

BRAID RIGIDITY FOR PATH ALGEBRAS

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ABSTRACT. Path algebras are a convenient way of describing decompositions of tensor powers of an object in a tensor category. If the category is braided, one obtains representations of the braid groups B_n for all $n \in \mathbb{N}$. We say that such representations are rigid if they are determined by the path algebra and the representations of B_2 . We show that besides the known classical cases also the braid representations for the path algebra for the 7-dimensional representation of G_2 satisfies the rigidity condition, provided B_3 generates $\text{End}(V^{\otimes 3})$. We obtain a complete classification of ribbon tensor categories with the fusion rules of $\mathfrak{g}(G_2)$ if this condition is satisfied.

We say that a simple object V in a semisimple tensor category \mathcal{C} has the multiplicity 1 property if its tensor product with any simple object in \mathcal{C} decomposes into a direct sum of mutually non-isomorphic simple objects. This provides a convenient canonical decomposition of $V^{\otimes n}$ into a direct sum of simple objects which are labeled by paths, and a description of $\text{End}(V^{\otimes n})$ via path algebras. E.g. for V the vector representation of $Gl(N)$, the paths correspond to standard tableaux of certain Young diagrams. If the category is braided, we also obtain representations of the braid group B_n with respect to a basis labeled by these paths. For $Gl(N)$, we would obtain Young's orthogonal representations of the symmetric groups.

It is a fundamental problem to classify all possible tensor categories for a given set of tensor product (or fusion) rules. For braided tensor categories, an important tool is to classify the corresponding representations of the braid group. This proved to be successful in classifying braided tensor categories whose fusion rules were the ones of a classical Lie group, see [8], [22], where the braid representations could be described in terms of Hecke algebras and *BMW* algebras. Unfortunately, there does not seem to be a convenient algebraic description (via relations) of braid representations appearing for exceptional Lie groups. This motivated our approach via path algebras.

One can abstractly define braid representations compatible with path algebras, see Eq 1.6 and 1.7. We say that a path algebra is *braid rigid*, if any compatible braid representation (with some mild additional conditions, see Definition 1.7) is already uniquely determined by the image of the first braid generator. This is the case for path algebras associated to classical Lie types. The main result in this paper states that also the path algebra associated to the 7-dimensional representation V of G_2 is braid rigid, provided that the image of the braid group B_3 generates $\text{End}(V^{\otimes 3})$. In particular, in these cases we obtain the same braid representations as for the quantum group $U_q\mathfrak{g}(G_2)$ of Lie type G_2 . This implies the classification of ribbon tensor categories whose fusion rules are the ones of $\mathfrak{g}(G_2)$ if they satisfy the condition about

$\text{End}(V^{\otimes 3})$ just stated. This result has already appeared before in [16], but our proof is quite different and does not use any computer calculations.

Our approach was inspired by our previous work [14] where we gave another proof that the braid groups generate $\text{End}(V^{\otimes n})$ for V the 7-dimensional representation V of $U_q\mathfrak{g}(G_2)$ for q not a root of unity. We did this by finding quite explicit formulas for the path representation of braid groups for certain types of paths. Our main new result in this paper is that the path representations for an abstract semisimple ribbon tensor category with the fusion rules of $\mathfrak{g}(G_2)$ have to be isomorphic to the ones for the quantum group $U_q\mathfrak{g}(G_2)$ for q not a root of unity, at least when the condition for the third tensor power of V is satisfied.

Here is the content of our paper in more detail. We review basic definitions concerning path algebras and braided tensor categories in the first section. We also define braid rigidity for path algebras there. We then give the necessary combinatorial and algebraic information about the Lie algebra $\mathfrak{g}(G_2)$ in the second section. Let V be the object in an abstract tensor category \mathcal{C} of type G_2 corresponding to the smallest nontrivial representation of $\mathfrak{g}(G_2)$. Then we show in the third section that if B_3 generates $\text{End}_{\mathcal{C}}(V^{\otimes 3})$, this braid representation has to be isomorphic to the corresponding representation in $\text{End}_{\mathcal{U}}(V^{\otimes 3})$, where $\mathcal{U} = \text{Rep}(U_q\mathfrak{g}(G_2))$ for some q not a root of unity. The main technical result is then proved in the last section. We show that the result in Section 3 can be extended to all tensor powers of V by proving that the corresponding braid representations are path rigid. We use this to classify all ribbon tensor categories \mathcal{C} with the fusion rules of G_2 , subject to the already mentioned condition concerning $\text{End}_{\mathcal{C}}(V^{\otimes 3})$.

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1. BASIC DEFINITIONS

1.1. Path algebras. The notion of path algebras (or equivalent versions of it) has been known in many contexts such as operator algebras, representation theory and algebra for a long time. We review some basic facts here, which will also help to fix notations.

Let Λ be a set of labels with distinguished label 0 together with a not necessarily symmetric relation \rightarrow . A path of length n is a map $t : \{0, 1, \dots, n\} \rightarrow \Lambda$ such that $t(0) = 0$ and $t(i) \rightarrow t(i+1)$ for $0 \leq i < n$. We denote by \mathcal{P}_n the set of all paths of length n . We define algebras C_n by

$$(1.1) \quad C_n \cong \bigoplus_{\nu} M_{m(\nu, n)},$$

where $m(\nu, n)$ is the number of paths t of length n with $t(n) = \nu$ and M_m are the $m \times m$ matrices. Let $W(\nu, n)$ be a simple C_n -module labeled by the label ν . It follows from the definitions that it has a basis labeled by the paths in \mathcal{P}_n which end in ν . Its decomposition into simple C_{n-1} modules is given by the map $t \mapsto t'$, where t' is the restriction of t to

$\{0, 1, \dots, n-1\}$. Hence we have the following isomorphism of C_{n-1} -modules:

$$(1.2) \quad W(\nu, n) \cong \bigoplus_{\mu} W(\mu, n-1),$$

where μ runs through all labels μ which are endpoint of a path of length $n-1$ such that $\mu \rightarrow \nu$.

Definition 1.1. The path algebra \mathcal{P} corresponding to the label set Λ with the relation \rightarrow is given by the sequence of algebras C_n with the embeddings $C_{n-1} \subset C_n$ defined by 1.2.

Example 1.2. The standard example for a path algebra is given by the labeling set Λ consisting of all Young diagrams with 0 being the empty Young diagram, and $\mu \rightarrow \nu$ if $\mu \subset \nu$ and $|\nu| = |\mu| + 1$, i.e. ν is obtained by adding a box to μ . Then each path corresponds to a Young tableau, and $C_n \cong \mathbb{C}S_n$. We refer to this path algebra as the path algebra from Young's lattice.

Remark 1.3. Our version is not the most general version of path algebra. It will be clear from the next subsection that there also exist interesting examples where one replaces our relation \rightarrow by nonnegative integers $k(\mu, \nu)$ for each ordered pair $(\mu, \nu) \in \Lambda \times \Lambda$. As this more complicated case will not be relevant in our paper, we stick to this simpler version.

1.2. Tensor categories. In this and the following subsection \mathcal{C} will denote a semisimple tensor category, whose Hom spaces are complex vector spaces. The reader not familiar with tensor categories can safely think of \mathcal{C} being the representation category of a Drinfeld-Jimbo quantum group, or just of the corresponding semisimple Lie algebra. Let Λ be a labeling set for the simple objects of \mathcal{C} , where 0 is the label for the trivial object. We also assume that V is a simple object of \mathcal{C} with the *multiplicity 1 property*, i.e. $V_{\mu} \otimes V$ is a direct sum of mutually non-isomorphic simple objects, for any simple object V_{μ} in \mathcal{C} . We then define the relation $\mu \rightarrow \nu$ if $V_{\nu} \subset V_{\mu} \otimes V$. This allows us to give a fairly simple description of $\text{End}_{\mathcal{C}}(V^{\otimes n})$ via paths.

Theorem 1.4. *We have a direct sum decomposition of objects in \mathcal{C} given by*

$$V^{\otimes n} = \bigoplus_{\nu} m(\nu, n) V_{\nu},$$

where the multiplicity $m(\nu, n)$ is given by the number of paths in \mathcal{P}_n which end in ν . In particular, we have

$$(1.3) \quad C_n = \bigoplus_{\nu} M_{m(\nu, n)} \cong \text{End}_{\mathcal{C}}(V^{\otimes n})$$

where M_k are the $k \times k$ matrices.

Remark 1.5. As indicated at the end of the previous subsection, it is not hard to give a path algebra description of $\text{End}_{\mathcal{C}}(V^{\otimes n})$ also if V does not have the multiplicity 1 property. This can be done in terms of Littelmann paths (see [12]). The simpler version here has been known much longer, see e.g. [20] and references there.

Corollary 1.6. *There exists an assignment $t \in \mathcal{P}_n \mapsto p_t \in C_n = \text{End}_{\mathcal{C}}(V^{\otimes n})$ such that $p_t V^{\otimes n}$ is an irreducible \mathcal{C} -object labeled by $t(n)$, and such that $p_t p_s = \delta_{ts} p_t$. The idempotents p_t are uniquely defined by the properties above and the following one: If $s \in \mathcal{P}_{n-1}$, we have*

$$p_s \otimes id = \sum_{t, t'=s} p_t.$$

1.3. Path representations. We denote by $\mathcal{P}_n(\nu)$ all paths of length n in \mathcal{P}_n which end in ν . One checks easily that $z_\nu^{(n)} = \sum_{t \in \mathcal{P}_n(\nu)} p_t$ is a central idempotent in $C_n = \text{End}_{\mathcal{C}}(V^{\otimes n})$.

By definition, we can define a basis $(v_t)_{t \in \mathcal{P}_n(\nu)}$ for the simple $C_n(\nu)$ -module $W(\nu, n)$; often we will just write t for v_t . Here the vector v_t spans the image of p_t for each $t \in \mathcal{P}_n(\nu)$ and it is uniquely determined up to scalar multiples. Let δ, ν be dominant weights for which $V_\delta \subset V^{\otimes n-k}$ and $V_\nu \subset V^{\otimes n}$, and let $\mathcal{P}_k(\delta, \nu)$ be the set of all paths of length k from δ to ν , with paths as defined in Section 1.1. Let $W_k(\delta, \nu)$ be the vector space spanned by these paths and let t be a fixed path in $\mathcal{P}_{n-k}(\delta)$. Then we obtain a representation of $\text{End}_{\mathcal{C}}(V^{\otimes k})$ on $W_k(\delta, \nu)$ by

$$(1.4) \quad a \in \text{End}_{\mathcal{C}}(V^{\otimes k}) \mapsto (p_t \otimes a) z_\nu^{(n)};$$

here we used the obvious bijection between elements $s \in \mathcal{P}_k(\delta, \nu)$ and paths $\tilde{s} \in \mathcal{P}_n(\nu)$ for which $\tilde{s}|_{[0, n-k]} = t$, i.e. \tilde{s} is the extension of t by s .

1.4. Braided tensor categories. We recall a few basic facts about braided and ribbon tensor categories, see e.g. [7], [23] for more details. This serves mostly as motivation for the definitions in the next subsection. A braided tensor category \mathcal{C} has canonical isomorphisms $c_{V,W} : V \otimes W \rightarrow W \otimes V$ for any objects V, W in \mathcal{C} . They satisfy the condition

$$(1.5) \quad c_{U, V \otimes W} = (1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W),$$

and a similar identity for $c_{U \otimes V, W}$. Let B_n be Artin's braid groups, given by generators σ_i , $1 \leq i \leq n-1$ and relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ as well as $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$. One can show that we obtain a representation of the braid group B_n into $\text{End}(V^{\otimes n})$ for any object V in \mathcal{C} via the map

$$(1.6) \quad \sigma_i \mapsto 1_{i-1} \otimes c_{V,V} \otimes 1_{n-1-i},$$

where 1_k is the identity morphism on $V^{\otimes k}$. Using a path basis (t) as in the last subsection, we can express the action of σ_i via a matrix A_i such that

$$(1.7) \quad \sigma_i \mapsto A_i : t \mapsto \sum_s a_{st}^{(i)} s,$$

where the summation goes over paths s for which $s(j) = t(j)$ for $j \neq i$; this follows from Eq 1.4 with $n = i+1$ and $k = 2$. As the vectors ts are uniquely determined up to rescaling, it also follows that the matrix entries of A_i are uniquely determined up to conjugation by a diagonal matrix.

An associated ribbon braid structure is given by maps $\Theta_W : W \rightarrow W$ satisfying

$$(1.8) \quad \Theta_{V \otimes W} = c_{W,V} c_{V,W} (\Theta_V \otimes \Theta_W).$$

Let $\Delta_n \in B_n$ be defined inductively by $\Delta_2 = \sigma_1$ and $\Delta_n = \Delta_{n-1} \sigma_{n-1} \sigma_{n-2} \dots \sigma_1$. Then it is well-known that $\Delta_n^2 = (\sigma_1 \sigma_2 \dots \sigma_{n-1})^n$ generates the center of B_n . One can then prove by induction on n that

$$(1.9) \quad \Theta_{V^{\otimes n}} = \Delta_n^2 \Theta_V^{\otimes n}.$$

If V_λ is a simple object, the ribbon map just acts via a scalar, which we will denote by Θ_λ . Let us also assume that the representation of B_n into $\text{End}(V^{\otimes n})$ is semisimple. Then the central element Δ_n^2 acts in the simple component labeled by α via a scalar denoted by $z_{\alpha,n}$. If the B_n -representation labeled by α acts nontrivially on the $\text{End}(V^{\otimes n})$ -module $W_\lambda^{(n)}$, then it follows from Eq 1.9 that

$$(1.10) \quad \Theta_\lambda = z_{\alpha,n} \Theta_V^n,$$

where we identified Θ_V with the scalar via which it acts on V .

1.5. Braid rigidity. We now translate the notions of the previous section into the language of path algebras.

Definition 1.7. Let \mathcal{P} be a path algebra.

(a) We call a system of representations of braid groups *representations of type \mathcal{P}* if the braid generators act on paths as in 1.7. Moreover, we also require that the central element $\Delta_n^2 \in B_n$ acts via a fixed scalar $z_{\lambda,n}$ on every path of length n which ends in λ .

(b) We call a path algebra \mathcal{P} *braid rigid* if any non-trivial braid representation of type \mathcal{P} is uniquely determined by the image of σ_1 ; see the example below for the description of trivial braid representations of type \mathcal{P} .

Example 1.8. 1. We can always define trivial braid representations for any path algebra \mathcal{P} as follows. We fix a non-zero number α and we assign to each path s of length 2 an eigenvalue $\alpha_s = \pm\alpha$ of σ_1 . Then we define the action of σ_i on a path t to be equal to multiplication by α_s if the restriction of t to $\{0, 1, 2\}$ is equal to s . It is easy to check that we obtain a path representation of B_n for any n which is a direct sum of 1-dimensional abelian representations.

2. Let \mathcal{P} be the path algebra given by Young's lattice. We claim that the corresponding path algebra is braid rigid provided that the ratio between the two eigenvalues of σ_1 is not a root of unity. Indeed, as we only have two paths of length two, the image of σ_1 has at most two eigenvalues. It is well-known that in this case we obtain representations of the Hecke algebras $H_n(q)$ of type A_{n-1} , where $q = -\alpha_1/\alpha_2$ for α_1 and α_2 being the eigenvalues of the image of σ_1 . If the braid representation is nontrivial, the representation of B_3 on the 2 paths ending in the Young diagram [21] has to be irreducible. Using the q -Jucys-Murphy approach, see e.g. [14], Lemma 1.8 for a review, we can then inductively compute all matrix entries (up to rescaling of basis vectors) for any path representation.

3. The same statement is also true if $-\alpha_1/\alpha_2$ is a primitive ℓ -th root of unity if we restrict the label set Λ to the set $\Lambda^{(k,\ell)}$ of so-called (k, ℓ) -diagrams, i.e. to Young diagrams with $\leq k$ rows such that $\lambda_1 - \lambda_k \leq \ell - k$, see [24] for details.

4. One can similarly also show that the path representations for the path algebra generated by the vector representation V of an orthogonal or symplectic group are braid rigid. This follows essentially from [22], where a complete classification of braided tensor categories was given for which the fusion rules are the ones of the representation category of an orthogonal or symplectic group. The main point of the proof there was to show that $\text{End}_{\mathcal{C}}(V^{\otimes n})$ was given by a quotient of the so-called *BMW*-algebra, see [1], [17].

1.6. Matrix blocks. We again assume \mathcal{C} to be a general ribbon tensor category as in Section 1.4. It follows from Eq 1.4 and 1.7 that the matrix A_{n-1} acts in blocks leaving invariant path spaces $W_2(\delta, \lambda)$ spanned by a basis (v_t) labeled by all paths of length 2 from δ to λ . It follows from the definitions that

$$(1.11) \quad W_2(\delta, \lambda) \cong \text{Hom}_{\mathcal{C}}(V_\lambda, V_\delta \otimes V^{\otimes 2}).$$

Indeed, as the image of σ_{n-1} commutes with $\text{End}(V^{\otimes n-2})$, the only relevant part of the path t for the action of A_{n-1} are the weights $t(i)$, $n-2 \leq i \leq n$. We will consider certain cases in which we can calculate the matrix entries of A_{n-1} . More precisely we consider the following cases:

- a) We have $\dim W_2(\delta, \lambda) = 2$, and A_{n-1} acts with two distinct eigenvalues on it.
- b) The action of A_{n-1} is diagonalizable on $W_2(\delta, \lambda)$ with exactly three distinct eigenvalues, with one of them having multiplicity 1. We define q such that the ratio of the other two eigenvalues is equal to $-q^2$.

The following proposition is a reformulation of results in [26] and [14]:

Lemma 1.9. *Assume that A_{n-1} and $W_2(\delta, \lambda)$ satisfy the conditions just stated. Then its entries with respect to the path basis of $W_2(\delta, \lambda)$ can be calculated in terms of the eigenvalues of A_{n-1} and the entries of the rank 1 eigenprojection P , up to conjugation by a diagonal matrix. In particular, for each such block all the off-diagonal matrix entries of A_{n-1} are nonzero if the corresponding entries of P are nonzero.*

Proof. This is a consequence of [14], Lemma 1.8, Prop. 1.6, Lemma 1.8 and Lemma 3.2. We give some details for the reader's convenience. If $\dim W_2(\delta, \lambda) = 2$, the claim follows from a well-known q -version of the Jucys-Murphy approach, see e.g. [14], Lemma 1.8 for details. If A_{n-1} has three eigenvalues, let P be the eigenprojection of A_{n-1} for the eigenvalue with multiplicity 1. Moreover, let q^α be a scalar such that $A' = q^\alpha A_{n-1}$ has eigenvalues q , $-q^{-1}$ and r^{-1} such that $A'P = r^{-1}P$. As, by construction we have

$$A' - (A')^{-1} = (q - q^{-1})I - (r - r^{-1} + q - q^{-1})P,$$

we can calculate the matrix entries of A' from the equation

$$(1 - q^{e(t)+e(s)})a'_{ts} = (q - q^{-1})\delta_{ts} - (r - r^{-1} + q - q^{-1})p_{ts},$$

where $q^{e(t)}$ is the scalar via which $\Delta_n^2 \Delta_{n-1}^{-2}$ acts on the path t , see [26], Lemma 4.1 for details.

2. THE EXAMPLE G_2

2.1. **Quantum groups.** The best known examples of ribbon categories are given by the representation categories $\mathcal{U} = \text{Rep}(\mathbf{U})$ of a Drinfeld-Jimbo quantum group $\mathbf{U} = U_q\mathfrak{g}$, where \mathfrak{g} is a semisimple Lie algebra. We assume as ground ring the field $\mathbb{C}(q)$ of rational functions in the variable q . It is well-known that in our setting the category \mathcal{U} of integrable representations of \mathbf{U} is semisimple, and it has the same Grothendieck semiring as the original Lie algebra. We shall need the following result due to Drinfeld [3].

Proposition 2.1. *Let $V_\lambda, V_\mu, V_\Lambda = V$ be simple \mathbf{U} -modules with highest weights λ, μ, Λ respectively, and such that V_μ is a submodule of $V_\lambda \otimes V_\Lambda$. Let us write $c_{\lambda,\mu}$ for the braiding morphism $V_\lambda \otimes V_\mu \rightarrow V_\mu \otimes V_\lambda$. Then*

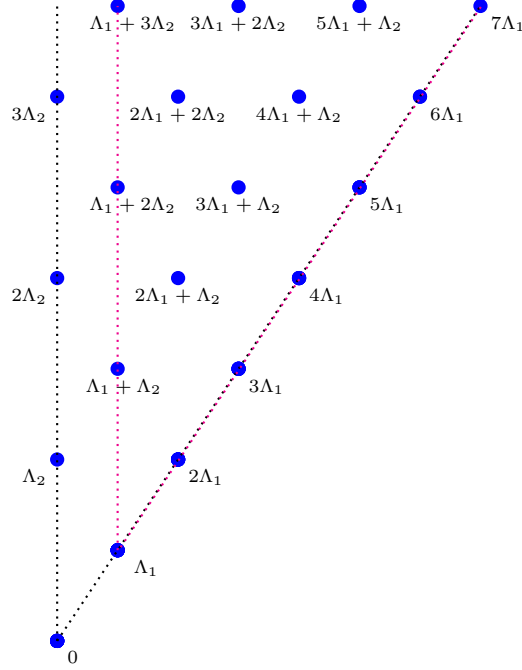
$$(c_{\lambda,\Lambda}c_{\Lambda,\lambda})|_{V_\mu} = q^{C_\mu - C_\lambda - C_\Lambda} 1_{V_\mu},$$

where for any weight γ the quantity C_γ is given by $C_\gamma = (\gamma + 2\rho, \gamma)$. Here, ρ is the Weyl vector. Moreover, the twisting factors Θ_λ are given by

$$\Theta_\lambda = q^{C_\lambda}.$$

2.2. **Path algebra for G_2 .** We will be particularly interested in the case with $\mathfrak{g} = \mathfrak{g}(G_2)$ and V its simple 7-dimensional representation. We first recall some basic facts about its roots and weights (see e.g. [4], [6]).

With respect to the orthonormal unit vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of \mathbb{R}^3 , the roots of \mathfrak{g} can be written $\Phi = \pm\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3, 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3, 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\}$. The base can be chosen $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3\}$. The Weyl vector is given by $\rho = 2\varepsilon_1 + \varepsilon_2 - 3\varepsilon_3$ and the Weyl group is D_6 . The fundamental dominant weights are given by $\{\Lambda_1 = \varepsilon_1 - \varepsilon_3, \Lambda_2 = \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3\}$. The following describes the dominant Weyl chamber:

Weyl Chamber for \mathfrak{g}

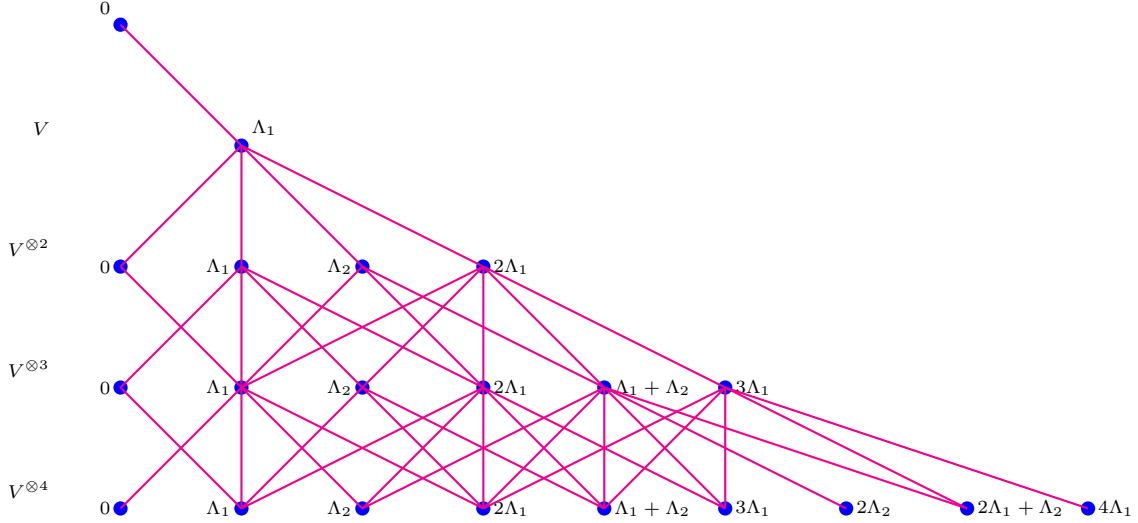
It follows from Weyl's dimension formula that the q -dimension of the $U_q\mathfrak{g}(G_2)$ -module V_λ with highest weight $\lambda = (\lambda_1, \lambda_2, -\lambda_1 - \lambda_2)$ is equal to

$$(2.1) \quad \dim_q V_\lambda = \frac{[\lambda_1 - \lambda_2 + 1][2\lambda_1 + \lambda_2 + 5][\lambda_1 + 2\lambda_2 + 4][3\lambda_1 + 6][3\lambda_2 + 3][3(\lambda_1 + \lambda_2) + 9]}{[1][5][4][6][3][9]}$$

2.3. Tensor product rules. We will study the path representations with respect to the smallest nontrivial \mathfrak{g} -module $V = V_{\Lambda_1}$. We will need to know how to tensor irreducible representations with V . We will review this here for the reader's convenience. The representation V has dimension 7, with its weights being the short roots of \mathfrak{g} together with the zero weight. The decomposition of the tensor product

$$(2.2) \quad V_\lambda \otimes V \cong \bigoplus_{\mu} V_\mu$$

with V_λ a simple module with highest weight $\lambda = a\Lambda_1 + b\Lambda_2$ can be described as follows (see e.g. [14], Prop. 2.1 and Remark 2.2): Consider the hexagon centered at λ and with corners $\lambda + \omega$, with ω running through the short roots of \mathfrak{g} . If this hexagon is contained in the dominant Weyl chamber C , then $V_\lambda \otimes V$ decomposes into the direct sum of irreducibles \mathfrak{g} -modules whose highest weights are given by the corners and the center of the hexagon. If it is not contained in C , leave out all the corners of the hexagon which are not in C ; moreover, if $\lambda = b\Lambda_2$, also leave out λ itself. Using this, we can draw the Bratteli diagram for $V^{\otimes n}$.



Bratteli diagram for $V^{\otimes n}$

2.4. **Braid representations for $U_q\mathfrak{g}(G_2)$.** We normalize the invariant product on the weight lattice such that $(\Lambda_1, \Lambda_1) = 2$ and $(\Lambda_2, \Lambda_2) = 6$. With these conventions we get the values $C_\nu = 0, 28, 12, 24$ for $\nu = 0, 2\Lambda_1, \Lambda_1, \Lambda_2$. Hence it follows from Proposition 2.1 that the eigenvalues of $R_{V,V}$ are given by $q^{-12}, q^2, -q^{-6}$ and -1 . So if $\alpha = q$, $A' = \alpha^{-1}R_{V,V}$ has the desired eigenvalues q and $-q^{-1}$ for the representations $V_{2\Lambda_1}$ and V_{Λ_2} . As we shall see in a moment, it will be convenient to associate with P the eigenprojection of A projecting onto $V = V_{\Lambda_1} \subset V^{\otimes 2}$, which corresponds to the eigenvalue $-q^{-6}$. Indeed, let $W = \text{Hom}_{\mathbf{U}}(V_\lambda, V_\delta \otimes V^{\otimes 2})$. Then P is the projection onto the subspace $\text{Hom}_{\mathbf{U}}(V_\lambda, V_\delta \otimes V_{\Lambda_1})$ of W , given by the embedding $V \subset V^{\otimes 2}$. As all weights of V have multiplicity 1, the multiplicity of V_λ in $V_\delta \otimes V_{\Lambda_1}$ is at most 1. Hence P has at most rank 1 in $\text{Hom}_{\mathbf{U}}(V_\lambda, V_\delta \otimes V^{\otimes 2})$. In particular, the conditions for Lemma 1.9 are satisfied. We can now refine the results of that lemma in our setting as follows, restating results which have already appeared before in [26] and [14]. Let

$$t : \delta \rightarrow \mu_t \rightarrow \lambda$$

be a path of length 2 and let

$$e(t) = C_{\mu_t} - \frac{1}{2}(C_\lambda + C_\delta) + 1,$$

where $C_\gamma = (\gamma + 2\rho, \gamma)$ for a weight γ .

Lemma 2.2. *Consider the space $W_2(\delta, \lambda)$ with a basis labeled by paths of length 2 from δ to λ . If $\lambda \neq \delta$, then the matrix A_{n-1} can be calculated up to conjugation by a diagonal matrix as*

in Lemma 1.9. In particular, all off-diagonal entries of A_{n-1} are well-defined and not equal to 0 if q is not a root of unity.

Proof. This result was essentially already shown in [14]. We give a proof here for the reader's convenience. Let P_γ be the eigenprojection of A_{n-1} corresponding to the representation $V_\gamma \subset V^{\otimes 2}$. It follows from the definitions that it acts on ${}_2W(\delta, \gamma)$ as a rank $c_{\delta, \gamma}^\lambda$ idempotent, where $c_{\delta, \gamma}^\lambda$ is the multiplicity of V_λ in $V_\delta \otimes V_\delta$. This rank is equal to 0 for $\gamma = 0$ unless $\delta = \lambda$. Hence A_{n-1} can only act with at most three distinct eigenvalues on $W_2(\delta, \gamma)$ for $\delta \neq \lambda$.

It was shown in [14] that A_{n-1} acts with two eigenvalues only if the dimension of $W_2(\delta, \lambda)$ is equal to 2. It is well-known how to calculate the matrix coefficients in this case via the Jucys-Murphy approach, see e.g. [14], Lemma 1.8 for a review and a precise statement.

If A_{n-1} acts with three distinct eigenvalues, the eigenprojection P for $\gamma = \Lambda_1$ has rank 1. It was shown in [14], Lemma 3.2 that its diagonal entry d_s for the path s is given by

$$d_s = \frac{[e(s) + 1]}{1 - [7]} \prod_{t \neq s} \frac{[(e(s) + e(t))/2]}{[(e(s) - e(t))/2]}.$$

It was shown in [14] Proposition 1.6 and Lemma 2.6 that these entries are nonzero and well-defined for q not a root of unity. As P is a rank 1 idempotent, this also shows that all of its matrix entries are nonzero. We can now calculate the matrix entries of A_{n-1} as shown in the proof of Lemma 1.9. In particular, this shows that also the off-diagonal entries of A_{n-1} are well-defined and nonzero for q not a root of unity.

The following result has first been shown in [11], with different proofs also given in [15] and [14]:

Theorem 2.3. (*First Fundamental Theorem*) *Let V be the 7-dimensional representation of $\mathbf{U} = U_q(\mathfrak{g}(G_2))$ with highest weight Λ_1 . Then $\text{End}_{\mathbf{U}}(V^{\otimes n})$ is generated by the image of the braid group B_n in $\text{End}_{\mathbf{U}}(V^{\otimes n})$ for q not a root of unity.*

3. TENSOR CATEGORIES OF TYPE G_2

In the rest of this paper, we let \mathcal{C} be a semisimple rigid ribbon tensor category of type G_2 . By this we mean that its simple objects X_λ are labeled by the dominant integral weights λ of G_2 , and the decomposition of tensor products of simple objects is given by the tensor product rules for G_2 . See e.g. [7], [23] for precise definitions of the other terms. We shall first study the braid representations corresponding to small tensor powers of the object corresponding to the 7-dimensional irreducible representation V . The main result of this section is that the eigenvalues of the braid generators are forced to be the same as in the quantum group case whenever the braid representations generate $\text{End}_{\mathcal{C}}(V^{\otimes n})$ for $n = 2, 3$.

3.1. Preparations. We shall use properties of ribbon categories, in particular Eq 1.10 to find constraints for the eigenvalues of $c_{V,V}$. The following elementary lemma will be useful:

Lemma 3.1. *Let W be a representation of B_3 of dimension m on which σ_1 acts with eigenvalues α_i , $1 \leq i \leq m$ and on which Δ_3^2 acts via the scalar $z_{3,m}$. Then we have*

$$z_{3,2} = -(\alpha_1\alpha_2)^3, \quad z_{3,3} = (\alpha_1\alpha_2\alpha_3)^2, \quad z_{3,4} = \sqrt{\alpha_1\alpha_2\alpha_3\alpha_4^3},$$

where there exist representations for both choices of the square root for $m = 4$.

Proof. Let us first assume we have a representation of B_n acting on an m -dimensional vector space such that Δ_n^2 acts via the scalar $z_{n,m}$. Calculating the determinant of the matrix representing $\Delta_n^2 = (\sigma_1\sigma_2 \dots \sigma_{n-1})^{n(n-1)}$ in two different ways, we obtain

$$(3.1) \quad z_{n,m}^m = \det(\sigma_1)^{n(n-1)};$$

This equation does not determine which m -th root of the determinant we have to take for $z_{n,m}$. But it was shown in [21] that for dimension $m \leq 5$ representations of B_3 as in the statement are essentially obtained by their eigenvalues. Using the explicit braid representations in [21], one can check the claim by a direct calculation.

3.2. Calculations of eigenvalues. We use the notations as in Section 2.2. The dominant integral weights λ are of the form $\lambda = (\lambda_1, \lambda_2, -\lambda_1 - \lambda_2)$ with $\lambda_1 \geq \lambda_2 \geq 0$. In the following we will just write $\lambda = (\lambda_1, \lambda_2)$ for brevity. So the highest weight of the 7-dimensional simple representation V of G_2 is given by $\Lambda_1 = (1, 0)$. The second fundamental weight is $\Lambda_2 = (1, 1)$. It follows from the decomposition of $V^{\otimes 2}$, see Eq 2.2, or [14], Example 2.3 that $c_{V,V}$ has four eigenvalues corresponding to the subrepresentations $\mathbf{1} = V_{(0,0)}$, $V = V_{(1,0)}$, $V_{(1,1)}$ and $V_{(2,0)}$ respectively. We will refer to the eigenvalue belonging to $V_\lambda \subset V^{\otimes 2}$ by α_λ . It follows from 1.10 for $n = 2$ that

$$(3.2) \quad \Theta_\lambda = \alpha_\lambda^2 \Theta_{(1,0)}^2.$$

As $\Theta_{(0,0)} = 1$ (which can be deduced from 1.8 and the braiding axioms), we obtain $1 = \alpha_{(0,0)}^2 \Theta_{(1,0)}^2$ and $\Theta_{(1,0)} = \alpha_{(1,0)}^2 \Theta_{(1,0)}^2$ from 3.2. Hence we have

$$(3.3) \quad \pm \alpha_{(0,0)} = 1/\Theta_{(1,0)} = \alpha_{(1,0)}^2.$$

Observe that the representations labeled by $(2, 0)$ and $(1, 1)$ appear with multiplicity 3 and 2 in $V^{\otimes 3}$. It follows from Lemma 3.1 and Eq. 1.10 that

$$\alpha_{(2,0)}^2 \Theta_{(1,0)}^2 = \Theta_{(2,0)} = (\alpha_{(1,0)}\alpha_{(1,1)}\alpha_{(2,0)})^2 \Theta_{(1,0)}^3,$$

from which we deduce, together with 3.3

$$(3.4) \quad \alpha_{(1,1)}^2 = 1.$$

Again using Lemma 3.1 and Eq. 3.2 for $\lambda = (1, 1)$ we obtain

$$\alpha_{(1,1)}^2 \Theta_{(1,0)}^2 = \Theta_{(1,1)} = -(\alpha_{(1,0)}\alpha_{(2,0)})^3 \Theta_{(1,0)}^3.$$

We deduce from this, again using $\alpha_{(1,0)}^2 \Theta_{(1,0)} = 1$, that

$$(3.5) \quad \alpha_{(1,0)}\alpha_{(2,0)}^3 = -1.$$

We have almost proved the following proposition:

Proposition 3.2. *Assume that the image of B_n in $\text{End}_{\mathcal{C}}(V^{\otimes n})$ generates these algebras for $n = 2, 3$. Then these representations are isomorphic to the corresponding ones appearing in $\mathcal{U} = \text{Rep}(U_q(\mathfrak{g}(G_2)))$ for some q .*

Proof. If we set $\alpha_{(2,0)} = q^2$, it follows $\alpha_{(1,0)} = -q^{-6}$ from 3.5, $\alpha_{(1,1)} = \pm 1$ from 3.4 and $\alpha_{(0,0)} = \pm q^{-12}$ from 3.3. Using Lemma 3.1 for the representation on $W((1,0), 3) = \text{Hom}(V_{(1,0)}, V^{\otimes 3})$, for which the dimension $m_\lambda = 4$, we obtain for the scalar $z_{(1,0),3}$ by which Δ_3^2 acts that

$$(\alpha_{(0,0)}\alpha_{(1,0)}\alpha_{(1,1)}\alpha_{(2,0)})^{6/4} = z_{(1,0),3} = \Theta_{(1,0)}^{-2} = \alpha_{(1,0)}^4 = q^{-24},$$

where we used Eq. 1.10. It follows that the product of the eigenvalues must be a power of q , i.e. the number of minus signs for the eigenvalues must be even. Otherwise we would not get a power of q from the radical. This implies $\alpha_{(1,1)}\alpha_{(0,0)} = -q^{-12}$, which forces the eigenvalues to be as in the statement, or as in the statement with opposite signs. Also observe that this also shows that the square root of the determinant of σ_1 is the same as in the quantum group case.

It only remains to show that the second option with $\alpha_{(1,1)} = 1$ and $\alpha_{(0,0)} = -q^{-12}$ can not occur for a ribbon tensor category. We obtain braid representations with such eigenvalues for the negative R -matrix in the quantum group case after substituting q^2 by $-q^2$. As the trivial representation $\mathbf{1}$ appears in $V^{\otimes 3}$, where $V = V_{(1,0)}$, the negative R -matrix violates the braiding axioms as follows: Recall that the R -matrix R_V for $V^{\otimes 3} \otimes V^{\otimes 3}$ is the image of the braid $\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3$. It has to act as identity on $\mathbf{1} \otimes \mathbf{1} \subset V^{\otimes 3} \otimes V^{\otimes 3}$. This is no longer the case if we replace R_V by $-R_V$. This finishes the proof.

3.3. Restriction of eigenvalues. It was shown in [21], Section 3 that dimensions of objects in ribbon tensor categories can be determined from braid representations under certain circumstances. More precisely, if Z is a selfdual object in a ribbon tensor category such that $Z^{\otimes 2} = \bigoplus_{i=1}^k Y_i$ with $k \leq 5$ and the Y_i mutually non-isomorphic simple objects, then the quotient $\dim(Y_i)/\dim(Z)^2$ can be determined from the representation of B_3 on $\text{Hom}(Z, Z^{\otimes 3})$, see the corollary in [21], Section 3.2. If one of the Y_i s is isomorphic to Z , this determines the dimension of Z , and hence also the dimensions of the objects Y_i , $1 \leq i \leq k$.

Theorem 3.3. *Let \mathcal{C} be a tensor category of type G_2 such that the image of B_3 in $\text{End}_{\mathcal{C}}(V^{\otimes 3})$ generates the whole algebra. Then the dimension of any object has to coincide with the corresponding object in $\mathcal{U} = \text{Rep}(U_q\mathfrak{g}(G_2))$. In particular, the eigenvalues of $c_{V,V}$ have to be as in the case of the category $\mathcal{U} = \text{Rep}(U_q\mathfrak{g}(G_2))$ with q not a root of unity.*

Proof. It follows from the discussion before this theorem for Z being the object V in \mathcal{C} and $k = 4$ that the dimensions of V , V_{Λ_2} and $V_{2\Lambda_1}$ are completely determined by the 4-dimensional irreducible representation of B_3 in $\text{End}_{\mathcal{C}}(V^{\otimes 3})$. Hence the dimensions of the objects $V = V_{\Lambda_1}$, V_{Λ_2} and $V_{2\Lambda_2}$ are the same as for the quantum group $U_q(\mathfrak{g}(G_2))$ by Proposition 3.2. As $\mathfrak{g}(G_2)$ has rank 2, it is well-known that the dimension of any object in \mathcal{C} is determined by the

dimensions of the fundamental objects V_{Λ_i} , $i = 1, 2$. This shows the first claim. For the second claim, it follows from the explicit dimension formula 2.1 that we would find a simple object V_λ whose dimension would be equal to 0 for q a root of unity $\neq \pm 1$. This would contradict rigidity of \mathcal{C} . If $q = \pm 1$, the braid representation factors through the symmetric group. As $\dim \mathbb{C}S_3 = 6 < \dim \text{End}_{\mathcal{C}}(V^{\otimes 3})$, this is not possible under our assumptions.

4. RIGIDITY OF PATH REPRESENTATIONS

4.1. Main result of section. We consider *path representations of the braid groups B_n of type G_2* , i.e. braid representations for the path algebra \mathcal{P} generated by the 7-dimensional irreducible representation V of the Lie algebra $\mathfrak{g}(G_2)$. Recall that if $\rho_{\nu, m}$ denotes the representation of B_m on the path space $W(\nu, m)$, we have the restriction rule

$$(4.1) \quad (\rho_{\lambda, n})|_{B_{n-1}} \cong \bigoplus_{\mu \leftrightarrow \lambda} \rho_{\mu, n-1},$$

where the summation goes over all μ for which $\lambda - \mu$ is a weight of $V = V_{\Lambda_1}$, with exceptions for the zero weight, see Section 2.3. We moreover assume the following:

- a) The eigenvalues of $\rho(\sigma_1)$ are as in Proposition 3.2 for q not a root of unity.
- b) The representations of B_3 are irreducible for all modules $W(\nu, 3)$.
- c) The braid Δ_n^2 acts as a scalar on each module $W(\nu, n)$ compatible with a ribbon braid structure, see Eq 1.9 and 1.10.

We can also obtain results for path representations with less restrictive conditions, see Corollary 4.12 and the remark after it. As we do not know any non-trivial examples for this more general setting, the goal of this section will be to prove the following theorem:

Theorem 4.1. *Any path representation of braid groups of type G_2 satisfying the conditions stated in this section is uniquely determined by the eigenvalues of σ_1 . More precisely, if we have two such path representations ρ_1 and ρ_2 such that their restrictions to B_2 coincide, then also their representations of B_n on any module $W(\nu, n)$ are isomorphic.*

4.2. Outline of proof. 1. We will show that if we have two path representations ρ_1 and ρ_2 of type G_2 on modules $W_i(\nu, n)$, $i = 1, 2$ which satisfy the conditions stated before Theorem 4.1, then we can also achieve that the matrices for $\rho_1(\sigma_i)$ and $\rho_2(\sigma_i)$, $1 \leq i < n$ coincide after rescaling the basis vectors of, say, the module $W_2(\nu, n)$. This will be done by induction on n , using the restriction rule 4.1 as follows (with $n = 2$ true by assumption):

2. To prove the claim for the module $W_2(\lambda, n)$, we can assume by induction assumption, using 4.1 that the matrices for σ_i with $1 \leq i < n - 1$ coincide for both ρ_1 and ρ_2 . Moreover these matrices do not change if we multiply the vectors for the basis for $\rho_{\mu, n-1}$ by a non-zero scalar, say c_μ , for each μ . Hence the claim will follow if we can show that we can find suitable scalars for the basis of the ρ_2 representation on $W_2(\lambda, n)$ such that $\rho_1(\sigma_{n-1}) = \rho_2(\sigma_{n-1})$.

3. If a matrix block for the new generator σ_{n-1} goes through the diagrams μ_1, \dots, μ_r at level $n - 1$, the matrix for σ_{n-1} in that block will be replaced by the same matrix conjugated

by the diagonal matrix $\text{diag}(c_{\mu_i})$ after the rescaling described in 2. So fixing this particular matrix block for σ_{n-1} will fix the scalars c_{μ_i} in 2 (up to a common multiple). Observe that our previous results show that such a matrix block is uniquely determined up to such a conjugation if the block has at most three distinct eigenvalues, see Prop. 1.9 and Lemma 2.2.

4. Given two extensions ρ_1, ρ_2 of $\bigoplus_{\mu \leftrightarrow \lambda} \rho_{\mu, n-1}$, we choose a block of σ_{n-1} of maximum size for which the matrix has at most three eigenvalues. As mentioned in 3, we can assume that the matrices in both extensions will be the same for this block. We will then first show that we can find a renormalization of basis vectors such that all blocks of σ_{n-1} with ≤ 3 eigenvalues coincide for both extensions. Finally, if there is a block with more than three eigenvalues, we will deduce the same result for it from the equality of all the other blocks, using the braid relations.

4.3. Checking braid relations. In order to check the braid relations for σ_{n-2} and σ_{n-1} , we consider submodules $W_3(\gamma, \lambda)$ whose basis is spanned by all paths of lengths 3 from γ to λ . As γ and λ are fixed by both σ_{n-2} and σ_{n-1} , the basis vectors are given by pairs (α, β) with α the weight on level $n-2$ and β the weight on level $(n-1)$. The block B_α of σ_{n-1} is determined by all paths with fixed first coordinate α and the block C_β of σ_{n-2} is determined by all paths with fixed weight second coordinate β .

Remark 4.2. In the following we will calculate a matrix entry $[\sigma_{n-1}]_{x,y}$ by exhibiting paths s' and s such that $[\sigma_{n-1}]_{x,y}$ is the only unknown entry in the calculation

$$[\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}]_{s',s} = [\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}]_{s',s},$$

where we only need to make sure that the other entries by which $[\sigma_{n-1}]_{x,y}$ is multiplied are nonzero for our choice of parameters.

Remark 4.3. An efficient strategy for calculating the matrix coefficients of, say, the left hand side, is as follows. We call any sequence of paths of the form

$$s' = (\alpha, \beta) \quad - \quad t' = (\alpha, \gamma) \quad - \quad t = (\kappa, \gamma) \quad - \quad s = (\kappa, \delta)$$

a *212 chain* from (α, β) to (κ, δ) . Then it is clear that

$$[\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}]_{s',s} = \sum_{\gamma} [\sigma_{n-1}]_{s',t'} [\sigma_{n-2}]_{t',t} [\sigma_{n-1}]_{t,s},$$

where the summation goes over all γ which generate a 212 chain from $s' = (\alpha, \beta)$ to $s = (\kappa, \delta)$. The calculation of $[\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}]_{s',s}$ can be similarly done via 121 chains, where we first change the first coordinate of $s' = (\alpha, \beta)$; see the proof of Lemma 4.4 for an example.

The following lemma is a fairly straightforward consequence of the braid relations. It is useful as the right hand side of the equation only includes diagonal entries of σ_{n-1} which are easy to calculate.

Lemma 4.4. *Let B_Λ and B_M be two blocks of σ_{n-1} , and let $s = (\Lambda, \beta)$ and $s' = (M, \beta)$. Then*

$$\sum_{\gamma} [\sigma_{n-1}]_{s',t'} [\sigma_{n-2}]_{t',t} [\sigma_{n-1}]_{t,s} = \sum_{u \in C_{\beta}} [\sigma_{n-2}]_{s',u} [\sigma_{n-1}]_{u,u} [\sigma_{n-2}]_{u,s},$$

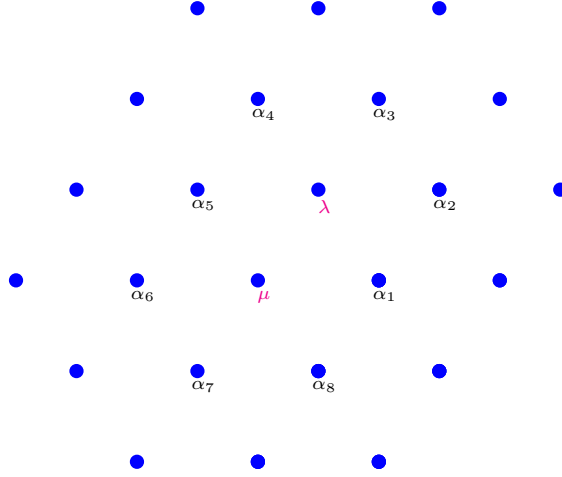
where γ in the first sum is such that $t = (\Lambda, \gamma) \in B_{\Lambda}$ and $t' = (M, \gamma) \in B_M$.

Proof. It follows from the discussion before the lemma that the left hand side is equal to $[\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}]_{s',s}$. To calculate $[\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}]_{s',s}$, just observe that the corresponding 121 chains have to be of the form

$$s' = (\alpha, \beta) \quad - \quad u = (\kappa, \beta) \quad - \quad v = (\kappa, \beta) \quad - \quad s = (M, \beta);$$

hence $u = v$, which implies the claim.

4.4. Set-up for calculating matrix entries. By definition of our representations, it suffices to check the relations for σ_{n-2} and σ_{n-1} on subspaces $W_3(\gamma, \lambda)$ spanned by paths of length 3 from a fixed diagram γ to λ , for each suitable γ . We will do it in detail for the most complicated case, for a weight $\lambda = a\Lambda_1 + b\Lambda_2$ sufficiently far from the walls of the Weyl chamber; this is satisfied if $a, b \geq 3$. For other cases see Section 4.8. One can check that we get the maximum number of paths for a subspace $W_3(\gamma, \lambda)$ if $\gamma = \mu$ has distance 1 from λ . Let us pick $\mu = (a-1)\Lambda_1 + b\Lambda_2$. Using the restriction rule for representations, we obtain 24 paths of length 3 from μ to λ . These will involve the following weights near λ and μ :



$$(4.2) \quad \begin{aligned} \alpha_1 &= (a+1)\Lambda_1 + (b-1)\Lambda_2, & \alpha_2 &= (a+2)\Lambda_1 + (b-1)\Lambda_2, \\ \alpha_3 &= (a+1)\Lambda_1 + b\Lambda_2, & \alpha_4 &= (a-1)\Lambda_1 + (b+1)\Lambda_2, \\ \alpha_5 &= (a-2)\Lambda_1 + (b+1)\Lambda_2, & \alpha_6 &= (a-3)\Lambda_1 + (b+1)\Lambda_2, \\ \alpha_7 &= (a-2)\Lambda_1 + b\Lambda_2, & \alpha_8 &= a\Lambda_1 + (b-1)\Lambda_2. \end{aligned}$$

In this case σ_{n-1} has 7 blocks given by path bases

$$\mathcal{B}_\lambda = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}, \text{ where } t_i = (\lambda, \rho) \text{ with } \rho = \lambda, \mu, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5;$$

$$\mathcal{B}_{\alpha_1} = \{t_8, t_9, t_{10}, t_{11}\}, \text{ where } t_i = (\alpha_1, \rho) \text{ with } \rho = \lambda, \mu, \alpha_1, \alpha_2;$$

$$\mathcal{B}_{\alpha_5} = \{t_{12}, t_{13}, t_{14}, t_{15}\}, \text{ where } t_i = (\alpha_5, \rho) \text{ with } \rho = \lambda, \mu, \alpha_4, \alpha_5;$$

$$\mathcal{B}_\mu = \{t_{16}, t_{17}, t_{18}, t_{19}\}, \text{ where } t_i = (\mu, \rho) \text{ with } \rho = \lambda, \mu, \alpha_1, \alpha_5;$$

$$\mathcal{B}_{\alpha_6} = \{t_{20}, t_{21}\}, \text{ where } t_i = (\alpha_6, \rho) \text{ with } \rho = \mu, \alpha_5;$$

$$\mathcal{B}_{\alpha_8} = \{t_{22}, t_{23}\}, \text{ where } t_i = (\alpha_8, \rho) \text{ with } \rho = \mu, \alpha_1;$$

$$\mathcal{B}_{\alpha_7} = \{t_{24}\}, \text{ where } t_i = (\alpha_7, \rho) \text{ with } \rho = \mu.$$

The corresponding blocks of σ_{n-2} are

$$C_\lambda = \{t_1, t_8, t_{12}, t_{16}\},$$

$$C_\mu = \{t_2, t_{10}, t_{13}, t_{17}, t_{20}, t_{22}, t_{24}\},$$

$$C_{\alpha_1} = \{t_3, t_9, t_{18}, t_{23}\},$$

$$C_{\alpha_2} = \{t_4, t_{11}\},$$

$$C_{\alpha_3} = \{t_5\},$$

$$C_{\alpha_4} = \{t_6, t_{14}\},$$

$$C_{\alpha_5} = \{t_7, t_{15}, t_{19}, t_{21}\}.$$

Remark 4.5. It is easy to see from the picture that we get similar block structures for the module $W_3(\alpha_1, \lambda)$, where we just reflect the paths for each block above at the vertical axis going through λ . Observe that the block B_λ remains unchanged by this, as the action of σ_{n-1} only depends on the labels $t(i)$, $n-2 \leq i \leq n$ of a path t .

4.5. Calculating diagonal entries. By Lemma 2.2, the blocks of σ_{n-1} of size ≤ 4 can be calculated up to conjugation by a diagonal matrix, i.e. up to rescaling of basis vectors. This determines the diagonal entries of all such blocks as well as the product of transposed entries, say $a_{st}a_{ts}$. We now show that we can also calculate the diagonal entries of the big 7×7 block of σ_{n-1} .

Lemma 4.6. *The diagonal entries of the matrices for σ_{n-1} are uniquely determined by the entries of σ_{n-2} and by the results in Lemma 2.2.*

Proof. By Lemma 2.2, we only need to consider blocks in which σ_{n-1} acts with more than three eigenvalues. The only such block in $W_3(\mu, \lambda)$ is the block B_λ . We first find the three diagonal matrix entries $[\sigma_{n-1}]_{u,u}$ with $u = (\lambda, \beta) \in B_\lambda$ such that $(\mu, \beta) \in B_\mu$ and $\beta \neq \mu$, i.e. for $\beta \in \{\lambda, \alpha_1, \alpha_5\}$. For this, we use Lemma 4.4 with $s = s' = (\mu, \beta)$, $\Lambda = M = \mu$, from which we get

$$\sum_{t \in \mathcal{B}_\mu} [\sigma_{n-1}]_{s,t} [\sigma_{n-2}]_{t,t} [\sigma_{n-1}]_{t,s} = \sum_{u \in C_\beta} [\sigma_{n-2}]_{s,u} [\sigma_{n-1}]_{u,u} [\sigma_{n-2}]_{u,s}.$$

The matrix entries on the left hand side are either known diagonal entries, or products of transposed entries which are known. The same applies to all entries on the right hand side, except one, namely $[\sigma_{n-1}]_{u,u}$ with $u = (\lambda, \beta) \in B_\lambda$. Moreover, the entry $[\sigma_{n-1}]_{u,u}$ is multiplied by matrix entries of σ_{n-2} which are nonzero for q not a root of unity, by Lemma 2.2. Thus, we find $[\sigma_{n-1}]_{t_1,t_1}$, $[\sigma_{n-1}]_{t_3,t_3}$, $[\sigma_{n-1}]_{t_7,t_7}$. Using the reflection symmetry, see Remark 4.5, we can similarly also calculate the diagonal entries of σ_{n-1} for the paths $t_1 = (\lambda, \lambda)$, $t_2 = (\lambda, \mu)$ and $t_4 = (\lambda, \alpha_2)$.

To calculate the diagonal entry for the path (λ, α_4) , we similarly use Lemma 4.4 for $s = s' = (\alpha_5, \alpha_4)$. The entry for (λ, α_3) is obtained by essentially the same calculation after using the reflection symmetry in Remark 4.5.

4.6. Determining most matrix blocks. We are dealing with the first part of point 4 of our outline. This means we are going to show that there exists a diagonal matrix which conjugates each block of $\rho_1(\sigma_{n-1})$ in which the matrix acts with ≤ 3 eigenvalues to the corresponding block of $\rho_2(\sigma_{n-1})$ and does not change the image of B_{n-1} .

Lemma 4.7. *Let ρ_1 and ρ_2 be two representations of B_n with the same path basis such that $\rho_1|_{B_{n-1}} = \rho_2|_{B_{n-1}}$ produce the same matrices. Then we can also make coincide all blocks for σ_{n-1} in which it has at most three different eigenvalues.*

Proof. By point three of our outline, we can assume that the matrices for block B_μ coincide for both representations. We will show that the braid relations will essentially determine the action of the generator σ_{n-1} for the blocks in the statement. We will frequently use the following two observations:

- a) The right hand side of Lemma 4.4 can always be calculated, by Lemma 4.6.

- b) If the matrix entry $[\sigma_{n-1}]_{x,y}$ can be calculated from the matrix entry $[\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}]_{s',s}$ as outlined in Remark 4.2, we can similarly calculate the entry $[\sigma_{n-1}]_{y,x}$ from the matrix entry $[\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}]_{s,s'}$.

Notation: We will say that a matrix entry *is known* if it can be expressed in terms of entries of σ_{n-2} and of the entries of σ_{n-1} from the fixed block B_μ . Observe that this means that such entries have to coincide in the two path representations ρ_1 and ρ_2 .

- a) We first find the matrix entries for the two 2×2 blocks B_α with $\alpha = \alpha_6$ or α_8 . We outline the calculation for $\alpha = \alpha_6$, where we use Lemma 4.4 with $s = (\alpha_6, \mu)$ and $s' = (\mu, \mu)$. The only unknown quantity on the left hand side is $[\sigma_{n-1}]_{t,s}$ for $t = (\alpha_6, \alpha_5)$; the only other possibility for t would be $t = s$ for which we get a known diagonal entry $[\sigma_{n-1}]_{s,s}$. Hence we can solve for $[\sigma_{n-1}]_{t,s}$, by Lemma 2.2.
- b) Let now B_α be one of the two other 4×4 blocks, i.e. $\alpha = \alpha_1$ or α_5 . We do the case with $\alpha = \alpha_1$. We first calculate the entry $[\sigma_{n-1}]_{t,s}$, where $t = (\alpha_1, \mu)$ and $s = (\alpha_1, \alpha_1)$ by using Lemma 4.4 with $s' = (\alpha_8, \mu)$. As B_{α_8} only has two paths, we only have two summands on the left hand side in that Lemma, with the only unknown quantity $[\sigma_{n-1}]_{t,s}$.

Next, we consider matrix entries involving the path $r = (\alpha_1, \lambda)$. We again use Lemma 4.4 with $s = (\alpha_1, \rho)$, $s' = (\mu, \rho)$, where $\rho \in \{\alpha_1, \mu\}$. There are three summands on the left hand side, for which all quantities are known except $[\sigma_{n-1}]_{r,s}$ (the other two quantities $[\sigma_{n-1}]_{t,s}$ are either a diagonal entry or known from the previous paragraph).

Recall that all matrix blocks where the braid generator has ≤ 3 eigenvalues are already determined by Lemma 1.9, up to conjugation by a diagonal matrix. Now we only need to observe that the path $t_{11} = (\alpha_1, \alpha_2)$ belongs to a block of σ_{n-2} which does not have paths appearing in B_μ . Hence we can rescale the paths in the block C_{α_2} so that the block B_{α_1} is the same in both representations ρ_1 and ρ_2 .

4.7. Calculating the unknown block. We have seen in Lemma 4.7 that we can make equal all but one matrix block of $\rho_1(\sigma_{n-1})$ with $\rho_2(\sigma_{n-1})$ on $W_3(\mu, \lambda)$ if all the blocks of $\rho_1(\sigma_{n-2})$ equal the ones of $\rho_2(\sigma_{n-2})$ and there is one 4×4 block for which $\rho_1(\sigma_{n-1})$ equals $\rho_2(\sigma_{n-1})$. Moreover, we also know that the diagonal entries have to coincide for all blocks. We now want to show that this also enables us to make the remaining matrix entries equal.

Lemma 4.8. *Let s_1 and s_2 be paths belonging to the unknown 7×7 block B_λ of σ_{n-1} . Then the matrix entry $[\sigma_{n-1}]_{s_1, s_2}$ is uniquely determined if none of the paths s_1 or s_2 is of the form (λ, μ) or (λ, α_3) .*

Proof. As neither of the paths s_1, s_2 belongs to the 1×1 block C_{α_3} , we can find paths $t_i \neq s_i$ in the same σ_{n-2} block. As neither path t_i belongs to the unknown block B_λ we can calculate

$$[\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}]_{t_2, t_1} = \sum_{u,v} [\sigma_{n-1}]_{t_2, u} [\sigma_{n-2}]_{u, v} [\sigma_{n-1}]_{v, t_1}.$$

As neither path s_i or t_i is in the block C_μ by assumption, we also know that the entries $[\sigma_{n-2}]_{t_i, s_i} \neq 0$ for $i = 1, 2$, by Lemma 2.2. Hence we can solve for $[\sigma_{n-1}]_{s_2, s_1}$, provided we can

show it is the only unknown entry in

$$[\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}]_{t_2, t_1} = \sum_{u', v'} [\sigma_{n-2}]_{t_2, u'} [\sigma_{n-1}]_{u', v'} [\sigma_{n-2}]_{v', t_1}.$$

Writing $\sigma_{n-2} \cdot t_i$ as a linear combination of paths t , there is only one of them in B_λ , namely s_i . Hence all other matrix coefficients $[\sigma_{n-1}]_{u', v'}$ in the sum above belong to blocks whose coefficients are known. This finishes the proof.

In order to calculate the additional matrix entries of B_λ , we will use Remark 4.5. Applying Lemma 4.7 to all paths in $W_3(\alpha_1, \lambda)$ we can similarly assume that all corresponding blocks B_γ of $\rho_i(\sigma_{n-1})$ of size ≤ 4 also coincide on $W_3(\alpha_1, \lambda)$ for $i = 1, 2$. Moreover, we can also assume that if a block B_γ appears in both $W_3(\mu, \lambda)$ and $W_3(\alpha_1, \lambda)$, we get the same matrices.

Lemma 4.9. *After suitable renormalizations of the path basis vectors for ρ_2 , the matrices $\rho_1(\sigma_{n-1})$ and $\rho_2(\sigma_{n-1})$ coincide.*

Proof. By Lemma 4.7, it only remains to check the claim for those matrix entries in B_λ which have not already been covered in Lemma 4.8. Observe that if we apply Lemma 4.8 to the block B_λ in $W_3(\alpha_1, \lambda)$, we have to exclude the paths (λ, α_1) and (λ, α_4) , by using the symmetry in Remark 4.5. Hence we can calculate all matrix entries in B_λ except those entries $[\sigma_{n-1}]_{u, v}$ where one path is $t_2 = (\lambda, \mu)$ or $t_5 = (\lambda, \alpha_3)$, and the other path is $t_3 = (\lambda, \alpha_1)$ or $t_6 = (\lambda, \alpha_4)$. There are eight such entries, which always come in pairs (u, v) and (v, u) . We proceed as outlined in Remark 4.2. As observed in the proof of Lemma 4.7, it suffices to indicate how to calculate one entry for each of these pairs. Here we use notation from Section 4.4:

- For $(u, v) = (t_2, t_3)$, we take $(s', s) = (t_2, t_{22})$;
- For $(u, v) = (t_2, t_6)$, we take $(s', s) = (t_2, t_{13})$;
- For $(u, v) = (t_5, t_3)$, we take $(s', s) = (t_5, t_{10})$;
- For $(u, v) = (t_5, t_6)$, we take $(s', s) = (t_5, t_{13})$.

4.8. Matrices for weights near the boundary of the Weyl chamber. If the weight λ is near the boundary of the Weyl chamber, we can calculate the block B_λ similarly as before. However, some of the arguments need to be changed. In particular, we can usually not apply Remark 4.5. The most involved case is if we take $\lambda = a\Lambda_1$ and $\mu = (a-1)\Lambda_1$. We can still use the picture for the weights in the general case, except we need to remove all weights below the line connecting λ and μ . One can see from this that the block B_λ has 5 paths. We proceed as in the general case, where we need to make the following changes (the necessary lemmas appear in the next section).

- a) We can calculate three diagonal entries of the block B_λ as in Lemma 4.6, and we can calculate the diagonal entry for the path (λ, μ) from Lemma 4.10. This also determines the fifth diagonal entry as we know the trace of σ_{n-1} in that block.
- b) We can then calculate all matrix blocks of size ≤ 4 as in Lemma 4.7, assuming we fix the matrix for block B_{α_5} .

- c) We calculate all matrix entries of the block B_λ not involving the paths (λ, μ) and (λ, α_3) as in Lemma 4.8.
- d) We calculate the matrix entries $[\sigma_{n-1}]_{u,v}$ involving the path $u = (\lambda, \mu)$ but not the path (λ, α_3) similarly as it was done at the end of the proof of Lemma 4.9 as follows (it is important to do it in the given order):
 - For $v = (\lambda, \alpha_5)$, we take $(s', s) = (u, (\alpha_6, \mu))$;
 - For $v = (\lambda, \lambda)$, we take $(s', s) = (u, (\mu, \mu))$;
 - For $v = (\lambda, \alpha_4)$, we take $(s', s) = (u, (\alpha_5, \mu))$.
- e) The only missing matrix entries of the block B_λ are the ones involving the path $t_5 = (\lambda, \alpha_3)$. These are determined in Lemma 4.11, where $t_0 = t_5$.

The blocks B_λ near the left boundary can be similarly calculated, where one has to make far fewer adjustments. E.g. if $\lambda = a\Lambda_2$ and $\mu = \Lambda_1 + (a-1)\Lambda_2$, the block B_λ only has three paths and can be calculated as in Lemma 2.2. If $\lambda = \Lambda_1 + a\Lambda_2$ and $\mu = a\Lambda_2$, we can again use Remark 4.5.

4.9. Additional lemmas. Let p_i be the eigenprojection of the image of σ_i corresponding to the eigenvalue $\alpha_{(0,0)}$. We see from the Bratteli diagram that $p_1\rho(\sigma_2)^mp_1 = \gamma_m p_1$ for all $m \in \mathbb{Z}$, where γ_m is a scalar. The same relation holds if we shift the indices to $n-2$ and $n-1$. The projection p_{n-2} only is nonzero in the block C_μ . For the path algebra for $U_q\mathfrak{g}(G_2)$ (in fact, for any self-dual representation of a quantum group) its diagonal entries can be calculated as it was done in e.g. [18] (2.15) in connection with orthogonal and symplectic groups. Indeed, the diagonal entry of p_{n-2} for the path $t = (\gamma, \mu)$ is given by

$$\frac{\dim_q V_\gamma}{\dim_q V \dim_q V_\mu},$$

with q -dimensions as given in Eq 2.1. In particular, we also obtain from this that the diagonal entries are non-zero for q not a root of unity. As p_{n-2} acts as a rank 1 idempotent in that block, all its matrix entries in that block are non-zero.

Lemma 4.10. *Assume we have a path representation of B_n such that its restriction to B_{n-1} is isomorphic to the one of $U_q\mathfrak{g}(G_2)$. Then the diagonal entry of σ_{n-1} for the path (λ, μ) can be calculated from the other diagonal entries of σ_{n-1} and the entries of p_{n-2} .*

Proof. We compare the $u = (\lambda, \mu)$ diagonal entries of the equation $p_{n-2}\rho(\sigma_{n-1})p_{n-2} = \gamma_1 p_{n-2}$ to obtain the equation

$$\sum_\nu p_{u,(\nu,\mu)}[\sigma_{n-1}]_{(\nu,\mu),(\nu,\mu)}p_{(\nu,\mu),u} = \gamma p_{u,u},$$

where $p_{u,v}$ is a matrix entry of p_{n-2} . Observe that the product of opposite off-diagonal entries of p_{n-2} is equal to the product of the corresponding diagonal entries, which are known and nonzero by assumption. Hence we can solve for the only unknown diagonal entry of σ_{n-1} with $\nu = \lambda$ in that equation.

Lemma 4.11. *Assume all matrix entries of $\rho_1(\sigma_{n-1})$ and $\rho_2(\sigma_{n-1})$ coincide except possibly the non-diagonal entries involving the path t_0 belonging to the 1×1 block C_{α_3} . Then we can renormalize the basis vector t_0 such that the matrices $\rho_1(\sigma_{n-1})$ and $\rho_2(\sigma_{n-1})$ coincide.*

Proof. Let t_1 and t_2 be paths in B_λ such that $t_i \neq t_0$ for both $i = 1, 2$. Then we can calculate

$$[\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}]_{t_1,t_2} = \sum_{u,v \neq t_0} [\sigma_{n-2}]_{t_1,u} [\sigma_{n-1}]_{u,v} [\sigma_{n-2}]_{v,t_2};$$

indeed as $u(n-1) = t_1(n-1) \neq t_0(n-1)$ and $v(n-1) = t_2(n-1) \neq t_0(n-1)$, $[\sigma_{n-1}]_{u,v}$ is known for all summands on the right hand side, and hence so is the whole sum. By Lemma 4.4

$$[\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}]_{t_1,t_2} = \sum_{s \text{ in } B_\lambda} [\sigma_{n-1}]_{t_1,s} [\sigma_{n-2}]_{s,s} [\sigma_{n-1}]_{s,t_2},$$

where we know all summands on the right hand side except the one with $s = t_0$. The 1×1 block entry $[\sigma_{n-2}]_{t_0,t_0} \neq 0$. Hence we can solve for $[\sigma_{n-1}]_{t_1,t_0} [\sigma_{n-1}]_{t_0,t_2}$, for any t_1, t_2 in the unknown block in terms of known matrix entries. Pick a $v = t_2 \neq t_0$ such that $[\sigma_{n-1}]_{v,t_0} [\sigma_{n-1}]_{t_0,v} \neq 0$. Then also the quotients $[\sigma_{n-1}]_{t_1,t_0} / [\sigma_{n-1}]_{v,t_0}$ are determined in terms of known matrix entries.

We can now rescale the basis vector t_0 for ρ_2 such that $\rho_1(\sigma_{n-1})_{v,t_0} = \rho_2(\sigma_{n-1})_{v,t_0}$. Then the claim follows from the statements of the results in the previous paragraph.

4.10. Proof of Theorem 4.1. We will prove the theorem for representations of B_n on paths of length n by induction on n . The statement is true by assumption for $n = 2$. For $n = 3$, we have irreducible representations of B_3 up to dimension 4. Here the statement follows from [21]: It is proven there that irreducible representations of B_3 up to dimension 3 are completely determined by the eigenvalues of σ_1 . For dimension 4, there are two possibilities for given eigenvalues of σ_1 , depending on the choice of the square root of the determinant, as the central element Δ_3^2 acts via the scalar $\det(\sigma_1)^{3/2}$. This square root is determined by the ribbon braid structure and has to be equal to q^{-8} , see the proof of Proposition 3.2.

For the general induction step, we proceed as sketched in the outline. By induction assumption, we can assume that the matrices $\rho_1(\sigma_i)$ and $\rho_2(\sigma_i)$ coincide for $i < n - 1$. By Lemmas 4.7 and 4.8 we can also achieve that the matrices $\rho_j(\sigma_{n-1})$ coincide for $j = 1, 2$.

Corollary 4.12. *The uniqueness result of Theorem 4.1 would also hold for any path representation of type G_2 for which the statement in Lemma 2.2 about certain matrix entries being nonzero holds.*

Proof. The proof of Theorem 4.1 only needed that certain matrix entries were well-defined and nonzero. Their explicit values were irrelevant.

Remark 4.13. It is easy to obtain additional path representations from a given one by multiplying the matrices for the standard generators σ_i by a common constant. We are not aware of any other path representations of type G_2 besides the ones already mentioned.

4.11. Application to tensor categories. Let \mathcal{C} be a semisimple ribbon tensor category whose fusion rules are the ones of $\mathfrak{g}(G_2)$, and let \mathcal{U} be equal to $\text{Rep}(U_q\mathfrak{g}(G_2))$. The following theorem has already appeared before in [16], where it was proved by a different method (see also the remarks below).

Theorem 4.14. *Let V be the object in \mathcal{C} corresponding to the 7-dimensional representation of $U_q\mathfrak{g}(G_2)$. Assume that the image of B_3 generates $\text{End}_{\mathcal{C}}(V^{\otimes 3})$. Then $\mathcal{C} \cong \text{Rep}(U_q\mathfrak{g}(G_2))$ for some q not a root of unity.*

Proof. It follows from Theorem 3.3 that the eigenvalues of the braiding element $c_{V,V}$ coincide with the ones for the corresponding braiding morphism in $\mathcal{U} = \text{Rep}(U_q\mathfrak{g}(G_2))$ for q not a root of unity. Theorem 4.1 and Theorem 2.3 now imply that $\text{End}_{\mathcal{C}}(V^{\otimes n}) \cong \text{End}_{\mathcal{U}}(V^{\otimes n})$ such that the tensor embeddings

$$\text{End}_{\mathcal{C}}(V^{\otimes n}) \otimes \text{End}_{\mathcal{C}}(V^{\otimes m}) \rightarrow \text{End}_{\mathcal{C}}(V^{\otimes n+m})$$

are compatible with these isomorphisms. As explained in [8], see also [22], this almost implies the equivalence of categories \mathcal{C} and \mathcal{U} . The only additional necessary information comes from the embeddings of the trivial object $\mathbf{1}$ into some tensor power $V^{\otimes N}$. If we have

$$\iota_N : \mathbf{1} \rightarrow V^{\otimes N} \quad \text{and} \quad \pi_N : V^{\otimes N} \rightarrow \mathbf{1}$$

such that $\pi_N \circ \iota_N$ is the identity of $\mathbf{1}$, we obtain a number $\Theta(N)$ defined by

$$\Theta(N)1_V = (\pi_N \otimes 1_V) c_{V,V^{\otimes N}} (1_V \otimes \iota_N).$$

One easily derives from this also that

$$c_{V,V^{\otimes N}} (1_V \otimes \iota_N) = \Theta(N)(\iota_N \otimes 1_V),$$

$$c_{V^{\otimes N},V}(\iota_N \otimes 1_V) = \Theta(N)^{-1}(1_V \otimes \iota_N),$$

see [22], Lemma 4.1 (where part (b) is correct if Θ is replaced by Θ^{-1}). It follows from [22], Proposition 4.5 that $\Theta(N)^N = 1$. Now observe that the trivial representation $\mathbf{1}$ appears both in the second and the third tensor power with multiplicity 1. Using the formulas above and the braiding axiom 1.5, we calculate the middle expression in the formula below in two different ways to obtain

$$\Theta(3)^2 1_{\mathbf{1}} = (\pi_3 \otimes \pi_2) c_{V^{\otimes 2},V^{\otimes 3}} (\iota_2 \otimes \iota_3) = \Theta(2)^{-3} 1_{\mathbf{1}}.$$

Hence it follows that $\Theta(3) = 1 = \Theta(2)$. The equivalence of \mathcal{C} and \mathcal{U} now follows e.g. from [22], Theorem 4.8 and Corollary 3.6 or [8], Proposition 2.1.

Remark 4.15. 1. The techniques in this paper should also work for fusion tensor categories of type G_2 , for q a root of unity, assuming that B_3 generates $\text{End}(V^{\otimes 3})$. Our approach might even be useful to prove analogous results for tensor products of type E_6 and E_7 . Indeed, results similar to the ones in Lemma 2.2 have already been proved in [26] for Lie types E_N in general.

2. Our methods can not be applied for symmetric tensor categories. However, it would seem reasonable to expect that analogous techniques, using infinitesimal braid relations and Casimirs, see e.g. [11] might still work for such cases.

3. We have results which would replace our surjectivity assumption for the third tensor power by somewhat weaker assumptions (e.g. just assuming semisimplicity of the representation of B_3 for $q \neq \pm 1$). We would obtain a complete classification of tensor categories of type G_2 for $q \neq \pm 1$ if we could show that the surjectivity assumptions would always hold in these cases.

4. Theorem 4.14 has already been proved in [16] with the same assumptions for $\text{End}(V^{\otimes 3})$ as in our paper, including a generalization to $q = \pm 1$. The authors used Kuperberg's spiders in [9] which do not play a role in our approach.

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