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Twisted loop transgression and higher Jandl gerbes over finite groupoids

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Given a double cover $\pi : \mathcal{G} \to \hat{\mathcal{G}}$ of finite groupoids, we explicitly construct cochain-level twisted loop transgression maps, τ_π and τ_π^{ref}, thereby associating to a Jandl n–gerbe $\hat{\lambda}$ on $\hat{\mathcal{G}}$ a Jandl $(n-1)$–gerbe $\tau_\pi(\hat{\lambda})$ on the quotient loop groupoid of $\mathcal{G}$ and an ordinary $(n-1)$–gerbe $\tau_\pi^{\mathrm{ref}}(\hat{\lambda})$ on the unoriented quotient loop groupoid of $\mathcal{G}$. For $n = 1, 2$, we prove that the character theory (resp. centre) of the category of Real $\hat{\lambda}$–twisted n–vector bundles over $\hat{\mathcal{G}}$ admits a natural interpretation in terms of flat sections of the $(n-1)$–vector bundle associated to $\tau_\pi^{\mathrm{ref}}(\hat{\lambda})$ (resp. the Real $(n-1)$–vector bundle associated to $\tau_\pi(\hat{\lambda})$). We relate our results to Real versions of twisted Drinfeld doubles of finite groups and fusion categories and to discrete torsion in orientifold string theory and M–theory.

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Introduction

The goal of this paper is, firstly, to construct and compute twisted versions of loop transgression maps on the cohomology of finite groupoids and their various loop groupoids and, secondly, to connect these maps to Real categorical representation theory, monoidal categories and discrete torsion in orientifold string theory and M–theory.

To put our results in context, recall that the ordinary loop transgression map τ is a fundamental construction which associates to a degree n cohomology class on a closed manifold X a degree $n-1$ class on its free loop space $\mathrm{Map}(S^1, X)$ using integration along the S^1–fibre of the projection $S^1 \times \mathrm{Map}(S^1, X) \to \mathrm{Map}(S^1, X)$. For a detailed discussion of loop transgression, see Brylinski [8]. When X is replaced with a finite groupoid $\mathcal{G}$, which is the focus of the present paper, Willerton [40] constructed and computed a cochain-level loop transgression map $\tau : C^\bullet(\mathcal{G}) \to C^{\bullet-1}(\Lambda\mathcal{G})$ for $\mathsf{U}(1)$–valued simplicial cochains. Here $\Lambda\mathcal{G} := \mathrm{Hom}_{\mathsf{Cat}}(B\mathbb{Z}, \mathcal{G})$ is the loop groupoid of $\mathcal{G}$,

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whose geometric realization $|\Lambda\mathcal{G}|$ is homotopy equivalent to $\mathrm{Map}(S^1, |\mathcal{G}|)$. In the setting of orbifolds, a differential refinement of this map was constructed by Lupercio and Uribe [31]. The map τ has found applications in many areas of mathematics, including oriented topological field theory and complex representation theory; see Freed [19], Brylinski and McLaughlin [9], Willerton [40], Fuchs, Schweigert and Valentino [22] and Kong, Tian and Zhou [30].

Suppose now that $\pi\colon \widehat{\mathcal{G}} \to B\mathbb{Z}_2$ is a finite $\mathbb{Z}_2$–graded groupoid, such as the classifying space of a $\mathbb{Z}_2$–graded finite group $\widehat{\mathsf{G}} \to \mathbb{Z}_2 = \{\pm 1\}$. Such $\mathbb{Z}_2$–gradings arise, for example, in real versions of representation theory and in mathematical approaches to unoriented topological field theory and orientifold string theory. Let $\mathcal{G} \to \widehat{\mathcal{G}}$ be the associated double cover and $C^{\bullet+\pi}(\widehat{\mathcal{G}})$ the π–twisted cochain complex of $\widehat{\mathcal{G}}$. In Section 1.4 we define quotient loop groupoids $\Lambda_\pi\widehat{\mathcal{G}} := \Lambda\mathcal{G}/\!/\mathbb{Z}_2$ and $\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}} := \Lambda\mathcal{G}/\!/\mathbb{Z}_2$ by the $\mathbb{Z}_2$–actions on $\Lambda\mathcal{G}$ given by deck transformations of $\mathcal{G}$ and the diagonal action of deck transformations and reflection of the circle $B\mathbb{Z}$, respectively. Our first result is as follows:

Theorem A (Theorems 2.6 and 2.8) *There exist twisted loop transgression maps*

$$\tau_\pi^{\mathrm{ref}}\colon C^{\bullet+\pi}(\widehat{\mathcal{G}}) \to C^{\bullet-1}(\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}) \quad \textit{and} \quad \tau_\pi\colon C^{\bullet+\pi}(\widehat{\mathcal{G}}) \to C^{\bullet-1+\pi_{\Lambda_\pi\widehat{\mathcal{G}}}}(\Lambda_\pi\widehat{\mathcal{G}}),$$

both of which are cochain maps and have explicit combinatorial expressions. Here $\pi_{\Lambda_\pi\widehat{\mathcal{G}}}\colon \Lambda\mathcal{G} \to \Lambda_\pi\widehat{\mathcal{G}}$ is the canonical double cover.

For example, let $\widehat{\theta} \in C^{2+\pi}(\widehat{\mathcal{G}})$. Morphisms $\gamma\colon x \to x$ and $\omega\colon x \to y$ in $\widehat{\mathcal{G}}$, the former of degree $+1$, define a morphism $\omega\colon \gamma \to \omega\gamma^{\pi(\omega)}\omega^{-1}$ in $\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}$. Denoting by $[\omega]\gamma \in C_1(\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}})$ the associated 1–chain, Theorem A gives

$$\tau_\pi^{\mathrm{ref}}(\widehat{\theta})([\omega]\gamma) = \widehat{\theta}\big([\gamma^{-1} \mid \gamma]\big)^{(\pi(\omega)-1)/2} \frac{\widehat{\theta}\big([\omega\gamma^{\pi(\omega)}\omega^{-1} \mid \omega]\big)}{\widehat{\theta}\big([\omega \mid \gamma^{\pi(\omega)}]\big)}.$$

The map τ_π^{ref} has already appeared, in the form of its explicit expressions in low degrees, in work of Young on unoriented topological field theory [43] and Real 2–representation theory [44], where its geometry was also foreshadowed. Theorem A gives an a priori geometric construction of τ_π^{ref} and τ_π, in all degrees, and establishes that they are cochain maps. The latter fact is crucial for applications in [44; 43] and is difficult to verify directly in all but the simplest cases. Upon restriction along $\mathcal{G} \to \widehat{\mathcal{G}}$, both maps τ_π^{ref} and τ_π recover Willerton's transgression map τ, so τ_π^{ref} and τ_π can be viewed as enhanced versions of τ which take into account certain $\mathbb{Z}_2$–actions. However, our proof of Theorem A is necessarily different from Willerton's proof for τ, since the latter relies on an explicit homotopy equivalence $|\Lambda\mathcal{G}| \sim \mathrm{Map}(S^1, |\mathcal{G}|)$ which is not equivariant for circle reflection. We instead work in a categorical setting, where all constructions are equivariant, and do not pass to geometric realizations.

In the remainder of the paper we connect Theorem A to Real (categorical) representation theory and related areas. Following Atiyah [1], the term "Real" indicates a $\mathbb{C}$–linear object (or theory) with a $\mathbb{C}$–antilinear involution. We take a geometric approach and, using terminology of Schreiber, Schweigert and Waldorf [35] and Willerton [40], interpret a twisted $(n+1)$–cocycle $\hat{\lambda} \in Z^{n+1+\pi_{\hat{\mathcal{G}}}}(\hat{\mathcal{G}})$ as a flat Jandl n–gerbe on $\hat{\mathcal{G}}$, that is, a $\mathsf{U}(1)$–bundle n–gerbe on $\mathcal{G}$ with complex conjugation-twisted equivariance data for the double cover $\mathcal{G} \to \hat{\mathcal{G}}$. For $n = -1, 0, 1$, this is a Real $\mathsf{U}(1)$–valued function, a Real $\mathsf{U}(1)$–bundle and a Jandl (or Real) gerbe on $\hat{\mathcal{G}}$, respectively. Jandl gerbes play a central role in unoriented topological field theory (see Kapustin and Turzillo [26] and Young [43]), orientifold string theory (see Schreiber, Schweigert and Waldorf [35] and Distler, Freed and Moore [13]), topological phases of matter with time reversal symmetry (see Barkeshli, Bonderson, Cheng, Jian and Walker [4]) and Real representation theory [43]. The twisted transgression maps associate to a Jandl n–gerbe $\hat{\lambda}$ an ordinary $(n-1)$–gerbe $\tau_\pi^{\mathrm{ref}}(\hat{\lambda})$ on $\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}$ and a Jandl $(n-1)$–gerbe $\tau_\pi(\hat{\lambda})$ on $\Lambda_\pi\hat{\mathcal{G}}$, both of which we regard as higher holonomy gerbes of $\hat{\lambda}$. Our results indicate that $\tau_\pi^{\mathrm{ref}}(\hat{\lambda})$ and $\tau_\pi(\hat{\lambda})$ control the character theory and centre, respectively, of the category of $\hat{\lambda}$–twisted n–vector bundles on $\hat{\mathcal{G}}$. Here we formulate precisely and prove these statements for $n \leq 2$.

In Section 3, after briefly treating the rather trivial case $n = 0$, we study the case $n = 1$. As is well known — see Freed, Hopkins and Teleman [20], Moutuou [32] and Fok [17] — complex vector bundles over $\hat{\mathcal{G}}$ can be twisted by a Jandl gerbe $\hat{\theta} \in Z^{2+\pi_{\hat{\mathcal{G}}}}(\hat{\mathcal{G}})$. The collection of such twisted bundles forms an $\mathbb{R}$–linear category $\mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(\hat{\mathcal{G}})$ whose Grothendieck group $K^{\hat{\theta}}(\hat{\mathcal{G}})$ is a twisted form of the KR–theory of $\mathcal{G}$. We prove that the character theory of $\mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(\hat{\mathcal{G}})$ is naturally described in terms of the transgressed $\mathsf{U}(1)$–bundle $\tau_\pi^{\mathrm{ref}}(\hat{\theta})$ on $\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}$.

Theorem B (Theorem 3.12) *The Real character map induces an isomorphism of complex inner product spaces*

$$\chi\colon K^{\hat{\theta}}(\hat{\mathcal{G}}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \Gamma_{\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}}(\tau_\pi^{\mathrm{ref}}(\hat{\theta})_{\mathbb{C}})$$

with target the space of flat sections of the associated complex line bundle $\tau_\pi^{\mathrm{ref}}(\hat{\theta})_{\mathbb{C}}$.

A key point in the proof of Theorem B is the use of the explicit combinatorial expressions in Theorem A to identify $\Gamma_{\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}}(\tau_\pi^{\mathrm{ref}}(\hat{\theta})_{\mathbb{C}})$ as the appropriate space of class functions. For certain very simple choices of $\widehat{\mathcal{G}}$ and $\widehat{\theta}$, Theorem B reduces to previously known results in the real and quaternionic representation theory of finite groups; see Reynolds [33] and Bröcker and tom Dieck [7]. One benefit of our geometric approach is that Theorem B, and all other results of this paper, applies uniformly to all $\mathbb{Z}_2$–gradings $\widehat{\mathcal{G}}$ and all twists $\widehat{\theta}$. As an application of Theorem B, we prove in Corollary 3.14 a Real generalization of a theorem of Schur, computing the number of simple objects of $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(B\widehat{\mathsf{G}})$ as a $\widehat{\theta}$–weighted count of Real conjugacy classes of G.

In contrast to the complex (ie non-Real) case, the centre $Z(\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}}))$ and Grothendieck group $K^{\widehat{\theta}}(\widehat{\mathcal{G}})$ of the category $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}})$ are not directly related. Instead, we prove that $Z(\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}}))$ can be understood in terms of the twisted transgression map τ_π.

Theorem C (Theorem 3.17) *There is a canonical $\mathbb{R}$–algebra isomorphism*

$$Z(\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}})) \simeq \Gamma_{\Lambda_\pi \widehat{\mathcal{G}}}(\tau_\pi(\widehat{\theta})_{\mathbb{C}}^{-1}),$$

where the right-hand side is the space of flat sections of the Real line bundle $\tau_\pi(\widehat{\theta})_{\mathbb{C}}^{-1}$.

In Section 4 we study 2–categorical analogues of the results of Section 3. For simplicity, we restrict attention to connected groupoids. Fix a Jandl 2–gerbe $\widehat{\eta} \in Z^{3+\pi}(B\widehat{\mathsf{G}})$. Motivated by Willerton's interpretation of the twisted Drinfeld double $D^{\eta}(\mathsf{G})$ for $\eta \in Z^3(B\mathsf{G})$, of a finite group G as the $\tau(\eta)$–twisted groupoid algebra of $\Lambda B\mathsf{G}$ [40], we use $\tau_\pi^{\mathrm{ref}}(\widehat{\eta})$ and $\tau_\pi(\widehat{\eta})$ to define thickened twisted Drinfeld doubles $\mathbb{D}^{\widehat{\eta}}(\widehat{\mathsf{G}})$ and $D^{\widehat{\eta}}(\widehat{\mathsf{G}})$, respectively. These are complex vector spaces with possibly sesquilinear associative multiplications which contain the twisted Drinfeld double of $\mathsf{G} = \ker(\pi)$ as a subalgebra. The methods of Section 3 allow to immediately describe the character theory of thickened Drinfeld doubles. For example, characters of $D^{\widehat{\eta}}(\widehat{\mathsf{G}})$–modules — which, because of their connection with two-dimensional topological field theory, we call twisted one-loop characters — are shown in Proposition 4.4 to give an interesting extension of twisted elliptic characters of G by a Klein bottle sector. Next, we construct from $\widehat{\eta}$ a Real fusion category $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}})$, which we view as a categorified twisted Real group algebra of G, and show in Proposition 4.8 that categories of this form exhaust Real pointed fusion categories. We also identify their Drinfeld centres, giving the following 2–categorical version of Theorem C:

Theorem D (Theorem 4.9) *There is a canonical $\mathbb{R}$–linear monoidal equivalence*

$$Z_D(\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}})) \simeq \mathsf{Vect}_{\mathbb{C}}^{\tau_\pi(\widehat{\eta})^{-1}}(\Lambda_\pi B\widehat{\mathsf{G}}).$$

Next, we establish a tighter connection between the ordinary Drinfeld double $D^{\eta}(\mathsf{G})$ and $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})$ by giving the latter the structure of a Real quasibialgebra. Moreover, we show that the category of Real $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})$–modules is monoidally equivalent to $\mathsf{Vect}_{\mathbb{C}}^{\tau_{\pi}(\hat{\eta})}(\Lambda_{\pi} B\hat{\mathsf{G}})$. In Section 4.4 we explain that Real $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}})$–module categories recover the bicategory $2\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(B\hat{\mathsf{G}})$ of $\hat{\eta}$–twisted 2–vector bundles over $B\hat{\mathsf{G}}$, as developed in [44] in the form of Real 2–representation theory. One of the main results of [44] is the existence of a Real categorical character theory. This theory is most naturally formulated in terms of $D^{\hat{\eta}}(\hat{\mathsf{G}})$–modules and twisted one-loop characters, thereby illustrating the character-theoretic meaning of $\tau_{\pi}^{\mathrm{ref}}(\hat{\eta})$. Because of the complicated form of $\tau_{\pi}^{\mathrm{ref}}(\hat{\eta})$, this result is a highly nontrivial check of our general proposal that $\tau_{\pi}^{\mathrm{ref}}$ controls the character theory of higher vector bundles. Finally, in Section 4.5 we interpret $\tau_{\pi}^{\mathrm{ref}}$ in terms of discrete torsion phases of nonorientable 2– and 3–manifolds in orientifold string theory and M–theory. In particular, we provide a representation-theoretic perspective on computations of Bantay [3] and Sharpe [38].

We have restricted our attention to finite groupoids, both for simplicity and because of our applications. In work in progress, we study more general twisted transgression maps in the geometric setting of topological stacks.

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1 Loop groupoids and their quotients

1.1 Essentially finite groupoids

We briefly recall some standard material about groupoids.

A groupoid is a category in which all morphisms are isomorphisms. A groupoid $\mathcal{G}$ is said to be essentially finite if its set of connected components, $\pi_0(\mathcal{G})$, is finite and each object of $\mathcal{G}$ has finitely many automorphisms. Unless mentioned otherwise, all groupoids in this paper are assumed to be essentially finite.

Example Let G be a group and X a (left) G–set. The action groupoid $X /\!/ \mathsf{G}$ has object set X and morphisms $\mathrm{Hom}_{X/\!/\mathsf{G}}(x_1, x_2) = \{g \in \mathsf{G} \mid gx_1 = x_2\}$. Morphisms are composed using multiplication in G. The groupoid $X /\!/ \mathsf{G}$ is essentially finite if and only if X has finitely many G–orbits and each orbit has finite stabilizer.

When X consists of a single point, we write $B\mathsf{G}$ in place of $X /\!/ \mathsf{G}$.

Let $C_\bullet(\mathcal{G}) := C_\bullet(\mathcal{G};\mathbb{Z})$ be the complex of simplicial chains on $\mathcal{G}$. Explicitly, $C_n(\mathcal{G})$ is the free abelian group generated by symbols $[g_n \mid \cdots \mid g_1]$, where $x_1 \xrightarrow{g_1} \cdots \xrightarrow{g_n} x_{n+1}$ is a diagram in $\mathcal{G}$. The differential of $C_\bullet(\mathcal{G})$ is

$$\partial[g_n|\cdots|g_1] = [g_{n-1}|\cdots|g_1] + \sum_{j=1}^{n-1}(-1)^{n-j}[g_n|\cdots|g_{j+1}g_j|\cdots|g_1] + (-1)^n[g_n|\cdots|g_2].$$

Let A be an abelian group, written multiplicatively, and $\kappa\colon \mathcal{G} \to B\mathrm{Aut}(\mathsf{A})$ a functor. The complex of κ–twisted cochains on $\mathcal{G}$ is the abelian group $\mathrm{Hom}_{\mathbb{Z}}(C_\bullet(\mathcal{G}),\mathsf{A})$. The differential d sends $\lambda \in \mathrm{Hom}_{\mathbb{Z}}(C_{n-1}(\mathcal{G}),\mathsf{A})$ to the n–cochain $d\lambda$ with values

$$\begin{aligned} &d\lambda\big([g_n \mid \cdots \mid g_1]\big) \\ &= \kappa(g_n)\cdot\lambda\big([g_{n-1}|\cdots|g_1]\big) \prod_{j=1}^{n-1} \lambda\big([g_n|\cdots|g_{j+1}g_j|\cdots|g_1]\big)^{(-1)^{n-j}} \cdot \lambda\big([g_n|\cdots|g_2]\big)^{(-1)^n}. \end{aligned}$$

The inclusion of the subcomplex

$$C^{\bullet+\kappa}(\mathcal{G};\mathsf{A}) := \mathrm{Hom}^{\mathrm{n}}_{\mathbb{Z}}(C_\bullet(\mathcal{G}),\mathsf{A}) \subset \mathrm{Hom}_{\mathbb{Z}}(C_\bullet(\mathcal{G}),\mathsf{A})$$

of normalized cochains is a quasi-isomorphism. Write $Z^{\bullet+\kappa}(\mathcal{G};\mathsf{A})$ and $H^{\bullet+\kappa}(\mathcal{G};\mathsf{A})$ for the cocycles and cohomology of $C^{\bullet+\kappa}(\mathcal{G};\mathsf{A})$, respectively. Write $C^{\bullet+\kappa}(\mathcal{G})$ for $C^{\bullet+\kappa}(\mathcal{G};\mathsf{A})$ if A is fixed and omit κ from the notation if it is trivial.

1.2 Essentially finite groupoids over $B\mathbb{Z}_2$

Let $\mathbb{Z}_2$ be the multiplicative group $\{\pm 1\}$. We sometimes denote its nonidentity element by ζ.

A groupoid over $B\mathbb{Z}_2$, or a $\mathbb{Z}_2$–graded groupoid, is a functor $\pi_{\hat{\mathcal{G}}}\colon \hat{\mathcal{G}} \to B\mathbb{Z}_2$. Write π for $\pi_{\hat{\mathcal{G}}}$ if it will not cause confusion. The degree of a morphism $\omega \in \mathrm{Mor}(\hat{\mathcal{G}})$ is $\pi(\omega) \in \mathbb{Z}_2$. When $\pi_{\hat{\mathcal{G}}}$ is trivial, all results below reduce to known results for ordinary groupoids. For this reason, we assume that π is strongly nontrivial in the sense that, for each $x \in \hat{\mathcal{G}}$, there exists a morphism of degree -1 with source x.

There is a canonical decomposition $\pi_0(\hat{\mathcal{G}}) = \pi_0(\hat{\mathcal{G}})_{-1} \sqcup \pi_0(\hat{\mathcal{G}})_1$, with $\pi_0(\hat{\mathcal{G}})_{-1}$ the subset of connected components of $\hat{\mathcal{G}}$ consisting of objects which have at least one automorphism of degree -1. By strong nontriviality, for each representative $x \in \pi_0(\widehat{\mathcal{G}})_1$, we can choose a morphism of degree -1 with source x, whose target we denote by $\bar{x}$. Note that $x \neq \bar{x}$. Let $\widehat{\mathcal{G}}_{\{x,\bar{x}\}} \subset \widehat{\mathcal{G}}$ be the full subcategory on $\{x, \bar{x}\}$.

Proposition 1.1 *There is an equivalence of $\mathbb{Z}_2$–graded groupoids*

$$\widehat{\mathcal{G}} \simeq \bigsqcup_{x \in \pi_0(\widehat{\mathcal{G}})_{-1}} B\mathrm{Aut}_{\widehat{\mathcal{G}}}(x) \sqcup \bigsqcup_{x \in \pi_0(\widehat{\mathcal{G}})_1} \widehat{\mathcal{G}}_{\{x,\bar{x}\}}.$$

Proof It suffices to prove the statement for connected $\widehat{\mathcal{G}}$. The case $\pi_0(\widehat{\mathcal{G}}) = \pi_0(\widehat{\mathcal{G}})_{-1}$ is left to the reader. Suppose that $\pi_0(\widehat{\mathcal{G}}) = \pi_0(\widehat{\mathcal{G}})_1$. For each $y \in \widehat{\mathcal{G}}$, there exists a morphism g_y of degree $+1$ from y to x or from y to $\bar{x}$, but not both. Fix a choice of such morphisms. Then the functor $\widehat{\mathcal{G}} \to \widehat{\mathcal{G}}_{\{x,\bar{x}\}}$ which sends an object y to the target of g_y and a morphism $\omega\colon y_1 \to y_2$ to $g_{y_2}\omega g_{y_1}^{-1}$ is quasi-inverse to the inclusion $\widehat{\mathcal{G}}_{\{x,\bar{x}\}} \hookrightarrow \widehat{\mathcal{G}}$ and is compatible with the structure maps to $B\mathbb{Z}_2$. □

The functor $\pi\colon \widehat{\mathcal{G}} \to B\mathbb{Z}_2$ classifies a double cover of groupoids, denoted by $\pi\colon \mathcal{G}_\pi \to \widehat{\mathcal{G}}$. The use of π for both the classifying map and the double cover should not cause confusion. We often write $\mathcal{G}$ for $\mathcal{G}_\pi$. The functor $\pi\colon \mathcal{G} \to \widehat{\mathcal{G}}$ admits the following explicit model [21, Section 10.4]. The objects and morphisms of $\mathcal{G}$ are $\mathrm{Obj}(\widehat{\mathcal{G}}) \times \mathbb{Z}_2$ and

$$\mathrm{Hom}_{\mathcal{G}}((x_1, \epsilon_1), (x_2, \epsilon_2)) = \{\omega \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(x_1, x_2) \mid \pi(\omega)\epsilon_1 = \epsilon_2\}.$$

Morphisms are composed as in $\widehat{\mathcal{G}}$. The functor π sends a morphism $(x_1, \epsilon_1) \xrightarrow{\omega} (x_2, \epsilon_2)$ in $\mathcal{G}$ to $x_1 \xrightarrow{\omega} x_2$. The (nontrivial) deck transformation $\sigma_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{G}$ is the strict involution which sends a morphism $(x_1, \epsilon_1) \xrightarrow{\omega} (x_2, \epsilon_2)$ to $(x_1, -\epsilon_1) \xrightarrow{\omega} (x_2, -\epsilon_2)$.

Example Let $1 \to \mathsf{G} \to \widehat{\mathsf{G}} \xrightarrow{\pi} \mathbb{Z}_2 \to 1$ be an exact sequence of groups. We call $\widehat{\mathsf{G}}$ a $\mathbb{Z}_2$–graded group. Let X be a $\widehat{\mathsf{G}}$–set. There is an induced $\mathbb{Z}_2$–grading $\pi\colon X/\!\!/\widehat{\mathsf{G}} \to B\mathbb{Z}_2$. The choice of an element $\varsigma \in \widehat{\mathsf{G}} \setminus \mathsf{G}$ identifies the canonical double cover $X/\!\!/\mathsf{G} \to X/\!\!/\widehat{\mathsf{G}}$ with $(X/\!\!/\widehat{\mathsf{G}})_\pi \to X/\!\!/\widehat{\mathsf{G}}$. Under this identification, the deck transformation $\sigma_{X/\!\!/\mathsf{G}}\colon X/\!\!/\mathsf{G} \to X/\!\!/\mathsf{G}$ is the weak involution given on morphisms by

$$(x_1 \xrightarrow{g} x_2) \mapsto \varsigma x_1 \xrightarrow{\varsigma g \varsigma^{-1}} \varsigma x_2.$$

The action of $\varsigma^2 \in \mathsf{G}$ defines a natural isomorphism $1_{X/\!\!/\widehat{\mathsf{G}}} \Rightarrow (\sigma_{X/\!\!/\mathsf{G}})^2$, thereby exhibiting $\sigma_{X/\!\!/\mathsf{G}}$ as an involution.

Fix an abelian group A and let $\mathbb{Z}_2 \to \mathrm{Aut}(\mathsf{A})$ be the morphism determined by the inversion automorphism of A. A $\mathbb{Z}_2$–grading $\pi_{\widehat{\mathcal{G}}} \colon \widehat{\mathcal{G}} \to B\mathbb{Z}_2$ therefore defines a twisted cochain complex $C^{\bullet+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$. A description of $C^{\bullet+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$ in terms of the double cover $\mathcal{G}$ is as follows. Let $[\omega_n \mid \cdots \mid \omega_1]_{\epsilon_1} \in C_n(\mathcal{G})$ be the chain associated to the diagram

$$(x_1, \epsilon_1) \xrightarrow{\omega_1} \cdots \xrightarrow{\omega_n} (x_{n+1}, \epsilon_{n+1})$$

in $\mathcal{G}$. The notation is unambiguous, since $\epsilon_i = \pi(\omega_{\leq i-1})\epsilon_1$, where $\omega_{\leq i} := \omega_i \cdots \omega_1$. The complex $C_\bullet(\mathcal{G})$ is a $\mathbb{Z}[\mathbb{Z}_2]$–module via

$$\zeta \cdot [\omega_n \mid \cdots \mid \omega_1]_{\epsilon_1} = [\omega_n \mid \cdots \mid \omega_1]_{-\epsilon_1}. \tag{1}$$

Write A and A_- for the trivial and nontrivial $\mathbb{Z}[\mathbb{Z}_2]$–module structures on A, respectively. For example, $c_0 + c_1\zeta \in \mathbb{Z}[\mathbb{Z}_2]$ acts on A_- by $z \mapsto z^{c_0 - c_1}$.

Lemma 1.2 *There are mutually inverse cochain isomorphisms*

$$\Phi_- \colon C^{\bullet+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}}) \leftrightarrows \mathrm{Hom}^{\mathrm{n}}_{\mathbb{Z}[\mathbb{Z}_2]}(C_\bullet(\mathcal{G}), \mathsf{A}_-) : \Psi_-$$

given by

$$\Phi_-(\widehat{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]_{\epsilon_1}\big) = \widehat{\lambda}\big([\omega_n \mid \cdots \mid \omega_1]\big)^{\epsilon_{n+1}}$$

and

$$\Psi_-(\widehat{\mu})\big([\omega_n \mid \cdots \mid \omega_1]\big) = \widehat{\mu}\big([\omega_n \mid \cdots \mid \omega_1]_{\pi(\omega_{\leq n})}\big).$$

Proof This is a direct calculation. □

There is an untwisted version of Lemma 1.2, in which A replaces A_- and maps Φ and Ψ are defined as for Φ_- and Ψ_-, but with the signs removed.

1.3 Loop groupoids

Following Willerton [40] (see also [31]), the loop groupoid of an essentially finite groupoid $\mathcal{G}$ is defined to be the functor category

$$\Lambda\mathcal{G} := \mathrm{Hom}_{\mathsf{Cat}}(B\mathbb{Z}, \mathcal{G}).$$

Objects of $\Lambda\mathcal{G}$ can be identified with loops (x, γ) in $\mathcal{G}$, that is, morphisms $\gamma \colon x \to x$. A morphism $(x_1, \gamma_1) \to (x_2, \gamma_2)$ is then a morphism $g \colon x_1 \to x_2$ in $\mathcal{G}$ which satisfies $\gamma_2 = g\gamma_1 g^{-1}$. Composition of morphisms is as in $\mathcal{G}$. Since $\mathcal{G}$ is essentially finite, so too is $\Lambda\mathcal{G}$. Moreover, $\Lambda\mathcal{G}$ coincides with the inertia groupoid of $\mathcal{G}$, whence our notation.

Denote by $|\mathcal{G}|$ the geometric realization of $\mathcal{G}$. Since $\mathcal{G}$ is essentially finite, the sets $\pi_0(|\mathcal{G}|)$ and $\pi_1(|\mathcal{G}|)$ are finite. In the other direction, any finite homotopy 1–type can be realized as the geometric realization of an essentially finite groupoid. The geometric realization of $\Lambda\mathcal{G}$ is homotopy equivalent to the free loop space of $|\mathcal{G}|$; that is, $|\Lambda\mathcal{G}| \sim \mathrm{Map}(S^1, |\mathcal{G}|)$. See [40, Theorem 2] or [31, Proposition 6.1.1].

1.4 Quotients of loop groupoids

Let $\pi\colon \widehat{\mathcal{G}} \to B\mathbb{Z}_2$ be an essentially finite $\mathbb{Z}_2$–graded groupoid with associated double cover $\mathcal{G} \to \widehat{\mathcal{G}}$. We introduce in this section two quotients of $\Lambda\mathcal{G}$.

Definition 1.3 The quotient loop groupoid of $\mathcal{G}$ is the groupoid $\Lambda_\pi\widehat{\mathcal{G}}$ with objects the loops (x,γ) of degree $+1$ in $\widehat{\mathcal{G}}$ and morphisms

$$\mathrm{Hom}_{\Lambda_\pi\widehat{\mathcal{G}}}((x_1,\gamma_1),(x_2,\gamma_2)) = \{\omega \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(x_1,x_2) \mid \gamma_2 = \omega\gamma_1\omega^{-1}\}.$$

There is a strongly nontrivial grading $\pi_{\Lambda_\pi\widehat{\mathcal{G}}}\colon \Lambda_\pi\widehat{\mathcal{G}} \to B\mathbb{Z}_2$ which sends a morphism ω to $\pi(\omega)$. To identify the associated double cover, let $p\colon \Lambda\mathcal{G} \to \Lambda_\pi\widehat{\mathcal{G}}$ be the functor which sends a morphism $((x_1,\epsilon_1),\gamma_1) \xrightarrow{\omega} ((x_2,\epsilon_2),\gamma_2)$ to $(x_1,\gamma_1) \xrightarrow{\omega} (x_2,\gamma_2)$.

Lemma 1.4 *The double cover classified by $\pi_{\Lambda_\pi\widehat{\mathcal{G}}}$ is equivalent to p.*

Proof The functor $\Lambda\mathcal{G} \to (\Lambda_\pi\widehat{\mathcal{G}})_{\pi_{\Lambda_\pi\widehat{\mathcal{G}}}}$ defined on morphisms by

$$[((x_1,\epsilon_1),\gamma_1) \xrightarrow{\omega} ((x_2,\epsilon_2),\gamma_2)] \mapsto ((x_1,\gamma_1),\epsilon_1) \xrightarrow{\omega} ((x_2,\gamma_2),\epsilon_2)$$

is an equivalence and is compatible with the structure maps to $\Lambda_\pi\widehat{\mathcal{G}}$. □

Because of Lemma 1.4, we henceforth write $\pi_{\Lambda_\pi\widehat{\mathcal{G}}}$ for p. Under the equivalence of Lemma 1.4 (and its obvious inverse functor), the deck transformation $\sigma_{\Lambda\mathcal{G}}\colon \Lambda\mathcal{G} \to \Lambda\mathcal{G}$ is the strict involution given on objects by $((x,\epsilon),\gamma) \mapsto ((x,-\epsilon),\gamma)$ and on morphisms by the identity. In terms of functors, this reads

$$\sigma_{\Lambda\mathcal{G}}\colon \mathrm{Hom}_{\mathsf{Cat}}(B\mathbb{Z},\mathcal{G}) \to \mathrm{Hom}_{\mathsf{Cat}}(B\mathbb{Z},\mathcal{G}), \qquad F \mapsto \sigma_{\mathcal{G}} \circ F.$$

Next, we define a modification of $\Lambda_\pi\widehat{\mathcal{G}}$ which incorporates reflection of the circle.

Definition 1.5 The unoriented quotient loop groupoid of $\mathcal{G}$ is the groupoid $\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}$ with objects the loops (x,γ) of degree $+1$ in $\widehat{\mathcal{G}}$ and morphisms

$$\mathrm{Hom}_{\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}}((x_1,\gamma_1),(x_2,\gamma_2)) = \{\omega \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(x_1,x_2) \mid \gamma_2 = \omega\gamma_1^{\pi(\omega)}\omega^{-1}\}.$$

There is a strongly nontrivial grading $\pi_{\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}}\colon \Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}} \to B\mathbb{Z}_2$. Let $p^{\mathrm{ref}}\colon \Lambda\mathcal{G} \to \Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}$ be the functor given by $[((x_1,\epsilon_1),\gamma_1) \xrightarrow{\omega} ((x_2,\epsilon_2),\gamma_2)] \mapsto (x_1,\gamma_1^{\epsilon_1}) \xrightarrow{\omega} (x_2,\gamma_2^{\epsilon_2})$.

Lemma 1.6 *The double cover classified by $\pi_{\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}}$ is equivalent to p^{ref}.*

Proof The equivalence $\Lambda\mathcal{G} \to (\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}})_{\pi_{\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}}}$ is defined by

$$[((x_1,\epsilon_1),\gamma_1) \xrightarrow{\omega} ((x_2,\epsilon_2),\gamma_2)] \mapsto ((x_1,\gamma_1^{\epsilon_1}),\epsilon_1) \xrightarrow{\omega} ((x_2,\gamma_2^{\epsilon_2}),\epsilon_2). \qquad \square$$

We henceforth write $\pi_{\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}}$ for p^{ref}. Under the equivalence of Lemma 1.6, the deck transformation $\sigma_{\Lambda\mathcal{G}}^{\mathrm{ref}}\colon \Lambda\mathcal{G} \to \Lambda\mathcal{G}$ is the strict involution $\sigma_{\Lambda\mathcal{G}}^{\mathrm{ref}}((x,\epsilon),\gamma) = ((x,-\epsilon),\gamma^{-1})$ or, in terms of functors, $\sigma_{\Lambda\mathcal{G}}^{\mathrm{ref}}(F) = \sigma_{\mathcal{G}} \circ F \circ B\iota$, where $\iota\colon \mathbb{Z} \to \mathbb{Z}$ is negation.

Example Let $\hat{\mathsf{G}}$ be a $\mathbb{Z}_2$–graded group. The Real conjugation action of $\hat{\mathsf{G}}$ on G is

$$\omega \cdot g = \omega g^{\pi(\omega)}\omega^{-1}, \quad (\omega, g) \in \hat{\mathsf{G}} \times \mathsf{G}.$$

With this notation, there are equivalences

$$\Lambda B\mathsf{G} \simeq \mathsf{G}/\!\!/\mathsf{G}, \quad \Lambda_\pi B\hat{\mathsf{G}} \simeq \mathsf{G}/\!\!/\hat{\mathsf{G}}, \quad \Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}} \simeq \mathsf{G}/\!\!/_{\mathrm{R}}\hat{\mathsf{G}},$$

where the group actions on G are G–conjugation, $\hat{\mathsf{G}}$–conjugation and Real $\hat{\mathsf{G}}$–conjugation, respectively. The choice of an element $\varsigma \in \hat{\mathsf{G}} \setminus \mathsf{G}$ identifies the double covers

$$\Lambda B\mathsf{G} \to \Lambda_\pi B\hat{\mathsf{G}}, \quad \Lambda B\mathsf{G} \to \Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$$

with the canonical functors

$$\mathsf{G}/\!\!/\mathsf{G} \to \mathsf{G}/\!\!/\hat{\mathsf{G}}, \quad \mathsf{G}/\!\!/\mathsf{G} \to \mathsf{G}/\!\!/_{\mathrm{R}}\hat{\mathsf{G}}.$$

Under these identifications, the deck transformations are given on objects by

$$\sigma_{\Lambda_\pi B\hat{\mathsf{G}}}(\gamma) = \varsigma\gamma\varsigma^{-1} \quad \text{and} \quad \sigma_{\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}}(\gamma) = \varsigma\gamma^{-1}\varsigma^{-1}$$

and on morphisms by $\sigma_{\Lambda_\pi^{(\mathrm{ref})} B\hat{\mathsf{G}}}(\omega) = \varsigma\omega\varsigma^{-1}$.

2 Twisted loop transgression

2.1 Oriented loop transgression

We begin by recalling the ordinary (oriented) loop transgression map in the setting of essentially finite groupoids. For detailed discussions, the reader is referred to [40], or [8; 31] in the geometric setting.

Let $\mathcal{G}$ be an essentially finite groupoid. The evaluation functor $\mathrm{ev}\colon B\mathbb{Z} \times \Lambda\mathcal{G} \to \mathcal{G}$ is given on morphisms by

$$[(x_1,\gamma_1) \xrightarrow{(n,g)} (x_2,\gamma_2)] \mapsto x_1 \xrightarrow{g\gamma_1^n = \gamma_2^n g} x_2.$$

Let $\mathrm{pr}_{\Lambda\mathcal{G}}\colon B\mathbb{Z}\times\Lambda\mathcal{G}\to\Lambda\mathcal{G}$ be the projection. Consider the span of groupoids

$$\begin{tikzcd}[column sep=small] & B\mathbb{Z}\times\Lambda\mathcal{G} \arrow[dl, "\mathrm{ev}"'] \arrow[dr, "\mathrm{pr}_{\Lambda\mathcal{G}}"] & \\ \mathcal{G} & & \Lambda\mathcal{G} \end{tikzcd} \tag{2}$$

Passing to geometric realizations gives a diagram which is homotopy equivalent to the standard evaluation–projection correspondence for $\mathrm{Map}(S^1,|\mathcal{G}|)$.

Let $[1]:=[\bullet\xrightarrow{1}\bullet]\in C_1(B\mathbb{Z})$, which we view as a fundamental cycle of $|B\mathbb{Z}|\sim S^1$. The composition

$$C_\bullet(\Lambda\mathcal{G})\xrightarrow{[1]\otimes-}C_1(B\mathbb{Z})\otimes_{\mathbb{Z}}C_\bullet(\Lambda\mathcal{G})\xrightarrow{\mathsf{EZ}}C_{\bullet+1}(B\mathbb{Z}\times\Lambda\mathcal{G})\xrightarrow{\mathrm{ev}_*}C_{\bullet+1}(\mathcal{G}) \tag{3}$$

defines the chain-level loop transgression map. Here EZ is the Eilenberg–Zilber shuffle map. Writing $\mathsf{ez}_{[1]}$ for the composition $\mathsf{EZ}\circ([1]\otimes-)$, the pushforward along $\mathrm{pr}_{\Lambda\mathcal{G}}$ is the map

$$\mathrm{pr}_{\Lambda\mathcal{G}!}\colon C^\bullet(B\mathbb{Z}\times\Lambda\mathcal{G})\to C^{\bullet-1}(\Lambda\mathcal{G}),\quad \lambda\mapsto\lambda\circ\mathsf{ez}_{[1]}. \tag{4}$$

The loop transgression map is then

$$\tau\colon C^\bullet(\mathcal{G})\xrightarrow{\mathrm{pr}_{\Lambda\mathcal{G}!}\circ\mathrm{ev}^*}C^{\bullet-1}(\Lambda\mathcal{G}).$$

The map τ anticommutes with the differentials, $d\tau(\lambda)=\tau(d\lambda)^{-1}$, and so descends to a map on cocycles and cohomologies.

To compute τ, let $[g_n\mid\cdots\mid g_1]\gamma\in C_n(\Lambda\mathcal{G})$ be the chain associated to the diagram $\gamma=\gamma_1\xrightarrow{g_1}\gamma_2\xrightarrow{g_2}\cdots\xrightarrow{g_n}\gamma_{n+1}$ in $\Lambda\mathcal{G}$. Then we have

$$\mathsf{ez}_{[1]}\big([g_n\mid\cdots\mid g_1]\gamma\big)=\sum_{i=0}^n(-1)^{n-i}[g_n\mid\cdots\mid g_{i+1}\mid 1\mid g_i\mid\cdots\mid g_1]\gamma_1,$$

where we have introduced the shorthand $g_i=(0,g_i)$ and $1=(1,\mathrm{id}_x)$ for morphisms in $B\mathbb{Z}\times\Lambda\mathcal{G}$. The result of applying the map (3) to $[g_n\mid\cdots\mid g_1]\gamma$ is thus

$$\sum_{i=0}^n(-1)^{(n-i)}[g_n\mid\cdots\mid g_{i+1}\mid\gamma_{i+1}\mid g_i\mid\cdots\mid g_1].$$

Dualizing and passing to A–coefficients gives for $\lambda\in C^{n+1}(\mathcal{G})$ the expression

$$\tau(\lambda)\big([g_n\mid\cdots\mid g_1]\gamma\big)=\prod_{i=0}^n\lambda\big([g_n\mid\cdots\mid g_{i+1}\mid\gamma_{i+1}\mid g_i\mid\cdots\mid g_1]\big)^{(-1)^{n-i}}, \tag{5}$$

which is precisely the result of Willerton [40, Theorem 3].

Willerton's derivation of (5) is rather different from that presented here, relying on a particular homotopy equivalence $|\Lambda\mathcal{G}| \sim \mathrm{Map}(S^1, |\mathcal{G}|)$, the so-called Parmesan map [40, Theorem 2]. The Parmesan map is not equivariant for reflection of the circle, and so is not well suited for the purposes of this paper.

2.2 Twisted pushforwards for groupoids

We begin by generalizing the pushforward map (4) so as to include a twist on the codomain.

Lemma 2.1 *Let $\mathcal{G}$ be an essentially finite groupoid and $\kappa\colon \mathcal{G} \to B\mathrm{Aut}(\mathsf{A})$ a functor. Then the abelian group homomorphism*

$$\mathrm{pr}_{\mathcal{G}!}\colon C^{\bullet+\mathrm{pr}_{\mathcal{G}}^{*}\kappa}(B\mathbb{Z}\times\mathcal{G}) \to C^{\bullet-1+\kappa}(\mathcal{G}), \quad \lambda \mapsto \lambda \circ \mathsf{ez}_{[1]},$$

anticommutes with the differentials.

Proof This follows from the equality $\partial \circ \mathsf{ez}_{[1]} = -\mathsf{ez}_{[1]} \circ \partial$. □

Now let $\widehat{\mathcal{G}}$ be an essentially finite $\mathbb{Z}_2$–graded groupoid. Consider the strict $\mathbb{Z}_2$–action on $B\mathbb{Z}$ which negates morphisms. Then $\mathrm{pr}_{\mathcal{G}}\colon B\mathbb{Z}\times\mathcal{G} \to \mathcal{G}$ is strictly equivariant for the diagonal $\mathbb{Z}_2$–action on $B\mathbb{Z}\times\mathcal{G}$ and so descends to a functor

$$\widetilde{\mathrm{pr}}_{\mathcal{G}}\colon B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G} \to \widehat{\mathcal{G}}.$$

Write $\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}\colon B\mathbb{Z}\times\mathcal{G} \to B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G} := (B\mathbb{Z}\times\mathcal{G})/\!/\mathbb{Z}_2$ for the canonical double cover. The objective of the remainder of this section is to construct a pushforward

$$\widetilde{\mathrm{pr}}_{\mathcal{G}!}\colon C^{\bullet+\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}+\pi^{*}_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}\kappa}(B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}) \to C^{\bullet-1+\kappa}(\widehat{\mathcal{G}}).$$

Note that we have used $\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}$ to denote both a classifying map, viewed as a cochain twist, and its associated double cover, used to pull back κ. In view of our later applications, we consider only the case in which κ is trivial or $\kappa = \pi_{\widehat{\mathcal{G}}}$. In the latter case, $\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}} + \pi^{*}_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}\kappa$ is the trivial twist.

We begin with some notation. Given morphisms $\omega_1, \dots, \omega_n$ in $\widehat{\mathcal{G}}$, set

$$\Delta_{\omega_n,\dots,\omega_1} = \begin{cases} 1 & \text{if } \pi(\omega_n) = \cdots = \pi(\omega_1) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

By convention, $\Delta_{\varnothing} = 1$. For each $\epsilon \in \mathbb{Z}_2$ and $i \geq 1$, let

$$s_i^{\epsilon} = [\epsilon \,|\, -\epsilon \,|\, \cdots \,|\, (-1)^{i+1}\epsilon] \in C_i(B\mathbb{Z}).$$

Define a $\mathbb{Z}$–linear map

$$f_n\colon C_n(\mathcal{G}) \to \bigoplus_{k=1}^{n+1} C_k(B\mathbb{Z}) \otimes_{\mathbb{Z}} C_{n+1-k}(\mathcal{G})$$

by

$$f_n\big([\omega_n \mid \cdots \mid \omega_1]_{\epsilon_1}\big)$$
$$= \epsilon_{n+1} s_1^{\epsilon_{n+1}} \otimes [\omega_n|\cdots|\omega_1]_{\epsilon_1} + \sum_{i=0}^{n-1} (-1)^i \epsilon_{n+1-i} \Delta_{\omega_n,\dots,\omega_{n-i}} s_{i+2}^{\epsilon_{n+1}} \otimes [\omega_{n-1-i}|\cdots|\omega_1]_{\epsilon_1}.$$

For notational simplicity, we often write s_i^j for $s_i^{\epsilon_j}$ and $\Delta_{n,\dots,1}$ for $\Delta_{\omega_n,\dots,\omega_1}$.

We regard $C_\bullet(B\mathbb{Z}) \otimes_{\mathbb{Z}} C_\bullet(\mathcal{G})$ as a $\mathbb{Z}[\mathbb{Z}_2]$–module with its standard tensor product differential. An element of $C_\bullet(B\mathbb{Z}) \otimes_{\mathbb{Z}} C_\bullet(\mathcal{G})$ is called degenerate if it can be written in the form $\sum_i a_i \otimes b_i$, where, for each i, at least one of a_i or b_i is a degenerate chain.

Proposition 2.2 *Let $c \in C_\bullet(\mathcal{G})$. The following equalities hold*:

(i) $f(\zeta \cdot c) = -\zeta \cdot f(c)$.

(ii) $f(\partial c) = -\partial f(c) +$ degenerate elements.

Proof The first statement follows from a direct calculation using (1).

Consider then the second statement. To begin, note that

$$\partial s_i^{\epsilon} = s_{i-1}^{-\epsilon} + (-1)^i s_{i-1}^{\epsilon} + \text{degenerate chains}.$$

Since we are not concerned with degenerate elements, we henceforth omit them from all equalities. Using this, we find that the terms of $\partial f_n\big([\omega_n \mid \cdots \mid \omega_1]\big)$ and $-f_{n-1}(\partial[\omega_n \mid \cdots \mid \omega_1])$ which involve $s_1^{\pm}$ are

$$-\epsilon_{n+1} s_1^{n+1} \otimes \partial[\omega_n \mid \cdots \mid \omega_1] + \epsilon_{n+1} \Delta_n (s_1^{-(n+1)} + s_1^{n+1}) \otimes [\omega_{n-1} \mid \cdots \mid \omega_1]$$

and

$$-\epsilon_n s_1^n \otimes [\omega_{n-1} \mid \cdots \mid \omega_1] - \epsilon_{n+1} \sum_{j=1}^{n-1} (-1)^{n-j} s_1^{n+1} \otimes [\omega_n \mid \cdots \mid \omega_{j+1}\omega_j \mid \cdots \mid \omega_1]$$
$$- \epsilon_{n+1}(-1)^n s_1^{n+1} \otimes [\omega_n \mid \cdots \mid \omega_2],$$

respectively. The term $[\omega_n \mid \cdots \mid \omega_2]$ appears with coefficient $(-1)^{n+1}\epsilon_{n+1} s_1^{n+1}$ in both of these expressions while $[\omega_{n-1} \mid \cdots \mid \omega_1]$ appears with coefficients

$$-\epsilon_{n+1} s_1^{n+1} + \epsilon_{n+1} \Delta_n (s_1^{-(n+1)} + s_1^{n+1}) \quad \text{and} \quad -\epsilon_n s_1^n.$$

When $\Delta_n = 0$, these quantities are plainly equal. When $\Delta_n = 1$, the first becomes

$$\epsilon_n s_1^{-n} - \epsilon_n (s_1^n + s_1^{-n}) = -\epsilon_n s_1^n.$$

Now fix $i \geq 1$. The terms of $\partial f_n\big([\omega_n \mid \cdots \mid \omega_1]\big)$ and $-f_{n-1}(\partial[\omega_n \mid \cdots \mid \omega_1])$ which involve $s_{i+1}^{\pm}$ are

$$(6) \quad (-1)^i \epsilon_{n+1-i} \Delta_{n,\dots,n-i} (s_{i+1}^{-(n+1)} + (-1)^i s_{i+1}^{n+1}) \otimes [\omega_{n-1-i} \mid \cdots \mid \omega_1] \\ + \epsilon_{n+2-i} \Delta_{n,\dots,n+1-i} s_{i+1}^{n+1} \otimes \partial[\omega_{n-i} \mid \cdots \mid \omega_1]$$

and

$$(7) \quad -(-1)^{i-1} \epsilon_{n+1-i} \Delta_{n-1,\dots,n-i} s_{i+1}^{n} \otimes [\omega_{n-1-i} \mid \cdots \mid \omega_1] \\ -(-1)^{n-1} \sum_{j<n-i} (-1)^{j+i} \epsilon_{n-i} \Delta_{n,\dots,n+1-i} s_{i+1}^{n+1} \otimes [\omega_{n-i} \mid \cdots \mid \omega_{j+1}\omega_j \mid \cdots \mid \omega_1] \\ -(-1)^{n-1+i} \epsilon_{j_*+2} \Delta_{n,\dots,(j_*+1)j_*} s_{i+1}^{n+1} \otimes [\omega_{j_*-1} \mid \cdots \mid \omega_1] \\ - \sum_{j>n-i} (-1)^{n-1+j+i} \epsilon_{n+1-i} \Delta_{n,\dots,(j+1)j,\dots,n-i} s_{i+1}^{n+1} \otimes [\omega_{n-i-1} \mid \cdots \mid \omega_1],$$

respectively, where $j_* = n-i$. The coefficients of $[\omega_{n-i} \mid \cdots \mid \omega_{k+1}\omega_k \mid \cdots \mid \omega_1]$ in (6) and (7) are both $(-1)^{n-i-k} \epsilon_{n+2-i} \Delta_{n,\dots,n+1-i} s_{i+1}^{n+1}$ while the coefficients of $[\omega_{n-1-i} \mid \cdots \mid \omega_1]$ are

$$(8) \quad (-1)^i \epsilon_{n+1-i} \Delta_{n,\dots,n-i} (s_{i+1}^{-(n+1)} + (-1)^i s_{i+1}^{n+1}) + \epsilon_{n+2-i} \Delta_{n,\dots,n+1-i} s_{i+1}^{n+1}$$

and

$$(9) \quad -(-1)^{i-1} \epsilon_{n+1-i} \Delta_{n-1,\dots,n-i} s_{i+1}^{n} - (-1)^{n-1+i+j_*} \epsilon_{j_*+2} \Delta_{n,\dots,(j_*+1)j_*} s_{i+1}^{n+1} \\ - \sum_{j>n-i} (-1)^{n-1+j+i} \epsilon_{n+1-i} \Delta_{n,\dots,(j+1)j,\dots,n-i} s_{i+1}^{n+1}.$$

The sum (8) is nonzero in exactly two cases:

- $\Delta_{n,\dots,n-i} = 1$, in which case (8) is

  $$(-1)^i \epsilon_{n+1-i} s_{i+1}^n + (-1)^i \epsilon_{n+1-i} ((-1)^i + (-1)^{i-1}) s_{i+1}^{-n} = (-1)^i \epsilon_{n+1-i} s_{i+1}^n,$$

  which is equal to (9).
- $\Delta_{n,\dots,n+1-i} = 1$ and $\pi(\omega_{n-i}) = 1$, in which case (8) is $\epsilon_{n+2-i} s_{i+1}^{n+1}$, which is equal to (9) (corresponding to the term with $j_* = n-i$).

It remains to consider the case in which (8) is zero. It suffices to assume that exactly one of $\omega_n, \dots, \omega_{n-i}$ has degree $+1$, as otherwise (9) is clearly zero. We can also assume that $\pi(\omega_{n-i})=-1$, the case $\pi(\omega_{n-i})=1$ having been treated above. We need to show that (9) is zero. If $\pi(\omega_n)=1$, then (9) is equal to (take $j=n-1$)

$$-(-1)^{i-1}\epsilon_{n+1-i}s_{i+1}^{n}-(-1)^{n-1+n-1+i}\epsilon_{n+1-i}s_{i+1}^{n+1}=0.$$

In all other cases, (9) is equal to

$$-\sum_{j>n-i}(-1)^{n-1+j+i}\epsilon_{n+1-i}\,\Delta_{n,\dots,(j+1)j,\dots,n-i}s_{i+1}^{n+1}\otimes[\omega_{n-i-1}\mid\cdots\mid\omega_1].$$

This sum vanishes, as its two nonzero terms have consecutive j indices. □

Let $\mathsf{ez}_f\colon C_\bullet(\mathcal{G})\to C_{\bullet+1}(B\mathbb{Z}\times\mathcal{G})$ be the composition $\mathsf{EZ}\circ f$.

Proposition 2.3 *Let $\kappa\colon\hat{\mathcal{G}}\to B\mathbb{Z}_2$ be either the trivial functor or the $\mathbb{Z}_2$–grading $\pi_{\hat{\mathcal{G}}}$. Then the map*

$$\mathrm{Hom}_{\mathbb{Z}}(C_\bullet(B\mathbb{Z}\times\mathcal{G}),\mathsf{A})\to\mathrm{Hom}_{\mathbb{Z}}(C_{\bullet-1}(\mathcal{G}),\mathsf{A}),\quad \hat{\phi}\mapsto\hat{\phi}\circ\mathsf{ez}_f,$$

defines an abelian group homomorphism

$$\widetilde{\mathrm{pr}}_{\mathcal{G}!}\colon C^{\bullet+\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}+\pi^*_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}\kappa}(B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G})\to C^{\bullet-1+\kappa}(\hat{\mathcal{G}})$$

which anticommutes with the differentials.

Proof We consider the case in which κ is trivial; the other case is analogous. Let $\hat{\phi}\in\mathrm{Hom}^{\mathrm{n}}_{\mathbb{Z}[\mathbb{Z}_2]}(C_\bullet(B\mathbb{Z}\times\mathcal{G}),\mathsf{A}_-)$ and $c\in C_{\bullet-1}(\mathcal{G})$. We compute

$$\begin{aligned}\widetilde{\mathrm{pr}}_{\mathcal{G}!}(\hat{\phi})(\zeta\cdot c)&=\hat{\phi}\big(\mathsf{EZ}(f(\zeta\cdot c))\big)=\hat{\phi}\big(\mathsf{EZ}(-\zeta\cdot f(c))\big)=\hat{\phi}\big(-\zeta\cdot\mathsf{EZ}(f(c))\big)\\&=\hat{\phi}\big(\mathsf{EZ}(f(c))\big)=\widetilde{\mathrm{pr}}_{\mathcal{G}!}(\hat{\phi})(c).\end{aligned}$$

The second, third and fourth equalities follow from Proposition 2.2(i), the naturality (and hence $\mathbb{Z}_2$–equivariance) of EZ and the $\mathbb{Z}[\mathbb{Z}_2]$–linearity of $\hat{\phi}$, respectively. Lemma 1.2 therefore implies that we obtain a map $C^{\bullet+\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}}(B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G})\to C^{\bullet-1}(\hat{\mathcal{G}})$.

To see that $\widetilde{\mathrm{pr}}_{\mathcal{G}!}$ anticommutes with the differentials, we compute

$$\begin{aligned}(d\,\widetilde{\mathrm{pr}}_{\mathcal{G}!}(\hat{\phi}))(c)&=\hat{\phi}\big(\mathsf{EZ}(f(\partial c))\big)=\hat{\phi}\big(\mathsf{EZ}(-\partial f(c))\big)=\hat{\phi}\big(-\partial\mathsf{EZ}(f(c))\big)\\&=(d\hat{\phi})\big(\mathsf{EZ}(f(c))\big)^{-1}=\widetilde{\mathrm{pr}}_{\mathcal{G}!}(d\hat{\phi})(c)^{-1}.\end{aligned}$$

The second and third equalities follow from Proposition 2.2(ii), the normalization of $\hat{\phi}$ and the fact that EZ is a chain map which sends degenerate elements to degenerate chains. □

Remark The map $[1]\otimes -\colon C_\bullet(\mathcal{G}) \to C_1(B\mathbb{Z})\otimes_{\mathbb{Z}} C_\bullet(\mathcal{G})$, used to define $\mathrm{pr}_{\mathcal{G}!}$ in Lemma 2.1, satisfies part (ii) of Proposition 2.2 (without working modulo degenerate elements), but not part (i). Indeed, $[-1]$ and $-[1]$ are homologous but not equal.

The map $\widetilde{\mathrm{pr}}_{\mathcal{G}!}$ has the expected functorial properties of a pushforward.

Proposition 2.4 *Consider a diagram of $\mathbb{Z}_2$–graded groupoids*

$$\begin{tikzcd} \widehat{\mathcal{G}} \arrow[rr, "\widehat{F}"] \arrow[dr, "\pi_{\widehat{\mathcal{G}}}"'] & \arrow[d, phantom, "\Leftarrow \eta"] & \widehat{\mathcal{H}} \arrow[dl, "\pi_{\widehat{\mathcal{H}}}"] \\ & B\mathbb{Z}_2 & \end{tikzcd}$$

with induced morphism $F\colon \mathcal{G}\to\mathcal{H}$ of double covers. Then there is a commutative diagram

$$\begin{tikzcd} C^{\bullet+\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{H}}+\pi^*_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{H}}\kappa}(B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{H}) \arrow[r, "\widetilde{\mathrm{pr}}_{\mathcal{H}!}"] \arrow[d, "(\mathrm{id}_{B\mathbb{Z}}\times F)^*"'] & C^{\bullet-1+\kappa}(\widehat{\mathcal{H}}) \arrow[d, "F^*"] \\ C^{\bullet+\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}+\pi^*_{B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}}F^*\kappa}(B\mathbb{Z}\times_{\mathbb{Z}_2}\mathcal{G}) \arrow[r, "\widetilde{\mathrm{pr}}_{\mathcal{G}!}"] & C^{\bullet-1+F^*\kappa}(\widehat{\mathcal{G}}) \end{tikzcd}$$

Proof The natural isomorphism η is a function $\mathrm{Obj}(\widehat{\mathcal{G}})\to\mathbb{Z}_2$ which satisfies

$$\pi_{\widehat{\mathcal{H}}}(F(\omega))\eta_{x_1} = \eta_{x_2}\pi_{\widehat{\mathcal{G}}}(\omega)$$

for all $\omega\colon x_1\to x_2$. The induced functor $F\colon\mathcal{G}\to\mathcal{H}$ is given on a morphism by

$$[(x_1,\epsilon_1)\xrightarrow{\omega}(x_2,\epsilon_2)]\mapsto (F(x_1),\eta_{x_1}\epsilon_1)\xrightarrow{F(\omega)}(F(x_2),\eta_{x_2}\epsilon_2).$$

Direct inspection shows that $(\mathrm{id}_{C_\bullet(B\mathbb{Z})}\otimes F_*)\circ f_{\mathcal{G}} = f_{\mathcal{H}}\circ F_*$. Now use the naturality of EZ and Proposition 2.3. □

2.3 Twisted loop transgression

We define and compute variants of the loop transgression map τ of Section 2.1, producing (possibly twisted) cochains on the (unoriented) quotient loop groupoid $\Lambda_\pi^{(\mathrm{ref})}\widehat{\mathcal{G}}$ from cochains on $\widehat{\mathcal{G}}$.

Let $\pi_{\widehat{\mathcal{G}}}\colon\widehat{\mathcal{G}}\to B\mathbb{Z}_2$ be an essentially finite $\mathbb{Z}_2$–graded groupoid. Consider $B\mathbb{Z}$ with its trivial $\mathbb{Z}_2$–action. Then (2) is a diagram of strictly equivariant functors between groupoids with strict $\mathbb{Z}_2$–actions. It follows that there is a strictly commutative diagram

$$
\begin{tikzcd}
& B\mathbb{Z}\times\Lambda\mathcal{G} \arrow[dl, "\mathrm{ev}"'] \arrow[dr, "\mathrm{pr}_{\Lambda\mathcal{G}}"] \arrow[dd, "\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G}}"'] & \\
\mathcal{G} \arrow[dd, "\pi_{\widehat{\mathcal{G}}}"'] & & \Lambda\mathcal{G} \arrow[dd, "\pi_{\Lambda_\pi\widehat{\mathcal{G}}}"] \\
& B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G} \arrow[dl, "\widetilde{\mathrm{ev}}"] \arrow[dr, "\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}}"'] & \\
\widehat{\mathcal{G}} & & \Lambda_\pi\widehat{\mathcal{G}}
\end{tikzcd}
\tag{10}
$$

whose squares are Cartesian. Under the equivalence $B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G}\simeq B\mathbb{Z}\times\Lambda_\pi\widehat{\mathcal{G}}$, the functor $\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}}$ is identified with $\mathrm{pr}_{\Lambda_\pi\widehat{\mathcal{G}}}$ and $\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G}}$ with $\mathrm{id}_{B\mathbb{Z}}\times\pi_{\Lambda_\pi\widehat{\mathcal{G}}}$.

Definition 2.5 The twisted loop transgression map is the composition

$$
\begin{aligned}
\tau_\pi : C^{\bullet+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}}) \xrightarrow{\widetilde{\mathrm{ev}}^*} C^{\bullet+\widetilde{\mathrm{ev}}^*\pi_{\widehat{\mathcal{G}}}}(B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G}) \simeq C^{\bullet+\widetilde{\mathrm{pr}}^*_{\Lambda\mathcal{G}}\pi_{\Lambda_\pi\widehat{\mathcal{G}}}}(B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G})& \\
\xrightarrow{\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}!}} C^{\bullet-1+\pi_{\Lambda_\pi\widehat{\mathcal{G}}}}(\Lambda_\pi\widehat{\mathcal{G}})&,
\end{aligned}
$$

where the middle isomorphism is constructed using the Cartesian squares of diagram (10) and the final map $\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}!}=\mathrm{pr}_{\Lambda_\pi\widehat{\mathcal{G}}!}$ is that of Lemma 2.1.

By construction, the map τ_π anticommutes with the differentials.

Theorem 2.6 *Let $\widehat{\lambda}\in C^{n+1+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$ and $[\omega_n\,|\cdots|\,\omega_1]\gamma\in C_n(\Lambda_\pi\widehat{\mathcal{G}})$. Then*

$$
\tau_\pi(\widehat{\lambda})\big([\omega_n\,|\cdots|\,\omega_1]\gamma\big)=\prod_{i=0}^{n}\widehat{\lambda}\big([\omega_n\,|\cdots|\,\omega_{i+1}\,|\,\gamma_{i+1}\,|\,\omega_i\,|\cdots|\,\omega_1]\big)^{(-1)^{n-i}}.
$$

Proof Consider the commutative diagram

$$
\begin{tikzcd}
& \mathrm{Hom}^{\mathrm{n}}_{\mathbb{Z}[\mathbb{Z}_2]}(C_\bullet(B\mathbb{Z}\times\Lambda\mathcal{G}),\mathsf{A}_-) \arrow[dr, "\mathrm{pr}_{\Lambda\mathcal{G}!}"] \arrow[dd, "\Psi_-"'] & \\
\mathrm{Hom}^{\mathrm{n}}_{\mathbb{Z}[\mathbb{Z}_2]}(C_\bullet(\mathcal{G}),\mathsf{A}_-) \arrow[ur, "\mathrm{ev}^*"] \arrow[dd, "\Psi_-"'] & & \mathrm{Hom}^{\mathrm{n}}_{\mathbb{Z}[\mathbb{Z}_2]}(C_{\bullet-1}(\Lambda\mathcal{G}),\mathsf{A}_-) \arrow[dd, "\Psi_-"] \\
& C^{\bullet+\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G}}}(B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G}) \arrow[dr, "\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}!}"'] & \\
C^{\bullet+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}}) \arrow[ur, "\widetilde{\mathrm{ev}}^*"'] & & C^{\bullet-1+\pi_{\Lambda_\pi\widehat{\mathcal{G}}}}(\Lambda_\pi\widehat{\mathcal{G}})
\end{tikzcd}
$$

The vertical maps are chain isomorphisms by Lemma 1.2. Setting $\epsilon_1 = +1$, we find

$$\begin{aligned}\tau_\pi(\hat{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big) &= (\Psi_- \circ \mathrm{pr}_{\Lambda\mathcal{G}!} \circ \mathrm{ev}^* \circ \Phi_-)(\hat{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big)\\ &= \Phi_-(\hat{\lambda})\big(\mathrm{ev}_*\mathsf{EZ}\big([1] \otimes [\omega_n \mid \cdots \mid \omega_1](\gamma, \epsilon_{n+1})\big)\big)\\ &= \prod_{i=0}^{n} \hat{\lambda}\big([\omega_n \mid \cdots \mid \omega_{i+1} \mid \gamma_{i+1} \mid \omega_i \mid \cdots \mid \omega_1]\big)^{(-1)^{n-i}}.\end{aligned}$$

The last equality follows from calculations similar to Section 2.1 and the equality $\pi(\omega_{\leq n}) = \epsilon_{n+1}$, which ensures that the sign introduced by Φ_- cancels with ϵ_{n+1}. □

Suppose now that $\mathbb{Z}_2$ acts by negation on $B\mathbb{Z}$. Then $\zeta \in \mathbb{Z}_2$ acts on morphisms in $B\mathbb{Z} \times \Lambda\mathcal{G}$ by

$$\zeta \cdot [((x_1, \epsilon_1), \gamma_1) \xrightarrow{(n,\omega)} ((x_2, \epsilon_2), \gamma_2)] = ((x_1, -\epsilon_1), \gamma_1^{-1}) \xrightarrow{(-n,\omega)} ((x_2, -\epsilon_2), \gamma_2^{-1}).$$

Again, both functors $\mathrm{ev}\colon B\mathbb{Z} \times \Lambda\mathcal{G} \to \mathcal{G}$ and $\mathrm{pr}_{\Lambda\mathcal{G}}\colon B\mathbb{Z} \times \Lambda\mathcal{G} \to \Lambda\mathcal{G}$ are strictly $\mathbb{Z}_2$–equivariant and we obtain a strictly commutative diagram of Cartesian squares similar to (10), but with $\Lambda_\pi\hat{\mathcal{G}}$ replaced by $\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}$. Passing to cochains gives the commutative diagram

$$\begin{tikzcd} & \mathrm{Hom}^{\mathrm{n}}_{\mathbb{Z}[\mathbb{Z}_2]}(C_\bullet(B\mathbb{Z} \times \Lambda\mathcal{G}), \mathsf{A}_-) \arrow[dr, "\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}!}"] \arrow[dd, "\Psi_-"'] & \\ \mathrm{Hom}^{\mathrm{n}}_{\mathbb{Z}[\mathbb{Z}_2]}(C_\bullet(\mathcal{G}), \mathsf{A}_-) \arrow[ur, "\mathrm{ev}^*"] \arrow[dd, "\Psi_-"'] & & \mathrm{Hom}^{\mathrm{n}}_{\mathbb{Z}[\mathbb{Z}_2]}(C_{\bullet-1}(\Lambda\mathcal{G}), \mathsf{A}) \arrow[dd, "\Psi"] \\ & C^{\bullet+\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G}}}(B\mathbb{Z} \times_{\mathbb{Z}_2} \Lambda\mathcal{G}) \arrow[dr, "\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}!}"'] & \\ C^{\bullet+\pi_{\hat{\mathcal{G}}}}(\hat{\mathcal{G}}) \arrow[ur, "\widetilde{\mathrm{ev}}^*"'] & & C^{\bullet-1}(\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}) \end{tikzcd} \tag{11}$$

By Lemma 1.2, the vertical arrows are isomorphisms. Using Proposition 2.3, we make the following definition:

Definition 2.7 The reflection twisted loop transgression map τ_π^{ref} is the composition

$$\tau_\pi^{\mathrm{ref}}\colon C^{\bullet+\pi_{\mathcal{G}}}(\hat{\mathcal{G}}) \xrightarrow{\widetilde{\mathrm{ev}}^*} C^{\bullet+\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G}}}(B\mathbb{Z} \times_{\mathbb{Z}_2} \Lambda\mathcal{G}) \xrightarrow{\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}!}} C^{\bullet-1}(\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}),$$

where $\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}!}$ is the map of Proposition 2.3 with trivial twist κ.

By construction, τ_π^{ref} anticommutes with the differentials.

To compute τ_π^{ref}, we introduce some notation. For each $1 \le i \le n+1$, let $\mathfrak{S}_{i,n+1} \subset \mathfrak{S}_{n+1}$ be the subset of $(i, n+1-i)$–shuffles. Given $\mathfrak{s} \in \mathfrak{S}_{i,n+1}$, denote by $\mathfrak{s} \cdot [\omega_n \mid \cdots \mid \omega_1]\gamma$ the $(n+1)$–simplex of $\widehat{\mathcal{G}}$ whose $\mathfrak{s}(j)^{\text{th}}$ entry for $1 \le j \le i$ is $\gamma_{\mathfrak{s}(j)-j}^{(-1)^{j+1}\pi(\omega_{\le n})}$ and whose remaining entries are $\omega_{n-i+1}, \dots, \omega_1$, with ω_{k+1} appearing after ω_k. In symbols,

$$\mathfrak{s} \cdot [\omega_n \mid \cdots \mid \omega_1]\gamma = [\cdots \mid \gamma_{\mathfrak{s}(2)-2}^{-\pi(\omega_{\le n})} \mid \omega_{\mathfrak{s}(2)-2} \mid \cdots \mid \omega_{\mathfrak{s}(1)} \mid \gamma_{\mathfrak{s}(1)-1}^{\pi(\omega_{\le n})} \mid \omega_{\mathfrak{s}(1)-1} \mid \cdots \mid \omega_1].$$

Given $\widehat{\lambda} \in C^{n+1+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$, put

$$\mathsf{sh}_i(\widehat{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big) := \prod_{\mathfrak{s} \in \mathfrak{S}_{i,n+1}} \widehat{\lambda}(\mathfrak{s} \cdot [\omega_n \mid \cdots \mid \omega_1]\gamma)^{\mathrm{sgn}(\mathfrak{s})}.$$

The map τ_π^{ref} can now be computed as follows:

Theorem 2.8 *Let $\widehat{\lambda} \in C^{n+1+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$ and $[\omega_n \mid \cdots \mid \omega_1]\gamma \in C_n(\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}})$. Then*

$$\tau_\pi^{\mathrm{ref}}(\widehat{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big) = \prod_{j=0}^{n} \big(\mathsf{sh}_{n+1-j}(\widehat{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big)\big)^{(-1)^{n+j}\Delta_{n,\dots,n-j}}. \tag{12}$$

Proof Setting $\epsilon_1 = +1$, we have

$$\begin{aligned}\tau_\pi^{\mathrm{ref}}(\widehat{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big) &= (\Psi \circ \widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}^!} \circ \mathrm{ev}^* \circ \Phi_-)(\widehat{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big) \\ &= \Phi_-(\widehat{\lambda})\big(\mathrm{ev}_*\mathsf{EZ}\big(f_n\big([\omega_n \mid \cdots \mid \omega_1](\gamma, \epsilon_1)\big)\big)\big),\end{aligned}$$

where

$$\begin{aligned}f_n\big([\omega_n|\cdots|\omega_1](\gamma,\epsilon_1)\big) = \epsilon_{n+1} s_1^{n+1} &\otimes [\omega_n|\cdots|\omega_1](\gamma,\epsilon_1) \\ &+ \sum_{i=0}^{n-1} (-1)^i \epsilon_{n+1-i}\,\Delta_{n,\dots,n-i}\, s_{i+2}^{n+1} \otimes [\omega_{n-1-i}|\cdots|\omega_1](\gamma,\epsilon_1).\end{aligned}$$

The definition of Φ_- shows that the first term of the right-hand side contributes to $\tau_\pi^{\mathrm{ref}}(\widehat{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big)$ with an overall sign of $\epsilon_{n+1}^2 = 1$, yielding the $j = n$ factor of the product (12), while the i^{th} term of the sum contributes, if at all, with an overall sign of

$$(-1)^i \epsilon_{n+1-i}\epsilon_{n-i} = (-1)^i \pi(\omega_{n-i}) = (-1)^{i+1}.$$

This gives the $j = n-1-i$ factor of the product (12). In each of these statements we have used that the map sh is defined exactly so that it realizes the composition $\mathrm{ev}_* \circ \mathsf{EZ}$. □

For example, when $\widehat{\alpha} \in C^{1+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$, Theorem 2.8 gives $\tau_\pi^{\mathrm{ref}}(\widehat{\alpha})([\cdot]\gamma) = \widehat{\alpha}([\gamma])$. If instead $\widehat{\theta} \in C^{2+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$ and $\widehat{\eta} \in C^{3+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$, then

$$\tau_\pi^{\mathrm{ref}}(\widehat{\theta})([\omega]\gamma) = \widehat{\theta}\big([\gamma^{-1} \mid \gamma]\big)^{-\Delta_\omega} \frac{\widehat{\theta}\big([\omega\gamma^{\pi(\omega)}\omega^{-1} \mid \omega]\big)}{\widehat{\theta}\big([\omega \mid \gamma^{\pi(\omega)}]\big)}$$

while $\tau_\pi^{\mathrm{ref}}(\widehat{\eta})\big([\omega_2 \mid \omega_1]\gamma\big)$ is equal to

$$\begin{aligned}
&\widehat{\eta}\big([\gamma \mid \gamma^{-1} \mid \gamma]\big)^{\Delta_{\omega_2,\omega_1}} \\
&\quad\times \left(\frac{\widehat{\eta}\big([\omega_1\gamma^{-\pi(\omega_1)}\omega_1^{-1} \mid \omega_1\gamma^{\pi(\omega_1)}\omega_1^{-1} \mid \omega_1]\big)\widehat{\eta}\big([\omega_1 \mid \gamma^{-\pi(\omega_1)} \mid \gamma^{\pi(\omega_1)}]\big)}{\widehat{\eta}\big([\omega_1\gamma^{-\pi(\omega_1)}\omega_1^{-1} \mid \omega_1 \mid \gamma^{\pi(\omega_1)}]\big)}\right)^{-\Delta_{\omega_2}} \\
&\quad\times \frac{\widehat{\eta}\big([\omega_2 \mid \omega_1 \mid \gamma^{\pi(\omega_2\omega_1)}]\big)\widehat{\eta}\big([\omega_2\omega_1\gamma^{\pi(\omega_2\omega_1)}\omega_1^{-1}\omega_2^{-1} \mid \omega_2 \mid \omega_1]\big)}{\widehat{\eta}\big([\omega_2 \mid \omega_1\gamma^{\pi(\omega_2\omega_1)}\omega_1^{-1} \mid \omega_1]\big)}.
\end{aligned}$$

It follows immediately from Theorem 2.8 that there is a commutative diagram

$$\begin{tikzcd}
C^{\bullet+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}}) \arrow[r, "\tau_\pi^{\mathrm{ref}}"] \arrow[d, "\pi^*_{\widehat{\mathcal{G}}}"'] & C^{\bullet-1}(\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}) \arrow[d, "\pi^*_{\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}}"] \\
C^\bullet(\mathcal{G}) \arrow[r, "\tau"'] & C^{\bullet-1}(\Lambda\mathcal{G})
\end{tikzcd} \tag{13}$$

This allows us to interpret the terms involving $\Delta_{n,\dots,n-j}$ for $1 \le j \le n$ in (12) as corrections to Willerton's expression (5) which take into account the failure of the map $\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}}$ to be orientable.

Continuing, we define a third twisted loop transgression map.

Definition 2.9 The reflection twisted loop transgression map $\widetilde{\tau}_\pi^{\mathrm{ref}}$ is the composition

$$\begin{aligned}
\widetilde{\tau}_\pi^{\mathrm{ref}} \colon C^\bullet(\widehat{\mathcal{G}}) \xrightarrow{\widetilde{\mathrm{ev}}^*} C^\bullet(B\mathbb{Z} \times_{\mathbb{Z}_2} \Lambda\mathcal{G}) \simeq C^{\bullet+2\pi_{B\mathbb{Z}\times_{\mathbb{Z}_2}\Lambda\mathcal{G}}}(B\mathbb{Z} \times_{\mathbb{Z}_2} \Lambda\mathcal{G}) \\
\xrightarrow{\widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}!}} C^{\bullet-1+\pi_{\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}}}(\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}),
\end{aligned}$$

where the final map is that of Proposition 2.3 with $\kappa = \pi_{\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}}$.

Again, $\widetilde{\tau}_\pi^{\mathrm{ref}}$ anticommutes with the differentials.

Theorem 2.10 *Let $\widetilde{\lambda} \in C^{n+1}(\widehat{\mathcal{G}})$ and $[\omega_n \mid \cdots \mid \omega_1]\gamma \in C_n(\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}})$. Then*

$$\widetilde{\tau}_\pi^{\mathrm{ref}}(\widetilde{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big) = \prod_{j=0}^{n} \big(\mathsf{sh}_{n+1-j}(\widetilde{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big)\big)^{\Delta_{n,\dots,n-j}}.$$

Proof Set $\epsilon_1 = +1$. The obvious analogue of diagram (11) gives

$$\begin{aligned}\widetilde{\tau}_{\pi}^{\mathrm{ref}}(\widetilde{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big) &= (\Psi_- \circ \widetilde{\mathrm{pr}}_{\Lambda\mathcal{G}!} \circ \mathrm{ev}^* \circ \Phi)(\widetilde{\lambda})\big([\omega_n \mid \cdots \mid \omega_1]\gamma\big) \\ &= \Phi(\widetilde{\lambda})\big(\mathrm{ev}_* \mathsf{EZ}\big(f_n\big([\omega_n \mid \cdots \mid \omega_1](\gamma, \epsilon_{n+1})\big)\big)\big),\end{aligned}$$

where, because of the initial object (γ, ϵ_{n+1}), we have

$$\begin{aligned}&f_n\big([\omega_n \mid \cdots \mid \omega_1](\gamma, \epsilon_{n+1})\big) \\ &\quad = s_1^1 \otimes [\omega_n \mid \cdots \mid \omega_1](\gamma, \epsilon_{n+1}) \\ &\qquad + \sum_{i=0}^{n-1} (-1)^i \pi(\omega_{\geq n-i+1}) \Delta_{n,\dots,n-i}\, s_{i+2}^{\pi(\omega_{\geq n-i+1})} \otimes [\omega_{n-1-i} \mid \cdots \mid \omega_1](\gamma, \epsilon_{n+1}).\end{aligned}$$

After noting that if $\Delta_{n,\dots,n-i} \neq 0$ then $\pi(\omega_{\geq n-i+1}) = (-1)^i$, the remainder of the proof is similar to that of Theorem 2.8. $\square$

For example, when $\widetilde{\theta} \in Z^2(\widehat{\mathcal{G}})$, Theorem 2.10 gives

$$\widetilde{\tau}_{\pi}^{\mathrm{ref}}(\widetilde{\theta})([\omega]\gamma) = \widetilde{\theta}\big([\gamma^{-1} \mid \gamma]\big)^{\Delta_\omega} \frac{\widetilde{\theta}\big([\omega\gamma^{\pi(\omega)}\omega^{-1} \mid \omega]\big)}{\widetilde{\theta}\big([\omega \mid \gamma^{\pi(\omega)}]\big)}.$$

3 Jandl twisted vector bundles

Throughout this section, $\mathcal{G}$ is an essentially finite groupoid and $\widehat{\mathcal{G}}$ is an essentially finite groupoid over $B\mathbb{Z}_2$. If $\widehat{\mathcal{G}}$ is the object of interest, then $\mathcal{G}$ is its associated double cover. The coefficient group is $\mathsf{A} = \mathsf{U}(1)$.

Given a complex vector space V, let $\overline{V}$ be its complex conjugate. Set ${}^{+1}V := V$ and ${}^{-1}V := \overline{V}$, with similar notation ${}^{\epsilon}z$ for $z \in \mathbb{C}$. A map $\varphi \colon V \to W$ between complex vector spaces is called ϵ–linear if $\varphi \colon {}^{\epsilon}V \to W$ is $\mathbb{C}$–linear.

3.1 Real functions and Real line bundles

Following Willerton [40], we describe simple geometric interpretations of $Z^{0+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$ and $Z^{1+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$.

A 0–cocycle $\beta \in Z^0(\mathcal{G})$ is a locally constant $\mathsf{U}(1)$–valued function on (the objects of) $\mathcal{G}$. The integral of β is

$$\int_{\mathcal{G}} \beta := \sum_{x \in \mathcal{G}} \frac{\beta(x)}{|x \to|} = \sum_{x \in \pi_0(\mathcal{G})} \frac{\beta(x)}{|\mathrm{Aut}_{\mathcal{G}}(x)|},$$

where $|x \to|$ is the number of morphisms in $\mathcal{G}$ with source x. The equality follows from the closedness of β.

Similarly, $\hat{\beta} \in Z^{0+\pi_{\hat{\mathcal{G}}}}(\hat{\mathcal{G}})$ is a Real $\mathsf{U}(1)$–valued function on $\hat{\mathcal{G}}$, that is, a $\mathsf{U}(1)$–valued function which satisfies $\hat{\beta}(x_2) = \hat{\beta}(x_1)^{\pi(\omega)}$ for each morphism $\omega\colon x_1 \to x_2$.

As explained in [40, Section 2.2], a 1–cocycle $\alpha \in Z^1(\mathcal{G})$ defines a trivialized flat complex line bundle $\alpha_{\mathbb{C}}$ over $\mathcal{G}$. This is the data of complex lines $L_x = \mathbb{C}$ for $x \in \mathcal{G}$ and linear (multiplication) maps

$$\alpha(x_1 \xrightarrow{g} x_2)\colon L_{x_1} \to L_{x_2}, \quad g \in \mathrm{Mor}(\mathcal{G}),$$

which satisfy the obvious associativity constraints. Note that $\alpha_{\mathbb{C}}$ is also the associated complex line bundle of a $\mathsf{U}(1)$–bundle with connection on $\mathcal{G}$ determined by α. Similar comments apply below. Flat sections of $\alpha_{\mathbb{C}}$, that is, collections of complex numbers $\{s_x \in L_x\}_{x\in\mathcal{G}}$ satisfying $\alpha(x_1 \xrightarrow{g} x_2)s_{x_1} = s_{x_2}$ for $g \in \mathrm{Mor}(\mathcal{G})$, form a complex vector space $\Gamma_{\mathcal{G}}(\alpha_{\mathbb{C}})$. Given $s_1, s_2 \in \Gamma_{\mathcal{G}}(\alpha_{\mathbb{C}})$, the fibrewise product $\bar{s}_1 s_2$ is in $Z^0(\mathcal{G})$. Using this observation, define an inner product on $\Gamma_{\mathcal{G}}(\alpha_{\mathbb{C}})$ by

$$\langle s_1, s_2 \rangle = \int_{\mathcal{G}} \bar{s}_1 s_2.$$

Similarly, $\hat{\alpha} \in Z^{1+\pi_{\hat{\mathcal{G}}}}(\hat{\mathcal{G}})$ defines a trivialized flat Real line bundle $\hat{\alpha}_{\mathbb{C}}$ over $\hat{\mathcal{G}}$. This is the data of trivialized complex lines L_x for $x \in \hat{\mathcal{G}}$ and linear maps

$$\hat{\alpha}(x_1 \xrightarrow{\omega} x_2)\colon {}^{\pi(\omega)}L_{x_1} \to L_{x_2}, \quad \omega \in \mathrm{Mor}(\hat{\mathcal{G}}),$$

which satisfy the associativity condition

$$\hat{\alpha}(x_1 \xrightarrow{\omega_2\omega_1} x_3) = \hat{\alpha}(x_2 \xrightarrow{\omega_2} x_3) \cdot {}^{\pi(\omega_2)}\hat{\alpha}(x_1 \xrightarrow{\omega_1} x_2).$$

A flat section of $\hat{\alpha}_{\mathbb{C}}$ is a collection of complex numbers $\{s_x \in L_x\}_{x\in\hat{\mathcal{G}}}$ which satisfies

$$\hat{\alpha}(x_1 \xrightarrow{\omega} x_2)({}^{\pi(\omega)}s_{x_1}) = s_{x_2}, \quad \omega \in \mathrm{Mor}(\hat{\mathcal{G}}).$$

Flat sections of $\hat{\alpha}_{\mathbb{C}}$ form a real inner product space $\Gamma_{\hat{\mathcal{G}}}(\hat{\alpha}_{\mathbb{C}})$ with $\langle s_1, s_2\rangle = \int_{\hat{\mathcal{G}}} \bar{s}_1 s_2$.

Proposition 3.1 *For each $\hat{\alpha} \in Z^{1+\pi_{\hat{\mathcal{G}}}}(\hat{\mathcal{G}})$, there is an equality*

$$\tfrac{1}{2}\dim_{\mathbb{R}} \Gamma_{\hat{\mathcal{G}}}(\hat{\alpha}_{\mathbb{C}}) = \int_{\Lambda_\pi^{\mathrm{ref}}\hat{\mathcal{G}}} \uptau_\pi^{\mathrm{ref}}(\hat{\alpha}).$$

Proof As both sides of the claimed equality are additive with respect to disjoint union and equivalence of groupoids over $B\mathbb{Z}_2$, it suffices to consider the model cases of

Proposition 1.1. When $\widehat{\mathcal{G}} = B\widehat{\mathsf{G}}$, a section $s \in \mathbb{C} \setminus 0$ of $\widehat{\alpha}_{\mathbb{C}}$ is flat if and only if $\widehat{\alpha}([g])s = s$ for $g \in \mathsf{G}$ and $\widehat{\alpha}([\omega])\bar{s} = s$ for $\omega \in \widehat{\mathsf{G}} \setminus \mathsf{G}$. The first condition implies $\widehat{\alpha}|_{\mathsf{G}} = 1$. The cocycle condition implies that $\widehat{\alpha}|_{\widehat{\mathsf{G}}\setminus\mathsf{G}}$ is constant. The second condition is

$$\operatorname{Arg}(s) \equiv \tfrac{1}{2}\operatorname{Arg}(\widehat{\alpha}|_{\widehat{\mathsf{G}}\setminus\mathsf{G}}) \mod \pi\mathbb{Z}.$$

It follows that $\Gamma_{B\widehat{\mathsf{G}}}(\widehat{\alpha}_{\mathbb{C}})$ is $\{0\}$ unless $\widehat{\alpha}|_{\mathsf{G}} = 1$, in which case $\Gamma_{B\widehat{\mathsf{G}}}(\widehat{\alpha}_{\mathbb{C}}) \simeq \mathbb{R}$. On the other hand, Theorem 2.8 gives

$$\int_{\Lambda_\pi^{\mathrm{ref}} B\widehat{\mathsf{G}}} \tau_\pi^{\mathrm{ref}}(\widehat{\alpha}) = \sum_{\gamma\in\mathsf{G}} \frac{\widehat{\alpha}([\gamma])}{2|\mathsf{G}|}.$$

As in [40, Theorem 6], the sum is zero unless $\widehat{\alpha}|_{\mathsf{G}} = 1$, in which case it is $\frac{1}{2}$.

When $\widehat{\mathcal{G}} = \widehat{\mathcal{G}}_{\{x,\bar{x}\}}$, a flat section of $\widehat{\alpha}_{\mathbb{C}}$ consists of $s_x, s_{\bar{x}} \in \mathbb{C}$. Now $\widehat{\alpha}(x \xrightarrow{\omega} \bar{x})\bar{s}_x = s_{\bar{x}}$ for $\pi(\omega) = -1$ implies that, if the section is nonzero, both s_x and $s_{\bar{x}}$ are nonzero. Hence, $\widehat{\alpha}$ is the identity on degree 1 morphisms and constant on degree -1 morphisms and $\Gamma_{\widehat{\mathcal{G}}}(\widehat{\alpha}_{\mathbb{C}}) \simeq \mathbb{C}$. On the other hand,

$$\int_{\Lambda_\pi^{\mathrm{ref}}\widehat{\mathcal{G}}} \tau_\pi^{\mathrm{ref}}(\widehat{\alpha}) = \sum_{\gamma\in\operatorname{Aut}^1_{\widehat{\mathcal{G}}}(x)} \frac{\widehat{\alpha}([\gamma])}{2|\operatorname{Aut}^1_{\widehat{\mathcal{G}}}(x)|} + \sum_{\gamma\in\operatorname{Aut}^1_{\widehat{\mathcal{G}}}(\bar{x})} \frac{\widehat{\alpha}([\gamma])}{2|\operatorname{Aut}^1_{\widehat{\mathcal{G}}}(\bar{x})|}$$

is zero unless $\widehat{\alpha}$ is the identity on degree 1 morphisms, in which case it is one. $\square$

Remark (i) As an alternative proof of Proposition 3.1, it can be shown that $\Gamma_{\widehat{\mathcal{G}}}(\widehat{\alpha}_{\mathbb{C}})$ defines a real structure on $\Gamma_{\mathcal{G}}(\alpha_{\mathbb{C}})$, in that $\Gamma_{\widehat{\mathcal{G}}}(\widehat{\alpha}_{\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \Gamma_{\mathcal{G}}(\alpha_{\mathbb{C}})$. It follows that $\dim_{\mathbb{R}} \Gamma_{\widehat{\mathcal{G}}}(\widehat{\alpha}_{\mathbb{C}}) = \dim_{\mathbb{C}} \Gamma_{\mathcal{G}}(\alpha_{\mathbb{C}})$, the right-hand side of which is computed in [40, Theorem 6]. This strategy is used in Proposition 3.19 below.

(ii) For later comparison, observe that Proposition 3.1 can also be stated as the equality $\frac{1}{2}\dim_{\mathbb{R}} \Gamma_{\widehat{\mathcal{G}}}(\widehat{\alpha}_{\mathbb{C}}) = \int_{\Lambda_\pi\widehat{\mathcal{G}}} \tau_\pi(\widehat{\alpha})^{-1}$. The freedom to use $\tau_\pi(\widehat{\alpha})^{-1}$ or $\tau_\pi^{\mathrm{ref}}(\widehat{\alpha})$ is an artefact of the low cohomological degree of $\widehat{\alpha}$.

3.2 Jandl twisted vector bundles

We explain how elements of $Z^{2+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$ can be used to twist complex vector bundles over $\widehat{\mathcal{G}}$, following the non-Real [40] and continuous cases [35; 23; 32; 21]. Since our groupoids are essentially finite, we are able to give a simplified treatment.

Let $\widehat{\theta} \in Z^{2+\pi_{\widehat{\mathcal{G}}}}(\widehat{\mathcal{G}})$. Define a groupoid ${}^{\widehat{\theta}}\widehat{\mathcal{G}}$ by

$$\operatorname{Obj}({}^{\widehat{\theta}}\widehat{\mathcal{G}}) = \operatorname{Obj}(\widehat{\mathcal{G}}), \quad \operatorname{Hom}_{{}^{\widehat{\theta}}\widehat{\mathcal{G}}}(x,y) = \mathsf{U}(1) \times \operatorname{Hom}_{\widehat{\mathcal{G}}}(x,y).$$

Morphisms in ${}^{\hat{\theta}}\hat{\mathcal{G}}$ are composed according to the rule

$$(a_2,\omega_2)\circ(a_1,\omega_1)=\big(\hat{\theta}\big([\omega_2\,|\,\omega_1]\big)a_2a_1^{\pi(\omega_2)},\omega_2\omega_1\big).$$

Associativity of composition follows from the cocycle condition on $\hat{\theta}$. The category ${}^{\hat{\theta}}\hat{\mathcal{G}}$ is a twisted central extension of $\hat{\mathcal{G}}$ by $B\mathsf{U}(1)$. Compare with [21, Section 1.2]. More precisely, there are canonical functors

$$\mathsf{Obj}(\hat{\mathcal{G}})\times B\mathsf{U}(1)\xrightarrow{i}{}^{\hat{\theta}}\hat{\mathcal{G}}\xrightarrow{p}\hat{\mathcal{G}}$$

with p surjective on objects and full and i an isomorphism onto the subgroupoid of morphisms which map to an identity in $\hat{\mathcal{G}}$. The twisted centrality condition is

$$\omega\circ i(x_1\xrightarrow{a}x_1)=i(x_2\xrightarrow{a^{\pi(\omega)}}x_2)\circ\omega,$$

where $a\in\mathsf{U}(1)$ and $\omega\colon x_1\to x_2$ in $\hat{\mathcal{G}}$. Following the geometric case [35], a twisted central extension of $\hat{\mathcal{G}}$ by $B\mathsf{U}(1)$ is called a Jandl gerbe over $\hat{\mathcal{G}}$. In the language of [32] (which would also be consistent with that of Section 3.1), this is a Real gerbe. By choosing sections of p, we verify that any Jandl gerbe over $\hat{\mathcal{G}}$ is equivalent to ${}^{\hat{\theta}}\hat{\mathcal{G}}$ for some $\hat{\theta}\in Z^{2+\pi_{\hat{\mathcal{G}}}}(\hat{\mathcal{G}})$. Moreover, such a $\hat{\theta}$ is well defined up to exact 2–cocycles. In this way, $H^{2+\pi_{\hat{\mathcal{G}}}}(\hat{\mathcal{G}})$ classifies equivalence classes of Jandl gerbes over $\hat{\mathcal{G}}$.

Consider $\mathsf{Vect}_{\mathbb{C}}$ as the defining 2–representation of $B\mathsf{U}(1)$. The (antilinear) complex conjugation functor $\mathsf{Vect}_{\mathbb{C}}\to\mathsf{Vect}_{\mathbb{C}}$ is compatible with the complex conjugation action on $B\mathsf{U}(1)$. We can therefore associate to ${}^{\hat{\theta}}\hat{\mathcal{G}}$ a Real 2–line bundle $p\colon\hat{\theta}_{\mathbb{C}}\to\hat{\mathcal{G}}$ as follows. Let $\mathsf{RVect}_{\mathbb{C}}$ be the category of finite-dimensional complex vector spaces and their complex linear or antilinear maps. There is a natural functor $\mathsf{RVect}_{\mathbb{C}}\to B\mathbb{Z}_2$ which records the linearity of morphisms. Then $\hat{\theta}_{\mathbb{C}}$ is the category with objects $\mathsf{Obj}(\mathsf{RVect}_{\mathbb{C}})\times\mathsf{Obj}(\hat{\mathcal{G}})$, morphisms

$$\begin{aligned}&\mathrm{Hom}_{\hat{\theta}_{\mathbb{C}}}((V_1,x_1),(V_2,x_2))\\&\qquad=\{(\varphi,\omega)\in\mathrm{Hom}_{\mathsf{RVect}_{\mathbb{C}}}(V_1,V_2)\times\mathrm{Hom}_{\hat{\mathcal{G}}}(x_1,x_2)\mid\varphi\text{ is }\pi(\omega)\text{–linear}\}\end{aligned}$$

and composition law

$$(\varphi_2,\omega_2)\circ(\varphi_1,\omega_1)=(\hat{\theta}\big([\omega_2\,|\,\omega_1]\big)\varphi_2\varphi_1,\omega_2\omega_1).$$

The functor $p\colon\hat{\theta}_{\mathbb{C}}\to\hat{\mathcal{G}}$ sends an object (V,x) to x and a morphism (φ,ω) to ω.

Definition 3.2 A $\hat{\theta}$–twisted vector bundle over $\hat{\mathcal{G}}$ is a functor $F\colon{}^{\hat{\theta}}\hat{\mathcal{G}}\to\mathsf{RVect}_{\mathbb{C}}$ over $B\mathbb{Z}_2$ such that, for each $(a,x)\in\mathsf{U}(1)\times\hat{\mathcal{G}}$, the map $F(a,\mathrm{id}_x)\colon F(x)\to F(x)$ is multiplication by a.

Let $\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})$ be the category of $\hat{\theta}$–twisted vector bundles over $\hat{\mathcal{G}}$ and their $\mathbb{C}$–linear natural transformations. The category $\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})$ is $\mathbb{R}$–linear and additive.

Lemma 3.3 *An object $F \in \mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})$ is equivalent to each of the following data*:

(i) *Complex vector spaces $F(x)$ for $x \in \hat{\mathcal{G}}$ together with complex linear maps*

$$F(x_1 \xrightarrow{\omega} x_2)\colon {}^{\pi(\omega)}F(x_1) \to F(x_2), \quad \omega \in \mathrm{Mor}(\hat{\mathcal{G}}),$$

which satisfy $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ for $x \in \hat{\mathcal{G}}$ and

$$F(\omega_2)F(\omega_1) = \hat{\theta}\big([\omega_2 \mid \omega_1]\big)F(\omega_2\omega_1), \quad (\omega_1, \omega_2) \in \mathrm{Mor}^{(2)}(\hat{\mathcal{G}}), \tag{14}$$

where $\mathrm{Mor}^{(2)}(\hat{\mathcal{G}})$ is the set pairs of composable morphisms in $\hat{\mathcal{G}}$.

(ii) *A section $F\colon \hat{\mathcal{G}} \to \hat{\theta}_{\mathbb{C}}$ of $p\colon \hat{\theta}_{\mathbb{C}} \to \hat{\mathcal{G}}$.*

Proof That the first data is equivalent to a $\hat{\theta}$–twisted vector bundle is a direct verification. The second data is clearly equivalent to the first. □

In view of Lemma 3.3(ii), we can regard $\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})$ as the category of sections of $p\colon \hat{\theta}_{\mathbb{C}} \to \hat{\mathcal{G}}$.

Example Let $\hat{\mathsf{G}} = \mathsf{G} \times \mathbb{Z}_2$ with π the projection. Then $\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(B\hat{\mathsf{G}})$ is equivalent to the category of real (resp. quaternionic) representations of G when $\hat{\theta} = 1$ (resp. $\hat{\theta}\big([\omega_2 \mid \omega_1]\big) = e^{\pi i \Delta_{\omega_2,\omega_1}}$). Similarly, $\mathsf{Vect}^{\hat{\theta}=1}_{\mathbb{C}}(B\hat{\mathsf{G}})$ consists of Real representations of G (with respect to $\mathbb{Z}_2$–graded group $\hat{\mathsf{G}}$), in the sense of Atiyah and Segal [2] and Karoubi [27]. In general, $\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(B\hat{\mathsf{G}})$ consists of $\hat{\theta}$–projective Real representations of G. For this reason, we often refer to $\hat{\theta}$–twisted vector bundles as $\hat{\theta}$–twisted representations.

To give a module-theoretic interpretation of $\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})$, let $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$ be the complex vector space with basis $\{l_\omega\}_{\omega \in \mathrm{Mor}(\hat{\mathcal{G}})}$ and associative $\mathbb{C}$–semilinear product

$$(c_2 l_{\omega_2}) \cdot (c_1 l_{\omega_1}) = \begin{cases} c_2({}^{\pi(\omega_2)}c_1)\hat{\theta}\big([\omega_2 \mid \omega_1]\big) l_{\omega_2\omega_1} & \text{if } (\omega_1, \omega_2) \in \mathrm{Mor}^{(2)}(\hat{\mathcal{G}}), \\ 0 & \text{otherwise.} \end{cases}$$

We call $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$ the $\hat{\theta}$–twisted Real groupoid algebra of $\hat{\mathcal{G}}$. We define a Real representation of $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$ to be a complex vector space V together with a $\mathbb{C}$–linear map $\rho\colon \mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}] \to \mathrm{End}_{\mathbb{R}}(V)$ which is an $\mathbb{R}$–algebra homomorphism and is such that $\rho(l_\omega)$ is $\pi(\omega)$–linear. The collection of Real representations and their $\mathbb{C}$–linear $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$–module homomorphisms form an $\mathbb{R}$–linear category $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]\text{–mod}^{\mathrm{R}}$.

Proposition 3.4 *There is an $\mathbb{R}$–linear equivalence* $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}}) \simeq \mathbb{C}^{\widehat{\theta}}[\widehat{\mathcal{G}}]\text{–mod}^{\mathrm{R}}$.

Proof This follows from Lemma 3.3(i). □

Proposition 3.5 (see [40, Theorem 18]) *Fix an equivalence as in Proposition 1.1. The restriction along this equivalence defines an equivalence*

$$\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}}) \xrightarrow{\sim} \bigoplus_{x\in\pi_0(\widehat{\mathcal{G}})_{-1}} \mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}|_x}(B\operatorname{Aut}_{\widehat{\mathcal{G}}}(x)) \oplus \bigoplus_{x\in\pi_0(\widehat{\mathcal{G}})_1} \mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}|_{\{x,\bar{x}\}}}(\widehat{\mathcal{G}}_{\{x,\bar{x}\}}).$$

Proof As in Proposition 3.1, it suffices to prove the statement for connected $\widehat{\mathcal{G}}$. Suppose that $\pi_0(\widehat{\mathcal{G}}) = \pi_0(\widehat{\mathcal{G}})_{-1}$ and let $\rho \in \mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}|_x}(B\operatorname{Aut}_{\widehat{\mathcal{G}}}(x))$ with fibre V_x. For each $y \in \widehat{\mathcal{G}}$, fix a degree 1 morphism $g_y\colon y \to x$ and define a $F_\rho \in \mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}})$ by $F_\rho(y) = V_x$ for $y \in \widehat{\mathcal{G}}$ and (see [40, Section 2.4.1])

$$F_\rho(y \xrightarrow{\omega} z) = \frac{\widehat{\theta}([g_z \mid \omega])}{\widehat{\theta}([g_z\omega g_y^{-1} \mid g_y])}\rho(g_z\omega g_y^{-1}). \tag{15}$$

This defines a functor $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}|_x}(B\operatorname{Aut}_{\widehat{\mathcal{G}}}(x)) \to \mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}})$ which is quasi-inverse to restriction to $\{x\}$. Indeed, let $F \in \mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}})$ with $F|_x = \rho$. Then $\Phi_y = F(g_y)$ are the components of a natural isomorphism $\Phi\colon F \Rightarrow F_\rho$.

If instead $\pi_0(\widehat{\mathcal{G}}) = \pi_0(\widehat{\mathcal{G}})_1$, let $\rho \in \mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}_{\{x,\bar{x}\}}}(\widehat{\mathcal{G}}_{\{x,\bar{x}\}})$ with fibres V_x and $V_{\bar{x}}$. Using notation from the proof of Proposition 1.1, define $F \in \mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}})$ by $F_\rho(y) = V_{\mathrm{target}(g_y)}$ for $y \in \widehat{\mathcal{G}}$ with $F_\rho(y \xrightarrow{\omega} z)$ as in (15). The resulting functor

$$\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}|_{\{x,\bar{x}\}}}(\widehat{\mathcal{G}}_{\{x,\bar{x}\}}) \to \mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}})$$

is quasi-inverse to restriction to $\{x, \bar{x}\}$. □

Proposition 3.6 *The category* $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}})$ *is semisimple.*

Proof This can be proved in the two model cases by first employing a variation of Weyl's unitary trick to unitarize a $\widehat{\theta}$–twisted vector bundle and then taking orthogonal complements of subbundles. Since semisimplicity is preserved under equivalences, the general case then follows from Proposition 3.5. □

We end this section with a fixed-point interpretation of $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\theta}}(\widehat{\mathcal{G}})$. For simplicity, take $\widehat{\mathcal{G}} = B\widehat{\mathsf{G}}$. Each $\varsigma \in \widehat{\mathsf{G}} \setminus \mathsf{G}$ determines an antilinear involution $(Q^\varsigma, \Theta^\varsigma)$ of the $\mathbb{C}$–linear category $\mathsf{Vect}_{\mathbb{C}}^{\theta}(B\mathsf{G})$ of θ–twisted vector bundles over $B\mathsf{G}$. The functor $Q^\varsigma\colon \mathsf{Vect}_{\mathbb{C}}^{\theta}(B\mathsf{G}) \to \mathsf{Vect}_{\mathbb{C}}^{\theta}(B\mathsf{G})$ is given on objects by $Q^\varsigma(V,\rho) = (\overline{V}, \rho^\varsigma)$, where

$\rho^{\varsigma}(g) = \tau_{\pi}(\widehat{\theta})([\varsigma]g) \cdot \overline{\rho(\varsigma g \varsigma^{-1})}$. That $Q^{\varsigma}(V,\rho)$ is indeed a θ–twisted representation follows from the identity

$$\frac{\widehat{\theta}\big([\omega g_2 \omega^{-1} \mid \omega g_1 \omega^{-1}]\big)}{\widehat{\theta}\big([g_2 \mid g_1]\big)^{\pi(\omega)}} = \frac{\tau_{\pi}(\widehat{\theta})([\omega]g_2)\tau_{\pi}(\widehat{\theta})([\omega]g_1)}{\tau_{\pi}(\widehat{\theta})([\omega]g_2 g_1)}, \quad g_i \in \mathsf{G},\ \omega \in \widehat{\mathsf{G}}. \tag{16}$$

This identity can be verified directly or, more conceptually, using the method of proof of [29, Proposition 8.1]. The natural isomorphism $\Theta^{\varsigma} \colon 1_{\mathsf{Vect}^{\theta}(\mathsf{G})} \Rightarrow (Q^{\varsigma})^2$ has components $\Theta^{\varsigma}_{\rho} = \widehat{\theta}\big([\varsigma \mid \varsigma]\big)^{-2}\rho(\varsigma^{-2})$ and satisfies $Q^{\varsigma}(\Theta_{\rho}) = \Theta_{Q^{\varsigma}(\rho)}$. Up to equivalence of categories with involution, $(\mathsf{Vect}^{\theta}_{\mathbb{C}}(B\mathsf{G}), Q^{\varsigma}, \Theta^{\varsigma})$ depends only on the pair $(\widehat{\mathsf{G}}, \widehat{\theta})$.

Proposition 3.7 *There is an equivalence* $\mathsf{Vect}^{\theta}_{\mathbb{C}}(B\mathsf{G})^{h(Q^{\varsigma},\Theta^{\varsigma})} \simeq \mathsf{Vect}^{\widehat{\theta}}_{\mathbb{C}}(B\widehat{\mathsf{G}})$ *of* $\mathbb{R}$*–linear categories, where the left-hand side denotes the homotopy fixed-point category.*

Proof At the level of objects, the equivalence $\mathsf{Vect}^{\theta}_{\mathbb{C}}(B\mathsf{G})^{h(Q^{\varsigma},\Theta^{\varsigma})} \to \mathsf{Vect}^{\widehat{\theta}}_{\mathbb{C}}(B\mathsf{G})$ assigns to a homotopy fixed point $\psi_{\rho} \colon (V,\rho) \to Q^{\varsigma}(V,\rho)$ the $\widehat{\theta}$–twisted representation ρ which is equal to ρ as a θ–twisted representation and has

$$\rho(\omega) = \widehat{\theta}\big([\omega \mid \varsigma^{-1}]\big)\rho(\omega\varsigma^{-1}) \circ \psi_{\rho}, \quad \omega \in \widehat{\mathsf{G}} \setminus \mathsf{G}.$$

On morphisms, F^{ς} is the identity. □

Using Proposition 3.7, define the hyperbolic induction (or Realification) functor

$$\mathrm{HInd}^{\widehat{\mathsf{G}}}_{\mathsf{G}} \colon \mathsf{Vect}^{\theta}_{\mathbb{C}}(B\mathsf{G}) \to \mathsf{Vect}^{\widehat{\theta}}_{\mathbb{C}}(B\widehat{\mathsf{G}})$$

so that it assigns to $(V,\rho) \in \mathsf{Vect}^{\theta}_{\mathbb{C}}(B\mathsf{G})$ the $\widehat{\theta}$–twisted representation with underlying θ–twisted representation $V \oplus Q^{\varsigma}(V)$ and on which $\omega \in \widehat{\mathsf{G}} \setminus \mathsf{G}$ acts on by

$$\rho_{\mathrm{HInd}(\rho)}(\omega)(v_1, v_2) = \big(\widehat{\theta}\big([\omega \mid \varsigma^{-1}]\big)\rho(\omega\varsigma^{-1})v_2, \widehat{\theta}\big([\varsigma \mid \omega]\big)\overline{\rho(\varsigma\omega)}v_1\big).$$

3.3 The character theory of $\mathsf{Vect}^{\widehat{\theta}}_{\mathbb{C}}(\widehat{\mathcal{G}})$

We continue to work in the setting of Section 3.2.

Definition 3.8 The $\widehat{\theta}$–twisted K–theory of $\widehat{\mathcal{G}}$ is the Grothendieck group $K^{\widehat{\theta}}(\widehat{\mathcal{G}}) := K_0(\mathsf{Vect}^{\widehat{\theta}}_{\mathbb{C}}(\widehat{\mathcal{G}}))$.

This is a special case of the twisted K–theory studied in [21] and reduces to the KR–theory of [2; 27] when $\widehat{\theta}$ is trivial. By Proposition 3.6, $K^{\widehat{\theta}}(\widehat{\mathcal{G}})$ is the free abelian group on isomorphism classes of simple $\widehat{\theta}$–twisted vector bundles.

The next result is crucial to what follows.

Proposition 3.9 *The assignment of an object $F \in \mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})$ to the function*

$$\chi_F : \mathrm{Obj}(\Lambda^{\mathrm{ref}}_{\pi}\hat{\mathcal{G}}) \to \mathbb{C}, \quad (x,\gamma) \mapsto \mathrm{tr}_{F(x)} F(\gamma),$$

defines an abelian group homomorphism $\chi\colon K^{\hat{\theta}}(\hat{\mathcal{G}}) \to \Gamma_{\Lambda^{\mathrm{ref}}_{\pi}\hat{\mathcal{G}}}(\uptau^{\mathrm{ref}}_{\pi}(\hat{\theta})_{\mathbb{C}})$.

Proof Let $\omega\colon x_1 \to x_2$ be a morphism in $\hat{\mathcal{G}}$. Equation (14) and the twisted 2–cocycle condition on $\hat{\theta}$ imply

$$F(\omega\gamma^{\pi(\omega)}\omega^{-1}) = \frac{\hat{\theta}([\omega\gamma^{\pi(\omega)}\omega^{-1} \mid \omega])}{\hat{\theta}([\omega \mid \gamma^{\pi(\omega)}])} F(\omega)F(\gamma^{\pi(\omega)})F(\omega)^{-1}. \tag{17}$$

When $\pi(\omega) = -1$, we can use (14) to replace $F(\gamma^{-1})$ with $\hat{\theta}([\gamma \mid \gamma^{-1}])F(\gamma)^{-1}$. Doing so and taking the trace of (17) gives

$$\mathrm{tr}_{F(x_2)} F(\omega\gamma^{\pi(\omega)}\omega^{-1}) = \uptau^{\mathrm{ref}}_{\pi}(\hat{\theta})([\omega]\gamma)\, \mathrm{tr}_{F(x_2)}(F(\omega)F(\gamma)^{\pi(\omega)}F(\omega)^{-1}).$$

Since $F(\omega)$ is $\pi(\omega)$–linear and $\mathrm{tr}_{F(x_1)}(F(\gamma)^{-1}) = \overline{\mathrm{tr}_{F(x_1)} F(\gamma)}$ (see the proof of [40, Proposition 10]), we arrive at

$$\mathrm{tr}_{F(x_2)} F(\omega\gamma^{\pi(\omega)}\omega^{-1}) = \uptau^{\mathrm{ref}}_{\pi}(\hat{\theta})([\omega]\gamma)\, \mathrm{tr}_{F(x_1)} F(\gamma);$$

that is, $\chi_F \in \Gamma_{\Lambda^{\mathrm{ref}}_{\pi}\hat{\mathcal{G}}}(\uptau^{\mathrm{ref}}_{\pi}(\hat{\theta})_{\mathbb{C}})$. Since $\chi_{F_1 \oplus F_2} = \chi_{F_1} + \chi_{F_2}$, this completes the proof. □

We call χ_F the Real character of F. The pullback of χ_F along $\Lambda\mathcal{G} \to \Lambda^{\mathrm{ref}}_{\pi}\hat{\mathcal{G}}$ is the character of the θ–twisted vector bundle which underlies F, as defined in [40, Section 2.3.3]. Proposition 3.9 is a twisted reality condition on χ_F. Indeed, when $\hat{\mathcal{G}} = B\hat{\mathsf{G}}$ with $\hat{\mathsf{G}} = \mathsf{G} \times \mathbb{Z}_2$ and $\hat{\theta}|_{\mathsf{G}} = 1$, Proposition 3.9 reduces to the reality of the character of a real or quaternionic representation of G.

Given a trivially twisted vector bundle $F \in \mathsf{Vect}^{\hat{\theta}=1}_{\mathbb{C}}(\hat{\mathcal{G}})$, let $\Gamma_{\hat{\mathcal{G}}}(F)$ be the real vector space of its flat sections. The following result generalizes Proposition 3.1:

Proposition 3.10 (see [40, Proposition 7]) *The equality $\frac{1}{2}\dim_{\mathbb{R}} \Gamma_{\hat{\mathcal{G}}}(F) = \int_{\Lambda^{\mathrm{ref}}_{\pi}\hat{\mathcal{G}}} \chi_F$ holds.*

Consider the Hom bifunctor

$$\langle -,-\rangle\colon \mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})^{\mathrm{op}} \times \mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}}) \to \mathsf{Vect}_{\mathbb{R}}, \quad (F_1, F_2) \mapsto \mathrm{Hom}_{\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})}(F_1, \hat{F}_2),$$

which we regard as a categorical inner product on $\mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(\hat{\mathcal{G}})$. There is a canonical real vector space isomorphism

$$\langle F_1, \hat{F}_2 \rangle \simeq \Gamma_{\hat{\mathcal{G}}}(F_1^{\vee} \otimes F_2), \tag{18}$$

the right-hand side being the space of flat sections of the trivially twisted bundle $F_1^{\vee} \otimes F_2 \in \mathsf{Vect}_{\mathbb{C}}^{1}(\hat{\mathcal{G}})$.

Lemma 3.11 *Let $F_1, F_2 \in \mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(\hat{\mathcal{G}})$. The equality $\dim_{\mathbb{R}} \langle F_1, \hat{F}_2 \rangle = \frac{1}{2} \langle \chi_{F_1}, \chi_{F_2} \rangle$ holds.*

Proof The proof is as in [40, Proposition 10]. We compute

$$\begin{aligned} \dim_{\mathbb{R}} \langle F_1, \hat{F}_2 \rangle &= \dim_{\mathbb{R}} \Gamma_{\hat{\mathcal{G}}}(F_1^{\vee} \otimes \hat{F}_2) && \text{(by (18))} \\ &= \frac{1}{2} \int_{\Lambda_{\pi}^{\mathrm{ref}} \hat{\mathcal{G}}} \chi_{F_1^{\vee} \otimes \hat{F}_2} && \text{(by Proposition 3.10).} \end{aligned}$$

Since $\chi_{F_1^{\vee} \otimes \hat{F}_2} = \overline{\chi}_{F_1} \cdot \chi_{\hat{F}_2}$, the right-hand side is equal to $\frac{1}{2} \langle \chi_{F_1}, \chi_{F_2} \rangle$. □

It is proved in [40, Theorem 11] that, for any $\theta \in Z^2(\mathcal{G})$, the (ordinary) character map induces an isomorphism

$$\chi \colon K^{\theta}(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \Gamma_{\Lambda \mathcal{G}}(\tau(\theta)_{\mathbb{C}}). \tag{19}$$

The next result is a Real generalization of this isomorphism. Special cases of this result, in terms of projective representations of G over $\mathbb{R}$ twisted by $Z^2(B\mathsf{G}; \mathbb{R}^{\times})$, can be found in [33, Theorem 6; 28, Section 10.2].

Theorem 3.12 *The Real character map induces an isomorphism*

$$\chi \colon K^{\hat{\theta}}(\hat{\mathcal{G}}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \Gamma_{\Lambda_{\pi}^{\mathrm{ref}} \hat{\mathcal{G}}}(\tau_{\pi}^{\mathrm{ref}}(\hat{\theta})_{\mathbb{C}})$$

of complex inner product spaces.

Proof Proposition 3.5 gives an isomorphism of abelian groups

$$K^{\hat{\theta}}(\hat{\mathcal{G}}) \simeq \bigoplus_{x \in \pi_0(\hat{\mathcal{G}})_{-1}} K^{\hat{\theta}|_x}(B\mathrm{Aut}_{\hat{\mathcal{G}}}(x)) \oplus \bigoplus_{x \in \pi_0(\hat{\mathcal{G}})_{1}} K^{\hat{\theta}|_{\{x,\bar{x}\}}}(\hat{\mathcal{G}}_{\{x,\bar{x}\}}). \tag{20}$$

Lemma 3.11, together with Schur's lemma for $\hat{\theta}$–twisted vector bundles, then implies that χ is injective.

In view of the isomorphism (20), it suffices to prove surjectivity of χ in the two model cases. Suppose that $\widehat{\mathcal{G}} = B\widehat{\mathsf{G}}$. Let $\mathrm{Irr}^{\theta}(\mathsf{G})$ be the set of isomorphism classes of simple objects of $\mathsf{Vect}^{\theta}_{\mathbb{C}}(B\mathsf{G})$. Let $s \in \Gamma_{\Lambda^{\mathrm{ref}}_{\pi} B\widehat{\mathsf{G}}}(\uptau^{\mathrm{ref}}_{\pi}(\widehat{\theta})_{\mathbb{C}})$. Using the isomorphism (19), we can write

$$s = \sum_{V \in \mathrm{Irr}^{\theta}(\mathsf{G})} \langle \chi_V, s \rangle \chi_V.$$

The additional symmetry conditions describing the subspace $\Gamma_{\Lambda^{\mathrm{ref}}_{\pi} B\widehat{\mathsf{G}}}(\uptau^{\mathrm{ref}}_{\pi}(\widehat{\theta})_{\mathbb{C}}) \subset \Gamma_{\Lambda B\mathsf{G}}(\uptau_{\pi}(\theta)_{\mathbb{C}})$ give

$$s(\omega\gamma^{-1}\omega^{-1}) = \uptau^{\mathrm{ref}}_{\pi}(\widehat{\theta})([\omega]\gamma)s(\gamma), \quad \gamma \in \mathsf{G},\ \omega \in \widehat{\mathsf{G}} \setminus \mathsf{G}.$$

It follows that, for any $\varsigma \in \widehat{\mathsf{G}} \setminus \mathsf{G}$, the function $V \mapsto \langle \chi_V, s \rangle$ descends to the orbit space $\mathrm{Irr}^{\theta}(\mathsf{G})/\langle Q^{\varsigma} \rangle$ of $\mathrm{Irr}^{\theta}(\mathsf{G})$ under the action of the functor Q^{ς}. Hence, we have

$$s = \sum_{\mathcal{O} \in \mathrm{Irr}^{\theta}(\mathsf{G})/\langle Q^{\varsigma} \rangle} a_{\mathcal{O}} \sum_{V \in \mathcal{O}} \chi_V = \frac{1}{2} \sum_{\mathcal{O} \in \mathrm{Irr}^{\theta}(\mathsf{G})/\langle Q^{\varsigma} \rangle} a_{\mathcal{O}} \sum_{V \in \mathcal{O}} (\chi_V + \chi_{\varsigma \cdot V})$$

for some $a_{\mathcal{O}} \in \mathbb{C}$. Noting that $\chi_{\mathrm{HInd}^{\widehat{\mathsf{G}}}_{\mathsf{G}}(V)} = \chi_V + \chi_{\varsigma \cdot V}$, we see that $\sum_{V \in \mathcal{O}} \chi_V$, and hence s, is in the image of $K^{\widehat{\theta}}(B\widehat{\mathsf{G}}) \otimes_{\mathbb{Z}} \mathbb{C}$.

Suppose instead that $\widehat{\mathcal{G}} = \widehat{\mathcal{G}}_{\{x,\bar{x}\}}$ and let $s \in \Gamma_{\Lambda^{\mathrm{ref}}_{\pi}\widehat{\mathcal{G}}}(\uptau^{\mathrm{ref}}_{\pi}(\widehat{\theta})_{\mathbb{C}}) \subset \Gamma_{\Lambda\mathcal{G}}(\uptau(\theta)_{\mathbb{C}})$. Write

$$s = \sum_{V \in \mathrm{Irr}^{\widehat{\theta}|_x}(\mathrm{Aut}_{\widehat{\mathcal{G}}}(x))} \langle \chi_V, s \rangle \chi_V + \sum_{V' \in \mathrm{Irr}^{\widehat{\theta}|_{\bar{x}}}(\mathrm{Aut}_{\widehat{\mathcal{G}}}(\bar{x}))} \langle \chi_{V'}, s \rangle \chi_{V'}.$$

Fix a morphism $\varsigma : x \to \bar{x}$ of degree -1 and use the isomorphism $\mathrm{Aut}_{\widehat{\mathcal{G}}}(x) \xrightarrow{\sim} \mathrm{Aut}_{\widehat{\mathcal{G}}}(\bar{x})$, $\gamma \mapsto \varsigma\gamma\varsigma^{-1}$, to identify $\mathrm{Irr}^{\widehat{\theta}|_x}(\mathrm{Aut}_{\widehat{\mathcal{G}}}(x))$ and $\mathrm{Irr}^{\widehat{\theta}|_{\bar{x}}}(\mathrm{Aut}_{\widehat{\mathcal{G}}}(\bar{x}))$. The additional symmetry condition on s implies that

$$s = \sum_{V \in \mathrm{Irr}^{\widehat{\theta}|_x}(\mathrm{Aut}_{\widehat{\mathcal{G}}}(x))} \langle \chi_{V_x}, s \rangle (\chi_{V_x} + \chi_{V_{\bar{x}}}),$$

where $V_{\bar{x}} \in \mathsf{Vect}^{\widehat{\theta}|_{\bar{x}}}_{\mathbb{C}}(B\mathrm{Aut}_{\widehat{\mathcal{G}}}(\bar{x}))$ is the pullback of $\overline{V}$ along ς. Explicitly, a loop $\varsigma\gamma\varsigma^{-1}$ at $\bar{x}$ acts on $V_{\bar{x}} = \overline{V}_x$ by

$$\frac{\widehat{\theta}([\varsigma \mid \gamma])\widehat{\theta}([\varsigma\gamma \mid \varsigma^{-1}])}{\widehat{\theta}([\varsigma\gamma\varsigma^{-1} \mid \varsigma])}\rho(\gamma).$$

The sum $V_x \oplus V_{\bar{x}}$ becomes a $\widehat{\theta}$–twisted vector bundle by taking $\rho(\varsigma) : V_x \to V_{\bar{x}}$ to be the identity map (which is $\mathbb{C}$–antilinear) and setting $\rho(\varsigma\gamma) := \widehat{\theta}([\varsigma \mid \gamma])^{-1}\rho(\varsigma)\rho(\gamma)$. Moreover, $\chi_{V_x \oplus V_{\bar{x}}} = \chi_{V_x} + \chi_{V_{\bar{x}}}$, whence s is in the image of $K^{\widehat{\theta}}(\widehat{\mathcal{G}}) \otimes_{\mathbb{Z}} \mathbb{C}$.

The statement about inner products follows from Lemma 3.11. □

Corollary 3.13 *The category $\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})$ has exactly $\int_{\Lambda\Lambda^{\mathrm{ref}}_{\pi}\hat{\mathcal{G}}} \uptau\uptau^{\mathrm{ref}}_{\pi}(\hat{\theta})$ simple objects.*

Proof By Proposition 3.6, the rank of $K^{\hat{\theta}}(\hat{\mathcal{G}})$ equals the number of simple $\hat{\theta}$–twisted vector bundles. The corollary now follows from Theorem 3.12 and the equality $\int_{\Lambda\Lambda^{\mathrm{ref}}_{\pi}\hat{\mathcal{G}}} \uptau\uptau^{\mathrm{ref}}_{\pi}(\hat{\theta}) = \dim_{\mathbb{C}} \Gamma_{\Lambda^{\mathrm{ref}}_{\pi}\hat{\mathcal{G}}}(\uptau^{\mathrm{ref}}_{\pi}(\hat{\theta})_{\mathbb{C}})$, which is implied by [40, Theorem 6]. □

Specializing Corollary 3.13 to $\hat{\mathcal{G}} = B\hat{\mathsf{G}}$ gives the following result:

Corollary 3.14 *The number of simple $\hat{\theta}$–twisted representations of $\hat{\mathsf{G}}$ is*

$$\int_{\Lambda\Lambda^{\mathrm{ref}}_{\pi}B\hat{\mathsf{G}}} \uptau\uptau^{\mathrm{ref}}_{\pi}(\hat{\theta}) = \frac{1}{2|\mathsf{G}|} \sum_{\substack{(\gamma,\omega)\in \mathsf{G}\times\hat{\mathsf{G}} \\ \gamma=\omega\gamma^{\pi(\omega)}\omega^{-1}}} \hat{\theta}([\gamma^{-1} \mid \gamma])^{-\Delta_{\omega}} \frac{\hat{\theta}([\gamma \mid \omega])}{\hat{\theta}([\omega \mid \gamma^{\pi(\omega)}])}. \tag{21}$$

The right-hand side of (21) decomposes into two sums, corresponding to $\pi(\omega) = 1$ and $\pi(\omega) = -1$. The former sum is one half the number of simple θ–twisted class functions of G, which, by a theorem of Schur (see [28, Section 3.6]), is one half the number of simple θ–twisted representations. The latter sum is one half the number of Real simple θ–twisted class functions, where Real means that $\chi(\omega\gamma^{\pi(\omega)}\omega^{-1}) = \uptau^{\mathrm{ref}}_{\pi}(\hat{\theta})([\omega]\gamma)\chi(\gamma)$ for some, and hence any, $\omega \in \hat{\mathsf{G}} \setminus \mathsf{G}$. Corollary 3.14 is therefore a Real version of Schur's result. When $\hat{\mathsf{G}} = \mathsf{G} \times \mathbb{Z}_2$ and $\hat{\theta}|_{\mathsf{G}} = 1$, this recovers standard results in the real/quaternionic representation theory of G. See for example [7, Theorem II.6.3].

3.4 The centre of $\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}})$

The centre $Z(\mathsf{Vect}^{\theta}_{\mathbb{C}}(\mathcal{G}))$, that is, the algebra of natural transformations of the identity functor of the category $\mathsf{Vect}^{\theta}_{\mathbb{C}}(\mathcal{G})$, is isomorphic to $\Gamma_{\Lambda\mathcal{G}}(\uptau(\theta)^{-1}_{\mathbb{C}})$ [40, Section 2.3.4]. In fact, the map

$$K^{\theta}(\mathcal{G}) \times Z(\mathsf{Vect}^{\theta}_{\mathbb{C}}(\mathcal{G})) \to \mathbb{C}, \quad (V,\eta) \mapsto \mathrm{tr}_V\, \eta_V,$$

induces a perfect pairing between $K^{\theta}(\mathcal{G})\otimes_{\mathbb{Z}}\mathbb{C}$ and $Z(\mathsf{Vect}^{\theta}_{\mathbb{C}}(\mathcal{G}))$, giving a compatibility between two decategorifications of $\mathsf{Vect}^{\theta}_{\mathbb{C}}(\mathcal{G})$. There is no analogous compatibility in the Real setting. Instead, in this section we will see that $Z(\mathsf{Vect}^{\hat{\theta}}_{\mathbb{C}}(\hat{\mathcal{G}}))$ can be described using $\uptau_{\pi}$.

Consider the $\mathbb{R}$–linear embedding

$$\Gamma_{\Lambda_{\pi}\mathcal{G}}(\uptau_{\pi}(\hat{\theta})^{-1}_{\mathbb{C}}) \to \mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}], \quad s \mapsto \sum_{\gamma\in\Lambda_{\pi}\hat{\mathcal{G}}} s_{\gamma} l_{\gamma}.$$

Its image is stable under multiplication of $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$, as follows from (16). In this way $\Gamma_{\Lambda_\pi \mathcal{G}}(\tau_\pi(\hat{\theta})_{\mathbb{C}}^{-1})$ inherits the structure of an $\mathbb{R}$–algebra.

Proposition 3.15 *The centre of the $\mathbb{R}$–algebra $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$ is isomorphic to $\Gamma_{\Lambda_\pi \hat{\mathcal{G}}}(\tau_\pi(\hat{\theta})_{\mathbb{C}}^{-1})$.*

Proof For each morphism $\omega\colon x_1 \to x_2$ in $\hat{\mathcal{G}}$ and $c_{x_1} \in \mathbb{C}$, we have equalities

$$l_\omega(c_{x_1} l_{\mathrm{id}_{x_1}}) = {}^{\pi(\omega)}c_{x_1} l_\omega, \quad (c_{x_1} l_{\mathrm{id}_{x_1}}) l_\omega = \delta_{x_1,x_2} c_{x_1} l_\omega$$

in $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$. Elements of the centre $Z(\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}])$ are therefore of the form $\sum_{\gamma \in \Lambda_\pi \hat{\mathcal{G}}} c_\gamma l_\gamma$. Requiring this element to commute with l_ω gives

$$\begin{aligned} l_\omega \sum_{\gamma \in \Lambda_\pi \hat{\mathcal{G}}} c_\gamma l_\gamma = \Big(\sum_{\gamma \in \Lambda_\pi \hat{\mathcal{G}}} c_\gamma l_\gamma\Big) l_\omega &= \sum_{\delta \in \Lambda_\pi \hat{\mathcal{G}}} \frac{\hat{\theta}([\omega\delta\omega^{-1} \mid \omega])}{\hat{\theta}([\omega \mid \delta])} c_{\omega\delta\omega^{-1}} l_\omega l_\delta \\ &= l_\omega \sum_{\delta \in \Lambda_\pi \hat{\mathcal{G}}} {}^{\pi(\omega)}\Big(\frac{\hat{\theta}([\omega\delta\omega^{-1} \mid \omega])}{\hat{\theta}([\omega \mid \delta])} c_{\omega\delta\omega^{-1}}\Big) l_\delta. \end{aligned}$$

It follows that

$$c_\gamma = {}^{\pi(\omega)}\Big(\frac{\hat{\theta}([\omega\gamma\omega^{-1} \mid \omega])}{\hat{\theta}([\omega \mid \gamma])} c_{\omega\gamma\omega^{-1}}\Big), \quad \gamma \in \Lambda_\pi \hat{\mathcal{G}}. \tag{22}$$

Conversely, the equalities (22) ensure that $\sum_{\gamma \in \Lambda_\pi \hat{\mathcal{G}}} c_\gamma l_\gamma$ commutes with l_ω. The map

$$Z(\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]) \to \Gamma_{\Lambda_\pi \hat{\mathcal{G}}}(\tau_\pi(\hat{\theta})_{\mathbb{C}}^{-1}), \qquad \sum_{\gamma \in \Lambda_\pi \hat{\mathcal{G}}} c_\gamma l_\gamma \mapsto (\gamma \mapsto c_\gamma),$$

is therefore well defined and gives the desired isomorphism. □

The $\mathbb{R}$–algebra $Z(\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}])$ is isomorphic to the centre of $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$–mod, the category of modules over the $\mathbb{R}$–algebra $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$. We want to relate this to the centre of the $\mathbb{R}$–linear category $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$–$\mathrm{mod}^{\mathrm{R}}$. To do so, let A be a finite-dimensional Real algebra, that is, a $\mathbb{Z}_2$–graded complex vector space which has the structure of a unital $\mathbb{R}$–algebra which satisfies

$$(c_2 a_2) \cdot (c_1 a_1) = c_2({}^{\pi(a_2)}c_1) a_2 a_1$$

for all $c_i \in \mathbb{C}$ and homogeneous $a_i \in A$. Let $Z(A)_1$ be the degree 1 subalgebra of $Z(A)$, the centre of A considered as an $\mathbb{R}$–algebra.

Lemma 3.16 *There is an $\mathbb{R}$–algebra isomorphism $Z(A\text{–}\mathrm{mod}^{\mathrm{R}}) \simeq Z(A)_1$.*

Proof This is a straightforward variation of the proof that the centre of the category of modules over a unital algebra is isomorphic to the centre of the algebra. □

Theorem 3.17 *There is a canonical $\mathbb{R}$–algebra isomorphism*

$$Z(\mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(\hat{\mathcal{G}})) \simeq \Gamma_{\Lambda_\pi \hat{\mathcal{G}}}(\tau_\pi(\hat{\theta})_{\mathbb{C}}^{-1}).$$

Proof By Proposition 3.4, the categories $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$–mod$^{\mathrm{R}}$ and $\mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(\hat{\mathcal{G}})$ are $\mathbb{R}$–linearly equivalent. Proposition 3.15 and Lemma 3.16 then give algebra isomorphisms

$$\Gamma_{\Lambda_\pi \hat{\mathcal{G}}}(\tau_\pi(\hat{\theta})_{\mathbb{C}}^{-1}) \simeq Z(\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]) = Z(\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}])_1 \simeq Z(\mathsf{Vect}^{\hat{\theta}}(\hat{\mathcal{G}})),$$

the middle equality following from the explicit description of $Z(\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}])$. □

Corollary 3.18 *For any finite $\mathbb{Z}_2$–graded group $\hat{\mathsf{G}}$, there is an equality*

$$\dim_{\mathbb{R}} Z(\mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(B\hat{\mathsf{G}})) = \frac{1}{|\mathsf{G}|} \sum_{\substack{(g_1,g_2)\in \mathsf{G}^2 \\ g_1 g_2 = g_2 g_1}} \frac{\hat{\theta}([g_1 \mid g_2])}{\hat{\theta}([g_2 \mid g_1])}.$$

Proof This follows by combining Proposition 3.1 and Theorem 3.17. □

In particular, the dimension of $Z(\mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(B\hat{\mathsf{G}}))$ is independent of the lift $(\hat{\mathsf{G}}, \hat{\theta})$ of (G, θ). In fact, by [40, Theorem 6] we have

$$\dim_{\mathbb{R}} Z(\mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(B\hat{\mathsf{G}})) = \dim_{\mathbb{C}} Z(\mathsf{Vect}_{\mathbb{C}}^{\theta}(B\mathsf{G})).$$

A conceptual explanation of this equality is as follows:

Proposition 3.19 *For each $\varsigma \in \hat{\mathsf{G}} \setminus \mathsf{G}$, the pair $(Q^{\varsigma}, \Theta^{\varsigma})$ induces a ς–independent $\mathbb{C}$–antilinear algebra involution*

$$q\colon Z(\mathsf{Vect}_{\mathbb{C}}^{\theta}(B\mathsf{G})) \to Z(\mathsf{Vect}_{\mathbb{C}}^{\theta}(B\mathsf{G}))$$

whose fixed-point set is $Z(\mathsf{Vect}_{\mathbb{C}}^{\hat{\theta}}(B\hat{\mathsf{G}}))$.

Proof Under the equivalence $\mathsf{Vect}_{\mathbb{C}}^{\theta}(B\mathsf{G}) \simeq \mathbb{C}^{\theta}[B\mathsf{G}]$–mod, the functor Q^{ς} becomes the antilinear algebra automorphism

$$q^{\varsigma} : \mathbb{C}^{\theta}[B\mathsf{G}] \to \mathbb{C}^{\theta}[B\mathsf{G}], \quad \sum_{g\in \mathsf{G}} c_g l_g \mapsto \sum_{g\in\mathsf{G}} \tau_\pi(\hat{\theta})([\varsigma]g)^{-1} \bar{c}_g l_{\varsigma g \varsigma^{-1}}.$$

Closedness of $\tau_\pi(\hat{\theta})$ implies that q^{ς} squares to $\mathrm{Ad}_{l_{\varsigma^2}}$. It follows that q^{ς} restricts to an antilinear algebra involution $q\colon \Gamma_{\Lambda B\mathsf{G}}(\tau(\theta)_{\mathbb{C}}^{-1}) \to \Gamma_{\Lambda B\mathsf{G}}(\tau(\theta)_{\mathbb{C}}^{-1})$ which, again by the closedness of $\tau_\pi(\hat{\theta})$, is independent of ς. The explicit form of q^{ς} shows that the fixed-point set of q is $\Gamma_{\Lambda_\pi B\hat{\mathsf{G}}}(\tau_\pi(\hat{\theta})_{\mathbb{C}}^{-1})$. To finish the proof, apply Theorem 3.17. □

4 Jandl twisted 2–vector bundles

We study representation-theoretic aspects of $Z^{3+\pi_{\hat{\mathcal{G}}}}(\hat{\mathcal{G}})$ and the corresponding 2–categorical analogues of the results of Section 3. For simplicity, we restrict attention in this section to $\mathbb{Z}_2$–graded groupoids of the form $\hat{\mathcal{G}} = B\hat{\mathsf{G}}$.

4.1 Thickened Drinfeld doubles

Let G be a finite group with 3–cocycle $\eta \in Z^3(B\mathsf{G})$. The η–twisted Drinfeld double $D^{\eta}(\mathsf{G})$ is a quasi-Hopf algebra with explicitly defined product and coproduct, introduced by Dijkgraaf, Pasquier and Roche in their study of orbifolds of rational conformal field theory [12]. The starting point of this section is Willerton's algebra isomorphism between $D^{\eta}(\mathsf{G})$ and the twisted groupoid algebra $\mathbb{C}^{\tau(\eta)}[\Lambda B\mathsf{G}]$ [40, Section 3.1]. This isomorphism both provides a conceptual definition of the algebra $D^{\eta}(\mathsf{G})$ and leads to short proofs of a number of its fundamental properties, such as a description of the characters of its finite-dimensional modules [40].

Turning to the Real setting, fix a finite $\mathbb{Z}_2$–graded group $\hat{\mathsf{G}}$. We use the notion of twisted Real groupoid algebras introduced in Section 3.2.

Definition 4.1 Let $\hat{\eta} \in Z^{3+\pi_{\hat{\mathsf{G}}}}(B\hat{\mathsf{G}})$ and $\tilde{\eta} \in Z^3(B\hat{\mathsf{G}})$ be lifts of $\eta \in Z^3(B\mathsf{G})$. Define

(i) Real algebras by $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}}) := \mathbb{C}^{\tau_\pi(\hat{\eta})}[\Lambda_\pi B\hat{\mathsf{G}}]$ and $\mathbb{D}^{\tilde{\eta}}(\hat{\mathsf{G}}) := \mathbb{C}^{\tilde{\tau}_\pi^{\mathrm{ref}}(\tilde{\eta})}[\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}]$;

(ii) a $\mathbb{C}$–algebra by $D^{\hat{\eta}}(\hat{\mathsf{G}}) := \mathbb{C}^{\tau_\pi^{\mathrm{ref}}(\hat{\eta})}[\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}]$.

In particular, $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})$ and $\mathbb{D}^{\tilde{\eta}}(\hat{\mathsf{G}})$ are $\mathbb{R}$–algebras. Compatibility of the oriented and twisted loop transgression maps, as in diagram (13) for τ_π^{ref}, implies that $D^{\eta}(\mathsf{G})$ embeds into each of $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})$, $\mathbb{D}^{\tilde{\eta}}(\hat{\mathsf{G}})$ and $D^{\hat{\eta}}(\hat{\mathsf{G}})$ as the complex subalgebra of degree 1 morphisms. For this reason, we refer to any of the algebras in the above definition as twisted thickened Drinfeld doubles.

The results of Section 3.3 can be applied to the representation theory of thickened Drinfeld doubles. To begin, we identify their representation groups. Motivated by [40, Section 3.2], we view these results as Real counterparts of the Freed–Hopkins–Teleman theorem [20, Theorem 1] in our (much simpler) finite setting. For a Real analogue for compact, connected and simply connected Lie groups, see [18, Theorem 5.12].

Proposition 4.2 *There are isomorphisms of abelian groups*

(i) $K^{\tau_\pi^{\mathrm{ref}}(\hat{\eta})}(\mathsf{G}/\!/_{\mathrm{R}}\hat{\mathsf{G}}) \simeq K_0(D^{\hat{\eta}}(\hat{\mathsf{G}})\text{–mod})$,

(ii) $K^{\tilde{\tau}_\pi^{\mathrm{ref}}(\tilde{\eta})}(\mathsf{G}/\!/_{\mathrm{R}}\hat{\mathsf{G}}) \simeq K_0(\mathbb{D}^{\tilde{\eta}}(\hat{\mathsf{G}})\text{–mod}^{\mathrm{R}})$, *and*

(iii) $K^{\tau_\pi(\hat{\eta})}(\mathsf{G}/\!/\hat{\mathsf{G}}) \simeq K_0(\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})\text{–mod}^{\mathrm{R}})$.

Proof Recall that $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}} \simeq \mathsf{G}/\!/_{\mathrm{R}}\hat{\mathsf{G}}$ and $\Lambda_\pi B\hat{\mathsf{G}} \simeq \mathsf{G}/\!/\hat{\mathsf{G}}$. The first isomorphism then follows from the equivalence $\mathsf{Vect}_{\mathbb{C}}^{\tau_\pi^{\mathrm{ref}}(\hat{\eta})}(\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}) \simeq \mathbb{C}^{\tau_\pi^{\mathrm{ref}}(\hat{\eta})}[\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}]$–mod (see [40, Proposition 8]) and the second from $\mathsf{Vect}_{\mathbb{C}}^{\tilde{\tau}_\pi^{\mathrm{ref}}(\tilde{\eta})}(\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}) \simeq \mathbb{C}^{\tilde{\tau}_\pi^{\mathrm{ref}}(\tilde{\eta})}[\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}]$–mod$^{\mathrm{R}}$ (see Proposition 3.4). The third isomorphism is proved in the same way as the second. □

Only the final two isomorphisms of Proposition 4.2 involve a form of Real equivariant K–theory of G. The first isomorphism still has a Real flavour, however, as it involves the unoriented loop groupoid $\Lambda_\pi^{\mathrm{ref}} B\hat{\mathsf{G}}$. The second isomorphism is the finite analogue of the result of Fok [18]. It would be interesting to combine these results to describe the twisted $\hat{\mathsf{G}}$–equivariant K–theory of G for $\hat{\mathsf{G}}$ an arbitrary $\mathbb{Z}_2$–graded compact Lie group.

Writing the appropriate loop groupoid as a disjoint union of standard models, as in Proposition 1.1, leads to a decomposition of the representation categories of thickened Drinfeld doubles. In this way we obtain Real generalizations of Dijkgraaf–Pasquier–Roche induction [12, Section 2.2; 40, Section 3.3]. For example,

$$D^{\hat{\eta}}(\hat{\mathsf{G}})\text{–mod} \simeq \bigoplus_{g \in \pi_0(\mathsf{G}/\!/_{\mathrm{R}}\hat{\mathsf{G}})} \mathsf{Vect}_{\mathbb{C}}^{\tau_\pi^{\mathrm{ref}}(\hat{\eta})|_g}(BZ_{\hat{\mathsf{G}}}^{\mathrm{R}}(g)),$$

where $Z_{\hat{\mathsf{G}}}^{\mathrm{R}}(g) = \{\omega \in \hat{\mathsf{G}} \mid \omega g^{\pi(\omega)}\omega^{-1} = g\}$ is the Real centralizer of g. Simple $D^{\hat{\eta}}(\hat{\mathsf{G}})$–modules are therefore labelled by a Real conjugacy class of G and a simple twisted representation of its Real centralizer. Similarly, Proposition 3.5 shows that $\mathbb{D}^{\tilde{\eta}}(\hat{\mathsf{G}})$–mod$^{\mathrm{R}}$ decomposes as

$$\bigoplus_{g \in \pi_0(\mathsf{G}/\!/_{\mathrm{R}}\hat{\mathsf{G}})_{-1}} \mathsf{Vect}_{\mathbb{C}}^{\tilde{\tau}_\pi^{\mathrm{ref}}(\tilde{\eta})|_g}(BZ_{\hat{\mathsf{G}}}^{\mathrm{R}}(g)) \oplus \bigoplus_{\{g,\bar{g}\} \in \pi_0(\mathsf{G}/\!/_{\mathrm{R}}\hat{\mathsf{G}})_1} \mathsf{Vect}_{\mathbb{C}}^{\tilde{\tau}_\pi^{\mathrm{ref}}(\tilde{\eta})|_{\{g,\bar{g}\}}}(\mathsf{G}/\!/_{\mathrm{R}}\hat{\mathsf{G}}|_{\{g,\bar{g}\}}).$$

The first sum is over conjugacy classes of G which are fixed by the involution determined by $\hat{\mathsf{G}}$ and the second is over the $\mathbb{Z}_2$–quotient of its complement. The previous two decompositions, and the quasi-inverse from the proof of Proposition 3.5, therefore yield a construction of a representation of the thickened Drinfeld double from a collection of twisted (Real) representations of the groupoids appearing on the right-hand side.

4.2 Twisted one-loop characters

The next definition is motivated by the definition of twisted elliptic characters of [40, Section 3.4], or [25, Section 6] in the untwisted case.

Definition 4.3 Let $\hat{\eta} \in Z^{3+\pi_{\hat{\mathsf{G}}}}(B\hat{\mathsf{G}})$. Elements of $\Gamma_{\Lambda\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}}(\uptau\uptau_\pi^{\mathrm{ref}}(\hat{\eta})_{\mathbb{C}})$ are called $\hat{\eta}$–twisted one-loop characters of G.

To make this definition explicit, let $\mathsf{G}^{(2)} \subset \mathsf{G}^2$ be the set of commuting pairs in G and let

$$\hat{\mathsf{G}}^{\langle 2\rangle} := \{(g,\omega) \in \mathsf{G}\times\hat{\mathsf{G}} \mid g\omega = \omega g^{\pi(\omega)}\}$$

be the set of graded commuting pairs in G. Then a twisted one-loop character is a function $\chi\colon \hat{\mathsf{G}}^{\langle 2\rangle} \to \mathbb{C}$ which satisfies

$$\chi(\sigma g^{\pi(\sigma)}\sigma^{-1}, \sigma\omega\sigma^{-1}) = \uptau\uptau_\pi^{\mathrm{ref}}(\hat{\eta})([\sigma]g \xrightarrow{\omega} g)\chi(g,\omega), \quad \sigma \in \hat{\mathsf{G}}.$$

Here $[\sigma]g \xrightarrow{\omega} g$ denotes the 1–chain on $\Lambda\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}$ corresponding to the morphism σ with source $(g \xrightarrow{\omega} g) \in \mathrm{Obj}(\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}})$.

The relevance of this definition to the representation theory of $D^{\hat{\eta}}(\hat{\mathsf{G}})$ is as follows:

Proposition 4.4 *The character map is an isometry*

$$\chi\colon K_0(D^{\hat{\eta}}(\hat{\mathsf{G}})\text{–mod}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \Gamma_{\Lambda\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}}(\uptau\uptau_\pi^{\mathrm{ref}}(\hat{\eta})_{\mathbb{C}}).$$

Proof Since $D^{\hat{\eta}}(\hat{\mathsf{G}}) = \mathbb{C}^{\uptau_\pi^{\mathrm{ref}}(\hat{\eta})}[\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}]$, this follows from [40, Theorem 11]. $\square$

Corollary 4.5 *The number of simple $D^{\hat{\eta}}(\hat{\mathsf{G}})$–modules is $\int_{\Lambda^2\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}} \uptau^2\uptau_\pi^{\mathrm{ref}}(\hat{\eta})$.*

Proof Proposition 4.4 implies that

$$\operatorname{rank} K_0(D^{\hat{\eta}}(\hat{\mathsf{G}})\text{–mod}) = \dim_{\mathbb{C}} \Gamma_{\Lambda\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}}(\uptau\uptau_\pi^{\mathrm{ref}}(\hat{\eta})_{\mathbb{C}}).$$

By [40, Theorem 6], the right-hand side is equal to $\int_{\Lambda^2\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}} \uptau^2\uptau_\pi^{\mathrm{ref}}(\hat{\eta})$. $\square$

The term one-loop character is motivated by two-dimensional unoriented topological (or conformal) field theory. The closed one-loop sector of such a theory comprises the 2–torus $\mathbb{T}^2$ and the Klein bottle $\mathbb{K}$. Consider the decomposition

$$\Gamma_{\Lambda\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}}(\uptau\uptau_\pi^{\mathrm{ref}}(\hat{\eta})_{\mathbb{C}}) = \Gamma^{\mathbb{T}^2}_{\Lambda\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}}(\uptau\uptau_\pi^{\mathrm{ref}}(\hat{\eta})_{\mathbb{C}}) \oplus \Gamma^{\mathbb{K}}_{\Lambda\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}}}(\uptau\uptau_\pi^{\mathrm{ref}}(\hat{\eta})_{\mathbb{C}}), \tag{23}$$

where the first and second summands consist of suitably $\hat{\mathsf{G}}$–equivariant functions on the subsets $\mathsf{G}^{(2)}$ and $(\mathsf{G}\times(\hat{\mathsf{G}}\setminus\mathsf{G}))\cap\hat{\mathsf{G}}^{\langle 2\rangle}$ of $\hat{\mathsf{G}}^{\langle 2\rangle} = \mathrm{Obj}(\Lambda\Lambda_\pi^{\mathrm{ref}}B\hat{\mathsf{G}})$, respectively. The first summand in (23) is a subspace of the space $\Gamma_{\Lambda^2 B\mathsf{G}}(\uptau^2(\eta)_{\mathbb{C}})$ of η–twisted elliptic characters. Note that

$$\Lambda^2 B\mathsf{G} \simeq \mathcal{B}un_{\mathsf{G}}(\mathbb{T}^2),$$

the groupoid of principal G–bundles over $\mathbb{T}^2$. The equivalence assigns to an element $(g_1, g_2) \in \mathsf{G}^{(2)} = \mathrm{Obj}(\Lambda^2 B\mathsf{G})$ the G–bundle whose holonomies along the two standard 1–cycles of $\mathbb{T}^2$ are g_1 and g_2. In this way, the first summand in (23) admits a natural interpretation in terms of $\mathcal{B}un_{\mathsf{G}}(\mathbb{T}^2)$. The second summand in (23) consists of what we call $\hat{\eta}$–twisted Klein characters. To explain its moduli-theoretic meaning, let $\mathrm{or}_{\mathbb{K}} \to \mathbb{K}$ be the orientation double cover of $\mathbb{K}$, so that $\mathrm{or}_{\mathbb{K}} \simeq \mathbb{T}$. Let $\mathcal{B}un^{\mathrm{or}}_{\hat{\mathsf{G}}}(\mathbb{K})$ be the groupoid of principal $\hat{\mathsf{G}}$–bundles $P \to \mathbb{K}$ together with the data of an isomorphism of double covers

$$P \times_{\hat{\mathsf{G}}} \mathbb{Z}_2 \simeq \mathrm{or}_{\mathbb{K}}. \tag{24}$$

We refer to such bundles as $\mathrm{or}_{\mathbb{K}}$–twisted G–bundles. Using holonomies, $\mathcal{B}un^{\mathrm{or}}_{\hat{\mathsf{G}}}(\mathbb{K})$ is seen to be equivalent to the full subgroupoid of $\Lambda\Lambda^{\mathrm{ref}}_{\pi} B\hat{\mathsf{G}}$ on $(\mathsf{G} \times (\hat{\mathsf{G}} \setminus \mathsf{G})) \cap \hat{\mathsf{G}}^{\langle 2 \rangle}$. At the level of objects, a point $(g, \omega) \in (\mathsf{G} \times (\hat{\mathsf{G}} \setminus \mathsf{G})) \cap \hat{\mathsf{G}}^{\langle 2 \rangle}$ defines a homomorphism

$$\pi_1(\mathbb{K}) \simeq \langle a, b \mid ab^{-1}a = b \rangle \to \hat{\mathsf{G}}^{\langle 2 \rangle}, \quad a \mapsto g,\ b \mapsto \omega,$$

which we interpret as the holonomy of a $\hat{\mathsf{G}}$–bundle $P \to \mathbb{K}$. That $\pi(\omega) = -1$ ensures that the isomorphism (24) holds. In this way, $\Gamma^{\mathbb{K}}_{\Lambda\Lambda^{\mathrm{ref}}_{\pi} B\hat{\mathsf{G}}}(\uptau\uptau^{\mathrm{ref}}_{\pi}(\hat{\eta})_{\mathbb{C}})$ is identified with the space of flat sections of a complex line bundle over $\mathcal{B}un^{\mathrm{or}}_{\hat{\mathsf{G}}}(\mathbb{K})$. Groupoids of orientation twisted G–bundles are developed in detail in [43, Section 3.2], uniformly for all compact manifolds, in the context of unoriented Dijkgraaf–Witten theory. See Section 4.5 below for another appearance of these groupoids.

Straightforward modifications of the previous discussion apply to $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})$ and $\mathbb{D}^{\tilde{\eta}}(\hat{\mathsf{G}})$. We limit ourselves to describing the character theory of $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})$, which is rather different from that of $D^{\hat{\eta}}(\hat{\mathsf{G}})$. Theorem 3.12 gives

$$K_0(\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})\text{–mod}^{\mathrm{R}}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \Gamma_{\Lambda^{\mathrm{ref}}_{\pi}\Lambda_{\pi} B\hat{\mathsf{G}}}(\uptau^{\mathrm{ref}}_{\pi}\uptau_{\pi}(\hat{\eta})_{\mathbb{C}}).$$

The right-hand side is the set of functions $\chi\colon \mathsf{G}^{(2)} \to \mathbb{C}$ which satisfy

$$\chi(\sigma g_1 \sigma^{-1}, \sigma g_2^{\pi(\sigma)} \sigma^{-1}) = \uptau^{\mathrm{ref}}_{\pi}\uptau_{\pi}(\hat{\eta})([\sigma] g_1 \xrightarrow{g_2} g_1) \cdot \chi(g_1, g_2), \quad \sigma \in \hat{\mathsf{G}}.$$

Characters of Real $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})$–representations therefore form a subspace of the space of η–twisted elliptic characters. In particular, there is no Klein bottle sector in the character theory of $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})$.

4.3 Real pointed fusion categories and their Drinfeld centres

In this section, we define a categorical version of the $\hat{\theta}$–twisted Real groupoid algebra $\mathbb{C}^{\hat{\theta}}[\hat{\mathcal{G}}]$ of Section 3.2. We use coefficients $\mathsf{A} = \mathbb{C}^{\times}$. We could also use $\mathsf{U}(1)$–coefficients at the expense of incorporating unitary structures in what follows. For background on monoidal categories, the reader is referred to [16].

To begin, we introduce a Real version of monoidal categories.

Definition 4.6 A Real monoidal category is a $\mathbb{C}$–linear abelian category $\mathcal{C}$ with

(i) a decomposition $\mathcal{C} = \mathcal{C}^{(1)} \oplus \mathcal{C}^{(-1)}$ into full abelian subcategories;

(ii) an additive functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which restricts to $\mathbb{C}$–bilinear functors
$$\otimes\colon \mathcal{C}^{(i)} \times {}^{i}\mathcal{C}^{(j)} \to \mathcal{C}^{(ij)}, \quad i, j \in \mathbb{Z}_2,$$
where ${}^{+1}\mathcal{C}^{(j)} = \mathcal{C}^{(j)}$ and ${}^{-1}\mathcal{C}^{(j)}$ is the complex conjugate category of $\mathcal{C}^{(j)}$;

(iii) an object $\mathbf{1} \in \mathcal{C}$, together with left and right unitors; and

(iv) for homogeneous objects $X_k \in \mathcal{C}$ for $k = 1, 2, 3$, natural $\mathbb{C}$–linear associativity isomorphisms $\alpha_{X_3,X_2,X_1}\colon (X_3 \otimes X_2) \otimes X_1 \xrightarrow{\sim} X_3 \otimes (X_2 \otimes X_1)$

such that the evident triangle and pentagon axioms hold.

In particular, underlying a Real monoidal category $\mathcal{C}$ is an $\mathbb{R}$–linear monoidal category. Moreover, the subcategory $\mathcal{C}^{(1)} \subset \mathcal{C}$ is a $\mathbb{C}$–linear monoidal category which contains $\mathbf{1}$. The Real monoidal category $\mathcal{C}$ can therefore be understood as a twisted $\mathbb{Z}_2$–graded extension of $\mathcal{C}^{(1)}$, where the twist involves complex conjugation.

Definition 4.7 (i) A Real fusion category is a finite semisimple Real monoidal category which is rigid and has a simple monoidal unit.

(ii) A Real fusion category is called pointed if its simple objects are invertible.

Real pointed fusion categories and their $\mathbb{C}$–linear monoidal equivalences form a groupoid RPFus.

Example Let $\widehat{\mathsf{G}}$ be a finite $\mathbb{Z}_2$–graded group with twisted 3–cocycle $\hat{\eta} \in Z^{3+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}})$. Let $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}})$ be the $\mathbb{C}$–linear category of finite-dimensional $\widehat{\mathsf{G}}$–graded complex vector spaces. We write objects of $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}})$ as $V = \bigoplus_{\omega \in \widehat{\mathsf{G}}} V_{\omega}$. Given $\omega \in \widehat{\mathsf{G}}$, let $\mathbb{C}_{\omega} \in \mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}})$ be the simple object which is a copy of $\mathbb{C}$ in degree ω. Any simple object of $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}})$ is isomorphic to one of this form. The $\mathbb{Z}_2$–grading $\pi_{\widehat{\mathsf{G}}}$ induces a decomposition
$$\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}}) = \mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}})^{(1)} \oplus \mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}})^{(-1)}.$$

Define a Real monoidal structure $\otimes$ on $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}})$ by

$$(V^{(2)} \otimes V^{(1)})_\omega = \bigoplus_{\substack{\omega_1,\omega_2 \in \widehat{\mathsf{G}} \\ \omega = \omega_2\omega_1}} V^{(2)}_{\omega_2} \otimes_{\mathbb{C}} {}^{\pi(\omega_2)}V^{(1)}_{\omega_1},$$

with a similar formula for morphisms. The associator component

$$(V^{(3)}_{\omega_3} \otimes_{\mathbb{C}} {}^{\pi(\omega_3)}V^{(2)}_{\omega_2}) \otimes_{\mathbb{C}} {}^{\pi(\omega_3\omega_2)}V^{(1)}_{\omega_1} \to V^{(3)}_{\omega_3} \otimes_{\mathbb{C}} ({}^{\pi(\omega_3)}V^{(2)}_{\omega_2} \otimes_{\mathbb{C}} {}^{\pi(\omega_3\omega_2)}V^{(1)}_{\omega_1})$$

is $\widehat{\eta}\big([\omega_3 \mid \omega_2 \mid \omega_1]\big)$ times the canonical associator. The pentagon axiom is equivalent to the twisted 3–cocycle condition on $\widehat{\eta}$. Let $\mathbf{1} = \mathbb{C}_e$ with right and left unitors

$$\rho_V : V \otimes \mathbb{C}_e \to V, \quad v_\omega \otimes c \mapsto {}^{\pi(\omega)}c v_\omega, \qquad \text{and} \qquad \lambda_V : \mathbb{C}_e \otimes V \to V, \quad c \otimes v_\omega \mapsto c v_\omega.$$

Define the dual V^* of V by $(V^*)_\omega = {}^{\pi(\omega)}V^\vee_{\omega^{-1}}$. The nonstandard evaluation and coevaluation maps are

$$\widetilde{\mathrm{ev}} : V \otimes V^* \to \mathbb{C}_e, \quad v_\omega \otimes {}^{\pi(\omega)}f_\sigma \mapsto \delta_{\omega,\sigma^{-1}} \widehat{\eta}\big([\omega \mid \omega^{-1} \mid \omega]\big) f_\sigma(v_\omega),$$

where $\delta_{?,?}$ is a delta function, and

$$\mathrm{coev}_\omega : \mathbb{C}_e \to \mathbb{C}_\omega \otimes \mathbb{C}^*_\omega, \quad 1 \mapsto \widehat{\eta}\big([\omega \mid \omega^{-1} \mid \omega]\big)^{-1} \mathrm{coev}(1), \quad \omega \in \widehat{\mathsf{G}}.$$

Then $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}})$ is a Real pointed fusion category. The subcategory $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}})^{(1)} \simeq \mathsf{Vect}_{\mathbb{C}}^{\eta}(\mathsf{G})$ is the $\mathbb{C}$–linear pointed fusion category associated to G and the restricted 3–cocycle $\eta \in Z^3(B\mathsf{G})$, as described in [16, Example 2.3.8].

The group $\mathrm{Aut}_{\mathsf{Grp}/\mathbb{Z}_2}(\widehat{\mathsf{G}})$ of $\mathbb{Z}_2$–graded group automorphisms of $\widehat{\mathsf{G}}$ acts by pullback on $H^{3+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}})$. The following result, which we include for completeness, describes the category RPFus. The corresponding $\mathbb{C}$–linear result is well known.

Proposition 4.8 (i) *Any object of* RPFus *is equivalent to a Real pointed fusion category of the form* $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}})$.

(ii) *There is a short exact sequence of groups*

$$1 \to H^{2+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}}) \to \pi_0 \mathrm{Aut}_{\mathsf{RPFus}}(\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}})) \to \mathrm{Stab}_{\mathrm{Aut}_{\mathsf{Grp}/\mathbb{Z}_2}(\widehat{\mathsf{G}})}(\widehat{\eta}) \to 1,$$

where $\pi_0 \mathrm{Aut}_{\mathsf{RPFus}}(\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}}))$ *is the group of isomorphism classes of autoequivalences of* $\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}}) \in \mathsf{RPFus}$.

Proof Let $\mathcal{C} \in \mathsf{RPFus}$. Its group $\widehat{\mathsf{G}}$ of isomorphism classes of simple objects inherits a natural $\mathbb{Z}_2$–grading from the $\otimes$–compatible decomposition $\mathcal{C} = \mathcal{C}^{(1)} \oplus \mathcal{C}^{(-1)}$. After choosing a representative simple object for each element of $\widehat{\mathsf{G}}$, the components of the associator at triples of simple objects define a cocycle $\hat{\eta} \in Z^{3+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}})$. By finite semisimplicity of $\mathcal{C}$, the full inclusion $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}}) \hookrightarrow \mathcal{C}$ is an equivalence.

Consider the second statement. Any element $\Phi \in \mathrm{Aut}_{\mathsf{RPFus}}(\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}}))$ preserves the set of simple objects with its $\mathbb{Z}_2$–grading. Using this observation, it easy to see that the sequence is right exact. Suppose then that Φ is the identity on objects. The component of the monoidal data

$$\Phi(\mathbb{C}_{\omega_2}) \otimes \Phi(\mathbb{C}_{\omega_1}) \xrightarrow{\sim} \Phi(\mathbb{C}_{\omega_2} \otimes \mathbb{C}_{\omega_1}), \quad \omega_i \in \widehat{\mathsf{G}},$$

is an endomorphism of $\mathbb{C}_{\omega_2\omega_1}$ and so is multiplication by a complex number, say $\hat{\theta}\big([\omega_2 \mid \omega_1]\big)$. Compatibility of the monoidal data with the associator is the cocycle condition $\hat{\theta} \in Z^{2+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}})$. Changing Φ within its isomorphism class changes $\hat{\theta}$ by an exact 2–cocycle. This completes the proof. $\square$

We can now prove a 2–categorical analogue of Theorem 3.17.

Theorem 4.9 *There is an $\mathbb{R}$–linear equivalence of categories*

$$Z_D(\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}})) \simeq \mathsf{Vect}_{\mathbb{C}}^{\tau_\pi(\hat{\eta})^{-1}}(\Lambda_\pi B\widehat{\mathsf{G}}),$$

where the left-hand side is the Drinfeld centre of $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}})$.

Proof Let $(V, \beta) \in Z_D(\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}}))$, so that $V \in \mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\widehat{\mathsf{G}})$ and $\beta\colon - \otimes V \Rightarrow V \otimes -$ is a $\mathbb{C}$–linear natural isomorphism which satisfies a hexagon axiom; see [16, Section 7.13]. The component of β at $\mathbb{C}_\omega$ is the data of $\mathbb{C}$–linear isomorphisms

$$\beta_{\delta,\omega}\colon \mathbb{C}_\omega \otimes_{\mathbb{C}} {}^{\pi(\omega)}V_\delta \xrightarrow{\sim} V_{\omega\delta\omega^{-1}} \otimes_{\mathbb{C}} {}^{\pi(\delta)}\mathbb{C}_\omega, \quad \delta \in \widehat{\mathsf{G}}.$$

If V_δ is nonzero for some $\delta \in \widehat{\mathsf{G}} \setminus \mathsf{G}$, then $\beta_{\delta,e}(c \otimes v_\delta) = v_\delta \otimes \bar{c}$. Consider the morphism $m_\lambda\colon \mathbb{C}_e \to \mathbb{C}_e$ given by multiplication by $\lambda \in \mathbb{C}$. Naturality of β in $\mathbb{C}_e$ requires commutativity of the diagram

$$\begin{tikzcd}
\mathbb{C}_e \otimes_{\mathbb{C}} V_\delta \arrow[r, "\beta_{\delta,e}"] \arrow[d, "m_\lambda \otimes \mathrm{id}_{V_\delta}"'] & V_\delta \otimes_{\mathbb{C}} \overline{\mathbb{C}}_e \arrow[d, "\mathrm{id}_{V_\delta} \otimes \bar{m}_\lambda"] \\
\mathbb{C}_e \otimes_{\mathbb{C}} V_\delta \arrow[r, "\beta_{\delta,e}"] & V_\delta \otimes_{\mathbb{C}} \overline{\mathbb{C}}_e
\end{tikzcd}$$

which is the case if and only if $\lambda \in \mathbb{R}$. It follows that V is supported on G. The hexagon axiom for (V, β) implies that the remaining structure maps $\beta_{g,\omega}$ for $(g, \omega) \in \mathsf{G} \times \widehat{\mathsf{G}}$, give V the structure of a $\tau_\pi(\widehat{\eta})^{-1}$–twisted vector bundle over $\Lambda_\pi B\widehat{\mathsf{G}}$. □

Being a Drinfeld double, $Z_D(\mathsf{Vect}_{\mathbb{C}}^{\widehat{\eta}}(\widehat{\mathsf{G}}))$ has canonical $\mathbb{R}$–linear monoidal structure. The category $\mathsf{Vect}_{\mathbb{C}}^{\tau_\pi(\widehat{\eta})^{-1}}(\Lambda_\pi B\widehat{\mathsf{G}})$ then inherits a monoidal structure via Theorem 4.9. For ease of notation, we replace η with η^{-1} in what follows. Under the equivalence $\mathsf{Vect}_{\mathbb{C}}^{\tau_\pi(\widehat{\eta})}(\Lambda_\pi B\widehat{\mathsf{G}}) \simeq \mathbb{D}^{\widehat{\eta}}(\widehat{\mathsf{G}})\text{–mod}^{\mathsf{R}}$, the monoidal structure of $\mathsf{Vect}_{\mathbb{C}}^{\tau_\pi(\widehat{\eta})^{-1}}(\Lambda_\pi B\widehat{\mathsf{G}})$ is induced by a quasi-coassociative coproduct

$$\Delta \colon \mathbb{D}^{\widehat{\eta}}(\widehat{\mathsf{G}}) \to \mathbb{D}^{\widehat{\eta}}(\widehat{\mathsf{G}}) \otimes_{\mathbb{C}} \mathbb{D}^{\widehat{\eta}}(\widehat{\mathsf{G}}).$$

To describe this, first recall that the subgroup $\mathrm{Inn}(\widehat{\mathsf{G}}) \leq \mathrm{Aut}_{\mathsf{Grp}/\mathbb{Z}_2}(\widehat{\mathsf{G}})$ of inner automorphisms acts trivially on $H^{3+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}})$. More precisely, for $g_i \in \mathsf{G}$ and $\omega \in \widehat{\mathsf{G}}$, we have

$$\frac{\widehat{\eta}\big([\omega g_3\omega^{-1} \mid \omega g_3\omega^{-1} \mid \omega g_3\omega^{-1}]\big)}{\widehat{\eta}\big([g_3 \mid g_2 \mid g_1]\big)^{\pi(\omega)}} = dc_\omega\big([g_3 \mid g_2 \mid g_1]\big), \tag{25}$$

where $c_\omega \in C^2(B\mathsf{G})$ is given by (see [29, Proposition 8.1])

$$c_\omega\big([g_2 \mid g_1]\big) = \frac{\widehat{\eta}\big([\omega g_2\omega^{-1} \mid \omega \mid g_1]\big)}{\widehat{\eta}\big([\omega \mid g_2 \mid g_1]\big)\widehat{\eta}\big([\omega g_2\omega^{-1} \mid \omega g_1\omega^{-1} \mid \omega]\big)}.$$

Let $l_{g\xrightarrow{\omega}} \in \mathbb{D}^{\widehat{\eta}}(\widehat{\mathsf{G}})$ be the basis vector corresponding to the morphism $\omega \colon g \to \omega g\omega^{-1}$ in $\Lambda_\pi B\widehat{\mathsf{G}}$. Define

$$\Delta(l_{g\xrightarrow{\omega}}) = \sum_{\substack{g_1,g_2\in\mathsf{G} \\ g_2g_1=g}} c_\omega\big([g_2 \mid g_1]\big) l_{g_2\xrightarrow{\omega}} \otimes l_{g_1\xrightarrow{\omega}}$$

with associator

$$\Phi = \sum_{g_1,g_2,g_3\in\mathsf{G}} \widehat{\eta}\big([g_3 \mid g_2 \mid g_1]\big) l_{g_3\xrightarrow{e}} \otimes l_{g_2\xrightarrow{e}} \otimes l_{g_1\xrightarrow{e}}.$$

Formally, these are the same definitions as for the quasibialgebra $D^\eta(\mathsf{G})$ [12]. The defining equation (25) implies that Φ is an associator; that is, Δ is coassociative up to conjugation by Φ. The equation

$$\frac{\tau_\pi(\widehat{\eta})\big([\omega_2 \mid \omega_1]g_2\big)\tau_\pi(\widehat{\eta})\big([\omega_2 \mid \omega_1]g_1\big)}{\tau_\pi(\widehat{\eta})\big([\omega_2 \mid \omega_1]g_2g_1\big)} = \frac{c_{\omega_2\omega_1}\big([g_2 \mid g_1]\big)}{c_{\omega_1}\big([g_2 \mid g_1]\big)^{\pi(\omega_2)} \cdot c_{\omega_2}\big([\omega_1 g_2\omega_1^{-1} \mid \omega_1 g_1\omega_1^{-1}]\big)},$$

which can be proved in the same way as [29, Corollary 8.3], implies that Δ is a morphism of Real algebras. We summarize the above structure by saying that $\mathbb{D}^{\hat{\eta}}(\hat{\mathsf{G}})$ is a Real quasibialgebra. Note that $D^{\eta}(\mathsf{G})$ is a complex subquasibialgebra of $D^{\hat{\eta}}(\hat{\mathsf{G}})_{\mathrm{R}}$. In this way, we obtain a quasibialgebraic interpretation of $Z_D(\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}}))$ and its $\mathbb{C}$–linear monoidal subcategory $Z_D(\mathsf{Vect}_{\mathbb{C}}^{\eta}(\mathsf{G}))$.

4.4 Twisted loop transgression and Real 2–representation theory

We use the categorified Real group algebra $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}})$ of Section 4.3 to formulate the Real 2–representation theory of G. The latter theory, which concerns the 2–categorical generalization of Sections 3.2 and 3.3, is developed in detail in [44], so we limit ourselves to explaining the connection to the twisted transgression map $\tau_{\pi}^{\mathrm{ref}}$.

We require a Real generalization of the bicategory of module categories over a $\mathbb{C}$–linear monoidal category, as described in [16, Section 7.2]. Let $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}})\text{–mod}^{\mathrm{R}}$ be the bicategory whose objects are left Real $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}})$–module categories, that is, pairs $(F,\mathcal{M})$ consisting of a finite semisimple $\mathbb{C}$–linear category $\mathcal{M}$ and a $\mathbb{C}$–linear monoidal functor $F\colon \mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}}) \to \mathrm{End}_{\mathbb{R}}(\mathcal{M})$ such that each functor

$$F(\mathbb{C}_{\omega})\colon {}^{\pi(\omega)}\mathcal{M} \to \mathcal{M}, \quad \omega \in \hat{\mathsf{G}},$$

is $\mathbb{C}$–linear. The 1–morphisms of $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}})\text{–mod}^{\mathrm{R}}$ are $\mathbb{C}$–linear functors with intertwining natural isomorphisms for the $\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}})$–actions while 2–morphisms are their compatible $\mathbb{C}$–linear natural transformations. With these definitions, there is an $\mathbb{R}$–linear biequivalence

$$\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}})\text{–mod}^{\mathrm{R}} \simeq 2\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(B\hat{\mathsf{G}}), \tag{26}$$

where the right-hand side is the bicategory of $\hat{\eta}$–twisted Real 2–representations of G on Kapranov–Voevodsky 2–vector spaces, as defined in [44, Section 5.4]. The biequivalence (26), which restricts to the previously known $\mathbb{C}$–linear biequivalence $\mathsf{Vect}_{\mathbb{C}}^{\eta}(\mathsf{G})\text{–mod} \simeq 2\mathsf{Vect}_{\mathbb{C}}^{\eta}(B\mathsf{G})$, is the 2–categorical analogue of Proposition 3.4.

One of the main outcomes of [44] is the existence of a categorified character theory for $2\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(B\hat{\mathsf{G}})$. Its basic structure is summarized by the following theorem:

Theorem 4.10 [44] *A Real 2–representation $\rho \in 2\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(B\hat{\mathsf{G}})$ has*

(i) *a Real categorical character $\mathbb{T}\mathrm{r}_{\rho} \in \mathsf{Vect}_{\mathbb{C}}^{\tau_{\pi}^{\mathrm{ref}}(\hat{\eta})}(\Lambda_{\pi}^{\mathrm{ref}} B\hat{\mathsf{G}})$, and*

(ii) *a Real 2–character $\chi_{\rho} \in \Gamma_{\Lambda\Lambda_{\pi}^{\mathrm{ref}} B\hat{\mathsf{G}}}(\tau\tau_{\pi}^{\mathrm{ref}}(\hat{\eta})_{\mathbb{C}})$.*

The proof of Theorem 4.10(i) can be seen as a 2–categorical upgrade of the proof of Proposition 3.9. In particular, $\tau_\pi^{\mathrm{ref}}(\hat{\eta})$ arises through its explicit expression from Section 2.3. Part (ii) follows from part (i) once it is known that $\tau_\pi^{\mathrm{ref}}(\hat{\eta})$ is closed. While it is possible (but unpleasant) to verify closedness of $\tau_\pi^{\mathrm{ref}}(\hat{\eta})$ directly, this follows immediately from Section 2.3. Finally, we note that the naive 2–categorical analogue of Theorem 3.12 fails, so that the equivalence class of a Real 2–representation is not, in general, determined by its Real categorical character and 2–character. A different description of equivalence classes of objects of $2\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(B\hat{\mathsf{G}})$ is given in [34]. It should be noted that this description involves, in a nonobvious way, Real 2–characters.

Remark It can be shown that there is an $\mathbb{R}$–linear monoidal equivalence

$$Z(2\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(B\hat{\mathsf{G}})) \simeq Z_D(\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(\hat{\mathsf{G}})), \tag{27}$$

the left-hand side being the monoidal category of pseudonatural transformations of the identity pseudofunctor of $2\mathsf{Vect}_{\mathbb{C}}^{\hat{\eta}}(B\hat{\mathsf{G}})$. In view of (26), the equivalence (27) can be proved in the same way as [5, Corollary 5.3]. The equivalence (27) leads to a reformulation of Theorem 4.9 as a direct 2–categorical analogue of Theorem 3.17.

4.5 Discrete torsion in string theory and M–theory with orientifolds

In this section, we show that τ_π^{ref} encodes one-loop discrete torsion phases in orientifold string theory and M–theory. At a practical level, the fact that τ_π^{ref} is a chain map implies the (higher) gauge invariance of explicit discrete torsion phases, something which is verified by hand in physical approaches. The appearance of τ_π^{ref} is natural from the geometric picture of discrete torsion, reviewed below, in which the B–field amplitude is interpreted as a pushforward in differential cohomology. Moreover, our results suggest new representation-theoretic structures in M–theory. There is a well-established connection between discrete torsion in orbifold string theory on $X/\!/\mathsf{G}$ and twisted representations of G on (the vector space fibres of) Chan–Paton bundles on D–branes [15; 37, Section V]. In the setting of orientifold strings, Real twisted representations of G, as discussed in Section 3.2, are instead relevant [6, Section 3.4]. The 3–form C–field, which plays the role of the 2–form B–field in string theory, leads to a categorification of the string theory setting, in which $M2$–branes are endowed with 2–vector bundles [10]. The appearance of τ_π^{ref} in M–theory discrete torsion, explained below, together with the central role of τ_π^{ref} in Real categorical representation theory, recalled in Section 4.4, suggests that for M–theory orbifolds the fibres of these 2–vector bundles carry the structure of a twisted Real categorical representation.

To begin, we recall the definition of the B–field amplitude in oriented string theory and its relation to discrete torsion. For detailed treatments, see [37; 14]. The oriented case is well studied in both mathematics and physics and forms the basis for the orientifold generalizations which follow.

The spacetime of oriented string theory is an orbifold $\mathfrak{X}$; there are additional data and conditions on $\mathfrak{X}$, depending on the form of string theory under consideration, but we do not require these for the present discussion. A worldsheet with target $\mathfrak{X}$ is a closed oriented surface Σ and a smooth map $\varphi\colon \Sigma \to \mathfrak{X}$. A (gauge equivalence class of a) B–field on $\mathfrak{X}$ is a differential cohomology class $\check{B} \in \check{H}^3(\mathfrak{X})$. Schematically, the partition function of the worldsheet theory takes the form

$$Z = \int_{\varphi \in \mathrm{Map}(\Sigma,\mathfrak{X})} \exp\biggl(2\pi i \int_{\Sigma} \varphi^* \check{B}\biggr) \cdots D\varphi,$$

where $\int_{\Sigma}\colon \check{H}^3(\Sigma) \to \check{H}^1(\mathrm{pt}) \simeq \mathbb{R}/\mathbb{Z}$ is pushforward along $\Sigma \to \mathrm{pt}$. The omitted terms in Z are not required in what follows. While Z is not mathematically defined, the B–field amplitude $\exp\bigl(2\pi i \int_{\Sigma} \varphi^* \check{B}\bigr)$ is; it is this quantity which leads to discrete torsion.

Let $\mathfrak{X} = X /\!\!/ \mathsf{G}$ be the orbifold quotient of a smooth manifold X by a finite group G. Postcomposition with the canonical map $p\colon X /\!\!/ \mathsf{G} \to B\mathsf{G}$ defines

$$p \circ (-)\colon \mathrm{Map}(\Sigma, X /\!\!/ \mathsf{G}) \to \mathrm{Map}(\Sigma, B\mathsf{G}). \tag{28}$$

The connected components $\pi_0\mathrm{Map}(\Sigma, B\mathsf{G})$ label isomorphism classes of G–bundles over Σ. Given $f \in \pi_0\mathrm{Map}(\Sigma, B\mathsf{G})$, the subgroupoid $\mathrm{Map}(\Sigma, X /\!\!/ \mathsf{G})_f \subset \mathrm{Map}(\Sigma, X /\!\!/ \mathsf{G})$ of maps $\varphi\colon \Sigma \to X /\!\!/ \mathsf{G}$ for which $p \circ \varphi$ is in the component f is called the f–twisted sector. Maps in this sector can be interpreted as G–equivariant maps $P_f \to X$, where $P_f \to \Sigma$ is a G–bundle classified by f. In particular, when f classifies the trivial G–bundle, this is a map $\Sigma \to X$, that is, a worldsheet in the theory with target X. A special class of B–fields on $X /\!\!/ \mathsf{G}$ arises from classes $\theta \in H^2(B\mathsf{G})$ via the composition

$$H^2(B\mathsf{G}) \xrightarrow{p^*} H^2(X /\!\!/ \mathsf{G}) \hookrightarrow \check{H}^3(X /\!\!/ \mathsf{G}), \quad \theta \mapsto \check{B}_\theta, \tag{29}$$

where the second map is the inclusion of flat B–fields. Then $\exp\bigl(2\pi i \int_{\Sigma} \varphi^* \check{B}_\theta\bigr)$ is constant on $\mathrm{Map}(\Sigma, X /\!\!/ \mathsf{G})_f$ and independent of X. The partition function Z is therefore expected to decompose as

$$Z = \sum_{f \in \pi_0\mathrm{Map}(\Sigma, B\mathsf{G})} \exp\biggl(2\pi i \int_{\Sigma} f^*\theta\biggr) \int_{\varphi \in \mathrm{Map}(\Sigma,\mathfrak{X})_f} \cdots D\varphi. \tag{30}$$

In this way, the discrete torsion phases $\exp\bigl(2\pi i \int_\Sigma f^*\theta\bigr)$ modify the contribution of each twisted sector to the partition function Z. Expressions for $\exp\bigl(2\pi i \int_\Sigma f^*\theta\bigr)$ for simple surfaces are given in [37; 31; 24].

Remark The discovery [39] of discrete torsion was via algebraic, rather than geometric, techniques. Indeed, discrete torsion was discovered as modularity-preserving phase ambiguities in the torus (or one-loop) partition function in orbifold conformal field theory. In the unoriented setting, the one-loop sector, which now includes a Klein bottle worldsheet, again plays a distinguished role [3].

Our interest is in discrete torsion in orientifold string theory. For detailed discussions of orientifold strings, see [38; 14; 23]. The spacetime in an orientifold theory is an orbifold double cover $\pi\colon \mathfrak{X} \to \hat{\mathfrak{X}}$. A worldsheet is a closed surface Σ, not assumed oriented or orientable, a smooth map $\varphi\colon \Sigma \to \hat{\mathfrak{X}}$ and a lift of φ to double covers $\tilde{\varphi}\colon \mathrm{or}_\Sigma \to \mathfrak{X}$. Denote by $\mathrm{Map}^{\mathrm{or}}(\Sigma, \hat{\mathfrak{X}})$ the space of worldsheets. The B–field $\check{B} \in \check{H}^{3+\pi}(\hat{\mathfrak{X}})$ lies in the differential cohomology of $\hat{\mathfrak{X}}$ twisted by the double cover $\mathfrak{X}$. The partition function of the worldsheet theory takes the schematic form

$$Z = \int_{\varphi \in \mathrm{Map}^{\mathrm{or}}(\Sigma, \hat{\mathfrak{X}})} \exp\biggl(2\pi i \int_\Sigma \varphi^* \check{B}\biggr) \cdots D\varphi,$$

where now $\int_\Sigma$ is the twisted pushforward $\check{H}^{3+\mathrm{or}_\Sigma}(\Sigma) \to \check{H}^1(\mathrm{pt})$ and $\tilde{\varphi}$ is used to identify $\check{H}^{3+\varphi^*\pi}(\Sigma)$ with $\check{H}^{3+\mathrm{or}_\Sigma}(\Sigma)$.

The global quotient setting is a double cover $\pi\colon X/\!\!/\mathsf{G} \to X/\!\!/\hat{\mathsf{G}}$ arising from the action of a finite $\mathbb{Z}_2$–graded group $\hat{\mathsf{G}}$ on X. The map $\hat{p}\colon X/\!\!/\hat{\mathsf{G}} \to B\hat{\mathsf{G}}$ defines

$$\hat{p} \circ (-)\colon \mathrm{Map}^{\mathrm{or}}(\Sigma, X/\!\!/\hat{\mathsf{G}}) \to \mathrm{Map}^{\mathrm{or}}(\Sigma, B\hat{\mathsf{G}}).$$

A component $f \in \pi_0 \mathrm{Map}^{\mathrm{or}}(\Sigma, B\hat{\mathsf{G}})$ classifies an object $P_f \in \mathcal{B}un^{\mathrm{or}}_{\hat{\mathsf{G}}}(\Sigma)$, in the notation of Section 4.2, and maps in the f–twisted sector admit an interpretation in terms of $\hat{\mathsf{G}}$–equivariant maps $P_f \to X$. In other words, topological types of or_Σ–twisted G–bundles label orientifold twisted sectors. The twisted analogue of the sequence (29) associates to a class $\hat{\theta} \in H^{2+\pi_{\hat{\mathsf{G}}}}(B\hat{\mathsf{G}})$ a flat B–field $\check{B}_{\hat{\theta}} \in \check{H}^{3+\pi}(X/\!\!/\hat{\mathsf{G}})$ whose amplitude $\exp\bigl(2\pi i \int_\Sigma \varphi^* \check{B}_{\hat{\theta}}\bigr)$ is constant on each twisted sector and independent of X, leading to a decomposition of Z analogous to (30).

Let $\hat{\theta} \in Z^{2+\pi_{\hat{\mathsf{G}}}}(B\hat{\mathsf{G}})$. Computations using conformal field theory [3, (16)] or equivariant structures on gerbes with connection [38, Section 5.1] give, for the Klein bottle discrete torsion phase,

$$\widehat{\theta}\big([g^{-1} \mid g]\big)^{-1} \frac{\widehat{\theta}\big([g \mid \omega]\big)}{\widehat{\theta}\big([\omega \mid g^{-1}]\big)}, \tag{31}$$

where $g \in \mathsf{G}$ and $\omega \in \widehat{\mathsf{G}} \setminus \mathsf{G}$ satisfy $\omega g^{-1} \omega^{-1} = g$. This expression is recovered, and its meaning clarified, by twisted transgression. The iterated transgression $\tau\tau_{\pi}^{\mathrm{ref}}(\widehat{\theta})$ is the locally constant function on $\Lambda\Lambda_{\pi}^{\mathrm{ref}} B\widehat{\mathsf{G}}$ whose value on $(g, \omega) \in \mathrm{Obj}(\Lambda\Lambda_{\pi}^{\mathrm{ref}} B\widehat{\mathsf{G}}) = \mathsf{G}^{\langle 2 \rangle}$ is

$$\widehat{\theta}\big([g^{-1} \mid g]\big)^{-\Delta_\omega} \frac{\widehat{\theta}\big([g \mid \omega]\big)}{\widehat{\theta}\big([\omega \mid g^{\pi(\omega)}]\big)}.$$

This specializes to (31) when $\pi(\omega) = -1$ and to the well-known expression for the discrete torsion phases of $\mathbb{T}^2$ when $\pi(\omega) = 1$ [39; 36, Section IV.B]. In other words, by interpreting $\Lambda\Lambda_{\pi}^{\mathrm{ref}} B\widehat{\mathsf{G}}$ as $\mathcal{B}un_{\mathsf{G}}(\mathbb{T}^2) \sqcup \mathcal{B}un_{\widehat{\mathsf{G}}}^{\mathrm{or}}(\mathbb{K})$, as described below Corollary 4.5, we see that the function $\tau\tau_{\pi}^{\mathrm{ref}}(\widehat{\theta})$ simply records to the discrete torsion phase associated to each one-loop orientifold twisted sector.

The discussion for M–theory is formally similar to that for string theory. We refer the reader to [36; 38; 11; 42, Section IV] for details and subtleties in the definition of a C–field. We take as the M–theory spacetime an orbifold $\mathfrak{Y}$. A basic field in M–theory is the C–field, (to a first approximation[1]) a class $\check{C} \in \check{H}^{4+\mathrm{or}_{\mathfrak{Y}}}(\mathfrak{Y})$. An $M2$–brane is a closed 3–manifold W with a smooth map $\varphi \colon W \to \mathfrak{Y}$ and a lift $\widetilde{\varphi}$ of φ to orientation double covers. Among other factors, the integrand of the partition function of the $M2$–brane theory on W contains the C–field amplitude $\exp\big(2\pi i \int_W \varphi^* \check{C}\big)$.

In the global quotient setting, let Y be an oriented manifold on which a finite $\mathbb{Z}_2$–graded group $\widehat{\mathsf{G}}$ acts by orientation-preserving/reversing diffeomorphisms according to the grading $\pi \colon \widehat{\mathsf{G}} \to \mathbb{Z}_2$. Then $\mathfrak{Y} = Y /\!/ \widehat{\mathsf{G}}$ is an M–theory spacetime and $\check{C} \in \check{H}^{4+\pi}(Y /\!/ \widehat{\mathsf{G}})$ is twisted by the double cover $\pi \colon Y /\!/ \mathsf{G} \to Y /\!/ \widehat{\mathsf{G}}$. Associated to a class $\widehat{\eta} \in H^{3+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}})$ is a flat C–field $\check{C}_{\widehat{\eta}} \in \check{H}^{4+\pi}(Y /\!/ \widehat{\mathsf{G}})$, leading to discrete torsion phases in the $M2$–brane partition function.

Consider $\widehat{\eta} \in Z^{3+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}})$ and its iterated transgression $\tau^2\tau_{\pi}^{\mathrm{ref}}(\widehat{\eta})$, a locally constant function on $\Lambda^2\Lambda_{\pi}^{\mathrm{ref}} B\widehat{\mathsf{G}}$. There is a bijection of $\mathrm{Obj}(\Lambda^2\Lambda_{\pi}^{\mathrm{ref}} B\widehat{\mathsf{G}})$ with

$$\widehat{\mathsf{G}}^{\langle 3 \rangle} := \{(g, \omega_1, \omega_2) \in \mathsf{G} \times \widehat{\mathsf{G}}^2 \mid (g, \omega_i) \in \widehat{\mathsf{G}}^{\langle 2 \rangle},\ (\omega_1, \omega_2) \in \widehat{\mathsf{G}}^{(2)}\}.$$

[1] Depending on the topology of $\mathfrak{Y}$, it may only be the difference of two C–fields that lies in $\check{H}^{4+\mathrm{or}_{\mathfrak{Y}}}(\mathfrak{Y})$; see [41, (1.2); 11, Section 2; 42, Section IV.A]. In this case, we would consider below the translate of a background C–field by $\widehat{\eta} \in H^{3+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}})$. The element $\widehat{\eta}$ is still the source of discrete torsion.

The set $\widehat{\mathsf{G}}^{\langle 3\rangle}$ decomposes according to the degrees of its elements. Holonomy considerations, analogous to those discussed in the dimensional case, interpret this decomposition as orientation twisted G–bundles over a closed 3–manifold W whose orientation double cover is $\mathbb{T}^3 \sqcup \mathbb{T}^3$ or $\mathbb{T}^3$. We view this manifold as comprising an M–theoretic analogue of the one-loop sector in string theory. When $\pi(\omega_1) = \pi(\omega_2) = 1$, we have $W \simeq \mathbb{T}^3$ and the value of $\tau^2\tau_\pi^{\mathrm{ref}}(\hat{\eta})$ on $(g, \omega_1, \omega_2) \in \widehat{\mathsf{G}}^{\langle 3\rangle}$ is

$$\frac{\hat{\eta}([\omega_2 \mid g \mid \omega_1])\hat{\eta}([\omega_1 \mid \omega_2 \mid g])\hat{\eta}([g \mid \omega_2 \mid \omega_1])}{\hat{\eta}([\omega_2 \mid \omega_1 \mid g])\hat{\eta}([g \mid \omega_2 \mid \omega_1])\hat{\eta}([\omega_1 \mid g \mid \omega_2])}. \tag{32}$$

The remaining cases, in which at least one ω_i has degree -1, have $W \simeq \mathbb{K} \times S^1$. For example, when $\pi(\omega_1) = 1 = -\pi(\omega_2)$, the value of $\tau^2\tau_\pi^{\mathrm{ref}}(\hat{\eta})$ on (g, ω_1, ω_2) is

$$\frac{\hat{\eta}([g^{-1}|g|\omega_1])\hat{\eta}([\omega_1|g^{-1}|g])}{\hat{\eta}([g^{-1}|\omega_1|g])}\frac{\hat{\eta}([\omega_1|\omega_2|g^{-1}])\hat{\eta}([g|\omega_1|\omega_2])\hat{\eta}([\omega_2|g^{-1}|\omega_1])}{\hat{\eta}([\omega_1|g|\omega_2])\hat{\eta}([\omega_2|\omega_1|g^{-1}])\hat{\eta}([g|\omega_2|\omega_1])}. \tag{33}$$

The function $\tau^2\tau_\pi^{\mathrm{ref}}(\hat{\eta})$ is a geometric encoding of one-loop discrete torsion phases in M–theory. Indeed, the phases (32) and (33) appear in the work of Sharpe[2] [38, Section 6.2] as discrete torsion phases of $\mathbb{T}^3$ and $\mathbb{K} \times S^1$, where they were derived using a Čech description of equivariant structures on 2–gerbes with connection. Since τ_π^{ref} is a chain map, it is immediate that (32) and (33) depend only on the class $[\hat{\eta}] \in H^{3+\pi_{\widehat{\mathsf{G}}}}(B\widehat{\mathsf{G}})$. In Sharpe's approach, this higher gauge invariance must be verified by lengthy direct calculations.

Discrete torsion phases for general 3–manifolds are computed by the three-dimensional unoriented Dijkgraaf–Witten theory associated to the pair $(\widehat{\mathsf{G}}, \hat{\eta})$ [43]; in the string-theoretic setting, one considers instead the two-dimensional theory associated to $(\widehat{\mathsf{G}}, \hat{\theta})$. These computations can be formulated in terms of orientation-twisted transgression along the 3–manifold [43, Section 4.2]. Only in particular geometric settings, such as the one-loop case considered above or 3–manifolds arising as S^1–fibrations, can one formulate the computations in terms of iterated transgression along simpler manifolds.

References

[1] **M F Atiyah**, *K–theory and reality*, Q. J. Math. 17 (1966) 367–386 MR Zbl

[2] **M F Atiyah**, **G B Segal**, *Equivariant K–theory and completion*, J. Differential Geom. 3 (1969) 1–18 MR Zbl

[2] In Sharpe's notation, $g_1 = \omega_2$, $g_2 = g$ and $g_3 = \omega_1$. The equality of the phase (33) with Sharpe's phase follows from the identity $\hat{\eta}([g^{-1} \mid g \mid \omega_1])\hat{\eta}([\omega_1 \mid g^{-1} \mid g])/\hat{\eta}([g^{-1} \mid \omega_1 \mid g]) = \hat{\eta}([\omega_1 \mid g \mid g^{-1}])\hat{\eta}([g \mid g^{-1} \mid \omega_1])/\hat{\eta}([g \mid \omega_1 \mid g^{-1}])$ for $(g, \omega_1) \in \mathsf{G}^2$.

[3] **P Bantay**, *Symmetric products, permutation orbifolds and discrete torsion*, Lett. Math. Phys. 63 (2003) 209–218 MR Zbl

[4] **M Barkeshli**, **P Bonderson**, **M Cheng**, **C-M Jian**, **K Walker**, *Reflection and time reversal symmetry enriched topological phases of matter: path integrals, non-orientable manifolds, and anomalies*, Comm. Math. Phys. 374 (2020) 1021–1124 MR Zbl

[5] **E Bernaschini**, **C Galindo**, **M Mombelli**, *Group actions on 2–categories*, Manuscripta Math. 159 (2019) 81–115 MR Zbl

[6] **V Braun**, *Orientifolds and K–theory*, from "Progress in string, field and particle theory" (L Baulieu, E Rabinovici, J Harvey, B Pioline, P Windey, editors), NATO Sci. Ser. II 104, Springer (2003) 369–372 Zbl

[7] **T Bröcker**, **T tom Dieck**, *Representations of compact Lie groups*, Graduate Texts in Math. 98, Springer (1985) MR Zbl

[8] **J-L Brylinski**, *Loop spaces, characteristic classes and geometric quantization*, Progr. Math. 107, Birkhäuser, Boston, MA (1993) MR Zbl

[9] **J-L Brylinski**, **D A McLaughlin**, *The geometry of degree-four characteristic classes and of line bundles on loop spaces, I*, Duke Math. J. 75 (1994) 603–638 MR Zbl

[10] **S Bunk**, **R J Szabo**, *Fluxes, bundle gerbes and 2–Hilbert spaces*, Lett. Math. Phys. 107 (2017) 1877–1918 MR Zbl

[11] **E Diaconescu**, **G Moore**, **D S Freed**, *The M-theory 3–form and E_8 gauge theory*, from "Elliptic cohomology" (H R Miller, D C Ravenel, editors), Lond. Math. Soc. Lect. Note Ser. 342, Cambridge Univ. Press (2007) 44–88 MR Zbl

[12] **R Dijkgraaf**, **V Pasquier**, **P Roche**, *Quasi Hopf algebras, group cohomology and orbifold models*, Nuclear Part. Phys. Proc. 18B (1990) 60–72 MR Zbl

[13] **J Distler**, **D S Freed**, **G W Moore**, *Orientifold précis*, from "Mathematical foundations of quantum field theory and perturbative string theory" (H Sati, U Schreiber, editors), Proc. Sympos. Pure Math. 83, Amer. Math. Soc., Providence, RI (2011) 159–172 MR Zbl

[14] **J Distler**, **D S Freed**, **G W Moore**, *Spin structures and superstrings*, from "Surveys in differential geometry, XV: Perspectives in mathematics and physics" (T Mrowka, S-T Yau, editors), Surv. Differ. Geom. 15, International, Somerville, MA (2011) 99–130 MR Zbl

[15] **M Douglas**, *D-branes and discrete torsion*, preprint (1998) arXiv hep-th/9807235

[16] **P Etingof**, **S Gelaki**, **D Nikshych**, **V Ostrik**, *Tensor categories*, Math. Surv. Monogr. 205, Amer. Math. Soc., Providence, RI (2015) MR Zbl

[17] **C-K Fok**, *The real K–theory of compact Lie groups*, Symmetry Integrability Geom. Methods Appl. 10 (2014) art. id. 022 MR Zbl

[18] **C-K Fok**, *Equivariant twisted real K–theory of compact Lie groups*, J. Geom. Phys. 124 (2018) 325–349 MR Zbl

[19] **D S Freed**, *Higher algebraic structures and quantization*, Comm. Math. Phys. 159 (1994) 343–398 MR Zbl

[20] **D S Freed**, **M J Hopkins**, **C Teleman**, *Loop groups and twisted K–theory, I*, J. Topol. 4 (2011) 737–798 MR Zbl

[21] **D S Freed**, **G W Moore**, *Twisted equivariant matter*, Ann. Henri Poincaré 14 (2013) 1927–2023 MR Zbl

[22] **J Fuchs**, **C Schweigert**, **A Valentino**, *A geometric approach to boundaries and surface defects in Dijkgraaf–Witten theories*, Comm. Math. Phys. 332 (2014) 981–1015 MR Zbl

[23] **K Gawędzki**, **R R Suszek**, **K Waldorf**, *Bundle gerbes for orientifold sigma models*, Adv. Theor. Math. Phys. 15 (2011) 621–687 MR Zbl

[24] **K Gomi**, **Y Terashima**, *Discrete torsion phases as topological actions*, Comm. Math. Phys. 287 (2009) 889–901 MR Zbl

[25] **M J Hopkins**, **N J Kuhn**, **D C Ravenel**, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc. 13 (2000) 553–594 MR Zbl

[26] **A Kapustin**, **A Turzillo**, *Equivariant topological quantum field theory and symmetry protected topological phases*, J. High Energy Phys. 2017 (2017) art. id. 006 MR Zbl

[27] **M Karoubi**, *Sur la K–théorie équivariante*, from "Séminaire Heidelberg–Saarbrücken–Strasbourg sur la K–théorie" (A Dold, B Eckmann, editors), Lecture Notes in Math. 136, Springer (1970) 187–253 MR Zbl

[28] **G Karpilovsky**, *Projective representations of finite groups*, Monogr. Text. Pure Appl. Math. 94, Dekker, New York (1985) MR Zbl

[29] **M Kim**, *Arithmetic Chern–Simons theory, I*, from "Galois covers, Grothendieck–Teichmüller theory and dessins d'enfants" (F Neumann, S Schroll, editors), Springer Proc. Math. Stat. 330, Springer (2020) 155–180 MR Zbl

[30] **L Kong**, **Y Tian**, **S Zhou**, *The center of monoidal 2–categories in $3+1$D Dijkgraaf–Witten theory*, Adv. Math. 360 (2020) art. id. 106928 MR Zbl

[31] **E Lupercio**, **B Uribe**, *Loop groupoids, gerbes, and twisted sectors on orbifolds*, from "Orbifolds in mathematics and physics" (A Adem, J Morava, Y Ruan, editors), Contemp. Math. 310, Amer. Math. Soc., Providence, RI (2002) 163–184 MR Zbl

[32] **E-k M Moutuou**, *Twisted groupoid KR–theory*, PhD thesis, Université de Lorraine (2012) Available at `https://hal.univ-lorraine.fr/tel-01749211`

[33] **W F Reynolds**, *Twisted group algebras over arbitrary fields*, Illinois J. Math. 15 (1971) 91–103 MR Zbl

[34] **D Rumynin**, **M B Young**, *Burnside rings for real 2–representation theory: the linear theory*, Commun. Contemp. Math. 23 (2021) art. id. 2050012 MR Zbl

[35] **U Schreiber**, **C Schweigert**, **K Waldorf**, *Unoriented WZW models and holonomy of bundle gerbes*, Comm. Math. Phys. 274 (2007) 31–64 MR Zbl

[36] **E Sharpe**, *Analogues of discrete torsion for the M-theory three-form*, Phys. Rev. D 68 (2003) art. id. 126004 MR

[37] **E Sharpe**, *Discrete torsion*, Phys. Rev. D 68 (2003) art. id. 126003 MR

[38] **E Sharpe**, *Notes on discrete torsion in orientifolds*, J. Geom. Phys. 61 (2011) 1017–1032 MR Zbl

[39] **C Vafa**, *Modular invariance and discrete torsion on orbifolds*, Nuclear Phys. B 273 (1986) 592–606 MR Zbl

[40] **S Willerton**, *The twisted Drinfeld double of a finite group via gerbes and finite groupoids*, Algebr. Geom. Topol. 8 (2008) 1419–1457 MR Zbl

[41] **E Witten**, *On flux quantization in M-theory and the effective action*, J. Geom. Phys. 22 (1997) 1–13 MR Zbl

[42] **E Witten**, *The "parity" anomaly on an unorientable manifold*, Phys. Rev. B 94 (2016) art. id. 195150

[43] **M B Young**, *Orientation twisted homotopy field theories and twisted unoriented Dijkgraaf–Witten theory*, Comm. Math. Phys. 374 (2020) 1645–1691 MR Zbl

[44] **M B Young**, *Real representation theory of finite categorical groups*, High. Struct. 5 (2021) 18–70 MR Zbl

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