CUBE SUMS OF FORM $3p$ AND $3p^2$ II

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Abstract. Let $p \equiv 2, 5 \mod 9$ be a prime. We prove that both $3p$ and $3p^2$ are cube sums. We also establish some explicit Gross-Zagier formulae and investigate the 3 part full BSD conjecture of the related elliptic curves.

1. Introduction

We call a nonzero rational number a cube sum if it is of the form $a^3 + b^3$ with $a, b \in \mathbb{Q}^\times$. For the history and background about this Diophiantin problem please refer to [DV09][DV18][HSY][SSY]. Up to now, only four family numbers with many prime factors are proved to be cube sums [Sat86][Cow00] and the Sylvester conjecture concerning primes p only has partial result [DV18]. In this paper, we mainly prove the following theorem completing our partial result in [SSY] using a different construction.

Theorem 1.1. *Let $p \equiv 2, 5 \mod 9$ be a prime. Then both $3p$ and $3p^2$ are cube sums.*

Since 2 is a cube sum, from now on, we may assume $p \equiv 2, 5 \mod 9$ is an odd prime number. Let E_n be the elliptic curve given by $x^3 + y^3 = nz^3$. It has the Weierstrass equation $y^2 = x^3 - 432n^2$. If $n > 2$ is not a cube, then $E_n(\mathbb{Q})_{\rm tor} = 0$ and n is a cube sum if and only $E_n(\mathbb{Q})$ has rank at least one. Following [Sat87], we use the Heegner points twisted from a fixed elliptic curve to prove the above theorem.

In second part of this paper, we establish some explicit Gross-Zagier formulae (Theorem 4.2) and use them to investigate the 3-part full BSD conjecture for E_{3p} and E_{3p^2}. More explicitly, let $\text{Ш}(E_n)$, $E_n(\mathbb{Q})_{\rm tor}$, Ω_n, $R(E_n)$ and $c_\ell(E_n)$ denote the Shafarevich-Tate group, the torsion subgroup, the minimal real period, the regulator and the Tamagawa number of E_n over $\mathbb{Q}$ respectively. Then the full BSD conjecture predicts that if $L(s, E)$ is of order r at $s = 1$, then

$$|\text{Ш}(E_n)| = \frac{L^{(r)}(1, E_n)}{\Omega_n \cdot R(E_n)} \cdot \frac{|E_n(\mathbb{Q})_{\rm tor}|^2}{\prod_\ell c_\ell(E_n)}.$$

Let P (resp. Q) be a generator of the free part of $E_{3p^2}(\mathbb{Q})$(resp. $E_{3p}(\mathbb{Q})$). We prove that

Theorem 1.2. *Let $p \equiv 2 \mod 9$ be a rational prime number. Then*

$$|\text{Ш}(E_p)| \cdot |\text{Ш}(E_{3p^2})| = \frac{L(1, E_p)}{\Omega_p \cdot \widehat{h}_{\mathbb{Q}}(P)} \cdot \frac{L'(1, E_{3p^2})}{\Omega_{3p^2}} \cdot \frac{|E_p(\mathbb{Q})_{\rm tor}|^2}{\prod_\ell c_\ell(E_p)} \cdot \frac{|E_{3p^2}(\mathbb{Q})_{\rm tor}|^2}{\prod_\ell c_\ell(E_{3p^2})}, \tag{1.1}$$

up to a power of $2p$.

Let $p \equiv 5 \mod 9$ be a rational prime number. Then

$$|\text{Ш}(E_{p^2})| \cdot |\text{Ш}(E_{3p})| = \frac{L(1, E_{p^2})}{\Omega_{p^2} \cdot \widehat{h}_{\mathbb{Q}}(Q)} \cdot \frac{L'(1, E_{3p})}{\Omega_{3p}} \cdot \frac{|E_{p^2}(\mathbb{Q})_{\rm tor}|^2}{\prod_\ell c_\ell(E_{p^2})} \cdot \frac{|E_{3p}(\mathbb{Q})_{\rm tor}|^2}{\prod_\ell c_\ell(E_{3p})}, \tag{1.2}$$

up to a power of $2p$.

Note that by the work of Perrin-Riou [PR87], Kobayashi [Kob13], the ℓ part full BSD conjecture of E_{3p} and E_{3p^2} is known for $\ell \nmid 6p$. But the prime 3 is very special in the Iwasawa theory for the elliptic curve family $E_D : y^2 = x^3 + D$ whose CM field $K = \mathbb{Q}(\sqrt{-3})$ has 6 roots of unity and 2 is special for all elliptic curves. In particular, there is no any general results about the 2 and 3 part full BSD conjecture of E_D.

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2. Modular Actions on Heegner Points

2.1. **Modular curves and modular actions.** We will use the notations as in [HSY, Section 2] for the related modular curves. Recall $X_0(3^5)$ is the classical modular curve over $\mathbb{Q}$ of level $\Gamma_0(3^5)$. Define N to be the normalizer of $\Gamma_0(3^5)$ in $\mathrm{GL}_2^+(\mathbb{Q})$. Then the linear fractional transformation action of N on $X_0(3^5)$ induces an isomorphism

$$N/\mathbb{Q}^\times\Gamma_0(3^5) \simeq \mathrm{Aut}_{\overline{\mathbb{Q}}}(X_0(3^5)).$$

The quotient group $N/\mathbb{Q}^\times\Gamma_0(3^5) \simeq S_3 \rtimes \mathbb{Z}/3\mathbb{Z}$, where S_3 denotes the symmetric group with 3 letters which is generated by the Atkin-Lehner operator $W = \begin{pmatrix} 0 & 1 \\ -3^5 & 0 \end{pmatrix}$ and the matrix $A = \begin{pmatrix} 28 & 1/3 \\ 3^4 & 1 \end{pmatrix}$, and the subgroup $\mathbb{Z}/3\mathbb{Z}$ is generated by the matrix $B = \begin{pmatrix} 1 & 0 \\ 3^4 & 1 \end{pmatrix}$.

Put

$$U = \langle U_0(3^5), W, A\rangle \subset \mathrm{GL}_2(\mathbb{A}_f),$$

and

$$\Gamma = \mathrm{GL}_2(\mathbb{Q})^+ \cap U = \langle \Gamma_0(3^5), W, A\rangle,$$

and let X_Γ be the modular curve over $\mathbb{Q}$ of level Γ whose underlying Riemann surface is

$$X_\Gamma(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})^+\backslash \left(\mathcal{H} \sqcup \mathbb{P}^1(\mathbb{Q})\right) \times \mathrm{GL}_2(\mathbb{A}_f)/U.$$

So the curve is the quotient of $X_0(3^5)$ by the actions of W and A. Then X_Γ is a smooth projective curve over $\mathbb{Q}$ of genus 1, and the infinity cusp $[\infty]$ is rational over $\mathbb{Q}$. We identify X_Γ with an elliptic curve over $\mathbb{Q}$ with $[\infty]$ as its zero element [HSY, Proposition 2.1]. Let N_Γ be the normalizer of Γ in $\mathrm{GL}_2(\mathbb{Q})^+$. Then we have a natural embedding

$$\Phi : N_\Gamma/\mathbb{Q}^\times\Gamma \hookrightarrow \mathrm{Aut}_{\overline{\mathbb{Q}}}(X_\Gamma) \simeq \mathcal{O}_K^\times \ltimes X_\Gamma(\overline{\mathbb{Q}}),$$

where $\mathcal{O}_K^\times$ embeds into $\mathrm{Aut}_{\overline{\mathbb{Q}}}(X_\Gamma)$ by complex multiplications and $X_\Gamma(\overline{\mathbb{Q}})$ embeds into $\mathrm{Aut}_{\overline{\mathbb{Q}}}(X_\Gamma)$ by translations. The matrices

$$B = \begin{pmatrix} 1 & 0 \\ 3^4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1/9 \\ -3^3 & -2 \end{pmatrix}$$

lie in N_Γ, and hence induce automorphisms of X_Γ.

The elliptic curves E_n are all endowed with complex multiplication by K and we fix the complex multiplication $[\cdot] : \mathcal{O}_K \simeq \mathrm{End}_K(E_n)$ by $[-\omega](x,y) = (\omega x, -y)$. We will always take the simple Weierstrass equation $y^2 = x^3 - 2^4 \cdot 3$ for the elliptic curve E_9. We quote [HSY, Proposition 2.1] as follows.

Proposition 2.1. *The elliptic curve $(X_\Gamma, [\infty])$ is isomorphic to E_9 over $\mathbb{Q}$. Moreover, for any point $P \in X_\Gamma$, we have*

$$\Phi(B)(P) = [\omega^2]P, \quad \Phi(C)(P) = [\omega^2]P + (0, 4\sqrt{-3}).$$

In particular, the automorphisms $\Phi(B)$ and $\Phi(C)$ are defined over K.

Note that there exists a unique isomorphism $X_\Gamma \to E_9$ over $\mathbb{Q}$ such that the cusp $[1/9]$ has coordinates $(0, 4\sqrt{-3})$. We use this isomorphism to identify X_Γ with E_9.

Let $V \subset U_0(3^5)$ be the subgroup consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \equiv d \bmod 3$, and put $U_0 = \langle V, W, A\rangle$. Let X_Γ^0 be the modular curve over $\mathbb{Q}$ whose underlying Riemann surface is

$$X_\Gamma^0(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})^+\backslash \left(\mathcal{H} \bigsqcup \mathbb{P}^1(\mathbb{Q})\right) \times \mathrm{GL}_2(\mathbb{A}_f)/U_0.$$

The modular curve X_Γ^0 is isomorphic to $X_\Gamma \times_\mathbb{Q} K$ as a curve over $\mathbb{Q}$. Usually, we denote by $[z,g]_{U_0}$ the point on X_Γ^0 which is represented by the pair (z,g) where $z \in \mathcal{H}$ and $g \in \mathrm{GL}_2(\mathbb{A}_f)$. Let $N_{\mathrm{GL}_2(\mathbb{A}_f)}(U_0)$ be the normalizer of U_0 in $\mathrm{GL}_2(\mathbb{A}_f)$. Then there is a natural homomorphism

$$N_{\mathrm{GL}_2(\mathbb{A}_f)}(U_0)/U_0 \longrightarrow \mathrm{Aut}_\mathbb{Q}(X_\Gamma^0)$$

induced by right translation on X_Γ^0: for $P = [z,g]_{U_0} \in X_\Gamma^0$ and $x \in N_{\mathrm{GL}_2(\mathbb{A}_f)}(U_0)$

$$P \mapsto P^x = [z, gx]_{U_0}.$$

2.2. **Modular actions on Heegner points.** Let $p \equiv 2, 5 \bmod 9$ be an odd prime number. Denote
$$\tau_i = M_i \omega \in \mathcal{H},$$
where
$$M_1 = \begin{pmatrix} p/9 & 0 \\ 2 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} p/9 & 0 \\ 5 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} p/9 & 0 \\ 2 & 4 \end{pmatrix}, \quad \omega = \frac{-1+\sqrt{-3}}{2}.$$
For $i = 1, 2, 3$, let $\rho_i : K \hookrightarrow \mathrm{M}_2(\mathbb{Q})$ be the normalised embedding (see [CST17, HSY]) with fixed point $\tau_i \in \mathcal{H}$. Then ρ_i are explicitly given by
$$\rho_1(\omega) = M_1 \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} M_1^{-1} = \begin{pmatrix} 1 & -p/9 \\ 27/p & -2 \end{pmatrix}.$$
$$\rho_2(\omega) = M_2 \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} M_2^{-1} = \begin{pmatrix} 4 & -p/9 \\ 187/p & -5 \end{pmatrix}.$$
$$\rho_3(\omega) = M_3 \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} M_3^{-1} = \begin{pmatrix} -1/2 & -p/36 \\ 27/p & -1/2 \end{pmatrix},$$
Let $R_0(3^5)$ bet the standard Eichler order of discriminant 3^5 in $\mathrm{M}_2(\mathbb{Q})$. Then $\rho_1(K) \cap R_0(3^5) = \mathcal{O}_{9p}$, $\rho_2(K) \cap R_0(3^5) = \mathcal{O}_{9p}$ and $\rho_3(K) \cap R_0(3^5) = \mathcal{O}_{36p}$. So τ_1, τ_2 is defined over H_{9p} and τ_3 is defined over H_{36p} by the complex multiplication theory. Here H_m is the ring class field of K with conductor m.

Remark 2.1. In order to prove Theorem 1.1, we just need τ_1 and τ_2. But we do not how to get rational points over $\mathbb{Q}$ from τ_1 and τ_2 since we do not know how the complex conjugation acts on them. In order to prove Theorem 1.2, we need the help of τ_3 which shares the same Galois action with τ_2 but will give us real point directly. This is also the reason why only prove half cases in Theorem 1.2.

Let $\mathcal{O}_{K,3}$ be the completion of $\mathcal{O}_K$ at the unique place above 3. Let $\mathcal{O}_{K,3}$ be the completion of $\mathcal{O}_K$ at the unique place above 3. We have
$$\mathcal{O}_{K,3}^\times / \mathbb{Z}_3^\times (1 + 9\mathcal{O}_{K,3}) = \langle \omega_3 \rangle \times \langle 1 + 3\omega_3 \rangle \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z},$$
where ω_3 is the image of ω into $\mathcal{O}_{K,3}^\times$. Under both the embeddings ρ_1 and ρ_2, it is straightforward to verify that ω_3 and $1 + 3\omega_3$ normalize U_0, and therefore they induce automorphisms of X_Γ^0.

Theorem 2.2. *Let P be an arbitrary point on X_Γ^0.*

1. *Under the embedding ρ_1, we have*
$$P^{1+3\omega_3} = [\omega]P, \ \textit{and} \ P^{\omega_3} = [\omega]P + (0, -4\sqrt{-3}).$$
2. *Under the embedding ρ_2 and ρ_3, we have*
$$P^{1+3\omega_3} = [\omega]P, \ \textit{and} \ P^{\omega_3} = [\omega^2]P + (0, -4\sqrt{-3}).$$

Proof. We give the proof of the first assertion in details and the case under the embedding ρ_2 is similar. Now suppose K is embedded in $\mathrm{M}_2(\mathbb{Q})$ under ρ_1. Since ω_3 and $1 + 3\omega_3$ have determinants $\equiv 1 \bmod 3$, as elements in $\mathrm{Aut}_{\mathbb{Q}}(X_\Gamma^0)$, they lie in the subgroup $\mathrm{Aut}_K(X_\Gamma)$. See [HSY, page 6] for the structure of the automorphism groups. Suppose $P = [z, 1]$, $z \in \mathcal{H}$, be a point on X_Γ^0. We have
$$A^2B^2(1 + 3\omega_3) = \left(\begin{pmatrix} 783/p + 9508 & -2377p/3 - 145/3 \\ 2268/p + 27540 & -2295p - 140 \end{pmatrix}_3, A^2B^2 \right) \in V,$$
where the subscript 3 denotes the 3-adic component of the adelic matrices. Then by Proposition 2.1,
$$P^{1+3\omega_3} = \Phi(B^2)(P) = [\omega]P.$$
Similarly, if $p \equiv 2 \bmod 9$, then $AC^2\omega_3 \in V$, and hence
$$P^{\omega_3} = \Phi(C^2)(P) = [\omega]P + (0, -4\sqrt{-3}).$$
If $p \equiv 5 \bmod 9$, then $A^2C^2\omega_3 \in V$, and hence
$$P^{\omega_3} = \Phi(C^2)(P) = [\omega]P + (0, -4\sqrt{-3}).$$

For the case under embbedding ρ_2 and ρ_3, it is straight to verify that $A^2B^2(1 + 3\omega_3) \in V$ for any odd prime $p \equiv 2, 5 \bmod 9$, and $AB^2C^2\omega_3 V$, when $p \equiv 2 \bmod 9$, and $A^2B^2C^2\omega_3 \in V$ when $p \equiv 5 \bmod 9$. Then the second assertion follows from Proposition 2.1.

□

2.3. **Galois actions on Heegner points.** Fix the Artin reciprocity law $\sigma : \widehat{K}^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ by sending local uniformizers to Frobenius automorphisms. Denote by σ_t the image of $t \in \widehat{K}^\times$. Let $P_i = [\tau_i, 1]_{U_0}$ be the CM points on X_Γ^0 for $i = 1, 2, 3$. In the following, when we consider the CM point P_i, we assume K is embedded in $\mathrm{M}_2(\mathbb{Q})$ under ρ_i.

Theorem 2.3. *For $i = 1, 2$, the point $P_i \in X_\Gamma^0(H_{9p})$ satisfies*

$$P_i^{\sigma_{1+3\omega_3}} = [\omega]P_i, \ \text{and } P_i^{\sigma_{\omega_3}} = [\omega^i]P_i + (0, -4\sqrt{-3}).$$

Similarly, $P_3 \in X_\Gamma^0(H_{36p})$ satiesfies

$$P_3^{\sigma_{1+3\omega_3}} = [\omega]P_3, \ \text{and } P_3^{\sigma_{\omega_3}} = [\omega^2]P_3 + (0, -4\sqrt{-3}).$$

Proof. By Shimura's reciprocity law [Shi94, Theorems 6.31 and 6.38], we have

$$P_i^{\sigma_t} = P_i^t = [\tau_i, t], \quad t \in \widehat{K}^\times.$$

Since $\widehat{K}^\times \cap U_0 = \widehat{\mathcal{O}_{9p}}^\times$, by class field theory, we see P_i is defined over the ring class field H_{9p}, and the Galois actions of σ_{ω_3} and $\sigma_{1+3\omega_3}$ are clear from Theorem 2.2. The proof for P_3 is similar. ☐

Remark 2.2. Since $\tau_3 = p\omega/9(2\omega+4) = p\sqrt{-3}/54$, $e^{2\pi i\tau_2/n}$ is real. So P_3 is in fact a real point on E_9.

3. Nontriviality of Heegner points

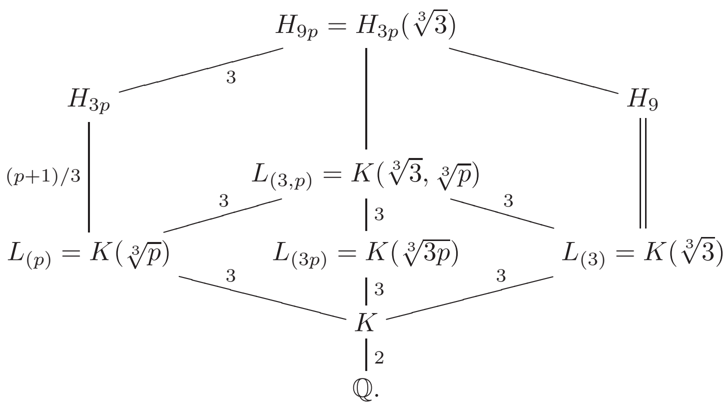
The elliptic curve E_3 has Weierstrass equation $y^2 = x^3 - 2^4 3^5$. Consider the isomorphism

$$\phi : E_9 \longrightarrow E_3, \quad (x, y) \mapsto (9x/\sqrt[3]{9}, 9y).$$

We have the following commutative diagram:

$$\begin{array}{ccc} E_9(H_{9p})^{\sigma_{1+3\omega_3}=\omega^2} & \xrightarrow{\mathrm{Tr}_{H_{9p}/L_{(3,p)}}} & E_9(L_{(3,p)})^{\sigma_{1+3\omega_3}=\omega^2} \\ \downarrow \phi & & \downarrow \phi \\ E_3(H_{3p}) & \xrightarrow{\mathrm{Tr}_{H_{3p}/L_{(p)}}} & E_3(L_{(p)}) \end{array}$$

where the field extension diagram is as follows (H_m is the ring class field of K with conductor m):



The following proposition on related field extensions is partially quoted from [SSY, Proposition 2.6].

Proposition 3.1. *Let $p \equiv 2, 5 \bmod 9$ be odd primes.*

1. *The field $H_{9p} = H_{3p}(\sqrt[3]{3})$ with Galois group $\mathrm{Gal}(H_{9p}/H_{3p}) \simeq \langle\sigma_{1+3\omega_3}\rangle^{\mathbb{Z}/3\mathbb{Z}}$, and*

$$\left(\sqrt[3]{3}\right)^{\sigma_{1+3\omega_3}-1} = \omega^2.$$

2. *We have $\left(\sqrt[3]{3}\right)^{\sigma_{\omega_3}-1} = 1$ and*

$$(\sqrt[3]{p})^{\sigma_{\omega_3}-1} = \begin{cases} \omega, & p \equiv 2 \bmod 9; \\ \omega^2, & p \equiv 5 \bmod 9. \end{cases}$$

3. *$[H_{36p} : H_{9p}] = 6$ and $H_{36p} = H_{9p}(\sqrt{-1}, \sqrt[3]{2})$ with Galois group*

$$\mathrm{Gal}(H_{36p}/H_{9p}) \simeq \langle\sigma_{1+2\omega_2}\rangle^{\mathbb{Z}/2\mathbb{Z}} \times \langle\sigma_{\omega_2}\rangle^{\mathbb{Z}/3\mathbb{Z}}.$$

Proof. (1) and (2) are contained in [SSY, Proposition 2.6]. We just need to prove (3), but the argument is similar to the proof of (1). Note that $\mathcal{O}_m = \mathbb{Z} + m\mathcal{O}_K$,

$$\mathrm{Gal}(H_{36p}/H_{9p}) \simeq K^\times \widehat{\mathcal{O}_{9p}}^\times / K^\times \widehat{\mathcal{O}_{36p}}^\times \simeq \widehat{\mathcal{O}_{9p}}^\times / (\widehat{\mathcal{O}_{9p}}^\times \cap K^\times \widehat{\mathcal{O}_{36p}}^\times) \simeq \mathcal{O}_{K,2}^\times / \mathbb{Z}_2^\times (1 + 4\mathcal{O}_{K,2})$$

is of order 6 and generated by ω_2 and $1 + 2\omega_2$. The ideal $\sqrt{-3}\mathcal{O}_K = (1 + 2\omega)$ and let v be the place corresponding to the prime ideal $(1 + 3\omega)$. Then by the local-global principle, we have

$$(\sqrt{-1})^{\sigma_{1+2\omega_2} - 1} = \left(\frac{1 + 2\omega_2, -1}{K_2; 2} \right) = \left(\frac{1 + 2\omega_v, -1}{K_v; 2} \right)^{-1} = (-1)^{-(3-1)/2} \bmod (1 + 2\omega) = -1,$$

where $\left(\frac{\cdot, \cdot}{K_w; 2} \right)$ denotes the second Hilbert symbol over K_w. Similarly,

$$\left(\sqrt[3]{2} \right)^{\sigma_{\omega_2} - 1} = \left(\frac{\omega_2, 2}{K_2; 3} \right) = \omega^{-(4-1)/3} \bmod 2 = \omega^2.$$

□

By Proposition 3.1 and 2.3, $\phi(P_1), \phi(P_2) \in E_3(H_{3p})$. Let

$$z_1 = \mathrm{Tr}_{H_{3p}/L_{(p)}} \phi(P_1), \text{ and } z_2 = \mathrm{Tr}_{H_{3p}/L_{(p)}} \phi(P_2), \tag{3.1}$$

then $z_1, z_2 \in E_3(L_{(p)})$.

Theorem 3.2. *Both z_1 and z_2 are nontorsion.*

Proof. By Theorem 2.3,

$$z_i^{\sigma_{\omega_3}} = [\omega^i] z_i + \frac{p+1}{3}(0, 36\sqrt{-3}) \tag{3.2}$$

for $i = 1, 2$. By [SSY, Proposition 2.3], the torsion points in $E_3(L_{(p)})$ are $O, (0, \pm 36\sqrt{-3})$ which can not satisfy (3.2). □

Let $\phi_p : E_3 \to E_{3p}$ and $\phi_{p^2} : E_3 \to E_{3p^2}$ be the map given by $(x, y) \mapsto (\sqrt[3]{p}x, py)$ and $(x, y) \mapsto (\sqrt[3]{p^2}x, p^2 y)$. Set

$$y_i = [\sqrt{-3}] z_i \in E_3(L_{(p)}).$$

By (3.2), we know that $(y_i)^{\sigma_{\omega_3}} = [\omega^i](y_i)$.

Proof of Theorem 1.1. By Proposition 3.1 and Theorem 3.2, if $p \equiv 2 \bmod 9$, then $\phi_p(y_2)$ is a nontorsion point in $E_{3p}(K)$ and $\phi_{p^2}(y_1)$ is a nontorsion point in $E_{3p^2}(K)$; if $p \equiv 5 \bmod 9$, then $\phi_p(y_1)$ is a nontorsion point in $E_{3p}(K)$ and $\phi_{p^2}(y_2)$ is a nontorsion point in $E_{3p^2}(K)$. Since $E_{3p}(\mathbb{Q})$ has the same rank with $E_{3p}(K)$ and $E_{3p^2}(\mathbb{Q})$ has the same rank with $E_{3p^2}(K)$, we finish the proof. □

Define

$$E_3(L_{(p)})^{\sigma_{\omega_3} = [\omega^i]} := \{ P \in E_3(L_{(p)}) : P^{\sigma_{\omega_3}} = [\omega^i] P \}.$$

Theorem 3.3. *The point y_i are not divisible by $\sqrt{-3}$ in $E_3(L_{(p)})^{\sigma_{\omega_3} = [\omega^i]}$.*

Proof. Assume the contrary that $y_i = \sqrt{-3}Q + T$ with

$$Q \in E_3(L_{(p)})^{\sigma_{\omega_3} = [\omega^i]} \text{ and } T \in E_3(L_{(p)})_{\mathrm{tor}}^{\sigma_{\omega_3} = [\omega^i]}.$$

Then

$$\sqrt{-3}(z_i - Q) = T.$$

Since $E_3(L_{(p)})_{\mathrm{tor}} = E_3[\sqrt{-3}]$, we conclude that $z_i - Q \in E_3[\sqrt{-3}]$. Suppose $z_i = Q + R$ with $R \in E_3[\sqrt{-3}]$. Taking Galois action of σ_{ω_3}, we obtain

$$0 = [1 - \omega^i] R = (0, \pm 36\sqrt{-3}),$$

which is a contradiction. □

Remark 3.1. As is seen if $p \equiv 2 \bmod 9$ resp. $p \equiv 5 \bmod 9$, y_1 may be identified with a point of infinite order in $E_{3p}(K)$ resp. $E_{3p^2}(K)$; and if $p \equiv 2 \bmod 9$ resp. $p \equiv 5 \bmod 9$, y_2 may be identified with a point of infinite order in $E_{3p^2}(K)$ resp. $E_{3p}(K)$.

4. The explicit Gross-Zagier formulae

In the rest of the paper we clarify the embedding of K into $\mathrm{M}_2(\mathbb{Q})$ as follows. As indicated in the proof of Theorem 1.1, in the case $p \equiv 2 \bmod 9$ and $\chi = \chi_{3p^2}$ or the case $p \equiv 5 \bmod 9$ and $\chi = \chi_{3p}$, we use the Heegner point z_1 to construct nontrivial points on elliptic curves, and hence we embed K into $\mathrm{M}_2(\mathbb{Q})$ under ρ_1. Otherwise, in the case $p \equiv 2 \bmod 9$ and $\chi = \chi_{3p}$ or the case $p \equiv 5 \bmod 9$ and $\chi = \chi_{3p^2}$, we use the Heegner point z_2, and we embed K into $\mathrm{M}_2(\mathbb{Q})$ under ρ_2.

4.1. **The explicit Gross-Zagier formulae.** Let π be the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ corresponding to $E_{9/\mathbb{Q}}$. Then π is only ramified at 3 with conductor 3^5. For $n \in \mathbb{Q}^\times$, let $\chi_n : \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathbb{C}^\times$ be the cubic character given by $\chi_n(\sigma) = (\sqrt[3]{n})^{\sigma-1}$. Define

$$L(s, E_9, \chi_n) := L(s - 1/2, \pi_K \otimes \chi_n), \quad \epsilon(E_9, \chi_n) := \epsilon(1/2, \pi_K \otimes \chi_n),$$

where π_K is the base change of π to $\mathrm{GL}_2(\mathbb{A}_K)$.

Let $p \equiv 2, 5 \bmod 9$ be an odd prime number, and put $\chi = \chi_{3p}$ resp. χ_{3p^2}. From the Artin formalism, we have

$$L(s, E_9, \chi) = L(s, E_p)L(s, E_{3p^2}) \text{ resp. } L(s, E_{p^2})L(s, E_{3p}).$$

By [Liv95], we have the epsilon factors $\epsilon(E_{3p^2})(\text{ resp. } \epsilon(E_{3p})) = -1$ and $\epsilon(E_p)(\text{ resp. } \epsilon(E_{p^2})) = +1$, and hence the epsilon factor

$$\epsilon(E_9, \chi) = \epsilon(E_p)\epsilon(E_{3p^2}) \text{ resp. } \epsilon(E_{p^2})\epsilon(E_{3p}) = -1.$$

For a quaternion algebra $\mathbb{B}_\mathbb{A}$, we define its ramification index $\epsilon(\mathbb{B}_v) = +1$ for any place v of $\mathbb{Q}$ if the local component $\mathbb{B}_v$ is split and $\epsilon(\mathbb{B}_v) = -1$ otherwise. The following proposition guarentees we are in the same setting as in [HSY, Theorem 4.3].

Proposition 4.1. *The incoherent quaternion algebra $\mathbb{B}$ over $\mathbb{A}$, which satisfies*

$$\epsilon(1/2, \pi_{K,v} \otimes \chi_v) = \chi_v(-1)\epsilon_v(\mathbb{B})$$

for all places v of $\mathbb{Q}$, is only ramified at the infinity place.

Proof. Since π is unramified at finite places $v \nmid 3$, χ is unramified at finite places $v \nmid 3p$ and p is inert in K, by [Gro88, Proposition 6.3] we get $\epsilon(1/2, \pi_{K,v} \otimes \chi_v) = +1$ for all finite $v \neq 3$. Again by [Gro88, Proposition 6.5], we also know that $\epsilon(1/2, \pi_{K,\infty} \otimes \chi_\infty) = -1$. Since $\epsilon(1/2, \pi_K \otimes \chi) = -1$, we see that $\epsilon(1/2, \pi_{K,3} \otimes \chi_3) = +1$. Since χ is a cubic character, $\chi_v(-1) = 1$ for any v. Hence $\mathbb{B}$ is only ramified at the infinity place. □

Recall we have defined the Heegner points z_1, z_2 in (3.1). We also define

$$z_3 = \mathrm{Tr}_{H_{3p}/L_{(p)}} \phi(\mathrm{Tr}_{H_{36p}/H_{9p}} P_3). \tag{4.1}$$

Since we use the same elliptic curve E_9 as in [HSY, Theorem 4.3], very little modification of the proof gives us the following explicit Gross-Zagier formulae once we verity the explicit local computation of toric integrals in Corollary 4.10.

Theorem 4.2. *One has the following explicit formulae of Heegner points*

$$\frac{L(1, E_p)L'(1, E_{3p^2})}{\Omega_p \Omega_{3p^2}} = \begin{cases} 2^{-1} \cdot 9 \cdot \widehat{h}_\mathbb{Q}(z_2), & \text{if } p \equiv 2 \bmod 9; \\ 9 \cdot \widehat{h}_\mathbb{Q}(z_1), & \text{if } p \equiv 5 \bmod 9. \end{cases}$$

And

$$\frac{L(1, E_{p^2})L'(1, E_{3p})}{\Omega_{p^2} \Omega_{3p}} = \begin{cases} 9 \cdot \widehat{h}_\mathbb{Q}(z_1), & \text{if } p \equiv 2 \bmod 9; \\ 2^{-1} \cdot 9 \cdot \widehat{h}_\mathbb{Q}(z_2), & \text{if } p \equiv 5 \bmod 9. \end{cases}$$

Theorem 4.3. *One also has the following explicit formulae of Heegner points*

$$\frac{L(1, E_p)L'(1, E_{3p^2})}{\Omega_p \Omega_{3p^2}} = 2^{-3} \cdot \widehat{h}_\mathbb{Q}(z_3), \quad \text{if } p \equiv 2 \bmod 9$$

And

$$\frac{L(1, E_{p^2})L'(1, E_{3p})}{\Omega_{p^2} \Omega_{3p}} = 2^{-3} \cdot \widehat{h}_\mathbb{Q}(z_3), \quad \text{if } p \equiv 5 \bmod 9.$$

Remark 4.1. The defference between the formulae of z_2 and z_3 is due to the fact that we take a trace from H_{36p} to H_{9p} for z_3 which is of degree 6. More explictely, we use $\frac{\#\mathrm{Pic}(\mathcal{O}_p)}{\#\mathrm{Pic}(\mathcal{O}_{36p})}$ instead of $\frac{\#\mathrm{Pic}(\mathcal{O}_p)}{\#\mathrm{Pic}(\mathcal{O}_{9p})}$ for z_3 when we modify the proof of [HSY, Theorem 4.3].

Corollary 4.4. *$z_3 \in E_3(L(p))$ is nontorsion and satisfies $z_3^{\sigma_{\omega_3}} = [\omega^2]z_3$. If $p \equiv 2 \bmod 9$, then $\phi_p(z_3)$ is a nontorsion point in $E_{3p}(\mathbb{Q})$. If $p \equiv 5 \bmod 9$, then $\phi_p^2(z_3)$ is a nontorsion point in $E_{3p^2}(\mathbb{Q})$. In both cases, $\phi_p(z_3)$ and $\phi_{p^2}(z_2)$ are not divisible by 3 over $\mathbb{Q}$.*

Proof. Since $[H_{36p} : H_{9p}] = 6$, by Theorem 2.3 and (4.1), we know that $z_3^{\sigma_{\omega_3}} = [\omega^2]z_3$. Since z_3 is a real point by Remark 2.2, by Proposition 3.1, if $p \equiv 2 \bmod 9$, $\phi_p(z_3) \in E_{3p}(\mathbb{Q})$ and if $p \equiv 5 \bmod 9$, $\phi_{p^2}(z_3) \in E_{3p^2}(\mathbb{Q})$. By Theorem 4.2 and Theorem 4.3, $\widehat{h}_{\mathbb{Q}}(z_3) = 36\widehat{h}_{\mathbb{Q}}(z_2) = 12\widehat{h}_{\mathbb{Q}}(y_2)$. This implies z_3 is nontorsion and $\phi_p(z_3)$, $\phi_{p^2}(z_3)$ can not be divisible by 3 over $\mathbb{Q}$. Otherwise there exists point z in $E_3(L(p))^{\sigma_{\omega_3}=[\omega^2]}$ such that $9\widehat{h}_{\mathbb{Q}}(z) = \widehat{h}_{\mathbb{Q}}(z_3) = 12\widehat{h}_{\mathbb{Q}}(y_2)$. But this is impossible since y_2 is not divisible by $\sqrt{-3}$ in $E_3(L_{(p)})^{\sigma_{\omega_3}=[\omega^2]}$ and $E_3(L(p))^{\sigma_{\omega_3}=[\omega^2]}$ is of rank 1 over K by Kolyvagin. □

4.2. **Waldspurger's local period integral.** This subsection is purely local and we shall compute the 3-adic period integral for the 3-adic local newform following [HSY19]. Recall π is the automorphic representation of $\mathrm{GL}_2(\mathbb{Q})$ corresponding to E_9 and π_3 the 3-adic part of π. Then the conductor $c(\pi_3) = 3^5$. Let $p \equiv 2, 5 \bmod 9$ be an odd prime and let $\chi : \mathrm{Gal}(\bar{K}/K) \to \mathcal{O}_K^\times$ be the character given by $\chi(\sigma) = \chi_{3p}(\sigma) = (\sqrt[3]{3p})^{\sigma-1}$ resp. $\chi(\sigma) = \chi_{3p^2}(\sigma) = (\sqrt[3]{3p^2})^{\sigma-1}$. We also view χ as a Hecke character on $\mathbb{A}_K^\times$ by the Artin map and the conductor the 3 part is $c(\chi_3) = (\sqrt{-3})^4$. Assume that f_3 is the standard newform of π_3. We shall compute the following normalized period integral

$$\beta_3^0(f_3, f_3) = \int_{t \in \mathbb{Q}_3^\times \backslash K_3^\times} \frac{(\pi(t)f_3, f_3)}{(f_3, f_3)} \chi_3(t)dt \tag{4.2}$$

which appears in the proof of the explicit Gross-Zagier formulae. Let $\Theta : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ be the unitary Hecke character associated to the base-changed CM elliptic curve $E_{9/K}$. Then Θ has conductor $9\mathcal{O}_K$. We denote Θ_3 the 3-adic component of Θ. Then π_3 is the local representation of $\mathrm{GL}_2(\mathbb{Q}_3)$ corresponding to Θ_3. Note

$$\mathcal{O}_{K,3}^\times/(1 + 9\mathcal{O}_{K,3}) \simeq \langle \pm 1 \rangle^{\mathbb{Z}/2\mathbb{Z}} \times \langle 1 + \sqrt{-3} \rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1 - \sqrt{-3} \rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1 + 3\sqrt{-3} \rangle^{\mathbb{Z}/3\mathbb{Z}}.$$

Lemma 4.5. *We have $c(\Theta_3) = 4$, and Θ_3 is given explicitly by*

$$\Theta_3(-1) = -1, \quad \Theta_3(1 + \sqrt{-3}) = \frac{-1 - \sqrt{-3}}{2}, \quad \Theta_3(\sqrt{-3}) = i,$$
$$\Theta_3(1 - \sqrt{-3}) = \frac{-1 + \sqrt{-3}}{2}, \quad \Theta_3(1 + 3\sqrt{-3}) = \frac{-1 + \sqrt{-3}}{2}.$$

Proof. This is [HSY19, Lemma 4.1]. □

The local character χ_3 has conductor $\mathbb{Z}_3^\times(1+9\mathcal{O}_{K,3})$, and hence it is in fact a character of the quotient group $K_3^\times/\mathbb{Q}_3^\times(1 + 9\mathcal{O}_{K,3})$. Note that

$$K_3^\times/\mathbb{Q}_3^\times(1 + 9\mathcal{O}_{K,3}) \simeq \langle \sqrt{-3} \rangle^{\mathbb{Z}} \times \langle 1 + \sqrt{-3} \rangle^{\mathbb{Z}/3\mathbb{Z}} \times \langle 1 + 3\sqrt{-3} \rangle^{\mathbb{Z}/3\mathbb{Z}}.$$

Lemma 4.6. *We have $c(\chi_3) = 4$ and χ_3 is given explicitly by the following tables:*

1. *if $\chi = \chi_{3p}$, then*

$p \bmod 9$	$\chi_3(1 + \sqrt{-3})$	$\chi_3(1 + 3\sqrt{-3})$	$\chi_3(\sqrt{-3})$
2	ω^2	ω	1
5	ω	ω	1

2. *if $\chi = \chi_{3p^2}$, then*

$p \bmod 9$	$\chi_3(1 + \sqrt{-3})$	$\chi_3(1 + 3\sqrt{-3})$	$\chi_3(\sqrt{-3})$
2	ω	ω	1
5	ω^2	ω	1

Proof. The proof is routine in class-field theory. See [HSY19, Lemma 4.2] for more details. □

Corollary 4.7. *If $p \equiv 2$ resp. $5 \bmod 9$, and $\chi = \chi_{3p}$ resp. χ_{3p^2}, then the local character $\Theta_3\overline{\chi}_3$ is given explicitly by*

$$\Theta_3\overline{\chi}_3(-1) = -1, \quad \Theta_3\overline{\chi}_3(1 + \sqrt{-3}) = 1,$$
$$\Theta_3\overline{\chi}_3(1 - \sqrt{-3}) = 1, \quad \Theta_3\overline{\chi}_3(1 + 3\sqrt{-3}) = 1, \quad \Theta_3\overline{\chi}_3(\sqrt{-3}) = i.$$

If $p \equiv 2$ resp. $5 \bmod 9$, and $\chi = \chi_{3p^2}$ resp. χ_{3p}, the local character $\Theta_3\overline{\chi}_3$ is given explicitly by

$$\Theta_3\overline{\chi}_3(-1) = -1, \quad \Theta_3\overline{\chi}_3(1+\sqrt{-3}) = \omega,$$
$$\Theta_3\overline{\chi}_3(1-\sqrt{-3}) = \omega^2, \quad \Theta_3\overline{\chi}_3(1+3\sqrt{-3}) = 1, \quad \Theta_3\overline{\chi}_3(\sqrt{-3}) = i.$$

Let θ_3 be the 3-adic character which parametrizes the supercuspidal representation π_3 via compact-induction construction as in [HSY19, Section 2.2]. The test vector issue for Waldspurger's period integral is closely related to $c(\theta_3\overline{\chi}_3)$ or $c(\theta_3\chi_3)$. We can work out these by using Lemma 4.5, 4.6 and Corollary 4.7, and the relation between θ_3 and Θ_3 in [HSY19, Theorem 2.8]. Now we can prove the following key Lemma.

Lemma 4.8. *If $p \equiv 2$ resp. $5 \bmod 9$, and $\chi = \chi_{3p}$ resp. χ_{3p^2}, we have $\theta_3\overline{\chi}_3 = 1$. If $p \equiv 2$ resp. $5 \bmod 9$, and $\chi = \chi_{3p^2}$ resp. χ_{3p}, we have $c(\theta_3\overline{\chi}_3) = 2$ and $\alpha_{\theta_3\overline{\chi}_3} = \frac{1}{3\sqrt{-3}}$. Moreover, in any cases, $c(\theta_3\overline{\chi}_3) \le c(\theta_3\chi_3)$.*

Proof. Let ψ_3 be the additive character such that $\psi_3(x) = e^{2\pi i\iota(x)}$ where $\iota : \mathbb{Q}_3 \to \mathbb{Q}_3/\mathbb{Z}_3 \subset \mathbb{Q}/\mathbb{Z}$ is the map given by $x \mapsto -x \bmod \mathbb{Z}_3$ which is compatible with the choice in [CST14]. Let $\psi_{K_3}(x) = \psi_3 \circ \mathrm{Tr}_{K_3/\mathbb{Q}_3}(x)$, be the additive character of K_3.

Recall that α_{Θ_3} is the number associated to Θ_3 as in [HSY19, Lemma 2.1] so that

$$\Theta_3(1+x) = \psi_{K_3}(\alpha_{\Theta_3}x),$$

for any x satisfying $v_{K_3}(x) \ge c(\Theta_3)/2 = 2$. By the definition of ψ_{K_3} and Lemma 4.5, we know that $\alpha_{\Theta_3} = \frac{1}{9\sqrt{-3}}$. Now let η_3 be the quadratic character associated to the quadratic field extension $K_3/\mathbb{Q}_3$. Then by [BH06, Proposition 34.3], $\lambda_{K_3/\mathbb{Q}_3}(\psi') = \tau(\eta_3, \psi_3')/\sqrt{3} = -i$, here $\tau(\eta_3, \psi_3')$ is the Gauss sum and $\psi_3'(x) = \psi_3(\frac{x}{3})$ is the additive character of level one. By [Lan, Lemma 5.1], $\lambda_{K_3/\mathbb{Q}_3}(\psi_3) = \eta_3(3)\lambda_{K_3/\mathbb{Q}_3}(\psi_3') = -i$. Then Δ_{θ_3} is the unique level one character of K_3 such that $\Delta_{\theta_3}|_{\mathbb{Z}_3^\times} = \eta_3$ and

$$\Delta_{\theta_3}(\sqrt{-3}) = \eta((\sqrt{-3})^3\alpha_{\Theta_3})\lambda_{K_3/\mathbb{Q}_3}(\psi_3)^3 = -i.$$

Recall that $\theta_3 = \Theta_3\Delta_{\theta_3}$. Then by Corollary 4.7 we can easily check that:

(1) If $p \equiv 2$ resp. $5 \bmod 9$, and $\chi = \chi_{3p}$ resp. χ_{3p^2}, $\theta_3\overline{\chi}_3$ is the trivial character.
(2) If $p \equiv 2$ resp. $5 \bmod 9$, and $\chi = \chi_{3p^2}$ resp. χ_{3p}, $\theta_3\overline{\chi}_3$ is of level 2 and by definition we can choose $\alpha_{\theta_3\overline{\chi}_3} = \frac{1}{3\sqrt{-3}}$.

Since χ_3 is a cubic character, $\theta_3\chi_3 = \theta_3\overline{\chi}_3^2$. Since $c(\chi_3) = c(\overline{\chi}_3) = 4$, $c(\theta_3\chi_3) = 4$ and the last assertion follows. □

To apply the results in [HSY19] to calculate the local period integral, we take $\mathbb{F} = \mathbb{Q}_3$, $\varpi = 3 = q$, $D = -3$, $K_3 \simeq \mathbb{E} \simeq \mathbb{L} \simeq \mathbb{Q}(\sqrt{-3})_3$, $c(\theta_3) = c(\chi_3) = 4$, $n = 2$. By [HSY19, Lemma 2.9], we have the minimal vector $\varphi_0 = \mathrm{Char}(\varpi^{-2}U_{\mathbb{F}}(1))$ in the Kirillov model. Recall K is embedded into $\mathrm{M}_2(\mathbb{Q})$ as in Section 2.2 which linearly extends the following map:

$$\sqrt{-3} \mapsto \begin{pmatrix} a & 3^{-2}b \\ 3^3c & -a \end{pmatrix} := \begin{cases} \begin{pmatrix} 3 & -2p/9 \\ 54/p & -3 \end{pmatrix}, & \text{if } K \text{ is embedded under } \rho_1; \\ \begin{pmatrix} 9 & -2p/9 \\ 374/p & -9 \end{pmatrix}, & \text{if } K \text{ is embedded under } \rho_2; \\ \begin{pmatrix} 0 & -p/18 \\ 54/p & 0 \end{pmatrix}, & \text{if } K \text{ is embedded under } \rho_3. \end{cases} \tag{4.3}$$

Proposition 4.9. *Suppose $\mathrm{Vol}(\mathbb{Z}_3^\times\backslash\mathcal{O}_{K,3}^\times) = 1$ so that $\mathrm{Vol}(\mathbb{Q}_3^\times\backslash K_3^\times) = 2$. For f_3 being the newform, K being embedded in $\mathrm{M}_2(\mathbb{Q})$ as in (4.3), we have*

(4.4)
$$\beta_3^0(f_3, f_3) = \begin{cases} 1, & \text{if } p \equiv 2 \text{ resp. } 5 \bmod 9,\ \chi = \chi_{3p} \text{ resp. } \chi_{3p^2} \text{ and } K \text{ is embedded under } \rho_2 \text{ or } \rho_3; \\ 1/2, & \text{if } p \equiv 2 \text{ resp. } 5 \bmod 9,\ \chi = \chi_{3p^2} \text{ resp. } \chi_{3p} \text{ and } K \text{ is embedded under } \rho_1. \end{cases}$$

Proof. We may assume f_3 to be L^2-normalized. To evaluate f_3 for the embedding in (4.3) is equivalent to use the standard embedding [HSY19, (2.13)] of $\mathbb{E}$ and use the corresponding translate of the newform. In particular the embedding in (4.3) is conjugate to the standard embedding by the following

$$\begin{pmatrix} a & 3^{-2}b \\ 3^3c & -a \end{pmatrix} = \begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix} \begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}. \tag{4.5}$$

Thus we have

$$\beta_3^0(f_3, f_3) = \int_{\mathbb{F}^\times\backslash\mathbb{E}^\times} \left(\pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}^{-1} t \begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3, f_3\right) \chi(t) dt \tag{4.6}$$
$$= \int_{\mathbb{F}^\times\backslash\mathbb{E}^\times} \left(\pi_3\left(t\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3, \pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix} f_3\right)\right) \chi(t) dt,$$

which is by definition $\left\{\pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3, \pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix} f_3\right)\right\}$ for the bilinear pairing as in [HSY19, (3.1)] and the standard embedding as in [HSY19, (2.13)]. Note that by [HSY19, Corollary 2.10],

$$\pi_3\left(\begin{pmatrix} -9c & a/3 \\ 0 & 1 \end{pmatrix}\right) f_3 = \frac{1}{\sqrt{2}} \sum_{x\in(\mathbb{Z}_3/3\mathbb{Z}_3)^\times} \pi_3\left(\begin{pmatrix} 1 & a/3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi_0$$

where φ_0 is the minimal test vector.

If $p \equiv 2$ resp. 5 mod 9 and $\chi = \chi_{3p}$ resp. χ_{3p^2}, we embed K into $\mathrm{M}_2(\mathbb{Q})$ under ρ_2 or ρ_3. In this case, $c(\theta_3\overline{\chi_3}) = 0$ and $a = 0$ or 9. By the $l = 0$ case in [HSY19, Section 2.4], we have a unique x mod ϖ for which $\{\varphi_x, \varphi_x\}$ is nonvanishing (In fact, we must have $x \equiv 1 \bmod 3$). According to [HSY19, Proposition 3.3], there are no off-diagonal terms, and we have

$$\beta_3^0(f_3, f_3) = \frac{1}{(q-1)q^{\lceil \frac{c(\theta_3)}{2e_{\mathbb{L}}}\rceil - 1}} \{\varphi_x, \varphi_x\} = \frac{1}{2}\cdot 2 = 1. \tag{4.7}$$

If $p \equiv 2$ resp. 5 mod 9, and $\chi = \chi_{3p^2}$ resp. χ_{3p}, we embed K into $\mathrm{M}_2(\mathbb{Q})$ under ρ_1. In this case, we have $c(\theta_3\overline{\chi}_3) = 2l = 2$ and $u = a/3 = 1$. The action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\varphi_x = \pi_3\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_0$ is by a simple character. This is the case $l = 1$ and $n - l = 1$ is odd. By the choice in [HSY19, Section 2.4],

$$D' = \frac{1}{\alpha_{\theta_3}^2 \varpi_{\mathbb{L}}^{2c(\theta_3)}} = -3,$$

noting $\alpha_{\theta_3} = \alpha_{\Theta_3} = \frac{1}{9\sqrt{-3}}$.

By Lemma 4.8, $\alpha_{\theta_3\chi_3^{-1}} = \frac{1}{3\sqrt{-3}}$ in this case, and we have

$$\begin{aligned} \Delta(1) &= 4\varpi^n\alpha_{\theta_3\overline{\chi}_3}\sqrt{D}\left(\varpi^n\alpha_{\theta_3\overline{\chi}_3}\sqrt{D} - 2\sqrt{\frac{D}{D'}}\right) + 4\frac{D}{D'}D \\ &\equiv 4\cdot 9\cdot\frac{1}{3\sqrt{-3}}\cdot\sqrt{-3}\cdot(-2) + 4\cdot(-3) \bmod \varpi^2 \\ &\equiv -8\cdot 3 - 4\cdot 3 \bmod \varpi^2 \\ &\equiv 0 \bmod \varpi^2. \end{aligned} \tag{4.8}$$

$\Delta(1)$ is indeed congruent to a square. Then we can get a unique solution of x mod ϖ from [HSY19, (2.17)], and again by [HSY19, Proposition 3.3],

$$\beta_3^0(f_3, f_3) = \frac{1}{(q-1)q^{\lceil \frac{c(\theta_3)}{2e_{\mathbb{L}}}\rceil - 1}}\frac{1}{q^{\lfloor l/2\rfloor}} = \frac{1}{2}. \tag{4.9}$$

□

Let f' be the admissible test vector of (π, χ) which is as defined in [CST14, Definition 1.4]. By definition, the 3-adic part f_3' is χ_3^{-1}-eigen under the action of $K_3^\times$. The following corollary is used in the proof of the explicit Gross-Zagier formulae.

Corollary 4.10. *For the admissible test vector f_3' and the newform f_3 we have*

$$\frac{\beta_3^0(f_3', f_3')}{\beta_3^0(f_3, f_3)} = \begin{cases} 2, & \text{if } p \equiv 2 \text{ resp. } 5 \bmod 9,\ \chi = \chi_{3p} \text{ resp. } \chi_{3p^2} \text{ and } K \text{ is embedded under } \rho_2 \text{ and } \rho_3, \\ 4, & \text{if } p \equiv 2 \text{ resp. } 5 \bmod 9,\ \chi = \chi_{3p^2} \text{ resp. } \chi_{3p} \text{ and } K \text{ is embedded under } \rho_1. \end{cases}$$

Proof. Keep the normalization of the volumes in Proposition 4.9. By definition of f', we have $\beta_3^0(f_3', f_3') = \mathrm{Vol}(\mathbb{Q}_3^\times\backslash K_3^\times) = 2$. Then the corollary follows from Proposition 4.9. □

5. The 3-part of the Birch and Swinnerton-Dyer conjectures

Let n be a positive cube-free integer and E'_n be the elliptic curve given by Weierstrass equation $y^2 = x^3 + (4n)^2$. Then there is an unique isogeny $\phi_n : E_n \to E'_n$ of degree 3 up to $[\pm 1]$ and denote ϕ'_n its dual isogeny.

Proposition 5.1. *Let $p \equiv 2, 5 \bmod 9$ be an odd prime. Then*

$$\dim_{\mathbb{F}_3} \mathrm{Sel}_3(E_{3p^2}(\mathbb{Q})) \leq 1, \quad \dim_{\mathbb{F}_3} \mathrm{Sel}_3(E_p(\mathbb{Q})) = 0.$$

$$\dim_{\mathbb{F}_3} \mathrm{Sel}_3(E_{3p}(\mathbb{Q})) \leq 1, \quad \dim_{\mathbb{F}_3} \mathrm{Sel}_3(E_{p^2}(\mathbb{Q})) = 0.$$

Proof. By [Sat86, Theorem 2.9], we know that

$$\mathrm{Sel}_{\phi_{3p^2}}(E_{3p^2}(\mathbb{Q})) = \mathrm{Sel}_{\phi'_{3p^2}}(E'_{3p^2}(\mathbb{Q})) = \mathbb{Z}/3\mathbb{Z},$$

and

$$\mathrm{Sel}_{\phi_p}(E_{3p^2}(\mathbb{Q})) = \mathbb{Z}/3\mathbb{Z}, \quad \mathrm{Sel}_{\phi'_p}(E'_{3p^2}(\mathbb{Q})) = 0.$$

Note that $E_p[3](\mathbb{Q})$ and $E_{3p^2}[3](\mathbb{Q})$ are trivial and $|E'_p[\phi'_p](\mathbb{Q})| = |E'_{3p^2}[\phi'_{3p^2}](\mathbb{Q})| = 3$. By [HSY, Lemma 5.1], we have

$$\dim_{\mathbb{F}_3} \mathrm{Sel}_3(E_{3p^2}(\mathbb{Q})) \leq 1, \quad \dim_{\mathbb{F}_3} \mathrm{Sel}_3(E_p(\mathbb{Q})) = 0.$$

Similarly we have

$$\dim_{\mathbb{F}_3} \mathrm{Sel}_3(E_{3p}(\mathbb{Q})) \leq 1, \quad \dim_{\mathbb{F}_3} \mathrm{Sel}_3(E_{p^2}(\mathbb{Q})) = 0.$$

□

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. We will give the proof of (1.1) when $p \equiv 2 \bmod 9$. One can verify (1.2) in case $p \equiv 5 \bmod 9$ similarly. Now assume $p \equiv 2 \bmod 9$. By [ZK87, Table 1], we know that $c_3(E_p) = 2$ and $c_\ell(E_p) = 1$ for any prime $\ell \neq 3$, while $c_\ell(E_{3p^2}) = 1$ for all primes ℓ.

Let P be the generator of the free part of $E_{3p^2}(\mathbb{Q})$. Then the BSD conjecture predicts that

$$\frac{L'(1, E_{3p^2})}{\Omega_{3p^2}} = |\text{Ш}(E_{3p^2})| \cdot \widehat{h}_{\mathbb{Q}}(P) \quad \text{and} \quad \frac{L(1, E_p)}{\Omega_p} = 2 \cdot |\text{Ш}(E_p)|.$$

Combining these two, we get

$$\frac{L(1, E_p)}{\Omega_p} \cdot \frac{L'(1, E_{3p^2})}{\Omega_{3p^2}} = 2 \cdot |\text{Ш}(E_p)| \cdot |\text{Ш}(E_{3p^2})| \cdot \widehat{h}_{\mathbb{Q}}(P). \tag{5.1}$$

By Theorem 4.3, we expect

$$|\text{Ш}(E_p)| \cdot |\text{Ш}(E_{3p^2})| = 2^{-4} \cdot \frac{\widehat{h}_{\mathbb{Q}}(z_3)}{\widehat{h}_{\mathbb{Q}}(P)}. \tag{5.2}$$

Note the RHS of (5.2) is a nonzero rational number.

By Proposition 5.1, $E_{3p^2}(\mathbb{Q})$ has rank 1, and form the exact sequence

$$0 \to E(\mathbb{Q})/3E(\mathbb{Q}) \to Sel_3(E(\mathbb{Q})) \to \text{Ш}(E)[3] \to 0,$$

we know directly that

$$|\text{Ш}(E_p)[3^\infty]| = |\text{Ш}(E_{3p^2})[3^\infty]| = 1.$$

In order to prove the 3-part of (5.2), it suffices to prove

$$\widehat{h}_{\mathbb{Q}}(P) = u \cdot \widehat{h}_{\mathbb{Q}}(z_3)$$

for some $u \in \mathbb{Z}_3^\times \cap \mathbb{Q}$. This is clear by Corollay 4.4.

□

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References

[BH06] C. Bushnell and G. Henniart. *The Local Langlands Conjecture for* GL(2). Springer-Verlag, Berlin, 2006.

[Cow00] Daniel R. Coward. Some sums of two rational cubes. *Q. J. Math.*, 51(4):451–464, 2000.

[CST14] Li Cai, Jie Shu, and Ye Tian. Explicit Gross-Zagier and Waldspurger formulae. *Algebra & Number Theory*, 8(10):2523–2572, 2014.

[CST17] L. Cai, J. Shu, and Y. Tian. Cube sum problem and an explicit Gross-Zagier formula. *Amer. Jour. of Math.*, 139(3):785–816, 2017.

[DV09] Samit Dasgupta and John Voight. Heegner points and sylvester's conjecture. *Arithmetic Geometry: Clay Mathematics Institute Summer School, Arithmetic Geometry, July 17-August 11, 2006, Mathematisches Institut, Georg-August-Universität, Göttingen, Germany*, 8:91, 2009.

[DV18] Samit Dasgupta and John Voight. Sylvester's problem and mock heegner points. *Proc. Amer. Math. Soc.*, 146:3257–3273, 2018.

[Gro88] Benedict H. Gross. Local orders, root numbers, and modular curves. *American Journal of Mathematics*, 110(6):1153–1182, 1988.

[HSY] Yueke Hu, Jie Shu, and Hongbo Yin. An explicit Gross-Zagier formula related to the Sylvester conjecture. *to appear in Trans. of Amer. Math. Soc.*

[HSY19] Yueke Hu, Jie Shu, and Hongbo Yin. Waldspurger's period integral for newforms. *arXiv:1907.11428*, 2019.

[Kob13] S. Kobayashi. The p-adic Gross-Zagier formula for elliptic curves at supersingular primes. *Invent. Math.*, 191:527–629, 2013.

[Lan] R.P. Langlands. On the functional equation of the artin L-functions. *Unpublished note. https://publications.ias.edu/sites/default/files/a-ps.pdf.*

[Liv95] E. Liverance. A formula for the root number of a family of elliptic curves. *Journal of Number Theory*, 51(2):288 – 305, 1995.

[PR87] B. Perrin-Riou. Points de heegner et derivées de fonctions L p-adiques. *Inventiones mathematicae*, 89:455–510, 1987.

[Sat86] P. Satgé. Groupes de Selmer et corps cubiques. *J. Number Theory*, 23(3):294–317, 1986.

[Sat87] Philippe Satgé. Un analogue du calcul de Heegner. *Invent. Math.*, 87(2):425–439, 1987.

[Shi94] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[SSY] Jie Shu, Xu Song, and Hongbo Yin. Cube sums of form $3p$ and $3p^2$. *arXiv:1804.02924*.

[ZK87] D. Zagier and G. Kramarz. Numerical investigations related to the L-series of certain elliptic curves. *J. Indian Math. Soc. (N.S.)*, 52:51–69 (1988), 1987.

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