A note on Selberg's Lemma and negatively curved Hadamard manifolds

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Abstract

We prove that the conclusion of Selberg's Lemma fails for discrete isometry groups of negatively curved Hadamard manifolds.

In this note we give a negative answer to the first question on Margulis' problem list [M, pg. 27]: Margulis asked if the conclusion of Selberg's Lemma holds for finitely generated isometry groups of Hadamard manifolds.

Theorem 1. *For every $\epsilon > 0$ and $n \geqslant 4$ there exists an n-dimensional Hadamard manifold X_ϵ of sectional curvature $-1-\epsilon \leqslant K_X \leqslant -1$ and a finitely generated discrete isometry group $\Gamma_\epsilon < \mathrm{Isom}(X_\epsilon)$ which has unbounded torsion.*

The idea of the proof is simple: We start with a complete hyperbolic n-manifold $M^n, n \geqslant 4$, with finitely-generated (actually, free) fundamental group and infinitely many rank one cusps $C^n_{\lambda_i}$: Such examples were constructed in [KP, K]. We then replace all but finitely many cusps $C^n_{\lambda_i}$ by metrically complete negatively curved (with pinching constants $(1+\epsilon)^{-1}$) orbifolds O^n_i with boundary, where $\pi_1(O^n_i)$ is cyclic of order i. The result of this "cusp closing" is a complete negatively curved orbifold O_ϵ; the action of $\Gamma_\epsilon := \pi_1(O_\epsilon)$ on the universal cover X_ϵ of O_ϵ provides the required examples.

The Riemannian metrics in Theorem 1 are C^∞ but not real-analytic. It is unclear if Theorem 1 holds in the real-analytic category.

Observe that the above question has positive answer for properly discontinuous group actions in dimension 3 (and, hence, 2, although the 2-dimensional case is elementary): Given a smooth contractible 3-manifold X and a faithful properly discontinuous smooth action $\Gamma \times X \to X$ of a finitely-generated group Γ, there exists an orbifold analogue of the Scott compact core O_c of the orbifold $O = X/\Gamma$; see [FM]. In particular, Γ is isomorphic to the fundamental group of the compact orbifold O_c. According to [H] and the geometrization theorem for good compact 3-dimensional orbifolds (see [BLP] or [KL]), the orbifold O_c is *very good*, i.e. Γ contains a torsion-free subgroup of finite index.

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1 Cusps in hyperbolic manifolds

We will use the upper half-space model $\mathbb{H}^n = \{\mathbf{x} : x_n > 0\}$ of the hyperbolic n-space. An isometry of $\mathbb{H}^n$ is *unipotent* if it is conjugate to a translation $\mathbf{x} \mapsto \mathbf{x} + a\mathbf{e}_1$ for $a \geqslant 0$. An isometry of $\mathbb{H}^n$ is called *parabolic* if it has a unique fixed point in the closed ball compactification $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ of $\mathbb{H}^n$. Here $\partial_\infty \mathbb{H}^n$ is the *ideal/visual boundary* of $\mathbb{H}^n$.

We let $\beta_\lambda : \mathbb{H}^n \to \mathbb{R}$ denote the *Busemann function* for the point $\lambda \in \partial_\infty \mathbb{H}^n$; this function is uniquely defined up to an additive constant. Sublevel sets of Buisemann functions are *horoballs* in $\mathbb{H}^n$.

Throughout the paper we will be using only closed horoballs and closed metric neighborhoods.

Let $\mathbb{H}^n \hookrightarrow \mathbb{H}^N$ denote an isometric totally-geodesic embedding, $N \geqslant n$. This embedding is equivariant under a canonical monomorphism $\mathrm{Isom}(\mathbb{H}^n) \to \mathrm{Isom}(\mathbb{H}^N)$: Each isometry ϕ of $\mathbb{H}^n$ extends to an isometry of $\mathbb{H}^N$ acting trivially on the normal bundle of $\mathbb{H}^n$ in $\mathbb{H}^N$.

For every hyperbolic subspace $X' = \mathbb{H}^n \subset X = \mathbb{H}^N$ we have the orthogonal projection $p_{X',X} : X \to X'$. Fibers of this projection are hyperbolic subspaces orthogonal to X'. In the case of nested hyperbolic subspaces

$$X'' \subset X' \subset X,$$

we have

$$p_{X'',X} = p_{X'',X'} \circ p_{X',X}. \tag{2}$$

Below we review the notion of *cusps* of hyperbolic manifolds/orbifolds; we refer to [Bo1, Bo2, Ra] for details.

Let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a discrete subgroup with the limit set $\Lambda = \Lambda(\Gamma) \subset \partial_\infty \mathbb{H}^n$. A *parabolic limit point* of Γ is a fixed point of a parabolic isometry $\gamma \in \Gamma$. The Γ-stabilizer $\Pi = \Gamma_\lambda < \Gamma$ of such $\lambda \in \Lambda$ is called *a maximal parabolic subgroup* of Γ. A parabolic limit point λ of Γ is called *bounded* (equivalently, *cusped*) if the quotient $(\Lambda - \{\lambda\})/\Gamma_\lambda$ is compact. Each bounded parabolic fixed point of Γ corresponds to a "cusp" of the quotient orbifold $M = \mathbb{H}^n/\Gamma$ defined as follows. Let $X'_\lambda \subset X = \mathbb{H}^n$ be a smallest Π-invariant hyperbolic subspace of X (such a subspace need not be unique). Then Π acts with finite covolume on the intersection $B_\lambda \cap X'_\lambda$ for every horoball $B_\lambda \subset \mathbb{H}^n$ centered at λ. The virtual rank of the virtually abelian group Π equals $r_\lambda = \dim(X'_\lambda) - 1$.

Let $p_\lambda := p_{X'_\lambda,X} : X = \mathbb{H}^n \to X'_\lambda$ be the orthogonal projection as above. Define

$$\tilde{C}'_\lambda := B_\lambda \cap X'_\lambda$$

and

$$\tilde{C}_\lambda := p_\lambda^{-1}(\tilde{C}'_\lambda) \subset \mathbb{H}^n$$

(both depend on B_λ and X'_λ, of course).

Definition 1.1. *If the orbi-covering map $\pi : \mathbb{H}^n/\Pi \to \mathbb{H}^n/\Gamma = M$ is injective on $\tilde{C}_\lambda/\Pi$, then the image $C_\lambda := \pi(\tilde{C}_\lambda/\Pi)$ is called a* cusp neighborhood *in M (or, simpy, a* cusp *in M) corresponding to Π. The domain $\tilde{C}_\lambda$ is then called a* cusped region *of the limit point $\lambda \in \Lambda(\Gamma)$. The number r_λ is the* rank *of the cusp C_λ.*

By abusing the notation, for a cusped region $\tilde{C}_\lambda$, we will denote $C_\lambda = \tilde{C}_\lambda/\Pi$ as well.

For each n-dimensional cusp C_λ we define its *core* $C'_\lambda \subset C_\lambda$ as the quotient $\tilde{C}'_\lambda/\Pi$. The core is unique up to an isometry $C_\lambda \to C_\lambda$.

A parabolic limit point $\lambda \in \Lambda(\Gamma)$ is a bounded if and only if for a sufficiently small horoball B_λ (depending, among other things, on the choice of X'_λ) $\tilde{C}_\lambda$ is a cusped region.

Remark 1.1. *Each maximal parabolic subgroup $\Gamma_\lambda < \Gamma$ (regardless of whether λ is a bounded parabolic limit point or not) corresponds to a* Margulis cusp *of $M = \mathbb{H}^n/\Gamma$: It is the projection to M of the region $U_\lambda \subset \mathbb{H}^n$ consisting of points x such that there exists a parabolic element $\gamma \in \Gamma_\lambda$ satisfying $d(x, \gamma(x)) \leqslant \mu_n$, the Margulis constant of $\mathbb{H}^n$. Margulis cusps should not be confused with the cusps defined above. Margulis cusps will not be used in this paper.*

If $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is *geometrically finite* then every parabolic limit point of Γ is bounded, M has only finitely many cusps and, after taking sufficiently small horoballs B_λ, we can assume that these cusps are pairwise disjoint. If $n = 3$ then every finitely-generated discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ has only finitely many cusps (and Margulis cusps), but this fails in dimensions $n \geqslant 4$ (see [KP]); the existence of such examples is critical for the proof of Theorem 1.

We will call a cusp *unipotent* if its fundamental group Π is unipotent, i.e. every element of Π is unipotent.

For each $r > 0$ we define the *r-collar* $C_{\lambda,r} \subset C_\lambda$ of the boundary of a cusp C_λ as the quotient by Π of $p_\lambda^{-1}(N_r(X'_\lambda \cap \partial B_\lambda))$, where $N_r(\cdot)$ denotes the r-neighborhood in $X'_\lambda \cap B_\lambda$. Then the minimal distance between the boundary components of the collar $C_{\lambda,r}$ equals r.

Given a hyperbolic subspace $\mathbb{H}^n \subset \mathbb{H}^N$, every horoball $B_\lambda \subset \mathbb{H}^n$ properly embeds in a horoball $B_\lambda^N \subset \mathbb{H}^N$. Accordingly, each cusp $C_\lambda^n \subset \mathbb{H}^n/\Pi$ properly embeds in a cusp $C_\lambda^N \subset \mathbb{H}^N/\Pi$.

In this paper we will be only interested in rank one unipotent cusps C_λ. Such a cusp is uniquely determined (up to isometry) by its dimension n and one more real parameter, the *core length* $\ell(C_\lambda)$, defined as the length of the boundary loop of the 2-dimensional core $C'_\lambda \subset C_\lambda$.

We will need the following example of a finitely generated discrete subgroup of $\mathrm{Isom}(\mathbb{H}^4)$ with infinitely many cusps:

Theorem 3 ([KP, K])**.** *There exists a discrete (geometrically infinite) subgroup $\Phi < \mathrm{Isom}(\mathbb{H}^4)$ isomorphic to a free group F_k of rank $k < \infty$, such that:*

1. The quotient manifold $M^4 = \mathbb{H}^4/\Phi$ contains an infinite collection of pairwise disjoint and isometric rank 1 unipotent cusps $C_{\lambda_i}, i \in \mathbb{N}$.

2. Φ is a normal subgroup of a geometrically finite group $\hat{\Phi} < \mathrm{Isom}(\mathbb{H}^4)$; every rank one cusp of M^4 injectively covers a rank one cusp of $\mathbb{H}^4/\hat{\Phi}$.

We let $L := \ell(C_{\lambda_i})$ denote the common core length of the cusps C_{λ_i} of M.

Remark 1.2. *Besides the cusps C_{λ_i}, the manifold M^4 also has finitely many Margulis cusps. These Margulis cusps project to rank two cusps of $\mathbb{H}^4/\hat{\Phi}$. The parabolic limit points of Φ corresponding to these Margulis cusps are not bounded.*

We retain the notation Φ for the image of Φ under the embedding $\mathrm{Isom}(\mathbb{H}^4) \to \mathrm{Isom}(\mathbb{H}^n)$ and define $M^n := \mathbb{H}^n/\Phi$. As noted above, each cusp $C_{\lambda_i}^4$ of M^4 embeds properly in a cusp $C_{\lambda_i}^n \subset M^n$; these n-dimensional cusps are also pairwise disjoint, since there exist 1-Lipschitz retracts $M^n \to M^4$, satisfying $C_{\lambda_i}^n \to C_{\lambda_i}^4$.

2 Warped products

The "cusp closing" procedure in the proof of Theorem 1 will use *warped products* of negatively curved manifolds. In this section we review this basic construction.

Let $(B, ds_B^2), (F, ds_F^2)$ be Riemannian manifolds and $f : B \to (0, \infty)$ a smooth function. The *warped product* of these manifolds, denoted $W = B \times_f F$, is the product $B \times F$ equipped with the Riemannian metric

$$ds^2 = ds_B^2 + f^2 ds_F^2,$$

see [BO, sect. 7] for a detailed discussion. In particular, the Riemannian manifold W is complete iff B and F both are; [BO, Lemma 7.2].

If Γ is a group acting isometrically on each factor $(B, ds_B^2), (F, ds_F^2)$ and $f : B \to \mathbb{R}$ is Γ-invariant, then the product action of Γ on W is also isometric. In particular, the notion of warped product extends to Riemannian orbifolds.

We will need the sectional curvature formula for the warped product in the case of $(B, d_B^2) = (\mathbb{R}, dt^2)$, given in [BO, pg. 27]:

$$K(\Pi) = -\frac{f''(t)}{f(t)}||x||^2 + \frac{L(v,w) - (f'(t))^2}{f^2(t)}||v||^2.$$

Here Π is plane in $T_{(t,q)}W$ with the orthonormal basis $\{x + v, w\}$, where $v, w \in T_qF$, x is a horizontal tangent vector (thus, $||x||^2 + ||v||^2 = ||x||^2 + f^2||v||_F^2 = 1$) and $L(v, w)$ is the sectional curvature of (F, ds_F^2) at q on the plane spanned by the vectors v, w. (Note that [BO, pg. 26] also contains the sectional curvature formula for general warped products).

In particular, if F is negatively curved with sectional curvature $-1 - \epsilon \leqslant L \leqslant -1$ and $f(t) = \cosh(t)$, then

$$-1 - \epsilon \leqslant K(\Pi) \leqslant -1$$

as well.

The hyperbolic space $\mathbb{H}^n$ is isometric to a warped product $\mathbb{H}^{n-2} \times_f \mathbb{H}^2$ with $f : \mathbb{H}^{n-2} \to \mathbb{R}_+$, $f(p) = \cosh(d(o, p))$, where $o \in \mathbb{H}^{n-2}$ is a basepoint. This warped product decomposition can be realized as follows. We let $\mathbb{H}^2$ be embedded in $\mathbb{H}^n$ as

$$\{(x_1, 0, 0, ..., 0, x_n) : x_n > 0\}.$$

Horizontal (totally-geodesic) leaves of the warped product $\mathbb{H}^{n-2} \times_f \mathbb{H}^2$ correspond to codimension two hyperbolic subspaces in $\mathbb{H}^n$ orthogonal to $\mathbb{H}^2$, while vertical leaves are obtained from $\mathbb{H}^2$ by rotating it via elements of $SO(n-1) < SO(n)$ fixing pointwise the coordinate line

$$\mathbb{R}\mathbf{e}_n = \{(x_1, 0, ..., 0, 0)\}.$$

The vertical projection $\mathbb{H}^{n-2} \times_f \mathbb{H}^2 \to \mathbb{H}^2$ is just the orthogonal projection $p_{\mathbb{H}^2,\mathbb{H}^n}$.

Yes another way to realize this decomposition of $\mathbb{H}^n$ is as the iterated warped product

$$\mathbb{H}^n = \mathbb{R} \times_f^{n-2} \mathbb{H}^2,$$

$$\mathbb{H}^3 = \mathbb{R} \times_f \mathbb{H}^2, \mathbb{H}^4 = \mathbb{R} \times_f \mathbb{H}^3, ..., \mathbb{H}^n = \mathbb{R} \times_f \mathbb{H}^{n-1},$$

where $f(t) = \cosh(t)$. The orthogonal projection $p_{\mathbb{H}^2,\mathbb{H}^n}$ equals the vertical projection $\eta : \mathbb{R} \times_f^{n-2} \mathbb{H}^2 \to \mathbb{H}^2$ given by iterating vertical projections in the warped product decompositions $\mathbb{H}^k = \mathbb{R} \times_f \mathbb{H}^{k-1}$, cf. (2).

We generalize this iterated warped product as follows. We let $\tilde{F}$ be a simply-connected complete negatively curved surface with sectional curvature in the interval $[-1-\epsilon, -1]$. Define the iterated warped product

$$\tilde{W} = \mathbb{R} \times_{\cosh}^{n-2} \tilde{F}. \tag{4}$$

It follows that $\tilde{W}$ is still a simply-connected complete negatively curved manifold with sectional curvature in $[-1-\epsilon, -1]$. We let $\tilde{\eta} : \tilde{W} \to \tilde{F}$ denote the vertical projection. Then for an open subset $\tilde{U} \subset \tilde{F}$, the preimage $\tilde{\eta}^{-1}(\tilde{U})$ is an iterated warped product $\mathbb{R} \times_f^{n-2} \tilde{U}$. In particular, if $\tilde{U}$ has constant curvature -1, so does $\tilde{\eta}^{-1}(\tilde{U})$.

3 Closing rank one cusps

We will apply iterated warped products (4) to surfaces $\tilde{F}$ (and their Riemannian orbifold quotients), constructed by splicing quotients of $\mathbb{H}^2$ by cyclic parabolic and by finite cyclic groups. The goal is "close" n-dimensional rank one unipotent cusps C^n_λ, converting them to orbifolds of variable negative curvature with finite cyclic fundamental groups, while leaving the Riemannian metric on a suitable r-collar of C_λ unchanged. The cusp-closing is a rather standard procedure, we describe it here in detail for the sake of completeness.

We start by describing cusp-closing in dimension 2. Let $\Sigma_0 < \mathrm{Isom}(\mathbb{H}^2)$ be a cyclic parabolic subgroup; the surface $T_0 := \mathbb{H}^2/\Sigma_0$ is foliated by projections of Σ_0-invariant horocycles in $\mathbb{H}^2$. Let $c_0 \subset T_0$ be the (unique) leaf of length $a > 0$. (The number a will be specified later on.)

Similarly, let $\Sigma_i < \mathrm{Isom}(\mathbb{H}^2)$ be a finite cyclic subgroup of order $i \geqslant 2$. The quotient-orbifold $T_i = \mathbb{H}^2/\Sigma_i$ is foliated by projections of Σ_i-invariant circles in $\mathbb{H}^2$. Let $c_i \subset T_i$ be the (unique) leaf of the same length a as above. The hyperbolic surfaces/orbifolds T_0 and T_i admit isometric $U(1)$-actions whose orbits are leaves of the above foliations. The distance from c_i to the singular point of T_i equals $R_i = \mathrm{arcsinh}(\frac{ai}{2\pi})$.

We let T_0' denote the closure of the infinite area component in $T_0 - c_0$ and let T_i'' denote the closure of the bounded component of $T_i - c_i$. Gluing T_0', T_i'' via an isometry of their boundaries results in a metric orbifold S_i; the metric on S_i is, of course, smooth away from $\bar{c}_i := c_0 \equiv c_i$ and is singular along that curve. (The group $U(1)$ still acts isometrically on S_i.) Below we smooth out the metric on S_i by modifying it near $\bar{c}_i$, so that the new metric has negative curvature with small pinching constant when i is large.

Fix $r > 0$ and let the nested annuli $A_{i,r} \subset A_{i,2r}$ denote the r- and $2r$-neighborhoods of $\bar{c}_i$ in S_i with respect to the singular metric on S_i. We will take r such that

$$r < \frac{1}{2}\,\mathrm{arcsinh}(a/\pi) \leqslant R_i,$$

hence, the annulus $A_{i,2r}$ is disjoint from the singular point of the orbifold S_i.

The next lemma follows from the geometric convergence of suitable conjugates of subgroups $\Sigma_i < \mathrm{Isom}(\mathbb{H}^2)$ to $\Sigma_0 < \mathrm{Isom}(\mathbb{H}^2)$, since the latter implies C^∞ Gromov–Hausdorff convergence

$$(T_i, t_i) \to (T_0, t_0),$$

where $t_i \in c_i, t_0 \in c_0$; see [BP, Ch. E].

Lemma 4. *For each $\epsilon > 0$ there is i_ϵ such that for all $i \geqslant i_\epsilon$ there exist $U(1)$-invariant Riemannian metrics g_i on the orbifolds S_i satisfying:*

1. g_i equals the restrictions of the metrics of T_0 and T_i respectively on the unbounded/bounded components of $S_i - A_{i,r}$.

2. The curvature of g_i lies in the interval $[-1-\epsilon, -1]$.

In what follows, we equip the orbifolds S_i with the above metrics g_i and denote the resulting Riemannian orbifold F_i. We let $\tilde{F}_i \to F_i$ denote the (degree i) universal cover of the orbifold F_i; then $\tilde{F}_i$ is a simply-connected negatively curved complete Riemannian surface. We let $O^2_{i,2r} \subset F_i$ denote the union of the annulus $A_{i,2r} \subset F_i$ and the suborbifold $T_i'' \subset F_i$ (equipped with the restriction of the metric g_i, of course). The boundary curve of $O^2_{i,2r}$ has length ae^{2r}.

Before extending this construction to higher dimensions we describe cusp-closing for hyperbolic surfaces. Let M^2 be a complete hyperbolic surface (possibly of infinite area) and $C_{\lambda_i} \subset M^2$ be pairwise disjoint cusps with equal core lengths $= L > a$. Each C_{λ_i}, of course embeds isometrically in the surface T_0 as above; we let $2r$ denote the distance between the boundary curve of C_{λ_i} and the loop $c_0 \subset T_0$. Thus, $L = ae^{2r}$.

The r-collar $C_{\lambda_i,r} \subset C_{\lambda_i}$ is isometric to the r-neighborhood $E_{i,r} = N_r(\partial O^2_{i,2r})$ of the boundary of $O^2_{i,2r}$ ($E_{i,r}$ is a component of $A_{i,2r} - A_{i,r}$ and has constant negative curvature). Hence, we can replace each cusp $C_{\lambda_i} \subset M^2$ with a Riemannian orbifold $O^2_{i,2r}$ by first removing $C_{\lambda_i} - C_{\lambda_i,r}$ and then gluing $O^2_{i,2r}$ via an isometry $E_{i,r} \to C_{\lambda_i,r}$. The resuling Riemannian orbifold O^2 is said to be obtained from M^2 by *cusp-closing.*

Remark 4.1. *The Riemannian orbifold O^2 is complete and has negative curvature, which equals -1 except for the annuli $A_{i,r}$ where the curvature is variable. In dimension 2 (at least if the group $\pi_1(M^2)$ is finitely generated) one can also accomplish cusp-closing via a metric of constant negative curvature (by perturbing the hyperbolic metric of M^2 globally rather than inside of the cusps C_{λ_i}), but this is not what we are interested in. See also the [K, Corollary 2] in the setting of hyperbolic 4-manifolds/orbifolds.*

We now proceed with cusp-closing in higher dimensions.

Applying the iterated warped product construction to the Riemannian surfaces $\tilde{F} = \tilde{F}_i$ as described above, we obtain n-dimensional Hadamard manifolds $\tilde{W}^n_i$ equipped with isometric Σ_i-actions; let $W^n_i := \tilde{W}^n_i/\Sigma_i$ be the Riemannian quotient-orbifolds. Thus, each W^n_i is the $n-2$-fold iterated warped product

$$W^n_i = \mathbb{R} \times^{n-2}_{\cosh} F_i$$

with the vertical projection $\eta_i : W_i \to F_i$. As noted above, W_i has constant curvature -1 away from $\eta^{-1}(A_{i,r})$.

As with the 2-dimensional cusp closing, we will only need the parts

$$O^n_{i,2r} := \eta^{-1}(O^2_{i,2r}) \subset W^n_i$$

of the orbifolds W^n_i. Each $O^n_{i,2r}$ can be regarded as an $\mathbb{H}^{n-2}$-bundle over the orbifold $O^2_{i,2r}$. The orbifolds $O^n_{i,2r}$ will be replacing rank one unipotent cusps of a hyperbolic n-manifold. This will be accomplished by gluing along constant curvature boundary collars

$$E^n_{i,r} := \eta^{-1}(E_{i,r}) \subset O^n_{i,2r}.$$

Let C^n_λ be an n-dimensional rank one unipotent cusp of core length $\ell(C^n_\lambda) = L = ae^{2r}$. In view of the iterated warped product decomposition

$$\mathbb{H}^n = \mathbb{R} \times^{n-2}_{\cosh} \mathbb{H}^2,$$

the cusp C^n_λ also decomposes as the iterated warped product

$$C^n_\lambda = \mathbb{R} \times^{n-2}_{\cosh} C^2_\lambda,$$

where $C^2_\lambda \subset C^n_\lambda$ is a 2-dimensional core of C^n_λ, with the projection $\eta : C^n_\lambda \to C^2_\lambda$. The boundary r-collar $C^n_{\lambda,r} \subset C^n_\lambda$ equals the preimage

$$\eta^{-1}(C^2_{\lambda,r})$$

of the r-collar of the core cusp C^2_λ.

Thus we obtain isometries of boundary collars

$$C^n_{\lambda,r} \to E^n_{i,r},$$

from the boundary collar of a cusp C^n_λ to the boundary collars of the orbifolds $O^n_{i,2r}$.

5 Proof of Theorem 1

We construct a Riemannian orbifold O^n_ϵ as follows. Given $\epsilon > 0$, we let $i_\epsilon \in \mathbb{N}$ be as in Lemma 4. Recall that $L = \ell(C^4_{\lambda_i})$ is the common core length of the cusps $C^4_{\lambda_i}$ of the hyperbolic 4-manifold M^4 from Theorem 3. We then let $r > 0$ and $a > 0$ be such that

$$L = e^{2r}a, \quad r < \frac{1}{2}\operatorname{arcsinh}(a/\pi),$$

which can be always accomplished by taking r to be sufficiently small.

From each cusp $C^n_{\lambda_i}, i \geqslant i_\epsilon$, of the hyperbolic n-manifold $M^n = \mathbb{H}^n/\Phi$ we remove the complement to the boundary r-collar $C^n_{\lambda_i,r}$; let M' denote the remaining manifold:

$$M' := M^n - \coprod_{i \geqslant i_\epsilon} (C^n_{\lambda_i} - C^n_{\lambda_i,r}).$$

Then for each $i \geqslant i_\epsilon$ we glue to M' the Riemannian orbifold $O^n_{i,2r}$ via an isometry of the collars

$$C^n_{\lambda_i,r} \to E^n_{i,r},$$

The result is an n-dimensional Riemannian orbifold $O_\epsilon := O^n_\epsilon$.

Remark 5.1. *Since $\pi_1(M^n) \cong \pi_1(M^4) \cong \Phi$ is free of rank k, the fundamental group $\Gamma_\epsilon = \pi_1(O_\epsilon)$ has the presentation*

$$\langle s_1, ..., s_k | w_i^i, i \geqslant i_\epsilon \rangle,$$

where the words w_i represent generators of fundamental groups of the cusps $C^4_{\lambda_i} \subset M^4$.

By the construction, the sectional curvature of O_ϵ lies in the interval $[-1-\epsilon, -1]$. Since M' and all orbifolds $O^n_{i,2r}$ are metrically complete and the minimal distance between the boundary components of each collar $C^n_{\lambda_i,r}$ equals $r > 0$, it follows that the Riemannian orbifold O_ϵ is also complete.

Since the orbifold O_ϵ is complete and negatively curved, it is good (developable); see [BH, pg. 603, Theorem 2.15] and also [Ra, Ch. 13]. Hence, the universal cover of O_ϵ is an n-dimensional Hadamard manifold $X = X_\epsilon$ of curvature $-1 - \epsilon_m \leqslant K_X \leqslant -1$. The fundamental group Γ_ϵ acts on X_ϵ faithfully, properly discontinuously and isometrically with $O_\epsilon \cong X_\epsilon/\Gamma_\epsilon$. In particular, Γ_ϵ has unbounded torsion: The fundamental group (cyclic group of order i) of each orbifold $O^n_{i,2r}$ $(i \geqslant i_\epsilon)$ embeds in Γ_ϵ. □

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