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Lower bounds for discrete negative moments of the Riemann zeta function

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We prove lower bounds for the discrete negative $2k$-th moment of the derivative of the Riemann zeta function for all fractional k. The bounds are in line with a conjecture of Gonek and Hejhal. Along the way, we prove a general formula for the discrete twisted second moment of the Riemann zeta function. This agrees with a conjecture of Conrey and Snaith.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta function. We are interested in the discrete negative moments

$$J_{-k}(T)=\sum_{0\leqslant\gamma\leqslant T}\frac{1}{|\zeta'(\rho)|^{2k}}$$

where $\rho=\beta+i\gamma$ are the nontrivial zeros of $\zeta(s)$ and k is a positive real number. A natural assumption when considering these moments is that all the nontrivial zeros of the zeta function are simple. We therefore assume this throughout the paper unless otherwise mentioned. From work of Ingham [1942], Titchmarsh [1986, Theorem 14.27], Odlyzko and te Riele [1985], Ng [2004] and Montgomery and Vaughan [2007, Theorem 15.5], we know that $J_{-k}(T)$ is closely related to the partial sums of the Möbius function. There are also some more recent works that relate $J_{-k}(T)$ to other arithmetic problems; see, e.g., [Saha and Sankaranarayanan 2019; Meng 2017; Humphries 2013; Suzuki 2011].

Gonek [1989] and Hejhal [1989] independently conjectured that

$$J_{-k}(T)\asymp T(\log T)^{(k-1)^2}$$

for all real k. However, the range of k in which this conjecture holds seems to be in doubt since Gonek (unpublished) has suggested that there exist infinitely many zeros ρ for which $\zeta'(\rho)^{-1}\gg|\gamma|^{1/3-\epsilon}$, in which case the conjecture would fail for $k>\frac{3}{2}$. Hughes, Keating and O'Connell [Hughes et al. 2000] used random matrix theory to predict a precise constant in this conjecture for general real k. Interestingly, their formulas on the random matrix theory side undergo a phase change at the point $k=\frac{3}{2}$ which gives alternative evidence that the conjecture may fail for $k>\frac{3}{2}$. The positive moments of this conjecture have been studied a lot. We only know the asymptotic behaviour of $J_1(T)$ due to work of Gonek [1984]. A sharp lower bound for J_k with $k\in\mathbb{N}$ was proved by Milinovich and Ng [2014] under the generalised

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Riemann hypothesis (GRH) for Dirichlet L-functions whilst a sharp upper bound for J_k with real $k > 0$ has been recently proved by Kirila [2020] under the Riemann hypothesis (RH), improving previous work of Milinovich [2010].

Little is known about the negative moments in this conjecture. For $J_{-1}(T)$, Gonek [1989] made the more precise conjecture (which agrees with the prediction in [Hughes et al. 2000]) that

$$J_{-1}(T) \sim \frac{3}{\pi^3} T$$

and showed under RH that

$$J_{-1}(T) \geq CT$$

with an unspecified constant C. Milinovich and Ng [2012] later proved that

$$J_{-1}(T) \geqslant (1 + o(1)) \frac{3}{2\pi^3} T$$

where the constant differs from the conjectured value by only a factor of 2. By an application of Hölder's inequality, Gonek [1989] showed that under RH

$$J_{-k}(T) \gg T(\log T)^{1-3k}$$

for all $k > 0$. In the special case $k = \frac{1}{2}$, Heath-Brown has shown that [Titchmarsh 1986, page 386]

$$J_{-1/2}(T) \gg T$$

via the connection with $\sum_{n \leqslant x} \mu(n)$ under RH. Our aim in this paper is to improve these lower bounds for all fractional k. In fact, we obtain the sharp lower bound for all fractional discrete negative moments conjectured by Gonek [1989] and Hejhal [1989].

Theorem 1. *Assume RH and that all zeros of $\zeta(s)$ are simple. Then*

$$J_{-k}(T) \gg T(\log T)^{(k-1)^2}$$

for all fractional $k \geqslant 0$.

Remarks. • With $k = \frac{1}{2}$, our theorem gives the following improvement to Heath-Brown's bound:

$$J_{-1/2}(T) \gg T(\log T)^{1/4}, \tag{1}$$

which has been obtained independently by Milinovich, Ng and Soundararajan using similar methods. In fact, they gave the same lower bound as in (1) even when the sum is restricted to the simple nontrivial zeros.

• The implicit constants depend on the height of the rational number k. This is a common feature of the method which we discuss in detail below.

• The work of Radziwiłł and Soundararajan [2013] gives lower bounds for the $2k$-th moment of the Riemann zeta function in the t-aspect for all *real* $k \geqslant 1$. This was recently extended to all real $k \geqslant 0$ by Heap and Soundararajan [2022] using a different argument. It is likely that one could use the latter methods to extend Theorem 1 to all *real* $k \geqslant 0$. A key ingredient would be a formula for the twisted second moment which we give in Theorem 4 below. In addition, the techniques used in [Heap and Soundararajan 2022] to compute real powers of Dirichlet polynomials in the t-aspect should be extended for the discrete averages considered here.

Now let us discuss the strategy in proving Theorem 1. Our proof utilises the method of Rudnick and Soundararajan [2005] and shares some similarities with the work of Chandee and Li [2013] on lower bounds for fractional moments of Dirichlet L-functions in the q-aspect. Here and throughout, let

$$k = a/b$$

with $a, b \in \mathbb{N}$. To prove Theorem 1, we apply Hölder's inequality in the form

$$\sum_{0\leqslant\gamma\leqslant T} |P(\rho)|^{2a} \leqslant \biggl(\sum_{0\leqslant\gamma\leqslant T} |\zeta'(\rho)|^2 |P(\rho)|^{2(a+b)}\biggr)^{a/(a+b)} \biggl(\sum_{0\leqslant\gamma\leqslant T} \frac{1}{|\zeta'(\rho)|^{2k}}\biggr)^{b/(a+b)}, \tag{2}$$

where

$$P(s) = \sum_{n\leqslant x} \frac{\tau_{-1/b}(n)}{n^s}\psi(n), \tag{3}$$

and $x = T^{\theta/(a+b)}$ with $\theta < \frac{1}{2}$. Here, $\tau_\alpha(*)$ denotes the Dirichlet series coefficients of $\zeta(s)^\alpha$, $\alpha \in \mathbb{C}$, and $\psi(*)$ is a smoothing weight which will be properly defined later (see formula (7) below). We then have the following two propositions.

Proposition 2. *Let $P(s)$ be given by* (3) *and let*

$$S_1 := \sum_{0\leqslant\gamma\leqslant T} P(\rho)^a P(1-\rho)^a.$$

Then for fixed $a, b \in \mathbb{N}$,

$$S_1 \sim c(a,b) T (\log T)^{a^2/b^2+1}$$

for some positive constant $c(a,b)$ as $T \to \infty$.

Proposition 3. *Let $P(s)$ be given by* (3) *and let*

$$S_2 := \sum_{0\leqslant\gamma\leqslant T} \zeta'(\rho)\zeta'(1-\rho) P(\rho)^{a+b} P(1-\rho)^{a+b}.$$

Then for fixed $a, b \in \mathbb{N}$,

$$S_2 \ll_{a,b} T(\log T)^{a^2/b^2+3}$$

as $T \to \infty$.

Note these propositions are unconditional. Assuming RH, the sums S_1 and S_2 become those in Hölder's inequality (2) giving

$$J_{-k}(T) \geqslant \frac{S_1^{k+1}}{S_2^k} \gg T\frac{(\log T)^{(k+1)(k^2+1)}}{(\log T)^{k(k^2+3)}} = T(\log T)^{(k-1)^2},$$

and Theorem 1 follows.

In proving Proposition 2, we first apply a result of Ng [2008a] which gives a formula for sums of the S_1-type when $P(s)$ is from a fairly general class of Dirichlet polynomials. To evaluate the resulting formula, we generalise and simplify the argument of Chandee and Li [2013]. Our method allows in some cases for an asymptotic evaluation (rather than only the order of the magnitude as in [Chandee and Li 2013]) of the multidimensional Mellin integrals which commonly feature in this area; see, e.g., [Kowalski et al. 2000]. For Proposition 3, we need a general formula for sums of the S_2-type, which we were not able to find in the literature when $P(s)$ has general coefficients. However, some specific cases for $P(s)$ have been dealt with in [Bui 2011; Bui and Heath-Brown 2013; Conrey et al. 1986; 1998; Feng and Wu 2012; Ng 2008b]. Following their methods, we derive the following result.

Theorem 4. *Let $\alpha, \beta \ll 1/\log T$ be sufficiently small shifts. Let $Q(s) = \sum_{n\leqslant y} a(n)n^{-s}$ with $y = T^\theta$ and $\theta < \frac{1}{2}$ and denote $\bar{Q}(s) = \sum_{n\leqslant y} \overline{a(n)}n^{-s}$. Suppose that there exist some fixed positive constants r and C such that $|a(mn)| \ll |a(m)a(n)|$ and $|a(n)| \ll \tau_r(n)(\log n)^C$. Then for any constant $A > 0$, it holds that*

$$\sum_{0\leqslant\gamma\leqslant T} \zeta(\rho+\alpha)\zeta(1-\rho+\beta)Q(\rho)\bar{Q}(1-\rho) = \mathcal{J}(\alpha,\beta,T) + \mathcal{L}(\alpha,\beta,T) + \overline{\mathcal{L}(\bar{\beta},\bar{\alpha},T)} + O(T(\log T)^{-A}), \quad (4)$$

where

$$\mathcal{J}(\alpha,\beta,T) = \sum_{g\leqslant y}\sum_{\substack{h,k\leqslant y/g\\(h,k)=1}} \frac{a(gh)\overline{a(gk)}}{ghk}\frac{1}{2\pi}\int_0^T \log\Big(\frac{t}{2\pi}\Big)\Big[\frac{\zeta(1+\alpha+\beta)}{h^\beta k^\alpha} + \Big(\frac{t}{2\pi}\Big)^{-\alpha-\beta}\frac{\zeta(1-\alpha-\beta)}{h^{-\alpha}k^{-\beta}}\Big]dt \quad (5)$$

and

$$\mathcal{L}(\alpha,\beta,T) = \frac{d}{d\gamma}\frac{1}{2\pi}\int_0^T\Big[S_{\alpha,\beta,\gamma}(T) + \Big(\frac{t}{2\pi}\Big)^{-\alpha-\beta}S_{-\beta,-\alpha,\gamma}(T) + \Big(\frac{t}{2\pi}\Big)^{-\beta-\gamma}S_{\alpha,-\gamma,-\beta}(T)\Big]dt\Big|_{\gamma=0}$$

with

$$S_{\alpha,\beta,\gamma}(T) = \sum_{h,k\leqslant y}\sum_{hm=kn}\frac{a(h)\overline{a(k)}f_{\alpha,\gamma}(m)n^{-\beta}}{(hkmn)^{1/2}}w(mn/T^2),$$
$$f_{\alpha,\gamma}(n) = \sum_{n_1n_2n_3=n}\mu(n_1)n_2^{-\alpha}n_3^{-\gamma}, \quad (6)$$

and

$$w(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{H(s)}{s}x^{-s}ds, \qquad c > 0.$$

Here $H(s) = H_{\alpha,\beta,\gamma}(s)$ is an analytic function such that $H(\sigma+it) \ll_\sigma e^{-C't^2}$ for large t for some constant $C' > 0$, and additionally satisfies $H(0) = 1$ and is zero at $2s = \beta - \alpha$ and $2s = \beta - \gamma$. The constant c in $w(x)$ can be any positive real number.

Remarks. • We note that this result is unconditional. Initially, results of this type required the assumption of GRH [Conrey et al. 1986] to analyse the error term. This condition was later weakened to the generalised Lindelöf hypothesis by Conrey, Ghosh and Gonek [Conrey et al. 1998] by an application of the large sieve inequality, and was finally made unconditional by Bui and Heath-Brown [2013] using Heath-Brown's identity. We follow the latter method when analysing our error terms. One of the important features of our formula is that we do not need to assume that the coefficients $a(n)$ are supported on squarefree integers, which was required in [Bui and Heath-Brown 2013]; see Section 6 for a detailed discussion.

• Our main term takes the form as predicted by the recipe method/ratios conjecture [Conrey et al. 2005; Conrey and Snaith 2007]. Indeed, the S terms are in a diagonal form which is preferable for applications. However, they do not arise as diagonal sums initially (see Theorem 5), and some work is needed to express them in the more applicable form (see Lemma 6). Another useful tool in applications is a contour integral representation for the sums over the permutations of the shifts α, β, γ. These can be found in formulas (16) and (21).

• If one assumes GRH, then one can allow for general complex coefficients satisfying $a(n) \ll n^\epsilon$. In this case the error term for small moduli in Proposition 10 below can be replaced by $O(y^{3/2+\epsilon}T^{1/2+\epsilon})$.

Before moving on to the proofs we give a brief heuristic justification for the choice of Dirichlet polynomial $P(s)$ in (3) since we could not find this elsewhere in the literature. In order for Hölder's inequality to be sharp, we need the summands to be approximately equal. Unfortunately, $1/\zeta'(s)$ has no representation in terms of a Dirichlet series and so there is no obvious choice for a polynomial approximation. However, we expect that our mean values will not change too much if we shift away from the half-line slightly with distance $\delta \asymp 1/\log T$. We also expect that in this region $\zeta'(s) \approx (\log T)\zeta(s)$, at least on average. Applying these two principles, the right-hand side of (2) becomes

$$\biggl(\sum_{0\leqslant\gamma\leqslant T} |\zeta(\rho+\delta)|^2 |P(\rho+\delta)|^{2(a+b)}\biggr)^{a/(a+b)} \biggl(\sum_{0\leqslant\gamma\leqslant T} \frac{1}{|\zeta(\rho+\delta)|^{2k}}\biggr)^{b/(a+b)},$$

and notice that the powers of $\log T$ cancel by homogeneity. We may now set our summands equal:

$$\zeta(s)^2 P(s)^{2(a+b)} \approx \frac{1}{\zeta(s)^{2a/b}},$$

and find that we should take $P(s) \approx \zeta(s)^{-1/b}$.

The rest of the paper is organised as follows. In Section 2 we prove Proposition 2. In Section 3 we prove Proposition 3 assuming Theorem 4. The remainder of the paper is then devoted to proving Theorem 4. In Section 4, we first reduce Theorem 4 to Theorem 5. To prove Theorem 5, we compute the main term and bound the error term using two propositions: Propositions 10 and 11. Then in Section 5 we prove Proposition 10, and in Section 6 we prove Proposition 11.

2. Proof of Proposition 2

We first choose the weight function $\psi(n)$ in (3). Let B be a positive integer and let

$$\psi(n) = \mathbb{1}_{n\leqslant x}\left(\frac{\log(x/n)}{\log x}\right)^B. \tag{7}$$

We will need to take B sufficiently large in terms of k at several points throughout the paper, so for the moment we keep it general. To be clear, in the end we will choose

$$B = 14 + 12k$$

but for the purposes of this section we only require $B \geqslant 1$. Note that by the Mellin inversion formula (or simply by a residue computation) we have

$$\psi(n) = \frac{B!}{2\pi i(\log x)^B}\int_{(c)}\left(\frac{x}{n}\right)^s\frac{ds}{s^{B+1}} \tag{8}$$

for $c > 0$ where, here and throughout, $\int_{(c)} = \int_{c-i\infty}^{c+i\infty}$.

We first write

$$R(s) = P(s)^a = \sum_{n\leqslant x^a}\frac{r(n,x)}{n^s}$$

with

$$r(n,x) = r_{a,b}(n,x) = \sum_{\substack{n_1\cdots n_a=n\\ n_j\leqslant x}}\tau_{-1/b}(n_1)\psi(n_1)\cdots\tau_{-1/b}(n_a)\psi(n_a).$$

Then

$$S_1 = \sum_{0\leqslant\gamma\leqslant T}R(\rho)R(1-\rho).$$

The mean value S_1 can be computed in a familiar manner; either by writing it as a contour integral or by, what amounts to the same thing, applying Gonek's uniform version Landau's formula. These details have been carried out by Ng [2008a] for a fairly general class of Dirichlet polynomial. By applying [Ng 2008a, Proposition 4(i)] we find that

$$S_1 = N(T)\sum_{n\leqslant x^a}\frac{r(n,x)^2}{n} - \frac{T}{\pi}\sum_{\ell m=n\leqslant x^a}\frac{\Lambda(\ell)r(m,x)r(n,x)}{n} + o(T) \tag{9}$$

where $N(T) = \frac{T}{2\pi}\log(T/2\pi e) + O(\log T)$ is the number of zeros of $\zeta(s)$ in the strip $0 < \sigma < 1$, $0 \leqslant t \leqslant T$.

Denote

$$S_{11} = \sum_{n\leqslant x^a}\frac{r(n,x)^2}{n}$$

and

$$S_{12} = \sum_{\ell m=n\leqslant x^a}\frac{\Lambda(\ell)r(m,x)r(n,x)}{n}.$$

In this section we will show that

$$S_{11} \sim c_1(a,b)(\log T)^{a^2/b^2}$$

and that

$$S_{12} \sim c_2(a,b)(\log T)^{a^2/b^2+1}$$

for some explicit constants $c_1(a,b)$, $c_2(a,b)$. We then show that $\frac{1}{2}c_1(a,b) - c_2(a,b) > 0$ and hence $S_1 \sim c_3(a,b)T(\log T)^{a^2/b^2+1}$ for some positive constant $c_3(a,b)$. The result will then follow.

2.1. *Computing S_{11}*. Unfolding the sum gives

$$S_{11} = \sum_{\substack{n_1\cdots n_a = n_{a+1}\cdots n_{2a} \\ n_j \leqslant x}} \frac{\tau_{-1/b}(n_1)\cdots\tau_{-1/b}(n_{2a})}{(n_1\cdots n_{2a})^{1/2}} \psi(n_1)\cdots\psi(n_{2a}).$$

By applying the Mellin inversion formula (8) in each n_j and interchanging the order of summation and integration, we obtain

$$S_{11} = \frac{B!^{2a}}{(2\pi i)^{2a}(\log x)^{2aB}} \int_{(c)^{2a}} \sum_{\substack{n_1\cdots n_a = \\ n_{a+1}\cdots n_{2a}}} \frac{\tau_{-1/b}(n_1)\cdots\tau_{-1/b}(n_{2a})}{n_1^{1/2+s_1}\cdots n_{2a}^{1/2+s_{2a}}} \prod_{\ell=1}^{2a} x^{s_\ell} \frac{ds_\ell}{s_\ell^{B+1}},$$

where we have taken $c = 1/\log x$. Here, we use the notation

$$\int_{(c)^{2a}} = \underbrace{\int_{(c)} \cdots \int_{(c)}}_{2a}.$$

After a short calculation with Euler products we find that the Dirichlet series in the integrand is given by

$$\mathscr{A}(\underline{s}) \prod_{i,j=1}^{a} \zeta(1+s_i+s_{a+j})^{1/b^2}$$

where

$$\mathscr{A}(\underline{s}) = \prod_p \prod_{i,j=1}^{a} \left(1 - \frac{1}{p^{1+s_i+s_{a+j}}}\right)^{1/b^2} \sum_{\substack{m_1+\cdots+m_a = m_{a+1}+\cdots+m_{2a} \\ m_j \geqslant 0}} \frac{\tau_{-1/b}(p^{m_1})\cdots\tau_{-1/b}(p^{m_{2a}})}{p^{m_1(1/2+s_1)+\cdots+m_{2a}(1/2+s_{2a})}}. \tag{10}$$

Note that $A(\underline{s})$ is holomorphic in the region $\sigma_j > -\frac{1}{4}$, $j = 1, \ldots, 2a$, since it is absolutely convergent there.

We will reproduce the following argument several times throughout the paper, so we take this opportunity to briefly describe the steps and give some justification. Note that the integrand has fractional powers of $\zeta(s)$. This coupled with the fact that we have a multidimensional integral means that shifting contours would be very messy. However, note that the integrand is largest when we simultaneously have $\Im(s_j) \approx 0$. We can therefore localise our integral around these points, expand the integrand in Taylor/Laurent series, and then extract the main term via the substitution $s_j \mapsto s_j/\log x$. The remaining integral then gives a combinatorial constant which we can compute as the weighted volume of a polytope using a trick from [Brevig and Heap 2019]; see Section 2.3 below. This method can essentially be thought of as a multidimensional version of the saddle point method, although in our case it is fairly easy to see where the saddle/main contribution is.

In practice it is simpler if we make the substitutions first, so let $s_j \mapsto s_j/\log x$ for each j. Then S_{11} becomes

$$S_{11} = \frac{B!^{2a}}{(2\pi i)^{2a}} \int_{(1)^{2a}} \mathscr{A}\biggl(\frac{\underline{s}}{\log x}\biggr) \prod_{i,j=1}^{a} \zeta\biggl(1+\frac{s_i+s_{a+j}}{\log x}\biggr)^{1/b^2} \prod_{\ell=1}^{2a} e^{s_\ell} \frac{ds_\ell}{s_\ell^{B+1}}.$$

Let us localise the integral. For each j we split the integral at the points $t_j = \Im(s_j) = \pm\sqrt{\log x}$; the main contribution will come from the integral over the region $s_j \in [1-i\sqrt{\log x},\, 1+i\sqrt{\log x}]$. To estimate the tail integrals we use the bound

$$\mathscr{A}(\underline{s}/\log x) \prod_{i,j=1}^{a} \zeta\biggl(1+\frac{s_i+s_{a+j}}{\log x}\biggr)^{1/b^2} \ll (\log x)^{a^2/b^2}$$

valid for $s_j = 1+it_j$ uniformly in $t_j \in \mathbb{R}$. Then,

$$\int_{1+i\sqrt{\log x}}^{1+i\infty} \int_{(1)^{2a-1}} \mathscr{A}\biggl(\frac{\underline{s}}{\log x}\biggr) \prod_{i,j=1}^{a} \zeta\biggl(1+\frac{s_i+s_{a+j}}{\log x}\biggr)^{1/b^2} \prod_{\ell=1}^{2a} e^{s_\ell} \frac{ds_\ell}{s_\ell^{B+1}} \\ \ll (\log x)^{a^2/b^2} \int_{1+i\sqrt{\log x}}^{1+i\infty} \frac{ds}{|s|^{B+1}} \ll (\log x)^{a^2/b^2-1/2}$$

by absolute convergence. Naturally, the tail integrals in the lower half plane satisfy the same bound as do those with respect to the other integration variables. Note that the smooth weights $\psi(n)$ have made the task of estimating these tails significantly easier compared to the usual Perron's formula with a sharp cut off.

Collecting the errors gives

$$S_{11} = \frac{B!^{2a}}{(2\pi i)^{2a}} \int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}} \cdots \int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}} \mathscr{A}\biggl(\frac{\underline{s}}{\log x}\biggr) \prod_{i,j=1}^{a} \zeta\biggl(1+\frac{s_i+s_{a+j}}{\log x}\biggr)^{1/b^2} \prod_{\ell=1}^{2a} e^{s_\ell} \frac{ds_\ell}{s_\ell^{B+1}} \\ + O((\log x)^{a^2/b^2-1/2}).$$

In this region of integration we have the expansions

$$\mathscr{A}(\underline{s}/\log x) = \mathscr{A}(\underline{0}) + O\biggl(\frac{1}{\log x}\sum_j |s_j|\biggr) = \mathscr{A}(\underline{0}) + O\biggl(\frac{1}{\sqrt{\log x}}\biggr) \tag{11}$$

and

$$\zeta\biggl(1+\frac{s_i+s_{a+j}}{\log x}\biggr)^{1/b^2} = \frac{(\log x)^{1/b^2}}{(s_i+s_{a+j})^{1/b^2}}\biggl(1+O\biggl(\frac{1}{\sqrt{\log x}}\biggr)\biggr). \tag{12}$$

Therefore,

$$S_{11} = \frac{\mathscr{A}(\underline{0}) B!^{2a} (\log x)^{a^2/b^2}}{(2\pi i)^{2a}} \int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}} \cdots \int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}} \prod_{i,j=1}^{a} \frac{1}{(s_i + s_{a+j})^{1/b^2}} \prod_{\ell=1}^{2a} e^{s_\ell} \frac{ds_\ell}{s_\ell^{B+1}} + O((\log x)^{a^2/b^2-1/2}).$$

On extending any given integral to $i\infty$ we obtain a multiplicative error of $O((\log x)^{-B/2})$ which leads to a total contribution of size $O((\log x)^{a^2/b^2-1/2})$. Therefore, we obtain the following asymptotic formula

$$S_{11} = \mathscr{A}(\underline{0}) \beta(a,b) (\log x)^{a^2/b^2} + O((\log x)^{a^2/b^2-1/2}), \tag{13}$$

where

$$\beta(a,b) = \frac{B!^{2a}}{(2\pi i)^{2a}} \int_{(1)^{2a}} \prod_{i,j=1}^{a} \frac{1}{(s_i + s_{a+j})^{1/b^2}} \prod_{\ell=1}^{2a} e^{s_\ell} \frac{ds_\ell}{s_\ell^{B+1}}.$$

We postpone the computation of these constants to Section 2.3.

2.2. *Computing S_{12}*. Recall

$$S_{12} = \sum_{\ell m = n \leqslant x^a} \frac{\Lambda(\ell) r(m,x) r(n,x)}{n}.$$

In order to have multiplicative coefficients we write

$$\Lambda(n) = \sum_{n_1 n_2 = n} \mu(n_1) \log n_2 = \frac{d}{d\gamma} \sum_{n_1 n_2 = n} \mu(n_1) n_2^{\gamma} |_{\gamma=0}.$$

Then unfolding the coefficients $r(n,x)$ and applying the above gives

$$\begin{aligned} S_{12} &= \sum_{\substack{\ell n_1 \cdots n_a = n_{a+1} \cdots n_{2a} \\ n_j \leqslant x}} \frac{\Lambda(\ell) \tau_{-1/b}(n_1) \cdots \tau_{-1/b}(n_{2a})}{(\ell n_1 \cdots n_{2a})^{1/2}} \cdot \psi(n_1) \cdots \psi(n_{2a}) \\ &= \frac{d}{d\gamma} \Bigg(\sum_{\substack{\ell_1 \ell_2 n_1 \cdots n_a = n_{a+1} \cdots n_{2a} \\ n_j \leqslant x}} \frac{\mu(\ell_1) \tau_{-1/b}(n_1) \cdots \tau_{-1/b}(n_{2a})}{\ell_2^{1/2-\gamma} (\ell_1 n_1 \cdots n_{2a})^{1/2}} \cdot \psi(n_1) \cdots \psi(n_{2a}) \Bigg) \Bigg|_{\gamma=0}. \end{aligned}$$

As before, we apply Mellin inversion (8) to find

$$S_{12} = \frac{d}{d\gamma} \frac{B!^{2a}}{(2\pi i)^{2a} (\log x)^{2aB}} \int_{(c)^{2a}} \sum_{\ell_1 \ell_2 n_1 \cdots n_a = n_{a+1} \cdots n_{2a}} \frac{\mu(\ell_1) \tau_{-1/b}(n_1) \cdots \tau_{-1/b}(n_{2a})}{\ell_1^{1/2} \ell_2^{1/2-\gamma} n_1^{1/2+s_1} \cdots n_{2a}^{1/2+s_{2a}}} \prod_{j=1}^{2a} x^{s_j} \frac{ds_j}{s_j^{B+1}} \Bigg|_{\gamma=0}.$$

Now a short calculation shows that the Dirichlet series in the integrand is given by

$$\mathscr{B}(\underline{s}, \gamma) \frac{\prod_{i,j=1}^{a} \zeta(1+s_i+s_{a+j})^{1/b^2} \prod_{j=1}^{a} \zeta(1+s_{a+j})^{1/b}}{\prod_{j=1}^{a} \zeta(1+s_{a+j}-\gamma)^{1/b}},$$

where

$$\mathcal{B}(\underline{s},\gamma) = \prod_p \frac{\prod_{i,j=1}^a \big(1-\frac{1}{p^{1+s_i+s_{a+j}}}\big)^{1/b^2} \prod_{j=1}^a \big(1-\frac{1}{p^{1+s_{a+j}}}\big)^{1/b}}{\prod_{j=1}^a \big(1-\frac{1}{p^{1+s_{a+j}-\gamma}}\big)^{1/b}} \sum_{\substack{\ell_1+\ell_2+m_1+\cdots+m_a\\ =m_{a+1}+\cdots+m_{2a}\\ m_j\geqslant 0}} \frac{\mu(p^{\ell_1})\tau_{-1/b}(p^{m_1})\cdots\tau_{-1/b}(p^{m_{2a}})}{p^{\ell_1+\ell_2-\gamma+m_1(1/2+s_1)+\cdots+m_{2a}(1/2+s_{2a})}}$$

is the corresponding holomorphic factor. Again, this is easily seen to be holomorphic in the region $\sigma_j > -\frac{1}{4}$, $j = 1, \ldots, 2a$. Then, taking the derivative inside the integral we obtain

$$S_{12} = \frac{B!^{2a}}{(2\pi i)^{2a}(\log x)^{2aB}} \int_{(c)^{2a}} \bigg[\mathcal{B}'(\underline{s},0)\prod_{i,j=1}^a \zeta(1+s_i+s_{a+j})^{1/b^2} + \mathcal{B}(\underline{s},0)\prod_{i,j=1}^a \zeta(1+s_i+s_{a+j})^{1/b^2}\frac{1}{b}\sum_{j=1}^a \frac{\zeta'(1+s_{a+j})}{\zeta(1+s_{a+j})}\bigg]\prod_{\ell=1}^{2a} x^{s_\ell}\frac{ds_\ell}{s_\ell^{B+1}}.$$

Now, the first integral can be treated as in the previous subsection (the only difference being the arithmetic factor $\mathcal{B}'(\underline{s},0)$ which is of no real consequence). In this way we find it is $O((\log x)^{a^2/b^2})$. In the remaining integral we first note that $\mathcal{B}(\underline{s},0)=\mathcal{A}(\underline{s})$ and then let $s_j \mapsto s_j/\log x$ for each j to give

$$S_{12}=\frac{B!^{2a}}{(2\pi i)^{2a}}\int_{(1)^{2a}} \mathcal{A}\bigg(\frac{\underline{s}}{\log x}\bigg)\prod_{i,j=1}^a \zeta\bigg(1+\frac{s_i+s_{a+j}}{\log x}\bigg)^{1/b^2}\frac{1}{b}\sum_{j=1}^a\frac{\zeta'\big(1+\frac{s_{a+j}}{\log x}\big)}{\zeta\big(1+\frac{s_{a+j}}{\log x}\big)}\prod_{j=1}^{2a} e^{s_j}\frac{ds_j}{s_j^{B+1}}+O((\log x)^{a^2/b^2}).$$

As before we may trivially bound the integrand, this time by $(\log x)^{a^2/b^2+1}$, and then truncate the integrals at height $t_j = \pm\sqrt{\log x}$ to give

$$S_{12} = \frac{B!^{2a}}{(2\pi i)^{2a}}\int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}}\cdots\int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}} \mathcal{A}\bigg(\frac{\underline{s}}{\log x}\bigg) \times\prod_{i,j=1}^a \zeta\bigg(1+\frac{s_i+s_{a+j}}{\log x}\bigg)^{1/b^2}\frac{1}{b}\sum_{j=1}^a\frac{\zeta'\big(1+\frac{s_{a+j}}{\log x}\big)}{\zeta\big(1+\frac{s_{a+j}}{\log x}\big)}\prod_{j=1}^{2a} e^{s_j}\frac{ds_j}{s_j^{B+1}} + O((\log x)^{a^2/b^2+1/2})$$

since the tail integrals result in an error $O((\log x)^{a^2/b^2+1-B/2})$ and $B\geqslant 1$. Then, applying the Taylor and Laurent expansions given in (11) and (12) along with

$$\frac{\zeta'\big(1+\frac{s_{a+j}}{\log x}\big)}{\zeta\big(1+\frac{s_{a+j}}{\log x}\big)} = -\frac{\log x}{s_{a+j}} + O\bigg(\frac{1}{\sqrt{\log x}}\bigg),$$

which is valid in the current region of integration, we find that

$$S_{12} = -\frac{1}{b}\frac{\mathcal{A}(\underline{0})B!^{2a}(\log x)^{a^2/b^2+1}}{(2\pi i)^{2a}}\int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}}\cdots\int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}}\prod_{i,j=1}^a\frac{1}{(s_i+s_{a+j})^{1/b^2}}\sum_{j=1}^a\frac{1}{s_{a+j}}\prod_{j=1}^{2a}e^{s_j}\frac{ds_j}{s_j^{B+1}} + O((\log x)^{a^2/b^2+1/2}).$$

Extending the integrals back to $\pm i\infty$ incurs an error of size $O((\log x)^{a^2/b^2+1/2})$. Also, by symmetry the sum $\sum_{j=1}^{a} s_{a+j}^{-1}$ results in a-copies of the integral with a factor of s_{2a}^{-1}, say. Hence, we obtain the asymptotic formula

$$S_{12} = -\frac{a}{b}\mathscr{A}(\underline{0})\gamma(a,b)(\log x)^{a^2/b^2+1} + O((\log x)^{a^2/b^2+1/2}), \tag{14}$$

where

$$\gamma(a,b) = \frac{B!^{2a}}{(2\pi i)^{2a}}\int_{(1)^{2a}}\prod_{i,j=1}^{a}\frac{1}{(s_i+s_{a+j})^{1/b^2}}\Bigg[\prod_{j=1}^{2a-1}e^{s_j}\frac{ds_j}{s_j^{B+1}}\Bigg]e^{s_{2a}}\frac{ds_{2a}}{s_{2a}^{B+2}}.$$

2.3. ***Computation of the constants.*** Applying (13) and (14) in Ng's formula (9) we find that

$$S_1 = \mathscr{A}(\underline{0})\frac{\beta(a,b)}{2}T(\log T)(\log x)^{a^2/b^2} + (a/b)\mathscr{A}(\underline{0})\gamma(a,b)T(\log x)^{a^2/b^2+1} + O(T(\log x)^{a^2/b^2+1/2}).$$

Since $x = T^{\theta/(a+b)}$ it remains to show that the constants $\mathscr{A}(\underline{0})$, $\beta(a,b)$ and $\gamma(a,b)$ are positive.

A short calculation using the definition of $\mathscr{A}(\underline{s})$ given in (10) shows that

$$\mathscr{A}(\underline{0}) = \prod_p\Big(1-\frac{1}{p}\Big)^{k^2}\sum_{m\geqslant 0}\frac{\tau_{-k}(p^m)^2}{p^m}$$

which is an absolutely convergent product. Thus $\mathscr{A}(\underline{0}) > 0$. For the combinatorial constants we use a trick from [Brevig and Heap 2019, Lemma 8]. Recall that

$$\beta(a,b) = \frac{B!^{2a}}{(2\pi i)^{2a}}\int_{(1)^{2a}}\prod_{i,j=1}^{a}\frac{1}{(s_i+s_{a+j})^{1/b^2}}\prod_{\ell=1}^{2a}e^{s_\ell}\frac{ds_\ell}{s_\ell^{B+1}}.$$

For each term in the double product, we write

$$\frac{1}{(s_i+s_{a+j})^{1/b^2}} = \frac{1}{\Gamma(1/b^2)}\int_0^\infty e^{-(s_i+s_{a+j})x_{ij}}x_{ij}^{1/b^2}\frac{dx_{ij}}{x_{ij}}$$

so that

$$\beta(a,b) = \frac{B!^{2a}}{\Gamma(1/b^2)^{a^2}}\frac{1}{(2\pi i)^{2a}}\int_{(1)^{2a}}\int_{[0,\infty]^{a^2}}\Bigg[\prod_{i=1}^{a}e^{s_i(1-\sum_{j=1}^{a}x_{ij})}\Bigg]\Bigg[\prod_{j=1}^{a}e^{s_{a+j}(1-\sum_{i=1}^{a}x_{ij})}\Bigg]\prod_{i,j=1}^{a}x_{ij}^{1/b^2}\frac{dx_{ij}}{x_{ij}}\prod_{j=1}^{2a}\frac{ds_j}{s_j^{B+1}}.$$

After interchanging the order of integration and using the formula

$$\frac{B!}{2\pi i}\int_{(c)}e^{s(1-X)}\frac{ds}{s^{B+1}} = \begin{cases}(1-X)^B & \text{if } X\leqslant 1,\\ 0 & \text{otherwise;}\end{cases}$$

we obtain

$$\beta(a,b) = \frac{1}{\Gamma(1/b^2)^{a^2}}\int_{\mathscr{P}_{a,b}}\prod_{i=1}^{a}\Big(1-\sum_{j=1}^{a}x_{ij}\Big)^B\prod_{j=1}^{a}\Big(1-\sum_{i=1}^{a}x_{ij}\Big)^B\prod_{i,j=1}^{a}x_{ij}^{1/b^2}\frac{dx_{ij}}{x_{ij}},$$

where

$$\mathscr{P}_{a,b} = \left\{ (x_{ij}) \in \mathbb{R}^{a^2} : x_{ij} \geqslant 0, \sum_{i=1}^{a} x_{ij} \leqslant 1, \sum_{j=1}^{a} x_{ij} \leqslant 1 \right\}.$$

Hence $\beta(a, b) > 0$. A similar formula holds for $\gamma(a, b)$. The only difference is that the factor $\big(1-\sum_i x_{ia}\big)^B$ is replaced by $\big(1-\sum_i x_{ia}\big)^{B+1}/(B+1)$, from which $\gamma(a, b) > 0$ follows easily. Thus we complete the proof of Proposition 2.

3. Proof of Proposition 3

In this section we shall prove Proposition 3 assuming Theorem 4. We start from the formula

$$S_2 = \frac{d}{d\alpha}\frac{d}{d\beta} \sum_{0 \leqslant \gamma \leqslant T} \zeta(\rho+\alpha)\zeta(1-\rho+\beta) Q(\rho) Q(1-\rho) \bigg|_{\alpha=\beta=0}$$

where

$$Q(s) = \bigg(\sum_{n \leqslant x} \frac{\tau_{-1/b}(n)\psi(n)}{n^s} \bigg)^{a+b} = \sum_{n \leqslant y} \frac{a(n)}{n^s}$$

with

$$a(n) = a(n, x) = \sum_{\substack{n_1 \cdots n_{a+b} = n \\ n_j \leqslant x}} \prod_{j=1}^{a+b} \tau_{-1/b}(n_j)\psi(n_j) \tag{15}$$

and $y = x^{a+b} = T^\theta$ with $\theta < \frac{1}{2}$. Theorem 4 then gives S_2 as a sum of three terms. We write this as

$$S_2 = S_{21} + S_{22} + S_{23}$$

with

$$S_{21} = \frac{d}{d\alpha}\frac{d}{d\beta} \sum_{g \leqslant y} \sum_{\substack{h,k \leqslant y/g \\ (h,k)=1}} \frac{a(gh)a(gk)}{ghk} \frac{1}{2\pi} \int_1^T \log\Big(\frac{t}{2\pi}\Big) \bigg[\frac{\zeta(1+\alpha+\beta)}{h^\beta k^\alpha} + \Big(\frac{t}{2\pi}\Big)^{-\alpha-\beta} \frac{\zeta(1-\alpha-\beta)}{h^{-\alpha}k^{-\beta}} \bigg] dt \bigg|_{\alpha=\beta=0},$$

$$S_{22} = \frac{d}{d\alpha}\frac{d}{d\beta}\frac{d}{d\gamma} \frac{1}{2\pi} \int_1^T \bigg[S_{\alpha,\beta,\gamma}(T) + \Big(\frac{t}{2\pi}\Big)^{-\alpha-\beta} S_{-\beta,-\alpha,\gamma}(T) + \Big(\frac{t}{2\pi}\Big)^{-\beta-\gamma} S_{\alpha,-\gamma,-\beta}(T) \bigg] dt \bigg|_{\alpha=\beta=\gamma=0}$$

and $\overline{S_{23}}$ is defined by replacing (α, β) by $(\bar{\beta}, \bar{\alpha})$ in the integrand in S_{22}. If the weight $w(x)$ is real (which follows if $\overline{H(s)} = H(\bar{s})$) then the formula for S_{23} simply has α and β interchanged in the integrand and by symmetry this is equal to S_{22}. If this is not the case, then we have the conjugated terms instead but of course this entails only small modifications to the argument for S_{22}. Thus we only need to consider S_{21} and S_{22}.

3.1. ***Computing S_{21}.*** Our aim is to show $S_{21} \ll T(\log T)^{k^2+3}$. It is helpful to express the integrand in a form in which the holomorphy is immediately visible, as in [Conrey et al. 2005]. For this purpose we use the formula

$$\frac{\zeta(1+\alpha+\beta)}{h^\beta k^\alpha}+\left(\frac{t}{2\pi}\right)^{-\alpha-\beta}\frac{\zeta(1-\alpha-\beta)}{h^{-\alpha}k^{-\beta}} = -\left(\frac{t}{2\pi}\right)^{-(\alpha+\beta)/2}\frac{1}{(2\pi i)^2}\oint\oint\left(\frac{t}{2\pi}\right)^{(z_2-z_1)/2}\frac{\zeta(1+z_1-z_2)(z_2-z_1)^2}{h^{-z_2}k^{z_1}\prod_{i=1}^2(z_i-\alpha)(z_i+\beta)}dz_1dz_2 \quad (16)$$

where the integrals are over circles of radii $\ll 1/\log T$ that enclose the shifts α and β. This formula follows from a short residue computation. Interchanging the sum and integral we obtain

$$S_{21} = -\frac{d}{d\alpha}\frac{d}{d\beta}\frac{1}{2\pi}\int_0^T\left(\frac{t}{2\pi}\right)^{-(\alpha+\beta)/2}\log\left(\frac{t}{2\pi}\right) \times\frac{1}{(2\pi i)^2}\oint\oint\left(\frac{t}{2\pi}\right)^{(z_2-z_1)/2}\frac{\zeta(1+z_1-z_2)(z_2-z_1)^2}{\prod_{i=1}^2(z_i-\alpha)(z_i+\beta)}F(x,z_1,z_2)\,dz_1\,dz_2\,dt\bigg|_{\alpha=\beta=0}$$

where

$$F(x,z_1,z_2)=\sum_{g\leqslant y}\sum_{\substack{h,k\leqslant y/g\\(h,k)=1}}\frac{a(gh)a(gk)}{gh^{1-z_2}k^{1+z_1}}=\sum_{h,k\leqslant y}\frac{a(h)a(k)(h,k)^{1+z_1-z_2}}{h^{1-z_2}k^{1+z_1}}.$$

We now perform the differentiation and estimate the contour integrals. The contour lengths and the factor of $(z_2-z_1)^2$ contribute at most $(\log T)^{-4}$ whilst the zeta function, negative powers of z_i and $\log(t/2\pi)$ term contribute at most $(\log T)^6$. The differentiation gives us a factor of $(\log T)^2$ and thus in total we obtain

$$S_{21} \ll T(\log T)^4 \max_{|z_1|,|z_2|\ll 1/\log T}|F(x,z_1,z_2)|.$$

It remains to show that

$$\max_{|z_1|,|z_2|\ll 1/\log T}|F(x,z_1,z_2)| \ll (\log T)^{a^2/b^2-1}.$$

We will show this by the methods of the previous section. Throughout the following we shall assume that $z_1, z_2 \ll 1/\log T$.

We proceed by first unfolding the sum using the formula for the coefficients $a(n)$ given in (15). Writing $a+b=N$ for brevity, we find

$$F(x,z_1,z_2)=\sum_{\substack{h_1,\dots,h_N\leqslant x\\k_1,\dots,k_N\leqslant x}}\frac{\left[\prod_{i=1}^N\tau_{-1/b}(h_i)\tau_{-1/b}(k_i)\psi(h_i)\psi(k_i)\right](h_1\cdots h_N,k_1\cdots k_N)^{1+z_1-z_2}}{(h_1\cdots h_N)^{1-z_2}(k_1\cdots k_N)^{1+z_1}}.$$

By the Mellin inversion formula (8) we obtain

$$F(x,z_1,z_2)=\frac{B!^{2N}}{(\log x)^{2NB}}\frac{1}{(2\pi i)^{2N}}\int_{(c)^{2N}}\mathcal{F}_{z_1,z_2}(\underline{s})\prod_{j=1}^{2N}x^{s_j}\frac{ds_j}{s_j^{B+1}}$$

where

$$\mathcal{F}_{z_1,z_2}(\underline{s})=\sum_{\substack{h_1,\dots,h_N\geqslant 1\\ k_1,\dots,k_N\geqslant 1}}\frac{\left[\prod_{i=1}^N \tau_{-1/b}(h_i)\tau_{-1/b}(k_i)\right](h_1\cdots h_N,k_1\cdots k_N)^{1+z_1-z_2}}{h_1^{1+s_1-z_2}\cdots h_N^{1+s_N-z_2}k_1^{1+s_{N+1}+z_1}\cdots k_N^{1+s_{2N}+z_1}}$$

and $c=1/\log x$. Now, a short computation with Euler products shows that

$$\mathcal{F}_{z_1,z_2}(\underline{s})=\mathcal{C}_{z_1,z_2}(\underline{s})\frac{\prod_{i,j=1}^N \zeta(1+s_i+s_{j+N})^{1/b^2}}{\prod_{j=1}^N \zeta(1+s_j-z_2)^{1/b}\zeta(1+s_{j+N}+z_2)^{1/b}} \tag{17}$$

where $\mathcal{C}_{z_1,z_2}(\underline{s})$ is an absolutely convergent Euler product in the region $\sigma_j>-\frac{1}{4}$, $j=1,\dots,N$. This gives us the trivial bound

$$\mathcal{F}_{z_1,z_2}(\underline{s})\ll(\log x)^{N^2/b^2+2N/b},\quad \Re(s_j)\asymp\frac{1}{\log x}. \tag{18}$$

Substituting $s_j\mapsto s_j/\log x$ for each j gives

$$F(x,z_1,z_2)=\frac{B!^{2N}}{(2\pi i)^{2N}}\int_{(1)^{2N}}\mathcal{F}_{z_1,z_2}(\underline{s}/\log x)\prod_{j=1}^{2N}e^{s_j}\frac{ds_j}{s_j^{B+1}}.$$

By (18), any given tail integral over the line from $1+i\sqrt{\log x}$ to $1+i\infty$ results in a total contribution of

$$(\log T)^{N^2/b^2+2N/b}\int_{1+i\sqrt{\log x}}^{1+i\infty}\frac{ds}{|s|^{B+1}}\ll(\log T)^{N^2/b^2+2N/b-B/2}=(\log T)^{k^2+4k+3-B/2}$$

since the other integrals are absolutely convergent. Therefore, on taking $B\geqslant 10+8k$ this term is bounded by $(\log T)^{k^2-2}$. Consequently, we may localise the integral

$$F(x,z_1,z_2)=\frac{B!^{2N}}{(2\pi i)^{2N}}\int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}}\cdots\int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}}\mathcal{F}_{z_1,z_2}(\underline{s}/\log x)\prod_{j=1}^{2N}e^{s_j}\frac{ds_j}{s_j^{B+1}}+O((\log T)^{k^2-2}).$$

In this new region of integration we have the bounds

$$\zeta\left(1+\frac{s_j}{\log x}\pm z_i\right)^{-1}\ll\frac{|s_j|}{\log x}+|z_j|\ll\frac{1}{\log T}(|s_j|+1) \tag{19}$$

and

$$\zeta(1+(s_i+s_j)/\log x)\ll\frac{\log T}{|s_i+s_j|} \tag{20}$$

as well as $\mathcal{C}_{z_1,z_2}(\underline{s}/\log x)\ll 1$. Applying these bounds in (17) we find that in the region of integration we have

$$\mathcal{F}_{z_1,z_2}(\underline{s}/\log x)\ll(\log T)^{N^2/b^2-2N/b}\frac{\prod_{j=1}^N(|s_j|+1)^{1/b}(|s_{j+N}|+1)^{1/b}}{\prod_{i,j=1}^N|s_i+s_{j+N}|^{1/b^2}}.$$

By the absolute convergence of the integrals we obtain

$$F(x,z_1,z_2)\ll(\log T)^{N^2/b^2-2N/b}=(\log T)^{k^2-1}$$

and the required bound for S_{21} follows.

3.2. *Computing S_{22}*. The goal is to show the bound $S_{22} \ll T(\log T)^{k^2+3}$. By a simple (but tedious) residue calculation, we find that

$$S_{22} = -\frac{1}{4\pi}\frac{d}{d\alpha}\frac{d}{d\beta}\frac{d}{d\gamma}\left(\frac{t}{2\pi}\right)^{-(\alpha+\beta+\gamma)/2}\int_1^T \frac{1}{(2\pi i)^3}\oint\oint\oint G(x,\underset{\sim}{z})\frac{\Delta(z_1,z_2,z_3)^2}{\prod_{i=1}^3 (z_i-\alpha)(z_i+\beta)(z_i-\gamma)} \times \left(\frac{t}{2\pi}\right)^{(z_1-z_2+z_3)/2} dz_1\,dz_2\,dz_3\,dt\bigg|_{\alpha=\beta=\gamma=0} + o(T) \quad (21)$$

where $\Delta(z_1, z_2, z_3) = \prod_{i<j}(z_j - z_i)$ is the Vandermonde determinant and the integrals are over circles of radii $\ll 1/\log T$ enclosing the shifts and

$$G(x,\underset{\sim}{z}) = S_{z_1,-z_2,z_3}(T).$$

As before, we plan to interchange the order of summation and integration and then compute the resulting sum. Performing the differentiation and then trivially estimating the z_j integrals gives

$$S_{22} \ll T(\log T)^{9+3-2\cdot 3-3} \max_{|z_j|\ll 1/\log T} |G(x,\underset{\sim}{z})|.$$

Thus, we are required to show that

$$\max_{|z_j|\ll 1/\log T} |G(x,\underset{\sim}{z})| \ll (\log T)^{k^2}.$$

Using the definition of the weight w, we have

$$G(x,\underset{\sim}{z}) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{H(w)}{w} T^{2w} \sum_{hm=kn} \frac{a(h)a(k)f_{z_1,z_3}(m)n^{z_2}}{(hk)^{1/2}(mn)^{1/2+w}}\,dw.$$

Applying the definitions of the coefficients $a(n)$ in (15) and $f_{z_1,z_2}(n)$ in (6) along with the Mellin inversion formula (8) for the weights $\psi(n)$ gives

$$\sum_{hm=kn} \frac{a(h)a(k)f_{z_1,z_3}(m)n^{z_2}}{(hk)^{1/2}(mn)^{1/2+w}} = \frac{B!^{2N}}{(\log x)^{2NB}(2\pi i)^{2N}}\int_{(\kappa)^{2N}} \mathcal{G}(w,\underset{\sim}{s},\underset{\sim}{z})\prod_{j=1}^{2N} x^{s_j}\frac{ds_j}{s_j^{B+1}}$$

where

$$\mathcal{G}(w,\underset{\sim}{s},\underset{\sim}{z}) = \sum_{h_1\cdots h_N m_1 m_2 m_3 = k_1\cdots k_N n} \frac{\left(\prod_{i=1}^N \tau_{-1/b}(h_i)\tau_{-1/b}(k_i)\right)\mu(m_1)}{\left(\prod_{i=1}^N h_i^{1/2+s_i}k_i^{1/2+s_{i+N}}\right)m_1^{1/2+w}m_2^{1/2+z_1+w}m_3^{1/2+z_3+w}n^{1/2-z_2+w}}$$

and $N = a + b$, as before. By considering its Euler product we find

$$\mathcal{G}(w,\underset{\sim}{s},\underset{\sim}{z}) = \mathcal{D}(w,\underset{\sim}{s},\underset{\sim}{z})\frac{\zeta(1+z_1-z_2+2w)\zeta(1-z_2+z_3+2w)}{\zeta(1-z_2+2w)} \times \frac{\prod_{i,j=1}^N \zeta(1+s_i+s_{j+N})^{1/b^2}\prod_{j=1}^N \zeta(1+s_{j+N}+w)^{1/b}}{\prod_{i=1}^N \zeta(1+s_i-z_2+w)^{1/b}\prod_{j=1}^N \zeta(1+s_{j+N}+z_1+w)^{1/b}\zeta(1+s_{j+N}+z_3+w)^{1/b}} \quad (22)$$

where $\mathcal{D}(w,\underset{\sim}{s},\underset{\sim}{z})$ is an absolutely convergent Euler product provided $\sigma_j > -\frac{1}{4}$, $j = 1, \ldots, 2N$, $w > -\frac{1}{4}$.

We would first like to shift the w contour. Set $\kappa = 1/\log\log T$ and note that

$$\prod_{j=1}^{2N} x^{s_j} \ll x^{2N\kappa} = T^{2\kappa\theta}$$

since $x = T^{\theta/(a+b)}$. By (22) we have $\mathcal{G}(w, \underline{s}, \underline{z}) \ll (\log T)^{N^2/b^2+4N/b+3}$ and hence we can truncate the w integral to $|\Im w| \le \sqrt{A \log T}$ for some sufficiently large A at the cost of at most

$$(\log T)^{N^2/b^2+4N/b+3-2NB} T^{2+2\kappa\theta} \exp(-AC' \log T) \int_{(\kappa)^{2N}} \prod_{j=1}^{2N} \frac{ds_j}{|s_j|^{B+1}} \ll T^{-A'}$$

using the fact that $H(\sigma+it) \ll e^{-C't^2}$. Then we can shift the w contour to the line with $\Re(w) = -2\kappa/3$ so that we only encounter a simple pole at $w=0$. Again, we have the bound $\mathcal{G}(w, \underline{s}, \underline{z}) \ll (\log T)^{N^2/b^2+4N/b+3}$ on the new line of integration since we have remained in the zero free region. This integral then contributes

$$\ll T^{-4\kappa/3+2\theta\kappa} (\log T)^{N^2/b^2+4N/b+3-2NB} \int_{(-2\kappa/3)} \frac{|H(w)|}{|w|} dw \int_{(\kappa)^{2N}} \prod_{j=1}^{2N} \frac{ds_j}{|s_j|^{B+1}} = o(1).$$

Thus we obtain

$$G(x, \underline{z}) = \frac{B!^{2N}}{(\log x)^{2NB}(2\pi i)^{2N}} \int_{(\kappa)^{2N}} \mathcal{G}(0, \underline{s}, \underline{z}) \prod_{j=1}^{2N} x^{s_j} \frac{ds_j}{s_j^{B+1}} + o(1).$$

We now shift κ to $1/\log x$ and substitute $s_j \mapsto s_j/\log x$ for each j to give

$$G(x, \underline{z}) = \frac{B!^{2N}}{(2\pi i)^{2N}} \int_{(1)^{2N}} \mathcal{G}(0, \underline{s}/\log x, \underline{z}) \prod_{j=1}^{2N} e^{s_j} \frac{ds_j}{s_j^{B+1}} + o(1).$$

By (22) we have the trivial estimate

$$\mathcal{G}(0, \underline{s}/\log x, \underline{z}) \ll (\log T)^{N^2/b^2+4N/b+1}, \quad \Re(s_j) \asymp 1$$

whilst for $|s_j| = o(\log x)$ we have

$$\mathcal{G}(0, \underline{s}/\log x, \underline{z}) \ll (\log T)^{N^2/b^2-2N/b+1} \frac{\prod_{i=1}^{N}(1+|s_i|)^{1/b} \prod_{j=1}^{N}(1+|s_{j+N}|)^{1/b}(1+|s_{j+N}|)^{1/b}}{\prod_{i,j=1}^{N}|s_i+s_{j+N}|^{1/b^2} \prod_{j=1}^{N}|s_{j+N}|^{1/b}} \tag{23}$$

where we have used the bounds for $\zeta(s)$ given in (19) and (20). As before, truncating any of the s_j integrals at height $t_j = \sqrt{\log x}$ leads to an error of

$$(\log T)^{N^2/b^2+4N/b+1} \int_{1+i\sqrt{\log x}}^{1+i\infty} \frac{ds}{|s|^{B+1}} \ll (\log T)^{k^2+6k+6-B/2},$$

which can be bounded by $(\log T)^{k^2-1}$ on choosing $B \geqslant 12k+14$. Performing this truncation in each variable gives

$$G(x,\underline{z}) = \frac{B!^{2N}}{(2\pi i)^{2N}} \int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}} \cdots \int_{1-i\sqrt{\log x}}^{1+i\sqrt{\log x}} \mathscr{G}(0, \underline{s}/\log x, \underline{z}) \prod_{j=1}^{2N} e^{s_j} \frac{ds_j}{s_j^{B+1}} + O((\log T)^{k^2-1}),$$

which is bounded by $(\log T)^{N^2/b^2-2N/b+1} = (\log T)^{k^2}$ using (23). The bound $S_{22} \ll (\log T)^{k^2+3}$ then follows. Combining this with the bound for S_{21}, we complete the proof of Proposition 3.

4. Proof of Theorem 4

We first show that Theorem 4 can be deduced from the following theorem.

Theorem 5. *Let $\alpha, \beta \ll 1/\log T$ be sufficiently small shifts. Let $Q(s) = \sum_{n\leqslant y} a(n)n^{-s}$ with $y = T^\theta$ and $\theta < \frac{1}{2}$ and denote $\bar{Q}(s) = \sum_{n\leqslant y} \overline{a(n)}n^{-s}$. Suppose that there exist some fixed positive constants r and C such that $|a(mn)| \ll |a(m)a(n)|$ and $|a(n)| \ll \tau_r(n)(\log n)^C$. Then for any constant $A > 0$, it holds that*

$$\sum_{0\leqslant\gamma\leqslant T} \zeta(\rho+\alpha)\zeta(1-\rho+\beta)Q(\rho)\bar{Q}(1-\rho) = \mathscr{J}(\alpha,\beta,T) + \mathscr{I}(\alpha,\beta,T) + \overline{\mathscr{I}(\bar{\beta},\bar{\alpha},T)} + O(T(\log T)^{-A}),$$

where $\mathscr{J}$ is defined in (5) *and*

$$\mathscr{I}(\alpha,\beta,T) = \sum_{g\leqslant y} \sum_{\substack{h,k\leqslant y/g \\ (h,k)=1}} \frac{a(gh)\overline{a(gk)}}{ghk} \frac{d}{d\gamma} \frac{1}{2\pi} \int_0^T \biggl[Z_{\alpha,\beta,\gamma,h,k} + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} Z_{-\beta,-\alpha,\gamma,h,k} + \left(\frac{t}{2\pi}\right)^{-\beta-\gamma} Z_{\alpha,-\gamma,-\beta,h,k} \biggr] dt \bigg|_{\gamma=0} \tag{24}$$

with

$$Z_{\alpha,\beta,\gamma,h,k} = \frac{1}{h^\beta} \frac{\zeta(1+\alpha+\beta)\zeta(1+\beta+\gamma)}{\zeta(1+\beta)} \prod_{p^{k_p}\,\|\,k} \frac{\sum_{m\geqslant 0} f_{\alpha,\gamma}(p^{m+k_p})p^{-m(1+\beta)}}{\sum_{m\geqslant 0} f_{\alpha,\gamma}(p^m)p^{-m(1+\beta)}}$$

and $f_{\alpha,\gamma}$ is defined in (6).

Lemma 6. *Let $H(s)$ be an analytic function such that $H(\sigma+it) \ll_\sigma e^{-Ct^2}$ for some constant C, and additionally satisfies $H(0) = 1$ and is zero at $2s = \beta-\alpha$ and $2s = \beta-\gamma$. For $c > 0$ let*

$$\tilde{Z}_{\alpha,\beta,\gamma,h,k}(T) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{H(s)}{s} T^{2s} \sum_{hm=kn} \frac{f_{\alpha,\gamma}(m)n^{-\beta}}{(hk)^{1/2}(mn)^{1/2+s}} ds$$

where $f_{\alpha,\gamma}(n)$ is given by (6)*. Then for $h,k \leqslant T$ with $(h,k) = 1$ and $\alpha,\beta,\gamma \ll 1/\log T$, we have*

$$Z_{\alpha,\beta,\gamma,h,k} = hk\tilde{Z}_{\alpha,\beta,\gamma,h,k}(T) + O\biggl(\tau(k)\left(\frac{T^2}{hk}\right)^{-1/\log_3 T} (\log T)^3\biggr) + O\biggl(\frac{\tau(k)}{(\log T)^A}\biggr) \tag{25}$$

where $A > 0$ is an arbitrary constant.

Proof. This is similar to the proof of Theorem 1.2 in [Bettin et al. 2020]. Since $(h,k)=1$ we have

$$\tilde{Z}_{\alpha,\beta,\gamma,h,k}(T)=\frac{1}{h^{z_2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{H(s)}{s}T^{2s}\frac{1}{(hk)^{1+s}}\sum_{\ell}\frac{f_{\alpha,\gamma}(k\ell)}{\ell^{1+\beta+2s}}ds.$$

Now, a short computation shows that

$$\sum_{\ell}\frac{f_{\alpha,\gamma}(k\ell)}{\ell^{1+\beta+2s}}=\frac{\zeta(1+\alpha+\beta+2s)\zeta(1+\beta+\gamma+2s)}{\zeta(1+\beta+2s)}\prod_{p^{k_p}\,\|\,k}\frac{\sum_{m\geqslant 0}f_{\alpha,\gamma}(p^{m+k_p})p^{-m(1+\beta+2s)}}{\sum_{m\geqslant 0}f_{\alpha,\gamma}(p^{m})p^{-m(1+\beta+2s)}}.$$

This is holomorphic for $\sigma>0$ so we may freely shift to the line $\Re(s)=1/\log T$. We then truncate the integral at height $t=\pm\sqrt{A'\log\log T}$ for some large constant A' incurring an error of size

$$\ll\frac{1}{hk}e^{-cA'\log\log T}(\log T)^3\tau(k)\ll\frac{\tau(k)}{hk}(\log T)^{-A}. \tag{26}$$

Here the factor of $\tau(k)$ owes to the fact that uniformly for $\sigma\geqslant-\frac{1}{4}$, the product over primes dividing k is

$$\ll\prod_{p\,|\,k}(1+O(p^{-1/2}))\ll\tau(k).$$

Then, we can shift the contour to $\Re(s)=-1/\log_3 T$ and encounter only a simple pole at $s=0$, since we remain in the zero-free region of $\zeta(1+s)$ and the zeros of $H(s)$ cancel the other poles. Using the bound $\zeta(s)^{\pm1}\ll\log T$ on the contour, the integral over the left edge is

$$\ll T^{-2/\log_3 T}(hk)^{-1+1/\log_3 T}\tau(k)(\log T)^3$$

whilst the horizontal integrals give a lower order contribution plus a contribution of size (26). Therefore,

$$\tilde{Z}_{\alpha,\beta,\gamma,h,k}(T)=\frac{1}{hk}Z_{\alpha,\beta,\gamma,h,k}+O\biggl(\frac{\tau(k)}{hk}\biggl(\frac{T^2}{hk}\biggr)^{-1/\log_3 T}(\log T)^3\biggr)+O\biggl(\frac{\tau(k)}{hk}(\log T)^{-A}\biggr)$$

and the result follows. □

Proof that Theorem 5 implies Theorem 4. Applying Lemma 6 to $Z_{\alpha,\beta,\gamma,h,k}$ in $\mathcal{I}(\alpha,\beta,T)$, we see that the first error term of (25) gives a total contribution

$$\ll\biggl(\frac{T^2}{y^2}\biggr)^{-1/\log_3 T}(\log T)^3\sum_{g\leqslant y}\sum_{\substack{h,k\leqslant y/g\\(h,k)=1}}\frac{|a(gh)||a(gk)|\tau(k)}{ghk}\int_1^T dt\ll T^{1-1/\log_3 T}(\log T)^{O(1)}$$

using $y\leqslant T^{1/2}$ and the conditions for the coefficients $a(n)$. The second error term contributes $\ll T/(\log T)^{O(1)}$, again from the divisor type bounds for the coefficients $a(n)$. For the sum of the main term $\tilde{Z}$, we ungroup the sums in terms of the gcd's g and then push the integral through to find

$$\sum_{g\leqslant y}\sum_{\substack{h,k\leqslant y/g\\(h,k)=1}}\frac{a(gh)\overline{a(gk)}}{g}\tilde{Z}_{\alpha,\beta,\gamma,h,k}(T)=\sum_{h,k\leqslant y}\sum_{hm=kn}\frac{a(h)\overline{a(k)}f_{\alpha,\gamma}(m)n^{-\beta}}{(hkmn)^{1/2}}w(mn/T^2)=S_{\alpha,\beta,\gamma}(T),$$

which completes the proof. □

Now it remains to prove Theorem 5. Let

$$S_3 = S_3(\alpha, \beta, T) = \sum_{0 \leqslant \gamma \leqslant T} \zeta(\rho+\alpha)\zeta(1-\rho+\beta)Q(\rho)\bar{Q}(1-\rho)$$

where α, β are small ($\ll 1/\log T$), complex shifts. These types of mean values have been considered before by several authors [Conrey et al. 1986; 1998; Ng 2008b]. Accordingly, we shall only briefly describe the initial steps using [Conrey et al. 1986] as our main reference.

We write S_3 as the integral over the positively oriented rectangular contour Γ with vertices $a+i$, $a+iT$, $1-a+iT$, $1-a+i$, $a = 1+1/\log T$:

$$S_3 = \frac{1}{2\pi i}\int_\Gamma \frac{\zeta'(s)}{\zeta(s)}\zeta(s+\alpha)\zeta(1-s+\beta)Q(s)\bar{Q}(1-s)\,ds.$$

Since $Q(s) \ll y^{1-\sigma+\epsilon}$, $\zeta(s) \ll t^{(1-\sigma)/2+\epsilon}$ and T can be chosen such that $(\zeta'/\zeta)(s) \ll (\log T)^2$ on the contour, we find that the horizontal sections contribute $O(y^{1+\epsilon}T^{1/2+\epsilon})$. For the contour on the left-hand side we apply the functional equation

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{\chi'(s)}{\chi(s)} - \frac{\zeta'(1-s)}{\zeta(1-s)}$$

where $\chi(s) = \pi^{s-1/2}\Gamma((1-s)/2)/\Gamma(s/2)$ is the factor appearing in the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$. The integral involving $-\zeta'(1-s)/\zeta(1-s)$ is given by

$$\begin{aligned}
-\frac{1}{2\pi i}\int_{1-a+iT}^{1-a+i} & \frac{\zeta'(1-s)}{\zeta(1-s)}\zeta(s+\alpha)\zeta(1-s+\beta)Q(s)\bar{Q}(1-s)\,ds \\
&= \frac{1}{2\pi}\int_1^T \frac{\zeta'(a-it)}{\zeta(a-it)}\zeta(1-a+it+\alpha)\zeta(a-it+\beta)Q(1-a+it)\bar{Q}(a-it)\,dt \\
&= \overline{\frac{1}{2\pi}\int_1^T \frac{\zeta'(a+it)}{\zeta(a+it)}\zeta(1-a-it+\bar{\alpha})\zeta(a+it+\bar{\beta})\bar{Q}(1-a-it)Q(a+it)\,dt} \\
&= \overline{\frac{1}{2\pi i}\int_{a+i}^{a+iT} \frac{\zeta'(s)}{\zeta(s)}\zeta(1-s+\bar{\alpha})\zeta(s+\bar{\beta})\bar{Q}(1-s)Q(s)\,ds}.
\end{aligned}$$

This integral can therefore be expressed in terms of the integral over the right edge of the contour. For the integral involving $\chi'(s)/\chi(s)$ we shift to the half-line and apply Stirling's formula in the form

$$\frac{\chi'\left(\frac{1}{2}+it\right)}{\chi\left(\frac{1}{2}+it\right)} = -\log\left(\frac{|t|}{2\pi}\right) + O(1/|t|), \quad t \geqslant 1.$$

In this way, we find that

$$S_3 = J(\alpha, \beta, T) + I(\alpha, \beta, T) + \overline{I(\bar{\beta}, \bar{\alpha}, T)} + O(yT^{1/2+\epsilon})$$

where

$$J(\alpha, \beta, T) = \frac{1}{2\pi}\int_1^T [\log(t/2\pi) + O(1/t)]\zeta\left(\tfrac{1}{2}+\alpha+it\right)\zeta\left(\tfrac{1}{2}+\beta-it\right)\left|Q\left(\tfrac{1}{2}+it\right)\right|^2 dt$$

and

$$I(\alpha,\beta,T)=\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'(s)}{\zeta(s)}\zeta(s+\alpha)\zeta(1-s+\beta)Q(s)\bar{Q}(1-s)\,ds.$$

It remains to show that $J(\alpha,\beta,T)=\mathcal{J}(\alpha,\beta,T)+O(T(\log T)^{-A})$ where $\mathcal{J}$ is given by (5) and that $I(\alpha,\beta,T)=\mathcal{I}(\alpha,\beta,T)+O(T(\log T)^{-A})$ where $\mathcal{I}$ is given by (24).

4.1. *Computing J.* The integral J can be computed by integrating by parts and using well known formulas for the twisted second moment of the zeta function. In our case (with the shifts α,β) these mean values have been considered by Pratt and Robles [2018]. After a slight rephrasing, their Theorem 1.1 states that

$$\int_1^T\zeta\big(\tfrac12+\alpha+it\big)\zeta\big(\tfrac12+\beta-it\big)\big|Q\big(\tfrac12+it\big)\big|^2\,dt$$
$$=\sum_{g\leqslant y}\sum_{\substack{h,k\leqslant y/g\\(h,k)=1}}\frac{a(gh)\overline{a(gk)}}{ghk}\int_1^T\bigg[\frac{\zeta(1+\alpha+\beta)}{h^\beta k^\alpha}+\Big(\frac{t}{2\pi}\Big)^{-\alpha-\beta}\frac{\zeta(1-\alpha-\beta)}{h^{-\alpha}k^{-\beta}}\bigg]dt$$
$$+O(T^{3/20}y^{33/20})+O(y^{1/2}T^{1/2+\epsilon}).$$

Thus, after integrating by parts we see that J is indeed given by $\mathcal{J}$ plus an acceptable error.

4.2. *Computing I: initial manipulations.* Instead of working directly with $I(\alpha,\beta,T)$ we work with the integral

$$\mathcal{K}=\mathcal{K}_{\alpha,\beta}(\gamma):=\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta(s+\gamma)}{\zeta(s)}\zeta(s+\alpha)\zeta(1-s+\beta)Q(s)\bar{Q}(1-s)ds \tag{27}$$

so that

$$I(\alpha,\beta,T)=\frac{d}{d\gamma}\mathcal{K}_{\alpha,\beta}(\gamma)\bigg|_{\gamma=0}.$$

As with the other shifts we will assume throughout that $\gamma\ll 1/\log T$ and derive our formula for $\mathcal{K}_{\alpha,\beta}(\gamma)$ with error terms uniform in γ. The differentiation can then be performed by applying Cauchy's formula over a circle of radius $\ll 1/\log T$.

In (27) we apply the functional equation $\zeta(1-s+\beta)=\chi(s-\beta)\zeta(s-\beta)$ and then expand each term as a Dirichlet series to give

$$\begin{aligned}\mathcal{K}&=\sum_{m_1,m_2,m_3,m_4,h,k}\frac{\mu(m_1)a(h)\overline{a(k)}}{m_2^\gamma m_3^\alpha m_4^{-\beta}k}\frac{1}{2\pi i}\int_{a+i}^{a+iT}\chi(s-\beta)\Big(\frac{m_1m_2m_3m_4h}{k}\Big)^{-s}ds\\&=\sum_{m_1,m_2,m_3,m_4,h,k}\frac{\mu(m_1)a(h)\overline{a(k)}}{m_1^\beta m_2^{\beta+\gamma}m_3^{\alpha+\beta}h^\beta k^{1-\beta}}\frac{1}{2\pi i}\int_{a-\beta+i}^{a-\beta+iT}\chi(s)\Big(\frac{m_1m_2m_3m_4h}{k}\Big)^{-s}ds\\&=\sum_{\substack{m\\k\leq y}}\frac{b(m)\overline{a(k)}}{m^\beta k^{1-\beta}}\frac{1}{2\pi i}\int_{a-\beta+i}^{a-\beta+iT}\chi(s)\Big(\frac{m}{k}\Big)^{-s}ds,\end{aligned}$$

where

$$b(m)=\sum_{\substack{m_1m_2m_3m_4h=m\\ h\leqslant y}}\mu(m_1)m_2^{-\gamma}m_3^{-\alpha}m_4^{\beta}a(h). \tag{28}$$

Note that we have

$$b(m)\ll\tau_4*a(m)\ll\tau_{r+4}(m)(\log m)^C.$$

This integral can be evaluated by the following lemma.

Lemma 7. *Suppose that $B(s)=\sum_n b(n)n^{-s}$ for $\sigma>1$ where $b(n)\ll\tau_{k_1}(n)(\log n)^{l_1}$ for some nonnegative integers k_1 and l_1. Let $A(s)=\sum_{n\leq y}a(n)n^{-s}$ where $a(n)\ll\tau_{k_2}(\log n)^{l_2}$ for some nonnegative integers k_2, l_2 and $T^{\epsilon}\ll y\leq T$ for some $\epsilon>0$. Then*

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT}\chi(1-s)A(1-s)B(s)\,ds=\sum_{k\leq y}\frac{a(k)}{k}\sum_{m\leq nT/2\pi}b(m)e(-m/k)+O(yT^{1/2}(\log T)^{k_1+k_2+l_1+l_2}),$$

where $c=1+1/\log T$.

Proof. See [Conrey et al. 1998, Lemma 2]. □

Applying Lemma 7 we obtain

$$\mathcal{K}=\sum_{k\leqslant y}\frac{\overline{a(k)}}{k^{1-\beta}}\sum_{m\leqslant Tk/2\pi}\frac{b(m)e(-m/k)}{m^{\beta}}+O(yT^{1/2+\epsilon}).$$

Following [Conrey et al. 1998], we now express the additive character $e(-m/k)$ in terms of multiplicative characters. We write $m'=m/(m,k)$ and $k'=k/(m,k)$ so that

$$e(-m/k)=e(-m'/k')=\frac{1}{\phi(k')}\sum_{\chi\pmod{k'}}\tau(\bar\chi)\chi(-m')=\frac{\mu(k')}{\phi(k')}+\frac{1}{\phi(k')}\sum_{\chi\neq\chi_0\pmod{k'}}\tau(\bar\chi)\chi(-m')$$

where $\tau(\chi)$ denotes the Gauss sum. The first term here will lead to the main term whilst the second term will give rise to the error. When computing the error term we wish to apply the large sieve and hence it is necessary to express the sum over characters in terms of primitive characters. To this end we write

$$\frac{1}{\phi(k')}\sum_{\chi\neq\chi_0\pmod{k'}}\tau(\bar\chi)\chi(-m')=\frac{1}{\phi(k')}\sum_{q\,|\,k',q>1}\sum_{\psi\pmod q}^{*}\mu\Big(\frac{k'}{q}\Big)\bar\psi\Big(\frac{k'}{q}\Big)\tau(\bar\psi)\psi(-m')$$

where the $*$ denotes that the sum is over primitive characters. After an application of Möbius inversion as in [Conrey et al. 1998, Formula (5.10)], we have

$$\begin{aligned}\frac{1}{\phi(k')}\sum_{\chi\neq\chi_0\pmod{k'}}\tau(\bar\chi)\chi(-m')&=\sum_{d\,|\,(m,k)}\sum_{e\,|\,d}\frac{\mu(d/e)}{\phi(k/e)}\sum_{q\,|\,k/e,q>1}\sum_{\psi\pmod q}^{*}\mu\Big(\frac{k}{eq}\Big)\bar\psi\Big(\frac{k}{eq}\Big)\tau(\bar\psi)\psi\Big(-\frac{m}{e}\Big)\\&=\sum_{q\,|\,k,q>1}\sum_{\psi\pmod q}^{*}\tau(\bar\psi)\sum_{d\,|\,(m,k)}\psi\Big(\frac{m}{d}\Big)\delta(q,k,d,\psi),\end{aligned}$$

where

$$\delta(q,k,d,\psi)=\sum_{\substack{e\mid d\\ e\mid k/q}}\frac{\mu(d/e)}{\phi(k/e)}\bar{\psi}\Big(-\frac{k}{eq}\Big)\psi\Big(\frac{d}{e}\Big)\mu\Big(\frac{k}{eq}\Big).$$

Therefore, we can write

$$\mathcal{K}=\mathcal{M}+\mathcal{E}+O(yT^{1/2+\epsilon}), \tag{29}$$

where

$$\mathcal{M}=\sum_{k\le y}\frac{\overline{a(k)}}{k^{1-\beta}}\sum_{m\le Tk/2\pi}\frac{b(m)}{m^{\beta}}\frac{\mu(k/(m,k))}{\phi(k/(m,k))},$$

$$\mathcal{E}=\sum_{k\le y}\frac{\overline{a(k)}}{k^{1-\beta}}\sum_{m\le Tk/2\pi}\frac{b(m)}{m^{\beta}}\sum_{\substack{q\mid k\psi\\ q>1}}\sideset{}{^*}\sum_{(\bmod q)}\tau(\bar{\psi})\sum_{d\mid(m,k)}\psi\Big(\frac{m}{d}\Big)\delta(q,k,d,\psi).$$

4.3. ***Computing the main term*** $\mathcal{M}$**.** We compute $\mathcal{M}$ essentially by applying Perron's formula to the inner sum although there are some arithmetic complications to deal with. We first unfold the definition of $b(m)$ to write

$$\mathcal{M}=\sum_{h,k\leqslant y}\frac{a(h)\overline{a(k)}}{h^{\beta}k^{1-\beta}}\sum_{n\leqslant Tk/2\pi h}\frac{c(n)}{n^{\beta}}\frac{\mu(k/(nh,k))}{\phi(k/(nh,k))},$$

where

$$c(n)=\sum_{n_1n_2n_3n_4=n}\mu(n_1)n_2^{-\gamma}n_3^{-\alpha}n_4^{\beta}. \tag{30}$$

We then group the terms h,k according to their greatest common divisor $g=(h,k)$ and obtain the formula

$$\mathcal{M}=\sum_{g\le y}\sum_{\substack{h,k\le y/g\\ (h,k)=1}}\frac{a(gh)\overline{a(gk)}}{gh^{\beta}k^{1-\beta}}\sum_{n\le Tk/2\pi h}\frac{c(n)}{n^{\beta}}\frac{\mu(k/(n,k))}{\phi(k/(n,k))}.$$

On grouping together terms for which $(n,k)=d$ we obtain

$$\mathcal{M}=\sum_{g\le y}\sum_{h\le y/g}\frac{a(gh)}{gh^{\beta}}\sum_{d\le y/g}\sum_{\substack{k\le y/dg\\ (h,dk)=1}}\frac{\overline{a(gdk)}}{dk^{1-\beta}}\frac{\mu(k)}{\phi(k)}\sum_{\substack{n\le Tk/2\pi h\\ (n,k)=1}}\frac{c(dn)}{n^{\beta}}.$$

To encode the dependence of d in the innermost sum, we use the following lemma.

Lemma 8. *Let $j,D\in\mathbb{N}$ and let $f_1,\dots,f_j$ be arithmetic functions. Given a decomposition of integers $D=d_1\cdots d_j$, define $D_i=\prod_{u=1}^{j-i}d_u$ for $1\le i\le j-1$ and $D_j=1$. The following identities hold*:

$$\sum_{\substack{m\le x\\ (m,k)=1}}(f_1*f_2*\cdots*f_j)(mD)=\sum_{d_1\cdots d_j=D}\sum_{\substack{m_1\cdots m_j\le x\\ (m_i,kD_i)=1}}f_1(m_1d_j)f_2(m_2d_{j-1})\cdots f(m_jd_1),$$

$$\sum_{(m,k)=1}\frac{(f_1*f_2*\cdots*f_j)(mD)}{m^s}=\sum_{d_1\cdots d_j=D}\prod_{i=1}^{j}\sum_{(m_i,kD_i)=1}\frac{f(m_id_{j+1-i})}{m_i^s}.$$

Proof. The second identity is [Conrey et al. 1998, Lemma 3]. The first identity follows from the same method of proof. □

From Lemma 8 we see that the innermost Dirichlet series in $\mathcal{M}$ may be written as

$$\begin{aligned}\sum_{(n,k)=1}\frac{c(nd)}{n^{s+\beta}} &= \sum_{d_1d_2d_3d_4=d}\ \sum_{(m_1,kd_1d_2d_3)=1}\mu(m_1d_4)\sum_{(m_2,kd_1d_2)=1}(m_2d_3)^{-\gamma}\sum_{(m_3,kd_1)=1}(m_3d_2)^{-\alpha}\\ &\qquad\times\sum_{(m_4,k)=1}(m_4d_1)^{\beta}(m_1m_2m_3m_4)^{-(s+\beta)}\\ &= \sum_{d_1d_2d_3d_4=d}\mu(d_4)d_3^{-\gamma}d_2^{-\alpha}d_1^{\beta}\sum_{(m_1,kd)=1}\mu(m_1)m_1^{-s-\beta}\sum_{(m_2,kd_1d_2)=1}m_2^{-\gamma-s-\beta}\\ &\qquad\times\sum_{(m_3,kd_1)=1}m_3^{-\alpha-s-\beta}\sum_{(m_4,k)=1}m_4^{-s}\\ &= \frac{\zeta(s)\zeta(s+\gamma+\beta)\zeta(s+\alpha+\beta)}{\zeta(s+\beta)}G(s,k,d)\end{aligned}$$

where

$$\begin{aligned}G(s,k,d) = \sum_{d_1d_2d_3d_4=d}\mu(d_4)d_3^{-\gamma}d_2^{-\alpha}d_1^{\beta}\prod_{p\,|\,kd}(1-p^{-s-\beta})^{-1}\\ \times\prod_{p\,|\,kd_1d_2}(1-p^{-s-\gamma-\beta})\prod_{p\,|\,kd_1}(1-p^{-\alpha-s-\beta})\prod_{p\,|\,k}(1-p^{-s}).\end{aligned}\tag{31}$$

Note that $G(s,k,d)$ is holomorphic in the region $\sigma>0$ and that for $\sigma\geqslant\frac{1}{2}$, we have the bound

$$G(s,k,d)\ll\tau_4(d)\prod_{p\,|\,kd}(1+10p^{-\sigma})\ll\tau_4(d)\tau(kd)\tag{32}$$

since the shifts are all bounded by $1/\log T$ and $d,k\ll T$. Also, by changing the role of d_2,d_3,d_4, we can write

$$\begin{aligned}G(s,k,d) = \sum_{d_1d_2d_3d_4=d}\mu(d_4)d_3^{-\gamma}d_2^{\beta}d_1^{-\alpha}\prod_{p\,|\,kd}(1-p^{-s-\beta})^{-1}\\ \times\prod_{p\,|\,kd_1d_2}(1-p^{-s-\gamma-\beta})\prod_{p\,|\,kd_1}(1-p^{-s})\prod_{p\,|\,k}(1-p^{-s-\alpha-\beta}).\end{aligned}\tag{33}$$

as well as

$$\begin{aligned}G(s,k,d) = \sum_{d_1d_2d_3d_4=d}\mu(d_4)d_3^{\beta}d_2^{-\alpha}d_1^{-\gamma}\prod_{p\,|\,kd}(1-p^{-s-\beta})^{-1}\\ \times\prod_{p\,|\,kd_1d_2}(1-p^{-s})\prod_{p\,|\,kd_1}(1-p^{-\alpha-s-\beta})\prod_{p\,|\,k}(1-p^{-s-\gamma-\beta}).\end{aligned}\tag{34}$$

These alternative formulations will be useful when recovering the second and third Z terms of (24).

4.3.1. *Perron's formula.* We employ the following version of Perron's formula to evaluate the innermost sum in $\mathcal{M}$.

Lemma 9. *Let $f(s)=\sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with abscissa of absolute convergence σ_a. Let*

$$B(\sigma)=\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma}},$$

for $\sigma>\sigma_a$. Then for $\kappa>\sigma_a$, $x\ge 2$, $U\ge 2$, and $H\ge 2$, we have

$$\sum_{n\le x} a_n=\frac{1}{2\pi i}\int_{\kappa-iU}^{\kappa+iU} f(s)\frac{x^s}{s}ds+O\biggl(\sum_{x-x/H\le n\le x+x/H}|a_n|\biggr)+O\biggl(\frac{x^{\kappa}HB(\kappa)}{U}\biggr).$$

Proof. See [Liu and Ye 2007, Theorem 2.1]. □

Applying Lemma 9 with $U=\exp(c\sqrt{\log T})$, $H=\sqrt{U}$ and $\kappa=1+1/\log T$ we find that

$$\sum_{\substack{n\leqslant Tk/2\pi h\\(n,k)=1}}\frac{c(dn)}{n^{\beta}}=\frac{1}{2\pi i}\int_{\kappa-iU}^{\kappa+iU}\frac{\zeta(s)\zeta(s+\alpha+\beta)\zeta(s+\beta+\gamma)}{\zeta(s+\beta)}G(s,k,d)\biggl(\frac{Tk}{2\pi h}\biggr)^{s}\frac{ds}{s}$$
$$+O\biggl(\sum_{kT/(2\pi h)-kT/(2\pi h\sqrt{U})\le n\le kT/(2\pi h)+kT/(2\pi h\sqrt{U})}\biggl|\frac{c(dn)}{n^{\beta}}\biggr|\biggr)+O\biggl(\frac{Tk}{h}\frac{(\log T)^C}{\sqrt{U}}\biggr).$$

From (30), we have $|c(n)|\ll\tau_4(n)$ for $n\leqslant T^{O(1)}$, and thus by Shiu's bound for short divisor sums [Shiu 1980, Theorem 2] we have

$$\sum_{kT/(2\pi h)-kT/(2\pi h\sqrt{U})\le n\le kT/(2\pi h)+kT/(2\pi h\sqrt{U})}\biggl|\frac{c(dn)}{n^{\beta}}\biggr|$$
$$\ll\sum_{kT/(2\pi h)-kT/(2\pi h\sqrt{U})\le n\le kT/(2\pi h)+kT/(2\pi h\sqrt{U})}\tau_4(d)\tau_4(n)\ll\tau_4(d)\frac{kT}{h\sqrt{U}}(\log T)^4.$$

Therefore the error terms contribute to $\mathcal{M}$ at most

$$\sum_{g\le y}\sum_{h\le y/g}\frac{|a(gh)|}{gh^{\beta}}\sum_{d\le y/g}\sum_{k\le y/dg}\frac{|a(gdk)|}{dk^{1-\beta}}\frac{\tau_4(d)}{\phi(k)}\frac{Tk}{h\sqrt{U}}(\log T)^C$$
$$\ll\sum_{g\le y}\frac{|a(g)|^2}{g}\sum_{h\le y}\frac{|a(h)|}{h}\sum_{d\le y/g}\frac{|a(d)|\tau_4(d)}{d}\sum_{k\le y/dg}\frac{|a(k)|}{k}\frac{T}{\sqrt{U}}(\log T)^{C'}$$
$$\ll\frac{T}{\sqrt{U}}(\log T)^{C''},$$

where we have used that $|a(mn)|\le|a(m)a(n)|$ and $|a(n)|\ll\tau_r(n)(\log n)^C$.

Moving the contour to the line $\sigma = 1 - c/\log U$, we encounter three poles. Using (32) and that $\zeta(s)^{\pm 1} \ll \log U$ in the zero free region, the integral over the left edge of the contour leads to a contribution

$$(\log U)^4 \sum_{g\le y}\sum_{h\le y/g} \frac{|a(gh)|}{gh^{\beta}} \sum_{d\le y/g}\sum_{k\le y/dg} \frac{|a(gdk)|}{dk^{1-\beta}} \frac{\tau_4(d)\tau(kd)}{\phi(k)} \left(\frac{Tk}{h}\right)^{1-c/\log U} \ll T\exp(-c\sqrt{\log T}).$$

The integral over the horizontal lines contributes at most

$$\frac{(\log U)^4}{U} \sum_{g\le y}\sum_{h\le y/g} \frac{|a(gh)|}{gh^{\beta}} \sum_{d\le y/g}\sum_{k\le y/dg} \frac{|a(gdk)|}{dk^{1-\beta}} \frac{\tau_4(d)\tau(kd)}{\phi(k)} \Big(Tk/h\Big) \ll T\exp(-c\sqrt{\log T}).$$

Therefore, we arrive at

$$\mathcal{M} = \sum_{g\le y}\sum_{h\le y/g} \frac{a(gh)}{gh^{\beta}} \sum_{d\le y/g} \sum_{\substack{k\le y/dg\\ (h,dk)=1}} \frac{\overline{a(gdk)}}{dk^{1-\beta}} \frac{\mu(k)}{\phi(k)} \sum_{\substack{z=1\\ z=1-\alpha-\beta\\ z=1-\beta-\gamma}} \mathrm{res}_{s=z}(\mathcal{F}(s)) + O(T\exp(-c\sqrt{\log T})), \quad (35)$$

where

$$\mathcal{F}(s) = \frac{\zeta(s)\zeta(s+\alpha+\beta)\zeta(s+\beta+\gamma)}{\zeta(s+\beta)} G(s,k,d)\frac{1}{s}\left(\frac{Tk}{2\pi h}\right)^{s}.$$

4.3.2. *Computing the residues.* Let us first analyse the contribution from the residue at $s = 1$. This is given by

$$\begin{aligned} &\frac{T}{2\pi} \sum_{g\leqslant y}\sum_{h\leqslant y/g} \frac{a(gh)}{gh^{1+\beta}} \sum_{d\leqslant y/g} \sum_{\substack{k\leqslant y/dg\\ (h,dk)=1}} \frac{\overline{a(gdk)}}{dk^{-\beta}} \frac{\mu(k)}{\phi(k)} \frac{\zeta(1+\alpha+\beta)\zeta(1+\beta+\gamma)}{\zeta(1+\beta)} G(1,k,d) \\ &= \sum_{g\leqslant y} \sum_{\substack{h,k'\leqslant y/g\\ (h,k')=1}} \frac{a(gh)\overline{a(gk')}}{ghk'} \int_0^{T/2\pi} \frac{1}{h^{\beta}} \frac{\zeta(1+\alpha+\beta)\zeta(1+\beta+\gamma)}{\zeta(1+\beta)} \sum_{kd=k'} k^{\beta}\frac{k}{\phi(k)}\mu(k)G(1,k,d)\,dt. \end{aligned}$$

We will show that the integrand is given by $Z_{\alpha,\beta,\gamma,h,k}$. From the definition of Z we are required to show that

$$\sum_{kd=k'} k^{\beta}\frac{k}{\phi(k)}\mu(k)G(1,k,d) = \prod_{p^{k'_p}\,\|\,k'} \frac{\sum_{m\geqslant 0} f_{\alpha,\gamma}(p^{m+k'_p})p^{-m(1+\beta)}}{\sum_{m\geqslant 0} f_{\alpha,\gamma}(p^m)p^{-m(1+\beta)}} \quad (36)$$

where we recall that

$$f_{\alpha,\gamma}(n) = \sum_{n_1n_2n_3=n} \mu(n_1)n_2^{-\alpha}n_3^{-\gamma}.$$

To prove this identity we first manipulate the right-hand side of (36). Note that

$$f_{\alpha,\gamma}(p^m) = \frac{p^{-m\gamma} - p^{-\alpha(m+1)+\gamma} - p^{-(m-1)\gamma} + p^{-\alpha m+\gamma}}{1-p^{\gamma-\alpha}}$$

and hence

$$\prod_{p\,|\,k'}\sum_{m\geq 0} f_{\alpha,\gamma}(p^m)p^{-m(1+\beta)} = \prod_{p\,|\,k'} \frac{(1-p^{-(1+\beta)})}{(1-p^{-(1+\alpha+\beta)})(1-p^{-(1+\gamma+\beta)})}$$

and

$$\prod_{p^{k'_p}\,\|\,k'}\sum_{m\geq 0} f_{\alpha,\gamma}(p^{m+k'_p})p^{-m(1+\beta)} = \prod_{p^{k'_p}\,\|\,k'} \frac{p^{\alpha+\beta+\gamma+1}\Big(\frac{(p^{\gamma}-1)p^{-\gamma k'_p}}{p^{\beta+\gamma+1}-1}-\frac{(p^{\alpha}-1)p^{-\alpha k'_p}}{p^{\alpha+\beta+1}-1}\Big)}{p^{\gamma}-p^{\alpha}}$$
$$= \prod_{p^{k'_p}\,\|\,k'}\bigg(\frac{p^{-\gamma k'_p}(1-p^{-\gamma})}{(1-p^{-(1+\gamma+\beta)})(p^{-\alpha}-p^{-\gamma})}-\frac{p^{-\alpha k'_p}(1-p^{-\alpha})}{(1-p^{-(1+\alpha+\beta)})(p^{-\alpha}-p^{-\gamma})}\bigg).$$

Therefore, the right-hand side of (36) is given by

$$\prod_{p\,|\,k'}\bigg(p^{-\alpha k'_p}\frac{(1-p^{-(1+\gamma+\beta)})(1-p^{-\alpha})}{(1-p^{-(1+\beta)})(p^{-\gamma}-p^{-\alpha})}+p^{-\gamma k'_p}\frac{(1-p^{-(1+\alpha+\beta)})(1-p^{-\gamma})}{(1-p^{-(1+\beta)})(p^{-\alpha}-p^{-\gamma})}\bigg). \tag{37}$$

For the left-hand side, we find by the definition of $G(s,k,d)$ in (31) that

$$\sum_{kd=k'} k^{\beta}\frac{k}{\phi(k)}\mu(k)G(1,k,d) = \prod_{p\,|\,k'}(1-p^{-1-\beta})^{-1}\sum_{kd=k'}\mu(k)\sum_{d_1d_2d_3d_4=d}\mu(d_4)d_3^{-\gamma}d_2^{-\alpha}(kd_1)^{\beta}$$
$$\times\prod_{p\,|\,kd_1d_2}(1-p^{-1-\gamma-\beta})\prod_{p\,|\,kd_1}(1-p^{-1-\alpha-\beta}).$$

We first combine the two sums and write them as a single sum over the condition $kd_1d_2d_3d_4 = k'$. We then write kd_1 as ℓ and obtain

$$\prod_{p\,|\,k'}(1-p^{-1-\beta})^{-1}\sum_{\ell d_2d_3d_4=k'}\mu(d_4)d_3^{-\gamma}d_2^{-\alpha}\ell^{\beta}\prod_{p\,|\,\ell d_2}(1-p^{-1-\gamma-\beta})\prod_{p\,|\,\ell}(1-p^{-1-\alpha-\beta})\sum_{kd_1=\ell}\mu(k)$$
$$= \prod_{p\,|\,k'}(1-p^{-1-\beta})^{-1}\sum_{d_2d_3d_4=k'}\mu(d_4)d_3^{-\gamma}d_2^{-\alpha}\prod_{p\,|\,d_2}(1-p^{-1-\gamma-\beta}).$$

For the sum over d_i we have

$$\sum_{d_2d_3d_4=k'}\mu(d_4)d_3^{-\gamma}d_2^{-\alpha}\prod_{p\,|\,d_2}(1-p^{-(1+\gamma+\beta)})$$
$$= \prod_{p^{k'_p}\,\|\,k'}\bigg(\sum_{1\leq m\leq k'_p}p^{-\gamma(k'_p-m)}p^{-\alpha m}(1-p^{-(1+\gamma+\beta)})-\sum_{1\leq m\leq k'_p-1}p^{-\gamma(k'_p-1-m)}p^{-\alpha m}(1-p^{-(1+\gamma+\beta)})$$
$$+p^{-\gamma k'_p}-p^{-\gamma(k'_p-1)}\bigg)$$
$$= \prod_{p^{k'_p}\,\|\,k'}\bigg(p^{-\alpha k'_p}\frac{(1-p^{-(1+\gamma+\beta)})(1-p^{-\alpha})}{(p^{-\gamma}-p^{-\alpha})}+p^{-\gamma k'_p}\frac{(1-p^{-(1+\alpha+\beta)})(1-p^{-\gamma})}{(p^{-\alpha}-p^{-\gamma})}\bigg)$$

and thus after multiplying by $\prod_{p\,|\,k'}(1-p^{-1-\beta})^{-1}$ this is equal to (37). Equation (36) then follows.

When computing the residue at $s = 1-\alpha-\beta$ in (35), we get a factor of

$$\frac{1}{1-\alpha-\beta}\left(\frac{T}{2\pi}\right)^{1-\alpha-\beta} = \int_0^{T/2\pi}\left(\frac{t}{2\pi}\right)^{-\alpha-\beta} dt.$$

In the arithmetic sums we see that the effect of changing s from 1 to $1-\alpha-\beta$ is to replace (α,β) by $(-\beta,-\alpha)$ after using the expression for $G(s,k,d)$ given in (33). This is precisely the behaviour of the second Z term in (4) and hence we obtain this term. Likewise, for the residue at $s = 1-\beta-\gamma$, the effect of changing s from 1 to $1-\beta-\gamma$ is to replace (γ,β) by $(-\beta,-\gamma)$ after using the expression (34) for $G(s,k,d)$. This gives the third Z term.

4.4. *Bounding the error term $\mathscr{E}$.* In this section we give unconditional bounds for $\mathscr{E}$ when $a(n)$ satisfies

$$|a(mn)| \ll |a(m)a(n)| \quad \text{and} \quad |a(n)| \ll \tau_r(n)(\log n)^C \quad \text{for some } r, C > 0.$$

The error term $\mathscr{E}$ in (29) can be rewritten as

$$\mathscr{E} = \sum_{2\le q\le y}\ \sideset{}{^*}\sum_{\psi \pmod q} \tau(\bar\psi) \sum_{k\le y/q} \frac{a(kq)}{(kq)^{1-\beta}} \sum_{d\,|\,kq} \delta(q,kq,d,\psi) \sum_{m\le Tkq/2\pi d} \frac{b(md)\psi(m)}{(md)^\beta} =: \mathscr{E}_1 + \mathscr{E}_2$$

where

$$\mathscr{E}_1 = \sum_{2\le q\le \eta}\ \sideset{}{^*}\sum_{\psi \pmod q} \tau(\bar\psi) \sum_{kq\le y} \frac{a(kq)}{(kq)^{1-\beta}} \sum_{d\,|\,kq} \delta(q,kq,d,\psi) \sum_{m\le Tkq/2\pi d} \frac{b(md)\psi(m)}{(md)^\beta},$$
$$\mathscr{E}_2 = \sum_{\eta\le q\le y}\ \sideset{}{^*}\sum_{\psi \pmod q} \tau(\bar\psi) \sum_{kq\le y} \frac{a(kq)}{(kq)^{1-\beta}} \sum_{d\,|\,kq} \delta(q,kq,d,\psi) \sum_{m\le Tkq/2\pi d} \frac{b(md)\psi(m)}{(md)^\beta}.$$

We use Siegel's theorem to bound $\mathscr{E}_1$ and the large sieve inequalities to bound $\mathscr{E}_2$.

Proposition 10 (small moduli). *Suppose there exist some positive constants r and C such that $|a(nm)| \le |a(m)a(n)|$ and $|a(n)| \le \tau_r(n)(\log n)^C$. Let $A > 0$ be any fixed constant. Then for $\eta \ll (\log T)^A$, we have*

$$\mathscr{E}_1 \ll T\exp(-c\sqrt{\log T})\eta^{3/2+\epsilon}.$$

Proposition 11 (large moduli). *Suppose there exist some positive constants r and C such that $|a(nm)| \le |a(m)a(n)|$ and $|a(n)| \le \tau_r(n)(\log n)^C$. Then there exists some $C' = C'(r, C)$ such that for any $0 < \eta \le y$, we have*

$$\mathscr{E}_2 \ll (\log T)^{C'} T\eta^{-1/2+\epsilon} + yT^{1/2+\epsilon} + y^{4/3}T^{1/3+\epsilon} + y^{1/3+\epsilon}T^{5/6+\epsilon}.$$

Proof of Theorem 4. Combining Proposition 10 and Proposition 11, with $\eta = (\log T)^{C''}$ for C'' large enough, we find that $\mathscr{E} \ll T(\log T)^{-A}$ provided $y = T^\theta$ for some fixed $\theta < \frac{1}{2}$. Theorem 4 follows. □

5. Proof of Proposition 10

To prove Proposition 10, we use the following lemma.

Lemma 12. *Let* $\psi \pmod q$ *be a nonprincipal character with* $q \ll (\log T)^A$ *for some constant A. Then for* $T \ll x \ll T^2$ *and* $d \ll T$*, we have*

$$\sum_{m \le x} \frac{b(md)\psi(m)}{m^\beta} \ll x \exp(-c\sqrt{\log x})(\tau_4 * |a|)(d) j(d),$$

where

$$j(d) = \prod_{p \mid d} (1 + 10p^{-1/2}).$$

Proof. After an application of Lemma 9 with $\kappa = 1 + O(1/\log x)$, $H = \sqrt{U}$ with U to be determined later, we have

$$\sum_{m \le x} \frac{b(md)\psi(m)}{m^\beta} = \frac{1}{2\pi i} \int_{\kappa - iU}^{\kappa + iU} \sum_m \frac{b(md)\psi(m)}{m^{\beta+s}} x^s \frac{ds}{s} + E$$

where

$$E \ll \sum_{x - x/\sqrt{U} \leqslant m \leqslant x + x/\sqrt{U}} |b(md)\psi(m)| + \frac{x}{\sqrt{U}} \sum_{n \geqslant 1} |b(md)\psi(m)| m^{-\kappa}.$$

From (28), we have

$$b(md) \ll (md)^\beta (\tau_4 * |a|)(md).$$

Therefore, the second term above is bounded by $(x/\sqrt{U})(\tau_4 * |a|)(d)(\log x)^C$ and in fact, the same bound holds for the first term. To see this we use [Ng 2007, Lemma 6.4] which states that

$$\sum_{t-u \le n \le t} (\tau_k * a)(n) \ll u (\log t)^{k-1} \|a(n)/n\|_1$$

for $x/2 \le t - u \le t \le x$, $T \ll x \le T^2$, $u = x/U$ with $\exp(c\sqrt{\log x}) \le U \le \log x / \log\log x$, and $a(n)$ supported on integers less than or equal to $y \le \sqrt{T}$. Therefore,

$$E \ll \frac{x}{\sqrt{U}} (\tau_4 * |a|)(d)(\log x)^C.$$

Now it remains to compute

$$\int_{\kappa - iU}^{\kappa + iU} \sum_m \frac{b(md)\psi(m)}{m^{\beta+s}} x^s \frac{ds}{s}.$$

We shall move the contour to the line $\Re(s) = 1 - c/\log(qU)$ for some absolute c. To do this we first express the function $\sum_m b(md)\psi(m) m^{-\beta-s}$ in terms of L-functions.

Applying Lemma 8, we have

$$
\begin{aligned}
\sum_m \frac{b(md)\psi(m)}{m^{s+\beta}}
&= \sum_{d_1d_2d_3d_4d_5=d} \sum_{(m_1,d_1d_2d_3d_4)=1} \frac{\mu(m_1d_5)\psi(m_1)}{m_1^{s+\beta}} \sum_{(m_2,d_1d_2d_3)=1} \frac{(m_2d_4)^{-\gamma}\psi(m_2)}{m_2^{s+\beta}} \\
&\qquad \times \sum_{(m_3,d_1d_2)=1} \frac{(m_3d_3)^{-\alpha}\psi(m_3)}{m_3^{s+\beta}} \sum_{(m_4,d_1)=1} \frac{(m_4d_2)^{\beta}\psi(m_4)}{m_4^{s+\beta}} \sum_{m_5} \frac{a(m_5d_1)\psi(m_5)}{m_5^{s+\beta}} \\
&= \sum_{\prod_{i=1}^5 d_i=d} \frac{\mu(d_5)}{L(s+\beta,\psi)} \prod_{p\,|\,d}\Big(1-\frac{\psi(p)}{p^{s+\beta}}\Big)^{-1} d_4^{-\gamma} L(s+\beta+\gamma,\psi) \prod_{p\,|\,d_1d_2d_3}\Big(1-\frac{\psi(p)}{p^{s+\beta+\gamma}}\Big) \\
&\qquad \times d_3^{-\alpha} L(s+\beta+\alpha,\psi) \prod_{p\,|\,d_1d_2}\Big(1-\frac{\psi(p)}{p^{s+\beta+\alpha}}\Big) d_2^{\beta} L(s,\psi) \prod_{p\,|\,d_1}\Big(1-\frac{\psi(p)}{p^{s}}\Big) \sum_{m_5} \frac{a(m_5d_1)\psi(m_5)}{m_5^{s+\beta}} \\
&= \frac{L(s+\beta+\gamma,\psi)L(s+\beta+\alpha,\psi)L(s,\psi)}{L(s+\beta,\psi)} \prod_{p\,|\,d}\Big(1-\frac{\psi(p)}{p^{s+\beta}}\Big)^{-1} \sum_{\prod_{i=1}^5 d_i=d} \mu(d_5) d_4^{-\gamma} d_3^{-\alpha} d_2^{\beta} \\
&\qquad \times \prod_{p\,|\,d_1d_2d_3}\Big(1-\frac{\psi(p)}{p^{s+\beta+\gamma}}\Big) \prod_{p\,|\,d_1d_2}\Big(1-\frac{\psi(p)}{p^{s+\beta+\alpha}}\Big) \prod_{p\,|\,d_1}\Big(1-\frac{\psi(p)}{p^{s}}\Big) A(s+\beta,d_1),
\end{aligned}
$$

where

$$A(s,r) = \sum_m \frac{a(mr)\psi(m)}{m^s} = \sum_{mr\le y} \frac{a(mr)\psi(m)}{m^s}.$$

To bound the horizontal integrals when moving the contour, we need some bounds on the sum

$$\sum_m b(md)\psi(m) m^{-\beta-s}$$

for $\Re(s) = 1 - O(1/\log q\Im s)$ with $\Im s \gg 1$. Assuming that

$$|a(mn)| \le |a(m)||a(n)|,$$

we have

$$A(s,r) \ll |a(r)| \sum_{m\le y} \frac{|a(m)|}{m^{\Re(s)}} \ll |a(r)| \|a(n)/n\|_1 y^{1-\Re(s)} \ll |a(r)| (\log x)^C y^{1-\Re(s)}.$$

We also have, for $1-\Re(s) \ll 1/\log q|\Im s|$ and $\Im s \gg 1$,

$$\frac{1}{(\log q|\Im s|)^c} \ll L(s,\psi) \ll (\log q|\Im s|)^c.$$

Therefore, when $1-\Re(s) \ll 1/\log q|\Im s|$ and $\Im s \gg 1$, we have

$$\sum_m \frac{b(md)\psi(m)}{m^{s+\beta}} \ll (\log q|\Im s|)^C j(d)(\tau_4 * |a|)(d)(\log x)^C y^{1-\Re(s)}. \tag{38}$$

There is at most one simple pole for $\sum_m b(md)\psi(m)m^{-s-\beta}$ for all nonprincipal characters $\psi \pmod q$ with $q \ll T$ in the region

$$\left\{ s=\sigma+it \mid \sigma \geq \sigma_1(t) := 1-\frac{c}{\log q(|t|+2)} \right\},$$

where c is some absolute constant. By Siegel's theorem, if this pole exists, then it is a real number β such that $1-\beta \gg q^{-\epsilon}$. Thus,

$$\sum_{m\leq x} \frac{b(md)\psi(s)}{m^\beta} \ll \int_{\sigma_1(U)-iU}^{\sigma_1(U)+iU} \sum_m \frac{b(md)\psi(m)}{m^{s+\beta}} x^s \frac{ds}{s} + \left| \operatorname{Res}_{s=\beta} \sum_m \frac{b(md)\psi(m)}{m^{s+\beta}} \frac{x^s}{s} \right| \\ + \frac{x}{U}(\tau_4 * |a|)(d)(\log(qUx))^C + \frac{x}{\sqrt{U}}(\tau_4 * |a|)(d)(\log x)^C, \tag{39}$$

where the third term is the contribution from the horizontal integrals using (38). Using (38) again for the first integral we have

$$\int_{\sigma_1(U)-iU}^{\sigma_1(U)+iU} \sum_m \frac{b(md)\psi(m)}{m^{s+\beta}} x^s \frac{ds}{s} \ll (\log(qUx))^C j(d)(\tau_4 * |a|)(d)x^{\sigma_1(U)} \\ \ll j(d)(\tau_4 * |a|)(d)x \exp\left(-c\frac{\log x}{\log qU}\right). \tag{40}$$

For the residue at β, we have

$$x^\beta \ll x^{1-q^{-\epsilon}} \ll x\exp\left(-\frac{\log x}{q^\epsilon}\right) \ll x\exp\left(-\frac{\log x}{(\log x)^{\epsilon A}}\right) \ll x\exp(-c'\sqrt{\log x}) \tag{41}$$

by choosing $\epsilon \leq 1/2A$. Combining (39), (40) (41) and choosing $U = \exp(c\sqrt{\log x})$, we have for $q \leq (\log x)^A$,

$$\sum_m \frac{b(md)\psi(m)}{m^\beta} \ll_A j(d)(\tau_4 * |a|)(d)x\exp(-c'\sqrt{\log x}). \qquad \square$$

Proof of Proposition 10. From [Ng 2007, Lemma 6.6], we have

$$|\delta(q,kq,d,\psi)| \ll \frac{(d,k)\log\log T}{\phi(k)\phi(q)} \tag{42}$$

for primitive characters ψ and $kq \ll T$. From [Ng 2007, Lemma 6.7], we also have

$$\sum_{d\,|\,kq} \frac{(d,k)h(d)}{d} \ll (1*h)(k)\|h(n)/n\|_1 \tag{43}$$

for positive multiplicative functions h. Using (42), (43) and properties of $a(n)$, we have

$$\begin{aligned}\mathscr{E}_1 &= \sum_{2\le q\le\eta}\ \sideset{}{^*}\sum_{\psi\ (\mathrm{mod}\ q)}\tau(\bar{\psi})\sum_{kq\le y}\frac{\overline{a(kq)}}{(kq)^{1-\beta}}\sum_{d\,|\,kq}\delta(q,kq,d,\psi)\sum_{m\le Tkq/2\pi d}\frac{b(md)\psi(m)}{(md)^{\beta}}\\
&\ll \sum_{q\le\eta}\phi(q)\sqrt{q}\sum_{kq\le y}\frac{|a(kq)|}{kq}\sum_{d\,|\,kq}\frac{(d,k)\log\log T}{\phi(k)\phi(q)}j(d)(\tau_4*|a|)(d)\frac{Tkq}{d}\exp(-c'\sqrt{\log T})\\
&\ll \sum_{q\le\eta}q^{3/2}\sum_{kq\le y}\frac{|a(kq)|}{kq}\sum_{d\,|\,kq}\frac{(d,k)j(d)(\tau_4*|a|)(d)}{d}T\exp(-c''\sqrt{\log T})\\
&\ll \sum_{q\le\eta}|a(q)|q^{1/2}\sum_{kq\le y}\frac{|a(k)|}{k}\sum_{d\,|\,kq}\frac{(d,k)\tau(d)(\tau_4*|a|)(d)}{d}T\exp(-c'''\sqrt{\log T})\\
&\ll \sum_{q\le\eta}|a(q)|\tau(q)q^{1/2}\sum_{k\le y}\frac{|a(k)|\tau(k)(\tau_5*|a|)(k)}{k}\|(\tau_4*|a|)(n)/n\|_1T\exp(-c'''\sqrt{\log T})\\
&\ll \eta^{3/2+\epsilon}T\exp(-c''''\sqrt{\log T})\end{aligned}$$

where we have used $j(d)\ll\tau(d)\leqslant\tau(k)\tau(q)$. □

6. Proof of Proposition 11

6.1. ***Initial cleaning.*** The proof of Proposition 11 is similar to [Bui and Heath-Brown 2013], and we give the exposition by considering Type I/II terms. One main difference is that our coefficients $a(n)$ are not supported on square-free integers. This affects the treatment of $\delta(q,kq,d,\psi)$ in the initial cleaning stage to remove the q-dependence on d in the sum $d\,|\,kq$. In our case, $a(n)$ is not supported on square-free integers, but we can still remove the condition $d\,|\,q$ by exploiting the fact that ψ has conductor q.

We write $k=k'k_q$, where $(k',q)=1$ and k_q is such that $p\,|\,k_q\implies p\,|\,q$. Then we have

$$\delta(q,kq,d,\psi)=\sum_{e\,|\,(d,k)}\frac{\mu(d/e)}{\phi(kq/e)}\bar{\psi}\Big(-\frac{k'k_q}{e}\Big)\psi\Big(\frac{d}{e}\Big)\mu\Big(\frac{k'k_q}{e}\Big).$$

Since ψ is a character modulo q, only the terms $e=k_qe'$ with $(e',q)=1$ contribute to $\delta(q,kq,d,\psi)$. Thus, only the terms with $d=k_qd'$ such that $(d',q)=1$ contribute to $\delta(q,kq,d,\psi)$, in which case

$$\delta(q,kq,d,\psi)=\sum_{e\,|\,(d',k')}\frac{\mu(d'/e)}{\phi(k'q/e)}\bar{\psi}\Big(-\frac{k'}{e}\Big)\psi\Big(\frac{d'}{e}\Big)\mu\Big(\frac{k'}{e}\Big)=\delta(q,k'q,d',\psi).$$

It follows that

$$\begin{aligned}\mathscr{E}_2 &= \sum_{\eta\le q\le y}\ \sideset{}{^*}\sum_{\psi\ (\mathrm{mod}\ q)}\tau(\bar{\psi})\sum_{kq\le y}\frac{a(kq)}{(kq)^{1-\beta}}\sum_{d\,|\,kq}\delta(q,kq,d,\psi)\sum_{m\le Tkq/2\pi d}\frac{b(md)\psi(m)}{(md)^{\beta}}\\
&= \sum_{\eta\le q\le y}\ \sideset{}{^*}\sum_{\psi\ (\mathrm{mod}\ q)}\tau(\bar{\psi})\sum_{\substack{k_q\le y\\ k_q\,|\,q^\infty}}\sum_{\substack{k_qkq\le y\\ (k,q)=1}}\frac{a(k_qkq)}{(k_qkq)^{1-\beta}}\sum_{d\,|\,k}\delta(q,kq,d,\psi)\sum_{m\le Tkq/2\pi d}\frac{b(mdk_q)\psi(m)}{(mdk_q)^{\beta}}.\end{aligned}$$

Note that

$$\delta(q, kq, d, \psi) \ll \sum_{e\,|\,d} \frac{1}{\phi(kq/e)} \ll \sum_{e\,|\,d} \frac{e}{kq}(\log\log T) \ll (\log\log T)^2 dk^{-1}q^{-1}$$

since $\phi(n) \gg n(\log\log n)^{-1}$ and $\sigma(n) \ll n\log\log n$. Applying this along with the bounds $|a(mn)| \ll |a(m)||a(n)|$ and $|\tau(\psi)| = q^{1/2}$ we have

$$\mathscr{E}_2 \ll \sum_{\eta\le q\le y} \frac{|a(q)|}{q^{3/2}} \sideset{}{^*}\sum_{\psi \pmod q} \sum_{\substack{k_q\le y\\ k_q\,|\,q^\infty}} \sum_{\substack{k\le y/qk_q\\ (k,q)=1}} \frac{|a(k_qk)|}{k_qk^2}\Big(\sum_{d\,|\,k} d\Big)\Big|\sum_{m\le Tkq/2\pi d} \frac{b(mdk_q)\psi(m)}{(mdk_q)^\beta}\Big|$$

$$\ll \sum_{\eta\le q\le y} \frac{|a(q)|}{q^{3/2}} \sideset{}{^*}\sum_{\psi \pmod q} \sum_{\substack{k_q\le y\\ k_q\,|\,q^\infty}} \sum_{\substack{d\leqslant y/qk_q\\ (d,q)=1}} \frac{|a(dk_q)|}{dk_q} \sum_{\substack{k\le y/qk_qd\\ (k,q)=1}} \frac{|a(k)|}{k^2}\Big|\sum_{m\le Tkq/2\pi} \frac{b(mdk_q)\psi(m)}{(mdk_q)^\beta}\Big|.$$

After grouping k_q and d together and removing the condition $(k, q) = 1$ by positivity, we obtain

$$\mathscr{E}_2 \ll \sum_{\eta\le q\le y} \frac{|a(q)|}{q^{3/2}} \sideset{}{^*}\sum_{\psi \pmod q} \sum_{d\leqslant y/q} \frac{|a(d)|}{d} \sum_{k\le y/qd} \frac{|a(k)|}{k^2}\Big|\sum_{m\le Tkq/2\pi} \frac{b(md)\psi(m)}{(md)^\beta}\Big|.$$

We divide the summation over k, q, d into dyadic intervals $K \leqslant k \leqslant 2K$, $Q \leqslant q \leqslant 2Q$, $D \leqslant d \leqslant 2D$ where

$$\eta < Q \leqslant y, \quad KQD \ll y \tag{44}$$

to obtain

$$\mathscr{E}_2 \ll \sideset{}{'}\sum_{D} \sideset{}{'}\sum_{Q} \sideset{}{'}\sum_{K} \sum_{q\sim Q} \frac{|a(q)|}{q^{3/2}} \sum_{d\sim D} \frac{|a(d)|}{d} \sum_{k\sim K} \frac{|a(k)|}{k^2} \sideset{}{^*}\sum_{\psi \pmod q}\Big|\sum_{m\le kqT/2\pi} \frac{b(md)\psi(m)}{(md)^\beta}\Big|.$$

Here $\sum'_N$ is used to indicate the summation of the dyadic partition, so that $\sum'_N 1 \ll \log T$, and $\sum_{n\sim N}$ means $\sum_{N\leqslant n\leqslant 2N}$. Upon bounding by the maximal dyadic sums we find that there exist some (K, Q, D) satisfying (44) such that

$$\mathscr{E}_2 \ll (\log T)^3 \sum_{d\asymp D} \frac{|a(d)|}{d} \sum_{k\asymp K} \frac{|a(k)|}{k^2} \sum_{q\sim Q} \frac{|a(q)|}{q^{3/2}} \sideset{}{^*}\sum_{\psi \pmod q}\Big|\sum_{m\le kqT/2\pi} \frac{b(md)\psi(m)}{(md)^\beta}\Big|.$$

Applying $a(n) \ll \tau_r(n)(\log n)^C$ and the crude bound $a(q) \ll q^\epsilon$ we arrive at the following

$$\mathscr{E}_2 \ll \frac{(\log T)^C}{KQ^{3/2-\epsilon}} \sum_{d\asymp D} \frac{|a(d)|}{d} \sum_{q\sim Q} \sideset{}{^*}\sum_{\psi \pmod q} \max_{x\leqslant 2KQT}\Big|\sum_{m\leqslant x} \frac{b(md)\psi(m)}{(md)^\beta}\Big|.$$

where $b(n)$ is defined in (28).

6.2. ***Combinatorial decomposition.*** To evaluate the sum over m, we apply Heath-Brown's identity to μ to decompose $b(n)$ into $O((\log T)^C)$ linear combinations of functions of the form $(f * g)(n)$ where g is supported on integers of short lengths and $g(n) = n^c\psi(n)$ (Type I) or both f and g are supported on integers of short lengths (Type II). For Type I terms, we obtain cancellation from the sum over ψ using

Pólya–Vinogradov inequality. For Type II terms, we obtain cancellation using the large sieve inequality on short Dirichlet polynomials.

Let $M(s) = \sum_{n \le z^{1/J}} \mu(n) n^{-s}$. From Heath-Brown's identity

$$\frac{1}{\zeta(s)} = \sum_{1 \le j \le J} (-1)^{j-1} \binom{J}{j} \zeta(s)^{j-1} M(s)^j + \frac{1}{\zeta(s)} (1 - M(s)\zeta(s))^J$$

we have for $n \le z$,

$$\mu(n) = \sum_{1 \le j \le J} (-1)^{j-1} \binom{J}{j} 1^{(*)(j-1)} * \mu 1_{[1, z^{1/J}]}^{(*)j}.$$

Since $md \ll KQT \ll yT \ll T^{3/2}$, we can take $z = T^{3/2}$. On splitting each range of summation into dyadic intervals, we see that $b(n)$ can be written as a linear combination of $O((\log T)^{2J+3})$ expressions of the form $f_1 * \cdots * f_{2J+3}(n)$, where f_i are supported on dyadic intervals $[F_i/2, F_i]$. For terms in which f_i is absent, we set $F_i = 1$, and take $f_i(1) = 1$, $f_i(n) = 0, n \ge 2$. For $F_i > 1$, we have

$$\begin{aligned} f_i(n) &= \mu 1_{[1, z^{1/J}]}(n), \quad i = 1, \ldots, J, \\ f_j(n) &= 1, \quad j = J+1, \cdots, 2J-1, \\ f_{2J}(n) &= n^{-\gamma}, \quad f_{2J+1}(n) = n^{-\alpha}, \quad f_{2J+2} = n^{\beta}, \quad f_{2J+3} = a(n). \end{aligned}$$

Note that $F_{2J+3} \le y$ and $F_i \ll T^{3/2J}$ for $i = 1, \ldots, J$. By Lemma 8, we write

$$\frac{b(md)}{(md)^{\beta}} = \frac{f_1 * \cdots * f_{2J+3}(md)}{(md)^{\beta}} = \sum_{d_1 \cdots d_{2J+3} = d} g_1 * \cdots * g_{2J+3}(m)$$

with

$$g_i(m) = g_i(m; d_1, \ldots, d_i) = \begin{cases} f_i(md_i)(md_i)^{-\beta} & \text{if } (m, D_i) = 1, \text{ where } D_i = d_1 \cdots d_{i-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathscr{E}_2 \ll \frac{(\log T)^C}{KQ^{3/2-\epsilon}} \sum_{F_i}{}' \sum_{d \asymp D} \frac{|a(d)|}{d} \sum_{d_1 \cdots d_{2J+3} = d} \sum_{q \sim Q} \sum_{\psi \pmod q}^{*} \max_{x \le 2KQT} \left| \sum_{m \le x} (g_1 * \cdots * g_{2J+3})(m) \psi(m) \right|.$$

If $x \ll (yT)^{1/2}$, then we can bound trivially

$$\begin{aligned} \frac{(\log T)^C}{KQ^{3/2-\epsilon}} \sum_{F_i}{}' \sum_{d \le y} \frac{|a(d)|}{d} \sum_{d_1 \cdots d_{2J+3} = d} \sum_{q \sim Q} \sum_{\psi \pmod q}^{*} \left| \sum_{m \le x} g_1 * \cdots * g_{2J+3}(m) \psi(m) \right| &\ll \frac{Q^2 y^{1/2} T^{1/2+\epsilon}}{KQ^{3/2-\epsilon}} \\ &\ll yT^{1/2+\epsilon}. \end{aligned} \tag{45}$$

Thus, we arrive at the bound

$$\mathscr{E}_2 \ll \frac{(\log T)^C}{KQ^{3/2-\epsilon}} \sum_{d \asymp D} \frac{|a(d)|}{d} S(Q, T, d) \tag{46}$$

where

$$\mathcal{S}(Q,T,d)=\sum_{F_i}{}' \sum_{d_1\cdots d_{2J+3}=d}\sum_{q\sim Q}\sum_{\psi \ (\mathrm{mod}\, q)}{}^{*} \max_{(yT)^{\frac{1}{2}}\le x\le 2KQT}\left|\sum_{m\le x}(g_1*\cdots*g_{2J+4})(m)\psi(m)\right| \tag{47}$$

and each g_i is supported on $[G_i/2, G_i]$ with $G_i = F_i/d_i$ and $\prod_i G_i \ll x$.

Let $x \gg W \gg x^{2/3}$ be a parameter to be chosen later. We see that there exists an i such that $G_i \gg W$ (Type I) or there exists a subset $S \subset \{1, \ldots 2J+3\}$ such that $x/W \ll \prod_{i\in S} G_i \ll W$ (Type II). Indeed, if there is an i such that $G_i \gg W$ or $x/W \ll G_i \ll W$ then we are done. Otherwise we may suppose $G_i \ll x/W$ for all i. Since $\prod_{i=1}^{2J+3} G_i \gg x \gg W$, there exists an i_0 such that $\prod_{i=1}^{i_0} G_i \gg x/W$ and $\prod_{i=1}^{i_0-1} G_i \ll x/W$, and thus

$$\frac{x}{W} \ll \prod_{i=1}^{i_0} G_i \ll \frac{x}{W}\frac{x}{W} \ll W.$$

6.3. ***Type I terms.*** For Type I terms, we make use of cancellations of character sums. When $x \gg (yT)^{1/2} \gg y^{3/2}$, we have $y \ll x^{2/3} \ll W$. By taking J large enough, we also have $z^{1/J} \ll T^{3/2J} \ll T^{1/3} \ll x^{2/3} \ll W$. Thus if there exists i such that $G_i \gg W$, we must have $i \in \{J+1, \ldots, 2J+2\}$.

By an application of Möbius inversion, the Pólya–Vinogradov inequality and partial summation, we see

$$\sum_{\substack{n_i\sim G_i\\ (n_i,D_i)=1}} n_i^c\psi(n_i) \ll \tau(D_i)q^{1/2}\log q$$

uniformly for $c \ll 1/\log T$. By grouping the rest of the functions in the $2J+3$ convolution to a function $\tilde{g}$, we see the sum over m in (47) becomes

$$\sum_{mn_i\le x}\tilde{g}(m)\psi(m)\sum_{\substack{n_i\sim F_i/d_i\\ (n_i,D_i)=1}}(n_id_i)^c\psi(n_i) \ll \frac{x}{W}Q^{1/2}T^{\epsilon}.$$

Therefore, the contribution from Type I terms to $\mathcal{E}_2$ is bounded by

$$\frac{(\log T)^C}{KQ^{3/2-\epsilon}}Q^2\frac{KQT}{W}Q^{1/2}T^{\epsilon} = \frac{Q^2T^{1+\epsilon}}{W}. \tag{48}$$

6.4. ***Type II terms.*** For Type II terms, we use the large sieve inequality to obtain cancellations. To start with, we have from Perron's formula

$$\sum_{m\le x}(g_1*\cdots*g_{2J+3})(m)\psi(m) = \frac{1}{2\pi i}\int_{\kappa-iU}^{\kappa+iU} B(s,\psi,\vec{d})x^s\frac{ds}{s} + O(T^{\epsilon}),$$

where $\vec{d} = (d_1, \cdots, d_{2J+3})$, $U = T^{20}$ and $\kappa \asymp 1/\log(KQT)$ and

$$B(s,\psi,\vec{d}) = \sum_m (g_1*\cdots*g_{2J+3})(m)\psi(m)m^{-s}.$$

The error from $O(T^\epsilon)$ can be bounded by

$$\frac{(\log T)^C}{KQ^{3/2-\epsilon}} Q^2 T^\epsilon \ll Q^{1/2+\epsilon} T^\epsilon \ll y^{1+\epsilon} \tag{49}$$

and thus

$$S(Q,T,d) \ll \sideset{}{'}\sum_{F_i} \sum_{d_1\cdots d_{2J+3}=d} \sum_{q\sim Q} \sideset{}{^*}\sum_{\psi \pmod q} \int_{-U}^{U} \frac{\log(KQT)}{1+|t|} |B(\kappa+it,\psi,\vec{d})|\,dt + y^{1+\epsilon}. \tag{50}$$

Let $H_j(\psi,t)=\sum_{n\sim G_j} g_j(n)\psi(n)n^{-\kappa-it}$ and write

$$B(\kappa+it,\psi,\vec{d}) = \prod_{j=1}^{2J+3} H_j(\psi,t) = \mathscr{A}(\psi,t)\mathscr{B}(\psi,t)$$

where $\mathscr{A}(\psi,t)=\prod_{i\in S} H_i(\psi,t)$, $\mathscr{B}(\psi,t)=\prod_{i\notin S} H_i(\psi,t)$ are Dirichlet polynomials of lengths A, B respectively with $A, B \ll W$ from the definition of S.

It is enough to bound uniformly for $1 \leqslant V \leqslant T^{20}$,

$$\frac{1}{V} \sum_{q\sim Q} \sideset{}{^*}\sum_{\psi \pmod q} \int_{-V}^{V} |\mathscr{A}(\psi,t)\mathscr{B}(\psi,t)|\,dt. \tag{51}$$

After an application of the Cauchy–Schwarz inequality and the large sieve inequality in the form (see [Iwaniec and Kowalski 2004, Theorem 7.17])

$$\sum_{q\sim Q} \sideset{}{^*}\sum_{\psi \pmod q} \int_{-V}^{V} \Big|\sum_{m\le H} h_m \psi(m) m^{-it}\Big|^2 dt \ll (Q^2V+H)\sum |h_m|^2,$$

we see that (51) is bounded by

$$\begin{aligned}
&\frac{1}{V}\bigg(\sum_{q\sim Q} \sideset{}{^*}\sum_{\psi \pmod q} \int_{-V}^{V} |\mathscr{A}(\psi,t)|^2\,dt\bigg)^{1/2}\bigg(\sum_{q\sim Q} \sideset{}{^*}\sum_{\psi \pmod q} \int_{-V}^{V} |\mathscr{B}(\psi,t)|^2\,dt\bigg)^{1/2}\\
&\qquad\ll \tau_R(d)(\log T)^C V^{-1}((Q^2V+A)A(Q^2V+B)B)^{1/2}\\
&\qquad\ll \tau_R(d)(\log T)^C (AB)^{1/2}(QV^{-1/2}(A+B)^{1/2}+Q^2) + (\log T)^C ABV^{-1}
\end{aligned}$$

for some positive integer R since $g_j(m) \ll \tau_r(md)(\log md)^{C'} \ll \tau_r(m)\tau_r(d)(\log T)^{C'}$. Applying this in (50) and then (46) we see these terms contribute to $\mathscr{E}_2$ at most

$$\begin{aligned}
&\frac{(\log T)^{C'}}{KQ^{3/2-\epsilon}}\bigg(\sum_{d\asymp D} \frac{|a(d)|\tau_{2J+3}(d)\tau_R(d)}{d}\bigg) \sup_{\substack{1\leqslant V\leqslant T^{20}\\ AB\ll KQT}} ((AB)^{1/2}(QV^{-1/2}(A+B)^{1/2}+Q^2)+ABV^{-1})\\
&\qquad\ll (\log T)^{C'} Q^\epsilon \sup_{1\leqslant V\leqslant T^{20}} (T^{1/2}W^{1/2}V^{-1/2}K^{-1/2} + QT^{1/2}K^{-1/2} + TQ^{-1/2}V^{-1})\\
&\qquad\ll (\log T)^{C'} Q^\epsilon (T^{1/2}W^{1/2}K^{-1/2} + QT^{1/2} + TQ^{-1/2}).
\end{aligned} \tag{52}$$

Let $W = (KQT)^{2/3}$. Combining (45), (48), (49) and (52) we have

$$\mathscr{E}_2 \ll y^{4/3}T^{1/3+\epsilon} + (\log T)^{C'}(yT^{1/2+\epsilon} + T\eta^{-1/2+\epsilon} + y^{1/3}T^{5/6+\epsilon})$$

and Proposition 11 follows.

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