

$$H^{s-r}(Z, I^{j-r}(i^*\mathcal{L} \otimes \omega_Z)) \cong H_Z^s(X, I^j(\mathcal{L})).$$

Proof. This is stated in [Fasel 2008, Remarque 9.3.5]. We add some details for the reader's convenience. As Z is regular and thus lci, it follows that the index sets $Z^{(s-r)}$ and $Z \cap X^{(s)}$ are in bijection. Using the notation as in [Fasel 2008, Définition 9.2.1 and Remarque 9.2.2], the direct sums in degree s of the relevant Gersten complexes have summands $I^{j-s}(\mathcal{O}_{Z,z})$ and $I^{j-s}(\mathcal{O}_{X,x})$. Under the above bijection, for corresponding points z and x there is a chain of isomorphisms

$$W^{\text{fl}}(\mathcal{O}_{Z,z}) \cong W(\kappa(z), \omega_{\kappa(z)/F}) \cong W(\kappa(x), \omega_{\kappa(x)/F}) \cong W^{\text{fl}}(\mathcal{O}_{X,x})$$

given by applying [Fasel 2008, Corollaire 6.2.8], and this isomorphism restricts to an isomorphism of the respective powers of fundamental ideals. To see that this is indeed an isomorphism of complexes, one has to check that these degreewise isomorphisms respect the differentials. For Witt groups, this follows using the two commutative squares in the diagram [Gille 2007a, page 339] in which the horizontal maps compose to the boundary of the Gersten complex. For the fact that the differentials respect I -powers (up to a shift of one) see [Gille 2007b, Corollary 7.3]. The proofs in [Gille 2007b] and in [Fasel 2008] rely on the identification of each term of the Gersten–Witt complex as a direct sum of Witt groups indexed on points

of codimension p ; see (2.37) above and [Gille 2007b, diagram (14) in Section 5.8]. We observe that the vertical maps Tr in the diagram of [Gille 2007a] yield the (zigzag) pushforward maps we described above. This can be seen combining the commutative diagram in the proof of [Gille 2007a, Lemma 3.4] with the commutative diagram in [Gille 2007b, Proposition 3.6]. This also shows that for $f = i$ as above, the map f_* of [Fasel 2008, 5.3.6, 6.3.10] is precisely the one we study above composed with forgetting the support. \square

2.C.3. Pullbacks.

2.42. Definition. Given a morphism of schemes $f : X \rightarrow Y$, closed subsets $V \subset X$ and $W \subset Y$ such that $f^{-1}W \subset V$ and a line bundle \mathcal{L} over Y , the *pullback in \mathbf{I} -cohomology*

$$H_V^*(X, \mathbf{I}^k(f^*\mathcal{L})) \xleftarrow{(f,\phi)^*} H_W^*(Y, \mathbf{I}^k(\mathcal{L})),$$

is defined as in Definition 2.8 from the morphism of coefficient data induced by f according to Definition 2.33. We will denote this pullback simply by f^* .

The pullback along a morphism of smooth schemes $f : X \rightarrow Y$ can also be constructed in terms of Gersten complexes. To this end, one factors f as $X \rightarrow X \times Y \rightarrow Y$, where the first morphism is the graph of f and the second is the projection onto Y . The graph of f is a regular embedding and the projection to Y is flat, so it suffices to treat these two classes of morphisms separately.

For flat f , it is possible to define a pullback directly on the level of Gersten complexes [Fasel 2008, Section 3]:

$$C^\bullet(X, \mathbf{I}^k(f^*\mathcal{L})) \xleftarrow{f^*} C^\bullet(Y, \mathbf{I}^k(\mathcal{L})).$$

To define a pullback with support, note that this construction is functorial and, in particular, compatible with restriction to open subsets. Thus we obtain a morphism of complexes of sheaves, $f_*C^\bullet(-, \mathbf{I}_X^k(f^*\mathcal{L})) \leftarrow C^\bullet(-, \mathbf{I}_Y^k(\mathcal{L}))$. For any closed subset $W \subset Y$, we have an induced morphism of complexes of abelian groups $C^\bullet(X, \mathbf{I}^k(\mathcal{L}))_{f^{-1}W} \leftarrow C^\bullet(Y, \mathbf{I}^k(\mathcal{L}))_W$. Passing to cohomology, we obtain a pullback with support as in Definition 2.42.

For a regular embedding of smooth schemes $i : Z \rightarrow X$, and a line bundle \mathcal{L} over X , a pullback via Gersten complexes is constructed in [Fasel 2007, Section 5]. This is more delicate: it does not seem possible to define i^* directly on the level of Gersten complexes. However, it is possible to define on the level of Gersten complexes a morphism Θ that induces the pullback $(iq)^*$ along the composition $iq : \mathcal{N}_Z X \rightarrow Z \rightarrow X$, where $q : \mathcal{N}_Z X \rightarrow Z$ is the bundle projection of the normal bundle of the embedding. As q is flat, we can then consider the following zigzag

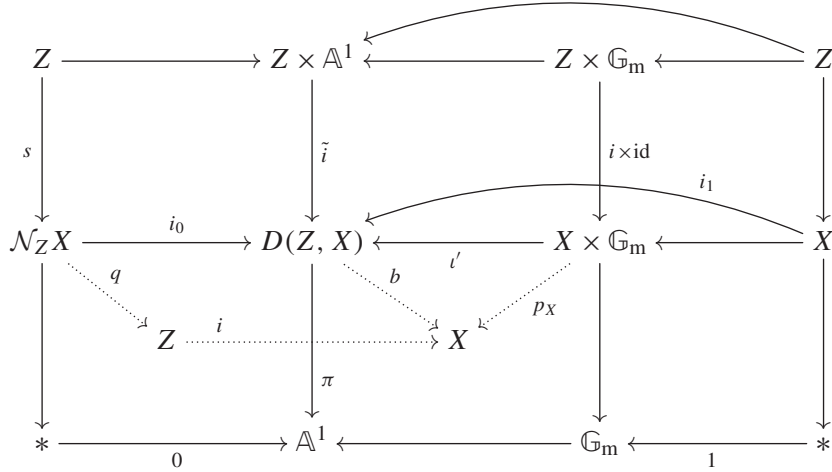


Figure 1. Deformation to the normal bundle.

of morphisms of complexes:

$$\begin{array}{ccc}
 & \Theta & \\
 & \curvearrowright & \\
 C^\bullet(\mathcal{N}_Z X, I^k(q^*i^*\mathcal{L})) & \xleftarrow{q^*} & C^\bullet(Z, I^k(i^*\mathcal{L})) \quad C^\bullet(X, I^k(\mathcal{L}))
 \end{array}$$

Once we pass to cohomology, q^* becomes an isomorphism by homotopy invariance [Fasel 2008, Théorème 11.2.9] and can be inverted, so we can define the pullback along i in I -cohomology as $i^* = (q^*)^{-1}H^\bullet(\Theta)$.

The construction of Θ is based on the deformation to the normal bundle, for which we summarize our notation in Figure 1. The deformation space $D(Z, X)$ is obtained by blowing up $X \times \mathbb{A}^1$ in $Z \times \{0\}$ and then removing the blowup of $X \times \{0\}$ along $Z \times \{0\}$. By construction, it comes with a natural morphism to $X \times \mathbb{A}^1$. Composing with the projections, we obtain morphisms $\pi : D(Z, X) \rightarrow \mathbb{A}^1$ and $b : D(Z, X) \rightarrow X$. The deformation space also comes with a natural embedding \tilde{i} of $Z \times \mathbb{A}^1$, and the composition of the central vertical arrows π and \tilde{i} is the projection onto \mathbb{A}^1 . The fiber of \tilde{i} over 0 is the zero section s of the normal bundle $\mathcal{N}_Z X$, and the fiber over all $t \neq 0$ is the given embedding $i : Z \rightarrow X$. In other words, the first, third and fourth vertical columns are obtained from the second column by base change along the inclusions of $\{0\}$, \mathbb{G}_m or $\{1\}$ into \mathbb{A}^1 , respectively. In particular, the restriction of the normal bundle $\mathcal{N}_{Z \times \mathbb{A}^1}(D(Z, X))$ to any fiber is isomorphic to $\mathcal{N}_Z X$.

The morphism of complexes Θ is constructed as a composition

$$\Theta = (i_{0*})^{-1} \partial m_t p_X^*,$$

the factors of which we now describe. The first factor is simply the pullback along the flat projection $p_X : X \times \mathbb{G}_m \rightarrow X$. For the second factor, let t denote the coordinate in $\mathbb{G}_m = \text{Spec}(k[t, t^{-1}])$. Left multiplication with the form $\langle t, -1 \rangle \in W^0(\mathbb{G}_m)$ induces a morphism of complexes

$$C^\bullet(X \times \mathbb{G}_m, \mathbf{I}^{k+1}(p_X^* \mathcal{L})) \xleftarrow{\langle t, -1 \rangle \cup -} C^\bullet(X \times \mathbb{G}_m, \mathbf{I}^k(p_X^* \mathcal{L})). \quad (2.43)$$

The required morphism m_t of complexes up to a sign is given in degree j by $(-1)^j \langle t, -1 \rangle \cup -$; see [Fasel 2007, Lemma 5.1]. For the third factor of Θ , consider the boundary map in the localization sequence associated with $\mathcal{N}_Z X \hookrightarrow D(Z, X) \hookrightarrow X \times \mathbb{G}_m$. As mentioned in Remark 2.40, this map can be realized as a morphism of complexes

$$C^{\bullet+1}(D(Z, X), \mathbf{I}^{k+1}(b^* \mathcal{L}))_{\mathcal{N}_Z X} \xleftarrow{\partial} C^\bullet(X \times \mathbb{G}_m, \mathbf{I}^{k+1}(p_X^* \mathcal{L}))$$

up to a sign. Hence, the composition ∂m_t becomes a morphism of complexes. Finally, since $\mathcal{N}_Z X$ is a Cartier divisor of $D(Z, X)$, we have the dévissage isomorphism described in Theorem 2.41:

$$C^\bullet(\mathcal{N}_Z X, \mathbf{I}^k(q^* i^* \mathcal{L})) \xrightarrow[\cong]{i_{0*}} C^{\bullet+1}(D(Z, X), \mathbf{I}^{k+1}(b^* \mathcal{L}))_{\mathcal{N}_Z X}.$$

2.44. Lemma. *Fasel's morphism Θ defines a morphism of complexes of sheaves on X :*

$$i_* q_* C^\bullet(-, \mathbf{I}_{\mathcal{N}_Z X}^k(q^* i^* \mathcal{L})) \xleftarrow{\Theta} C^\bullet(-, \mathbf{I}_X^k(\mathcal{L})).$$

Proof. Consider an open subscheme $U \subset X$. Pull back the spaces and morphisms in the upper half of Figure 1 along the inclusion of U into the copy of X that is the target of p_X and b . This yields another such diagram in which X is replaced by U and Z by $Z \cap U$; the restriction of $D(Z, X)$ to U can be identified with $D(Z \cap U, U)$ since blowing-up commutes with flat base change [Stacks 2005–, 0805], and the restriction of $\mathcal{N}_Z X$ to $U \cap Z$ can be identified with $\mathcal{N}_{Z \cap U} U$. We need to check that each factor of Θ is compatible with the restriction to U . For the first factor p_X^* , this has already been discussed. For the second factor m_t , see [Fasel 2007, Lemma 5.4]. The boundary map ∂ commutes with flat pullback, hence in particular with the two inclusions $U \times \mathbb{G}_m \subset X \times \mathbb{G}_m$ and $D(Z \cap U, U) \subset D(Z, X)$. Finally, i_{0*} commutes with the inclusions $D(Z \cap U, U) \subset D(Z, X)$ and $\mathcal{N}_{Z \cap U} U \subset \mathcal{N}_Z X$ by the base change formula [Fasel 2013, Corollaire 12.2.8]. \square

The lemma implies that, for any closed subset $V \subset X$, we have an induced morphism of complexes $\Theta_V : C^\bullet(\mathcal{N}_Z X, \mathbf{I}^k(\mathcal{L}_{\mathcal{N}}))_{\mathcal{N}_Z X|_V} \leftarrow C^\bullet(X, \mathbf{I}^k(\mathcal{L}))_V$. We can thus define the pullback along i with support in V as we did for support in X , as $i^* = (q^*)^{-1} H^\bullet(\Theta_V)$.

2.45. Proposition. *Let $f : X \rightarrow Y$ be a morphism between smooth schemes over F . Suppose we have closed subsets $V \subset X$ and $W \subset Y$ such that $f^{-1}W \subset V$, and a line bundle \mathcal{L} on Y . The induced pullback on \mathbf{I} -cohomology in terms of Gersten complexes described above agrees with the sheaf-theoretic pullback with support of Definition 2.42 under the identifications of Proposition 2.39.*

Proof. As noted, a morphism between smooth schemes can be factored into a regular embedding followed by a flat morphism. Compatibility of the pullbacks can be checked separately for each factor.

For a flat morphism f , it suffices by Proposition 2.9 to check that the degree-zero part of the pullback morphism constructed on the level of Gersten complexes of sheaves is compatible with the usual pullback on \mathbf{I}^k -sheaves. This is indeed the case, as in degree zero this pullback is the usual pullback on function fields.

For a regular embedding $i : Z \rightarrow X$, a proof of the desired compatibility in the context of Milnor–Witt sheaves is given in [Asok and Fasel 2016, Section 2]. The argument for \mathbf{I} -cohomology that we need here is analogous. Let us provide some details, concentrating for simplicity of notation on the case $\mathbf{I}^k = \mathbf{W}$. In the notation above, it suffices to check that $H^*(\Theta_V)$ is the usual pullback along the composition iq . By Proposition 2.9, it suffices to verify that the degree-zero part of the morphism of complexes of sheaves Θ constructed in Lemma 2.44 is compatible with the usual pullback q^*i^* . To verify this, we construct another, auxiliary, morphism of presheaves Θ^{pre} . We then show firstly that Θ^{pre} is indeed compatible with the degree-zero part of Θ , and secondly that Θ^{pre} agrees with the usual pullback q^*i^* on Witt presheaves. The compatibility of Θ with the sheaf pullback q^*i^* then follows.

The auxiliary morphism $\Theta^{\text{pre}} : i_*q_*W(-; q^*i^*\mathcal{L}) \leftarrow W(-; \mathcal{L})$ of presheaves on X is defined over an open subset $U \subset X$ as the composition along the top row of the following diagram:

$$\begin{array}{ccccccc}
 W(\mathcal{N}_{Z_U}U) & \xleftarrow{i_{0*}^{-1}} & W_{\mathcal{N}_{Z_U}U}^1(D(Z_U, U)) & \xleftarrow{\partial} & W(U \times \mathbb{G}_m) & \xleftarrow{m_t} & W(U \times \mathbb{G}_m) & \xleftarrow{p_U^*} & W(U) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 W(\mathcal{N}_0) & \xleftarrow{i_{0*}^{-1}} & W_{\mathcal{N}_0}^1(D_0) & \xleftarrow{\partial} & W(U_0) & & & & \\
 \downarrow \text{(id)} & & \downarrow \text{(c)} & & \downarrow \text{(id)} & & \downarrow & & \downarrow \\
 W(k(\mathcal{N}_Z X)) & \xleftarrow{i_{0*}^{-1}} & W(k(\mathcal{N}_Z X)) & \xleftarrow{\partial} & W(k(X \times \mathbb{G}_m)) & \xleftarrow{m_t} & W(k(X \times \mathbb{G}_m)) & \xleftarrow{p_X^*} & W(k(X))
 \end{array} \tag{2.46}$$

We have suppressed all twists in this diagram, but it should be clear what the correct twists are by comparison with the definition of Θ . The notation Z_U is short for $Z \cap U$, the morphism p_U is the obvious projection, m_t again denotes multiplication with the form $(1, -t)$, the boundary map ∂ in the top row is the boundary map of [Balmer 2000, Theorem 6.2] in the localization sequence of Witt groups

associated with $\mathcal{N}_{Z \cap U} U \hookrightarrow D(Z \cap U, U) \hookrightarrow U \times \mathbb{G}_m$, and i_{0*} in the top row is the dévissage isomorphism of [Gille 2007a, Theorem 4.4]. The composition of i_{0*} with the morphism that forgets support agrees with the pushforward considered in [Calmès and Hornbostel 2011] (compare [Gille 2007a, Remark 4.5] and [Calmès and Hornbostel 2011, §7.2]), so we can use the base change and projection formulas of [Calmès and Hornbostel 2011] in the following.

The composition along the lower row of diagram (2.46) is the degree-zero part of $\Theta(U)$, while the vertical morphisms/compositions are the natural morphisms from the Witt presheaves to the Gersten complexes. We need to show that this diagram commutes. This is clear for the two squares on the right. For the big square on the left, we introduce the auxiliary central row. We denote by D_0 the local ring of $D(Z, X)$ at the generic point of $\mathcal{N}_Z X$, by \mathcal{N}_0 the local ring of $\mathcal{N}_Z X$ at its generic point, and by U_0 the field of fractions of D_0 . Clearly $\mathcal{N}_0 = k(\mathcal{N}_Z X)$ and $U_0 = k(X \times \mathbb{G}_m)$. On the other hand, as $\mathcal{N}_Z X$ is a Cartier divisor, D_0 is a discrete valuation ring with \mathcal{N}_0 as residue field, and the spectrum of U_0 is the open complement of the spectrum of \mathcal{N}_0 in the spectrum of D_0 . Thus, square (b) commutes by naturality of the localization sequence, while square (a) commutes by the base change formula [Calmès and Hornbostel 2011, Theorem 6.9]. The composition $i_{0*}^{-1} \partial$ along the central row can be identified with the residue map using [Balmer and Walter 2002, Lemma 8.1], and hence agrees with the composition $i_{0*}^{-1} \partial$ in the lower row that is used in the construction of Θ .

To prove that Θ^{pre} agrees with the usual pullback $q^* i^*$ on Witt presheaves, we need to evaluate over each open subset $U \subset X$. For simplicity of notation, we spell this out only for $U = X$. We first check that

$$i_{0*}(1) = \partial(m_t(1)) \tag{2.47}$$

when $\mathcal{L} = \mathcal{O}_X$. To see this, we can pass to coherent Witt groups using [Gille 2002, Lemma 3.2]. We leave it as an exercise to chase through Balmer’s cone construction of ∂ and the construction of i_{0*} in [Gille 2007a, Theorem 3.2]; the main additional input is the short exact sequence $0 \rightarrow \mathcal{O}_{D(Z, X)} \xrightarrow{t} \mathcal{O}_{D(Z, X)} \rightarrow i_{0*} \mathcal{O}_{\mathcal{N}_Z X} \rightarrow 0$ on $D(Z, X)$.

Using (2.47), we now find for arbitrary $\alpha \in W^0(X, \mathcal{L})$:

$$\begin{aligned} i_{0*} q^* i^* \alpha &= i_{0*}(1 \cup i_0^* b^* \alpha) \\ &= i_{0*} 1 \cup b^* \alpha && \text{(by the projection formula)} \\ &= \partial(m_t(1)) \cup b^* \alpha && \text{(by (2.47))} \\ &= \partial(m_t(1) \cup t'^* b^* \alpha) && \text{(by the Leibniz rule)} \\ &= \partial(m_t(p_X^* \alpha)) \\ &= i_{0*} \Theta^{\text{pre}}(\alpha). \end{aligned}$$

(For the projection formula, see [Calmès and Hornbostel 2011, Theorem 6.5], and for the Leibniz rule, see [Gille and Nenashev 2003, Theorem 2.10].) This shows that Θ^{pre} agrees with q^*i^* , for arbitrary \mathcal{L} , as claimed. \square

2.C.4. Products. Taking X to be a scheme, $\mathcal{A} := \mathbf{W}$ to be the Witt sheaf and \mathcal{M} and \mathcal{N} to be powers of \mathbf{I} , possibly twisted by line bundles \mathcal{L} and \mathcal{L}' over X , Godement's construction described in Theorem 2.14 furnishes us with a product

$$\begin{aligned} H^p(X, \mathbf{I}^k(\mathcal{L})) \otimes_{\Gamma \mathbf{W}} H^q(X, \mathbf{I}^l(\mathcal{L}')) &\rightarrow H^{p+q}(X, \mathbf{I}^k(\mathcal{L}) \otimes_{\mathbf{W}} \mathbf{I}^l(\mathcal{L}')) \\ &\rightarrow H^{p+q}(X, \mathbf{I}^{k+l}(\mathcal{L} \otimes \mathcal{L}')). \end{aligned}$$

In particular, for trivial twists we obtain:

2.48. Corollary. *Godement's cup product*

$$H^p(X, \mathbf{I}^k) \otimes H^q(X, \mathbf{I}^l) \rightarrow H^{p+q}(X, \mathbf{I}^{k+l})$$

turns $\bigoplus_{p,k} H^p(X, \mathbf{I}^k)$ into a bigraded ring. It is graded commutative with respect to the p -grading and commutative with respect to the k -grading.

We can also construct a cross product for \mathbf{I} -cohomology in this way, as follows. For schemes X and Y over F , Godement's cross product (Definition 2.19) gives us a map $H^*(X, \mathbf{I}^k) \times H^*(Y, \mathbf{I}^l) \rightarrow H^*(X \times Y, \mathbf{I}^k \boxtimes \mathbf{I}^l)$. We suppress the subscript F here to lighten notation, but of course all products considered are scheme-theoretic and taken over F . On the other hand, on $X \times Y$, we have a morphism of sheaves

$$\mathbf{I}_X^k \boxtimes \mathbf{I}_Y^l \rightarrow \mathbf{I}_{X \times Y}^k \otimes \mathbf{I}_{X \times Y}^l \xrightarrow{\sim} \mathbf{I}_{X \times Y}^{k+l}.$$

So by changing coefficients along this morphism, we obtain a product

$$H^p(X, \mathbf{I}^k) \times H^q(Y, \mathbf{I}^l) \rightarrow H^{p+q}(X \times Y, \mathbf{I}^{k+l}).$$

More generally, given line bundles \mathcal{L} on X and \mathcal{L}' on Y , this product takes the form

$$H^p(X, \mathbf{I}^k(\mathcal{L})) \times H^q(Y, \mathbf{I}^l(\mathcal{L}')) \rightarrow H^{p+q}(X \times Y, \mathbf{I}^{k+l}(\mathcal{L} \boxtimes \mathcal{L}')). \quad (2.49)$$

We now compare these products to the products defined by Fasel via Gersten complexes. Fasel first defines a cross product, and then pulls back along the diagonal.

2.50. Theorem. *Consider a product of smooth F -schemes $X \times Y$. Let \mathcal{L}_X and \mathcal{L}_Y be line bundles on X and Y respectively. Define $\mathcal{L}_{XY} := \mathcal{L}_X \boxtimes \mathcal{L}_Y$. For open subsets $U \subset X$ and $V \subset Y$, the exterior product on Witt groups induces a morphism of complexes of abelian groups,*

$$\star : C^\bullet(U, \mathbf{I}^i(\mathcal{L}_X)) \otimes C^\bullet(V, \mathbf{I}^j(\mathcal{L}_Y)) \rightarrow C^\bullet(U \times V, \mathbf{I}^{i+j}(\mathcal{L}_{XY})),$$

as well as a morphism of the corresponding complexes of flasque sheaves on $X \times Y$.

Then the following square (in which the vertical arrows are the canonical augmentations) commutes:

$$\begin{array}{ccc} \mathbf{I}_{\mathcal{L}_X}^i \boxtimes \mathbf{I}_{\mathcal{L}_Y}^j & \xrightarrow{\quad \times \quad} & \mathbf{I}_{\mathcal{L}_{XY}}^{i+j} \\ \downarrow & & \downarrow \\ \mathbf{C}^\bullet(-, \mathbf{I}^i(\mathcal{L}_X)) \boxtimes \mathbf{C}^\bullet(-, \mathbf{I}^j(\mathcal{L}_Y)) & \xrightarrow{\quad \star \quad} & \mathbf{C}^\bullet(-, \mathbf{I}^{i+j}(\mathcal{L}_{XY})) \end{array}$$

Proof. The product on the Gersten complexes is constructed in [Fasel 2007, Theorem 4.16] by replacing G -cohomology with \mathbf{I} -cohomology. This can be done by following the argument in [Fasel 2007, Lemma 4.1 to Proposition 4.7]. By construction, the product is compatible with the augmentation $W(-, \mathcal{L}_X) \rightarrow \mathbf{C}^\bullet(-, \mathbf{I}^0(\mathcal{L}_X))$, so the same is true after restriction to \mathbf{I}^j . \square

2.51. Definition. The product of Gersten complexes in Theorem 2.50 induces a product on cohomology of the form

$$\star : H^p(\mathbf{C}^\bullet(X, \mathbf{I}^i(\mathcal{L}_X))) \otimes H^q(\mathbf{C}^\bullet(Y, \mathbf{I}^j(\mathcal{L}_Y))) \rightarrow H^{p+q}(\mathbf{C}^\bullet(X \times Y, \mathbf{I}^{i+j}(\mathcal{L}_{XY}))).$$

2.52. Proposition. *Fasel’s cross product on \mathbf{I} -cohomology is compatible with Godement’s cross product. Likewise, Fasel’s cup product on \mathbf{I} -cohomology is compatible with Godement’s cup product.*

Proof. The compatibility of the cross products follows from Theorem 2.22: the necessary assumption, namely the compatibility of the cross product on Gersten complexes with the cross product on the \mathbf{I}^j -sheaves, has already been verified in Theorem 2.50. The compatibility of the cup products follows from the compatibility of the cross products and Proposition 2.45, since both cup products can be obtained from the corresponding cross product by pulling back along the diagonal; see (2.21) before Theorem 2.22. \square

2.53. Remark. As the Witt ring is commutative, the only signs appearing in the ring structure on \mathbf{I} -cohomology are those arising from point (iii) of Theorem 2.14. For \mathbf{J} -cohomology we have additional signs coming from the graded commutativity of $\bigoplus_k \mathbf{J}^k \cong \bigoplus_k \mathbf{K}_k^{\text{MW}}$: for the cup product

$$H^p(X, \mathbf{J}^k) \otimes H^q(X, \mathbf{J}^l) \rightarrow H^{p+q}(X, \mathbf{J}^{k+l}),$$

one finds that $a \cup b = (-1)^{pq} \epsilon^{kl} b \cup a$ with $\epsilon = -\langle -1 \rangle$.

2.C.5. Thom class and pushforward. Let $i : Z \hookrightarrow X$ be a regular embedding of regular schemes of codimension r , with normal bundle $p : \mathcal{N} \rightarrow Z$. Denote the dévissage isomorphism of Theorem 2.41 as

$$i_* : H^s(Z, \mathbf{I}^j(i^* \mathcal{L} \otimes \det \mathcal{N})) \xrightarrow{\cong} H_Z^{s+r}(X, \mathbf{I}^{j+r}(\mathcal{L})).$$

The composition of i_* with extension of support agrees with the usual pushforward map in \mathbf{I} -cohomology, as pointed out at the end of the proof of Theorem 2.41.

However, a more indirect definition of i_* will be more useful for the comparison with topology. For this alternative definition, we first consider the case when i is the zero section of a vector bundle:

2.54. Definition. Let $p : V \rightarrow X$ be a rank d vector bundle with zero section $s : X \hookrightarrow V$. The *Thom class* $\text{th}(V) \in H_X^d(V, \mathbf{I}^d(p^* \det V^\vee))$ is the image of the unit element $1 \in H^0(X, \mathbf{W})$ under the isomorphism of Theorem 2.41 and the canonical trivialization of $s^* p^* \det V^\vee \otimes \det V$.

We now define the pushforward in \mathbf{I} -cohomology as follows:

2.55. Definition. Let $i : Z \hookrightarrow X$ be a regular embedding of codimension r between regular F -schemes, with normal bundle $p : \mathcal{N} \rightarrow Z$. The pushforward in \mathbf{I} -cohomology is defined as the composition

$$i_* : H^s(Z, \mathbf{I}^j(i^* \mathcal{L} \otimes \det \mathcal{N})) \xrightarrow{\text{th}(\mathcal{N}) \cup p^*(-)} H_Z^{s+r}(\mathcal{N}, \mathbf{I}^{j+r}(p^* i^* \mathcal{L})) \xrightarrow[\cong]{d(Z, X)} H_Z^{s+r}(X, \mathbf{I}^{j+r}(\mathcal{L})),$$

where $d(Z, X)$ is the deformation isomorphism induced by deformation to the normal bundle. The same name and notation are used for the composition of this map with the extension of support $H_Z^{s+r}(X, \mathbf{I}^{j+r}(\mathcal{L})) \rightarrow H^{s+r}(X, \mathbf{I}^{j+r}(\mathcal{L}))$.

For the deformation to the normal bundle, see Section 2.C.3 above, in particular Figure 1 on page 256. The deformation isomorphism $d(Z, X)$ is defined as the zigzag $d(Z, X) = i_1^* \circ (i_0^*)^{-1}$, in view of the fact that both pullbacks with support i_0^* and i_1^* are isomorphisms; see [Fasel 2009] or [Asok and Fasel 2016, Theorem 2.12]. In the context of \mathbb{A}^1 -homotopy theory, the existence of $d(Z, X)$ is a manifestation of homotopy purity. See [Asok and Fasel 2016, Section 3.5.2] for a more detailed discussion.

In any case, the definition of pushforward using this deformation isomorphism and the Thom class is known to agree with the usual one:

2.56. Lemma. *The dévissage isomorphism of Theorem 2.41 agrees with the composition in Definition 2.55.*

Proof. In this proof, we reserve the notation s_* and i_* for the dévissage isomorphisms of Theorem 2.41. Let $s : Z \rightarrow \mathcal{N}$ denote the zero section of the normal bundle $p : \mathcal{N} \rightarrow Z$. The projection formula [Fasel 2013, §2.2] shows that $s_* = \text{th}(V) \cup p^*(-)$. It remains to show that $d(Z, X)s_* = i_*$. This follows from the excess intersection formula of [loc. cit.], applied to the top two squares in the description of deformation to the normal bundle in Figure 1 on page 256 above Proposition 4.12 below. (Note that the normal bundles of the inclusions $Z \rightarrow Z \times \mathbb{A}^1$ at 0 and 1 are trivial, so the Euler classes appearing in the excess intersection formula are also trivial.) \square

2.D. Reminders on real-étale cohomology. Recall from [Scheiderer 1994] that the real spectrum of a scheme X is given as follows. For a commutative ring A , the *real spectrum* $\text{sper } A$ is the topological space of pairs $(x, <_x)$ where $x \in \text{Spec } A$ is a point and $<_x$ is an ordering of the residue field $\kappa(x)$. A subbasis for the topology is given by domains of positivity $D(a) = \{(x, >_x) \in \text{sper } A \mid a >_x 0 \text{ in } \kappa(x)\}$ for $a \in A$. These data can be glued giving rise to the *real spectrum* X_r of a scheme X . Forgetting the orderings of the residue fields yields a continuous map $\text{supp} : X_r \rightarrow X$ of topological spaces, called the *support map*.

The real-étale topology for a scheme X is then obtained by defining *real-étale coverings* to be those families $\{f_i : U_i \rightarrow X\}_{i \in I}$ of morphisms where each f_i is étale and the induced family $(f_i)_r : (U_i)_r \rightarrow X_r$ of morphisms on real spectra is jointly surjective. For a scheme X , the real-étale site $X_{\text{rét}}$ associated to X is the small étale site of X equipped with the real-étale coverings.

By [Scheiderer 1994, Theorem 1.3], the topos of sheaves on the real spectrum X_r is equivalent to the topos of sheaves on the real-étale site $X_{\text{rét}}$. The equivalence is induced by sending a sheaf \mathcal{F} on X_r to the real-étale sheaf

$$(f : Y \rightarrow X) \in X_{\text{rét}} \mapsto H^0(Y_r, f_r^* \mathcal{F}).$$

In particular, sheaf cohomology can be computed either using the real spectrum or the real-étale site.

For a scheme X , the support map $\text{supp} : X_r \rightarrow X$ induces a direct image functor

$$\text{supp}_* : \text{Ab}(X_r) \rightarrow \text{Ab}(X_{\text{Zar}}) : \mathcal{F} \mapsto ((U \subseteq X) \mapsto \mathcal{F}(\text{supp}^{-1}(U))).$$

By [Scheiderer 1994, Theorem 19.2], this functor is exact and faithful. For any scheme X and any sheaf \mathcal{F} on X_r , [Jacobson 2017, Lemma 4.6] provides natural isomorphisms

$$H^p(X_r, \mathcal{F}) \xleftarrow{\cong} H_{\text{Zar}}^p(X, \text{supp}_* \mathcal{F}) \quad \text{for } p \geq 0$$

which arise as edge maps for the Grothendieck spectral sequence for the composition of supp_* and the global sections functor. (In [loc. cit.] the above arrow is drawn in the wrong direction.) It can be shown that this arrow is given by supp_* constructed as in Definition 2.5 above.

2.E. Reminders on singular cohomology. We write $H^*(-)$ for singular cohomology with integral coefficients and $h^*(-)$ for singular cohomology with $\mathbb{Z}/2$ -coefficients. Similarly, $\text{CH}^*(-)$ denotes Chow groups, and $\text{Ch}^*(-)$ denotes Chow groups modulo two.

There are several well-known related definitions of cohomology for topological spaces, using singular cochains, sheaf cohomology, Čech cohomology, Eilenberg–Mac Lane spectra, etc. Even better, each of these comes with at least one and

often several different constructions of fundamental structures such as pullbacks, pushforwards along closed embeddings, or (cup) products. In this article, we will always use the sheaf cohomology variant and usually restrict to locally constant sheaves. We will not discuss the comparison with all the other wonderful constructions in topology in detail. Standard textbook references for sheaf cohomology include [Bredon 1997; Godement 1958; Iversen 1986].

We briefly recall the definition and key properties of Thom classes, notably the Thom isomorphism; see [Iversen 1986, Sections VIII.2–3] for sheaf cohomology and [Milnor and Stasheff 1974, Sections 8–10] for singular cohomology. Other textbook references on Thom classes in singular cohomology include [Spanier 1966; Switzer 1975; Hatcher 2002; tom Dieck 2008].

2.57. Definition. For any real rank d topological vector bundle $V \rightarrow X$, a *Thom class* $\text{th}(V) \in H_X^d(V, \mathbb{Z}(p^* \det V^\vee))$ is a cohomology class $\text{th}(V)$ such that for any point $x \in X$ its restriction to the fiber V_x is a generator of $H_x^d(V_x, \mathbb{Z})$.

Most references only consider the case of an oriented vector bundle. In this case, or when passing to $\mathbb{Z}/2\mathbb{Z}$ -coefficients, the local coefficient system $\mathbb{Z}(p^* \det V^\vee)$ in Definition 2.57 is constant. For topological Thom classes with twisted coefficients for nonoriented vector bundles, we refer to [Spanier 1966, Chapter 5, Exercise J.4]. In the present article, twisted coefficients in classical topology are described by twisting the locally constant sheaf \mathbb{Z} with a line bundle \mathcal{L} as in Definition 2.23. As explained in Section 2.E.1, these $\mathbb{Z}(\mathcal{L})$ may be translated to other descriptions — e.g., back to local coefficient systems — of twisted coefficients as used, e.g., in [Spanier 1966] and [Milnor and Stasheff 1974] whenever necessary.

Thom classes and local generators — if they exist — are unique up to sign, and this ambiguity disappears if we choose compatible orientations; see, e.g., [Milnor and Stasheff 1974, Theorem 9.1]. Finally, observe that some topology textbooks — but not [Adams 1974, Chapter III.10] and [Switzer 1975, Chapter 14] — work with disk and sphere bundles, rather than vector bundles and complements of zero sections. By excision and homotopy invariance, this doesn't change the relevant cohomology groups with support: we have

$$H_{\{0\}}^t(\mathbb{R}^d, \mathbb{Z}) = H^t(\mathbb{R}^d, \mathbb{R}^d - \{0\}, \mathbb{Z}) \cong H^t(D^d, S^{d-1}, \mathbb{Z}),$$

and the “topological Thom space” of the free real rank d bundle over a point is $D^d/S^{d-1} = S^d$.

The existence of a Thom class for a rank d vector bundle $V \rightarrow X$ leads to a *Thom isomorphism*

$$\begin{aligned} H^*(X, \mathbb{Z}) &\rightarrow H_X^{*+d}(V, \mathbb{Z}(p^* \det V^\vee)), \\ u &\mapsto \text{th}(V) \cup p^*(u), \end{aligned}$$

as explained in any of the references above; see, e.g., [Spanier 1966, Theorem 5.7.10], [Hatcher 2002, Corollary 4D.9] or [tom Dieck 2008, Theorem 17.9.1]. In particular, the image of $1 \in H^0(X, \mathbb{Z})$ under the Thom isomorphism is $\text{th}(V)$.

Composing the Thom isomorphism $H^*(Z) \rightarrow H_Z^{*+r}(\mathcal{N})$ for a closed embedding $i : Z \rightarrow X$ of codimension r with an oriented normal bundle \mathcal{N} with a “deformation to the normal bundle” or “tubular neighborhood” isomorphism and then forgetting the support $H_Z^*(X) \rightarrow H^*(X)$ is one of the standard definitions of the pushforward both for cohomology of abelian sheaves and for singular cohomology, which in this setting is sometimes called the “Gysin map”. Some further details are provided in the proofs of Theorem 4.7 and Proposition 4.12 below.

2.E.1. Reminders on line bundles and local systems. Recall that a *local system* on a topological space X is a functor from the fundamental groupoid of X to the category of abelian groups; see [Spanier 1966, Exercises 1.F]. For path-connected spaces, local systems with values in a fixed abelian group A are equivalent to group homomorphisms $f : \pi_1(X, x) \rightarrow \text{Aut}(A)$. For reasonable spaces like topological manifolds, local systems are moreover equivalent to locally constant sheaves of abelian groups; see [Spanier 1966, Exercise 6.F] or [Iversen 1986, Theorem IV.9.7].

We will be mainly interested in locally constant sheaves of rank one, i.e., sheaves whose stalks at all points are isomorphic to \mathbb{Z} . These correspond to rank-one local systems (or, for path-connected X , to homomorphisms $f : \pi_1(X, x) \rightarrow \text{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$).

Any real line bundle \mathcal{L} over X determines a rank-one locally constant sheaf $\mathbb{Z}(\mathcal{L})$, as follows. Let \mathcal{C}_X denote the sheaf of continuous real-valued functions on X . For any nowhere-vanishing function f on an open subset $U \subset X$, the sign map defines a locally constant map $\text{sign}(f) : U \rightarrow \{\pm 1\}$. We thus obtain a morphism of abelian sheaves $\mathcal{C}_X^\times \rightarrow \{\pm 1\}$. As $\{\pm 1\}$ acts on any sheaf of abelian groups \mathcal{A} in a canonical way, we obtain an induced action of \mathcal{C}_X^\times on \mathcal{A} . So any sheaf of abelian groups \mathcal{A} on X can be viewed as a $\mathbb{Z}[\mathcal{C}_X^\times]$ -module. We can therefore apply the construction of Definition 2.23 to \mathcal{A} and any line bundle \mathcal{L} over X to obtain a twisted sheaf $\mathcal{A}(\mathcal{L}) := \mathcal{A} \otimes_{\mathbb{Z}[\mathcal{C}(X)^\times]} \mathbb{Z}[\mathcal{L}^\times]$. In particular, taking \mathcal{A} to be the constant sheaf \mathbb{Z} , we obtain a locally constant sheaf $\mathbb{Z}(\mathcal{L})$, as promised, and we write $\mathbb{Z}(\mathcal{L})$ for the associated local system.

2.58. Proposition. *For any paracompact Hausdorff space X , the assignment that associates with a line bundle \mathcal{L} the locally constant sheaf $\mathbb{Z}(\mathcal{L})$ induces a bijection between the set of isomorphism classes of real line bundles over X and the set of isomorphism classes of rank-one locally constant sheaves on X . Both sets can moreover be identified with the (sheaf) cohomology group $H^1(X, \mathbb{Z}/2\mathbb{Z})$ in a natural way, such that the tensor product of line bundles/sheaves corresponds to the addition of cohomology classes. \square*

Over a scheme X/\mathbb{R} , an algebraic line bundle \mathcal{L} has an underlying real line bundle $\mathcal{L}_{\mathbb{R}}$ over $X(\mathbb{R})$. Let $\gamma : \text{Pic}(X)/2 \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ denote the natural homomorphism that sends the isomorphism class of an algebraic line bundle \mathcal{L} (up to squares) to the isomorphism class of the real topological line bundle $\mathcal{L}_{\mathbb{R}}$.

2.59. Proposition. *For any scheme X/\mathbb{R} , the map*

$$\gamma : \text{Pic}(X)/2 \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$$

described above is a group homomorphism. The map is an isomorphism for smooth projective cellular varieties.

Proof. The assignment that maps an algebraic line bundle \mathcal{L} over X/\mathbb{R} to the real line bundle $\mathcal{L}_{\mathbb{R}}$ over $X(\mathbb{R})$ is functorial and hence gives rise to a well-defined morphism on sets of isomorphism classes. The tensor product of algebraic line bundles is induced from the multiplication in \mathbb{G}_m , and the tensor product of real line bundles is induced from the multiplication in \mathbb{R}^{\times} . The above map from algebraic to real line bundles takes the algebraic transition functions to the corresponding continuous functions on real points. This is obviously compatible with the multiplication, and consequently with the tensor product of line bundles. This proves the first statement.

The second statement is a special case of Proposition 5.3 below. \square

2.60. Remark. The natural homomorphism $\gamma : \text{Pic}(X)/2 \rightarrow H^1(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is in general not surjective. For example, when $X = \mathbb{A}^2 \setminus \{0\}$ we have $\text{Pic}(\mathbb{A}^2 \setminus \{0\})/2 \cong \text{Ch}^1(\mathbb{A}^2 \setminus \{0\}) = 0$ by the localization sequence for Chow groups. On the other hand, $(\mathbb{A}^2 \setminus \{0\})(\mathbb{R}) \simeq S^1$ and $H^1(S^1, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Thus there exists a nontrivial local system on the real points of $\mathbb{A}^2 \setminus \{0\}$. In particular, the isomorphism for smooth projective cellular varieties does not extend to “linear” varieties.

To see that in general γ is not injective either, consider an elliptic curve E/\mathbb{R} such that $E(\mathbb{R})$ is isomorphic to $S^1 \times \mathbb{Z}/2\mathbb{Z}$, e.g., the completion of the affine curve given by $Y^2 = X^3 - X$. Then there are rational points on E which are not 2-divisible, and hence $\text{Pic}^0(E)/2$ will be nontrivial. Any corresponding nontrivial line bundle will map to 0 under γ .

2.E.2. Comparing real-étale and singular cohomology. For $F = \mathbb{R}$, we wish to identify real-étale cohomology with singular cohomology over \mathbb{R} . A general reference is [Scheiderer 1994]; some further discussion can be found in [Jacobson 2017, Remark 4.4].

The inclusion $\iota : X(\mathbb{R}) \rightarrow X_r$ sending x to $(x, \mathbb{R}_{\geq 0})$ is continuous. Let \mathcal{F} be a locally constant sheaf on X_r . Consider the sheaf $\iota^*\mathcal{F}$ on $X(\mathbb{R})$. This is a locally constant sheaf. If X is separated of finite type over \mathbb{R} and M is an abelian group, then we have a canonical isomorphism $H^i(X_r, M) \cong H_{\text{sing}}^i(X(\mathbb{R}), M)$; see, e.g., [Jacobson 2017, Remark 4.4]. This is generalized by the following lemma.

2.61. Lemma. *If X is a scheme over $F = \mathbb{R}$, then the map $\iota^* : H^i(X_r, \iota_*\mathcal{F}) \rightarrow H^i(X(\mathbb{R}), \mathcal{F})$ is an isomorphism for any **locally** constant sheaf \mathcal{F} on $X(\mathbb{R})$. Moreover, ι_* defines an equivalence of categories between the category of constant sheaves on X_r and the category of constant sheaves on $X(\mathbb{R})$. For an abelian group M and associated constant sheaves M_r and $M_{\mathbb{R}}$ on X_r and $X(\mathbb{R})$, respectively, we have $\iota_*M_{\mathbb{R}} = M_r$.*

Proof. For the first statement, see [Delfs 1991, Chapter II, Theorem 5.7]. For the second statement, see [Jacobson 2017, Remark 4.4]. \square

To establish the desired compatibility results of the above isomorphism ι^* with various functors (see Remark 4.2), one employs sheaf-theoretic arguments. In particular, given a map $f : X \rightarrow Y$ of schemes, we have a commutative square of continuous maps $f_r \circ \iota_X = \iota_Y \circ f(\mathbb{R})$ which still commutes when applying $(-)^*$.

3. Cycle class maps for real varieties

In the following section, we will discuss real cycle class maps for real varieties and schemes. We fix a scheme X over \mathbb{R} , and we write $X(\mathbb{R})$ for the set of real points of X equipped with the analytic topology, and $\rho : X(\mathbb{R}) \hookrightarrow X$ for the inclusion of the real points into the underlying topological space of the variety X .

To define a cycle class map with target the singular cohomology of $X(\mathbb{R})$, there are three approaches one could pursue:

- (A) A sheaf-theoretic approach to the definition of cycle class maps has been developed, for mod 2 coefficients in [Scheiderer 1995] and for integral coefficients in [Jacobson 2017]. This factors through real or real-étale cohomology as an intermediate step.
- (B) A more “geometric” approach would be through a definition of fundamental classes in Borel–Moore homology of the real points. This is similar to the cycle class map for complex varieties [Fulton 1998] and was done for mod 2 homology in [Borel and Haefliger 1961].
- (C) On a formal homotopy-theoretic level, one can also define a cycle class map as follows. The cohomology theory $\bigoplus_{i,j} H^i(-, \mathbf{I}^j)$ (here with a full bigrading) is representable in the stable homotopy category $\mathcal{SH}(F)$ by the Eilenberg–Mac Lane spectrum for the homotopy module \mathbf{I}^\bullet . Associated with the real embedding $\sigma : F \hookrightarrow \mathbb{R}$ there is a real realization functor $\text{Real} : \mathcal{SH}(F) \rightarrow \mathcal{SH}$ which maps a smooth F -scheme X to $X(\mathbb{R})$ equipped with the usual euclidean topology; see [Bachmann 2018, Section 10]. Formally, the real realization functor will induce a cycle class map

$$H^i(X, \mathbf{I}^j) = [X, \mathbb{H}\mathbf{I}^j[i]]_{\mathcal{SH}(F)} \rightarrow [X(\mathbb{R}), \text{Real}(\mathbb{H}\mathbf{I}^j)[i]]_{\mathcal{SH}}.$$

In this section, we will consider the first two of these possible definitions of real cycle class maps and show that they agree. (See Remark 3.9 for the third approach.) We explain the sheaf-theoretic construction of the cycle class map in Section 3.A, along with modifications involving twists and cohomology with supports in Section 3.B. Then we will discuss a more geometric description of the cycle class map (related to fundamental classes in Borel–Moore homology) in Section 3.C. Finally, we will discuss an extension of the cycle class map to the Chow–Witt ring in Section 3.D. Compatibilities of these identifications and the basic operations in sheaf cohomology (pullbacks, pushforwards, products etc) will be established in Section 4.

3.A. Sheaf-theoretic description of cycle class map. We first consider the sheaf-theoretic approach where the real cycle class map is obtained as the following composition:

$$\begin{array}{ccccc}
 & & H^n(X, \mathbf{I}^t) & & \\
 & \swarrow \text{can} & \downarrow & \searrow \text{real cycle class} & \\
 H^n(X, \text{colim}_j \mathbf{I}^j) & \xrightarrow[\text{sign}_\infty]{\cong} & H^n(X, \rho_* \mathbb{Z}) & \xrightarrow[\rho^*]{\cong} & H^n(X(\mathbb{R}), \mathbb{Z})
 \end{array} \tag{3.1}$$

We first discuss the map can in the diagram which is induced by the canonical map of Zariski sheaves $\text{can} : \mathbf{I}^t \rightarrow \text{colim}_j \mathbf{I}^j$.

3.2. Definition. Multiplication with $\langle\langle -1 \rangle\rangle$ induces morphisms $\langle\langle -1 \rangle\rangle : \mathbf{I}^j \rightarrow \mathbf{I}^{j+1}$. These maps give rise to a diagram,

$$\dots \rightarrow \mathbf{I}^j \xrightarrow{\langle\langle -1 \rangle\rangle} \mathbf{I}^{j+1} \xrightarrow{\langle\langle -1 \rangle\rangle} \mathbf{I}^{j+2} \rightarrow \dots$$

We denote the colimit of this system by $\text{colim}_j \mathbf{I}^j$.

3.3. Proposition. *The morphisms $\langle\langle -1 \rangle\rangle : \mathbf{I}^j \rightarrow \mathbf{I}^{j+1}$ as well as the canonical map $\text{can} : \mathbf{I}^t \rightarrow \text{colim}_j \mathbf{I}^j$ are \mathbb{G}_m -equivariant morphisms of sheaves with \mathbb{G}_m -action on the big site $\text{Sm}_{F, \text{Zar}}$. In particular, the results of Section 2.A.4 apply.*

Proof. Multiplication by $\langle\langle -1 \rangle\rangle$ is compatible with residue maps for \mathbf{I}^j because it is a global unramified symbol. This implies that $\langle\langle -1 \rangle\rangle : \mathbf{I}^j \rightarrow \mathbf{I}^{j+1}$ induces a morphism of unramified Zariski sheaves. It is also compatible with the \mathbb{G}_m -action, because both $\langle\langle -1 \rangle\rangle$ and multiplication by $\langle u \rangle$ arise from the \mathbf{K}_*^{MW} -module structure of \mathbf{I}^\bullet . Consequently, the colimit $\text{colim}_j \mathbf{I}^j$ is a sheaf of abelian groups with \mathbb{G}_m -action on the big site $\text{Sm}_{F, \text{Zar}}$, and the canonical map $\text{can} : \mathbf{I}^i \rightarrow \text{colim}_j \mathbf{I}^j$ is compatible with these structures. \square

3.4. Remark. There is a more sophisticated interpretation of this map. Namely, the collection of \mathbb{G}_m -sheaves \mathbf{I}^t may be interpreted as a homotopy module, as studied in detail in [Morel 2012]. See Examples 3.33 and 3.34 as well as Lemma 3.35

there; the relevant contractions are computed in [Asok and Fasel 2014, Lemma 2.7, Proposition 2.8]. Then the map $I^t \rightarrow \operatorname{colim}_j I^j$ is nothing but the localization of the homotopy module I^t with respect to $-[-1] \in K_1^{\text{MW}}(F)$. Using results and techniques from [Morel 2012] and [Bachmann 2018], it is possible to obtain many of the results in the following sections using the theory of homotopy modules. We do not pursue this approach here.

3.5. Remark. The product $I^k \otimes I^l \rightarrow I^{k+l}$ induces a product on $\operatorname{colim}_j I^j$ such that the following diagram is commutative:

$$\begin{array}{ccc} I^k \otimes I^l & \longrightarrow & I^{k+l} \\ \downarrow \text{can} \otimes \text{can} & & \downarrow \text{can} \\ \operatorname{colim}_j I^j \otimes \operatorname{colim}_j I^j & \longrightarrow & \operatorname{colim}_j I^j \end{array}$$

Next, following [Scheiderer 1994] and [Jacobson 2017], we recall the map of Zariski sheaves which the latter reference denotes by $\operatorname{colim}_j I^j \rightarrow \operatorname{supp}_* \circ \iota_* \mathbb{Z}$. Here $\iota : X(\mathbb{R}) \rightarrow X_r$ is the natural morphism from the real points of X to the real spectrum, and $\operatorname{supp} : X_r \rightarrow X_{\text{Zar}}$ is the natural morphism from the real spectrum to the Zariski topology of X . Since the real spectrum and real-étale topology are not relevant for our considerations, we will only deal with the composition $\rho = \operatorname{supp} \circ \iota : X(\mathbb{R}) \rightarrow X_{\text{Zar}}$ taking a real point to the corresponding closed point in X .

3.6. Definition. As in [Jacobson 2017, Section 8], the signature of Definition 2.35 induces a ring homomorphism $\operatorname{sign} : W(X) \rightarrow H^0(X_r, \mathbb{Z}) \cong H^0(X(\mathbb{R}), \mathbb{Z})$. After Zariski sheafification and restriction to fundamental ideals, this induces a morphism of sheaves $\operatorname{sign} : I^n \rightarrow \rho_* 2^n \mathbb{Z}$ which we can interpret as morphisms of coefficient data:

$$(X, I^j) \xleftarrow{(\rho, \operatorname{sign})} (X(\mathbb{R}), 2^j \mathbb{Z}).$$

Since $\langle\langle -1 \rangle\rangle$ realizes to $2 \in W(\mathbb{R}) \cong \mathbb{Z}$, we get a morphism of diagrams inducing a morphism

$$\operatorname{sign}_\infty : \operatorname{colim}_j I^j \rightarrow \rho_* \mathbb{Z}.$$

The following proposition is a direct consequence of the fundamental work of Jacobson [2017]:

3.7. Proposition. *The morphism $\operatorname{sign}_\infty : \operatorname{colim}_n I^n \rightarrow \rho_* \mathbb{Z}$ is an isomorphism of sheaves of abelian groups with \mathbb{G}_m -action on the big site $\operatorname{Sm}_{F, \text{Zar}}$. In particular, the results of Section 2.A.4 can be applied.*

We have a commutative diagram of sheaves with \mathbb{G}_m -action on the big site:

$$\begin{array}{ccc}
 \mathbf{I}^j & \xrightarrow{\text{can}} & \text{colim}_j \mathbf{I}^j \\
 \text{sign} \downarrow & & \downarrow \text{sign}_\infty \\
 \rho_* 2^j \mathbb{Z} & \xrightarrow[\text{can}]{\cong} & \rho_*(\text{colim}_j 2^j \mathbb{Z}) \cong \rho_* \mathbb{Z}
 \end{array}$$

A similar statement is true for the mod 2 identification $\text{colim}_n \bar{\mathbf{I}}^n \cong \rho_*(\mathbb{Z}/2\mathbb{Z})$.

Proof. The isomorphism follows from [Jacobson 2017, Theorem 8.6], keeping in mind that $\rho_* = \text{supp}_* \circ \iota_*$ and $\iota_* \mathbb{Z} = \mathbb{Z}$. We now explain where the \mathbb{G}_m -action on $\rho_* \mathbb{Z}$ comes from. For X a scheme over \mathbb{R} , $U \subseteq X$ open and $u \in \mathcal{O}_X(U)^\times$ a unit, we get a continuous invertible function $U(\mathbb{R}) \rightarrow \mathbb{R}^\times$. The \mathbb{G}_m -action is then given by multiplying a section $\sigma \in (\rho_* \mathbb{Z})(U) \cong \mathbb{Z}(U(\mathbb{R}))$ by the composition $U(\mathbb{R}) \rightarrow \mathbb{R}^\times \rightarrow \pm 1$. The compatibility of the signature maps sign and sign_∞ with the \mathbb{G}_m -actions is then clear. The commutativity of the diagram is then also clear; see also [Jacobson 2017, Section 8]. \square

3.8. Remark. The identification $\text{colim}_n \mathbf{I}^n \xrightarrow{\cong} \rho_* \mathbb{Z}$ is also compatible with transfers on fields; see [Bachmann 2018, Lemma 20]. This ultimately implies compatibility with pushforwards along finite morphisms, but we won't discuss this in the present paper.

The identification is also compatible with residue maps, hence induces a morphism of Gersten complexes. The residue maps for $\rho_* \mathbb{Z}$ are denoted by β and their compatibility with the residue maps on \mathbf{I}^n and $\text{colim}_j \mathbf{I}^j$ are discussed in [Jacobson 2017, Section 3].

It should be noted at this point that the left triangle in diagram (3.1) commutes by definition because the identification $\text{sign}_\infty : \text{colim}_j \mathbf{I}^j \xrightarrow{\cong} \rho_* \mathbb{Z}$ is induced from the signature map. The morphism $H^s(X, \mathbf{I}^t) \rightarrow H^s(X, \rho_* \mathbb{Z})$ can either be viewed as $\text{sign}_\infty \circ \text{can}$ or $\text{can} \circ \text{sign}$ in the diagram of Proposition 3.7.

The final morphisms in diagram (3.1) are simply pullback morphisms induced by the morphism $\rho : X(\mathbb{R}) \rightarrow X_{\text{Zar}}$, as in Example 2.3 and Definition 2.5. The fact that these pullback morphisms are in fact isomorphisms is established in [Jacobson 2017, Remark 4.4 and Lemma 4.6] and [Delfs 1991, Chapter II, Theorem 5.7].

3.9. Remark. We can already outline how the formal homotopy-theoretic definition of the cycle class map agrees with the sheaf-theoretic definition. A central point in the sheaf-theoretic approach to cycle class maps is Jacobson's identification $\text{colim}_n \mathbf{I}^n \cong \rho_* \mathbb{Z}$. Combining this with [Bachmann 2018, Corollary 38], we see that the Eilenberg–Mac Lane spectrum for \mathbf{I}^\bullet is mapped by real realization to the

Eilenberg–Mac Lane spectrum $\mathbb{H}\mathbb{Z}$ representing integral cohomology. Moreover, it is possible to trace through the definitions to see that the formal homotopy-theoretic cycle class map agrees with the sheaf-theoretic, essentially by virtue of the definition (or [Bachmann 2018, Proposition 36]). For a comparison of Thom classes obtained this way with ours, the reader may consult [Asok and Fasel 2016, Section 3.5].

3.10. Remark. Our discussion of cycle classes above makes no assumptions about singularities of the schemes involved, similarly to the discussion in [Scheiderer 1995] and [Jacobson 2017] for cycle classes with mod 2 and integral coefficients, respectively. However, some of the compatibility results discussed later will require restriction to smooth schemes.

3.B. Remarks on twisting and supports. In this section, we discuss the modifications necessary to add twists by line bundles and supports to the real cycle class map discussed above. For a scheme X/\mathbb{R} with closed subvariety $Z \hookrightarrow X$ and a line bundle \mathcal{L} over X , this extension of the real cycle class map should have the form

$$H_Z^s(X, \mathbf{I}^t(\mathcal{L})) \rightarrow H_{Z(\mathbb{R})}^s(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})).$$

As previously, the cycle class map is obtained as a composition in a diagram:

$$\begin{array}{ccccc}
 & & H_Z^n(X, \mathbf{I}^t(\mathcal{L})) & & \\
 & \swarrow \text{can} & \downarrow & \searrow \text{real cycle class} & \\
 H_Z^n(X, \text{colim}_j \mathbf{I}^j(\mathcal{L})) & \xrightarrow[\text{sign}_\infty]{\cong} & H_Z^n(X, \rho_* \mathbb{Z}(\mathcal{L})) & \xrightarrow[\rho^*]{\cong} & H_{Z(\mathbb{R})}^n(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))
 \end{array} \tag{3.11}$$

We will spend the rest of the section explaining why the maps in the above diagram exist, and discuss some of their properties.

By Proposition 3.3, we know that $\text{can} : \mathbf{I}^t \rightarrow \text{colim}_j \mathbf{I}^j$ is a \mathbb{G}_m -equivariant morphism of sheaves on the big site. In particular, Definition 2.23 provides, for any real scheme X with a line bundle \mathcal{L} , a twisted canonical map $\text{can} : \mathbf{I}^t(\mathcal{L}) \rightarrow \text{colim}_j \mathbf{I}^j(\mathcal{L})$ of sheaves on the small Zariski site of X . Note also that the isomorphism of Proposition 2.32 is compatible with the \mathbb{G}_m -action, and hence we have an induced isomorphism $\text{colim}_n \mathbf{I}_\mathcal{L}^n \cong \text{colim}_n \mathbf{I}^n(\mathcal{L})$. Consequently, there is no difference between the two possible ways to twist the map $\text{can} : \mathbf{I}^t \rightarrow \text{colim}_j \mathbf{I}^j$ discussed in Definitions 2.31 and 2.23. Now the first map in the diagram,

$$\text{can} : H_Z^s(X, \mathbf{I}^t(\mathcal{L})) \rightarrow H_Z^s(X, \text{colim}_j \mathbf{I}^j(\mathcal{L})),$$

is then given by the pullback with supports (see Definition 2.8) for the twisted canonical map $\mathbf{I}^t(\mathcal{L}) \rightarrow \text{colim}_j \mathbf{I}^j(\mathcal{L})$. The same argument applies to the identification $\text{sign}_\infty : \text{colim}_j \mathbf{I}^j \xrightarrow{\cong} \rho_* \mathbb{Z}$, by Proposition 3.7.

We briefly discuss the middle vertical arrow

$$H_Z^s(X, \mathbf{I}^t(\mathcal{L})) \rightarrow H_Z^n(X, \rho_*\mathbb{Z}(\mathcal{L})),$$

which can be viewed as a twisted version of the signature map. In fact, this morphism is the pullback on cohomology with supports for the morphism of sheaves $\mathbf{I}^t(\mathcal{L}) \rightarrow \rho_*\mathbb{Z}(\mathcal{L})$, and the latter is simply the composition $\text{sign}_\infty \circ \text{can} = \text{can} \circ \text{sign}$ of Proposition 3.7. It would also be possible to write out in detail how to obtain this twisted version of the signature from Definition 2.35, but we won't need this.

The signature is equivariant with respect to the natural \mathbb{G}_m -actions on both sides, so we can twist this morphism by an arbitrary line bundle \mathcal{L} over X and get the following twisted morphism of ringed coefficient data:

$$(X(\mathbb{R}), \mathbb{Z}(\rho^*\mathcal{L})) \xrightarrow{(\rho, \text{sign}_\infty)} (X, \text{colim}_j \mathbf{I}^j(\mathcal{L})).$$

For a closed subscheme $Z \subset X$, Definition 2.8 provides an induced pullback morphism on cohomology with supports

$$(\rho, \text{sign}_\infty)^* : H_Z^s(X, \text{colim}_j \mathbf{I}^j(\mathcal{L})) \rightarrow H_{Z(\mathbb{R})}^s(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})).$$

The real cycle class map in diagram (3.11) is the composition of the previous twisted signature map with the above pullback with supports.

We briefly mention the following twisted version of Jacobson's result [2017, Theorem 8.6(i) and Corollary 8.9(i)]. As in the untwisted case, it holds for schemes X with 2 invertible in \mathcal{O}_X rather than only schemes over \mathbb{R} .

3.12. Theorem. *Let \mathcal{L} be a line bundle on X .*

- (1) *The twisted signature map $\text{sign}_\infty : \text{colim}_n \mathbf{I}^n(\mathcal{L}) \rightarrow (\rho_*\mathbb{Z})(\mathcal{L}) \cong \rho_*(\mathbb{Z}(\rho^*\mathcal{L}))$ is an isomorphism. (Here we use the canonical isomorphism of Lemma 2.24.)*
- (2) *The map*

$$\text{sign}_\mathcal{L} : H^i(X, \mathbf{I}^j(\mathcal{L})) \rightarrow H^i(X, \rho_*\mathbb{Z}(\mathcal{L})) \cong H^i(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}(\mathbb{R})))$$

induced from the twisted signature map is an isomorphism of groups for $j \geq \dim(X) + 1$.

Proof. (1) The map is obtained using Definition 2.23 applied to the map sign_∞ , noting that Proposition 3.7 provides the required \mathbb{G}_m -equivariance. Moreover, it follows from Definition 2.23 that twisting an isomorphism of sheaves by a line bundle produces an isomorphism. The claim then follows locally from the untwisted isomorphism $\text{sign}_\infty : \text{colim}_j \mathbf{I}^j \xrightarrow{\cong} \rho_*\mathbb{Z}$ of [Jacobson 2017, Theorem 8.6].

(2) This follows from (1), as Corollary 8.9(i) follows from Theorem 8.6(i) in [Jacobson 2017]. \square

3.13. Remark. It follows easily from the above that the real cycle class maps are compatible with the maps $H_Z^n(X, I^t) \rightarrow H^n(X, I^t)$ forgetting the support.

3.C. Geometric description of cycle class maps. In this section, we discuss a more geometric and explicit description of the real cycle class map which is mostly based on the coniveau spectral sequence. It produces Gersten–Rost–Schmid-type complexes which compute the integral singular cohomology of $X(\mathbb{R})$ for a smooth \mathbb{R} -scheme X . This is an integral version of the Gersten-type resolutions in [Scheiderer 1995].

First, we note that the second part of the real cycle class map is given by an isomorphism

$$H^s(X, \rho_*\mathbb{Z}(\mathcal{L})) \xrightarrow{\cong} H^s(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})).$$

To make this isomorphism more explicit, we consider the coniveau spectral sequence computing $H^s(X, \rho_*\mathbb{Z}(\mathcal{L}))$ arising from the filtration by codimension of supports.

3.14. Proposition. *Let X be a smooth scheme over \mathbb{R} and let \mathcal{L} be a line bundle over X . Then there is a complex of abelian groups*

$$0 \rightarrow \bigoplus_{x \in X^{(0)}} H_x^0(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \rightarrow \bigoplus_{x \in X^{(1)}} H_x^1(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \rightarrow \bigoplus_{x \in X^{(2)}} H_x^2(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \rightarrow \dots$$

whose q -th cohomology is canonically isomorphic to $H^q(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$. Here, we use the notation

$$H_x^*(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) := \operatorname{colim}_{(\bar{x} \cap U)(\mathbb{R})} H_{(\bar{x} \cap U)(\mathbb{R})}^*(U(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$$

with the colimit taken over Zariski open neighborhoods U of x .

Proof. The argument is the standard one for coniveau spectral sequences; see [Scheiderer 1995, Theorem 2.1]. As in that work, let \mathcal{Z}^p be the set of closed reduced subschemes of X of codimension $\geq p$, and $\mathcal{Z}^p/\mathcal{Z}^{p+1}$ be the set of pairs (Y, Z) with $Y \in \mathcal{Z}^p$, $Z \in \mathcal{Z}^{p+1}$ and $Z \subset Y$. Defining

$$H_{\mathcal{Z}^p/\mathcal{Z}^{p+1}}^n(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) := \operatorname{colim}_{(Y,Z) \in \mathcal{Z}^p/\mathcal{Z}^{p+1}} H_{(Y \setminus Z)(\mathbb{R})}^n((X \setminus Z)(\mathbb{R}), \mathbb{Z}(\mathcal{L})),$$

the arguments in [Scheiderer 1995] (including the well-known analogue of the semipurity statement [Scheiderer 1995, Corollary 1.12] for singular cohomology of real points) imply the existence of a spectral sequence,

$$E_1^{p,q} = H_{\mathcal{Z}^p/\mathcal{Z}^{p+1}}^{p+q}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \Rightarrow H^{p+q}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})).$$

For the rest of the argument, the isomorphism

$$\phi : H_{\mathcal{Z}^p/\mathcal{Z}^{p+1}}^*(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \cong \bigoplus_{x \in X^{(p)}} H_x^*(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$$

is established as in [Scheiderer 1995].

The claim then follows from the vanishing of $H_x^n(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$ for $n \geq \text{codim } x$. By considering small enough Zariski neighborhoods of x , we can assume that $\bar{x} \cap U$ is smooth in U , and the required vanishing of such local cohomology groups follows from the cohomological dimension properties of singular cohomology. \square

3.15. Remark. Since the complex is a row in the E_1 -term of a spectral sequence arising from an exact couple, we can describe the canonical morphisms from the cohomology of the complex in Proposition 3.14 to $H^q(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$. We use the notation from the proof of the proposition. Consider the relevant piece of the complex:

$$\begin{aligned} \cdots \rightarrow \bigoplus_{x \in X^{(q-1)}} H_x^{q-1}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) &\rightarrow \bigoplus_{x \in X^{(q)}} H_x^q(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \\ &\xrightarrow{\partial} \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \rightarrow \cdots \end{aligned}$$

The boundary morphism is given by the boundary in the localization sequences. For a pair $(Y, Z) \in \mathcal{Z}^q/\mathcal{Z}^{q+1}$ it maps a class in $H_{(Y \setminus Z)(\mathbb{R})}^q((X \setminus Z)(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$ to its boundary in $H_{Z(\mathbb{R})}^{q+1}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$. Consequently, a class in the kernel of ∂ is actually closed in $X(\mathbb{R})$, and the canonical map $\ker \partial \rightarrow H^q(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$ is given by lifting the class to $H_{Y(\mathbb{R})}^q(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$ and then forgetting the support.

In the complex of Proposition 3.14, we can further identify the local cohomology groups. The result is actually the Rost–Schmid complex in the sense of Morel [2012] for the strictly \mathbb{A}^1 -invariant sheaf of abelian groups $\text{colim}_j \mathbf{I}^j \cong \rho_* \mathbb{Z}$. For details on the construction of Rost–Schmid complexes (in particular, the description of the differential, which we won’t reproduce here), see [Morel 2012, Section 5.1].

3.16. Definition. For a point x of a smooth \mathbb{R} -scheme X with residue field $\kappa(x)$, we define

$$H^0(x, \rho_* \mathbb{Z}) := \text{colim}_U H^0((\bar{x} \cap U)(\mathbb{R}), \mathbb{Z}),$$

with the colimit taken over the Zariski neighborhoods U of $x \in X$. This colimit has a natural action of $\kappa(x)^\times$ defined as follows: A section σ of $\rho_* \mathbb{Z}$ on a neighborhood U of x is a section of \mathbb{Z} on $U(\mathbb{R})$. For a sufficiently small neighborhood U of x , an element $f \in \kappa(x)^\times$ yields an invertible \mathbb{R} -valued function $f : U(\mathbb{R}) \rightarrow \mathbb{R}$, and the action is given by $(f, \sigma) \mapsto \text{sgn}(f) \cdot \sigma$. More generally,

for a one-dimensional $\kappa(x)$ -vector space L , we define

$$H^0(x, \rho_*\mathbb{Z}(L)) := H^0(x, \rho_*\mathbb{Z}) \otimes_{\mathbb{Z}[\kappa(x)^\times]} \mathbb{Z}[L^\times].$$

3.17. Remark. Given a line bundle \mathcal{L} over X , or over a Zariski open neighborhood of x in X , let \mathcal{L}_x denote the fiber of \mathcal{L} at x . For this one-dimensional $\kappa(x)$ -vector space, we have canonical isomorphisms

$$H^0(x, \rho_*\mathbb{Z}(\mathcal{L}_x)) \cong \operatorname{colim}_U H^0((\bar{x} \cap U)(\mathbb{R}), \mathbb{Z}(\mathcal{L})),$$

where the colimit is again taken over the Zariski neighborhoods U of $x \in X$.

3.18. Proposition. *Let X be a smooth \mathbb{R} -scheme and let \mathcal{L} be a line bundle over X . For a point $x \in X$ of codimension p , we have $H_x^n(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) = 0$ for $n \neq p$, and there is a canonical isomorphism*

$$H_x^p(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \cong H^0(x, \rho_*\mathbb{Z}(\mathcal{L}_x \otimes \Lambda_x^X)),$$

where $\Lambda_x^X := \bigwedge_{\kappa(x)}^p (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ is the fiber of the conormal sheaf of x in X .

Proof. Note that the colimits on both sides have the same indexing sets, the Zariski neighborhoods U of the point $x \in X$. In particular, for a Zariski neighborhood U of x and its closed subspace $Y := \bar{x} \cap U$, it suffices to produce canonical isomorphisms

$$H_{Y(\mathbb{R})}^p(U(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \cong H^0(Y(\mathbb{R}), \mathbb{Z}(\mathcal{L} \otimes \omega_{Y/U})),$$

where $\omega_{Y/U}$ is the conormal sheaf of Y in U . If they are canonical, they'll assemble into an isomorphism of diagrams and induce a canonical isomorphism of colimits. In fact, we only need to establish such canonical isomorphisms for a cofinal system of such neighborhoods. In particular, we can assume that $Y := \bar{x} \cap U$ is smooth and a complete intersection inside U , defined by p equations. The choice of such equations provides a section of $\omega_{Y/U}$, and equips every one of the connected components of $Y(\mathbb{R})$ with a choice of orientation. Now we can apply cohomological purity: the arguments in [Scheiderer 1995, Section 1] go through for singular cohomology of the real points and show that $R^p i^! \mathcal{F} \cong i^* \mathcal{F}$ for a locally constant sheaf \mathcal{F} on $Y(\mathbb{R})$. This isomorphism is not canonical, as it depends on the choice of orientations of the components of $Y(\mathbb{R})$. But we obtain a canonical isomorphism $R^p i^! \mathcal{F} \cong i^* \mathcal{F} \otimes \omega_{Y/U}$ by tensoring with the section of $\omega_{Y/U}$. For $\mathcal{F} = \mathbb{Z}(\mathcal{L})$, this canonical isomorphism of sheaves induces an isomorphism

$$H_{Y(\mathbb{R})}^p(U(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \rightarrow H^0(Y(\mathbb{R}), \mathbb{Z}(\mathcal{L} \otimes \omega_{Y/U})).$$

This map does not depend on the choice of orientation (or presentation of Y as a complete intersection inside U), and hence provides the required canonical isomorphisms and proves the claim. In a sense, the twisting by Λ_x^X takes care of the ambiguity of choices of orientations, as discussed in [Scheiderer 1995, Remark 1.8]. \square

3.19. Corollary. *Let X be a smooth \mathbb{R} -scheme, let Y be a closed subscheme and let \mathcal{L} be a line bundle on X .*

- (1) *Via the identifications of Proposition 3.18, the complex of Proposition 3.14 is canonically isomorphic to the Rost–Schmid complex for $\rho_*\mathbb{Z}$ twisted by \mathcal{L} :*

$$0 \rightarrow \bigoplus_{x \in X^{(0)}} H^0(x, \rho_*\mathbb{Z}(\mathcal{L}_x \otimes \Lambda_x^X)) \rightarrow \bigoplus_{y \in X^{(1)}} H^0(y, \rho_*\mathbb{Z}(\mathcal{L}_y \otimes \Lambda_y^X)) \rightarrow \cdots$$

In particular, the q -th cohomology of the Rost–Schmid complex above is canonically isomorphic to $H^q(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$.

- (2) *The q -th cohomology of the subcomplex*

$$0 \rightarrow \bigoplus_{x \in X^{(0)} \cap Y} H^0(x, \rho_*\mathbb{Z}(\mathcal{L}_x \otimes \Lambda_x^X)) \rightarrow \bigoplus_{y \in X^{(1)} \cap Y} H^0(y, \rho_*\mathbb{Z}(\mathcal{L}_y \otimes \Lambda_y^X)) \rightarrow \cdots$$

of the Rost–Schmid complex appearing in (1) is canonically isomorphic to $H_{Y(\mathbb{R})}^q(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$.

- (3) *For a quasiprojective \mathbb{R} -scheme Y , the q -th homology of the homological Rost–Schmid complex*

$$\cdots \rightarrow \bigoplus_{y \in Y_{(1)}} H^0(y, \rho_*\mathbb{Z}(\mathcal{L}_y \otimes \Lambda_y^Y)) \rightarrow \bigoplus_{x \in Y_{(0)}} H^0(x, \rho_*\mathbb{Z}(\mathcal{L}_x \otimes \Lambda_x^Y)) \rightarrow 0$$

computes the Borel–Moore homology $H_n^{\text{BM}}(Y(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$.

Proof. Claim (1) follows from the computation of Rost–Schmid complexes with the Gersten-type complexes coming out of the coniveau spectral sequence machinery, as in Section 5.3, especially Corollary 5.44, of [Morel 2012]. Claims (2) and (3) follow from (1) as in [Scheiderer 1995, Section 3]. \square

Now we can give a more explicit description of the real cycle class map

$$H^s(X, \mathbf{I}^t(\mathcal{L})) \rightarrow H^s(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$$

for a smooth \mathbb{R} -scheme X with a line bundle \mathcal{L} . The first morphism $H^s(X, \mathbf{I}^t(\mathcal{L})) \rightarrow H^s(X, \rho_*\mathbb{Z}(\mathcal{L}))$ simply changes the coefficients; on the level of Gersten–Rost–Schmid complexes, it is induced from the twisted signature maps

$$\text{sgn} : W(\kappa(x), \mathcal{L}_x \otimes \Lambda_x^X) \rightarrow \rho_*\mathbb{Z}(\mathcal{L}_x \otimes \Lambda_x^X).$$

The second morphism

$$H^s(X, \rho_*\mathbb{Z}(\mathcal{L})) \rightarrow H^s(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$$

has an explicit description in terms of Rost–Schmid complexes as discussed above. The real cycle class map is the composite of these two maps.

3.20. Remark. Although the above results are close parallels of [Scheiderer 1995, Section 3], we cannot obtain a definition of fundamental classes as in the definition before Proposition 3.3 there: First of all, an irreducible closed subscheme $Z \subset X$ of codimension q with generic point z does not come with a canonical section “1” of $H^0(z, \mathbb{Z}(\mathcal{L}_z \otimes \Lambda_z^X))$. Secondly, and more importantly, there is no reason to expect any such nontrivial section to be unramified, i.e., to define a cocycle in the complex of Corollary 3.19(2) and hence a class in $H_{\mathbb{Z}(\mathbb{R})}^0(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$. When Z is smooth, the line bundle $\mathcal{L}|_Z \otimes \omega_{Z/X}$ could be nontrivial; worse problems arise in the singular case. It’s the same orientability problem as always: closed subvarieties in X don’t necessarily give rise to cohomology classes for $X(\mathbb{R})$ — this only works with $\mathbb{Z}/2\mathbb{Z}$ -coefficients. On the other hand, even though fundamental classes generally don’t exist for subvarieties, this is no obstruction for the cycle class map $H^s(X, \mathbf{I}^t(\mathcal{L})) \rightarrow H^s(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$, since the cycles in \mathbf{I} -cohomology have the solutions to the orientability problem built into their coefficients.

For later use, and for the sake of example, we provide a computation of the cycle class map for $\mathbb{A}^n \setminus \{0\}$ over \mathbb{R} .

3.21. Proposition. *For $\mathbb{A}^n = \mathbb{A}_{\mathbb{R}}^n$, the real cycle class map $H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{I}^n) \rightarrow H^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z})$ is an isomorphism for all $n \geq 0$.*

More precisely, equip $\mathbb{A}^n \setminus \{0\}$ with the orientation given by trivializing $\omega_{\mathbb{A}^n \setminus \{0\}}$ by $\bigwedge_{i=1}^n dx_i$, and similarly equip $\mathbb{R}^n \setminus \{0\}$ with the orientation given by the standard coordinate functions. For $n \geq 2$, these choices of orientations provide isomorphisms $H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{I}^n) \cong W(\mathbb{R}) \cong \mathbb{Z}$ and $H^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}$ such that the following composition is the identity:

$$\mathbb{Z} \cong H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{I}^n) \rightarrow H^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}.$$

Proof. When $n = 1$, we need to compare $H^0(\mathbb{A}^1 \setminus \{0\}, \mathbf{W})$ and $H^0(\mathbb{R} \setminus \{0\}, \mathbb{Z})$, both of which are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. We leave it as an exercise to the reader to verify that the cycle class map is an isomorphism in this case, and assume that $n \geq 2$ for the remainder of the proof. We first investigate the cohomology group $H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{I}^n)$. The following statements can essentially be found in [Fasel 2011, Section 3.3]; see also [Asok et al. 2017, Lemma 4.2.5]. A cycle in the cohomology group consists of codimension- $(n-1)$ points x , i.e., generic points of curves in $\mathbb{A}^n \setminus \{0\}$, and the multiplicities are elements in $I(\kappa(x))$. By the standard computations of cohomology of spheres, $H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{I}^n) \cong W(F)$, and the canonical morphism sends a cycle to its residue at the origin in \mathbb{A}^n . In particular, an explicit generator for $H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{I}^n)$ is given by the form $\langle 1, X_1 \rangle$ on the line $X_2 = \dots = X_n = 0$ through the origin; the better way of writing this which takes proper care of the choices of orientation is using the Koszul complex as in [Fasel 2011, Section 3.3]. Note that by definition of the boundary map in the

localization sequence, this class corresponds to the class in $H_{\{0\}}^n(\mathbb{A}^n, \mathbf{I}^n)$ of the form $\langle 1 \rangle$ supported on the origin. Moreover, the choice of orientation formulated in the statement of the proposition is such that the above generator is mapped to 1 under the isomorphism $H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{I}^n) \cong W(F)$.

The other cohomology group we want to consider is the one arising from the Gersten-type complex for integral cohomology of $\mathbb{R}^n \setminus \{0\}$. The relevant piece of the complex is

$$H_{\mathbb{Z}^{n-2}/\mathbb{Z}^{n-1}}^{n-2}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z}) \rightarrow H_{\mathbb{Z}^{n-1}/\mathbb{Z}^n}^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z}) \xrightarrow{\partial} H_{\mathbb{Z}^n}^n(\mathbb{R}^n \setminus \{0\}, \mathbb{Z}).$$

The boundary morphism ∂ is given by the boundary in the localization sequence, that is, for a pair (Y, Z) of irreducible closed subschemes of $\mathbb{A}^n \setminus \{0\}$ of codimensions $n - 1$ and n , respectively, it maps a class in $H_{Y \setminus Z(\mathbb{R})}^{n-1}(\mathbb{R}^n \setminus (\{0\} \cup Z(\mathbb{R})), \mathbb{Z})$ to its boundary, which is a class in $H_{Z(\mathbb{R})}^n(\mathbb{R}^n \setminus \{0\}, \mathbb{Z})$. The canonical map from the cohomology of the above complex at H^{n-1} to the sheaf cohomology of $\mathbb{R}^n \setminus \{0\}$ is the morphism $\ker \partial \rightarrow H^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z})$ which forgets the support. In this picture, we can also identify a generator of the cohomology in the Gersten complex: take a generator of $H^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z})$ and use the canonical identification

$$H^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z}) \cong H^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n \setminus \mathbb{R}^1, \mathbb{Z}) \cong H_{\mathbb{R}^1 \setminus \{0\}}^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z}),$$

where the embedding $\mathbb{R}^1 \hookrightarrow \mathbb{R}^n$ is given by the first coordinate axis. As before, the choice of orientation used here is such that the isomorphism $H_{\mathbb{R}^1 \setminus \{0\}}^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}$ takes the section $\text{sgn}(X_1)$ of the sheaf \mathbb{Z} on the line $X_2 = \dots = X_n = 0$ to $1 \in \mathbb{Z}$.

Now we can trace through the explicit description of the cycle class map, using the signature to go from the Gersten complex for \mathbf{I}^n to the Gersten complex for the sheaf cohomology of \mathbb{Z} on $X(\mathbb{R})$. The generator $\langle 1, X_1 \rangle$ in the Gersten complex for \mathbf{I}^n has signature 2 on the half-line $X_1 > 0$ and -2 on the half-line $X_1 < 0$, viewed as a generator of the global sections of the sheaf $\rho_* 2\mathbb{Z}$ on the line $X_2 = \dots = X_n = 0$. Its image under the canonical map $\text{can} : \rho_* 2\mathbb{Z} \rightarrow \rho_*(\text{colim}_j 2^j \mathbb{Z}) \cong \rho_* \mathbb{Z}$ is then exactly the generator of the global sections of $\rho_* \mathbb{Z}$ discussed above; see also the notational triangle (3.1) at the beginning of Section 3.A. \square

3.22. Remark. We can also provide a quick and easy description of the cycle class map $H_{\{0\}}^n(\mathbb{A}^n, \mathbf{I}^n) \rightarrow H_{\{0\}}^n(\mathbb{R}^n, \mathbb{Z})$. The complex computing \mathbf{I} -cohomology with support is simply

$$\dots \rightarrow 0 \rightarrow W(\mathbb{R})(\Lambda_0^{\mathbb{A}^n}) \rightarrow 0 \rightarrow \dots,$$

where the nontrivial group is placed in degree n , coming from the closed point $0 \in \mathbb{A}^n$. By Corollary 3.19, the complex computing $H_{\{0\}}^n(\mathbb{R}, \mathbb{Z})$ is similarly given by

$$\dots \rightarrow 0 \rightarrow \mathbb{Z}(\Lambda_0^{\mathbb{A}^n}) \rightarrow 0 \rightarrow \dots.$$

The orientations used are in both cases $\bigwedge_{i=1}^n X_i$ (which is an element of $\Lambda_0^{\mathbb{A}^n}$ and therefore provides an isomorphism $\Lambda_0^{\mathbb{A}^n} \cong \mathbb{R}$), so that the morphism $W(\mathbb{R})(\Lambda_0^{\mathbb{A}^n}) \rightarrow \mathbb{Z}(\Lambda_0^{\mathbb{A}^n})$ is simply given by the signature morphism $W(\mathbb{R}) \rightarrow \mathbb{Z}$ mapping the form (1) to 1.

In slight variation of the discussion in [Scheiderer 1995, Section 3], the mod 2 cycle class map $\text{Ch}^n(X) \rightarrow H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ of Borel and Haefliger [1961] has the following sheaf-theoretic description: $H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ can be computed in terms of a Gersten resolution

$$0 \rightarrow \bigoplus_{x \in X^{(0)}} H^0(x, \rho_* \mathbb{Z}/2\mathbb{Z}) \rightarrow \bigoplus_{y \in X^{(1)}} H^0(y, \rho_* \mathbb{Z}/2\mathbb{Z}) \rightarrow \bigoplus_{z \in X^{(2)}} H^0(z, \rho_* \mathbb{Z}/2\mathbb{Z}) \rightarrow \dots$$

The cycle class map is then induced from mapping an irreducible closed subvariety $Y \subset X$ with generic point y to the constant section $1 \in H^0(y, \rho_* \mathbb{Z}/2\mathbb{Z})$.

3.23. Proposition. *Let X be a smooth variety over \mathbb{R} . Then there is a commutative triangle*

$$\begin{array}{ccc} \text{Ch}^n(X) & \xrightarrow{\quad} & H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \\ & \searrow \cong & \nearrow \\ & H^n(X, \mathbf{I}^n / \mathbf{I}^{n+1}) & \end{array}$$

where the top horizontal map is the Borel–Haefliger cycle class map, the downward isomorphism is induced from the identification $\mathbf{K}_n^M / 2 \cong \mathbf{I}^n / \mathbf{I}^{n+1}$ and the upward arrow is Jacobson’s cycle class map.

Proof. As mentioned above, [Scheiderer 1995, Section 3] implies that the horizontal map is induced on Gersten complexes by mapping an irreducible closed subvariety $Y \subset X$ with generic point y to $1 \in H^0(y, \rho_* \mathbb{Z}/2\mathbb{Z})$. The isomorphism $\text{Ch}^n(X) \cong H^n(X, \mathbf{I}^n / \mathbf{I}^{n+1})$ is also induced from a morphism on Gersten complexes: in the relevant degree, we have

$$\bigoplus_{y \in X^{(n)}} \mathbb{Z}/2\mathbb{Z} \cong \bigoplus_{y \in X^{(n)}} W(\kappa(y)) / I(\kappa(y))$$

with the identification (going from right to left) given by mapping a quadratic form to its rank mod 2. Finally, Jacobson’s cycle class map $H^n(X, \mathbf{I}^n / \mathbf{I}^{n+1}) \rightarrow H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is induced on the level of Gersten complexes by the direct sum of signature maps

$$\bigoplus_{y \in X^{(n)}} W(\kappa(y)) / I(\kappa(y)) \rightarrow \bigoplus_{y \in X^{(n)}} H^0(y, \rho_* \mathbb{Z}/2\mathbb{Z}).$$

The commutativity of the triangle then follows from the fact that for a real closed field F , the two morphisms $\mathbb{Z} \cong W(F) \rightarrow \mathbb{Z}/2\mathbb{Z}$ given by rank mod 2 and signature mod 2 agree. □

3.24. Remark. Kirsten Wickelgren has informed us that the above identification of the reduction of Jacobson’s cycle class map with the one of Borel and Haefliger was already established by Jacobson in unpublished work.

3.25. Remark. At some point we wondered if there could be a real analogue of Totaro’s factorization [1997] of the cycle class map

$$\mathrm{CH}^n(X) \rightarrow \mathrm{MU}^*(X(\mathbb{C})) \otimes_{\mathrm{MU}^*} \mathbb{Z} \rightarrow H^{2n}(X(\mathbb{C}), \mathbb{Z})$$

for a smooth complex variety X . It now seems that a comparable factorization of the real cycle class map $H^n(X, \mathbf{I}^n) \rightarrow H^n(X(\mathbb{R}), \mathbb{Z})$ through MSO would not be possible (and a factorization of the mod 2 cycle class map through MO would be useless because MO splits as a wedge of Eilenberg–Mac Lane spaces).

The basic reason lies in orientability issues. To map a cycle in $H^n(X, \mathbf{I}^n)$ to an element of $\mathrm{MSO}^*(X(\mathbb{R})) \otimes_{\mathrm{MSO}^*} \mathbb{Z}$, we would take an irreducible subvariety $Z \subset X$, take a resolution of singularities $\tilde{Z} \rightarrow Z$ and take the realization $\tilde{Z}(\mathbb{R}) \rightarrow X(\mathbb{R})$ of the composition $\tilde{Z} \rightarrow Z \rightarrow X$. For this to yield a class in $\mathrm{MSO}^*(X)$, the map needs to be orientable, but we don’t have enough control over the resolution of singularities to guarantee that. Moreover, when we want to show independence of resolutions, further blowups are required to be able to compare two given resolutions $\tilde{Z}_1 \rightarrow Z$ and $\tilde{Z}_2 \rightarrow Z$. Depending on the (co)dimensions, the real realization of a blowup of a smooth subvariety isn’t orientable, and then there are no pushforward maps to compare fundamental classes. So the construction of an MSO -factorization seems to break down at a rather fundamental level.

3.D. Cycle class maps for Chow–Witt rings. In this section, we will discuss a cycle class map for Chow–Witt rings. We first recall the equivariant cycle class map on Chow groups of real varieties introduced in [BW 2020] and then combine this with the cycle class map on \mathbf{I}^n -cohomology discussed above.

Based on earlier work of Krasnov and van Hamel, Benoist and Wittenberg [2020] discuss a cycle class map $\mathrm{CH}^*(X) \rightarrow H_{C_2}^*(X(\mathbb{C}), \mathbb{Z})$ for X a real variety which maps from the Chow groups to C_2 -equivariant cohomology of the complex points with integral coefficients. For an irreducible closed smooth subvariety $i : Z \subset X$ of codimension k , the cycle class map takes $[Z]$ to the fundamental class $[Z] \in H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))$, which is given as the image of $1 \in \mathbb{Z} = H_{C_2}^0(Z(\mathbb{C}), \mathbb{Z})$ under the proper pushforward map

$$i_* : H_{C_2}^0(Z(\mathbb{C}), \mathbb{Z}) \cong H_{C_2, Z(\mathbb{C})}^{2k}(X(\mathbb{C}), \mathbb{Z}(k)) \rightarrow H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k)).$$

The general definition of the cycle class map for a potentially singular irreducible closed subvariety $Z \subset X$ is obtained by dévissage. The equivariant cycle class map $\mathrm{CH}^*(X) \rightarrow H_{C_2}^*(X(\mathbb{C}), \mathbb{Z})$ is then compatible with proper pushforward, products and pullbacks; see [Krasnov 1991, Propositions 2.1.3 and 2.3.3]. It is also easy to

see that the equivariant cycle class map is compatible with the classical cycle class map via complexification (or, alternatively, forgetting the C_2 -equivariance), since the latter also takes $Z \subset X$ to i_*1 with

$$i_* : H^0(Z(\mathbb{C}), \mathbb{Z}) \cong H_{Z(\mathbb{C})}^{2k}(X(\mathbb{C}), \mathbb{Z}) \rightarrow H^{2k}(X(\mathbb{C}), \mathbb{Z}).$$

The images of algebraic cycles under the equivariant cycle class maps satisfy additional constraints related to the Steenrod squares. Benoist and Wittenberg consider the composition

$$H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k)) \rightarrow H_{C_2}^{2k}(X(\mathbb{R}), \mathbb{Z}(k)) \rightarrow \bigoplus_{0 \leq p \leq 2k, p \equiv k \pmod{2}} H^p(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}),$$

where the second map arises from the decomposition

$$H_{C_2}^i(X(\mathbb{R}), \mathbb{Z}) \cong \bigoplus_{p+q=i} H^p(X(\mathbb{R}), H^q(C_2, \mathbb{Z}(j))).$$

For a class $\alpha \in H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))$, the component of the image in $H^p(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is denoted by α_p . Then the restricted equivariant cohomology is defined as

$$H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))_0 = \{\alpha \mid \alpha_{k+i} = \text{Sq}^i(\alpha_k) \text{ for all } i \in 2\mathbb{Z}\} \subseteq H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k)).$$

Due to results of Kahn and Krasnov, the equivariant cycle class map factors through a morphism $\text{CH}^k(X) \rightarrow H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))_0$.

The following compatibility of cycle class maps is noted in [BW 2020], after Definition 1.19.

3.26. Proposition. *Let X be a smooth quasiprojective variety over \mathbb{R} (or more generally over a real closed field F). Then there is a commutative diagram*

$$\begin{array}{ccc} \text{CH}^n(X) & \longrightarrow & H_{C_2}^{2n}(X(\mathbb{C}), \mathbb{Z}(n))_0 \\ \downarrow & & \downarrow \\ \text{Ch}^n(X) & \longrightarrow & H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \end{array}$$

where the top horizontal is the refined equivariant cycle class map discussed above, and the bottom horizontal is the Borel–Haefliger cycle class map [1961], which we may reinterpret in our terms using Proposition 3.23 above. The left vertical map is the natural reduction mod 2, and the right-hand vertical map is the reduction map [BW 2020, Equation (1.58)].

Finally, we can combine the cycle class maps considered above into a cycle class map defined on the Chow–Witt ring. Recall that there is a fiber product description of the Milnor–Witt K-theory sheaves (see [Morel 2004]):

$$K_n^{\text{MW}} \cong K_n^M \times_{I^n/I^{n+1}} I^n,$$

where the maps are the natural projection $\mathbf{I}^n \rightarrow \mathbf{I}^n/\mathbf{I}^{n+1}$ and the composition of the projection $\mathbf{K}_n^M \rightarrow \mathbf{K}_n^M/2$ with Milnor’s homomorphism $\mathbf{K}_n^M/2 \rightarrow \mathbf{I}^n/\mathbf{I}^{n+1}$. (By the solution to the Milnor conjecture, the homomorphism of Milnor that appears here is an isomorphism.) The fiber product presentation above induces a commutative square

$$\begin{CD} \widetilde{\text{CH}}^n(X, \mathcal{L}) @>>> \ker \partial \\ @VVV @VVV \\ H^n(X, \mathbf{I}^n(\mathcal{L})) @>>> \text{Ch}^n(X) \end{CD}$$

where $\partial : \text{CH}^n(X) \rightarrow H^{n+1}(X, \mathbf{I}^n(\mathcal{L}))$ is the boundary map in sheaf cohomology associated to the short exact sequence of sheaves $0 \rightarrow \mathbf{I}^{n+1}(\mathcal{L}) \rightarrow \mathbf{K}_n^{\text{MW}}(\mathcal{L}) \rightarrow \mathbf{K}_n^M \rightarrow 0$, an analogue of integral Bockstein maps in classical topology. By [HW 2019, Proposition 2.11], the map $\widetilde{\text{CH}}^n(X, \mathcal{L}) \rightarrow H^n(X, \mathbf{I}^n(\mathcal{L})) \times_{\text{Ch}^n(X)} \ker \partial$ is surjective, and injective whenever the 2-torsion in $\text{CH}^n(X)$ is trivial.

By Proposition 3.26, the composition of the equivariant cycle class map of [BW 2020] $\text{CH}^n(X) \xrightarrow{\cong} H_{C_2}^{2n}(X(\mathbb{C}), \mathbb{Z}(n))_0$ of Proposition 5.8 with the inclusion $\ker \partial \subseteq \text{CH}^n(X)$ factors through the kernel of the composition

$$\psi : H_{C_2}^{2n}(X(\mathbb{C}), \mathbb{Z}(n))_0 \rightarrow H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta} H^{n+1}(X(\mathbb{R}), \mathbb{Z}).$$

3.27. Corollary. *Let X be a smooth quasiprojective variety over \mathbb{R} (or more generally over a field F with a real embedding $\sigma : F \hookrightarrow \mathbb{R}$). Then there is a well-defined cycle class map*

$$\widetilde{\text{CH}}^n(X, \mathcal{L}) \rightarrow H^n(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \times_{H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})} \ker \psi.$$

Proof. The cycle class map is the composition of the above natural projection map $\widetilde{\text{CH}}^n(X, \mathcal{L}) \rightarrow H^n(X, \mathbf{I}^n(\mathcal{L})) \times_{\text{Ch}^n(X)} \ker \partial$ with the map

$$H^n(X, \mathbf{I}^n(\mathcal{L})) \times_{\text{Ch}^n(X)} \ker \partial \rightarrow H^n(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \times_{H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})} \ker \psi$$

which is the fiber product of Jacobson’s cycle class map

$$H^n(X, \mathbf{I}^n(\mathcal{L})) \rightarrow H^n(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$$

discussed above, and the refined cycle class map $\text{CH}^n(X) \rightarrow H_{C_2}^{2n}(X(\mathbb{C}), \mathbb{Z}(n))_0$ of Benoist and Wittenberg [2020] (restricted to $\ker \partial \rightarrow \ker \psi$ as discussed above). The result follows from Propositions 3.26 and 3.23 and the commutative diagram

$$\begin{CD} H^n(X, \mathbf{I}^n(\mathcal{L})) @>>> H^n(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \\ @VVV @VVV \\ H^n(X, \mathbf{I}^n/\mathbf{I}^{n+1}) @>>> H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \end{CD}$$

where the horizontal maps are Jacobson’s cycle class maps and the vertical maps are the morphisms induced from the natural projection maps $I^n \rightarrow I^n/I^{n+1}$ and $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$, respectively. The latter commutative diagram follows directly from [Jacobson 2017] since both cycle class maps are induced from the signature on quadratic forms, hence compatible with reduction mod 2. \square

3.28. Remark. It would be interesting if this could be refined to a cycle class map whose target is the cohomology of a fiber product of sheaves in the same way as $K_n^{\text{MW}} \cong I^n \times_{I^n/I^{n+1}} K_n^M$. This would require that we state and prove a topological version of the Milnor conjecture. Then we could hope to detect the 2-torsion in the kernel of the natural projection

$$\widetilde{\text{CH}}^\bullet(X, \mathcal{L}) \rightarrow H^\bullet(X, I^\bullet(\mathcal{L})) \times_{\text{Ch}^\bullet(X)} \text{CH}^\bullet(X).$$

Of course, for cellular varieties as discussed in Section 5, this doesn’t play any role at all.

4. Compatibilities

In this section, we prove the main results on compatibility of the real cycle class maps with pullbacks, pushforwards and the multiplicative structure.

4.A. Compatibility with pullbacks.

4.1. Proposition. *Let $f : X \rightarrow Y$ be a morphism of smooth schemes over \mathbb{R} , let $i : Z \hookrightarrow Y$ be a (reduced) closed subscheme and let \mathcal{L} be a line bundle over Y . The morphisms in diagram (3.11) are compatible with pullback along f . In particular, the following diagram commutes for all nonnegative integers s and t :*

$$\begin{array}{ccc} H_{f^{-1}(Z)}^s(X, I^t(f^*\mathcal{L})) & \longrightarrow & H_{f^{-1}(Z)(\mathbb{R})}^s(X(\mathbb{R}), \mathbb{Z}(f^*\mathcal{L})) \\ \uparrow f^* & & \uparrow f^* \\ H_Z^s(Y, I^t(\mathcal{L})) & \longrightarrow & H_{Z(\mathbb{R})}^s(Y(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \end{array}$$

Proof. Recall from Section 2.A.2 that we define the cohomology groups with support $H_Z^s(Y, I^t(\mathcal{L}))$ as the derived functor of $\Gamma_Z := \Gamma i_* i^!$ evaluated on the sheaf $I^t(\mathcal{L})$, and similarly for all the other cohomology groups and sheaves involved. We denote by $i' : f^{-1}(Z) \hookrightarrow X$ the base change of i along f . Below, all cohomology groups are sheaf cohomology groups, and the pullback maps are those defined in Definitions 2.8 and 2.42. The claim will follow from three commutative diagrams, involving the maps can , sign_∞ and ρ^* in the definition of the real cycle class map of diagram (3.11). The commutativity of the diagrams below will be checked using the functoriality statement for pullbacks; see Proposition 2.10.

(1) We first discuss the map $\text{can} : \mathbf{I}^t \rightarrow \text{colim}_j \mathbf{I}^j$. By Proposition 2.45, the sheaf pullback $f^* : H_Z^s(Y, \mathbf{I}^t(\mathcal{L})) \rightarrow H_{f^{-1}(Z)}^s(X, \mathbf{I}^t(f^*\mathcal{L}))$ agrees with the usual pullback defined in terms of Gersten complexes. By Proposition 2.10 and Corollary 2.28, it suffices to show that the following diagram of morphisms of coefficient data is commutative:

$$\begin{array}{ccc} (X, \mathbf{I}^t(f^*\mathcal{L})) & \xleftarrow{(\text{id}, \text{can})} & (X, \text{colim}_j \mathbf{I}^j(f^*\mathcal{L})) \\ (f, \phi) \downarrow & & \downarrow (f, \phi_{\text{colim}}) \\ (Y, \mathbf{I}^t(\mathcal{L})) & \xleftarrow{(\text{id}, \text{can})} & (Y, \text{colim}_j \mathbf{I}^j(\mathcal{L})) \end{array}$$

Hopefully the direction of the horizontal arrows following the conventions in the definition of coefficient data at the beginning of Section 2 is not too confusing. The commutativity of the above diagram then is equivalent to the commutativity of the following diagram of morphisms of sheaves on Y :

$$\begin{array}{ccc} \mathbf{I}^t(\mathcal{L}) & \xrightarrow{\phi} & f_* \mathbf{I}^t(f^*\mathcal{L}) \\ \text{can} \downarrow & & \downarrow f_* \text{can} \\ \text{colim}_j \mathbf{I}^j(\mathcal{L}) & \xrightarrow{\phi_{\text{colim}}} & f_* \text{colim}_j \mathbf{I}^j(f^*\mathcal{L}) \end{array}$$

Here, the horizontal morphisms ϕ and ϕ_{colim} are the structure morphisms discussed before Definition 2.33. The commutativity of this diagram follows from Example 2.29 since the morphism $\text{can} : \mathbf{I}^t \rightarrow \text{colim}_j \mathbf{I}^j$ is a morphism of sheaves on the big site.

(2) Now we consider the identification $\text{sign}_\infty : \text{colim}_j \mathbf{I}^j \cong \rho_* \mathbb{Z}$. As in step (1), using Proposition 2.10 and Corollary 2.28, it suffices to prove commutativity of the following diagram of sheaves on Y :

$$\begin{array}{ccc} \text{colim}_n \mathbf{I}_Y^n(\mathcal{L}) & \xrightarrow{\text{sign}_\infty} & (\rho_Y)_* \mathbb{Z}(\mathcal{L}) \\ \phi_{\text{colim}} \downarrow & & \downarrow \tau \\ f_* \text{colim}_n \mathbf{I}_X^n(f^*\mathcal{L}) & \xrightarrow{\text{sign}_\infty} & f_*(\rho_X)_* \mathbb{Z}(f^*\mathcal{L}) \end{array}$$

The commutativity of this square is again a consequence of Example 2.29 since Jacobson’s isomorphism $\text{sign}_\infty : \text{colim}_j \mathbf{I}^j \cong \rho_* \mathbb{Z}$ is a morphism of sheaves with \mathbb{G}_m -action on the big site and consequently can be twisted; see Theorem 3.12.

(3) The last square concerns the compatibility of pullback morphisms with the pullback isomorphism ρ^* to singular cohomology. As before, using Proposition 2.10 and Corollary 2.28, it suffices to show that the following square of morphisms of

coefficient data is commutative:

$$\begin{array}{ccc}
 (X, \rho_* \mathbb{Z}(f^* \mathcal{L})) & \xleftarrow{\rho} & (X(\mathbb{R}), \mathbb{Z}(f^* \mathcal{L})) \\
 (f, \tau) \downarrow & & \downarrow (f, \tau) \\
 (Y, \rho_* \mathbb{Z}(\mathcal{L})) & \xleftarrow{\rho} & (Y(\mathbb{R}), \mathbb{Z}(\mathcal{L}))
 \end{array}$$

Note that here and below we are also tacitly using the canonical isomorphism of line bundles $(f^* \mathcal{L})(\mathbb{R}) \cong f_{\mathbb{R}}^* \mathbb{Z}(\mathcal{L})$ on $X(\mathbb{R})$.

On the level of continuous maps of topological spaces, the relevant input is the obvious commutativity of the diagram

$$\begin{array}{ccc}
 X(\mathbb{R}) & \xrightarrow{\rho} & X_{\text{Zar}} \\
 f \downarrow & & \downarrow f \\
 Y(\mathbb{R}) & \xrightarrow{\rho} & Y_{\text{Zar}}
 \end{array}$$

On the level of morphisms of sheaves on Y , this commutative diagram also implies the commutativity of the triangle

$$\begin{array}{ccc}
 \rho_* \mathbb{Z}(\mathcal{L}) & \xrightarrow{\tau} & f_* \rho_* \mathbb{Z}(f^* \mathcal{L}) \\
 \searrow \rho_*(\tau) & & \downarrow \cong \\
 & & \rho_* f_* \mathbb{Z}(f^* \mathcal{L})
 \end{array}$$

and this implies the claim. □

4.2. Remark. The results and techniques of this subsection obviously apply also to real cohomology (with pullbacks defined as in Section 2.A.1). More precisely, they apply to the map $\text{supp} : X_r \rightarrow X$ for X over any base field F , instead of $\rho : X(\mathbb{R}) \rightarrow X$ and X over \mathbb{R} . If F has a real embedding $F \hookrightarrow \mathbb{R}$, the results also apply to the induced map $\iota : X_{\mathbb{R}}(\mathbb{R}) \rightarrow (X_{\mathbb{R}})_r$. Note that for $F = \mathbb{R}$, the map ρ is simply the composition $\rho = \text{supp} \circ \iota$. Similar statements apply to the compatibility results on products in the next subsection. For pushforwards, the real variant is less obvious, as one first would have to establish a theory of Thom classes in this setting. For other results related to descriptions using Gersten complexes in the real setting as, e.g., in [Scheiderer 1995], we refer to the Appendix.

4.B. Compatibility with ring structures.

4.3. Proposition. *Let X be a smooth scheme over \mathbb{R} , let $V, W \subset X$ be two closed subsets, and let \mathcal{L}_1 and \mathcal{L}_2 be line bundles over X . The real cycle class map with supports as defined in diagram (3.11) is compatible with the cup product in the*

sense that the following diagram commutes for all integers p, q, k and l :

$$\begin{array}{ccc}
 H_V^p(X, \mathbf{I}^k(\mathcal{L}_1)) \otimes H_W^q(X, \mathbf{I}^l(\mathcal{L}_2)) & \xrightarrow{\cup} & H_{V \cap W}^{p+q}(X, \mathbf{I}^{k+l}(\mathcal{L}_1 \otimes \mathcal{L}_2)) \\
 \downarrow & & \downarrow \\
 H_{V(\mathbb{R})}^p(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}_1)) \otimes H_{W(\mathbb{R})}^q(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}_2)) & \xrightarrow{\cup} & H_{V(\mathbb{R}) \cap W(\mathbb{R})}^{p+q}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}_1 \otimes \mathcal{L}_2))
 \end{array}$$

Proof. Both cup products here can be obtained using Godement’s construction; see Theorem 2.14. For the cup product in \mathbf{I} -cohomology, we use the multiplicative structure of the sheaves \mathbf{I}^j as in the discussion above Corollary 2.48. Recall that this agrees with Fasel’s definition of cup product via Gersten complexes by Proposition 2.52.

As before, the result is the consequence of the commutativity of diagrams involving the maps can , sign_∞ and the sheaf pullback ρ^* . We ignore the supports in the following more detailed argument, as these only complicate the notation without changing the argument.

(1) The map $\text{can} : \mathbf{I}^t \rightarrow \text{colim}_j \mathbf{I}^j$ is multiplicative in the sense that we have a commutative square as in Remark 3.5. This square still commutes after twisting by line bundles. Using the naturality of Godement’s cup product in Theorem 2.14 with respect to morphisms of sheaves, we thus find that the following square commutes:

$$\begin{array}{ccc}
 H^p(X, \mathbf{I}^k(\mathcal{L}_1)) \otimes H^q(X, \mathbf{I}^l(\mathcal{L}_2)) & \xrightarrow{\cup} & H^{p+q}(X, \mathbf{I}^{k+l}(\mathcal{L}_1 \otimes \mathcal{L}_2)) \\
 \downarrow \text{can} \otimes \text{can} & & \downarrow \text{can} \\
 H^p(X, \text{colim}_j \mathbf{I}^j(\mathcal{L}_1)) \otimes H^q(X, \text{colim}_j \mathbf{I}^j(\mathcal{L}_2)) & \xrightarrow{\cup} & H^{p+q}(X, \text{colim}_j \mathbf{I}^j(\mathcal{L}_1 \otimes \mathcal{L}_2))
 \end{array}$$

(2) In the second step, we establish the compatibility of the cup product with the identification $\text{sign}_\infty : \text{colim}_j \mathbf{I}^j \cong \rho_* \mathbb{Z}$. We claim that the following diagram is commutative:

$$\begin{array}{ccc}
 H^p(X, \text{colim}_j \mathbf{I}^j(\mathcal{L}_1)) \otimes H^q(X, \text{colim}_j \mathbf{I}^j(\mathcal{L}_2)) & \xrightarrow{\cup} & H^{p+q}(X, \text{colim}_j \mathbf{I}^j(\mathcal{L}_1 \otimes \mathcal{L}_2)) \\
 \downarrow & & \downarrow \\
 H^p(X, \rho_* \mathbb{Z}(\mathcal{L}_1)) \otimes H^q(X, \rho_* \mathbb{Z}(\mathcal{L}_2)) & \xrightarrow{\cup} & H^{p+q}(X, \rho_* \mathbb{Z}(\mathcal{L}_1 \otimes \mathcal{L}_2))
 \end{array}$$

The vertical maps are each the identification sign_∞ , and the horizontal maps are cup products. Again, the naturality of Godement’s cup product with respect to morphisms of sheaves implies the commutativity of the diagram once we verify that the identification sign_∞ preserves the multiplicative structures on both sides. This follows because the signature map $\text{sign} : W(X) \rightarrow \rho_* \mathbb{Z}$ is a ring homomorphism.

(3) Finally, we need to check the compatibility of the following diagram, encoding the compatibility of the sheaf pullback with the cup products:

$$\begin{array}{ccc}
 H^p(X, \rho_*\mathbb{Z}(\mathcal{L}_1)) \otimes H^q(X, \rho_*\mathbb{Z}(\mathcal{L}_2)) & \xrightarrow{\cup} & H^{p+q}(X, \rho_*\mathbb{Z}(\mathcal{L}_1 \otimes \mathcal{L}_2)) \\
 \downarrow \rho^* \otimes \rho^* & & \downarrow \rho^* \\
 H^p(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}_1)) \otimes H^q(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}_2)) & \xrightarrow{\cup} & H^{p+q}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}_1 \otimes \mathcal{L}_2))
 \end{array}$$

The commutativity of this diagram follows from the naturality of the cup product with respect to continuous maps; see Proposition 2.15. \square

4.4. Corollary. *For any smooth scheme X over \mathbb{R} , the real cycle class map*

$$\bigoplus_{s,t} H^s(X, \mathbf{I}^t) \rightarrow \bigoplus_s H^s(X(\mathbb{R}), \mathbb{Z})$$

is a morphism of graded rings.

4.C. Compatibility with pushforwards and the localization sequence. In this section, we will now establish the compatibility of the real cycle class map with supports with pushforwards. We restrict to the case of pushforwards along closed immersions, which is enough for our intended applications. This will also imply the compatibility of the real cycle class maps with the localization sequence.

4.5. Remark. We want to point out that the real cycle class maps should actually be compatible with arbitrary proper pushforwards; tracing through the Gersten-type definition of pushforwards for \mathbf{I} -cohomology, the key point is the compatibility of the identification $\text{colim } \mathbf{I}^m \cong \rho_*\mathbb{Z}$ with transfers as established in [Bachmann 2018, Lemma 20].

4.6. Remark. Another thing to point out before we get started is that there are differences in the treatment of pushforward maps in algebra and topology, which could be a potential source for confusion. On the algebraic side, for any point $i : \text{Spec}(\mathbb{R}) \rightarrow \mathbb{P}^1$, there is a canonical pushforward $i_* : H^0(\text{Spec}(\mathbb{R}), \mathbf{I}^0(\omega_{\mathbb{R}})) \rightarrow H^1(\mathbb{P}^1, \mathbf{I}^1(\omega_{\mathbb{P}^1}))$. On the topological side, pushforwards generally don't appear as frequently, and there is no canonical pushforward $i_* : H^0(\text{pt}, \mathbb{Z}) \rightarrow H^1(\mathbb{R}\mathbb{P}^1, \mathbb{Z})$ in topology. This is not a contradiction: real realization maps untwisted coefficients \mathcal{O} to untwisted local coefficients \mathbb{Z} canonically. But the isomorphism $\omega_{\mathbb{P}^1} \cong \mathcal{O}$ of line bundles on \mathbb{P}^1 involves a choice, and this choice corresponds to the choice of an orientation on $\mathbb{R}\mathbb{P}^1$, hence of a topological fundamental class, and the latter uniquely determines the topological pushforward i_* . This means that on both the algebraic and topological sides, the pushforward maps depend on choices of orientations, but these are treated differently in algebra and topology.

The main result to be established in this subsection is the following compatibility of real cycle class maps with pushforwards:

4.7. Theorem. *Let X be a smooth real variety and let $Z \subset X$ be a smooth closed subvariety of codimension c . We denote the normal bundle of the inclusion $i : Z \rightarrow X$ by $\mathcal{N} = \mathcal{N}_Z X$ and set $\mathcal{L} := \det \mathcal{N}_Z X$. Then the following diagram is commutative:*

$$\begin{array}{ccccc}
 H^{n-c}(Z, I^{j-c}(\mathcal{L})) & \longrightarrow & H^{n-c}(Z, \rho_* \mathbb{Z}(\mathcal{L})) & \xrightarrow[\cong]{\rho^*} & H^{n-c}(Z(\mathbb{R}), \mathbb{Z}(\mathcal{L})) = H^{n-c}(Z(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \\
 \downarrow i_* & & & & \downarrow i_*^{\text{Th}} \\
 H^n(X, I^j) & \longrightarrow & H^n(X, \rho_* \mathbb{Z}) & \xrightarrow[\cong]{\rho^*} & H^n(X(\mathbb{R}), \mathbb{Z}) = H^n(X(\mathbb{R}), \mathbb{Z}) \\
 & & & & \downarrow i_*^{\text{PD}}
 \end{array}$$

The horizontal compositions are the cycle class maps, the left vertical morphism is the pushforward defined on Gersten complexes as in Definition 2.55, the vertical morphism i_*^{Th} is the pushforward defined in terms of the topological Thom class; see Definition 2.57 and the discussion at the end of Section 2.E. The vertical morphism i_*^{PD} is the Poincaré dual of the pushforward in singular Borel–Moore homology, which is defined only if both $X(\mathbb{R})$ and $Z(\mathbb{R})$ are oriented and hence $\mathbb{Z}(\mathcal{L}) = \mathbb{Z}$ is constant.

Proof. The commutativity of the left rectangle is the core of the result, and will be proved in Proposition 4.12 below. The basic idea is to show that the algebraic Thom class induces generators of the cohomology with support of the fibers of the normal bundle and use the characterization of the topological Thom class to deduce that the algebraic Thom class maps to a topological Thom class. To get a comparison of the pushforward maps, we need a discussion of the relation between deformation to the normal bundle and the topological tubular neighborhood construction, which is the main technical point in Proposition 4.12.

Concerning the square on the right-hand side, the morphism i_*^{Th} is defined using a Thom class in $H_{Z(\mathbb{R})}^n(X(\mathbb{R}), \mathbb{Z})$ combined with the tubular neighborhood theorem; compare, e.g., [tom Dieck 2008, Section 15.6; Hirsch 1976, Chapter 4; Iversen 1986, VIII.2.4]. On the other hand i_*^{PD} is defined using Poincaré duality for both Z and X and i_* in Borel–Moore homology. The commutativity of the square on the right-hand side for oriented manifolds follows from [Davis and Kirk 2001, Corollary 10.33] or [Iversen 1986, IX.7.5]. For the latter, note that we have a Thom isomorphism $i_* : H^{*-c}(Z(\mathbb{R})) \rightarrow H_{Z(\mathbb{R})}^*(X(\mathbb{R}))$ given by the (cup) product with the Iversen Thom class τ_Z ; see [loc. cit., VIII.2.3]. We have another Thom isomorphism $s_* : H^{*-c}(Z(\mathbb{R})) \rightarrow H_{Z(\mathbb{R})}^*(\mathcal{N}_{Z(\mathbb{R})} X(\mathbb{R}))$ given by the (cup) product with the usual Thom class $\text{th}(\mathcal{N}(\mathbb{R}))$ as in Definition 2.57. There is an isomorphism d between the two targets (see below for possible constructions of it). As in degree $* = c$ — where both Thom classes live — all three groups are free abelian

of rank one (we may assume $Z(\mathbb{R})$ connected), and the isomorphisms i_* , s_* and d map generators to generators. Hence d maps τ_Z to $\text{th}(\mathcal{N})$, possibly up to sign if a preferred sign was chosen. Now the claim follows from equality IX.7.5 of [Iversen 1986] (one cap there has to be a cup). \square

4.8. Proposition. *Let $i : Z \hookrightarrow X$ be a closed immersion between smooth \mathbb{R} -schemes with open complement $U := X \setminus Z$. Then there is a ladder of long exact sequences defining cohomology with supports,*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_Z^s(X, I^t(i^*\mathcal{L})) & \longrightarrow & H^s(X, I^t(\mathcal{L})) & \longrightarrow & H^s(U, I^t(\mathcal{L})) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & H_{Z(\mathbb{R})}^s(X(\mathbb{R}), \mathbb{Z}(i^*\mathcal{L})) & \longrightarrow & H^s(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) & \longrightarrow & H^s(U(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \longrightarrow \cdots \end{array}$$

Proof. The upper long exact sequence is the localization sequence of [Fasel 2008, Théorème 9.3.4], which coincides by Proposition 2.39 with the localization sequence from Proposition 2.11. The morphisms $\text{can} : I^t \rightarrow \text{colim}_j I^j$ and $\text{sign}_\infty : \text{colim}_j I^j \cong \rho_*\mathbb{Z}$ induce morphisms of the Gersten resolutions. From the (absent) proof of Proposition 2.11, it is then clear that these morphisms induce ladders of appropriate localization sequences because the localization sequence arises from the short exact sequence of complexes arising from the Gersten resolutions by applying Γ_Z or Γ_j^* :

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^\bullet(X, I^t(\mathcal{L}))_Z & \longrightarrow & C^\bullet(X, I^t(\mathcal{L})) & \longrightarrow & C^\bullet(U, I^t(\mathcal{L})) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C^\bullet(X, \rho_*\mathbb{Z}(\mathcal{L}))_Z & \longrightarrow & C^\bullet(X, \rho_*\mathbb{Z}(\mathcal{L})) & \longrightarrow & C^\bullet(U, \rho_*\mathbb{Z}(\mathcal{L})) \longrightarrow 0 \end{array}$$

Now we want to compare the lower exact sequence to a short exact sequence arising, as in Proposition 2.11, from the Godement resolution of $\mathbb{Z}(\mathcal{L})$ on X . There is a morphism of short exact sequence of complexes, from the Gersten-complex sequence to the Godement-complex sequence; it is induced by the zigzag of quasi-isomorphisms

$$C^\bullet(X, \rho_*\mathbb{Z}(\mathcal{L})) \leftarrow \mathbb{G}\rho_*\mathbb{Z}(\mathcal{L}) \rightarrow \rho_*\mathbb{G}\mathbb{Z}(\mathcal{L}). \quad \square$$

4.9. Remark. In other words, the localization sequence is compatible with morphisms of coefficient data.

4.10. Proposition. *There is a commutative diagram*

$$\begin{array}{ccc} H^0(\text{Spec } \mathbb{R}, I^0) & \longrightarrow & H^0(\text{pt}, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H_{\{0\}}^n(\mathbb{A}^n, I^n) & \longrightarrow & H_{\{0\}}^n(\mathbb{R}^n, \mathbb{Z}) \end{array}$$

with maps given as follows. The left vertical map is the algebraic pushforward in \mathbf{I} -cohomology of Definition 2.55, the right vertical map is the topological pushforward for sheaf cohomology with \mathbb{Z} -coefficients, and the horizontal maps are the respective real cycle class maps, where we use the cycle class map with support from Section 3.B for the lower morphism. The orientations are chosen as in Proposition 3.21.

Proof. We have $H^0(\mathrm{Spec} \mathbb{R}, \mathbf{I}^0) \cong W(\mathbb{R}) \cong \mathbb{Z}$. The algebraic pushforward

$$H^0(\mathrm{Spec} \mathbb{R}, \mathbf{I}^0) \rightarrow H_{\{0\}}^n(\mathbb{A}^n, \mathbf{I}^n)$$

is an isomorphism by Theorem 2.41 and maps the form $\langle 1 \rangle$ to the algebraic Thom class, which consequently is a generator of $H_{\{0\}}^n(\mathbb{A}^n, \mathbf{I}^n)$. The real realization

$$H^0(\mathrm{Spec} \mathbb{R}, \mathbf{I}^0) \rightarrow H^0(\mathrm{pt}, \mathbb{Z})$$

maps the form $\langle 1 \rangle$ to its signature, which is a generator of $H^0(\mathrm{pt}, \mathbb{Z})$. On the right-hand side, we have the topological Thom isomorphism $H^0(\mathrm{pt}, \mathbb{Z}) \cong H_{\{0\}}^n(\mathbb{R}^n, \mathbb{Z})$. The lower horizontal map is also an isomorphism: we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} H^{n-1}(\mathbb{A}^n, \mathbf{I}^n) & \longrightarrow & H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{I}^n) & \longrightarrow & H_{\{0\}}^n(\mathbb{A}^n, \mathbf{I}^n) & \longrightarrow & H^n(\mathbb{A}^n, \mathbf{I}^n) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{n-1}(\mathbb{R}^n, \mathbb{Z}) & \longrightarrow & H^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z}) & \longrightarrow & H_{\{0\}}^n(\mathbb{R}^n, \mathbb{Z}) & \longrightarrow & H^n(\mathbb{R}^n, \mathbb{Z}) \end{array}$$

where the vertical arrows are the real cycle class maps (with support) and the rows are the exact sequences defining cohomology with supports. The ladder above is commutative by Proposition 4.8. Using the five lemma, we are reduced (if $n \neq 0$, otherwise everything follows trivially) to show that the real cycle class map $H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{I}^n) \rightarrow H^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z})$ is an isomorphism, which is established in Proposition 3.21. More precisely, Proposition 3.21 actually shows that the real cycle class map takes the algebraic Thom class to the topological Thom class (provided we use the canonical choices of orientations induced from the standard coordinates on \mathbb{A}^n). This proves the commutativity of the diagram as claimed. \square

4.11. Proposition. *Let X be a smooth real scheme and let $p : V \rightarrow X$ be a rank d vector bundle on X . Then the algebraic Thom class*

$$\mathrm{th}^{\mathrm{alg}}(V) \in H_X^d(V, \mathbf{I}^d(p^* \det V^\vee))$$

of Definition 2.54 maps to a topological Thom class

$$\mathrm{th}^{\mathrm{top}}(V(\mathbb{R})) \in H_{X(\mathbb{R})}^d(V(\mathbb{R}), \mathbb{Z}(p^* \det V^\vee))$$

as in Definition 2.57. In other words, there is a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathbf{I}^0) & \longrightarrow & H^0(X(\mathbb{R}), \mathbb{Z}) \\ \downarrow & & \downarrow \\ H_X^d(V, \mathbf{I}^d(p^* \det V^\vee)) & \longrightarrow & H_{X(\mathbb{R})}^d(V(\mathbb{R}), \mathbb{Z}(p^* \det V^\vee)) \end{array}$$

where the horizontal maps are the real cycle class maps of Section 3 and the vertical maps are the respective pushforward maps, or equivalently, the Thom isomorphisms.

Proof. Denoting the zero section of V by s , we have a natural isomorphism

$$s^* : H^0(V, \mathbf{I}^0) \rightarrow H^0(X, \mathbf{I}^0),$$

making the top left group an $H^0(V, \mathbf{I}^0)$ -module; the bottom left group is an $H^0(V, \mathbf{I}^0)$ -module via the module structure for cohomology with supports induced by the cup product of Theorem 2.14 (which agrees with the algebraically defined intersection product by Proposition 2.52). The left-hand vertical morphism is then an $H^0(V, \mathbf{I}^0)$ -linear map, and hence the left-hand vertical map is completely determined by the image of $\langle 1 \rangle$ (which by Definition 2.54 is the algebraic Thom class). A similar statement holds for the right-hand vertical map by definition; see Section 2.E. Since the real cycle class map is compatible with the product by Proposition 4.3, the upper horizontal morphism is a ring homomorphism. This induces an $H^0(V, \mathbf{I}^0)$ -module structure on $H_{X(\mathbb{R})}^d(V(\mathbb{R}), \mathbb{Z}(\det V))$ and, by Proposition 4.3, the lower horizontal cycle class map is an $H^0(V, \mathbf{I}^0)$ -module homomorphism. As a consequence, the lower horizontal map is completely determined by the image of the algebraic Thom class $\text{th}^{\text{alg}}(V)$, which is a generator of the cyclic $H^0(V, \mathbf{I}^0)$ -module $H_X^d(V, \mathbf{I}^d(\det V))$ by Definition 2.54. Consequently, the commutativity of the diagram follows from the claim on Thom classes.

To prove the claim about the Thom classes, we use the characterization of the topological Thom class found in Definition 2.57, namely, that a class in $H_{X(\mathbb{R})}^d(V(\mathbb{R}), \mathbb{Z}(\det V))$ is a Thom class for the vector bundle $V(\mathbb{R}) \rightarrow X(\mathbb{R})$ if and only for every point $x \in X(\mathbb{R})$ the restriction of the class to $H_{\{x\}}^d(V_x(\mathbb{R}), \mathbb{Z}) \cong \mathbb{Z}$ is a generator. Consider the diagram

$$\begin{array}{ccc} H_X^d(V, \mathbf{I}^d(p^* \det V^\vee)) & \longrightarrow & H_{X(\mathbb{R})}^d(V(\mathbb{R}), \mathbb{Z}(p^* \det V^\vee)) \\ i_x^* \downarrow & & \downarrow i_{x(\mathbb{R})}^* \\ H_{\{x\}}^d(V_x, \mathbf{I}^d) & \longrightarrow & H_{\{x\}}^d(V_x(\mathbb{R}), \mathbb{Z}) \end{array}$$

where again the horizontal morphisms are the respective real cycle class maps and the vertical morphisms are restrictions to the fiber of the respective vector bundle.

In particular, the diagram is commutative by Proposition 4.1 and it suffices to show that the algebraic Thom class maps to a generator of the lower-right group.

Under the left vertical morphism, the algebraic Thom class $\text{th}^{\text{alg}}(V)$ restricts to the algebraic Thom class of the trivial vector bundle on the point x , in particular to a generator of the group $H_{\{x\}}^d(V_x, \mathbf{I}^d)$. This follows from the obvious transversal and cartesian diagram of smooth \mathbb{R} -schemes

$$\begin{array}{ccc} \text{Spec } \mathbb{R} & \xrightarrow{x} & X \\ \downarrow & & \downarrow s \\ \mathbb{A}^d \cong V_x & \longrightarrow & V \end{array}$$

and the base change theorem [Asok and Fasel 2016, Theorem 2.12]. By Proposition 4.10, the local algebraic Thom class in $H_{\{x\}}^d(V_x, \mathbf{I}^d)$ maps to a generator of $H_{\{x\}}^d(V_x(\mathbb{R}), \mathbb{Z})$, as required. We have proved that the image of the algebraic Thom class in $H_{X(\mathbb{R})}^d(V(\mathbb{R}), \mathbb{Z}(p^* \det V^\vee))$ restricts to a generator of the local cohomology group for every fiber. Therefore the algebraic Thom class for $V \rightarrow X$ maps to a topological Thom class for $V(\mathbb{R}) \rightarrow X(\mathbb{R})$ as claimed. \square

Before the final step in the proof of compatibility of pushforwards, we need to relate the deformation to the normal bundle with tubular neighborhoods. Recall our notation for the deformation to the normal cone from Figure 1 on page 256 in Section 2.C.3, and the construction of the deformation isomorphism

$$d(Z, X) : H_Z^p(\mathcal{N}_Z X, \mathbf{I}^q) \rightarrow H_Z^p(X, \mathbf{I}^q)$$

from Section 2.C.5. The next proposition shows that this isomorphism is compatible under the real cycle class map with a similar isomorphism arising from a tubular neighborhood argument in topology.

4.12. Proposition. *Let $i : Z \hookrightarrow X$ be a closed immersion of codimension c between smooth \mathbb{R} -schemes. Then there is a commutative diagram*

$$\begin{array}{ccc} H^i(Z, \mathbf{I}^j(\det \mathcal{N}_Z X)) & \longrightarrow & H^i(Z(\mathbb{R}), \mathbb{Z}(\det \mathcal{N}_{Z(\mathbb{R})} X(\mathbb{R}))) \\ \downarrow & & \downarrow \\ H^{i+c}(X, \mathbf{I}^{j+c}) & \longrightarrow & H^{i+c}(X(\mathbb{R}), \mathbb{Z}) \end{array}$$

where the horizontal morphisms are real cycle class maps and the vertical morphisms are the respective pushforward maps, defined using deformation to the normal bundle and a tubular neighborhood, respectively.

Proof. Consider the following diagram:

$$\begin{array}{ccc}
 H^i(Z, \mathbf{I}^j(\det \mathcal{N}_Z X)) & \longrightarrow & H^i(Z(\mathbb{R}), \mathbb{Z}(\det \mathcal{N}_{Z(\mathbb{R})} X(\mathbb{R}))) \\
 \text{th}^{\text{alg}}(\mathcal{N}_Z X) \downarrow & & \downarrow \text{th}^{\text{top}}(\mathcal{N}_{Z(\mathbb{R})} X(\mathbb{R})) \\
 H_Z^{i+c}(\mathcal{N}_Z X, \mathbf{I}^{j+c}) & \longrightarrow & H_{Z(\mathbb{R})}^{i+c}(\mathcal{N}_{Z(\mathbb{R})} X(\mathbb{R}), \mathbb{Z}) \\
 d(Z, X) \cong \downarrow & & \downarrow d(Z, X)(\mathbb{R}) \\
 H_Z^{i+c}(X, \mathbf{I}^{j+c}) & \longrightarrow & H_{Z(\mathbb{R})}^{i+c}(X(\mathbb{R}), \mathbb{Z}) \\
 \downarrow & & \downarrow \\
 H^{i+c}(X, \mathbf{I}^{j+c}) & \longrightarrow & H^{i+c}(X(\mathbb{R}), \mathbb{Z})
 \end{array}$$

The left vertical composition is the pushforward for \mathbf{I} -cohomology, by Definition 2.55. The maps in the first square are given by multiplication with the respective Thom classes, and hence the square is commutative by Propositions 4.11 and 4.3. The last square is commutative since real realization with supports commutes with forgetting the support; see Remark 3.13.

We recalled the construction of $d(Z, X)$ before the statement of Proposition 4.12. We now discuss the map $d(Z, X)(\mathbb{R})$ on the topological side. The topological variants of the proofs in [Nenashev 2007] and [Fasel 2009, Lemma 2.8] show that $i_0(\mathbb{R})^*$ and $i_1(\mathbb{R})^*$ are isomorphisms in singular cohomology with supports and appropriate local coefficient systems. Note that classical excision is sufficient because the real realization of a Nisnevich cover is a surjective local diffeomorphism, and hence can be refined by an honest open cover. As we define the map $d(Z, X)(\mathbb{R}) = i_1(\mathbb{R})^* \circ (i_0(\mathbb{R})^*)^{-1}$, the commutativity of the middle square follows from the compatibility of real realization and pullback morphisms; see Proposition 4.1.

Finally, it remains to prove that the right vertical composition in our big diagram is the pushforward in singular cohomology. For that, we consider the real realization of the diagram for the deformation to the normal bundle just before the statement of Proposition 4.12. The normal bundle $\mathcal{N}_{Z(\mathbb{R}) \times \mathbb{R}^1}(D(Z, X)(\mathbb{R}))$ is extended from $Z(\mathbb{R})$ by homotopy invariance. Moreover, since its restriction to $Z(\mathbb{R}) \times \{t\}$ (for any $t \in \mathbb{R}$) is the normal bundle of $i(\mathbb{R})$ we know that $\mathcal{N}_{Z(\mathbb{R}) \times \mathbb{R}^1}(D(Z, X)(\mathbb{R}))$ is the pullback of $\mathcal{N}_{Z(\mathbb{R})}(X(\mathbb{R}))$ along the projection $Z(\mathbb{R}) \times \mathbb{R} \rightarrow Z(\mathbb{R})$. We now choose a tubular neighborhood of $Z(\mathbb{R})$ in $\mathcal{N}_{Z(\mathbb{R})}(X(\mathbb{R}))$. By definition (see, e.g., [Hirsch 1976, Chapter 4.5]), this is a smooth map $T : \mathcal{N}_{Z(\mathbb{R}) \times \mathbb{R}^1}(D(Z, X)(\mathbb{R})) \rightarrow \mathcal{N}_{Z(\mathbb{R})}(X(\mathbb{R}))$ such that $T \circ s = s$ which restricts to a diffeomorphism from an open neighborhood of $Z(\mathbb{R})$ in $\mathcal{N}_{Z(\mathbb{R}) \times \mathbb{R}^1}(D(Z, X)(\mathbb{R}))$ to an open neighborhood of $Z(\mathbb{R})$

in $\mathcal{N}_{Z(\mathbb{R})}(X(\mathbb{R}))$. In our case, we may simply choose $T = \text{id}$ and choose the open disc bundle \mathcal{D} as the open neighborhood. Since the above conditions are stable under pullback along $Z \times \mathbb{R}^1 \rightarrow Z$, up to the above trivializing isomorphism for $\mathcal{N}_{Z(\mathbb{R}) \times \mathbb{R}^1}(D(Z, X)(\mathbb{R}))$ above, the fiber product $\mathcal{T} \cong \mathcal{D} \times \mathbb{R}^1$ and its restriction $\mathcal{T}|_1$ over $Z \times \{1\}$ are tubular neighborhoods as well.

To summarize, we obtain the diagram

$$\begin{array}{ccccc}
 Z(\mathbb{R}) & \longrightarrow & Z(\mathbb{R}) \times \mathbb{R}^1 & \longleftarrow & Z(\mathbb{R}) \\
 \downarrow s & & \downarrow & & \downarrow f \\
 \mathcal{D} & \xrightarrow{e_0} & \mathcal{T} & \xleftarrow{e_1} & \mathcal{T}|_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{N}_{Z(\mathbb{R})} X(\mathbb{R}) & \xrightarrow{i_0(\mathbb{R})} & D(Z, X)(\mathbb{R}) & \xleftarrow{i_1(\mathbb{R})} & X(\mathbb{R})
 \end{array}$$

where the first and third rows are the real points of the diagram for the deformation to the normal bundle and the second row contains the above tubular neighborhood \mathcal{T} and its restrictions to the fibers over 0 and 1, respectively. The diagram is obviously commutative. By excision, the lower vertical maps induce isomorphisms in cohomology with supports

$$\begin{aligned}
 H_{Z(\mathbb{R})}^i(\mathcal{N}_{Z(\mathbb{R})} X(\mathbb{R}), \mathbb{Z}) &\cong H_{Z(\mathbb{R})}^i(\mathcal{D}, \mathbb{Z}), \\
 H_{Z(\mathbb{R}) \times \mathbb{R}^1}^i(D(Z, X)(\mathbb{R}), \mathbb{Z}) &\cong H_{Z(\mathbb{R}) \times \mathbb{R}^1}^i(\mathcal{T}, \mathbb{Z}), \\
 H_{Z(\mathbb{R})}^i(X(\mathbb{R}), \mathbb{Z}) &\cong H_{Z(\mathbb{R})}^i(\mathcal{T}|_1, \mathbb{Z}).
 \end{aligned}$$

Consequently, the morphisms e_0^* and e_1^* of the appropriate cohomology groups with supports are also isomorphisms. Therefore, we can safely replace the composition $d(Z, X)(\mathbb{R}) = i_1^* \circ (i_0^*)^{-1}$ by the composition $e_1^* \circ (e_0^*)^{-1}$. Now we recall that the normal bundle $\mathcal{N}_{Z(\mathbb{R}) \times \mathbb{R}^1}(D(Z, X)(\mathbb{R}))$ is extended from Z and we can choose a global isomorphism to $\mathcal{N}_{Z(\mathbb{R})}(X(\mathbb{R})) \times \mathbb{R}$. In particular, there is a natural projection morphism $\mathcal{T} \rightarrow \mathcal{T}|_1$ fitting into a commutative diagram

$$\begin{array}{ccc}
 Z(\mathbb{R}) \times \mathbb{R} & \xrightarrow{\text{pr}} & Z(\mathbb{R}) \\
 \downarrow & & \downarrow \\
 \mathcal{T} & \xrightarrow{\text{pr}'} & \mathcal{T}|_1
 \end{array}$$

such that $\text{pr}' \circ e_0 = \text{id}$ and hence $(\text{pr}')^* : H_{Z(\mathbb{R})}^i(\mathcal{T}|_1, \mathbb{Z}) \rightarrow H_{Z(\mathbb{R}) \times \mathbb{R}^1}^i(\mathcal{T}, \mathbb{Z})$, is the inverse of e_1^* . Using these maps, we can rewrite the upper two squares of the above

diagram involving the tubular neighborhoods to

$$\begin{array}{ccc} Z(\mathbb{R}) & \xrightarrow{\text{id}} & Z(\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{T}|_1 \subseteq X(\mathbb{R}) \end{array}$$

where the lower horizontal morphism pr' is the inclusion of the disc bundle \mathcal{D} in $\mathcal{N}_{Z(\mathbb{R})}(X(\mathbb{R}))$ as the tubular neighborhood $\mathcal{T}|_1$ of $i(Z(\mathbb{R}))$. Moreover, the pull-back $e_0^* \circ (\text{pr}')^*$ is the inverse of the composition $e_1^* \circ (e_0^*)^{-1}$. Consequently, the identification

$$H_{Z(\mathbb{R})}^i(\mathcal{N}_{Z(\mathbb{R})}(X(\mathbb{R})), \mathbb{Z}) \cong H_{Z(\mathbb{R})}^i(X(\mathbb{R}), \mathbb{Z})$$

coming from the deformation to the normal bundle is inverse to the identification

$$H_{Z(\mathbb{R})}^i(X(\mathbb{R}), \mathbb{Z}) \rightarrow H_{Z(\mathbb{R})}^i(\mathcal{D}, \mathbb{Z}) \cong H_{Z(\mathbb{R})}^i(\mathcal{N}_{Z(\mathbb{R})}(X(\mathbb{R})), \mathbb{Z})$$

coming from the tubular neighborhood theorem. □

The following compatibility of real cycle class maps with the localization sequences is now a direct consequence of Theorem 4.7 and Proposition 4.8.

4.13. Corollary. *Let $i : Z \hookrightarrow X$ be a closed immersion between smooth \mathbb{R} -schemes of codimension c and let \mathcal{L} be a line bundle on X . Denote by $U := X \setminus Z$ the complement of Z and by ω the determinant of the normal bundle of the inclusion i . Then there is a ladder of long exact localization sequences,*

$$\begin{array}{ccccccc} \dots \rightarrow & H^{s-c}(Z, \mathbf{I}^{t-c}(i^* \mathcal{L} \otimes \omega)) & \longrightarrow & H^s(X, \mathbf{I}^t(\mathcal{L})) & \longrightarrow & H^s(U, \mathbf{I}^t(\mathcal{L})) & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ \dots \rightarrow & H^{s-c}(Z(\mathbb{R}), \mathbb{Z}(i^* \mathcal{L} \otimes \omega)) & \longrightarrow & H^s(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) & \longrightarrow & H^s(U(\mathbb{R}), \mathbb{Z}(\mathcal{L})) & \longrightarrow \dots \end{array}$$

4.D. Compatibility with the Bär sequence. The short exact sequence of sheaves $0 \rightarrow \mathbf{I}^{n+1}(\mathcal{L}) \rightarrow \mathbf{I}^n(\mathcal{L}) \rightarrow \bar{\mathbf{I}}^n \rightarrow 0$ gives rise to a long exact sequence of cohomology groups, the so-called Bär sequence. The boundary map of this sequence is usually denoted by $\beta_{\mathcal{L}}$ and referred to as the Bockstein homomorphism. We briefly discuss the compatibility of the real cycle class map with this homomorphism. By [Jacobson 2017, Theorem 8.5(4)], there is an isomorphism of short exact sequences,

$$\begin{array}{ccccccc} 0 \longrightarrow & \text{colim}_n \mathbf{I}^{n+1} & \xrightarrow{\eta} & \text{colim}_n \mathbf{I}^n & \xrightarrow{\rho} & \text{colim}_n \bar{\mathbf{I}}^n & \longrightarrow 0 \\ & \cong \downarrow & & \cong \downarrow & & \downarrow \cong & \\ 0 \longrightarrow & \rho_* \mathbb{Z} & \xrightarrow{2} & \rho_* \mathbb{Z} & \longrightarrow & \rho_* \mathbb{Z} / 2\mathbb{Z} & \longrightarrow 0 \end{array}$$

By the discussion in Section 3, the horizontal arrows in the above diagram are in fact isomorphisms of sheaves on the big site Sm_F and equivariant for the \mathbb{G}_m -actions on $\mathrm{colim} \mathbf{I}^n$ and $\rho_*\mathbb{Z}$. In particular, for a smooth scheme X and a line bundle \mathcal{L} on X , we can immediately obtain an isomorphism of short exact sequences of sheaves on the small site of X , twisted by \mathcal{L} as in Definition 2.23:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{colim}_n \mathbf{I}^{n+1}(\mathcal{L}) & \xrightarrow{\eta} & \mathrm{colim}_n \mathbf{I}^n(\mathcal{L}) & \xrightarrow{\rho} & \mathrm{colim}_n \bar{\mathbf{I}}^n & \longrightarrow & 0 \\ & & \cong \downarrow & & \cong \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & \rho_*\mathbb{Z}(\mathcal{L}) & \xrightarrow{2} & \rho_*\mathbb{Z}(\mathcal{L}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

It should be noted here that the \mathbb{G}_m -action on $\bar{\mathbf{I}}^n$ and $\rho_*\mathbb{Z}/2\mathbb{Z}$ is trivial, so that twisting has no effect.

4.14. Proposition. *Let X be a smooth scheme and let \mathcal{L} be a line bundle on X . Then there is a commutative diagram,*

$$\begin{array}{ccc} \mathrm{Ch}^n(X) & \xrightarrow{\beta_{\mathcal{L}}} & H^{n+1}(X, \mathbf{I}^{n+1}(\mathcal{L})) \\ \downarrow & & \downarrow \\ H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\beta_{\mathcal{L}}} & H^{n+1}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \end{array}$$

In particular, the algebraic and topological Bockstein maps correspond under the real cycle class maps.

Proof. The key point is the isomorphism of exact sequences discussed above, resulting from [Jacobson 2017, Theorem 8.5(4)]. There is another commutative ladder of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{I}^{n+1}(\mathcal{L}) & \xrightarrow{\eta} & \mathbf{I}^n(\mathcal{L}) & \longrightarrow & \bar{\mathbf{I}}^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{colim}_j \mathbf{I}^{j+1}(\mathcal{L}) & \xrightarrow{\eta} & \mathrm{colim}_j \mathbf{I}^j(\mathcal{L}) & \longrightarrow & \mathrm{colim}_j \bar{\mathbf{I}}^j & \longrightarrow & 0 \end{array}$$

which arise by twisting from the obvious commutative diagram involving the canonical maps $\mathbf{I}^n \rightarrow \mathrm{colim}_j \mathbf{I}^j$. Combining these commutative ladders of exact sequences implies that we have a commutative square for the boundary morphisms

$$\begin{array}{ccc} H^n(X, \bar{\mathbf{I}}^n) \cong \mathrm{Ch}^n(X) & \xrightarrow{\beta_{\mathcal{L}}} & H^{n+1}(X, \mathbf{I}^{n+1}(\mathcal{L})) \\ \downarrow & & \downarrow \\ H^n(X, \rho_*\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\beta_{\mathcal{L}}} & H^{n+1}(X, \rho_*\mathbb{Z}(\mathcal{L})) \end{array}$$

The claim follows by combining this with the commutative diagram

$$\begin{array}{ccc} H^n(X, \rho_*\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\beta_{\mathcal{L}}} & H^{n+1}(X, \rho_*\mathbb{Z}(\mathcal{L})) \\ \downarrow & & \downarrow \\ H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\beta_{\mathcal{L}}} & H^{n+1}(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \end{array}$$

where the vertical arrows are pullbacks along ρ . The lower horizontal arrow is the boundary map for the short exact sequence of sheaves

$$0 \rightarrow \mathbb{Z}(\mathcal{L}(\mathbb{R})) \rightarrow \mathbb{Z}(\mathcal{L}(\mathbb{R})) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

and the upper horizontal arrow is the boundary map of the pushforward of the above sequence along ρ . So the commutativity of the final square above is a consequence of the compatibility of pullbacks in sheaf cohomology with long exact sequences of sheaves; see Proposition 2.12. \square

5. Cellular varieties

5.1. Definition [Fulton 1998, Example 1.9.1]. A variety X over a base S is *cellular* if it has a filtration over S by closed subvarieties

$$X = Z_0 \supset Z_1 \supset \cdots \supset Z_N = \emptyset$$

such that, for each k , we have an isomorphism of S -varieties $Z_{k-1} \setminus Z_k \cong \mathbb{A}_S^{n_k}$ for some integer n_k .

5.2. Remark. For a smooth cellular variety, it is sometimes more useful to consider the complementary filtration by open subvarieties; see [Wendt 2012, §2.1] or [Zibrowius 2011, page 485]. This allows us to “piece the variety together without leaving the smooth world”.

5.3. Proposition. *For a smooth cellular variety X over \mathbb{R} , the complex, mod 2 complex and mod 2 real cycle class maps define ring isomorphisms:*

$$\begin{aligned} \text{CH}^n(X) &\xrightarrow{\cong} H^{2n}(X(\mathbb{C})), \\ \text{Ch}^n(X) &\xrightarrow{\cong} h^{2n}(X(\mathbb{C})), \\ \text{Ch}^n(X) &\xrightarrow{\cong} h^n(X(\mathbb{R})). \end{aligned}$$

Proof. The first two maps are ring homomorphisms by [Fulton 1998, Corollary 19.2(b)]; see also [Totaro 1997, Theorem 4.1]. The statement for the (integral) complex cycle class map is well known; see for example [Fulton 1998, Example 19.1.11(b)]. That it also defines an isomorphism $\text{Ch}^n(X) \cong h^{2n}(X(\mathbb{C}))$ follows immediately since $\text{Ch}^n(X) = \text{CH}^n(X)/2$ for any X and $h^{2n}(Y) = H^{2n}(Y)/2$ for any topological space Y whose cohomology is concentrated in even degrees.

For the real cycle class maps, compatibility with the ring structure is established in Corollary 4.4. That it is an isomorphism can be proved similarly as in the complex case, by cellular induction. Indeed, consider the main induction step (see Remark 5.2): we assume that X is the union of a regularly embedded codimension- c cell Z and its open complement U , and we know that the real cycle class map is an isomorphism for Z . We need to compare the following two exact sequences:

$$\begin{array}{ccccccc} \mathrm{Ch}^{n-c}(Z) & \longrightarrow & \mathrm{Ch}^n(X) & \longrightarrow & \mathrm{Ch}^n(U) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow & & \downarrow & & \\ h^{n-c}(Z(\mathbb{R})) & \longrightarrow & h^n(X(\mathbb{R})) & \longrightarrow & h^n(U(\mathbb{R})) & & \end{array}$$

The square on the left is commutative by Theorem 4.7, and the second square commutes by Proposition 4.1. The corresponding diagram for the complex cycle class map has (top and) lower row a short exact sequence, because $H^*(X(\mathbb{C}))$ is concentrated in even degrees only. In contrast, $h^*(X(\mathbb{R}))$ can be nonzero in any degree. Nonetheless, assuming by induction that the real cycle class map is surjective for U , we find that the restriction map $h^n(X(\mathbb{R})) \rightarrow h^n(U(\mathbb{R}))$ is also surjective in all degrees n , and hence that the pushforward $h^{n-c}(Z(\mathbb{R})) \rightarrow h^n(X(\mathbb{R}))$ is injective for all n . Thus, assuming by induction that the real cycle class map is an isomorphism for U , we may deduce inductively that the real cycle class map for X is also an isomorphism. \square

5.4. Proposition. *For a smooth cellular variety X over \mathbb{R} , multiplication by $\langle\langle -1 \rangle\rangle \in I(\mathbb{R})$ induces isomorphisms*

$$H^i(X, \bar{I}^j) \xrightarrow{\langle\langle -1 \rangle\rangle} H^i(X, \bar{I}^{j+1}) \quad \text{for all } j \geq i.$$

Note that $H^i(X, \bar{I}^j) = 0$ for all $j < i$, for any smooth X over any field. This follows directly from the definition/computation of \bar{I}^j -cohomology in terms of (quotients) of Gersten complexes. The proposition says that for real cellular X , the nonzero values of \bar{I}^j -cohomology are independent of j .

Proof. We have $W(\mathbb{R}) = \mathbb{Z}$ with $\langle\langle -1 \rangle\rangle = 2$ and $I^j(\mathbb{R}) = 2^j\mathbb{Z}$. So the conclusion holds for $X = \mathrm{Spec}(\mathbb{R})$. We now use homotopy invariance, the localization sequence and dévissage for \bar{I}^j -cohomology, which even hold for I^j -cohomology; see Section 2.C.

We also use that each of these isomorphisms/exact sequences is compatible with the $W(\mathbb{R})$ -module structure on \bar{I}^j -cohomology. First, by homotopy invariance, the claim follows for a “cell”, i.e., for \mathbb{A}^n , for any n . By Remark 5.2, it now suffices to consider the situation that X is the union of a regularly embedded codimension- c cell Z and its open complement U , and to assume that the hypothesis already holds

for U . To deduce that the claim also holds for X , we compare the localization sequences for \bar{I}^j and \bar{I}^{j+1} -cohomology:

$$\begin{array}{ccccccccc}
 H^{i-1}(U, \bar{I}^j) & \xrightarrow{\partial} & H^{i-c}(Z, \bar{I}^{j-c}) & \longrightarrow & H^i(X, \bar{I}^j) & \longrightarrow & H^i(U, \bar{I}^j) & \xrightarrow{\partial} & H^{i+1-c}(Z, \bar{I}^{j-c}) \\
 \downarrow \cong & & \downarrow \cong & & \downarrow ? & & \downarrow \cong & & \downarrow \\
 H^{i-1}(U, \bar{I}^{j+1}) & \xrightarrow{\partial} & H^{i-c}(Z, \bar{I}^{j+1-c}) & \longrightarrow & H^i(X, \bar{I}^{j+1}) & \longrightarrow & H^i(U, \bar{I}^{j+1}) & \xrightarrow{\partial} & H^{i+1-c}(Z, \bar{I}^{j+1-c})
 \end{array}$$

The diagram commutes since the localization sequences are induced from short exact sequences of $W(F)$ -modules, and all vertical arrows are given by multiplication by $\langle\langle -1 \rangle\rangle$. The arrow we are interested in is the central arrow. The first and fourth arrows are isomorphisms by the induction hypothesis. The second arrow is an isomorphism since Z is a cell. The fifth arrow is either an isomorphism (when $j \geq i + 1$) or the inclusion of 0 (when $j = i$), so in any case it is injective. This shows that the central arrow is an isomorphism, as desired. \square

5.5. Proposition. *Suppose X is a smooth variety over \mathbb{R} for which the conclusion of Proposition 5.4 holds, i.e., such that multiplication by $\langle\langle -1 \rangle\rangle$ induces isomorphisms*

$$H^i(X, \bar{I}^j) \cong H^i(X, \bar{I}^{j+1})$$

for all $j \geq i$. Then for any line bundle \mathcal{L} over X multiplication by $\langle\langle -1 \rangle\rangle$ also induces isomorphisms

$$H^i(X, I^j(\mathcal{L})) \xrightarrow{\langle\langle -1 \rangle\rangle} H^i(X, I^{j+1}(\mathcal{L})) \quad \text{for all } j \geq i.$$

5.6. Remark. The inclusions $I^j(\mathcal{L}) \hookrightarrow W(\mathcal{L})$ induce isomorphisms

$$H^i(X, I^j(\mathcal{L})) \cong H^i(X, W(\mathcal{L}))$$

for all $j < i$. This follows easily from the Bär sequence. So for X and \mathcal{L} as in Proposition 5.5 and a fixed cohomological degree i , the groups $H^i(X, I^j(\mathcal{L}))$ take only two different values according to whether $j \geq i$ or $j < i$.

Proof. As explained by Jacobson [2017, proof of Corollary 8.11], there is an integer d such that multiplication by $\langle\langle -1 \rangle\rangle$ induces isomorphisms of sheaves $I^j \cong I^{j+1}$ for all $j \geq d + 1$. In fact, we can take d to be exactly the Krull dimension of X , but we do not need to know this. Twisting with \mathcal{L} , we see that $\langle\langle -1 \rangle\rangle$ also induces isomorphisms $I^j(\mathcal{L}) \cong I^{j+1}(\mathcal{L})$ in this range. Thus, the conclusion of the proposition holds for $j \geq d + 1$. We now run a downward induction over j . Suppose the conclusion holds for some particular value j . We now compare the Bär sequences

for $\mathbf{I}^{j-1}(\mathcal{L})$ and $\mathbf{I}^j(\mathcal{L})$ around some cohomological degree $i \leq j - 1$:

$$\begin{array}{ccccccccc} H^{i-1}(X, \bar{\mathbf{I}}^{j-1}) & \xrightarrow{\partial} & H^i(X, \mathbf{I}^j(\mathcal{L})) & \rightarrow & H^i(X, \mathbf{I}^{j-1}(\mathcal{L})) & \rightarrow & H^i(X, \bar{\mathbf{I}}^{j-1}) & \xrightarrow{\partial} & H^{i+1}(X, \mathbf{I}^j(\mathcal{L})) \\ \downarrow \cong & & \downarrow \cong & & \downarrow ? & & \downarrow \cong & & \downarrow \cong \\ H^{i-1}(X, \bar{\mathbf{I}}^j) & \xrightarrow{\partial} & H^i(X, \mathbf{I}^{j+1}(\mathcal{L})) & \rightarrow & H^i(X, \mathbf{I}^j(\mathcal{L})) & \rightarrow & H^i(X, \bar{\mathbf{I}}^j) & \xrightarrow{\partial} & H^{i+1}(X, \mathbf{I}^{j+1}(\mathcal{L})) \end{array}$$

Again the diagram commutes because the long exact sequences are induced from a short sequence of sheaves of $W(F)$ -modules and all vertical arrows are multiplication by $\langle\langle -1 \rangle\rangle$. We need to show that the central arrow is an isomorphism. The first and fourth arrows are isomorphisms by our assumption on X . The second and fifth arrows are isomorphisms by our induction hypothesis. So the claim follows. \square

5.7. Theorem. *Let X be a smooth cellular variety over \mathbb{R} , and let \mathcal{L} be a line bundle over X . Consider the real cycle class maps*

$$H^i(X, \mathbf{I}^j(\mathcal{L})) \rightarrow H^i(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})).$$

- (a) *For $j \geq i$, the real cycle class map is a group isomorphism. In particular, in diagonal bidegrees the homomorphism of graded commutative rings considered in Corollary 4.4 is an isomorphism:*

$$\bigoplus_i H^i(X, \mathbf{I}^i) \cong \bigoplus_i H^i(X(\mathbb{R}), \mathbb{Z}).$$

- (b) *For $j < i$, the image of the real cycle class map is the subgroup*

$$2^{i-j} \cdot H^i(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \subset H^i(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})).$$

When $j = i - 1$, the real cycle class map is moreover injective. Thus, by composing the isomorphisms of Remark 5.6 with the real cycle class map, we obtain an isomorphism

$$H^i(X, \mathbf{I}^j(\mathcal{L})) \cong 2 \cdot H^i(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \quad \text{for all } j < i.$$

Proof. (a) Consider first the case $j \geq i$. It suffices to show that

$$\text{sign} : H^i(X, \mathbf{I}^j(\mathcal{L})) \rightarrow H^i(X(\mathbb{R}), 2^j \mathbb{Z}(\mathcal{L}))$$

is an isomorphism. Consider the following commutative square of morphisms of coefficient data (see [Jacobson 2017, Definition 8.4]):

$$\begin{array}{ccc} (X, \mathbf{I}^j) & \xleftarrow{(\rho, \text{sign})} & (X(\mathbb{R}), 2^j \mathbb{Z}) \\ \uparrow (\text{id}, \langle\langle -1 \rangle\rangle) & & \uparrow (\text{id}, 2) \cong \\ (X, \mathbf{I}^{j+1}) & \xleftarrow{(\rho, \text{sign})} & (X(\mathbb{R}), 2^{j+1} \mathbb{Z}) \end{array}$$

By Corollary 2.28, this square remains commutative if we twist by \mathcal{L} . We thus obtain the following commutative square in cohomology:

$$\begin{array}{ccc} H^i(X, \mathbf{I}^j(\mathcal{L})) & \longrightarrow & H^i(X(\mathbb{R}), 2^j\mathbb{Z}(\mathcal{L})) \\ \llcorner -1 \llcorner \downarrow & & \downarrow 2 \\ H^i(X, \mathbf{I}^{j+1}(\mathcal{L})) & \xrightarrow{\cong} & H^i(X(\mathbb{R}), 2^{j+1}\mathbb{Z}(\mathcal{L})) \end{array}$$

By Proposition 5.5, the left vertical morphism is an isomorphism for all $j \geq i$. One of the main results in [Jacobson 2017] is that the horizontal arrows are isomorphisms for sufficiently large j and trivial twists; see once again (the proof of) Corollary 8.11 in that work. The proof given there generalizes to arbitrary twists; see Theorem 3.12. So the proof can be completed by backward induction over j . The statement concerning the ring structure is immediate from Corollary 4.4.

(b) The claim concerning $j < i$ follows from the isomorphism for $j \geq i$ and the compatibility of the Bär sequence with the Bockstein sequence (Proposition 4.14). \square

5.8. Proposition. *For any smooth cellular variety X over \mathbb{R} , the restricted equivariant cycle class map $\mathrm{CH}^\bullet(X) \rightarrow H_{C_2}^\bullet(X(\mathbb{C}), \mathbb{Z})_{(0)}$ of [BW 2020] is an isomorphism.*

Proof. For $X = \mathrm{Spec} \mathbb{R}$, we obtain by [BW 2020, 1.2.2]

$$H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k)) = \begin{cases} 0 & \text{for } k \text{ odd,} \\ \mathbb{Z}/2\mathbb{Z} & \text{for } k > 0 \text{ even,} \end{cases}$$

and the map [BW 2020, (1.57)] has the form

$$H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k)) \rightarrow \bigoplus_{0 \leq p \leq 2k, p \equiv 0 \pmod{2}} H^p(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}),$$

where the direct sum in the target has its only nontrivial term for $p = 0$. In particular, for a class $\alpha \in H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))$ the only potentially nontrivial component is α_0 , in the notation of [BW 2020, 1.6.3]. Moreover, since $H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k)) \rightarrow H^0(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is an isomorphism, $\alpha_0 = 0$ if and only if $\alpha = 0$. This implies that $H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))_0$ as defined in [BW 2020, Definition 1.19] is trivial: the generator α of $H_{C_2}^{2k}(X(\mathbb{C}), \mathbb{Z}(k))$ has the only nontrivial component α_0 but then the condition $\alpha_0 = \mathrm{Sq}^{-2k}(\alpha_k)$ cannot be satisfied. In particular, for $X = \mathrm{Spec} \mathbb{R}$, the equivariant cycle class map $\mathrm{CH}^\bullet(X) \rightarrow H_{C_2}^{2\bullet}(X(\mathbb{C}), \mathbb{Z}(\bullet))_{(0)}$ of [BW 2020] is an isomorphism. By homotopy invariance, this also implies that it is an isomorphism for $X = \mathbb{A}_{\mathbb{R}}^n$.

We now want to argue by induction on the number of cells of X . As before, we have a diagram,

$$\begin{array}{ccccccc} \mathrm{CH}^{n-c}(Z) & \longrightarrow & \mathrm{CH}^n(X) & \longrightarrow & \mathrm{CH}^n(U) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow & & \downarrow & & \\ H_{C_2}^{2n-2c}(Z(\mathbb{C}), \mathbb{Z}(n-c))_0 & \longrightarrow & H_{C_2}^{2n}(X(\mathbb{C}), \mathbb{Z}(n))_0 & \longrightarrow & H_{C_2}^{2n}(U(\mathbb{C}), \mathbb{Z}(n))_0 & & \end{array}$$

The top row is the localization sequence for Chow groups. The vertical arrows are the restricted cycle class maps of [BW 2020] and therefore commutativity of the diagram follows by the compatibility statements in Sections 1.6.1 and 1.6.4 of [BW 2020]. However, we don't know exactness of the lower row in general. In our special case where Z is an affine space, we can argue as follows: we have the localization sequence for equivariant cohomology

$$H_{C_2}^{2n-2c}(Z(\mathbb{C}), \mathbb{Z}(n-c)) \rightarrow H_{C_2}^{2n}(X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow H_{C_2}^{2n}(U(\mathbb{C}), \mathbb{Z}(n)),$$

which is exact. The problem with exactness for the restricted cohomology is that a restricted class on $X(\mathbb{C})$ which vanishes on $U(\mathbb{C})$ may only come from an equivariant cohomology class on $Z(\mathbb{C})$ which may not be restricted. However, if Z is an affine space, we know what the nonrestricted classes look like: essentially, they are 2-torsion classes from the cohomology of C_2 . By [BW 2020, Theorem 1.20], the Steenrod square condition is still going to be violated for the pushforward of such a 2-torsion class. This implies the exactness in the middle of the lower row.

Now we can use the inductive assumption that the right-hand vertical map is an isomorphism. This implies that the restriction morphism $H_{C_2}^{2n}(X(\mathbb{C}), \mathbb{Z}(n))_0 \rightarrow H_{C_2}^{2n}(U(\mathbb{C}), \mathbb{Z}(n))_0$ is surjective. The injectivity of the pushforward morphism $\mathbb{Z} \cong H_{C_2}^0(Z(\mathbb{C}), \mathbb{Z})_0 \rightarrow H_{C_2}^{2c}(X(\mathbb{C}), \mathbb{Z}(c))_0$ also follows directly from nontriviality, which can be checked on the mod 2 reduction. The diagram chase then implies that the middle vertical morphism must be an isomorphism. \square

5.9. Corollary. *For any smooth cellular variety X over \mathbb{R} , the cycle class map in Corollary 3.27 is an isomorphism.*

Proof. By Proposition 3.26, the isomorphism $\mathrm{CH}^n(X) \xrightarrow{\cong} H_{C_2}^{2n}(X(\mathbb{C}), \mathbb{Z}(n))_0$ of Proposition 5.8 induces an isomorphism between the kernel of

$$\partial : \mathrm{CH}^n(X) \rightarrow \mathrm{Ch}^n(X) \xrightarrow{\beta} H^{n+1}(X, \mathbf{I}^{n+1})$$

and the kernel of the composition

$$H_{C_2}^{2n}(X(\mathbb{C}), \mathbb{Z}(n))_0 \rightarrow H^n(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta} H^{n+1}(X(\mathbb{R}), \mathbb{Z}).$$

Using the compatibility of Proposition 3.23, the above isomorphism combined with the isomorphisms $H^n(X, \mathbf{I}^n) \xrightarrow{\cong} H^n(X(\mathbb{R}), \mathbb{Z})$ of Theorem 5.7 imply that the

map in Corollary 3.27 is an isomorphism because $\mathrm{CH}^n(X)$ is 2-torsion-free for cellular X . \square

5.10. Remark. We could choose base fields F slightly more general than $F = \mathbb{R}$. For Proposition 5.4, we just need that the conclusion holds for $\mathrm{Spec}(F)$, i.e., that $\langle\langle -1 \rangle\rangle : \bar{I}^j(F) \rightarrow \bar{I}^{j+1}(F)$ is an isomorphism for all $j \geq 0$.

For Proposition 5.5, it should similarly suffice to assume that $\langle\langle -1 \rangle\rangle : I^j(F) \rightarrow I^{j+1}(F)$ is an isomorphism for all $j \geq 0$. By Lemma 5.11 below, this (stronger) condition implies that $\mathrm{vcd}_2(F) = 0$. The latter condition ensures that, in the proof of Proposition 5.5, multiplication by $\langle\langle -1 \rangle\rangle$ still induces isomorphisms $I^j \cong I^{j+1}$ for sufficiently large j (as in the proof of [Jacobson 2015, Theorem 4.5]).

Theorem 5.7 also follows for smooth cellular X over any field F satisfying the conditions of Lemma 5.11, provided we replace singular by real cohomology.

5.11. Lemma. *For a field F of characteristic $\neq 2$, the following conditions are equivalent (as always, $I^0(F) = W(F)$ by convention):*

- (1) $\mathrm{vcd}_2(F) = 0$ and F is not quadratically closed.
- (2) $\langle\langle -1 \rangle\rangle : I^j(F) \rightarrow I^{j+1}(F)$ is an isomorphism for all $j \geq 0$.

Proof. In the proof, we identify $\langle\langle -1 \rangle\rangle$ with $2 \in W(F)$. As the torsion in the Witt ring of a field is 2-primary [Elman et al. 2008, §31], $2 : I^j(F) \rightarrow I^{j+1}(F)$ is injective if and only if $I^j(F)$ is torsion-free. By [Jacobson 2015, Lemma 2.4], $\mathrm{vcd}_2(F) = 0$ if and only if $I(F)$ is torsion-free and $2 : I^j(F) \rightarrow I^{j+1}(F)$ is surjective for all $j \geq 0$. The field F is quadratically closed if and only if $I(F) \neq 0$ [Elman et al. 2008, Lemma 31.3]. Thus, under either condition, $2 : W(F) \rightarrow I(F)$ is surjective, $2 : I^j(F) \rightarrow I^{j+1}(F)$ is an isomorphism for all $j \geq 1$, and $I(F)$ is torsion-free. We need to show that, in this situation, $I(F)$ is nonzero if and only if $2 : W(F) \rightarrow I(F)$ is injective. This can be checked using the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I(F) & \longrightarrow & W(F) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \cong \downarrow 2 & & \downarrow 2 & & \downarrow 2 & & \\
 0 & \longrightarrow & I^2(F) & \longrightarrow & I(F) & \longrightarrow & I(F)/I^2(F) & \longrightarrow & 0
 \end{array}$$

In our situation, the first vertical arrow is an isomorphism, the second and (hence) the third vertical arrows are surjective, and $I(F)/I^2(F) = I(F)/2I(F)$. Given that $I(F)$ is torsion-free, $I(F)$ is nonzero if and only if $I(F)/2I(F)$ is nonzero, which is equivalent to the third vertical map being injective, which is equivalent to the central vertical map being injective, as claimed. \square

6. Characteristic classes

In this section, we discuss the compatibility of the real cycle class maps with characteristic classes of vector bundles. For spaces like BGL_n or the finite Grassmannians this will imply a strengthening of Theorem 5.7: not only is the real cycle class map a ring isomorphism, it also maps the natural generators (characteristic classes of the tautological bundles) in the algebraic setting of I -cohomology to their topological counterparts. This also settles a point which was only mentioned but not fully proved in [HW 2019] and [Wendt 2018].

6.1. Proposition. *Let X be a smooth scheme over \mathbb{R} and let $\mathcal{E} \rightarrow X$ be a vector bundle of rank n over X . Then the real cycle class map*

$$H^n(X, I^n(p^* \det \mathcal{E}^\vee)) \rightarrow H^n(X(\mathbb{R}), \mathbb{Z}(p^* \det \mathcal{E}^\vee))$$

maps the algebraic Euler class to the topological Euler class.

Proof. The algebraic Euler class is defined as $e_n(\mathcal{E}) := (p^*)^{-1}s_*(1)$, where $p : \mathcal{E} \rightarrow X$ is the structure morphism of the vector bundle, $s : X \rightarrow \mathcal{E}$ is the zero section and $\langle 1 \rangle$ is the image of the natural element of $W(F)$ under the morphism $W(F) \cong H^0(\mathrm{Spec} \mathbb{R}, I^0) \rightarrow H^0(X, I^0)$ induced from the structure morphism $X \rightarrow \mathrm{Spec} \mathbb{R}$.

The topological definition is the same; see, for example, the end of the proof of Theorem 4D.10 in [Hatcher 2002] in the case of orientable vector bundles. The claim then follows from the compatibility of real cycle class maps with pullbacks and pushforwards; see Proposition 4.1 and Theorem 4.7. \square

6.2. Remark. A similar statement is true for the characteristic classes in the real-étale cohomology, as well as the characteristic classes in semialgebraic cohomology over real closed fields. We won't need these statements here.

6.3. Theorem. *Let X be one of the spaces BSp_{2n} , BSL_n , BGL_n , or $\mathrm{Gr}(k, n)$ over the base field \mathbb{R} . The real cycle class map induces an isomorphism*

$$\bigoplus_{j, \mathcal{L}} H^j(X, I^j(\mathcal{L})) \xrightarrow{\cong} \bigoplus_{j, \mathcal{L}} H^j(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})),$$

and this identification maps the algebraic characteristic classes (Euler class, Pontryagin classes, Bockstein classes) to their topological counterparts.

Proof. Compatibility of the Euler class with the identification with singular cohomology under the real cycle class map is established in Proposition 6.1. Compatibility of the Pontryagin classes with the real cycle class map then follows from this: the Pontryagin (or Borel) classes for the symplectic groups are obtained from the Euler class by stabilization; see [HW 2019, Proposition 4.3]. The Pontryagin classes for SL_n and GL_n are then simply obtained as characteristic classes of the

symplectification. For the Stiefel–Whitney classes, we note that these also are uniquely defined by stabilization and the fact that the top class should be the reduction of the Euler class. Then Theorem 5.7 (or better its \bar{I}^\bullet -version) implies that the real cycle class maps induce isomorphisms

$$\bigoplus_i \text{Ch}^i(X) \xrightarrow{\cong} \bigoplus_i H^i(X, \bar{I}^i) \xrightarrow{\cong} \bigoplus_i H^i(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$$

for the spaces X in the statement. The compatibility of Bockstein classes with the real realization maps then follows from this and the compatibility with the Bockstein maps in Proposition 4.14. \square

6.4. Remark. As a special case, the ring structure for $\bigoplus_{j,\mathcal{L}} H^j(\mathbb{P}^n, \mathbf{I}^j(\mathcal{L}))$ determined in [Fasel 2013]—see also [Wendt 2018, Propositions 4.5 and 6.5]—completely agrees with the presentation known in topology. The natural generators in both cases are the Euler classes e_1 and e_n^\perp of the tautological line bundle and its complement (and the orientation class whenever \mathbb{P}^n is orientable); and the above result shows that these natural generators correspond under the real realization maps. The same statement is true for the Grassmannians.

Appendix: A Gersten complex for real cohomology

The purpose of this section is to give an alternative proof of Theorem 4.7.

A.A. The Rost–Schmid complex. The references for this section are [Schmid 1997] and [Fasel 2008]. Let X be a smooth integral scheme of Krull dimension n over a field k . The Rost–Schmid complex $C^{\text{RS}}(X, \mathbf{W})$ for Witt groups is defined as

$$\bigoplus_{x \in X^{(0)}} W(k(x), \omega_x) \rightarrow \bigoplus_{x \in X^{(1)}} W(k(x), \omega_x) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(n)}} W(k(x), \omega_x),$$

where ω_x denotes for any point $x \in X^{(j)}$ the vector space

$$\text{Hom}_{\mathcal{O}_{X,x}}(\det \mathfrak{m}_{X,x} / \mathfrak{m}_{X,x}^2, \mathcal{O}_{X,x}),$$

which, here, is canonically isomorphic to $\text{Ext}_{\mathcal{O}_{X,x}}^j(k(x), \mathcal{O}_{X,x})$. The differential

$$\partial_{\text{RS}}^i : \bigoplus_{x \in X^{(i)}} W(k(x), \omega_x) \rightarrow \bigoplus_{y \in X^{(i+1)}} W(k(y), \omega_y)$$

may be described geometrically by the composition (see, for example, [Fasel 2008, Section 7.3])

$$W(k(x), \omega_{k(x)/k}) \xrightarrow{\oplus \partial} \bigoplus_{\tilde{y}} W(k(\tilde{y}), \omega_{k(\tilde{y})/k}) \xrightarrow{\Sigma \text{tr}} W(k(y), \omega_{k(y)/k}),$$

where the sum is taken over all points \tilde{y} that dominate y and live inside the normalization $\widetilde{\{x\}}$ of $\overline{\{x\}}$ in its residue field $k(x)$. Note that $\mathcal{O}_{\widetilde{\{x\}}, \tilde{y}}$ is the integral

closure of the one-dimensional local domain $\mathcal{O}_{\{\bar{x}\},y}$ in its field of fractions, which is isomorphic to $k(x)$. Therefore, $\mathcal{O}_{\{\bar{x}\},\tilde{y}}$ is a discrete valuation ring with maximal ideal $\mathfrak{m}_{\{\bar{x}\},\tilde{y}}$ and the uniformizing parameter π . The first map ∂ is induced by the Rost–Schmid residue map

$$\delta : W(k(x)) = W(k(\tilde{x})) \rightarrow W(k(\tilde{y}), (\mathfrak{m}_{\{\bar{x}\},\tilde{y}}/\mathfrak{m}_{\{\bar{x}\},\tilde{y}}^2)^*)$$

defined by sending a form $\langle a \rangle$ on $k(x)$ to $\delta_2^{\pi\tilde{y}}(\langle a \rangle) \otimes \pi^*$, where δ_2 is Milnor’s second residue and π^* is the dual basis of the basis π in the free rank-one $k(\tilde{y})$ vector space $\mathfrak{m}_{\{\bar{x}\},\tilde{y}}/\mathfrak{m}_{\{\bar{x}\},\tilde{y}}^2$. The map ∂ is obtained by twisting δ with the rank-one free $k(x)$ vector space $\omega_{k(x)/k}$ in view of the canonical isomorphism

$$\omega_{k(\tilde{y})/k} \cong \omega_{k(\tilde{x})/k} \otimes (\mathfrak{m}_{\{\bar{x}\},\tilde{y}}/\mathfrak{m}_{\{\bar{x}\},\tilde{y}}^2)^*.$$

Note that the differential ∂ does not depend on the choice of uniformizing parameter. The map tr is the twisted transfer on Witt groups, for the field extension $k(\tilde{y})$ of $k(y)$; see [Fasel 2008, 6.4].

For any line bundle \mathcal{L} on X , the twisted Rost–Schmid complex $C^{\text{RS}}(X, \mathbf{W}, \mathcal{L})$ for Witt groups is defined as

$$\bigoplus_{x \in X^{(0)}} W(k(x), \omega_x^{\mathcal{L}}) \rightarrow \bigoplus_{x \in X^{(1)}} W(k(x), \omega_x^{\mathcal{L}}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(n)}} W(k(x), \omega_x^{\mathcal{L}})$$

in a natural way, similar to the untwisted case. It is known that the Rost–Schmid complex coincides with the Gersten complex defined by Balmer and Walter (re-viewed in Section 2.C.1) [Fasel 2008, Section 7].

A.B. The Jacobson–Scheiderer complex and a generalization. Suppose that X is a smooth integral scheme of Krull dimension d over a field k with a line bundle \mathcal{L} . Let $x \in X^{(i)}$ and $y \in X^{(i+1)}$. Our goal in this section is to define a twisted differential,

$$\partial_{\text{re}} : C(k(x)_r, \mathbb{Z}(\omega_{k(x)/k})) \rightarrow C(k(y)_r, \mathbb{Z}(\omega_{k(y)/k}));$$

see Section A.B.1 and Definition A.4 below. This will be defined by a twisted residue map followed by a twisted transfer map on the real spectrum. Here and in all that follows, we use the shorthand $F_r := \text{sp}er F$ for a field F .

A.B.1. Twisted residue. Let R be a discrete valuation ring with field of fractions F and let \mathfrak{m} be its maximal ideal. Let $k_{\mathfrak{m}}$ be its residue field. Choose a uniformizing parameter π of R . Suppose P is an ordering on F . We say R is *convex* on (F, P) whenever for all $x, y, z \in F$, $x \leq_P z \leq_P y$ and $x, y \in R$ implies $z \in R$. For any ordering $\bar{\xi} \in k_{\mathfrak{m}}$, the set

$$Y_{\bar{\xi}} := \{P \in \text{sp}er F : R \text{ is convex in } (F, P) \text{ and } \bar{\xi} = \bar{P} \text{ on } k_{\mathfrak{m}}\}$$

maps bijectively to $\{\pm 1\}$ by sending P to $\text{sign}_P(\pi)$ [Jacobson 2017, §3]. Denote by ξ_{\pm}^{π} the ordering on F such that $\text{sign}_{\xi_{\pm}^{\pi}}(\pi) = \pm 1$. Jacobson [2017, §3] defines a residue map (a group homomorphism)

$$\begin{aligned} \beta_{\pi} : C(F_r, \mathbb{Z}) &\rightarrow C((k_{\mathfrak{m}})_r, \mathbb{Z}), \\ s &\mapsto (\bar{\xi} \mapsto s(\xi_{+}^{\pi}) - s(\xi_{-}^{\pi})). \end{aligned}$$

This map depends on the choice of uniformizing parameter. We modify it slightly in the spirit of Rost and Schmid. For a field Q and a rank-one Q -module H , we define $C(Q_r, \mathbb{Z}(H))$ as the group

$$C(Q_r, \mathbb{Z}) \otimes_{\mathbb{Z}[Q^{\times}]} \mathbb{Z}[H^{\times}],$$

where the underlying map

$$\zeta : \mathbb{Z}[Q^{\times}] \rightarrow C(Q_r, \mathbb{Z})$$

is the ring homomorphism given by

$$\left(\sum n_a a \right) \mapsto \left(\xi \mapsto \sum n_a \text{sign}_{\xi} a \right).$$

A.1. Definition. The twisted residue map is the group homomorphism

$$\begin{aligned} \beta : C(F_r, \mathbb{Z}) &\rightarrow C((k_{\mathfrak{m}})_r, \mathbb{Z}((\mathfrak{m}/\mathfrak{m}^2)^{*})), \\ s &\mapsto \beta_{\pi}(s) \otimes \pi^{*}, \end{aligned}$$

where π^{*} is the dual basis of the basis π of the free rank-one $k_{\mathfrak{m}}$ -vector space $\mathfrak{m}/\mathfrak{m}^2$.

A.2. Lemma. *The twisted residue map β does not depend on the choice of the uniformizing parameter.*

Proof. Let $a\pi$ be another uniformizing parameter with $a \in R - \mathfrak{m}$. Note that $\beta_{a\pi}(s)(\bar{\xi}) = \text{sign}_P(a)\beta_{\pi}(s)(\bar{\xi})$ and $\text{sign}_P(a) = \text{sign}_{\bar{\xi}}(a)$. It follows that

$$\beta_{a\pi}(s) \otimes a\pi^{*} = \beta_{a\pi}(s)\zeta(a) \otimes \pi^{*}.$$

Now, $\beta_{a\pi}(s)\zeta(a)(\bar{\xi}) = \text{sign}_{\bar{\xi}}(a)^2\beta_{\pi}(s)(\bar{\xi}) = \beta_{\pi}(s)(\bar{\xi})$. The result follows. \square

A.B.2. Twisted transfer. Let L be a finite field extension of F . Assume that F has an ordering P . Hence, F has characteristic zero. We say an ordering R (on L) is an extension of P if $P \subset R$ under the injection $F \hookrightarrow L$. Define a map

$$\begin{aligned} t : C(L_r, \mathbb{Z}) &\rightarrow C(F_r, \mathbb{Z}), \\ \phi &\mapsto \left(P \mapsto \sum_{R \supset P} \phi(R) \right), \end{aligned}$$

where the sum runs through all extensions R of P . This map is well defined, because the sum is finite. (Note that the number of extensions equals $\text{sign}_P(\text{Tr}_*(1))$;

see [Scharlau 1985, Chapter 3, Theorem 4.5]). This map will be called the transfer map.

Now we twist the transfer. Note $C(L_r, \mathbb{Z}) = C(L_r, \mathbb{Z}(L))$. Suppose $k \subset F \subset L$ with L a separable finite extension of F . There is a canonical isomorphism (see [Fasel 2008, 6.4]),

$$\sigma : \omega_{F/k} \otimes_F L \cong \omega_{L/k}.$$

A.3. Definition. We define the twisted transfer

$$t : C(L_r, \mathbb{Z}(\omega_{L/k})) \rightarrow C(F_r, \mathbb{Z}(\omega_{F/k}))$$

by

$$\phi \mapsto t(\phi) : P \mapsto \sum_{R \supset P} \sigma \phi(R),$$

which is the composition

$$C(L_r, \mathbb{Z}(\omega_{L/k})) \xrightarrow[\cong]{\text{id} \otimes \sigma^{-1}} C(L_r, \mathbb{Z}(\omega_{F/k})) \xrightarrow{t \otimes \text{id}} C(F_r, \mathbb{Z}(\omega_{F/k})).$$

A.B.3. The differential. Let X be a smooth integral scheme of Krull dimension d over a field k with a line bundle \mathcal{L} . Let $x \in X^{(i)}$ and $y \in X^{(i+1)}$. Now, we are ready to define the map.

A.4. Definition. The differential in the Gersten complex of the real spectrum

$$\partial_{\text{re}} : C(k(x)_r, \mathbb{Z}(\omega_{k(x)/k})) \rightarrow C(k(y)_r, \mathbb{Z}(\omega_{k(y)/k}))$$

is defined by

$$C(k(x)_r, \mathbb{Z}(\omega_{k(x)/k})) \xrightarrow{\oplus \beta} \bigoplus_{\tilde{y}} C(k(\tilde{y})_r, \mathbb{Z}(\omega_{k(\tilde{y})/k})) \xrightarrow{\Sigma t} C(k(y)_r, \mathbb{Z}(\omega_{k(y)/k})),$$

where the sum is taken over all points \tilde{y} that dominate y and live inside the normalization $\{\tilde{x}\}$ of $\{x\}$ in its residue field $k(x)$. The maps β and t are defined as above.

A.5. Theorem. For any smooth integral scheme X over a field k , the sequence

$$\bigoplus_{x \in X^{(0)}} C(k(x)_r, \mathbb{Z}(\omega_x^{\mathcal{L}})) \rightarrow \bigoplus_{x \in X^{(1)}} C(k(x)_r, \mathbb{Z}(\omega_x^{\mathcal{L}})) \rightarrow \dots \rightarrow \bigoplus_{x \in X^{(n)}} C(k(x)_r, \mathbb{Z}(\omega_x^{\mathcal{L}})) \rightarrow 0$$

is a cochain complex. Its cohomology is the sheaf cohomology of X with coefficients in the Zariski sheaf $\text{supp}_* \mathbb{Z}(\mathcal{L})$.

We denote this complex by $C^{\text{re}}(X, \mathbb{Z}, \mathcal{L})$. The proof will be given at the end of Section A.C.

A.C. Comparison via the signature.

A.C.1. Signature. Let H be a free rank-one vector space over a field F . Choose a trivialization $s : F \rightarrow H$.

A.6. Definition. The twisted signature is the map

$$\begin{aligned} \text{sign} : W(F, H) &\rightarrow C(F_r, \mathbb{Z}(H)), \\ \phi &\mapsto \text{sign}_s(\phi) \otimes s(1), \end{aligned}$$

where $\text{sign}_s(\phi)(P) = \text{sign}_P(s^{-1}\phi)$.

A.7. Lemma. *The twisted signature does not depend on the choice of trivialization $s : F \rightarrow H$.*

Proof. Let $u : F \rightarrow H$ be another trivialization. We may assume the rank of ϕ is one. Then $s^{-1}\phi = \langle a \rangle$ and $u^{-1}\phi = \langle b \rangle$ for some $a, b \in F^\times$, and we find that

$$\text{sign}_s(\phi) \otimes s(1) = \text{const}_1 \otimes s(a) = \text{const}_1 \otimes \phi(1, 1) = \text{const}_1 \otimes u(b) = \text{sign}_u(\phi) \otimes u(1),$$

where const_1 denotes the constant map $\text{sp} F \rightarrow \mathbb{Z}$ with value 1. □

A.8. Lemma. *Consider the following diagram, in which the top horizontal morphism is given by composition with s and the lower horizontal morphism is given by tensoring with $s(1)$:*

$$\begin{array}{ccc} W(F) & \xrightarrow{s_*} & W(F, H) \\ \downarrow \text{sign} & & \downarrow \text{sign} \\ C(F_r, \mathbb{Z}) & \xrightarrow{s_*} & C(F_r, \mathbb{Z}(H)) \end{array}$$

This diagram commutes.

A.C.2. Comparison. Let R be a one-dimensional local domain containing a field k and essentially of finite type over k . Let F be its field of fractions and $k_{\mathfrak{m}}$ its residue field. Let \bar{R} denote the integral closure of R in F , let \tilde{R} denote the discrete valuation ring obtained by localizing \bar{R} at a maximal ideal \tilde{m} , and let $k_{\tilde{m}} := \tilde{R}/\tilde{m}$ denote the residue field at \tilde{m} . As R is essentially of finite type over a field, R is excellent, and hence the normalization $\text{Spec } \bar{R} \rightarrow \text{Spec } R$ is a finite morphism [EGA IV₂ 1965, 7.8.6(ii)]. In particular, \bar{R} contains only finitely many maximal ideals \tilde{m} , and the residue fields $k_{\tilde{m}}$ are finite extensions of $k_{\mathfrak{m}} = R/\mathfrak{m}$.

A.9. Theorem. *The following square commutes:*

$$\begin{array}{ccc} W(F, \omega_{F/k}) & \xrightarrow{\partial_{\text{RS}}} & W(k_{\mathfrak{m}}, \omega_{k_{\mathfrak{m}}/k}) \\ \downarrow \text{sign} & & \downarrow 2 \text{ sign} \\ C(F_r, \mathbb{Z}(\omega_{F/k})) & \xrightarrow{\partial_{\text{re}}} & C((k_{\mathfrak{m}})_r, \mathbb{Z}(\omega_{k_{\mathfrak{m}}/k})) \end{array}$$

Proof. By the discussion above, we can assume the integral closure \bar{R} of R in F is a finitely generated R -module. There are finitely many discrete valuation rings \tilde{R} dominating R , namely the localizations of \bar{R} at its maximal ideals. Let \tilde{m} denote

the maximal ideal of such an \tilde{R} . It is well known that $k_{\tilde{m}} := \tilde{R}/\tilde{m}$ is a finite field extension of $k_m = R/\mathfrak{m}$. The square

$$\begin{array}{ccc} W(F) & \xrightarrow{\delta} & W(k_{\tilde{m}}, (\tilde{m}/\tilde{m}^2)^*) \\ \downarrow \text{sign} & & \downarrow 2 \text{ sign} \\ C(F_r, \mathbb{Z}) & \xrightarrow{\beta} & C((k_{\tilde{m}})_r, \mathbb{Z}((\tilde{m}/\tilde{m}^2)^*)) \end{array} \quad (\text{A.10})$$

is commutative by [Jacobson 2017, Lemmas 3.1 and 3.2] and Lemma A.8. (Choose an arbitrary uniformizing parameter π and the trivialization $s : k_m \rightarrow (\mathfrak{m}/\mathfrak{m}^2)^*$ given by $1 \mapsto \pi^*$.) Now, twisting the square above, we conclude that the square

$$\begin{array}{ccc} W(k_{\tilde{m}}, \omega_{k_{\tilde{m}}/k}) & \xrightarrow{\text{tr}} & W(k_m, \omega_{k_m/k}) \\ \downarrow \text{sign} & & \downarrow \text{sign} \\ C((k_{\tilde{m}})_r, \mathbb{Z}(\omega_{k_{\tilde{m}}/k})) & \xrightarrow{t} & C((k_m)_r, \mathbb{Z}(\omega_{k_m/k})) \end{array}$$

also commutes. Without twists, this is [Scharlau 1985, Chapter 3, Theorem 4.5]. To see that it commutes with arbitrary twists, choose a trivialization $s : k_m \rightarrow \omega_{k_m/k}$ and define a trivialization $\tilde{s} : k_{\tilde{m}} \rightarrow \omega_{k_{\tilde{m}}/k}$ as the composition

$$k_{\tilde{m}} \cong k_m \otimes_{k_m} k_{\tilde{m}} \xrightarrow{s \otimes \text{id}} \omega_{k_m/k} \otimes_{k_m} k_{\tilde{m}} \xrightarrow{\sigma} \omega_{k_{\tilde{m}}/k}.$$

Then use Lemma A.8 again. \square

A.11. Corollary. *The signature induces a commutative diagram,*

$$\begin{array}{ccccccc} \bigoplus_{x \in X^{(0)}} W(k(x), \omega_x^{\mathcal{L}}) & \xrightarrow{\partial_{\text{RS}}} & \bigoplus_{x \in X^{(1)}} W(k(x), \omega_x^{\mathcal{L}}) & \xrightarrow{\partial_{\text{RS}}} & \dots & \xrightarrow{\partial_{\text{RS}}} & \bigoplus_{x \in X^{(n)}} W(k(x), \omega_x^{\mathcal{L}}) \longrightarrow 0 \\ \downarrow \text{sign} & & \downarrow 2 \text{ sign} & & & & \downarrow 2^n \text{ sign} \\ \bigoplus_{x \in X^{(0)}} C(k(x)_r, \mathbb{Z}(\omega_x^{\mathcal{L}})) & \xrightarrow{\partial_{\text{re}}} & \bigoplus_{x \in X^{(1)}} C(k(x)_r, \mathbb{Z}(\omega_x^{\mathcal{L}})) & \xrightarrow{\partial_{\text{re}}} & \dots & \xrightarrow{\partial_{\text{re}}} & \bigoplus_{x \in X^{(n)}} C(k(x)_r, \mathbb{Z}(\omega_x^{\mathcal{L}})) \longrightarrow 0 \end{array}$$

The upper horizontal sequence is the Rost–Schmid complex, which coincides with the Gersten complex defined by Balmer and Walter [Fasel 2008, Section 7]. Moreover, the upper sequence restricts from \mathbf{W} to powers I^j of the fundamental ideal, and this complex is indeed a flasque resolution for all j computing the Zariski sheaf cohomology of I^j by the proof of [Fasel 2007, Theorem 3.26].

The restriction of the signature to powers of the fundamental ideal $I^j(F)$ takes values in the subgroup $C(F_r, 2^j \mathbb{Z}) \subset C(F_r, \mathbb{Z})$. Indeed, $I(F)$ is additively generated by Pfister forms $\langle\langle a \rangle\rangle$, and $\text{sign}(\langle\langle a \rangle\rangle)(P) = \text{sign}_P(1) + \text{sign}_P(-a)$ is always even. So we obtain a map

$$\text{sign} : I^j(F) \rightarrow C(F_r, 2^j \mathbb{Z}).$$

Jacobson’s commutative square (A.10), displayed as the far side of the following cube, thus restricts to the square displayed as the front side of the cube:

$$\begin{array}{ccccc}
 & & W(F) & \xrightarrow{\delta_\pi} & W(k_m) \\
 & \nearrow & \downarrow & & \nearrow \\
 I^{j+1}(F) & \xrightarrow{\delta_\pi} & I^j(k_m) & & \\
 \downarrow \text{sign} & & \downarrow \text{sign} & & \downarrow 2 \text{ sign} \\
 & & C(F_r, \mathbb{Z}) & \xrightarrow{\beta_\pi} & C(k_m, \mathbb{Z}) \\
 \nearrow & & \downarrow 2 \text{ sign} & & \nearrow \\
 C(F_r, 2^{j+1}\mathbb{Z}) & \xrightarrow{\beta_\pi} & C((k_m)_r, 2^{j+1}\mathbb{Z}) & &
 \end{array}$$

Denoting by 2^{-j} sign the obvious composition

$$I^j(F) \xrightarrow{\text{sign}} C(F_r, 2^j\mathbb{Z}) \xrightarrow{\cong} C(F_r, \mathbb{Z}),$$

we can rewrite the commutative square at the front as

$$\begin{array}{ccc}
 I^{j+1}(F) & \xrightarrow{\delta_\pi} & I^j(k_m) \\
 \downarrow 2^{-(j+1)} \text{ sign} & & \downarrow 2^{-j} \text{ sign} \\
 C(F_r, \mathbb{Z}) & \xrightarrow{\beta_\pi} & C((k_m)_r, \mathbb{Z})
 \end{array}$$

The arguments used in the proof of Theorem A.9 therefore yield the following generalization.

A.12. Corollary. *The following square commutes:*

$$\begin{array}{ccc}
 I^{j+1}(F, \omega_{F/k}) & \xrightarrow{\partial_{\text{RS}}} & I^j(k_m, \omega_{k_m/k}) \\
 \downarrow 2^{-(j+1)} \text{ sign} & & \downarrow 2^{-j} \text{ sign} \\
 C(F_r, \mathbb{Z}(\omega_{F/k})) & \xrightarrow{\partial_{\text{re}}} & C((k_m)_r, \mathbb{Z}(\omega_{k_m/k}))
 \end{array}$$

Next, we want to pass to the colimit in the diagram in Corollary A.11. Note that multiplication by 2 on the Witt group is multiplication by the Pfister form $\langle\langle -1 \rangle\rangle$. Clearly, the following triangle commutes:

$$\begin{array}{ccc}
 I^j(F) & \xrightarrow{\cdot 2} & I^{j+1}(F) \\
 \searrow & & \swarrow \\
 & C(F_r, \mathbb{Z}) &
 \end{array}$$

The induced morphism

$$\text{sign}_\infty : \text{colim}_j I^j(F) \rightarrow C(F_r, \mathbb{Z})$$

is an isomorphism by a result of Arason and Knebusch (see [Jacobson 2017, Proposition 2.7]).

A.13. Corollary. *The signature induces an isomorphism of complexes:*

$$\operatorname{colim}_j C^{\text{RS}}(X, \mathbf{I}^j, \mathcal{L}) \xrightarrow[\operatorname{sign}_\infty]{\cong} C^{\text{re}}(X, \mathbb{Z}(\mathcal{L})).$$

(In particular, $C^{\text{re}}(X, \mathbb{Z}(\mathcal{L}))$ indeed is a complex.)

Proof of Theorem A.5. By replacing X with open subsets of X , we can view the complexes appearing in Corollary A.13 as complexes of flasque Zariski sheaves on X . The Rost–Schmid complexes of sheaves $C^{\text{RS}}(-, \mathbf{I}^j, \mathcal{L})$ are known to be exact in positive degrees over any smooth scheme X over a field k . Passing to the colimit, and using the isomorphism $\operatorname{sign}_\infty$, we find that the complex of sheaves $C^{\text{re}}(-, \mathbb{Z}, \mathcal{L})$ is likewise exact in positive degrees over such X , and hence provides a flasque resolution of the kernel of its first differential. The identification of this kernel with $\operatorname{supp}_* \mathbb{Z}$ follows directly from [Jacobson 2017, Proposition 4.7 and Lemma 4.9]. \square

Now we consider cohomology with support in a closed subset $Z \subset X$. Recall that, given an abelian sheaf \mathcal{F} on X with a flasque resolution $\mathcal{F} \rightarrow \mathcal{F}^\bullet$, the cohomology with support $H_Z^s(X, \mathcal{F})$ can be computed as the homology of the kernel of the restriction morphism $\mathcal{F}^\bullet(X) \rightarrow \mathcal{F}^\bullet(X \setminus Z)$. In our situation, we obtain:

A.14. Corollary. *Let X be a smooth scheme over a field k , and let Z be a closed subset of X . The signature map*

$$\operatorname{sign}_\infty : H_Z^i(X, \operatorname{colim} \mathbf{I}^j(\mathcal{L})) \rightarrow H_Z^i(X, \operatorname{supp}_* \mathbb{Z}(\mathcal{L}))$$

is an isomorphism.

In the following corollary, we write $\operatorname{rcl}_\infty$ for the composition

$$\begin{aligned} H_Z^i(X, \operatorname{colim} \mathbf{I}^j(\mathcal{L})) &\xrightarrow{\operatorname{sign}_\infty} H_Z^i(X, \operatorname{supp}_* \mathbb{Z}(\mathcal{L})) \xrightarrow{\operatorname{supp}^*} H_{Z_r}^i(X_r, \mathbb{Z}(\mathcal{L})) \\ &\xrightarrow{\iota^*} H_{Z(\mathbb{R})}^i(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})). \end{aligned} \quad (\text{A.15})$$

The real cycle class map $H_Z^i(X, \mathbf{I}^j) \rightarrow H_{Z(\mathbb{R})}^i(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$ is obtained by composing $\operatorname{rcl}_\infty$ with the canonical map $H^i(X, \operatorname{colim} \mathbf{I}^j) \rightarrow H_Z^i(X, \mathbf{I}^j)$ (see diagram (3.1)).

A.16. Corollary. *For any a smooth scheme X over \mathbb{R} , and any closed subset Z of X , the map $\operatorname{rcl}_\infty : H_Z^i(X, \operatorname{colim} \mathbf{I}^j(\mathcal{L})) \rightarrow H_{Z(\mathbb{R})}^i(X(\mathbb{R}), \mathbb{Z}(\mathcal{L}))$ is an isomorphism.*

Proof. The first map in (A.15) is an isomorphism by Corollary A.14. The last map is an isomorphism by Lemmas 2.61 and 2.24: $\iota_* \mathbb{Z}(\mathcal{L}) \cong \mathbb{Z}(\mathcal{L})$. It remains to see that the central map supp^* is an isomorphism. This is proved for unrestricted support in [Jacobson 2017, Lemma 4.6], and the argument given there can be generalized to arbitrary supports. Indeed, recall that the cohomology groups with support in Z are

defined via the right derived functors of Γ_Z . We thus have a Grothendieck spectral sequence,

$$H_Z^p(X, R^q \text{supp}_* \mathbb{Z}(\mathcal{L})) \Rightarrow R^{p+q}(\Gamma_Z \text{supp}_*)(\mathbb{Z}(\mathcal{L})).$$

By Lemma 2.7, we can identify $\Gamma_Z \text{supp}_*$ with Γ_{Z_r} , and hence we may identify the target of the spectral sequence with $H_{Z_r}^{p+q}(X_r, \mathbb{Z}(\mathcal{L}))$. On the other hand, by [Scheiderer 1994, Theorem 19.2] the higher direct images $R^q \text{supp}_*$ for $q > 0$ vanish, so the spectral sequence collapses. As in the case without supports (see Section 2.D), one may check that the edge map is given by supp^* . \square

A.D. Compatibility with pushforwards: an alternative proof.

A.D.1. The real Thom class and an alternative proof of Theorem 4.7.

A.17. Lemma. *Let $p : V \rightarrow X$ be a vector bundle of rank d over a smooth variety X over \mathbb{R} . The image of the algebraic Thom class $\text{th}(V)$ of Definition 2.54 under the real cycle class map*

$$H_X^d(V, \mathbf{I}^d(p^* \det V^\vee)) \rightarrow H_{X(\mathbb{R})}^d(V(\mathbb{R}), \mathbb{Z}(p^* \det V^\vee))$$

is a Thom class of the continuous vector bundle $p(\mathbb{R}) : V(\mathbb{R}) \rightarrow X(\mathbb{R})$ in the sense of Definition 2.57.

Proof. Let $\text{th}^{\text{cl}}(V)$ be the image in $H_{X(\mathbb{R})}^d(V(\mathbb{R}), \mathbb{Z}(p^* \det V^\vee))$ of the algebraic Thom class under the real cycle class map. Consider the following square:

$$\begin{array}{ccc} H^i(X, \text{colim } \mathbf{I}^j) & \xrightarrow{\text{th}(V) \cup p^*(-)} & H_X^{i+d}(V, \text{colim } \mathbf{I}^{j+d}(p^* \det V^\vee)) \\ \downarrow \text{rcl}_\infty & & \downarrow \text{rcl}_\infty \\ H^i(X(\mathbb{R}), \mathbb{Z}) & \xrightarrow{\text{th}^{\text{cl}}(V) \cup p^*(-)} & H_{X(\mathbb{R})}^{i+d}(V(\mathbb{R}), \mathbb{Z}(p^* \det V^\vee)) \end{array}$$

The square commutes as the real cycle class map is compatible with products (Proposition 4.3) and pullbacks (Proposition 4.1). The vertical maps are isomorphisms by Corollary A.16. Multiplication with the Thom class $\text{th}(V)$ can be identified with the dévissage isomorphism, as noted in the proof of Lemma 2.56, so by Lemma 2.13 the top horizontal map is a colimit of isomorphisms. It follows that the top horizontal map is itself an isomorphism, and hence the lower horizontal map is an isomorphism, too.

This applies, in particular, to the restricted bundle $V_x \rightarrow \text{Spec}(\mathbb{R})$ over any real point $x : \text{Spec}(\mathbb{R}) \rightarrow X$. So all maps in the following commutative square are

isomorphisms:

$$\begin{array}{ccc} H^i(x, \operatorname{colim} \mathbf{I}^j) & \xrightarrow{\operatorname{th}(V_x) \cup p^*(-)} & H_x^{i+d}(V_x, \operatorname{colim} \mathbf{I}^{j+d}(p^* \det V_x^\vee)) \\ \downarrow \operatorname{rcl}_\infty & & \downarrow \operatorname{rcl}_\infty \\ H^i(x(\mathbb{R}), \mathbb{Z}) & \xrightarrow{\operatorname{th}^{\operatorname{rcl}}(V_x) \cup p^*(-)} & H_{x(\mathbb{R})}^{i+d}(V_x(\mathbb{R}), \mathbb{Z}(p^* \det V_x^\vee)) \end{array}$$

As the Thom class $\operatorname{th}(V_x)$ of the restricted bundle is the pullback of $\operatorname{th}(V)$ along the inclusion $V_x \hookrightarrow V$, and as the real cycle class map is compatible with pullbacks, we find that $\operatorname{th}^{\operatorname{rcl}}(V_x)$ is likewise the pullback of $\operatorname{th}^{\operatorname{rcl}}(V)$ along the inclusion $V_x(\mathbb{R}) \hookrightarrow V(\mathbb{R})$. By taking $i = 0$ in the above diagram, we see that the lower horizontal map sends 1 to a generator. So, $\operatorname{th}^{\operatorname{rcl}}(V)$ is a Thom class in the sense of Definition 2.57, as claimed. \square

Alternative proof of Proposition 4.11. Proposition 4.11 follows from Lemma A.17 and the compatibility of the canonical map $H_Z^i(X, \mathbf{I}^j(\mathcal{L})) \rightarrow H_Z^i(X, \operatorname{colim} \mathbf{I}^j(\mathcal{L}))$ with the cup product. \square

Theorem 4.7 now follows exactly as before.

A.18. Remark. Given Proposition 4.11, which we have just reproved, the main remaining point is to see that the real cycle class map is compatible with deformation to the normal bundle. Using the same notation as in the first proof of Proposition 4.12, we have the following commutative diagram:

$$\begin{array}{ccccc} H_Z^i(\mathcal{N}_Z X, \operatorname{colim} \mathbf{I}^j(\mathcal{L})) & \xrightarrow{i_0^*} & H_{Z \times \mathbb{A}^1}^i(D(Z, X), \operatorname{colim} \mathbf{I}^j(\mathcal{L})) & \xleftarrow{i_1^*} & H_Z^i(X, \operatorname{colim} \mathbf{I}^j(\mathcal{L})) \\ \downarrow \operatorname{sign} & & \downarrow \operatorname{sign} & & \downarrow \operatorname{sign} \\ H_{Z(\mathbb{R})}^i(\mathcal{N}_Z X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) & \xrightarrow{i_0^*} & H_{(Z \times \mathbb{A}^1)(\mathbb{R})}^i(D(Z, X)(\mathbb{R}), \mathbb{Z}(\mathcal{L})) & \xleftarrow{i_1^*} & H_{Z(\mathbb{R})}^i(X(\mathbb{R}), \mathbb{Z}(\mathcal{L})) \end{array}$$

As in the proof of Lemma A.17, the horizontal maps are isomorphisms by the well-known construction of the deformation to the normal bundle (see the discussion above Proposition 4.12) and by Lemma 2.13, and the vertical maps are isomorphisms by Corollary A.16. We conclude that the lower horizontal maps are also isomorphisms. This is independent of any comparison of the deformation to the normal bundle with tubular neighborhood constructions.

We could have *defined* the pushforward in topology as the cup product with a Thom class followed by the deformation to the normal bundle, and this would have been enough for our purposes since the two localization sequences in topology (with values in the locally constant sheaf $\mathbb{Z}(\mathcal{L})$) and in algebraic geometry (with values in the sheaf $\operatorname{colim} \mathbf{I}^j(\mathcal{L})$) can be identified via the cycle class map. However, the identification of the pushforward defined in this way with more common constructions of pushforwards in singular cohomology requires arguments of the kind presented in the proof of Proposition 4.12.

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