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PERIODIC POINTS AND TOPOLOGICAL RESTRICTION HOMOLOGY

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Dedicated to Bruce Williams (1945-2018)

ABSTRACT. We answer in the affirmative two conjectures made by Klein and Williams. First, in a range of dimensions, the equivariant Reidemeister trace defines a complete obstruction to removing n-periodic points from a self-map f. Second, this obstruction defines a class in topological restriction homology.

We prove these results using duality and trace for bicategories. This allows for immediate generalizations, including a corresponding theorem for the fiberwise Reidemeister trace.

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1. INTRODUCTION

For a finite simplicial complex X and a continuous map $f: X \to X$, the **Lefschetz number** $L(f) \in \mathbb{Z}$ is a weighted sum of the fixed points of f. This invariant admits many generalizations. In this paper, we focus on generalizations that count the fixed points of f^n, or the n-periodic points of f.

Since it is a weighted sum, the Lefschetz number detects the presence of fixed points for any endomorphism in the homotopy class of f. However it does not give a sharp lower

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bound on the number of fixed points. For that we need to refine the Lefschetz number to the **Reidemeister trace** $R(f)$. This invariant takes values in the 0th homology group of the twisted free loop space of f,

$$\Lambda^f X := \{\, \gamma \in X^I \mid f(\gamma(1)) = \gamma(0) \,\}.$$

If X is a compact manifold of dimension at least 3, the Reidemeister trace is a complete obstruction to the removal of fixed points [Jia80, Gh66, Wec42].

In this paper we compare several refinements of the Lefschetz number and Reidemeister trace for periodic points, the weakest of which are the Lefschetz number and Reidemeister trace for f^n. To build the others, we use Fuller's observation that the fixed points of the map

$$\begin{aligned} X \times \cdots \times X &\xrightarrow{\Psi^n(f)} X \times \cdots \times X \\ (x_1, x_2, \ldots, x_n) &\longmapsto (f(x_n), f(x_1), \ldots, f(x_{n-1})) \end{aligned}$$

are precisely the periodic points of f of period n [Ful53, Kom88, Dol83]. If f is homotopic to a map that has no n-periodic points, then $\Psi^n(f)$ is homotopic to a map with no fixed points.

The map $\Psi^n(f)$ is equivariant with respect to the action of $C_n = \mathbb{Z}/n\mathbb{Z}$ that rotates coordinates. We can refine the observation above to say if f is homotopic to a map that has no n-periodic points, then $\Psi^n(f)$ is C_n-equivariantly homotopic to a map with no fixed points. Therefore the equivariant Reidemeister trace of $\Psi^n(f)$, which we also call the nth **Fuller trace**, is an obstruction to removing the n-periodic points from f. The Fuller trace is a map of equivariant spectra

$$R_{C_n}(\Psi^n(f)) \colon \mathbb{S} \to \Sigma^\infty_+ \Lambda^{\Psi^n(f)} X^n$$

or equivalently a map of spectra

$$R(\Psi^n(f))^{C_n} : \mathbb{S} \to \left(\Sigma^\infty_+ \Lambda^{\Psi^n(f)} X^n\right)^{C_n}.$$

The following comparison theorem is the main result of the paper.

Theorem 1.1 (Theorems A and B)**.** *Let X be a finitely dominated space. Then the following diagram commutes up to homotopy for each $k \mid n$.*

$$\begin{tikzcd}
 & & & \mathbb{S} \arrow[dlll, "R(\Psi^n(f))^{C_n}"'] \arrow[dll, "R(\Psi^k(f))^{C_k}"] \arrow[d, "R(\Psi^k(f))"] \arrow[dr, "R(f^k)"] & \\
(\Sigma^\infty_+ \Lambda^{\Psi^n(f)} X^n)^{C_n} \arrow[r, "R"] & (\Sigma^\infty_+ \Lambda^{\Psi^k(f)} X^k)^{C_k} \arrow[r, "F"] & \Sigma^\infty_+ \Lambda^{\Psi^k(f)} X^k \arrow[r, "\simeq"] & \Sigma^\infty_+ \Lambda^{f^k} X
\end{tikzcd}$$

Here "finitely dominated" means that X is a retract up to homotopy of a finite CW complex. This is essentially the most general case in which $R(f)$ is defined. The maps R and F are the natural analogs of the "restriction" and "Frobenius" maps from the theory of topological Hochschild homology, defined as in [Mad95, §2.5]. The homotopy equivalence equivalence at the bottom-right is given by the maps

$$\begin{aligned} \{\, \gamma_1, \ldots, \gamma_k \in X^I \mid f(\gamma_i(1)) = \gamma_{i+1}(0) \,\} &\longrightarrow \{\, \gamma \in X^I \mid f^k(\gamma(1)) = \gamma(0) \,\} \\ (\gamma_1, \ldots, \gamma_k) &\longmapsto f^{k-1}(\gamma_2) \cdot f^{k-2}(\gamma_3) \cdot \ldots \cdot f(\gamma_k) \cdot \gamma_1 \\ (\gamma, c_{f(\gamma(1))}, c_{f^2(\gamma(1))}, \ldots, c_{f^{k-1}(\gamma(1))}) &\longleftarrow\!\shortmid \gamma \end{aligned} \tag{1.2}$$

where c denotes the constant path at x.

Theorem 1.1 gives the following answer to a conjecture of Klein and Williams [KW10].

Corollary 1.3. *The Reidemeister traces $\{R(f^k)\}$ can be recovered from the Fuller trace $R(\Psi^n(f))^{C_n}$. The vanishing of $R(\Psi^n(f))^{C_n}$ implies the vanishing of $R(f^k)$ for all $k|n$.*

When combined with the main result of [Jez01], this implies

Corollary 1.4. *If X is a compact manifold of dimension at least 3, the Fuller trace $R(\Psi^n(f))^{C_n}$ vanishes in the homotopy category of spectra if and only if f is homotopic to a map with no n-periodic points.*

In other words, for high-dimensional manifolds the Fuller trace is a complete obstruction to the removal of n-periodic points.

Though our motivation for Theorem 1.1 comes mainly from dynamics, it also has important implications for algebraic K-theory. These implications can be succinctly expressed by the slogan "topological restriction homology (TR) is the most natural home for periodic-point invariants".

More precisely, recall that the **topological restriction homology** $\mathrm{TR}(A)$ of a ring spectrum A is the homotopy limit of $\mathrm{THH}(A)^{C_n}$ along restriction maps

$$R\colon \mathrm{THH}(A)^{C_n} \to \mathrm{THH}(A)^{C_k}$$

for $k \mid n$, see [BHM93, Mad95] for details. If $A = \Sigma^\infty_+ \Omega X$ for a connected CW complex X, $\mathrm{THH}(A) \simeq \Sigma^\infty_+ \Lambda X$, the suspension spectrum of the free loop space of X. The topological restriction homology of this ring, denoted $\mathrm{TR}(X)$, is the homotopy limit of the fixed point spectra $(\Sigma^\infty_+ \Lambda X)^{C_n}$ along the restriction maps

$$R : (\Sigma^\infty_+ \Lambda X)^{C_n} \to (\Sigma^\infty_+ \Lambda X)^{C_k}.$$

Concretely, the restriction map takes each C_n-equivariant map $S^V \to S^V \wedge (\Lambda X)_+$ to its $C_{n/k}$-fixed points:

$$S^{V^{C_{n/k}}} \longrightarrow S^{V^{C_{n/k}}} \wedge (\Lambda X)_+^{C_{n/k}} \xleftrightarrow{\cong} S^{V^{C_{n/k}}} \wedge (\Lambda X)_+$$

If X is equipped with an endomorphism f, we define the **twisted topological restriction homology** of X as

$$TR(X,f) := \operatorname*{holim}_{n,R} (\Sigma^\infty_+ \Lambda^{\Psi^n(f)} X^n)^{C_n}$$

where the restriction maps R are the maps of Theorem 1.1. Concretely, they take each C_n-equivariant map $S^V \to S^V \wedge (\Lambda^{\Psi^n(f)} X^n)_+$ to its $C_{n/k}$-fixed points:

$$S^{V^{C_{n/k}}} \longrightarrow S^{V^{C_{n/k}}} \wedge (\Lambda^{\Psi^n(f)} X^n)_+^{C_{n/k}} \xleftrightarrow{\cong} S^{V^{C_{n/k}}} \wedge (\Lambda^{\Psi^k(f)} X^k)_+$$

The justification for the name is that $\mathrm{TR}(X;f)$ agrees with a more general definition of $\mathrm{TR}(A;M)$ for any ring spectrum A and bimodule M, defined in a similar way to Lindenstrauss and McCarthy's W-theory [LM12] for ordinary rings. Details of this construction will appear in [CLM+20].

By the tom Dieck splitting theorem, each of the spectra in the homotopy limit system for $\mathrm{TR}(X;f)$ splits as a finite product of homotopy orbit spectra

$$(\Sigma^\infty_+ \Lambda^{\Psi^n(f)} X^n)^{C_n} \simeq \prod_{k|n} (\Sigma^\infty_+ \Lambda^{\Psi^k(f)} X^k)_{hC_k}, \tag{1.5}$$

and the restriction map R simply projects onto a subset of the factors. It follows that the homotopy limit $\mathrm{TR}(X;f)$ is an infinite product of homotopy orbit spectra

$$\mathrm{TR}(X;f) \simeq \prod_{n\geq 1} (\Sigma^\infty_+ \Lambda^{\Psi^n(f)} X^n)_{hC_n}. \tag{1.6}$$

Therefore, to define a class in $\pi_0 \mathrm{TR}(X;f)$, it is enough to give a class in π_0 of the spectrum (1.5) for each $n \geq 1$, agreeing along the restriction maps.

Theorem 1.1 says that the Fuller traces $R(\Psi^n(f))^{C_n}$ give such a collection of classes. In particular, the commutativity of the left-hand triangle implies they agree along the restriction maps. Therefore they define a class in $\pi_0 \mathrm{TR}(X;f)$ that we might call the "infinite Fuller trace" $R(\Psi^\infty(f))^{C_\infty}$. Concretely, this is the element of the product (1.6) whose nth term is recovered from $R(\Psi^n(f))^{C_n}$ using the tom Dieck splitting (1.5). By Corollary 1.4, the infinite Fuller trace is a complete obstruction to removing n-periodic points for any value of n. This is the precise interpretation of the slogan, "periodic point invariants most naturally live in TR."

This slogan has been articulated before. Klein, McCarthy, Williams, and others have remarked that one should be able to construct a trace map from endomorphism K-theory $K(\mathrm{End}_A(M))$ to $\mathrm{TR}(A;M)$ for any ring spectrum, as in [LM12]. Furthermore, there should be a class $[f] \in K(\mathrm{End}_A(M))$ whose image in $\pi_0 \mathrm{TR}(X,f)$ recovers the Reidemeister traces $R(f^n)$ of all the composites. Earlier results in this spirit can be found in [Gra77, GN94, Iwa99, Lüc99], but this particular result will be developed in [CLM+20]. Granting this, this defines a periodic point invariant in $\pi_0 \mathrm{TR}(X,f)$ without reference to the Fuller construction. From this point of view, the additional insight provided by Theorem 1.1 is that this class can be explicitly described as the trace of the Fuller map.

Fiberwise invariants. Following [DP80, GN94, Nic05, Pon10, Pon16], we interpret the Lefschetz number and Reidemeister trace as stable homotopy classes of maps rather than numbers. One of the primary advantages of this approach is that it allows for easy generalizations to the fiberwise and equivariant settings. Using this perspective, the following result has the same proof as its classical analog.

Theorem 1.7. *The variants of Theorem 1.1 and Corollary 1.3 for a family of fiberwise endomorphisms $f : E \to E$ over B also hold, provided $E \to B$ is a fibration with finitely dominated fiber.*

On the other hand, the fiberwise version of Corollary 1.4 is the following conjecture. It will require a very different set of techniques, and we plan to take it up in future work.

Conjecture 1.8. *The fiberwise Fuller trace $R_B(\Psi^n(f))^{C_n}$ is the complete obstruction to the removal of n-periodic points from a family of endomorphisms $f : E \to E$ over B, when B is a finite-dimensional cell complex and $E \to B$ is a smooth closed manifold bundle whose fiber M has dimension at least $3 + \dim B$.*

Note that the special case of $n = 1$ is proven in [KW07, Cor 10.5].

For higher values of n, we could have instead formulated the conjecture using the collection of Reidemeister traces $\{R_B(f^k) : k \mid n\}$, but we expect that version of the conjecture to be false. In other words, we expect that the Reidemeister traces of the iterates do not form a complete obstruction to removing n-periodic points from bundles, in contrast to the case of a single endomorphism [Jez01]. The reason for our expectation is that the product over $k \mid n$ of the maps in the bottom row of Theorem 1.1 is injective on π_0, but fails to be injective above π_0. As a result, once we start measuring the higher homotopy groups by looking at families of endomorphisms, we might find a Fuller trace that lies in the kernel, so that the corresponding Reidemeister traces are all zero. A counterexample of the following form would help settle this question.

Conjecture 1.9. *There is a family of endomorphisms $f : E \to E$ over some base B for which $R_B(f)$ and $R_B(f^2)$ are zero, but the fiberwise Fuller trace $R_B(\Psi^2(f))^{C_2}$ is nonzero.*

Organization. We first give a short proof of Theorem 1.1 in the special case where X is a compact ENR. We then proceed with the general case. The proof splits into two pieces, and these proofs are the first two parts of this paper. In Part 1 we prove that the right triangle commutes using the string diagram calculus developed in [MP18]. In Part 2 we

prove the left two triangles commute, by extending certain functors on the category of equivariant spectra to shadow functors on the bicategory of equivariant parameterized spectra. In Part 3 we prove Theorem 1.7.

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2. The case of a single smooth manifold or compact ENR

There are two approaches to proving Theorem 1.1. The first is a more classical and geometrically motivated path starting from an explicit descriptions of the Reidemeister trace. This builds on ideas both explicit and implicit in [CJ98, Dol74, Dol76], and a complete description of this version of the Reidemiester trace can be found in [Mal19]. Alternatively, there is a more formal and category theoretic approach that follows the understanding of fixed point invariants as traces in symmetric monoidal categories or bicategories. [DP80, Pon10].

These approaches both require significant effort to implement but the work in each case is very different. We have chosen to follow the second approach since we find it does not require the same level of outsourcing to papers such as [MS06, Mal19]. Despite this preference, we find the geometric approach provides very useful intuition. To benefit from these insights we first sketch the alternative proof of Theorem 1.1 for a single manifold or compact ENR.

Let X be a compact topological space, with a topological embedding $i\colon X \to V$ into an open subset $V \subseteq \mathbb{R}^N$ and a retract $p\colon V \to X$, making X into a compact ENR. Choose $\epsilon > 0$ so that the closed ϵ-tube about X is completely contained in V. The Reidemeister trace $R(f)$ is the map of spectra obtained by formal de-suspension of the following map of spaces.

$$\begin{array}{rcl} S^N & \longrightarrow & S^N_\epsilon \wedge (\Lambda^f X)_+ \\ v & \longmapsto & \begin{cases} (v - f(p(v))) \wedge \gamma_{f(p(v)),v} & \text{if } v \in V \text{ and } \|v - f(p(v))\| \leq \epsilon \\ * & \text{otherwise} \end{cases} \end{array}$$

Here S^N_ϵ is a sphere of radius ϵ, obtained by quotienting the complement of an open ball of radius ϵ in $\mathbb{R}^N$:

$$S^N_\epsilon = \mathbb{R}^N/(\mathbb{R}^N - B_\epsilon) \cong \overline{B_\epsilon}/\partial\overline{B_\epsilon}.$$

The path $\gamma_{f(p(v)),v)}$ is defined by the formula

$$\gamma_{f(p(v)),v}(t) = p[(1-t)f(p(v)) + tv].$$

The condition on ϵ guarantees that p is defined on the line segment from $f(p(v))$ to v, so that this path is well-defined. See [Mal19, §7.7] for a discussion of how this description of the Reidemeister trace arises from the more categorical descriptions later in this paper.

Note that the homotopy class of this map does not depend on the choice of ϵ.

The C_n-space $X^{\times n}$ becomes a C_n-equivariant ENR using the product embedding $i^{\times n}$ into the C_n-representation $\mathbb{R}^{nN} = \mathrm{Ind}^{C_n}\mathbb{R}^N$, and the product projection $p^{\times n}$. The Fuller trace $R_{C_n}(\Psi^n(f))$ is given by the equivariant version of the above map, de-suspended by the C_n-representation $\mathbb{R}^{nN}$ [Mal19, §9.5]. It is a map

$$S^{nN} \longrightarrow S^N_\epsilon \wedge \ldots \wedge S^N_\epsilon \wedge (\Lambda^{\Psi^n(f)} X^n)_+$$

and tor tuples $(v_1,\dots,v_n)$ where $v_i \in V$ and $\|v_{i+1} - f(p(v_i))\| \le \epsilon$ for every i, it is given by

$$\begin{aligned}(v_1,\dots,v_n) \quad &\longmapsto \quad (v_1 - f(p(v_n))) \wedge (v_2 - f(p(v_1))) \wedge (v_3 - f(p(v_2))) \wedge \dots \\ &\qquad \wedge \left(\gamma_{f(p(v_n)),v_1}, \gamma_{f(p(v_1)),v_2}, \gamma_{f(p(v_2)),v_3}, \dots\right)\end{aligned}$$

Everywhere else it is zero.

To prove that the left-hand triangle of Theorem 1.1 commutes, it is enough to observe that taking $C_{n/k}$-fixed points of this map replaces the n by k. The middle triangle of Theorem 1.1 commutes since forgetting the C_n action, we have the formula for the non-equivariant Reidemeister trace of $\Psi^n(f)$. The right-hand triangle, on the other hand, does not follow from such a simple observation. We have to show that if we take the above formula, then apply the equivalence $\Lambda^{\Psi^k(f)} X^k \xrightarrow{\sim} \Lambda^{f^k} X$ (1.2), the map we get is homotopic to the formula for $R(f^n)$.

Applying the equivalence in (1.2) to the path in $\Lambda^{\Psi^k(f)} X^k$, gives the path

$$f^{n-1}(\gamma_{f(p(v_1)),v_2}) \cdot \dots \cdot f(\gamma_{f(p(v_{n-1})),v_n}) \cdot \gamma_{f(p(v_n)),v_1}.$$

We now change this path by a homotopy. As observed above, replacing ϵ by $\delta < \epsilon$ does not change the Reidemeister trace in the homotopy category. Since f is a continuous function on a compact space it is uniformly continuous, and therefore there is a $\delta > 0$ so that when every v_i is within δ of X, the diameter of the paths $\gamma_{f(p(v_i)),v_{i+1}}$ and their images under f, f^2, …, and f^{n-1} are less than $\frac{\epsilon}{2n}$. (We measure all of these diameters as subsets of $\mathbb{R}^n$.) Then the sum of n of these diameters is less than $\frac{\epsilon}{2}$. This is small enough to guarantee that if we compose n such paths together, the straight-line homotopy in $\mathbb{R}^n$ between their composite and $\gamma_{f(p(v_n)),v_1}$ lies entirely in V, and can therefore be projected to X. This gives a continuous homotopy of paths in X rel endpoints

$$f^{n-1}(\gamma_{f(p(v_1)),v_2}) \cdot f^{n-2}(\gamma_{f(p(v_2)),v_3}) \cdot \dots \cdot f(\gamma_{f(p(v_{n-1})),v_n}) \cdot \gamma_{f(p(v_n)),v_1} \quad \sim \quad \gamma_{f^n(p(v_1)),v_1}.$$

Therefore our original formula is homotopic to:

$$\begin{aligned} S^{nN} \quad &\longrightarrow \quad S^N_\epsilon \wedge \dots \wedge S^N_\epsilon \wedge (\Lambda^{\Psi^n(f)} X^n)_+ \\ (v_1,\dots,v_n) \quad &\longmapsto \quad v_1 - f(p(v_n)) \wedge v_2 - f(p(v_1)) \wedge v_3 - f(p(v_2)) \wedge \dots \wedge \gamma_{f^n(p(v_1)),v_1}.\end{aligned}$$

The path now matches the path we would get for $R(f^n)$, but the sphere coordinates are different, so we apply a homotopy to those next. In the $(i+1)$st coordinate of the output, we apply a homotopy of the form

$$v_{i+1} - f(p(v_i)) \quad \sim \quad v_{i+1} - f^2(p(v_{i-1})) \quad \sim \quad \dots \quad \sim \quad v_{i+1} - f^i(p(v_1))$$

by dragging the second term along the path $f(\gamma_{f(p(v_{i-1})),v_i})$, then the path $f(\gamma_{f^2(p(v_{i-2})),v_{i-1}})$, and so on. Note that before we start this homotopy, our map is supported on the region where the distance from each v_i to X is less than or equal to δ, and throughout the homotopy, the boundary of this region is sent to the basepoint. In other words, if $d(v_{i+1}, X) \ge \delta$ then throughout the homotopy the size of the sphere coordinate is always $\ge \delta$, because every path we use is a path contained in X. This guarantees that we get a well-defined homotopy of maps on all of S^{nN}. Performing this for each $1 \le i \le n$ gives a homotopic map with the formula

$$\begin{aligned} S^{nN} \quad &\longrightarrow \quad S^N_\delta \wedge \dots \wedge S^N_\delta \wedge (\Lambda^{\Psi^n(f)} X^n)_+ \\ (v_1,\dots,v_n) \quad &\longmapsto \quad v_1 - f^n(p(v_1)) \wedge v_2 - f(p(v_1)) \wedge v_3 - f^2(p(v_1)) \wedge \dots \wedge \gamma_{f^n(p(v_1)),v_1}.\end{aligned}$$

This is almost the formula for $R(f^n)$ using the embedding $(i, f \circ i, f^2 \circ i, \dots, f^{n-1} \circ i) \colon X \to \mathbb{R}^{nN}$, except that f^n is only being applied to the first coordinate. So for the final step, we examine the above formula and observe that it still makes sense if we relax the assumptions and allow v_i to be any point in $\mathbb{R}^n$ when $i \ge 2$. With this change we can

then remove the $f^k(p(v_1))$ term from the second through nth coordinates by a homotopy, arriving at

$$\begin{array}{rcl} S^{nN} & \longrightarrow & S^N_\delta \wedge \ldots \wedge S^N_\delta \wedge (\Lambda^{\Psi^n(f)} X^n)_+ \\ (v_1,\ldots,v_n) & \longmapsto & v_1 - f^n(p(v_1)) \wedge v_2 \wedge v_3 \wedge \ldots \wedge \gamma_{f^n(p(v_1)),v_1} \end{array}$$

This agrees with the formula for $R(f^n)$ using the embedding $(i,0,0,\ldots,0)\colon X \to \mathbb{R}^{nN}$. Equivalently, it is the $(n-1)N$-fold suspension of the formula for $R(f^n)$ using $i\colon X \to \mathbb{R}^N$. This concludes the proof that the third triangle commutes in the homotopy category.

Part 1. Unwinding the Fuller trace

In this part we give a very general proof that the last triangle of Theorem 1.1 commutes:

Theorem A. *For any finitely dominated space X, the following diagram commutes up to homotopy.*

$$\begin{tikzcd} \mathbb{S} \arrow[d, "R(\Psi^k(f))"'] \arrow[dr, "R(f^k)"] & \\ \Sigma^\infty_+ \Lambda^{\Psi^k(f)} X^k \arrow[r, "\simeq"] & \Sigma^\infty_+ \Lambda^{f^k} X \end{tikzcd}$$

The argument is formal and based on the observation from [Pon10] that $R(f)$ is a bicategorical trace. To motivate this argument, we first describe the analogous argument for symmetric monoidal categories in Section 3. We then recall how to define fixed point invariants using bicategories in Sections 4 and 5, and finally prove the bicategorical version of the argument in Sections 6 and 7. In this part, we black-box all of the needed properties of parametrized spectra.

Remark on string diagrams. Even in simple cases, conventional notation choices obfuscate some of the central ideas in this paper. In an attempt to make these ideas more visible, we use the string diagrams calculus of [JSV96]. As shown in [JSV96], string diagram calculations are a rigorous alternative to traditional diagram chasing in symmetric monoidal categories and string diagrams manipulations can be translated into more conventional diagrams. (The corresponding result for bicategories with shadows can be found in [PS13].) Together [JSV96, PS13] put all string diagram manipulations in this paper on rigorous footing. The one exception is Figure 6.5 which should be regarded as motivation.

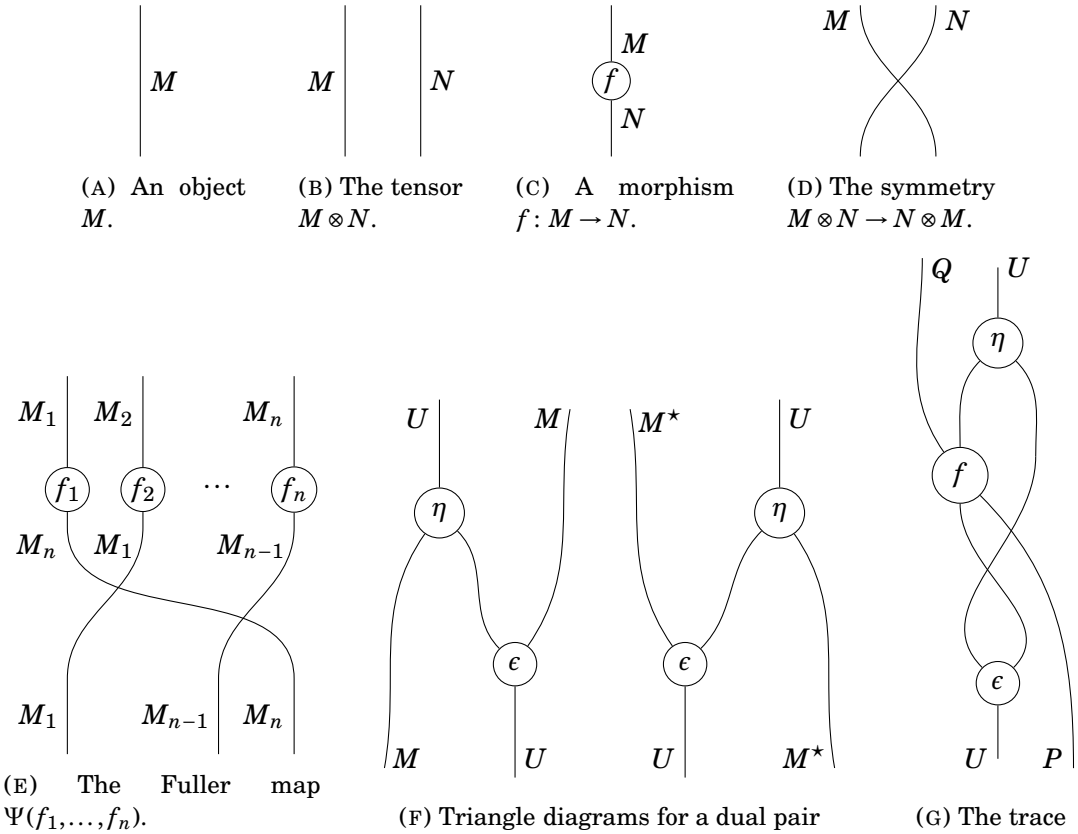
The building blocks for the symmetric monoidal string diagram calculus are the first four figures in Figure 3.1. They are "Poincaré dual" to the usual graphical representation of symmetric monoidal categories.

Finally, there are no string diagram calculations after Theorem 6.1 since this would require the development of a new calculus and that is beyond the scope of this paper.

3. Traces and multitraces in symmetric monoidal categories

In this section we consider the special case of Theorem A in a symmetric monoidal category $\mathscr{C}$ with monoidal product $\otimes$ and unit object U. Recall that:

- An object M of $\mathscr{C}$ is **dualizable** if there is an object $M^\star$ of $\mathscr{C}$ and morphisms
$$\eta\colon U \to M \otimes M^\star \text{ and } \epsilon\colon M^\star \otimes M \to U$$
so that the composites in Figure 3.1f are identity maps. These are the **triangle identities**.



(A) An object M.

(B) The tensor $M \otimes N$.

(C) A morphism $f\colon M \to N$.

(D) The symmetry $M \otimes N \to N \otimes M$.

(E) The Fuller map $\Psi(f_1,\dots,f_n)$.

(F) Triangle diagrams for a dual pair

(G) The trace

FIGURE 3.1. String diagrams for a symmetric monoidal category. We view the strings as oriented from top to bottom.

- If M is dualizable, the **trace** of a morphism $f\colon P \otimes M \to M \otimes Q$ is the composite in Figure 3.1g. If P and Q are units, $f\colon M \to M$.

We then define the **Fuller construction** of an n-tuple of maps $f_i\colon M_i \to M_{i-1}$ to be the composite

$$\Psi(f_1,\dots,f_n)\colon M_1 \otimes \cdots \otimes M_n \xrightarrow{f_1 \otimes \cdots \otimes f_n} M_n \otimes M_1 \otimes \cdots \otimes M_{n-1} \xrightarrow{\gamma} M_1 \otimes \cdots \otimes M_n \tag{3.2}$$

This is illustrated by the string diagram in Figure 3.1e. When all the M_i and f_i are equal, this is the nth Fuller map $\Psi^n(f)$ described in the introduction.

Theorem 3.3 (Symmetric monoidal version of Theorem A)**.** *For an n-tuple of dualizable objects $\{M_i\}_{i=1}^n$ in a symmetric monoidal category $(\mathscr{C},\otimes,U)$ and maps $f_i\colon M_i \to M_{i-1}$*

$$\operatorname{tr}(\Psi(f_1,\dots,f_n)) = \operatorname{tr}(f_1 \circ f_2 \circ \cdots \circ f_n)$$

as maps $U \to U$ in $\mathscr{C}$.

Proof. The trace of the Fuller construction

$$\Psi(f_1,\dots,f_n)\colon \bigotimes M_i \to \bigotimes M_i$$

of maps $f_i\colon M_i \to M_{i-1}$ is depicted in figure Figure 3.4a. Symmetry isomorphisms transform this trace to Figure 3.4b. Canceling as in Figure 3.1f transforms this to Figure 3.4c. This completes the proof of this theorem using string diagrams.

Alternatively, a diagram chase shows that if $X \otimes Y$ and Z are dualizable and $g\colon X \otimes Y \to X \otimes Z$ and $f\colon Z \to Y$, the trace of

$$X \otimes Y \otimes Z \xrightarrow{g \otimes f} X \otimes Z \otimes Y \xrightarrow{\mathrm{id} \otimes \gamma} X \otimes Y \otimes Z$$

is the trace of

$$X \otimes Y \xrightarrow{g} X \otimes Z \xrightarrow{\mathrm{id} \otimes f} X \otimes Y.$$

Since the map in (3.2) is the composite

$$\begin{aligned} M_1 \otimes \cdots \otimes M_{n-2} \otimes M_{n-1} \otimes M_n &\xrightarrow{f_1 \otimes \cdots \otimes f_{n-1} \otimes f_n} M_n \otimes M_1 \otimes \cdots M_{n-2} \otimes M_{n-1} \\ &\xrightarrow{\gamma_{1,n-2} \otimes \mathrm{id}} M_1 \otimes \cdots M_{n-2} \otimes M_n \otimes M_{n-1} \\ &\xrightarrow{\mathrm{id}^{\otimes n-2} \otimes \gamma} M_1 \otimes \cdots M_{n-2} \otimes M_{n-1} \otimes M_n \end{aligned}$$

the trace of $\Psi(f_1, \ldots, f_n)$ is the trace of $\Psi((f_n \circ f_1), \ldots, f_{n-1})$. Then the result follows by induction and the cyclic invariance of the trace. □

Example 3.5. If X is a finite or finitely dominated complex and $f\colon X \to X$, the **Lefschetz number** $L(f)$ is the trace of

$$\Sigma^\infty_+ f\colon \Sigma^\infty_+ X \to \Sigma^\infty_+ X$$

in the stable homotopy category. This trace is a self-map of the sphere spectrum $\mathbb{S} = \Sigma^\infty_+ *$. The above theorem implies that $L(\Psi^k(f)) = L(f^k)$, which is the Lefschetz version of Theorem A, see also [Ful67, 4.4].

Our proof of Theorem A will essentially be a generalization of the above proof. In the more general setting of a shadowed bicategory, we will re-arrange the Fuller trace into a map as in Figure 3.4b that we call the **multitrace**, and then re-arrange the multitrace into the trace of the composite. The latter step does not require us to re-order objects, so we can do it in any shadowed bicategory. The former does require us to re-order the objects, so we need to ask for more structure beyond that of a bicategory. We will show that this step can be performed anytime we have a "shadowed n-Fuller structure," defined in §6. In [MP18] we show that parametrized spectra have such a structure.

Example 3.6. The multitrace defined by Figure 3.4b coincides with the multitrace from [Sch04, §4], cf. [Mad95, (2.6.4),(2.6.5)]. To recall it explicitly, let $A_0, \ldots, A_k$ be $n \times n$ matrices with coefficients in a field, and V an n dimensional vector space with basis $\{e_i\}$. The coevaluation map of V is given by linearly extending $1 \mapsto \sum e_i \otimes e_i^*$, and the evaluation map is given by linearly extending $(\phi, v) \mapsto \phi(v)$.

The image of $(1, \ldots 1)$ under the multitrace of $(A_0, \ldots, A_k)$ is then

$$\sum_{0 \le i_1, \ldots, i_k \le n} a^0_{i_k i_0} \otimes a^1_{i_0 i_1} \otimes \ldots \otimes a^k_{i_{k-1} i_k}$$

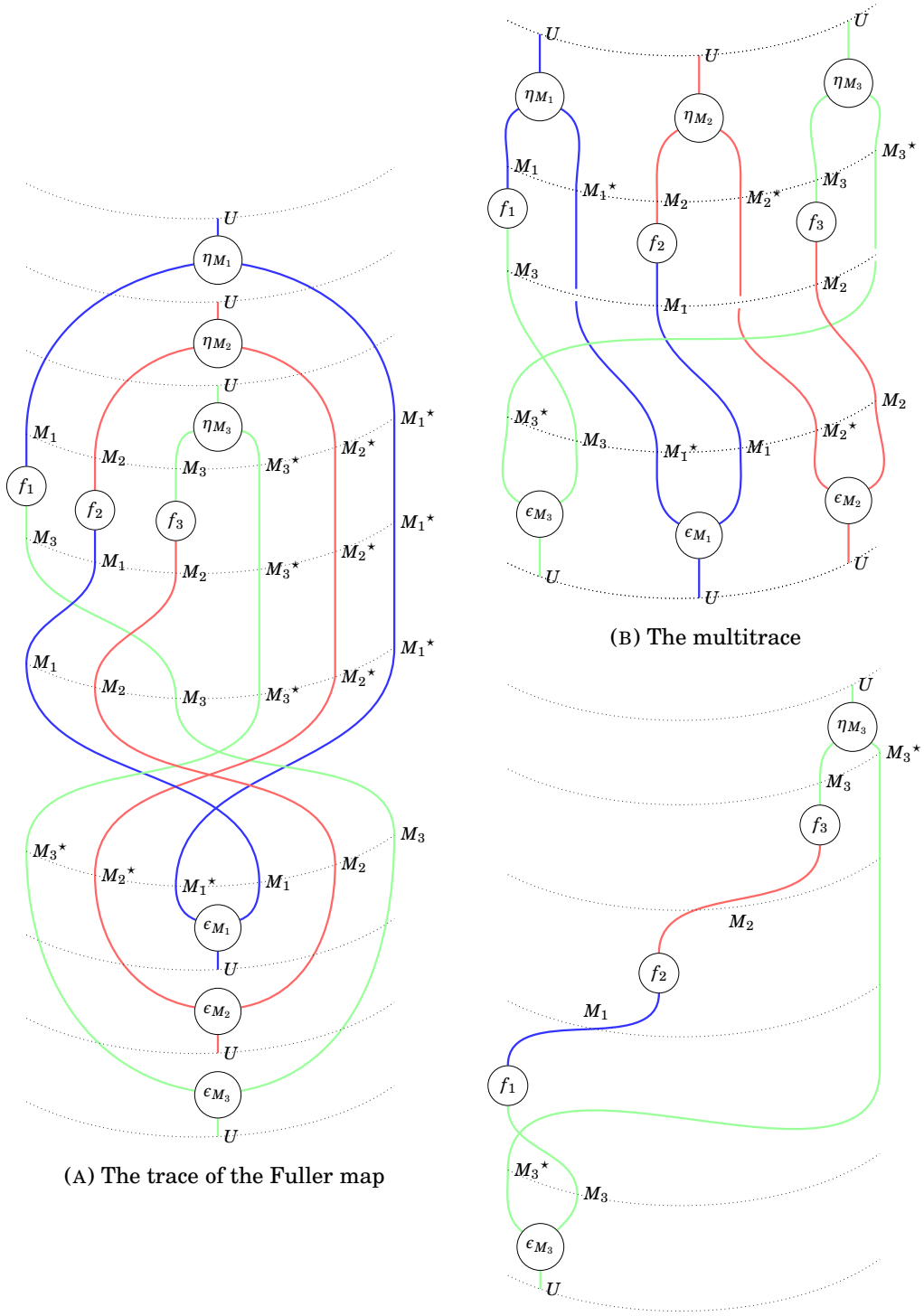
where $a^m_{j,l}$ is the (j,l) entry of A_m.

4. Bicategories and shadows

A **bicategory** $\mathscr{B}$ consists of the following data:

- A collection of *objects* or *0-cells* $R, S, T, \ldots$.
- For each pair of objects, a category $\mathscr{B}(R,S)$.
- For each object, a *unit* $U_R \in \mathscr{B}(R,R)$.
- For each triple of objects, a *composition* functor

$$\odot\colon \mathscr{B}(R,S) \times \mathscr{B}(S,T) \to \mathscr{B}(R,T).$$



(A) The trace of the Fuller map

(B) The multitrace

(C) The trace of f^3.

FIGURE 3.4. Untwisting the Fuller trace

- Associator and unit isomorphisms

$$\mathfrak{a}\colon M \odot (N \odot P) \xrightarrow{\sim} (M \odot N) \odot P$$

$$\mathfrak{l}\colon U_R \odot M \xrightarrow{\sim} M$$

$$\mathfrak{r}\colon M \odot U_S \xrightarrow{\sim} M$$

satisfying the same coherence axioms as for a monoidal category.

The objects of $\mathscr{B}(R,S)$ are called *1-cells* and the morphisms are *2-cells*. We think of these as "monoidal categories with many objects" and the operation $\odot$ as a tensor product. The coherence theorem for bicategories [Pow89] allows us to tensor a string of several 1-cells in a well-defined way up to canonical isomorphism, hence we often omit parentheses from expressions such as $M \odot N \odot P$.

A **shadow functor** on a bicategory $\mathscr{B}$ is a 1-category $\mathbf{T}$, a functor $\langle\!\langle - \rangle\!\rangle \colon \mathscr{B}(R,R) \to \mathbf{T}$ for each 0-cell R, and natural isomorphisms

$$\theta \colon \langle\!\langle M \odot N \rangle\!\rangle \xrightarrow{\sim} \langle\!\langle N \odot M \rangle\!\rangle$$

satisfying following two coherence conditions.

$$\begin{tikzcd}
\langle\!\langle (M \odot N) \odot P \rangle\!\rangle \arrow[r, "\theta"] \arrow[d, "\langle\!\langle \mathfrak{a} \rangle\!\rangle"'] & \langle\!\langle P \odot (M \odot N) \rangle\!\rangle \arrow[r, "\langle\!\langle \mathfrak{a} \rangle\!\rangle"] & \langle\!\langle (P \odot M) \odot N \rangle\!\rangle \\
\langle\!\langle M \odot (N \odot P) \rangle\!\rangle \arrow[r, "\theta"] & \langle\!\langle (N \odot P) \odot M \rangle\!\rangle \arrow[r, "\langle\!\langle \mathfrak{a} \rangle\!\rangle"] & \langle\!\langle N \odot (P \odot M) \rangle\!\rangle \arrow[u, "\theta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle\!\langle M \odot U_R \rangle\!\rangle \arrow[r, "\theta"] \arrow[dr, "\langle\!\langle \mathfrak{r} \rangle\!\rangle"'] & \langle\!\langle U_R \odot M \rangle\!\rangle \arrow[r, "\theta"] \arrow[d, "\langle\!\langle \mathfrak{l} \rangle\!\rangle"] & \langle\!\langle M \odot U_R \rangle\!\rangle \arrow[dl, "\langle\!\langle \mathfrak{r} \rangle\!\rangle"] \\
& \langle\!\langle M \rangle\!\rangle &
\end{tikzcd}$$

This makes $\mathscr{B}$ into a **bicategory with shadow**.

The point of a shadowed bicategory is that the 1-cells can be tensored along a circle. Given 1-cells $M_i \in \mathscr{B}(R_{i-1},R_i)$, indices taken mod n, define their **circular product** by

$$\langle\!\langle M_1,\dots,M_n \rangle\!\rangle := \langle\!\langle (\dots((M_1 \odot M_2) \odot M_3)\dots \odot M_n) \rangle\!\rangle.$$

The following allows us to work with such products without worrying about parenthesization.

Theorem 4.1 (Coherence for shadowed bicategories [MP18, Theorem 9.12])**.** *If a functor is naturally isomorphic to the circular product by a composition of isomorphisms* $\mathfrak{a},\mathfrak{l},\mathfrak{r},\theta$*, then there is only one such isomorphism.*

Example 4.2 (Examples of bicategories and shadows)**.**

(1) If $\mathscr{C}$ is a monoidal category, it is also a bicategory with one object. If $\mathscr{C}$ is a symmetric monoidal category, this is a shadowed bicategory in which $\mathbf{T} = \mathscr{C}$, the shadow functor is $\mathrm{id} \colon \mathscr{C} \to \mathscr{C}$, and θ is the symmetry isomorphism in $\mathscr{C}$.

(2) There is a bicategory of bimodules and homomorphisms where the 0-cells are rings R, the 1-cells are bimodules ${}_RM_S$ and the 2-cells are bimodule homomorphisms ${}_RM_S \to {}_RN_S$. The composition functor $\odot$ is the tensor product and the unit U_R is R as a bimodule over itself. There is a shadow functor that assigns ${}_RM_R$ to the quotient $M/\langle rm - mr \rangle$. This can be generalized by taking the 1-cells to be chain complexes and the 2-cells to be maps in the derived category.

(3) There is a point-set bicategory of parameterized spaces $\mathcal{U}_{\mathbf{S}}$. The 0-cells are spaces A and $\mathcal{U}_{\mathbf{S}}(A,B)$ is the category of spaces $X \to A \times B$. The composition of $X \to A \times B$ and $Y \to B \times C$ is the fiber product $X \times_B Y$ and the unit U_B is the diagonal $B \to B \times B$. The shadow sends $X \to B \times B$ to the pullback along the diagonal $B \to B \times B$, which gives an unbased space.

(4) There is a homotopy bicategory of spaces $\mathrm{ho}(\mathcal{U}_{\mathbf{S}})$. It is obtained by inverting the weak homotopy equivalences in each of the categories $\mathcal{U}_{\mathbf{S}}(A,B)$, and replacing $\odot$ and $\langle\!\langle\rangle\!\rangle$ by their right-derived functors. In particular, the shadow of the 1-cell in $\mathcal{U}_{\mathbf{S}}(X,X)$ given by $X \xrightarrow{(f,\mathrm{id})} X \times X$ is the twisted free loop space $\Lambda^f X$.

(5) There is a bicategory $\mathcal{E}\mathrm{x}$ of parameterized spectra [MS06, Ch. 17]. Its 0-cells are spaces A, and the category $\mathcal{E}\mathrm{x}(A,B)$ is the homotopy category of spectra parametrized by the product space $A \times B$. Each parametrized space $X \to A \times B$ in $\mathrm{ho}(\mathcal{U}_{\mathbf{S}}(A,B))$ has a suspension spectrum $\Sigma^\infty_{+(A\times B)} X$ in $\mathcal{E}\mathrm{x}(A,B)$. The shadow

functor from $\mathcal{E}\mathrm{x}$ to spectra agrees with the one in $\mathrm{ho}(\mathcal{U}_{\mathbf{S}})$ along the suspension spectrum functor.

(6) The last three examples admit generalizations $G\mathcal{U}_{G\mathbf{S}}, \mathrm{ho}G\mathcal{U}_{G\mathbf{S}}, G\mathcal{E}\mathrm{x}$ by allowing the action of a finite group G. When forming the homotopy category $\mathrm{ho}G\mathcal{U}_{G\mathbf{S}}$, we invert those maps of G-spaces that are equivalences on the H-fixed points for every $H \leq G$.

A **pseudofunctor** is a homomorphism of bicategories $F\colon \mathscr{C} \to \mathscr{D}$. It consists of the following data.

- A function $\mathrm{ob}\mathscr{C} \to \mathrm{ob}\mathscr{D}$ of 0-cells, denoted by F.
- A functor $\mathscr{C}(R,S) \to \mathscr{D}(F(R),F(S))$ for each pair of 0-cells in $\mathscr{C}$, denoted by F.
- Natural isomorphisms

$$m\colon F(M) \odot F(N) \xrightarrow{\sim} F(M \odot N)$$
$$i\colon U_{F(R)} \xrightarrow{\sim} F(U_R)$$

satisfying the same coherence axioms as for a strong monoidal functor.

A **strong shadow functor** is a homomorphism of shadowed bicategories $F\colon (\mathscr{C}, \mathbf{T}_{\mathscr{C}}) \to (\mathscr{D}, \mathbf{T}_{\mathscr{D}})$. It consists of a pseudofunctor and the following additional data.

- A functor of shadow categories $F_{\mathrm{tr}}\colon \mathbf{T}_{\mathscr{C}} \to \mathbf{T}_{\mathscr{D}}$.
- Natural isomorphisms $s\colon \langle\!\langle F(M)\rangle\!\rangle \xrightarrow{\sim} F_{\mathrm{tr}}\langle\!\langle M\rangle\!\rangle$ such that

$$\begin{tikzcd}
\langle\!\langle F(M) \odot F(N)\rangle\!\rangle \arrow[r, "\theta"] \arrow[d, "\langle\!\langle m\rangle\!\rangle"'] & \langle\!\langle F(N) \odot F(M)\rangle\!\rangle \arrow[d, "\langle\!\langle m\rangle\!\rangle"] \\
\langle\!\langle F(M \odot N)\rangle\!\rangle \arrow[d, "s"'] & \langle\!\langle F(N \odot M)\rangle\!\rangle \arrow[d, "s"] \\
F_{\mathrm{tr}}\langle\!\langle M \odot N\rangle\!\rangle \arrow[r, "F_{\mathrm{tr}}(\theta)"'] & F_{\mathrm{tr}}\langle\!\langle N \odot M\rangle\!\rangle.
\end{tikzcd}$$

commutes whenever it makes sense.

If $F, G\colon \mathscr{C} \to \mathscr{D}$ are strong shadow functors that are the same function of 0-cells, an **isomorphism of strong shadow functors** from F to G consists of natural isomorphisms $F \cong G$ and $F_{\mathrm{tr}} \cong G_{\mathrm{tr}}$ that commute with m, i, and s. We will often implicitly work with these functors up to isomorphism.

5. Duality and trace for bicategories

A 1-cell $M\colon R \nrightarrow S$ in a bicategory is **right dualizable**, or **dualizable over** S, if there is a 1-cell $M^\star\colon S \nrightarrow R$, and coevaluation and evaluation 2-cells

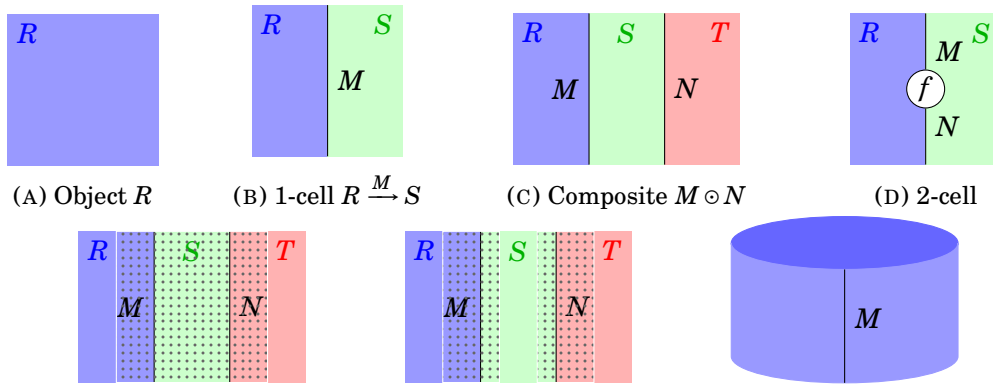
$$\eta\colon U_R \to M \odot M^\star \text{ and } \epsilon\colon M^\star \odot M \to U_S$$

satisfying the triangle identities. We say that $(M, M^\star)$ is a **dual pair**, that $M^\star$ is **left dualizable** or **dualizable over** S.

Example 5.1 (Dualizable objects)**.**

(1) An object M is dualizable in the symmetric monoidal category $\mathscr{C}$ if and only if it is right (or left) dualizable in the bicategory associated to $\mathscr{C}$.

(2) If A and R are rings, a bimodule ${}_AM_R$ is right dualizable precisely when it is finitely generated and projective as a right R-module, in which case the dual is $\mathrm{Hom}_R(M,R)$. Of course, M left dualizable when it is finitely generated and projective as a left A-module.



(A) Object R (B) 1-cell $R \xrightarrow{M} S$ (C) Composite $M \odot N$ (D) 2-cell

(E) $F(M \odot N)$ (F) $F(M) \odot F(N)$ (G) $\langle\!\langle M \rangle\!\rangle$

FIGURE 5.4. String diagrams for bicategories

Example 5.2. The graph $A \xrightarrow{\mathrm{id}_A, f} A \times B$ of a map $f : A \to B$ of unbased spaces defines 1-cells in $\mathrm{ho}(\mathcal{U}_{\mathbf{S}})(A,B)$ and $\mathrm{ho}(\mathcal{U}_{\mathbf{S}})(B,A)$. Taking suspension spectra gives two different 1-cells in $\mathcal{E}\mathrm{x}$, which we call the **base-change** 1-cells associated to f:

$$\left[A \xrightarrow{f} B\right] := \Sigma^\infty_{+(A\times B)} A, \qquad \left[B \xleftarrow{f} A\right] := \Sigma^\infty_{+(B\times A)} A.$$

- The 1-cell $\left[A \xrightarrow{f} B\right]$ is always right dualizable [MS06, 17.3.1].
- If $p : E \to B$ is a perfect fibration, i.e. a fibration whose fibers are finitely dominated, then $\left[B \xleftarrow{p} E\right]$ is right dualizable [PS14, 4.7]. When there is a G-action, the same is true if the fiber is equivariantly finitely dominated, meaning it is a retract in the homotopy category of G-spaces of a finite G-CW complex [PS14, 4.3][MS06, 18.2.1].

Base change objects define a pseudofunctor $\mathbf{S} \to \mathrm{ho}(\mathcal{U}_{\mathbf{S}})$. In particular, there are coherent isomorphisms

$$m_{[]} : \left[C \xleftarrow{g} B\right] \odot \left[B \xleftarrow{f} A\right] \xrightarrow{\sim} \left[C \xleftarrow{g\circ f} A\right], \qquad i_{[]} : U_A \xrightarrow{\sim} \left[A \xleftarrow{\mathrm{id}} A\right]. \tag{5.3}$$

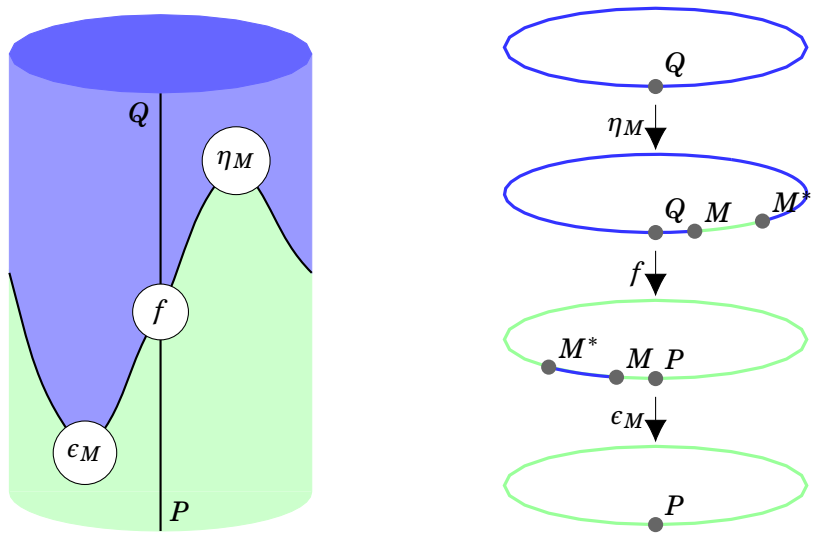
for each pair of composable maps $A \xrightarrow{f} B \xrightarrow{g} C$ and each 0-cell A. The same discussion applies with G-equivariant spaces as well.

Let $\mathscr{B}$ be a bicategory with a shadow functor to $\mathbf{T}$, and M a right dualizable 1-cell of $\mathscr{B}$. The **trace** of a 2-cell $f : Q \odot M \to M \odot P$ is the morphism in $\mathbf{T}$ is the composite:

$$\langle\!\langle Q \rangle\!\rangle \xrightarrow{\eta} \langle\!\langle Q, M, M^\star \rangle\!\rangle \xrightarrow{f} \langle\!\langle M, P, M^\star \rangle\!\rangle \xrightarrow{\epsilon} \langle\!\langle P \rangle\!\rangle.$$

When $\mathscr{B}$ comes from a symmetric monoidal category, this is the trace as defined in Section 3.

As in the symmetric monoidal case, it is helpful to visualize these traces using the string diagram calculus for bicategories from [PS13]. (The rigor of this approach is established in the appendix of that paper. As in the symmertic monoidal case any string diagram can be translated into a conventional commutative diagram.) We represent 0-cells by 2-dimensional regions, 1-cells by strings, and 2-cells by vertices; see Figure 5.4. Pasting pictures together corresponds to horizontal (tensoring) and vertical composition of the resulting expressions in the bicategory. To extend this visual language to bicategories with shadow, we represent the shadow by closing a planar string diagram into a



(A) Cylinder (B) Slices

FIGURE 5.5. The bicategorical trace

cylindrical string diagram. See Figure 5.4g. See Figure 5.5 for the string diagram for the bicategorical trace, cf. the earlier string diagram in Figure 3.1g.

Example 5.6. Suppose that X is a finite or finitely dominated complex and $f : X \to X$.

- The **Reidemeister trace** $R(f)$ is the trace of the canonical isomorphism
$$\left[* \xleftarrow{p} X\right] \xrightarrow{\sim} \left[* \xleftarrow{p} X\right] \odot \left[X \xleftarrow{f} X\right].$$
This trace is a map in the homotopy category
$$R(f) \colon \mathbb{S} \simeq \Sigma^\infty_+ \left\langle\!\left\langle [* \leftarrow *] \right\rangle\!\right\rangle \to \Sigma^\infty_+ \left\langle\!\left\langle \left[X \xleftarrow{f} X\right] \right\rangle\!\right\rangle \simeq \Sigma^\infty_+ \Lambda^f X$$
(cf [CP19, Appendix A]), which can be regarded as an element of $H_0(\Lambda^f X)$.
- The n**th Fuller trace** $R_{C_n}(\Psi^n(f))$ is the trace of the canonical isomorphism
$$\left[* \xleftarrow{p} X^n\right] \xrightarrow{\sim} \left[* \xleftarrow{p} X^n\right] \odot \left[X^n \xleftarrow{\Psi^n(f)} X^n\right].$$
It is a map in the C_n-equivariant homotopy category
$$R_{C_n}(\Psi^n(f)) \colon \mathbb{S} \simeq \Sigma^\infty_+ \left\langle\!\left\langle [* \leftarrow *] \right\rangle\!\right\rangle \to \Sigma^\infty_+ \left\langle\!\left\langle \left[X^n \xleftarrow{\Psi^n(f)} X^n\right] \right\rangle\!\right\rangle \simeq \Sigma^\infty_+ \Lambda^{\Psi^n(f)} X^n.$$

These are not the standard definitions of Lefschetz number or the Reidemeister trace. The definition here for the Lefschetz number is shown to agree with more classical descriptions in [DP80]. This description of the Reidemeister trace is compared to more classical versions in [Pon16] and to the description in [KW07] in [Pon10, 6.3.2]. The fact that different constructions of $\mathcal{E}$x give the same base-change isomorphisms is handled carefully in [Mal19], so we refrain from commenting on it here.

We end this section by recalling a fundamental functoriality result for the trace.

Theorem 5.7. [PS13, 8.3] *Let $F : \mathscr{B} \to \mathscr{C}$ be a strong shadow functor and suppose $M \in \mathscr{B}(R,S)$ is right dualizable.*

(1) Then $F(M)$ is right dualizable with dual $F(M^\star)$.

(2) For any $f : Q \odot M \to M \odot P$, the following square commutes:

$$\begin{tikzcd}
\langle\!\langle F(Q)\rangle\!\rangle \arrow[r, "\mathrm{tr}(m_{M,P}^{-1}\circ F(f)\circ m_{Q,M})"] \arrow[d, "s"'] & \langle\!\langle F(P)\rangle\!\rangle \arrow[d, "s"] \\
F_{\mathrm{tr}}\langle\!\langle Q\rangle\!\rangle \arrow[r, "F_{\mathrm{tr}}(\mathrm{tr}(f))"] & F_{\mathrm{tr}}\langle\!\langle P\rangle\!\rangle.
\end{tikzcd}$$

6. The multitrace for bicategories and Fuller bicategories

Now that we have defined the Reidemeister trace for f^n and for $\Psi^n(f)$, we may begin the formal work of relating them together.

Suppose in a shadowed bicategory $\mathscr{B}$ we select right dualizable 1-cells $M_i \in \mathscr{B}(A_i, B_i)$, 1-cells $Q_i \in \mathscr{B}(A_{i-1}, A_i)$, $P_i \in \mathscr{B}(B_{i-1}, B_i)$ (subscripts taken mod n), and 2-cells

$$\phi_i : Q_i \odot M_i \to M_{i-1} \odot P_i.$$

Then we define the "composite" $\phi_1 \circ \ldots \circ \phi_n$ to be the composite of the 2-cells

$$\begin{aligned} Q_1 \odot \ldots \odot Q_n \odot M_n &\xrightarrow{\mathrm{id}^{n-1}\odot\phi_n} Q_1 \odot \ldots \odot Q_{n-1} \odot M_{n-1} \odot P_n \xrightarrow{\mathrm{id}^{n-2}\odot\phi_{n-1}\odot\mathrm{id}} \\ & Q_1 \odot \ldots \odot Q_{n-2} \odot M_{n-2} \odot P_{n-1} \odot P_n \to \ldots \to M_n \odot P_1 \odot \ldots \odot P_n. \end{aligned}$$

If the modules Q_i and P_i are all units, this is canonically isomorphic to the composite of the maps ϕ_i.

On the other hand, we define the **multitrace** of the maps ϕ_i, denoted $\mathrm{tr}(\phi_1, \ldots, \phi_n)$, as the composite in $\mathbf{T}$:

$$\begin{tikzcd}
\langle\!\langle Q_1, \ldots, Q_n\rangle\!\rangle \arrow[r, "\langle\!\langle \mathrm{id}, \eta_1, \mathrm{id}, \ldots, \mathrm{id}, \eta_n\rangle\!\rangle"] & \langle\!\langle Q_1, M_1, M_1^\star, Q_2, M_2, M_2^\star, \ldots, M_n, M_n^\star\rangle\!\rangle \arrow[d, "\langle\!\langle \phi_1, \mathrm{id}, \ldots, \mathrm{id}, \phi_n, \mathrm{id}\rangle\!\rangle"] \\
\langle\!\langle P_1, \ldots, P_n\rangle\!\rangle & \langle\!\langle M_n, P_1, M_1^\star, M_1, P_2, M_2^\star, \ldots, P_n, M_n^\star\rangle\!\rangle \arrow[l, "\langle\!\langle \mathrm{id}, \epsilon_1, \mathrm{id}, \ldots, \mathrm{id}, \epsilon_n\rangle\!\rangle"']
\end{tikzcd}$$

Theorem 6.1 (Step 1 of Theorem A)**.** *The multitrace equals the trace of the composite,*

$$\mathrm{tr}(\phi_1, \ldots, \phi_n) = \mathrm{tr}(\phi_1 \circ \ldots \circ \phi_n),$$

as maps $\langle\!\langle Q_1, \ldots, Q_n\rangle\!\rangle \to \langle\!\langle P_1, \ldots, P_n\rangle\!\rangle$.

Proof. Using the string diagram calculus of [PS13], Figure 6.4 provides a full proof.

Alternatively, if X is dualizable, the composite

$$A \odot C \odot Y \xrightarrow{\mathrm{id}\odot\eta\odot\mathrm{id}^2} A \odot X \odot X^\star \odot C \odot Y \xrightarrow{f\odot\mathrm{id}\odot g} Y \odot B \odot X^\star \odot X \odot D \xrightarrow{\mathrm{id}^2\odot\epsilon\odot\mathrm{id}} Y \odot B \odot D \tag{6.2}$$

for 2-cells $f : A \odot X \to Y \odot B$ and $g : C \odot Y \to X \odot D$ is

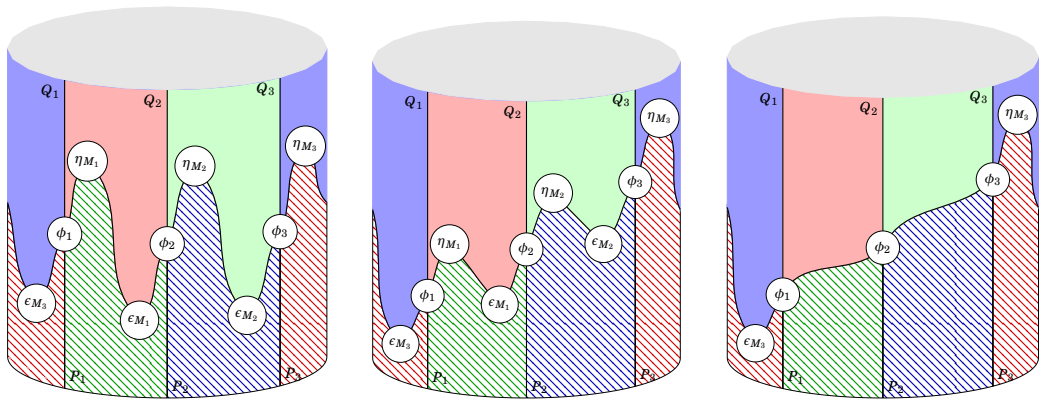
$$A \odot C \odot Y \xrightarrow{\mathrm{id}\odot g} A \odot X \odot D \xrightarrow{\mathrm{id}\odot\eta\odot\mathrm{id}^2} A \odot X \odot X^\star \odot X \odot D \xrightarrow{\mathrm{id}^2\odot\epsilon\odot\mathrm{id}} A \odot X \odot D \xrightarrow{f\odot\mathrm{id}} Y \odot B \odot D$$

Canceling the center evaluation and coevaluation, (6.2) is the composite

$$A \odot C \odot Y \xrightarrow{\mathrm{id}\odot g} A \odot X \odot D \xrightarrow{f\odot\mathrm{id}} Y \odot B \odot D \tag{6.3}$$

If Y is also dualizable the multitrace of f and g is the trace of (6.2) and so the multitrace of f and g is the trace of (6.3). Then the theorem follows by induction. □

Now we turn to the rest of the proof of Theorem A. As discussed in Section 3, we need to break free of the bicategory structure on $\mathscr{B}$ and use some additional structure that can reorder tensored objects. Here we give an axiomatic description of this extra structure and use it to prove Theorem A. The existence of examples of this structure (other than symmetric monoidal categories) is established in [MP18].



(A) The multitrace (B) Sliding morphisms (C) Applying triangle identities

FIGURE 6.4. The multitrace is isomorphic to the trace of the composition.

To motivate the following definitions, it is useful to think of the trace of $\Psi^n(f)$ as n nested circles, with an extra twist owing to the fact that $\Psi^n(f)$ rotates the factors around. See Figure 6.5. If we re-interpret this picture as a single circle winding around n times, we get precisely the multitrace pictured in Figure 6.4a. So we just need to formally understand the process of "unwinding the coil" in Figure 6.5, in other words lifting it to the n-fold cover of the circle.

A **shadowed n-Fuller structure** on a bicategory with shadow $\mathscr{B}$ consists of the following.

(1) A strong functor (pseudofunctor) of bicategories

$$\boxtimes\colon \underbrace{\mathscr{B}\times\ldots\times\mathscr{B}}_{n}\to\mathscr{B}.$$

Here $\mathscr{B}\times\ldots\times\mathscr{B}$ is the bicategory whose 0-cells are tuples of 0-cells $\mathscr{B}$ and

$$(\mathscr{B}\times\ldots\times\mathscr{B})((A_1,\ldots,A_n),(B_1,\ldots,B_n)) := \mathscr{B}(A_1,B_1)\times\ldots\times\mathscr{B}(A_n,B_n).$$

The product, shadow, associator, and so on are all defined componentwise.

More explicitly, this is a function that assigns a 0-cell $\boxtimes A_i$ to every tuple of 0-cells A_i, functors $\boxtimes\colon \prod\mathscr{B}(A_i,B_i)\to\mathscr{B}(\boxtimes A_i,\boxtimes B_i)$, and natural isomorphisms

$$m_{\boxtimes}\colon (\boxtimes M_i)\odot(\boxtimes N_i)\cong\boxtimes(M_i\odot N_i)$$
$$i_{\boxtimes}\colon U_{\boxtimes A_i}\cong\boxtimes U_{A_i}$$

satisfying the same coherence axioms as for a monoidal functor including (6.14), (6.12), (6.13).

(2) A pseudonatural transformation

$$\vartheta\colon \boxtimes\circ\gamma\to\boxtimes$$

where γ is the strong functor $\mathscr{B}\times\ldots\times\mathscr{B}\to\mathscr{B}\times\ldots\times\mathscr{B}$ that permutes the leftmost $\mathscr{B}$ to the right.

More explicitly, for each n tuple of objects $(A_1,\ldots,A_n)$ in $\mathscr{B}$ there is an object $T_{A_i}\in\mathscr{B}(A_2\times\ldots\times A_n\times A_1, A_1\times\ldots\times A_n)$ and natural isomorphisms

$$\vartheta\colon T_{A_i}\odot(\boxtimes M_i)\xrightarrow{\cong}(\boxtimes M_{i+1})\odot T_{B_i}$$

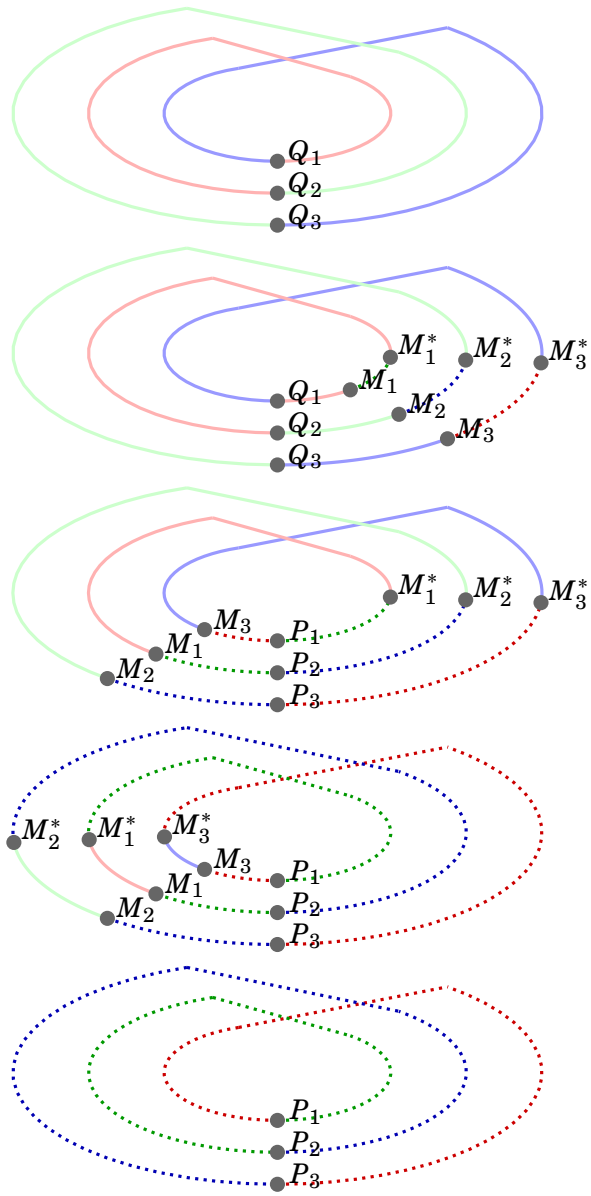


FIGURE 6.5. The trace of $\Psi^3(f)$

for all $M_i \in \mathscr{B}(A_i, B_i)$ that are compatible with $m_\boxtimes$ and $i_\boxtimes$.[1]

(3) A natural isomorphism

$$\tau : \langle\!\langle T_{A_{i-1}}, \boxtimes Q_i \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle Q_1, \ldots, Q_n \rangle\!\rangle$$

so that

$$\begin{tikzcd}
\langle\!\langle T_{A_{i-1}}, \boxtimes M_i, \boxtimes N_i \rangle\!\rangle \arrow[r, "\sim"'] \arrow[d, "\sim"', "\vartheta"] & \langle\!\langle T_{A_{i-1}}, \boxtimes (M_i \odot N_i) \rangle\!\rangle \arrow[r, "\tau", "\sim"'] & \langle\!\langle M_1, N_1, M_2, \ldots, M_n, N_n \rangle\!\rangle \arrow[dd, "\theta", "\sim"'] \\
\langle\!\langle \boxtimes M_{i+1}, T_{B_i}, \boxtimes N_i \rangle\!\rangle \arrow[d, "\theta"] & & \\
\langle\!\langle T_{B_i}, \boxtimes N_i, \boxtimes M_{i+1} \rangle\!\rangle \arrow[r, "\sim"'] & \langle\!\langle T_{B_i}, \boxtimes (N_i \odot M_{i+1}) \rangle\!\rangle \arrow[r, "\tau", "\sim"'] & \langle\!\langle N_1, M_2, \ldots, M_n, N_n, M_1 \rangle\!\rangle
\end{tikzcd} \tag{6.6}$$

[1]In fact, the compatibility with $i_\boxtimes$ is entirely optional, because our arguments below do not use it.

commutes for all $M_i \in \mathscr{B}(A_{i-1}, B_i)$ and $N_i \in \mathscr{B}(B_i, A_i)$.

Example 6.7. If $\mathscr{C}$ is a symmetric monoidal category, it has a canonical n-Fuller structure in which $\boxtimes$ is the n-fold tensor product, T_{A_i} is the unit, and the rest of the isomorphisms are the canonical ones that come from the coherence theorem for symmetric monoidal categories.

Example 6.8. The bicategory $\mathcal{E}\mathrm{x}$ has a shadowed n-Fuller structure. This can be deduced from our foundational work in Theorem 9.9 below, and the formal work from [MP18] summarized in Theorem 8.4 below.

The following statement is an immediate consequence of Theorem 5.7.

Lemma 6.9. *If $M_i \in \mathscr{B}(A_i, B_i)$ are right dualizable with duals $N_i \in \mathscr{B}(B_i, A_i)$ and*

$$\boxtimes \colon \mathscr{B} \times \ldots \times \mathscr{B} \to \mathscr{B}$$

is a strong functor of bicategories then $\boxtimes M_i \in \mathscr{B}(\prod A_i, \prod B_i)$ is dualizable with dual $\boxtimes N_i$.

For dualizable $M_i \in \mathscr{B}(A_i, B_i)$ and $Q_i \in \mathscr{B}(A_{i-1}, A_i)$ and $P_i \in \mathscr{B}(B_{i-1}, B_i)$ (subscripts are taken mod n) the **abstract Fuller map** of

$$\phi_i \colon Q_i \odot M_i \to M_{i-1} \odot P_i,$$

denoted $\Psi(\phi_1, \ldots, \phi_n) \in \mathscr{B}(\prod A_i, \prod B_i)$, is the composite

$$\begin{tikzcd}
T_{A_{i-1}} \odot \boxtimes Q_i \odot \boxtimes M_i \arrow[d, "\mathrm{id} \odot m_{\boxtimes}"] & T_{A_{i-1}} \odot \boxtimes M_{i-1} \odot \boxtimes P_i \arrow[r, "\vartheta \odot \mathrm{id}", "\sim"'] \arrow[d, "\mathrm{id} \odot m_{\boxtimes}"] & \boxtimes M_i \odot T_{B_{i-1}} \odot \boxtimes P_i \\
T_{A_{i-1}} \odot \boxtimes (Q_i \odot M_i) \arrow[r, "\mathrm{id} \odot \boxtimes \phi_i"] & T_{A_{i-1}} \odot \boxtimes (M_{i-1} \odot P_i) &
\end{tikzcd}$$

Essentially, it is $\boxtimes \phi_i$, but written in a form that allows us to use the dualizability of $\boxtimes M_i$ to take its trace.

Theorem 6.10 (Step 2 of Theorem A)**.** *If $\mathscr{B}$ is a bicategory with a shadowed Fuller structure, then for each tuple of maps ϕ_i as above the following diagram commutes.*

$$\begin{tikzcd}
\langle\!\langle T_{A_i} \odot \boxtimes Q_i \rangle\!\rangle \arrow[r, "\mathrm{tr}(\Psi(\phi_1, \ldots, \phi_n))"] \arrow[d, "\sim"', "\tau"] & \langle\!\langle T_{B_i} \odot \boxtimes P_i \rangle\!\rangle \arrow[d, "\sim"', "\tau"] \\
\langle\!\langle Q_1, \ldots, Q_n \rangle\!\rangle \arrow[r, "\mathrm{tr}(\phi_1, \ldots, \phi_n)"] & \langle\!\langle P_1, \ldots, P_n \rangle\!\rangle
\end{tikzcd}$$

Proof. This is a modification of the compatibility between trace and shadow functors (Theorem 5.7). The required commutative diagram is Figure 6.15. This is a very large diagram and so we have labeled the small regions so we can more easily indicate why they commute.

The right column of commutative diagrams in Figure 6.15 are mostly examples of the naturality of τ:

$$\begin{tikzcd}
\langle\!\langle T_{C_i} \odot \boxtimes X_i \rangle\!\rangle \arrow[r, "\tau"] \arrow[d, "\langle\!\langle \mathrm{id} \odot \boxtimes f_i \rangle\!\rangle"] & \langle\!\langle X_1 \odot \cdots \odot X_n \rangle\!\rangle \arrow[d, "\langle\!\langle f_1 \odot \cdots \odot f_n \rangle\!\rangle"] \\
\langle\!\langle T_{C_i} \odot \boxtimes Y_i \rangle\!\rangle \arrow[r, "\tau"] & \langle\!\langle Y_1 \odot \cdots \odot Y_n \rangle\!\rangle
\end{tikzcd} \tag{6.11}$$

This diagram commutes for 2-cells $f_i \colon X_i \to Y_i$. The remaining region in the right column is the assumed compatibility between θ and ϑ.

Many of the left column regions are the result of applying the functor $\langle\!\langle T_{C_i} \otimes - \rangle\!\rangle$ to coherence axioms for $\boxtimes$. These include:

- naturality of $m_{\boxtimes}$

$$\begin{tikzcd}
(\boxtimes X_i)\odot(\boxtimes Y_i) \arrow[r, "m_{\boxtimes}"] \arrow[d, "(\boxtimes f_i)\odot(\boxtimes g_i)"] & \boxtimes(X_i\odot Y_i) \arrow[d, "\boxtimes(f_i\odot g_i)"] \\
(\boxtimes W_i)\odot(\boxtimes Z_i) \arrow[r, "m_{\boxtimes}"] & \boxtimes(W_i\odot Z_i)
\end{tikzcd} \tag{6.12}$$

- associativity of $m_{\boxtimes}$

$$\begin{tikzcd}
(\boxtimes X_i)\odot(\boxtimes Y_i)\odot(\boxtimes Z_i) \arrow[r, "m_{\boxtimes}\odot\mathrm{id}"] \arrow[d, "\mathrm{id}\odot m_{\boxtimes}"] & (\boxtimes(X_i\odot Y_i))\odot(\boxtimes Z_i) \arrow[d, "m_{\boxtimes}"] \\
(\boxtimes X_i)\odot(\boxtimes(Y_i\odot Z_i)) \arrow[r, "m_{\boxtimes}"] & \boxtimes(X_i\odot Y_i\odot Z_i)
\end{tikzcd} \tag{6.13}$$

- compatibility of $i_{\boxtimes}$ and $m_{\boxtimes}$

$$\begin{tikzcd}
(\boxtimes X_i)\odot U_{\boxtimes C_i} \arrow[r] \arrow[d, "\mathrm{id}\odot i_{\boxtimes}"] & \boxtimes X_i \\
(\boxtimes X_i)\odot(\boxtimes U_{C_i}) \arrow[r, "m_{\boxtimes}"] & \boxtimes(X_i\odot U_{C_i}) \arrow[u]
\end{tikzcd} \tag{6.14}$$

In (6.14) the unlabeled arrows are unit isomorphisms.

The dotted and dashed arrows are defined to be the composites of the remaining arrows bounding the relevant region. For the two remaining regions:

(1) This square commutes by applying $\langle\!\langle - \rangle\!\rangle$ to a square that commutes by the functoriality of $\odot$.
(2) This square commutes by the naturality of the shadow isomorphism.

□

Together Theorems 6.1 and 6.10 prove a very abstract and general form of the "unwinding" argument, that the trace of a Fuller construction is isomorphic to the trace of the composite. To recover Theorem A from this, we have to further develop the case where the maps ϕ_i are canonical isomorphisms of base-change objects associated to maps f_i in some 1-category $\mathbf{S}$.

7. Base change

If $\mathscr{B}$ is a shadowed bicategory with an n-Fuller structure and $\mathbf{S}$ is a cartesian monoidal 1-category, a **system of base-change objects for $\mathscr{B}$ indexed by $\mathbf{S}$** is the following data and conditions.

(1) A pseudofunctor $[]\colon \mathbf{S}\to\mathscr{B}$.
In particular, natural isomorphisms
$$m_{[]}\colon \left[B_n \xleftarrow{f_n} B_{n-1}\right]\odot\ldots\odot\left[B_2\xleftarrow{f_2}B_1\right]\odot\left[B_1\xleftarrow{f_1}B_0\right]\cong\left[B_n\xleftarrow{f_n\circ\ldots\circ f_1}B_0\right]$$
compatible with composition. (The unit isomorphism $i_{[]}$ is not necessary.)
(2) A vertical natural isomorphism π filling the square of pseudofunctors
$$\begin{tikzcd}
\mathbf{S}^{\times n} \arrow[r, "\prod"] \arrow[d, "[]"'] & \mathbf{S} \arrow[d, "[]"] \\
\mathscr{B}^{\times n} \arrow[r, "\boxtimes"] & \mathscr{B}
\end{tikzcd}$$
where $\prod$ denotes a fixed model for the n-fold product in $\mathbf{S}$.

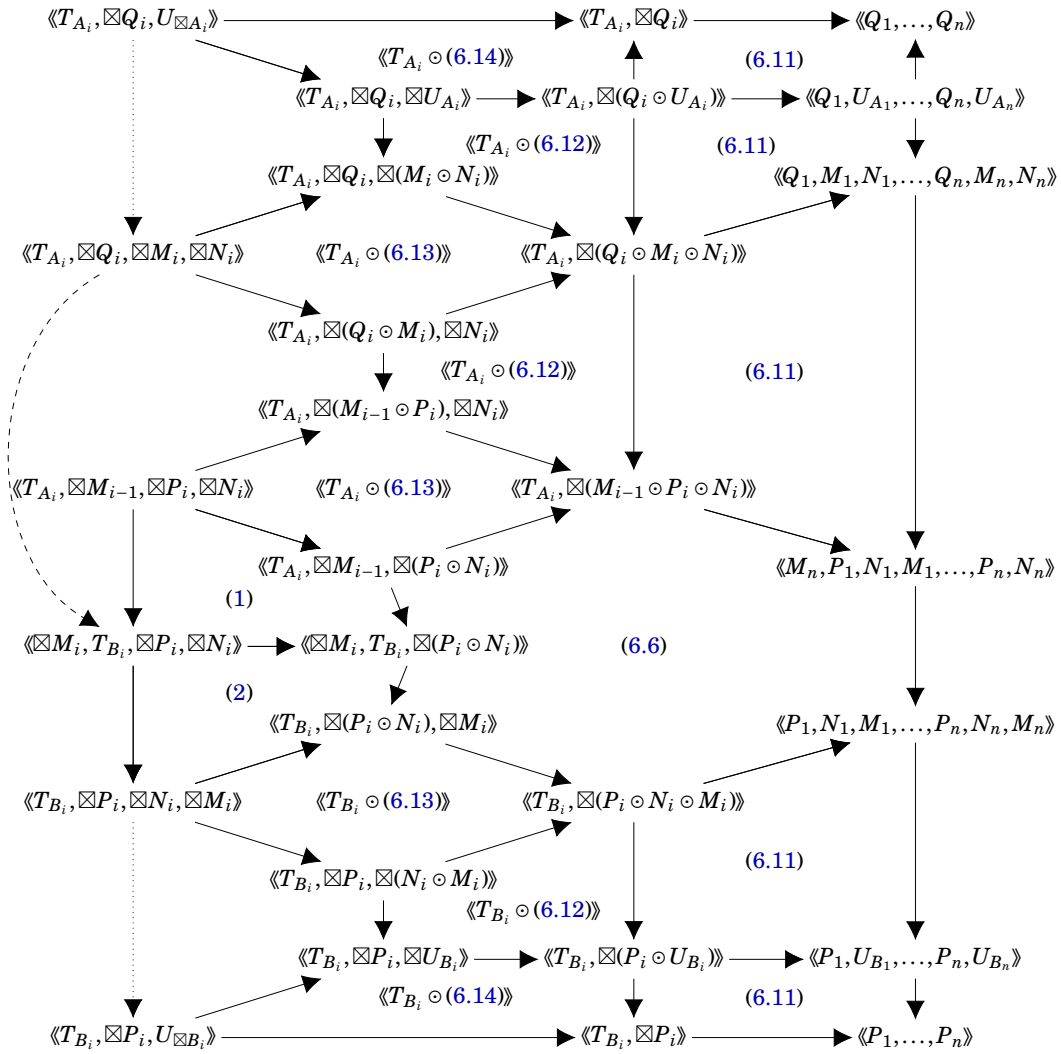


FIGURE 6.15. Proof of Theorem 6.10

This implies $\boxtimes A_i = \prod A_i$ for a tuple of 0-cells A_i, and for each n-tuple of maps $A_i \xrightarrow{f_i} B_i$ there is an isomorphism of 1-cells

$$\pi\colon \boxtimes\left[B_i \xleftarrow{f_i} A_i\right] \xrightarrow{\cong} \left[\prod B_i \xleftarrow{\prod f_i} \prod A_i\right]$$

so that for any n-tuple of composable maps $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$, the following pentagon of isomorphisms commutes.

$$\begin{tikzcd}
\left(\boxtimes\left[C_i \xleftarrow{g_i} B_i\right]\right) \odot \left(\boxtimes\left[B_i \xleftarrow{f_i} A_i\right]\right) \arrow[r, "m_\boxtimes", "\cong"'] \arrow[d, "\cong"', "\pi\odot\pi"] & \boxtimes\left(\left[C_i \xleftarrow{g_i} B_i\right] \odot \left[B_i \xleftarrow{f_i} A_i\right]\right) \arrow[d, "\cong"', "\boxtimes m_{[]}"] \\
 & \boxtimes\left[C_i \xleftarrow{g_i\circ f_i} A_i\right] \arrow[d, "\cong"', "\pi"] \\
\left[\prod C_i \xleftarrow{\prod g_i} \prod B_i\right] \odot \left[\prod B_i \xleftarrow{\prod f_i} \prod A_i\right] \arrow[r, "m_{[]}", "\cong"'] & \left[\prod C_i \xleftarrow{\prod(g_i\circ f_i)} \prod A_i\right]
\end{tikzcd} \tag{7.1}$$

(Again, the corresponding map for the unit of $\boxtimes$ is not necessary.)

(3) An equality $T_{B_i} = \left[\prod B_{i+1} \xleftarrow{\cong} \prod B_i\right]$ so that the following diagram relating ϑ, π and the pseudofunctor structure commutes.

(7.2)

$$\begin{tikzcd}
\left[\prod B_{i+1} \xleftarrow{\cong} \prod B_i\right] \odot \left(\boxtimes\left[B_i \xleftarrow{p_i} E_i\right]\right) \arrow[r, "\vartheta", "\cong"'] \arrow[dd, "\mathrm{id}\odot\pi", "\cong"'] & \left(\boxtimes\left[B_{i+1} \xleftarrow{p_{i+1}} E_{i+1}\right]\right) \odot \left[\prod E_{i+1} \xleftarrow{\cong} \prod E_i\right] \arrow[d, "\cong", "\pi\odot\mathrm{id}"'] \\
& \left[\prod B_{i+1} \xleftarrow{\prod p_{i+1}} \prod E_{i+1}\right] \odot \left[\prod E_{i+1} \xleftarrow{\cong} \prod E_i\right] \arrow[d, "\cong", "m_{[]}"'] \\
\left[\prod B_{i+1} \xleftarrow{\cong} \prod B_i\right] \odot \left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right] \arrow[r, "m_{[]}", "\cong"'] & \left[\prod B_{i+1} \xleftarrow{\mathrm{shift}\circ\prod p_i} \prod E_i\right]
\end{tikzcd}$$

Example 7.3. The bicategory $\mathcal{E}\mathrm{x}$ has a system of base-change objects from the category of unbased spaces. As with the n-Fuller structure, this follows from Theorem 9.9 and the results from [MP18] summarized in Theorem 8.4.

If $\mathscr{B}$ is a shadowed n-Fuller category with base change objects, any commuting square in $\mathbf{S}$ of the form

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "p"'] & E \arrow[d, "p"] \\
B \arrow[r, "\overline{f}"] & B
\end{tikzcd}$$

gives an isomorphism of base-change objects

$$\left[B \xleftarrow{\overline{f}} B\right] \odot \left[B \xleftarrow{p} E\right] \to \left[B \xleftarrow{p} E\right] \odot \left[E \xleftarrow{f} E\right].$$

If the base-change object $\left[B \xleftarrow{p} E\right]$ is right-dualizable in $\mathscr{B}$, then we can take the trace of this map. This is the Reidemeister trace associated to the above commuting square. Note that in $\mathcal{E}\mathrm{x}$, when $B = *$, it agrees with the definition of $R(f)$ we gave in Section 5.

If we instead have an n-tuple of commuting squares

$$\begin{tikzcd}
E_i \arrow[r, "f_i"] \arrow[d, "p_i"'] & E_{i-1} \arrow[d, "p_{i-1}"] \\
B_i \arrow[r, "\overline{f_i}"] & B_{i-1}
\end{tikzcd}$$

in $\mathbf{S}$ then we can define a commuting square

$$\begin{tikzcd}
\prod E_i \arrow[r, "\Psi(f_1,\ldots,f_n)"] \arrow[d, "\prod p_i"'] & \prod E_i \arrow[d, "\prod p_i"] \\
\prod B_i \arrow[r, "\Psi(\overline{f_1},\ldots,\overline{f_n})"] & \prod B_i.
\end{tikzcd}$$

The first squares define maps

$$\phi_i\colon \left[B_{i-1} \xleftarrow{\overline{f_i}} B_i\right] \odot \left[B_i \xleftarrow{p_i} E_i\right] \to \left[B_{i-1} \xleftarrow{p_{i-1}} E_{i-1}\right] \odot \left[E_{i-1} \xleftarrow{f_i} E_i\right]$$

for each i, and the second square defines a map

$$\phi\colon \left[\prod B_i \xleftarrow{\Psi(\overline{f}_1,\ldots,\overline{f}_n)} \prod B_i\right] \odot \left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right] \to \left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right] \odot \left[\prod E_i \xleftarrow{\Psi(f_1,\ldots,f_n)} \prod E_i\right].$$

Proposition 7.4 (Step 3 of Theorem A)**.** *In a shadowed n-Fuller category $\mathscr{B}$ with a system of base-change objects from* $\mathbf{S}$*, for any n-tuple of commuting squares in* $\mathbf{S}$

$$\begin{tikzcd} E_i \arrow[r, "f_i"] \arrow[d, "p_i"'] & E_{i-1} \arrow[d, "p_{i-1}"] \\ B_i \arrow[r, "\overline{f_i}"] & B_{i-1} \end{tikzcd}$$

there is a commuting diagram

$$\begin{tikzcd} \left\langle\!\!\left\langle T_{B_i} \odot \boxtimes \left[B_{i-1} \xleftarrow{\overline{f}_i} B_i\right]\right\rangle\!\!\right\rangle \arrow[r, "\langle\!\langle\cong\rangle\!\rangle"] \arrow[d, "\mathrm{tr}(\Psi(\phi_1,\ldots,\phi_n))"'] & \left\langle\!\!\left\langle\left[\prod B_i \xleftarrow{\Psi(\overline{f}_1,\ldots,\overline{f}_n)} \prod B_i\right]\right\rangle\!\!\right\rangle \arrow[d, "\mathrm{tr}(\phi)"] \\ \left\langle\!\!\left\langle T_{E_i} \odot \boxtimes \left[E_{i-1} \xleftarrow{f_i} E_i\right]\right\rangle\!\!\right\rangle \arrow[r, "\langle\!\langle\cong\rangle\!\rangle"] & \left\langle\!\!\left\langle\left[\prod E_i \xleftarrow{\Psi(f_1,\ldots,f_n)} \prod E_i\right]\right\rangle\!\!\right\rangle \end{tikzcd}$$

Proof. The first step in this proof is to compare $\Psi(\phi_1,\ldots,\phi_n)$ and ϕ. For this we use the commutative diagram in Figure 7.5. We have already encountered all of the small regions in the diagram. The regions not labeled by an equation number commute by:

(1) The functoriality of $\odot$.
(2) Naturality of $m_{[]}$.

The left composite in Figure 7.5 is $\Psi(\phi_1,\ldots,\phi_n)$ and the right composite is ϕ.

A straightforward diagram chase shows that for a diagram of the form below on the left, where β is an isomorphism and M_1 and M_2 are dualizable, the corresponding diagram of traces on the right commutes.

$$\begin{tikzcd} Q_1 \odot M_1 \arrow[r, "\alpha\odot\beta"] \arrow[d, "f_1"] & Q_2 \odot M_2 \arrow[d, "f_2"] \\ M_1 \odot P_1 \arrow[r, "\beta\odot\gamma"] & M_2 \odot P_2 \end{tikzcd} \qquad \begin{tikzcd} \langle\!\langle Q_1\rangle\!\rangle \arrow[r, "\langle\!\langle\alpha\rangle\!\rangle"] \arrow[d, "\mathrm{tr}(f_1)"] & \langle\!\langle Q_2\rangle\!\rangle \arrow[d, "\mathrm{tr}(f_2)"] \\ \langle\!\langle P_1\rangle\!\rangle \arrow[r, "\langle\!\langle\gamma\rangle\!\rangle"] & \langle\!\langle P_2\rangle\!\rangle \end{tikzcd}$$

Looking only at the outside edges of Figure 7.5 we have a commutative diagram of exactly this form,

$$\begin{tikzcd} \left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right] \odot \boxtimes \left[B_{i-1} \xleftarrow{\overline{f}_i} B_i\right] \odot \boxtimes \left[B_i \xleftarrow{p_i} E_i\right] \arrow[r, "(m_{[]}\circ(\mathrm{id}\odot\pi))\odot\pi"] \arrow[d, "\Psi(\phi_1,\ldots,\phi_n)"] & \left[\prod B_i \xleftarrow{\Psi(\overline{f}_1,\ldots,\overline{f}_n)} \prod B_i\right] \odot \left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right] \arrow[d, "\phi"] \\ \boxtimes \left[B_i \xleftarrow{p_i} E_i\right] \odot \left[\prod E_i \xleftarrow{\cong} \prod E_{i-1}\right] \odot \boxtimes \left[E_{i-1} \xleftarrow{f_i} E_i\right] \arrow[r, "\pi\odot(m_{[]}\circ(\mathrm{id}\odot\pi))"] & \left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right] \odot \left[\prod E_i \xleftarrow{\Psi(f_1,\ldots,f_n)} \prod E_i\right]. \end{tikzcd}$$

This completes the proof. □

$$
\begin{tikzcd}
\left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \boxtimes\left[B_{i-1} \xleftarrow{\overline{f_i}} B_i\right]\\ \boxtimes\left[B_i \xleftarrow{p_i} E_i\right]\end{array}\right) \arrow[r, "\begin{pmatrix}\mathrm{id}\\ \mathrm{id}\\ \pi\end{pmatrix}"] \arrow[dr, "\begin{pmatrix}\mathrm{id}\\ \pi\\ \pi\end{pmatrix}"] \arrow[d, "\begin{pmatrix}\mathrm{id}\\ m_{\boxtimes}\end{pmatrix}"'] \arrow[dr, phantom, "(1)"] & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \boxtimes\left[B_{i-1} \xleftarrow{\overline{f_i}} B_i\right]\\ \left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right]\end{array}\right) \arrow[r, "\begin{pmatrix}m_{[]}\circ(\mathrm{id}\odot\pi)\\ \mathrm{id}\end{pmatrix}"] \arrow[d, "\begin{pmatrix}\mathrm{id}\\ \pi\\ \mathrm{id}\end{pmatrix}"] \arrow[dr, phantom, "(1)"] & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\Psi(\overline{f_1},\dots,\overline{f_n})} \prod B_i\right]\\ \left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right]\end{array}\right) \arrow[d, "m_{[]}"] \\
 & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \left[\prod B_{i-1} \xleftarrow{\prod \overline{f_i}} \prod B_i\right]\\ \left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right]\end{array}\right) \arrow[ur, "\begin{pmatrix}m_{[]}\\ \mathrm{id}\end{pmatrix}"] \arrow[d, "\begin{pmatrix}\mathrm{id}\\ m_{[]}\end{pmatrix}"] \arrow[dr, phantom, "(6.13)"] & \\
\left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \boxtimes\left(\left[B_{i-1} \xleftarrow{\overline{f_i}} B_i\right]\odot\left[B_i \xleftarrow{p_i} E_i\right]\right)\end{array}\right) \arrow[d, "\begin{pmatrix}\mathrm{id}\\ \boxtimes\phi_i\end{pmatrix}"] \arrow[dr, phantom, "(2)"] & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \boxtimes\left[B_{i-1} \xleftarrow{\overline{f}_i\circ p_i} E_i\right]\end{array}\right) \arrow[ur, "\begin{pmatrix}\mathrm{id}\\ \boxtimes m_{[]}\end{pmatrix}"] \arrow[r, "\begin{pmatrix}\mathrm{id}\\ \pi\end{pmatrix}"] \arrow[d, equal] \arrow[dr, phantom, "(1)"] & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \left[\prod B_{i-1} \xleftarrow{\prod(\overline{f}_i\circ p_i)} \prod E_i\right]\end{array}\right) \arrow[r, "m_{[]}"] \arrow[d, equal] \arrow[dr, phantom, "(2)"] & \left[\prod B_i \xleftarrow{\prod(\overline{f}_i\circ p_i)\circ\cong} \prod E_i\right] \arrow[d, equal] \\
\left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \boxtimes\left(\left[B_{i-1} \xleftarrow{p_{i-1}} E_{i-1}\right]\odot\left[E_{i-1} \xleftarrow{f_i} E_i\right]\right)\end{array}\right) & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \boxtimes\left[B_{i-1} \xleftarrow{p_{i-1}\circ f_i} E_i\right]\end{array}\right) \arrow[ur, "\begin{pmatrix}\mathrm{id}\\ \boxtimes m_{[]}\end{pmatrix}"] \arrow[r, "\begin{pmatrix}\mathrm{id}\\ \pi\end{pmatrix}"] \arrow[dr, phantom, "(7.1)"] & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \left[\prod B_{i-1} \xleftarrow{\prod(p_{i-1}\circ f_i)} \prod E_i\right]\end{array}\right) \arrow[r, "m_{[]}"] & \left[\prod B_i \xleftarrow{\prod(p_{i-1}\circ f_i)\circ\cong} \prod E_i\right] \\
 & & & \\
\left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \boxtimes\left[B_{i-1} \xleftarrow{p_{i-1}} E_{i-1}\right]\\ \boxtimes\left[E_{i-1} \xleftarrow{f_i} E_i\right]\end{array}\right) \arrow[uu, "\begin{pmatrix}\mathrm{id}\\ m_{\boxtimes}\end{pmatrix}"] \arrow[r, "\begin{pmatrix}\mathrm{id}\\ \pi\\ \pi\end{pmatrix}"] \arrow[dr, "\begin{pmatrix}\mathrm{id}\\ \pi\\ \mathrm{id}\end{pmatrix}"'] \arrow[dr, phantom, "(1)"] \arrow[dd, "\begin{pmatrix}\vartheta\\ \mathrm{id}\end{pmatrix}"'] & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \left[\prod B_{i-1} \xleftarrow{\prod p_{i-1}} \prod E_{i-1}\right]\\ \left[\prod E_{i-1} \xleftarrow{\prod f_i} \prod E_i\right]\end{array}\right) \arrow[uu, "\begin{pmatrix}\mathrm{id}\\ m_{[]}\end{pmatrix}"] \arrow[dd, "\begin{pmatrix}m_{[]}\\ \mathrm{id}\end{pmatrix}"] \arrow[dr, phantom, "(6.13)"] \arrow[dddr, "m_{[]}"] & \\
 & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\cong} \prod B_{i-1}\right]\\ \left[\prod B_{i-1} \xleftarrow{\prod p_{i-1}} \prod E_{i-1}\right]\\ \boxtimes\left[E_{i-1} \xleftarrow{f_i} E_i\right]\end{array}\right) \arrow[ur, "\begin{pmatrix}\mathrm{id}\\ \mathrm{id}\\ \pi\end{pmatrix}"] \arrow[d, "\begin{pmatrix}m_{[]}\\ \mathrm{id}\end{pmatrix}"] \arrow[dr, phantom, "(1)"] & & \\
 & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\prod p_i\circ\cong} \prod E_{i-1}\right]\\ \boxtimes\left[E_{i-1} \xleftarrow{f_i} E_i\right]\end{array}\right) \arrow[r, "\begin{pmatrix}\mathrm{id}\\ \pi\end{pmatrix}"] \arrow[dr, phantom, "(7.2)"] & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\prod p_i\circ\cong} \prod E_{i-1}\right]\\ \left[\prod E_{i-1} \xleftarrow{\prod f_i} \prod E_i\right]\end{array}\right) \arrow[dr, phantom, "(6.13)"] & \\
 & & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right]\\ \left[\prod E_i \xleftarrow{\cong} \prod E_{i-1}\right]\\ \left[\prod E_{i-1} \xleftarrow{\prod f_i} \prod E_i\right]\end{array}\right) \arrow[u, "\begin{pmatrix}m_{[]}\\ \mathrm{id}\end{pmatrix}"] \arrow[dr, "\begin{pmatrix}\mathrm{id}\\ m_{[]}\end{pmatrix}"] \arrow[dr, phantom, "(1)"] & \\
\left(\begin{array}{c}\boxtimes\left[B_i \xleftarrow{p_i} E_i\right]\\ \left[\prod E_i \xleftarrow{\cong} \prod E_{i-1}\right]\\ \boxtimes\left[E_{i-1} \xleftarrow{f_i} E_i\right]\end{array}\right) \arrow[r, "\begin{pmatrix}\pi\\ \mathrm{id}\\ \mathrm{id}\end{pmatrix}"] & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right]\\ \left[\prod E_i \xleftarrow{\cong} \prod E_{i-1}\right]\\ \boxtimes\left[E_{i-1} \xleftarrow{f_i} E_i\right]\end{array}\right) \arrow[ur, "\begin{pmatrix}m_{[]}\\ \mathrm{id}\end{pmatrix}"] \arrow[ur, "\begin{pmatrix}\mathrm{id}\\ \mathrm{id}\\ \pi\end{pmatrix}"'] \arrow[rr, "\begin{pmatrix}\mathrm{id}\\ m_{[]}\circ(\mathrm{id}\odot\pi)\end{pmatrix}"] & & \left(\begin{array}{c}\left[\prod B_i \xleftarrow{\prod p_i} \prod E_i\right]\\ \left[\prod E_i \xleftarrow{\Psi(f_1,\dots,f_n)} \prod E_i\right]\end{array}\right)
\end{tikzcd}
$$

FIGURE 7.5. Comparing Fuller maps.
Stacked entries inside a single pair of large parentheses are combined with $\odot$.

Combining Theorems 6.1 and 6.10 and Proposition 7.4 in the setting of Proposition 7.4, in other words the first three steps of Theorem A, we get a commutative diagram

$$\begin{tikzcd}
\left\langle\!\!\left\langle \left[\prod B_i \xleftarrow{\Psi(\overline{f}_1,\ldots,\overline{f}_n)} \prod B_i\right]\right\rangle\!\!\right\rangle \arrow[r, "\mathrm{tr}(\phi)"] & \left\langle\!\!\left\langle \left[\prod E_i \xleftarrow{\Psi(f_1,\ldots,f_n)} \prod E_i\right]\right\rangle\!\!\right\rangle \\
\left\langle\!\!\left\langle T_{B_i} \odot \boxtimes \left[B_{i-1} \xleftarrow{\overline{f}_i} B_i\right]\right\rangle\!\!\right\rangle \arrow[u, "\langle\!\langle\cong\rangle\!\rangle"] \arrow[r, "\mathrm{tr}(\Psi(\phi_1,\ldots,\phi_n))"] \arrow[d, "\cong"'] & \left\langle\!\!\left\langle T_{E_i} \odot \boxtimes \left[E_{i-1} \xleftarrow{f_i} E_i\right]\right\rangle\!\!\right\rangle \arrow[u, "\langle\!\langle\cong\rangle\!\rangle"] \arrow[d, "\cong"'] \\
\left\langle\!\!\left\langle \left[B_n \xleftarrow{\overline{f}_1} B_1\right],\ldots,\left[B_{n-1} \xleftarrow{\overline{f}_n} B_n\right]\right\rangle\!\!\right\rangle \arrow[r, "\mathrm{tr}(\phi_1,\ldots,\phi_n)"] \arrow[d, "\langle\!\langle\cong\rangle\!\rangle"'] & \left\langle\!\!\left\langle \left[E_n \xleftarrow{f_1} E_1\right],\ldots,\left[E_{n-1} \xleftarrow{f_n} E_n\right]\right\rangle\!\!\right\rangle \arrow[d, "\langle\!\langle\cong\rangle\!\rangle"'] \\
\left\langle\!\!\left\langle \left[B_n \xleftarrow{\overline{f}_1} B_1\right] \odot \ldots \odot \left[B_{n-1} \xleftarrow{\overline{f}_n} B_n\right]\right\rangle\!\!\right\rangle \arrow[r, "\mathrm{tr}(\phi_1\circ\ldots\circ\phi_n)"] \arrow[d, "\langle\!\langle\cong\rangle\!\rangle"'] & \left\langle\!\!\left\langle \left[E_n \xleftarrow{f_1} E_1\right] \odot \ldots \odot \left[E_{n-1} \xleftarrow{f_n} E_n\right]\right\rangle\!\!\right\rangle \arrow[d, "\langle\!\langle\cong\rangle\!\rangle"'] \\
\left\langle\!\!\left\langle \left[B_n \xleftarrow{\overline{f}_1\circ\ldots\circ\overline{f}_n} B_n\right]\right\rangle\!\!\right\rangle \arrow[r, dashed] & \left\langle\!\!\left\langle \left[E_n \xleftarrow{f_1\circ\ldots\circ f_n} E_n\right]\right\rangle\!\!\right\rangle
\end{tikzcd}$$

relating the Reidemeister trace of the Fuller construction to the trace of the composite of base-change isomorphisms $\phi_1\circ\ldots\circ\phi_n$. To fill in the remaining dashed arrow, we observe that $\phi_1\circ\ldots\circ\phi_n$ arises by pasting the base-change isomorphisms that bring us from the lower to the upper route in the following diagram.

$$\begin{tikzcd}
E_n \arrow[r, "f_n"] \arrow[d, "p_n"'] & E_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "p_{n-1}"] & \cdots \arrow[r, "f_2"] & E_1 \arrow[r, "f_1"] \arrow[d, "p_1"] & E_n \arrow[d, "p_n"] \\
B_n \arrow[r, "\overline{f_n}"] & B_{n-1} \arrow[r, "\overline{f_{n-1}}"] & \cdots \arrow[r, "\overline{f_2}"] & B_1 \arrow[r, "\overline{f_1}"] & B_n
\end{tikzcd}$$

Using one last time the fact that [] is a pseudofunctor, along the canonical maps this is identified with the isomorphism provided by [] for the composite square. Therefore the dashed arrow is the Reidemeister trace for the square

$$\begin{tikzcd}
E_n \arrow[r, "f_1\circ\ldots\circ f_n"] \arrow[d, "p_n"'] & E_n \arrow[d, "p_n"] \\
B_n \arrow[r, "\overline{f_1}\circ\ldots\circ\overline{f_n}"] & B_n.
\end{tikzcd}$$

Taking B to be the terminal object, this gives the fourth and final piece of the proof of the following.

Corollary 7.6. *In a shadowed n-Fuller category $\mathscr{B}$ with a system of base-change objects from $\mathbf{S}$, for any n-tuple of composable maps $f_i\colon X_i \to X_{i-1}$ in $\mathbf{S}$, the Reidemeister trace of the Fuller construction $\Psi(f_1,\ldots,f_n)$ is isomorphic to the Reidemeister trace of the composite $f_1\circ\ldots\circ f_n$.*

Since the bicategory $\mathcal{E}\mathrm{x}$ has a shadowed n-Fuller structure and a system of base-change objects, this proves Theorem A. Our motivation for stating the proof at this level of generality is that the same argument will establish a more general result for the fiberwise Reidemeister trace and Fuller trace. See Part 3.

Part 2. Varying the group G

In this section we prove the first two triangles of Theorem 1.1 commute:

Theorem B. *The following diagram commutes up to homotopy.*

$$\begin{tikzcd}
 & & \mathbb{S} \arrow[dll, "R(\Psi^n(f))^{C_n}"'] \arrow[dl, "R(\Psi^k(f))^{C_k}"] \arrow[d, "R(\Psi^k(f))"] \\
(\Sigma^\infty_+ \Lambda^{\Psi^n(f)} X^n)^{C_n} \arrow[r, "R"] & (\Sigma^\infty_+ \Lambda^{\Psi^k(f)} X^k)^{C_k} \arrow[r, "F"] & \Sigma^\infty_+ \Lambda^{\Psi^k(f)} X^k
\end{tikzcd}$$

The essential idea is to show that the geometric fixed point functor Φ^H, and the functor ι_H^* that forgets group actions, are strong shadow functors, so that they preserve Reidemeister traces by Theorem 5.7. In contrast to the previous part where we black-boxed all needed properties of parameterized spectra, in this part we work directly with these spectra. In Section 8 we recall some general theory about passing between symmetric monoidal bifibrations (smbfs) and bicategories, and in Section 9 we apply these ideas to the smbf of parametrized G-spectra. We finish the proof of Theorem B in Section 10.

8. Symmetric monoidal bifibrations

Strong shadow functors such as Φ^H are difficult to construct on $G\mathcal{E}\mathrm{x}$ because the operations $\odot$ and $\langle\!\langle - \rangle\!\rangle$ are composites of left and right derived functors. It is far easier to show that the constituent pieces of $\odot$ are separately preserved by Φ^H, and then assemble those pieces back together. The structure of these constituent pieces is captured formally by the idea of a symmetric monoidal bifibration (smbf).

In this paper, a **bifibration** is a functor $\pi\colon \mathscr{C} \to \mathbf{S}$ from a category $\mathscr{C}$ to a cartesian monoidal category $\mathbf{S}$ with the following properties.

- For every pair of an object $X \in \mathscr{C}$ and an arrow $A \xrightarrow{f} \pi(X)$ in $\mathbf{S}$, there is **cartesian arrow** $f^*X \to X$ satisfying a universal property given in shorthand in Figure 8.2a.
- For every pair of an object $X \in \mathscr{C}$ and an arrow $\pi(X) \xrightarrow{f} A$ in $\mathbf{S}$, there is a **cocartesian arrow** $X \to f_!X$ satisfying a universal property given in shorthand in Figure 8.2b.
- There is a class of **Beck-Chevalley squares** in $\mathbf{S}$,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\
C \arrow[r, "k"'] & D,
\end{tikzcd}$$

  such that in each one the natural transformation of functors $\mathscr{C}^C \to \mathscr{C}^B$

$$f_!h^* \to f_!h^*k^*k_! \xrightarrow{\sim} f_!f^*g^*k_! \to g^*k_! \tag{8.1}$$

  is an isomorphism.
- The class of Beck-Chevalley squares can be chosen to include the following squares.
  - For any pair of composable maps $A \xrightarrow{f} B \xrightarrow{g} C$ and $A' \xrightarrow{f'} B'$,

$$\begin{tikzcd}
A \times A' \arrow[r, "1\times f'"] \arrow[d, "f\times 1"'] & A \times B' \arrow[d, "f\times 1"] \\
B \times A' \arrow[r, "1\times f'"] & B \times B'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "(1,g\circ f)"] \arrow[d, "(1,f)"'] & A \times C \arrow[d, "(1,f)\times 1"] \\
A \times B \arrow[r, "1\times(1,g)"] & A \times B \times C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "(1,g\circ f)"] \arrow[d, "f"'] & A \times C \arrow[d, "f\times 1"] \\
B \arrow[r, "(1,g)"] & B \times C.
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\exists!"] \arrow[drr] \arrow[dd, squiggly, "\Phi"] & & \\
& f^*X \arrow[r] \arrow[d, squiggly, "\Phi"] & X \arrow[d, squiggly, "\Phi"] \\
\Phi(Y) \arrow[r] & A \arrow[r, "f"] & \Phi(X)
\end{tikzcd}$$

(A) Cartesian arrows

$$\begin{tikzcd}
& & Y \\
X \arrow[r] \arrow[urr] \arrow[d, squiggly, "\Phi"] & f_!X \arrow[ur, dashed, "\exists!"'] \arrow[d, squiggly, "\Phi"] & \\
\Phi(X) \arrow[r, "f"] & B \arrow[r] & \Phi(Y) \arrow[uu, squiggly, "\Phi"']
\end{tikzcd}$$

(B) Cocartesian arrows

FIGURE 8.2. Cartesian and Cocartesian arrows

- Any square isomorphic to a Beck-Chevalley square. (This includes commuting squares with two parallel isomorphisms.)
- Any product of a Beck-Chevalley square and an object of $\mathbf{S}$.

A **fibration** is merely a functor $\Phi\colon \mathscr{C} \to \mathbf{S}$ that has cartesian arrows, while an **op-fibration** only has cocartesian arrows. A **map of bifibrations** is a strictly commuting square of functors

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "\pi_{\mathscr{C}}"'] & \mathscr{D} \arrow[d, "\pi_{\mathscr{D}}"] \\
\mathbf{S} \arrow[r, "F_\flat"'] & \mathbf{T}
\end{tikzcd}$$

such that F preserves cartesian arrows and cocartesian arrows, while $F_\flat$ preserves products and Beck-Chevalley squares.

A **symmetric monoidal bifibration (smbf)** is a bifibration $\pi\colon \mathscr{C} \to \mathbf{S}$ and a symmetric monoidal structure on $\mathscr{C}$ with monoidal product denoted $\boxtimes$ so that

- π is a strict symmetric monoidal functor,
- $\boxtimes$ is a map of fibrations, i.e. a tensor of two cartesian arrows is cartesian, and
- $\boxtimes$ is a map of op-fibrations, i.e. a tensor of two cocartesian arrows is cocartesian.

We think of this monoidal product as "external" and we denote the unit by I. A **map of symmetric monoidal bifibrations** is a map of bifibrations together with a strong symmetric monoidal structure on the functor $F\colon \mathscr{C} \to \mathscr{D}$, so that $F(X)\boxtimes F(Y) \to F(X \boxtimes Y)$ lies over the canonical map $F_\flat(A) \times F_\flat(B) \cong F_\flat(A \times B)$.

Intuitively, an smbf has three operations $\boxtimes$, f^*, $f_!$ that "commute" along canonical isomorphisms. For each pair of maps $f\colon A \to B$, $g\colon A' \to B'$, there is a canonical isomorphism

$$f^*X \boxtimes g^*Y \cong (f \times g)^*(X \boxtimes Y),$$

of functors $\mathscr{C}^B \times \mathscr{C}^{B'} \to \mathscr{C}^{A \times A'}$, and a similar canonical isomorphism for pushforwards.

Example 8.3. Let $\mathbf{S}$ be the category of unbased spaces. The objects of $\mathcal{U}$ are the arrows $X \to A$ in $\mathbf{S}$, and maps are commuting squares. The projection $\mathcal{U} \to \mathbf{S}$ sends $X \to A$ to A. A Cartesian arrow over $A \to B$ is a pullback square of spaces. The pushforward of $X \to A$ along $A \to B$ is the composite $X \to B$. It satisfies the Beck-Chevalley condition for all pullback squares. $\mathcal{U}$ is a symmetric monoidal bifibration, with tensor product given by the Cartesian product, sending $X \to A$ and $Y \to B$ to $X \times Y \to A \times B$.

As we have already mentioned, an smbf contains all the raw ingredients needed to form a bicategory with a system of base-change objects. We assemble the operations $\odot$,

$\langle\!\rangle$, U_B, and $\left[B \xleftarrow{f} A\right]$ from these more basic pieces as follows.

$$-\odot_B -\colon \mathscr{C}^{A\times B}\times\mathscr{C}^{B\times C}\to\mathscr{C}^{A\times C} \qquad \langle\!-\!\rangle_B\colon \mathscr{C}^{B\times B}\to\mathscr{C}^{*}$$

$$M\odot_B N=(\mathrm{id}_A\times\pi_B\times\mathrm{id}_C)_!(\mathrm{id}_A\times\Delta_B\times\mathrm{id}_C)^*(M\boxtimes N) \qquad \langle\!M\!\rangle_B=(\pi_B)_!(\Delta_B)^*M$$

$$U_B\colon *\to\mathscr{C}^{B\times B} \qquad \left[B\xleftarrow{f}A\right]\colon *\to\mathscr{C}^{B\times A}$$

$$U_B=(\Delta_B)_!\pi_B^*I \qquad \left[B\xleftarrow{f}A\right]=(f,\mathrm{id}_A)_!\pi_A^*I$$

Theorem 8.4. *Let $\mathscr{C}\to\mathbf{S}$ be a symmetric monoidal bifibration.*

- *The operations $-\odot-$ and U_B above and the maps $\mathfrak{a},\mathfrak{l},\mathfrak{r}$ defined in* [MP18, Figures 5.6 and 5.7] *define a bicategory $\mathscr{C}/\!\!/\mathbf{S}$.* [Shu08, 14.4, 14.11]
- *The operation $\langle\!-\!\rangle$ above and θ defined in* [MP18, Figure 5.8] *define a shadow on $\mathscr{C}/\!\!/\mathbf{S}$.* [PS12, 5.2]
- *There is a pseudofunctor $[]\colon\mathbf{S}\to\mathscr{C}/\!\!/\mathbf{S}$ that sends each morphism to the base-change object $\left[B\xleftarrow{f}A\right]$.* [MP18, Theorems 3.4 and 3.6]
- *The bicategory $\mathscr{C}/\!\!/\mathbf{S}$ has a shadowed n-Fuller structure and a system of base-change objects given by $[]$.* [MP18, Theorem 3.6]
- *Each map of symmetric monoidal bifibrations $F\colon(\mathscr{C},\mathbf{S})\to(\mathscr{D},\mathbf{T})$ induces a strong shadow functor $F\colon\mathscr{C}/\!\!/\mathbf{S}\to\mathscr{D}/\!\!/\mathbf{T}$, and an isomorphism $F\circ[]\cong[]\circ F_\flat$.* [MP18, Theorem 14.1]

The last bullet point in particular reduces the problem of building strong shadow functors Φ^H and ι_H^* to the problem of building maps of symmetric monoidal bifibrations.

Finally we discuss how to invert weak equivalences in a bifibration. Suppose $\pi\colon\mathscr{C}\to\mathbf{S}$ is a fibration, each fiber category $\mathscr{C}^A$ has a subcategory of weak equivalences, and $\mathrm{ho}\mathscr{C}$ is the category formally obtained by inverting these equivalences. By the universal property of $\mathrm{ho}\mathscr{C}$ there is a functor $\mathrm{ho}\mathscr{C}\to\mathbf{S}$, that is in general not a fibration.

We say that π is a **right-deformable fibration** if for each $\mathscr{C}^A$ there is

- a full subcategory $\mathscr{F}^A$,
- a functor $R_A\colon\mathscr{C}^A\to\mathscr{C}^A$ with image in $\mathscr{F}^A$, and
- a weak equivalence $r_A\colon\mathrm{id}_{\mathscr{C}^A}\xrightarrow{\sim}R_A$,

such that

- $f^*\colon\mathscr{C}^B\to\mathscr{C}^A$ preserves weak equivalences on $\mathscr{F}^B$, and
- $f^*(\mathscr{F}^B)\subseteq\mathscr{F}^A$.

The following two results are proven by an elementary but tedious diagram-chase, that compares $\mathrm{ho}\mathscr{C}$ to the Grothendieck construction formed from the right-derived pullback functors $f^*R_B\colon\mathrm{ho}(\mathscr{C}^B)\to\mathrm{ho}(\mathscr{C}^A)$. The full proof appears in [Mal19].

Theorem 8.5. *If π is a right-deformable fibration then $\mathrm{ho}\mathscr{C}\to\mathbf{S}$ is a fibration, and the canonical maps $\mathrm{ho}(\mathscr{C}^A)\to(\mathrm{ho}\mathscr{C})^A$ are isomorphisms of categories. Dually, if π a left-deformable op-fibration then $\mathrm{ho}\mathscr{C}\to\mathbf{S}$ is an op-fibration.*

We call the cartesian arrows in $\mathrm{ho}\mathscr{C}$ **homotopy cartesian** when we want to distinguish from the cartesian arrows in $\mathscr{C}$.

Proposition 8.6. *Suppose $\mathscr{C}$ is a right deformable fibration. Then an arrow in $\mathrm{ho}\mathscr{C}$ is homotopy cartesian if and only if it is isomorphic to a cartesian arrow in $\mathscr{C}$ with fibrant target. The dual statement applies to cocartesian arrows in a left-deformable op-fibration.*

9. Parametrized G-spectra and fixed point functors

By the last bullet point of Theorem 8.4, it now remains to construct the smbf of parametrized G-spectra, and to prove that ι_H^* and Φ^H give smbf maps.

9.1. **On the nose.** Fix a finite group G and an unbased left G-space B. Recall from [MS06] that there is a category $G\mathcal{S}(B)$ of **orthogonal G-spectra over** B, or equivalently **$\mathscr{J}_G$-spaces over** B. The objects are sequences that assign to each integer $n \geq 0$ a retractive $G \times O(n)$ space X_n over B, together with G-equivariant structure maps $\Sigma_B X_n \to X_{1+n}$, satisfying the condition that the composite map $\Sigma_B^p X_q \to X_{p+q}$ is $O(p) \times O(q)$-equivariant. We always assume the base space B is compactly generated weak Hausdorff, while X_n only has to be compactly generated.

For each G-equivariant map of base spaces $f\colon A \to B$, a map of orthogonal G-spectra over f consists of commuting diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ X_n \arrow[r, "\phi_n"] \arrow[d] & Y_n \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

in which ϕ_n is $G \times O(n)$-equivariant and commutes with the structure maps of X and Y. This defines a larger category $G\mathcal{S}$ of all orthogonal G-spectra over all base spaces, whose fiber category over B is $G\mathcal{S}(B)$. The projection functor to the category $G\mathbf{S}$ of unbased G-spaces is a bifibration, with Beck-Chevalley along strict pullback squares [MS06, 11.4.8]. We therefore have adjoint pullback and pushforward functors

$$f^*\colon G\mathcal{S}(B) \to G\mathcal{S}(A),$$
$$f_!\colon G\mathcal{S}(A) \to G\mathcal{S}(B).$$

The pullback f^* is also a left adjoint, and therefore preserves all colimits.

There is an **external smash product** functor

$$\overline{\wedge}\colon G\mathcal{S}(A) \times G\mathcal{S}(B) \to G\mathcal{S}(A \times B)$$

defined for retractive G-spaces by the formula

$$\begin{tikzcd} (X \times B) \cup_{A \times B} (A \times Y) \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ A \times B \arrow[r] & X \overline{\wedge} Y \end{tikzcd} \tag{9.1}$$

and then extended to parametrized G-spectra using the Day convolution along $\mathscr{J}_G$. This can be regarded as a functor on the entire category of G-spectra, $G\mathcal{S} \times G\mathcal{S} \to G\mathcal{S}$. It preserves cartesian and cocartesian arrows, and extends to a symmetric monoidal structure, hence it makes the category $G\mathcal{S}$ a symmetric monoidal bifibration. The unit of $\overline{\wedge}$ is the sphere spectrum, regarded as a parametrized spectrum over the one-point space $*$.

Let $F_V K$ denote the free parametrized $\mathscr{J}_G$-space on a retractive G-space K over B. Concretely, this is the $\mathscr{J}_G$-space whose value at W is the external smash product $\mathscr{J}_G(V,W) \overline{\wedge} K$, where $\mathscr{J}_G(V,W)$ is regarded as a retractive space over $*$. Since pullback and pushforward commute with $\overline{\wedge}$, they also commute with free spectra ([MS06, 11.4.7]):

$$f^* F_V K \cong F_V f^* K, \qquad f_! F_V K \cong F_V f_! K.$$

We will frequently use the class of "freely f-cofibrant" orthogonal spectra over B. A map of retractive spaces is a (closed, equivariant) **f-cofibration** if it is closed and has

the fiberwise, unbased, equivariant version of the homotopy extension property. A spectrum is **freely f-cofibrant** if it is isomorphic to a cell complex spectrum built from maps of the form $F_V K \to F_V L$, where $K \to L$ is a (closed, equivariant) f-cofibration of f-cofibrant spaces over B. By "cell complex" we mean a sequential colimit of pushouts of arbitrary coproducts of such maps.

Lemma 9.2. *Freely f-cofibrant spectra are preserved by the pullback functor f^*.*

Proof. This follows because f^* preserves colimits, free spectra, and f-cofibrations of retractive spaces. □

Next we define three functors that change the group G. First, for each subgroup $H \leq G$, we can forget the G-action and remember the action of H. This gives the **forgetful functor to** H. Second, if H has Weyl group $WH = NH/H$, the H-fixed point subspaces X_n^H form a WH-equivariant spectrum over the fixed point subspace B^H. This defines the **categorical fixed points** functor. Finally, we define the **geometric fixed points** functor by the following coequalizer of WH-spectra.

$$\bigvee_{V,W} F_{W^H} S^0 \overline{\wedge} \mathscr{J}_G^H(V,W) \overline{\wedge} X(V)^H \rightrightarrows \bigvee_V F_{V^H} S^0 \overline{\wedge} X(V)^H \longrightarrow \Phi^H X$$

In total, this gives three functors

$$\begin{aligned} \iota_H^* &: G\mathcal{S}(B) \to H\mathcal{S}(B) \\ (-)^H &: G\mathcal{S}(B) \to WH\mathcal{S}(B^H) \\ \Phi^H &: G\mathcal{S}(B) \to WH\mathcal{S}(B^H). \end{aligned}$$

The **restriction map** $r : X^H \to \Phi^H X$ assigns each level X_n^H to the $\mathbb{R}^n$-term on the right-hand side of the coequalizer system. (A little diagram-chasing shows this gives a well-defined map of spectra.)

Each of these definitions extends to maps in $G\mathcal{S}$, giving commuting squares of functors

$$\begin{tikzcd} G\mathcal{S} \arrow[r, "\iota_H^*"] \arrow[d] & H\mathcal{S} \arrow[d] \\ G\mathbf{S} \arrow[r, "\iota_H^*"] & H\mathbf{S} \end{tikzcd} \qquad \begin{tikzcd} G\mathcal{S} \arrow[r, "(-)^H"] \arrow[d] & WH\mathcal{S} \arrow[d] \\ G\mathbf{S} \arrow[r, "(-)^H"] & WH\mathbf{S} \end{tikzcd} \qquad \begin{tikzcd} G\mathcal{S} \arrow[r, "\Phi^H"] \arrow[d] & WH\mathcal{S} \arrow[d] \\ G\mathbf{S} \arrow[r, "(-)^H"] & WH\mathbf{S} \end{tikzcd}$$

and r lives over the identity transformation of $(-)^H : G\mathbf{S} \to WH\mathbf{S}$.

Proposition 9.3. *Each of these functors is a map of bifibrations, if we restrict to freely f-cofibrant spectra. We can give each one a lax symmetric monoidal structure, commuting with the same structure on the other three functors in its square. Furthermore ι_H^* is strong symmetric monoidal, and Φ^H is strong on the subcategory of freely f-cofibrant spectra.*

Proof. It is elementary to check that the first two functors preserve cartesian and cocartesian arrows. For Φ^H this reduces to the same claim for external smash products, and the fact that f^* preserves all colimits [MS06, 11.4.1].

The symmetric monoidal structure on ι_H^* is given by the identity map on the underlying spectra, and the coherences are obviously satisfied. For $(-)^H$ the symmetric monoidal structure map $X^H \overline{\wedge} Y^H \to (X \overline{\wedge} Y)^H$ is given by noticing that the inclusion to $X \overline{\wedge} Y$ lands in the H-fixed points, and $(\mathbb{S}_{(G)})^H \cong \mathbb{S}_{(WH)}$ is the unique isomorphism. The coherences are also straightforward. For Φ^H the map commuting it with smash product is constructed by the method of [MM02, 4.7], applied verbatim with smash products replaced by external smash products. We also use the same argument to prove this map is an isomorphism on free spectra, and therefore on freely f-cofibrant spectra by an induction

up the skeleton of the cell complex. To check the coherence of this symmetric monoidal structure, it suffices to restrict attention to one fiber. Then it follows immediately from [Mal17b, 1.2]. □

9.2. **The homotopy category.** A map of orthogonal G-spectra $X \to Y$ over B is a **level equivalence** if each map $X(V) \to Y(V)$ is an equivalence on the H-fixed points for all subgroups $H \leq G$. There is a **level fibrant replacement** functor R^{lv} that replaces each X by a level equivalent spectrum $X \xrightarrow{\sim} R^{lv}X$ so that $(R^{lv}X(V))^H \to B^H$ is a quasifibration [MS06, 6.5.1 and 12.1.7]. A map $X \to Y$ is a **stable equivalence** if on each fiber over $b \in B$ the map $R^{lv}X \to R^{lv}Y$ is an isomorphism on the H-equivariant stable homotopy groups for every $H \leq \mathrm{stab}_b \leq G$. This definition is independent of the choice of functor R^{lv}.

Theorem 9.4. [MS06, 12.3.10] *There is a model structure on $G\mathcal{S}(B)$ where the weak equivalences are stable equivalences.*

Proposition 9.5. [MS06, 12.6.7] *The pullback functor f^* is a Quillen right adjoint, and a Quillen equivalence if f is a weak equivalence of G-spaces (i.e. $A^H \xrightarrow{\sim} B^H$ is a weak equivalence for all $H \leq G$). In fact, f^* preserves all stable equivalences between spectra whose levels $X(V)^H$ are quasifibrations over B^H.*

This is the **stable qf-model structure**. The generating cofibrations are the free spectra on the **qf-cells**, i.e. those maps $G/H \times (S^{k-1} \to D^k)$ over B that are f-cofibrations. The generating acyclic cofibrations are the free spectra on maps of the form $G/H \times (D^{k-1} \to D^k)$ with a different cofibration condition, and also the $\overline{\wedge}$-pushout-products of generating cofibrations over B and the maps $k_{V,W}$ over $*$ from [MM02, III.4.6]. We will not spell out the condition on $D^{k-1} \to D^k$ because it will not matter; it only matters that we fix the definition once and for all.

In particular, every qf-cofibrant spectrum is also freely f-cofibrant. The next two lemmas therefore show that Φ^H and $\overline{\wedge}$ preserve stable equivalences between pullbacks of such spectra. This was already done nonequivariantly for $\overline{\wedge}$ in [Mal17a], but here we give a different argument that is easier to make equivariant.

Lemma 9.6. *Φ^H preserves cofibrations, acyclic cofibrations, and stable equivalences between freely f-cofibrant spectra.*

Proof. The proof that it preserves cofibrations and acyclic cofibrations is identical to the proof in the non-parametrized case [MM02], so we focus on the last claim.

We freely use the fact that a pushout-product of f-cofibrations of retractive spaces is again an f-cofibration, and that the external smash product $K \overline{\wedge} K'$ of f-cofibrant spaces preserves weak equivalences. This implies that $F_V K \to F_V L$ is an f-cofibration on each spectrum level when $K \to L$ is an f-cofibration, and also that $F_V K \to F_V K'$ is a level equivalence when $K \to K'$ is an equivalence of f-cofibrant spaces.

It suffices to show that for a freely f-cofibrant spectrum X, there is some qf-cofibrant spectrum X' and stable equivalence $X' \to X$ such that $\Phi^H X' \to \Phi^H X$ is an equivalence. Let $X^{(n)}$ denote the n-skeleton of X, meaning the target of the nth map in the sequential colimit system that defines X. By induction on n, we build two cofibrant spectra $X^{[n-1/2]}$ and $X^{[n]}$, fitting into a diagram

$$\begin{tikzcd}
X^{[n-1]} \arrow[r, "\sim"] \arrow[d, "\sim"] & X^{[n-1/2]} \arrow[r] \arrow[d, "\sim"] & X^{[n]} \arrow[d, "\sim"] \\
X^{(n-1)} \arrow[r, equal] & X^{(n-1)} \arrow[r] & X^{(n)}
\end{tikzcd}$$

where the $\sim$ maps are level equivalences and the top row consists of qf-cofibrations of qf-cofibrant spectra. The colimit over n is a homotopy colimit on each spectrum level and therefore $\operatorname{colim}_n X^{[n]} \to X$ is a level equivalence of spectra. Then we prove that $\Phi^H X^{[n]} \to \Phi^H X^{(n)}$ is an equivalence. Since Φ^H preserves free f-cofibrations, and pushouts and sequential colimits along such, this implies that $\Phi^H \operatorname{colim}_n X^{[n]} \to \Phi^H X$ is an equivalence, as desired.

Now we build these spectra. For each of the maps $K \to L$ appearing at stage n of the colimit system for X, factor $B \to K$ into a qf-cell complex $B \to K' \xrightarrow{\sim} K$, then factor $K' \to L$ into another qf-cell complex $K' \to L' \xrightarrow{\sim} L$. Because K' is a cell complex relative to B and $X_V^{[n-1]} \to X_V^{(n-1)}$ is a weak equivalence of G-spaces, the map $K' \to K \to X_V^{(n-1)}$ can be modified by a homotopy rel B to a map that lifts to $X_V^{[n-1]}$. This data together gives a map from the mapping cylinder of $K' \to L'$ rel B into $X_V^{(n-1)}$ for which the front end lifts to a map $K' \to X_V^{[n-1]}$. We define a projection map from the mapping cylinder back to B, by composing this map with the projection $X_V^{(n-1)} \to B$ so that we have a map of retractive spaces over B.

Form $X^{[n-1/2]}$ by attaching the cylinder part of this mapping cylinder to $X^{[n-1]}$, and $X^{[n]}$ by attaching the entire mapping cylinder. The maps in the diagram above are then clear. The levels of $X^{[n-1/2]}$ deformation retract onto $X^{[n-1]}$. Using the fact that external smash product preserves equivalences of f-cofibrant spaces, $X^{[n]} \to X^{(n)}$ is also a level equivalence.

Now apply Φ^H to the construction of $X^{[n]}$. We get the same equivalences as before, except possibly the one all the way on the right. Before Φ^H, it is a map of pushouts of the form

$$\begin{tikzcd}
X^{[n-1/2]} \arrow[d, "\sim"] & \coprod F_{V_\alpha} K'_\alpha \arrow[l] \arrow[r] \arrow[d, "\sim"] & \coprod F_{V_\alpha} L'_\alpha \arrow[d, "\sim"] \\
X^{(n-1)} & \coprod F_{V_\alpha} K_\alpha \arrow[l] \arrow[r] & \coprod F_{V_\alpha} L_\alpha .
\end{tikzcd}$$

After Φ^H, it is a map of pushouts of the form

$$\begin{tikzcd}
\Phi^H X^{[n-1/2]} \arrow[d, "\sim"] & \coprod F_{V_\alpha^H} (K'_\alpha)^H \arrow[l] \arrow[r] \arrow[d, "\sim"] & \coprod F_{V_\alpha^H} (L'_\alpha)^H \arrow[d, "\sim"] \\
\Phi^H X^{(n-1)} & \coprod F_{V_\alpha^H} K_\alpha^H \arrow[l] \arrow[r] & \coprod F_{V_\alpha^H} L_\alpha^H .
\end{tikzcd}$$

The left-hand vertical map is an equivalence by inductive hypothesis. The middle vertical map is an equivalence because $(K'_\alpha)^H \to K_\alpha^H$ is an equivalence of equivariantly f-cofibrant WH-spaces, and similarly for the right-hand vertical map. The horizontal maps on the right-hand side are also f-cofibrations on each spectrum level, because $K_\alpha^H \to L_\alpha^H$ and $(K'_\alpha)^H \to (L'_\alpha)^H$ are both WH-equivariant f-cofibrations. Therefore the map of pushouts $\Phi^H X^{[n]} \to \Phi^H X^{(n)}$ is a level equivalence of WH-spectra, completing the induction. $\square$

Lemma 9.7. *$\overline{\wedge}$ preserves stable equivalences between freely f-cofibrant spectra.*

Proof. The proof is essentially the same as the previous lemma. It suffices to take two freely f-cofibrant spectra X and Y, build the spectra $X^{[n-1/2]}$, $X^{[n]}$ as in that argument, and to show that $\left(\operatorname{colim}_n X^{[n]}\right) \overline{\wedge} Y \to X \overline{\wedge} Y$ is a level equivalence. Then we could do the same with the roles of X and Y swapped, and conclude that $QX \overline{\wedge} QY \to X \overline{\wedge} Y$ is an equivalence.

We first observe that pushout-products of spectra constructed with $\overline{\wedge}$ preserve free f-cofibrations; this follows from the same statement for spaces and the formal fact that smash products of free spectra are free. We already used in the previous proof that free f-cofibrations are level f-cofibrations. Using this, we can prove that if $K' \to K$ is an equivalence of f-cofibrant spaces then $F_{V_\alpha}K' \wedge Y \to F_{V_\alpha}K \wedge Y$ is an equivalence. We observe that $F_{V_\alpha}K \wedge -$ turns the cell complex structure of Y into a new cell complex structure in which the representations all have V_α added to them, and the spaces all have K smashed into them. By the pushout-product property, the pushout squares are all along free f-cofibrations of spectra, which are f-cofibrations on each level. For each space A occurring in the cell complex structure of Y, the equivalence of f-cofibrant spaces $K' \overline{\wedge} A \to K \overline{\wedge} A$ gives a level equivalence of free spectra. Hence every one of the pushout squares is changed by a level equivalence when we pass from K' to K; hence

$$F_{V_\alpha}K' \wedge Y \xrightarrow{\sim} F_{V_\alpha}K \wedge Y$$

is an equivalence.

Since free f-cofibrations are level f-cofibrations, $\operatorname{colim}_n(X^{[n]} \overline{\wedge} Y) \cong \left(\operatorname{colim}_n X^{[n]}\right) \overline{\wedge} Y$ is a homotopy colimit. It therefore suffices to prove by induction that $X^{[n]} \overline{\wedge} Y \to X^{(n)} \overline{\wedge} Y$ is an equivalence. For the inductive step we have the diagram

$$\begin{tikzcd}
X^{[n-1]} \wedge Y \arrow[r, "\sim"] \arrow[d, "\sim"] & X^{[n-1/2]} \wedge Y \arrow[r] \arrow[d, "\sim"] & X^{[n]} \wedge Y \arrow[d] \\
X^{(n-1)} \wedge Y \arrow[r, equal] & X^{(n-1)} \wedge Y \arrow[r] & X^{(n)} \wedge Y
\end{tikzcd}$$

where the marked $\sim$ on the top is the deformation retract of $X^{[n-1/2]}$ onto $X^{[n-1]}$. We just need to see that the vertical on the right is an equivalence. Before smashing with Y, it is a map of pushouts of the form

$$\begin{tikzcd}
X^{[n-1/2]} \arrow[d, "\sim"] & \coprod F_{V_\alpha}K'_\alpha \arrow[l] \arrow[r] \arrow[d, "\sim"] & \coprod F_{V_\alpha}L'_\alpha \arrow[d, "\sim"] \\
X^{(n-1)} & \coprod F_{V_\alpha}K_\alpha \arrow[l] \arrow[r] & \coprod F_{V_\alpha}L_\alpha.
\end{tikzcd}$$

After $\overline{\wedge} Y$, the horizontal maps of the right-hand square are level cofibrations, by the pushout-product property for free f-cofibrations. The vertical maps are equivalences by inductive hypothesis and the intermediate lemma we established earlier in the proof. Therefore the map of pushouts is an equivalence, and the induction is complete. □

Now we pass to the homotopy category by inverting all the stable equivalences in $G\mathcal{S}$. By Theorem 8.5 the resulting category ho$G\mathcal{S}$ is a fibration and op-fibration whose base category is the category $G\mathbf{S}$ of G-spaces. By Proposition 8.6, an arrow in ho$G\mathcal{S}$ is homotopy cocartesian if and only if it is isomorphic to a cocartesian arrow $X \to f_! X$ in which X is cofibrant. An arrow is homotopy cartesian if and only if it is isomorphic to a cartesian arrow $f^* Y \to Y$ with Y fibrant.

Remark 9.8. This homotopy category is the homotopy category of the "integral model structure" of [HP15], but taking the weak equivalences in the base category to be the isomorphisms, rather than the weak homotopy equivalences. In other words, it retains information about the base space up to homeomorphism, but the fiber spectra are remembered only up to stable equivalence.

Since every cell and acyclic cell in $G\mathcal{S}(A)$ pushes forward along f to a cell or acyclic cell in $G\mathcal{S}(B)$, the cofibrant replacements Q_A built using the small-object argument assemble into a single functor $Q : G\mathcal{S} \to G\mathcal{S}$. Using Q, we can left-derive the external smash

product functor

$$\overline{\wedge}\colon G\mathcal{S} \times G\mathcal{S} \to G\mathcal{S}.$$

This makes ho$G\mathcal{S}$ into a symmetric monoidal category. More concretely, the tensor product is $\overline{\wedge}^{\mathbf{L}} = (Q-)\overline{\wedge}(Q-)$, with associator, unitor, and symmetry isomorphism given by deleting all copies of Q that are not applied to the inputs (for instance one that is applied to the output of $\overline{\wedge}$), applying the corresponding isomorphism for $\overline{\wedge}$, and then re-inserting the extra copies of Q. Since any two left-derivations of a functor are canonically isomorphic, we can be assured that if we had chosen a different model structure we would get an isomorphic symmetric monoidal category.

Theorem 9.9. *This symmetric monoidal structure makes* ho$G\mathcal{S}$ *into a symmetric monoidal bifibration, with Beck-Chevalley for every homotopy pullback square of G-spaces.*

Proof. The projection to the base category $G\mathbf{S}$ is still strict symmetric monoidal because the map $QX \xrightarrow{\sim} X$ lies over the identity of $G\mathbf{S}$. The Beck-Chevalley property for pullback squares with one leg a fibration is [Shu11, 9.9], building on [MS06, Thm 13.7.7]; see also [Mal19]. For a commuting square of spaces where two of the parallel sides are weak equivalences, we also get the Beck-Chevalley property because each component of the Beck-Chevalley map is an isomorphism as functors of homotopy categories. We then deduce the Beck-Chevalley property for an arbitrary homotopy pullback square using the usual pasting lemma.

It remains to show that $\overline{\wedge}^{\mathbf{L}}$ preserves cocartesian arrows and cartesian arrows. In principle, this should be citable away to [MS06], but it is difficult to work directly with their construction of the symmetric monoidal structure on the pullback functors f^*. We instead start with the "canonical" one defined just above.[2]

Take any two homotopy cocartesian arrows in ho$G\mathcal{S}$. Up to isomorphism, they are cocartesian arrows $X \to f_!X$ and $Y \to g_!Y$ in the point-set category $G\mathcal{S}$ with X and Y cofibrant. On these inputs we have an equivalence $(Q-)\overline{\wedge}(Q-) \simeq -\overline{\wedge}-$, so the derived product $\overline{\wedge}^{\mathbf{L}}$ of these arrows in the homotopy category is isomorphic to their actual product $\overline{\wedge}$, which is cocartesian in $G\mathcal{S}$ and still has a cofibrant source, hence is homotopy cocartesian. Therefore $\overline{\wedge}^{\mathbf{L}}$ preserves homotopy cocartesian arrows.

Now take any two homotopy cartesian arrows in ho$G\mathcal{S}$. Up to isomorphism, they are cartesian arrows in the point-set category $G\mathcal{S}$ whose targets are both cofibrant and fibrant. Let us call them $f^*X \to X$ and $g^*Y \to Y$. Form the following commuting diagram

[2]It is clear that this has the expected behavior on suspension spectra, which is all we need for the applications anyway.

in $G\mathcal{S}$.

$$\begin{tikzcd}
Qf^*X \overline{\wedge} Qg^*Y \arrow[r] \arrow[d, "\sim"', "(1)"] & QX \overline{\wedge} QY \arrow[d, "\sim"] \\
f^*X \overline{\wedge} g^*Y \arrow[r] \arrow[d, "\sim"', "(5)"] & X \overline{\wedge} Y \arrow[d, "(2)"', "\sim"] \\
f^*PX \overline{\wedge} g^*PY \arrow[r] & PX \overline{\wedge} PY \\
(f \times g)^*P(X \overline{\wedge} Y) \arrow[r] \arrow[u, leftrightarrow, "\cong"] \arrow[d, "\sim"', "(3)"] & P(X \overline{\wedge} Y) \arrow[u, leftrightarrow, "\cong"'] \arrow[d, "\sim"] \\
(f \times g)^*PR(X \overline{\wedge} Y) \arrow[r] & PR(X \overline{\wedge} Y) \\
(f \times g)^*R(X \overline{\wedge} Y) \arrow[r] \arrow[u, "\sim", "(4)"'] & R(X \overline{\wedge} Y) \arrow[u, "\sim"']
\end{tikzcd}$$

Here P is the functor from [Mal17a]; it pulls back X and then pushes it forward along the two evaluation maps $B^I \rightrightarrows B$. The maps (1) and (2) are equivalences by Lemma 9.7 and the fact that f^* and P preserve freely f-cofibrant spectra. The equivalences (3), (4) are because f^* preserves the stable equivalences between spectra whose levels are quasifibrant, and this class includes both RX and PX when X is cofibrant. The equivalence (5) uses all of these facts together. Therefore $\overline{\wedge}^{\mathbf{L}}$ preserves homotopy cartesian arrows. □

Proposition 9.10. *The functor* $\iota_H^* \colon \mathrm{ho}G\mathcal{S}(B) \to \mathrm{ho}H\mathcal{S}(B)$ *extends to a map of symmetric monoidal bifibrations* $\mathrm{ho}G\mathcal{S} \to \mathrm{ho}H\mathcal{S}$ *over the functor* $\iota_H^* \colon G\mathbf{S} \to H\mathbf{S}$.

Proof. The functor ι_H^* clearly makes sense on all of $G\mathcal{S}$ and preserves all equivalences, therefore directly passes to a functor $\mathrm{ho}G\mathcal{S} \to \mathrm{ho}H\mathcal{S}$. The point-set functor preserves cartesian arrows, cocartesian arrows, cofibrant objects, and fibrant objects (because it is right Quillen). Therefore when viewed as a derived functor, it preserves homotopy cartesian arrows and homotopy cocartesian arrows. Since ι_H^* strictly commutes with $\overline{\wedge}$, it commutes with $\overline{\wedge}^{\mathbf{L}}$ up to isomorphism by deleting the Qs, applying the commutation, then re-inserting the Qs. (Notice we have to do this because cofibrant replacement for H-spectra may not be ι_H^* of the cofibrant replacement functor for G-spectra.) Again since $QX \xrightarrow{\sim} X$ is over the identity in the base, this isomorphism lies over the corresponding isomorphism of spaces $\iota_H^*(A) \times \iota_H^*(A') \cong \iota_H^*(A \times A')$ in the base category. This gives ι_H^* the structure of a symmetric monoidal functor of homotopy categories, whose coherences follow from the same coherences on the point-set level. □

Although Φ^H is not a left adjoint, by Lemma 9.6 it preserves cofibrations and acyclic cofibrations. It therefore also has a left-derived functor $\mathbf{L}\Phi^H = \Phi^H Q$.

Proposition 9.11. *The functor* $\mathbf{L}\Phi^H \colon \mathrm{ho}G\mathcal{S}(B) \to \mathrm{ho}WH\mathcal{S}(B)$ *extends to a map of symmetric monoidal bifibrations* $\mathrm{ho}G\mathcal{S} \to \mathrm{ho}WH\mathcal{S}$ *over the functor* $(-)^H \colon G\mathbf{S} \to WH\mathbf{S}$.

Proof. As above, the structure of $\mathbf{L}\Phi^H$ as a symmetric monoidal functor is obtained by deleting all extraneous copies of Q, applying the same structure for Φ^H, then re-inserting Q. Again, this gives a symmetric monoidal structure that lies over the canonical one on $(-)^H \colon G\mathbf{S} \to WH\mathbf{S}$, and its coherences follow from the same coherences in the point-set category for Φ^H.

The functor Φ^H preserves both cocartesian and cartesian arrows because the same is true for the smash products, fixed points, and colimits that make up its definition.[3] Since it also preserves cofibrant objects, and $\mathbf{L}\Phi^H \cong \Phi^H$ on the homotopy category of cofibrant objects, this implies $\mathbf{L}\Phi^H$ preserves homotopy cocartesian arrows.

However Φ^H does not preserve fibrant objects, so for homotopy cartesian arrows we have to work a little harder. Start with a point-set cartesian arrow $f^*X \to X$ in which X is cofibrant and fibrant.

$$\begin{tikzcd}
\Phi^H Q f^* X \arrow[r] \arrow[d, "\sim"'] & \Phi^H Q X \arrow[d, "\sim"] \\
\Phi^H f^* X \arrow[r] \arrow[d, "\sim"'] & \Phi^H X \arrow[d, "\sim"] \\
\Phi^H f^* P X \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & \Phi^H P X \arrow[d, leftrightarrow, "\cong"] \\
f^* P \Phi^H X \arrow[r] \arrow[d, "\sim"'] & P \Phi^H X \arrow[d, "\sim"] \\
f^* P R \Phi^H X \arrow[r] & P R \Phi^H X \\
f^* R \Phi^H X \arrow[r] \arrow[u, "\sim"] & R \Phi^H X \arrow[u, "\sim"']
\end{tikzcd}$$

The weak equivalences in the top half follow from Lemma 9.6, because the class of cofibrant spectra described in that lemma is preserved by pullback and by P. The isomorphisms in the middle follow because P is a composition of a pullback and a pushforward. The arrow at the bottom is homotopy cartesian, hence so is the arrow at the top. Therefore $\mathbf{L}\Phi^H$ preserves homotopy cartesian arrows. □

10. Change of groups for the Reidemeister trace

Combining Theorem 8.4 and Propositions 9.10 and 9.11 gives the following result.

Theorem 10.1. *If H is a subgroup of a finite group G, ι_H^* and Φ^H are strong shadow functors on $G\mathcal{E}\mathrm{x}$.*

Theorems 5.7 and 10.1 imply that if X is any finitely dominated G-CW complex and $f : X \to X$ any G-equivariant self-map, there are isomorphisms in the homotopy category

$$\iota_H^* R_G(f) \cong R_H(\iota_H^* f)$$
$$\Phi^H R_G(f) \cong R_{WH}(f^H).$$

Tracing through the constructions shows that these come about through familiar isomorphisms on the source and target, for instance the isomorphism $\Phi^H \Sigma^\infty_+ \Lambda^f X \cong \Sigma^\infty_+ \Lambda^{f^H} X^H$.

[3]This is under the convention that parametrized spectra are built from compactly generated spaces (k-spaces) that are not necessarily weak Hausdorff. The argument still works if we work entirely in weak Hausdorff spaces, but it takes longer to argue that f^* preserves the colimits that make up $\Phi^H X$ when X is freely f-cofibrant.

Corollary 10.2. *If G is a finite group, H is a normal subgroup of G, Y is a G-space and $\phi\colon Y \to Y$ is a G-equivariant map, then*

$$\begin{tikzcd}
 & & \mathbb{S} \arrow[dll, bend right, "R_G(\phi)^G"'] \arrow[dl, "R_{WH}(\phi^H)^{G/H}"] \arrow[d, "R(\phi^H)"] \\
(\Sigma^\infty_+ \Lambda^\phi Y)^G \arrow[r, "R"] & (\Sigma^\infty_+ \Lambda^{\phi^H}(Y^H))^{G/H} \arrow[r, "F"] & \Sigma^\infty_+ \Lambda^{\phi^H}(Y^H)
\end{tikzcd}$$

commutes up to homotopy.

Proof. We describe the argument for the left triangle. The right triangle is similar and more straightforward.

Let E and E' be fibrant replacements of $\Sigma^\infty_+ \Lambda^\phi Y$ and $\Sigma^\infty_+ \Lambda^{\phi^H} Y^H$, respectively, in orthogonal G-spectra. The underived versions of $(-)^H$ and Φ^H define a diagram of orthogonal spectra

$$\begin{tikzcd}
\mathbb{S} \arrow[rrr, "\cong"] \arrow[d, "\cong"'] & & & \Phi^H \mathbb{S} \arrow[d, "\cong"] & \\
\mathbb{S}^G \arrow[r, equal] \arrow[d, "(R_G(\phi))^G"'] & (\mathbb{S}^H)^{G/H} \arrow[r, "r", "\cong"'] \arrow[d, "((R_G(\phi))^H)^{G/H}"] & (\Phi^H \mathbb{S})^{G/H} \arrow[r, "\cong"] \arrow[d, "(\Phi^H R_G(\phi))^{G/H}"] & (\mathbb{S})^{G/H} \arrow[d, "(R_{WH}(\phi^H))^{G/H}"] \\
E^G \arrow[r, equal] \arrow[rrr, bend right, "R"'] & (E^H)^{G/H} \arrow[r, "r"] & (\Phi^H E)^{G/H} \arrow[r, "\sim"] & (E')^{G/H}
\end{tikzcd}$$

The unlabeled $\cong$s exist and the top region commutes because $\mathbb{S}$ has a unique automorphism. The right-hand region is the agreement of $\Phi^H R_G(\phi)$ with $R_{WH}(\phi^H)$ along $\Phi^H E \simeq E'$ lying under $\Sigma^\infty_+ \Lambda^\phi Y \cong \Sigma^\infty_+ \Lambda^{\phi^H} Y^H$ in the homotopy category. The bottom region is the definition of R for equivariant suspension spectra, cf. [DMP+19, §6], [Mad95, §2.5]. □

Theorem B follows by taking $G = C_n$, $H = C_k$, $Y = X^n$ and $\phi = \Psi^n(f)$.

Part 3. The fiberwise generalization

One of the primary strengths of our approach to Theorem A and Theorem B is that it applies in a range of categories. We will illustrate this by extending the results to the fiberwise setting.

11. Spectra over fibrations over B

Fix an unbased space B and let $\mathbf{S}_B$ be the category whose objects are Hurewicz fibrations $A \to B$ and whose maps are maps of spaces over B. This has a forgetful functor to spaces $\mathbf{S}_B \to \mathbf{S}$ that forgets the map to B.

We construct a new symmetric monoidal bifibration $\mathrm{ho}\mathcal{S}_{(B)}$ by pulling back $\mathrm{ho}\mathcal{S}$ along this functor $\mathbf{S}_B \to \mathbf{S}$. It is standard that this gives a bifibration, in which an arrow is (co)cartesian if and only if its image in $\mathrm{ho}\mathcal{S}$ is (co)cartesian. The symmetric monoidal structure is a little more subtle, but follows from the following lemma.

Lemma 11.1. *Suppose $F\colon \mathbf{T} \to \mathbf{S}$ is a functor of cartesian monoidal categories, and that $\mathbf{S}$ and $\mathbf{T}$ are endowed with a class of Beck-Chevalley squares, preserved by F, such that for any two maps $A \to A'$, $B \to B'$ in $\mathbf{T}$ the square*

$$\begin{tikzcd} F(A \times B) \arrow[r] \arrow[d] & F(A) \times F(B) \arrow[d] \\ F(A' \times B') \arrow[r] & F(A') \times F(B') \end{tikzcd}$$

is Beck-Chevalley in $\mathbf{S}$. Then for any smbf $\mathcal{A}$ over $\mathbf{S}$, the pullback category $F^*\mathcal{A}$ can be naturally given the structure of an smbf.

Proof. This is essentially a generalization of the proof of [Shu08, 12.8]: the product $\otimes$ in $F^*\mathcal{A}$ is defined as a pullback of the product $\boxtimes$ in $\mathcal{A}$ along the canonical map

$$F(A \times B) \to F(A) \times F(B).$$

The proof that this preserves cocartesian arrows reduces to the Beck-Chevalley condition in the statement of the lemma, and the proof that it preserves cartesian arrows is easier. We produce the rest of the symmetric monoidal structure for $\otimes$ by lifting the same structure from $\boxtimes$, using the universal property of cartesian arrows. A more explicit treatment appears in [Mal19]. □

Example 11.2. The functor $\mathbf{S}_B \to \mathbf{S}$ satisfies the statement of the lemma because for any two maps $A \to A'$ and $E \to E'$ of fibrations over B, the following square is homotopy pullback.

$$\begin{tikzcd} A \times_B E \arrow[r] \arrow[d] & A \times E \arrow[d] \\ A' \times_B E' \arrow[r] & A' \times E' \end{tikzcd}$$

- Pulling back $\mathrm{ho}\mathcal{U}$ along $\mathbf{S}_B \to \mathbf{S}$ gives an smbf $\mathrm{ho}\mathcal{U}_{(B)}$ whose objects are pairs of maps $X \to A \to B$ where $A \to B$ is a fibration, and morphisms are maps over B. The product with $Y \to A' \to B$ is the fiber product $X \times_B Y \to A \times_B A' \to B$.
- Pulling back $\mathrm{ho}\mathcal{S}$ along $\mathbf{S}_B \to \mathbf{S}$ gives an smbf $\mathrm{ho}\mathcal{S}_{(B)}$ whose objects are pairs of a fibrations $A \to B$ and a spectrum X over A. Morphisms are a map $A \to A'$ over B and a map of spectra $X \to Y$ over $A \to A'$. The pullback and pushforward are defined as in $\mathrm{ho}\mathcal{S}$, and the smash product is the relative external smash product, given by pulling back $X \overline{\wedge}^{\mathbf{L}} Y$ from $A \times A'$ to the fiber product $A \times_B A'$.
- Both of these generalize to G-spaces and G-spectra, giving $\mathrm{ho}G\mathcal{U}_{(B)}$ and $\mathrm{ho}G\mathcal{S}_{(B)}$. We always assume that $A \to B$ is a Hurewicz fibration whose path-lifting function is G-equivariant.

The bicategory $\mathcal{E}\mathrm{x}_B^{\mathrm{fib}}$ of spectra over fibrations over B, is the bicategory associated to $\mathrm{ho}\mathcal{S}_{(B)}$, compare [MS06, 19.2.6, 19.3.4]. Performing the same operation for G-equivariant Hurewicz fibrations $A \to B$ and G-equivariant spectra gives another bicategory $G\mathcal{E}\mathrm{x}_B^{\mathrm{fib}}$. The following is a corollary of Propositions 9.10 and 9.11, Lemma 11.1, and Theorem 8.4.

Corollary 11.3. *If H is a subgroup of a finite group G, ι_H^* and Φ^H are strong shadow functors on $G\mathcal{E}\mathrm{x}_B^{\mathrm{fib}}$.*

The bicategory $\mathcal{E}\mathrm{x}_B^{\mathrm{fib}}$ has an n-Fuller structure and a system of base-change objects by Theorems 8.4 and 9.9 and Lemma 11.1. In particular there is a pseudofunctor $[]_B\colon \mathbf{S}_B \to \mathcal{U}_B/\mathbf{S}_B$ and coherent isomorphisms

$$m_{[]}\colon \left[Z \xleftarrow{g} Y\right]_B \odot \left[Y \xleftarrow{f} X\right]_B \xrightarrow{\sim} \left[Z \xleftarrow{g\circ f} X\right]_B, \qquad i_{[]}\colon U_X \xrightarrow{\sim} \left[X \xleftarrow{\mathrm{id}} X\right]_B. \tag{11.4}$$

The same applies with G-equivariant spaces as well.

If $p\colon E \to B$ is a perfect fibration, i.e. a (Hurewicz) fibration with finitely dominated fibers, $\left[B \xleftarrow{p} E\right]_B$ is right dualizable as a 1-cell in $\mathcal{E}\mathrm{x}_B^{\mathrm{fib}}$. The same is true equivariantly if B has a trivial action and the fibers of p are equivariantly finitely dominated. Therefore for each commuting square

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

we can define fiberwise versions of the traces from Section 5.

- The **fiberwise Lefschetz number** $L_B(f)$ is the trace of f as a map in the symmetric monoidal category of spectra over B, in other words $\mathcal{E}\mathrm{x}_B^{\mathrm{fib}}(B,B)$. It is a self-map of the fiberwise sphere spectrum $\mathbb{S}_B = \Sigma^\infty_{+B} B$ in the homotopy category of spectra over B.
- The **pretransfer** is the trace of $f \times_B \mathrm{id}\colon E \to E \times_B E$, which is a slight refinement of $L_B(f)$. This gives a map $\mathbb{S}_B \to \Sigma^\infty_{+B} E$. When $f = \mathrm{id}$, this is the Becker-Gottlieb pretransfer [BG76, §5].
- The **fiberwise Reidemeister trace** $R_B(f)$ is the trace of the canonical isomorphism in $\mathcal{E}\mathrm{x}_B^{\mathrm{fib}}$

$$\left[B \xleftarrow{p} E\right]_B \xrightarrow{\sim} \left[B \xleftarrow{p} E\right]_B \odot \left[E \xleftarrow{f} E\right]_B. \tag{11.5}$$

  It gives a map in the homotopy category of spectra over B,

$$R_B(f)\colon \mathbb{S}_B \simeq \Sigma^\infty_{+B}\left\langle\!\!\left\langle \left[B \xleftarrow{=} B\right]_B \right\rangle\!\!\right\rangle \to \Sigma^\infty_{+B}\left\langle\!\!\left\langle \left[E \xleftarrow{f} E\right]_B \right\rangle\!\!\right\rangle \simeq \Sigma^\infty_{+B}\Lambda^f_B E$$

  which is $R(f_b)$ on each fiber.

  The fiberwise Reidemeister trace $R_B(f)$ is a complete obstruction to the removal of fixed points from a family of maps f, provided B is a cell complex of dimension d, and p is a fiber bundle whose fibers X are compact manifolds of dimension at least $d+3$ [KW07].
- The **fiberwise nth Fuller trace** $R_{B,C_n}(\Psi^B_n(f))$ is the trace of the map

$$\left[B \xleftarrow{p^{\times_B n}} E^{\times_B n}\right]_B \xrightarrow{\sim} \left[B \xleftarrow{p^{\times_B n}} E^{\times_B n}\right]_B \odot \left[E^{\times_B n} \xleftarrow{\Psi^B_n(f)} E^{\times_B n}\right]_B$$

  in $C_n\mathcal{E}\mathrm{x}_B^{\mathrm{fib}}$ arising from the commuting square

$$\begin{tikzcd} E^{\times_B n} \arrow[r, "\Psi^B_n(f)"] \arrow[d, "p^{\times_B n}"] & E^{\times_B n} \arrow[d, "p^{\times_B n}"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

  It is a map in the homotopy category of C_n-equivariant spectra over B

$$R_{B,C_n}(\Psi^B_n(f))\colon \mathbb{S}_B \simeq \Sigma^\infty_{+B}\left\langle\!\!\left\langle \left[B \xleftarrow{=} B\right]_B \right\rangle\!\!\right\rangle \to \Sigma^\infty_{+B}\left\langle\!\!\left\langle \left[E^n \xleftarrow{\Psi^B_n(f)} E^n\right]_B \right\rangle\!\!\right\rangle \simeq \Sigma^\infty_{+B}\Lambda^{\Psi^B_n(f)}_B E^{\times_B n}$$

  which is $R_{C_n}(\Psi^n(f_b))$ on each fiber.

We can now state the promised fiberwise version of Theorem 1.1.

Theorem 11.6 (Fiberwise version of Theorem 1.1)**.** *The following diagram commutes up to fiberwise homotopy.*

$$\begin{tikzcd}
 & & & \mathbb{S}_B \arrow[dlll, "R_B(\Psi_n^B(f))^{C_n}"'] \arrow[dl, "R_B(\Psi_k^B(f))^{C_k}"'] \arrow[d, "R_B(\Psi_k^B(f))"] \arrow[dr, "R_B(f^k)"] & \\
(\Sigma^\infty_{+B}\Lambda_B^{\Psi_n^B(f)}E^{\times_B n})^{C_n} \arrow[r, "R"] & (\Sigma^\infty_{+B}\Lambda_B^{\Psi_k^B(f)}E^{\times_B k})^{C_k} \arrow[r, "F"] & \Sigma^\infty_{+B}\Lambda_B^{\Psi_k^B(f)}E^{\times_B k} \arrow[r, "\simeq"] & \Sigma^\infty_{+B}\Lambda_B^{f^k}E
\end{tikzcd}$$

Note these are all maps of fibrations over B that on each fiber capture the simpler maps we constructed earlier.

Proof. The right-hand triangle is just Corollary 7.6 applied to the bicategory $\mathcal{E}\mathrm{x}_B^{\mathrm{fib}}$. The remaining two triangles are proven by restating the proof of Corollary 10.2 in the category of G-spectra over B, and then taking $G = C_n$, $H = C_k$, $Y = X^{\times_B n}$ and $\phi = \Psi_n^B(f)$. □

Our list of fixed point invariants that can be identified using this approach is far from exhaustive. We leave the adaptation of this theorem to the remaining generalizations of $L(f)$ and $R(f)$ to the interested reader.

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