TORSION MODELS FOR TENSOR-TRIANGULATED CATEGORIES: THE ONE-STEP CASE

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ABSTRACT. Given a suitable stable monoidal model category $\mathcal{C}$ and a specialization closed subset V of its Balmer spectrum one can produce a Tate square for decomposing objects into the part supported over V and the part supported over V^c spliced with the Tate object. Using this one can show that $\mathcal{C}$ is Quillen equivalent to a model built from the data of local torsion objects, and the splicing data lies in a rather rich category. As an application, we promote the torsion model for the homotopy category of rational circle-equivariant spectra from [18] to a Quillen equivalence. In addition, a close analysis of the one step case highlights important features needed for general torsion models which we will return to in future work.

CONTENTS

1. INTRODUCTION

The goal of this series of papers is to prove the existence of a model for a tensor-triangulated category T which is built from local torsion objects. When T is the derived category $\mathsf{D}(R)$ of a commutative Noetherian ring R, this corresponds to building a module from its local cohomology at each prime, and when working in chromatic homotopy theory this corresponds to building a spectrum from its monochromatic pieces. The main observation is that the splicing data lies over completed rings, making it more algebraic. The motivating example is the torsion model of [18] which is an algebraic counterpart of reconstructing a rational circle-equivariant spectrum from pieces with a single geometric isotropy group, and this work began with a view to extending this model to higher dimensional tori.

1.1. **Overview.** Given a tensor-triangulated category T one can consider the Balmer spectrum $\mathrm{Spc}^\omega(\mathsf{T})$ of its compact objects, which is a spectral topological space categorifying the Zariski spectrum of a commutative ring. This allows one to apply methods from commutative algebra in more general contexts. The reader may wish to consider the special case $\mathsf{T} = \mathsf{D}(R)$ for a commutative Noetherian ring R whilst reading the motivation. We will give the general details later.

In reconstructing an R-module M there are several approaches. The *Zariski standard model* reconstructs M from its localizations $L_\wp M := M_\wp$ at primes $\wp$, giving the equivalence between R-modules and quasi-coherent sheaves over $\mathrm{Spc}^\omega(\mathsf{T})$ where the rather complicated reconstruction data is encoded in the topology on the étale space of the sheaf. This has the feature that $L_\wp M$ will usually have support spread over several primes.

We may instead assemble a module from its torsion pieces supported at individual primes; in effect, we reconstruct M from its local cohomology $L_\wp \Gamma_\wp M$ (essentially $H^*_\wp(M_\wp)$) at the various primes. In other words, we reconstruct a module M from a version of its Cousin complex. We call this the *Zariski torsion model*.

As an alternative to using localizations, we may reconstruct M from its localized completions $(M^\wedge_\wp)_\wp$ as in [3], which we call the *adelic standard model* (properly speaking, if M is not compact we use $L_\wp \Lambda_\wp \mathbb{1} \otimes M$; the model based on $L_\wp \Lambda_\wp M$ is the *adelic categorical model* which will not concern us here). The advantage is two fold. First, we may use adelic structure to stick the pieces together, which is simpler and more algebraic than for the Zariski standard model, and second the localized completed rings are more likely to be algebraic, or even formal.

Finally, there is the *adelic torsion model*, which is the subject of this paper: we are once again reconstructing M from its local cohomology, but now working over completed rings. Since local cohomology does not notice completions, the pieces themselves are essentially the same as for the torsion version of the Zariski model, but now the reconstruction takes place in a more algebraic environment: it is this richer version that we call the adelic torsion model. Henceforth we will refer to the adelic standard and torsion models simply as the standard and torsion models, unless we wish to emphasize the distinction between the adelic and the Zariski approach.

The standard and torsion approaches have different advantages and disadvantages. The standard approach behaves well for monoidal structures. Since completion is exact on finitely generated modules, the splicing data in the standard approach is often represented at the abelian category level and one hopes to have lower homological dimension in algebraic versions. On the other hand objects tend to be quite large from a naive point of view. Objects in the torsion model tend to be smaller, but monoidal information is lost. Furthermore, the local cohomology of nice modules is in the top cohomological degree, so the splicing information in the torsion model tends to be only visible at the derived level and the homological dimension in algebraic versions is larger.

At this point, it is worth considering the simple example of the derived category $\mathsf{D}(\mathbb{Z}_{(p)})$ of the p-local integers. We consider the reconstruction of a $\mathbb{Z}_{(p)}$-module M in each of the four models.

- In the Zariski standard model, since $\mathbb{Z}_{(p)}$ is a local ring, the module itself occurs as the stalk over p, so the Zariski model does not simplify anything.
- In the Zariski torsion model we consider M as built from $M \otimes \mathbb{Q}$ (the contribution at (0)) and $M \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$ (derived tensor, the contribution at (p)).
- In the adelic standard model we consider M built from $M \otimes \mathbb{Q}$ (the contribution at (0)) and $M \otimes \mathbb{Z}^\wedge_p$ (the contribution at (p)).
- In the adelic categorical model we consider M built from $M \otimes \mathbb{Q}$ (the contribution at (0)) and $\Lambda_p M$ (the contribution at (p)).

- In the adelic torsion model we again consider M as built from $M \otimes \mathbb{Q}$ (the contribution at (0)) and $\Lambda_p M \otimes \mathbb{Q}/\mathbb{Z}_{(p)} = M \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$, but now the latter is viewed as a module over the p-adic integers.

Looking at an example from equivariant homotopy illustrates the distinction between the Zariski torsion model and the adelic torsion model more starkly, but we will return to that when we have more language.

1.2. The dimension filtration.

It is natural to mimic algebraic geometry, filtering the space $\mathrm{Spc}^\omega(\mathsf{T})$ by the dimension of its primes

$$\emptyset = \mathrm{Spc}^\omega(\mathsf{T})_{\leq -1} \subseteq \mathrm{Spc}^\omega(\mathsf{T})_{\leq 0} \subseteq \mathrm{Spc}^\omega(\mathsf{T})_{\leq 1} \subseteq \cdots \subseteq \mathrm{Spc}^\omega(\mathsf{T})$$

where $\mathrm{Spc}^\omega(\mathsf{T})_{\leq s}$ consists of those primes of dimension $\leq s$. Following [6] the category T has a corresponding filtration $\mathsf{T}_0 \subseteq \mathsf{T}_1 \subseteq \mathsf{T}_2 \subseteq \cdots \subseteq \mathsf{T}$ by dimension of support. In any case, the set $\mathrm{Spc}^\omega(\mathsf{T})_{\leq s} \setminus \mathrm{Spc}^\omega(\mathsf{T})_{\leq s-1}$ consists of primes of exact dimension s. With the torsion approach, the Verdier quotient $\mathsf{T}_s/\mathsf{T}_{s-1}$ splits as a sum of contributions from primes of exact dimension s. For the torsion model, the part over a prime $\mathfrak{p}$ is $\mathfrak{p}$-local and torsion.

In view of this splitting, our main focus is on how to splice together the part of T coming from primes of dimension s to the part of T supported on the generalization closed subset $U = \mathrm{Spc}^\omega(\mathsf{T})_{>s}$ of primes of dimension $> s$. It is in this splicing that the distinction between the Zariski and adelic torsion models appears. We will focus our attention on categories where the Balmer spectrum is Noetherian since the splicing is significantly simpler in that case.

In the present paper we consider the case when there is just one step in the assembly process. We will follow this process through for the torsion model in a higher dimensional Noetherian setting in [4]. Ensuring that the splicing data is algebraic requires us to refine the naive iterative approach.

When the Balmer spectrum is not Noetherian, the Zariski topology is no longer generated by the closures of points, bringing a significant additional layer of complication: the general framework for assembling models from a good filtration of the Balmer spectrum in this context is studied in [2].

1.3. The one step context.

When we are dealing with just one step, it is helpful to consider a more general context so we can identify particular features which lead to good behaviour.

We begin with a smashing localization L giving the context of [19] from which we can construct our basic framework. The associated torsion functor is the fibre Γ, and this then also gives the associated completion $\Lambda = \mathrm{Hom}(\Gamma\mathbb{1}, -)$. Altogether we have the diagram

$$\begin{array}{ccccc} \Gamma\mathbb{1} & \longrightarrow & \mathbb{1} & \longrightarrow & L\mathbb{1} \\ \simeq\downarrow & & \downarrow & & \downarrow \\ \Gamma\Lambda\mathbb{1} & \longrightarrow & \Lambda\mathbb{1} & \longrightarrow & L\Lambda\mathbb{1} \end{array}$$

in which the right hand square is a homotopy pullback.

For the motivating example in our general programme, we choose a subset U of the spectrum $\mathrm{Spc}^\omega(\mathsf{T})$ closed under passage to larger primes, such as the set of primes of dimension $> s$. One may then form a smashing localization $X \to LX$ supported at U.

It may be helpful to consider three running examples to illustrate the discussion.

(1) The derived category $\mathsf{D}(\mathbb{Z}_{(p)})$ of the p-local integers, where we take L to be rationalization. Equivalently this is given by choosing U to consist of the zero ideal.

(2) The $E(n)$-local category of spectra $L_n\mathrm{Sp}$ (implicitly p-local). Here we take L to be the smashing localization at the Johnson–Wilson theory $E(n-1)$, which corresponds to setting U to be the collection of primes of dimension > 0.

(3) The category Sp_G of G-equivariant spectra for G a compact Lie group. Here we choose a family $\mathcal{F}$ of subgroups and take L to be the smashing localization making the underlying spectrum $\mathcal{F}$-contractible.

T	$\mathbb{1}$	$L\mathbb{1}$	$\Gamma\mathbb{1}$	$\Lambda\mathbb{1}$	$L\Lambda\mathbb{1}$
$\mathsf{D}(\mathbb{Z}_{(p)})$	$\mathbb{Z}_{(p)}$	$\mathbb{Q}$	$\Sigma^{-1}\mathbb{Z}/p^\infty$	$\mathbb{Z}_p^\wedge$	$\mathbb{Q}\otimes\mathbb{Z}_p^\wedge$
$L_n\mathrm{Sp}$	L_nS^0	$L_{n-1}S^0$	M_nS^0	$L_{K(n)}S^0$	$L_{n-1}L_{K(n)}S^0$
Sp_G	S^0	$\widetilde{E}\mathcal{F}$	$E\mathcal{F}_+$	$DE\mathcal{F}_+$	$\widetilde{E}\mathcal{F}\wedge DE\mathcal{F}_+$

For more details on these examples, see Section 5.

The above diagram shows that the *Tate square*

$$
\begin{array}{ccc}
\mathbb{1} & \longrightarrow & L\mathbb{1} \\
\downarrow & & \downarrow \\
\Lambda\mathbb{1} & \longrightarrow & L\Lambda\mathbb{1}
\end{array}
$$

expresses the unit $\mathbb{1}$ of T as the homotopy pullback of localized completed pieces. Moreover, the objects occurring in the Tate square are still monoid objects in T, and the square yields the standard model of T in terms of modules over the localized-completed pieces [3]. The price to pay for having monoids is that the completion functor involves some 'blurring' around V so that the objects may have contributions from open sets containing V (for example $\mathbb{Z}_p^\wedge$ has non-trivial rationalization). Indeed, if the completion had support precisely at V, then the Tate object $L\Lambda\mathbb{1}$ would be equivalent to the zero object and $\mathbb{1}$ would be a product. The fact that the Tate object $L\Lambda\mathbb{1}$ is non-zero is because of the blurring appearing in the completion Λ.

The current paper gives a related model built instead from pieces which are torsion in the sense of the functor Γ. When we use the dimension filtration in [4] the pieces will each be supported at a single prime.

The diagram

$$
\begin{array}{c}
L\mathbb{1} \\
\downarrow \\
\Lambda\mathbb{1} \longrightarrow L\Lambda\mathbb{1} \longrightarrow \Sigma\Gamma\Lambda\mathbb{1} \simeq \Sigma\Gamma\mathbb{1}
\end{array}
$$

shows the relationship between the standard model and the torsion model. The standard model comes from the left hand three objects. The torsion model comes from the right hand three objects. This is called the torsion model because the whole category relative to the local category is controlled by the torsion object $\Gamma\mathbb{1}$.

The two models determine each other. The three objects in the standard model give the Tate square by taking pullbacks, and hence the whole diagram by taking fibres. The three objects in the torsion model give the left hand object by taking the fibre of the horizontal and then one reconstructs the whole diagram as before.

1.4. **The torsion model.** Suppose that T is a tensor-triangulated category which is the homotopy category of a well behaved stable monoidal model category $\mathcal{C}$, see Section 3.2 for detailed conditions.

We now give an informal statement of the one-step torsion model. The model is in terms of the sections of the diagram

$$
\begin{array}{c}
\mathrm{Mod}_{L\mathbb{1}} \\
\vdots \\
\bullet \\
\downarrow \\
\mathrm{Mod}_{L\Lambda\mathbb{1}} \longrightarrow \mathrm{Mod}_{\Lambda\mathbb{1}}.
\end{array}
$$

This consists of an object in each vertex category together with maps relating these (after suitable extension of scalars). The dot on the vertical map indicates that the object in the bottom left category is obtained from that in the top left by extension of scalars, see Section 2.2. More precisely, an object of this diagram is a $\Lambda\mathbb{1}$-module map $\beta\colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to N$ where V is a $L\mathbb{1}$-module and N is a $\Lambda\mathbb{1}$-module.

The main theorem states that the model category $\mathcal{C}$ is Quillen equivalent to a cellularization of the category of sections which introduces the torsion character of the model. The following theorem specializes Theorem 8.4 and Theorem 8.7 in the main body of the paper.

Theorem. *There is a Quillen equivalence*

$$
\mathcal{C} \simeq_Q \text{cell-}\left(
\begin{array}{c}
\mathrm{Mod}_{L\mathbb{1}} \\
\vdots \\
\bullet \\
\downarrow \\
\mathrm{Mod}_{L\Lambda\mathbb{1}} \longrightarrow \mathrm{Mod}_{\Lambda\mathbb{1}}
\end{array}
\right).
$$

The effect of the cellularization is that an object of the tensor-triangulated category $h\mathcal{C}$ is equivalent to one specified by:

- *an $L\mathbb{1}$-module V;*
- *a $\Lambda\mathbb{1}$-module T which is torsion in the sense that $\Gamma T \simeq T$;*
- *a $\Lambda\mathbb{1}$-module map $L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to T$.*

It is rare for the monoids appearing in the model to have formal homotopy. However, one can hope to use the torsion model as a first step towards an algebraic classification, such as the construction of an Adams spectral sequence.

In very favourable cases the torsion model may be equivalent to the derived category of an abelian category. Currently the best examples of this come from rational equivariant cohomology theories. With this example in mind we can highlight the advantage of the adelic torsion model over the Zariski torsion model.

The Zariski torsion model corresponds to the usual isotropy separation filtration (constructing X from $X \wedge E\mathcal{F}_+$ and $X \wedge \widetilde{E}\mathcal{F}$). This is a very powerful method, but it is not very algebraic because the sphere is very far from being formal, and its coefficient ring is very complicated. By contrast, the adelic torsion model does the same thing relative to the Tate construction on the sphere (constructing X from $X \wedge E\mathcal{F}_+$ and $X \wedge DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}$). These spectra are both modules over the ring $DE\mathcal{F}_+$. If we are working rationally, then $DE\mathcal{F}_+$ has well understood homotopy and is formal, so working over this ring is a big advantage.

1.5. **Rational $\mathbb{T}$-spectra.** For any compact Lie group G, the second author conjectured the existence of a graded abelian category $\mathcal{A}(G)$ together with a Quillen equivalence

$$
\mathrm{Sp}_G/\mathbb{Q} \simeq_Q d\mathcal{A}(G)
$$

between rational G-spectra and differential objects in $\mathcal{A}(G)$. We call the category $\mathcal{A}(G)$ an *abelian model*, and $d\mathcal{A}(G)$ the *algebraic model*.

For the circle group $G = \mathbb{T}$, two abelian categories $\mathcal{A}(\mathbb{T})$ and $\mathcal{A}_t(\mathbb{T})$ are defined in [18] as follows. Let $\mathcal{O}_{\mathcal{F}} = \prod_{F \in \mathcal{F}} H^*(B\mathbb{T}/F)$ where $\mathcal{F}$ is the family of finite subgroups of $\mathbb{T}$. Inverting the Euler classes $\mathcal{E}$ gives a ring $t_{\mathcal{F}} = \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$. We define

$$\mathcal{A}(\mathbb{T}) = \{\beta \colon N \to \mathcal{O}_{\mathcal{F}} \otimes V \mid \beta \text{ is inversion of } \mathcal{E}\} \qquad \mathcal{A}_t(\mathbb{T}) = \{t_{\mathcal{F}} \otimes V \to T\}$$

where V is a graded $\mathbb{Q}$-vector space, N is an $\mathcal{O}_{\mathcal{F}}$-module and T is an Euler torsion $\mathcal{O}_{\mathcal{F}}$-module. For more details see Section 10. It is also shown in [18] that the derived categories of $\mathcal{A}(\mathbb{T})$ and $\mathcal{A}_t(\mathbb{T})$ are both equivalent to the homotopy category of rational $\mathbb{T}$-spectra. The category $\mathcal{A}(\mathbb{T})$ is called the *standard model* for rational $\mathbb{T}$-spectra, and the category $\mathcal{A}_t(\mathbb{T})$ is called the *torsion model* for rational $\mathbb{T}$-spectra. The standard model has better monoidal properties and lower homological dimension, but the torsion model is often more approachable (see [18, Part II]).

The equivalence $h\mathrm{Sp}_{\mathbb{T}} \simeq \mathsf{D}(\mathcal{A}(\mathbb{T}))$ between rational $\mathbb{T}$-spectra and the derived category of the standard model has since been promoted to a Quillen equivalence [27], shown to be monoidal in [9]. The general theory developed in this paper shows the derived equivalence $h\mathrm{Sp}_{\mathbb{T}} \simeq \mathsf{D}(\mathcal{A}_t(\mathbb{T}))$ between rational $\mathbb{T}$-spectra and the torsion model can be promoted to a Quillen equivalence, which moreover proves that the derived equivalence is triangulated. The following theorem appears in the main body of the paper as Theorem 10.8.

Theorem. *There is a Quillen equivalence*

$$\mathrm{Sp}_{\mathbb{T}} \simeq_Q d\mathcal{A}_t(\mathbb{T})$$

and hence a triangulated equivalence $h\mathrm{Sp}_{\mathbb{T}} \simeq_{\triangle} \mathsf{D}(\mathcal{A}_t(\mathbb{T}))$.

The proof of the equivalence $h\mathrm{Sp}_{\mathbb{T}} \simeq \mathsf{D}(\mathcal{A}_t(\mathbb{T}))$ given in [18] passes through the equivalence $h\mathrm{Sp}_{\mathbb{T}} \simeq \mathsf{D}(\mathcal{A}(\mathbb{T}))$ to the standard model, obscuring the relation between the topology and the abelian torsion model $\mathcal{A}_t(\mathbb{T})$. Our proof is more direct: it proceeds by splitting $\mathbb{T}$-spectra into its $\mathcal{F}$-local and $\mathcal{F}$-torsion parts and showing the resulting model is formal.

In future work [4], we will describe a torsion model for rational G-spectra when G is a torus of arbitrary rank.

Conventions. We will follow the convention of writing the left adjoint on the top in adjoint pairs displayed horizontally, and on the left in adjoint pairs displayed vertically. Weak equivalences have a direction, but weak equivalence of objects is the resulting equivalence relation. Similarly for Quillen equivalences. Given a (graded or ungraded) ring R we write Mod_R for the category of dg-R-modules. All cellularizations are at stable sets of objects.

2. Diagrams of module categories

Throughout the paper we freely use the language of model categories and use [32] as a reference. We briefly recall some results on categories of modules and on diagrams of model categories as they will be used repeatedly throughout.

2.1. **Module categories.** Let $\mathcal{C}$ be a monoidal model category and let R be a monoid in $\mathcal{C}$. Denote by $\mathrm{Mod}_R(\mathcal{C})$ the category of (left) R-modules in $\mathcal{C}$. If the ambient category is clear, we will omit it from the notation. We will be interested in two model structures on $\mathrm{Mod}_R(\mathcal{C})$:

- *projective model structure:* the weak equivalences and fibrations are determined in the underlying category $\mathcal{C}$. This exists under mild conditions on $\mathcal{C}$, see [35].
- *injective model structure:* the weak equivalences and cofibrations are determined in the underlying category $\mathcal{C}$. This exists under more stringent hypotheses on $\mathcal{C}$, see [29].

We will be particularly interested in the following algebraic cases.

Example 2.1. Let R be a graded ring. The category of dg-modules over R supports a projective model structure where the weak equivalences are the quasi-isomorphisms, the fibrations are the surjections, and the cofibrations are the monomorphisms which are split on the underlying graded modules and which have dg-projective cokernel. See [1] for details.

Example 2.2. The category of dg-modules over R also supports an injective model structure where the weak equivalences are the quasi-isomorphisms and the cofibrations are the monomorphisms, see [32, 2.3.13] for example. This is important for our paper for the following reason. For a finitely generated homogeneous ideal I, let us consider the full subcategory of I-power torsion modules. One observes that the category of torsion modules is abelian but does not have enough projectives, in particular, it does not support a projective model structure. However, one can show that it has enough injectives and so it does support a injective model structure, see [32, 2.3.13] and [23, 8.6].

Given a map of monoids $\theta\colon S \to R$ in $\mathcal{C}$ there is an adjoint triple

$$\mathrm{Mod}_S \underset{\xrightarrow{\hspace{1em} \theta_! \longrightarrow}}{\overset{\xleftarrow{\hspace{1em} \theta_* \longrightarrow}}{\xleftarrow{\hspace{1em} \theta^* \hspace{1em}}}} \mathrm{Mod}_R$$

given by restriction of scalars θ^*, extension of scalars $\theta_* = R \otimes_S -$ and coextension of scalars $\theta_! = \mathrm{Hom}_S(R, -)$. This triple of functors is not always Quillen as the following result shows.

Lemma 2.3 ([42, 3.8]). *Let $\theta\colon S \to R$ be a map of monoids in a monoidal model category $\mathcal{C}$. The extension of scalars functor θ_* is always left Quillen in the projective model structure. If the unit of $\mathcal{C}$ is cofibrant, then the restriction of scalars functor θ^* is left Quillen in the projective model structure if and only if R is cofibrant as an S-module.*

2.2. **Diagrams of model categories.** We recall the necessary background from [26].

Definition 2.4. Let D be a small category. A *diagram of model categories* of shape D is a collection of model categories $\mathcal{C}(d)$ for each $d \in \mathsf{D}$ together with a left Quillen functor $\mathcal{C}(f)\colon \mathcal{C}(d) \to \mathcal{C}(d')$ for each morphism $f\colon d \to d'$ in D.

Definition 2.5. Let $\mathcal{C}$ be a diagram of model categories of shape D. The *category of sections* of $\mathcal{C}$ consists of an object $X_d \in \mathcal{C}(d)$ for each $d \in \mathsf{D}$ together with maps $\mathcal{C}(f)(X_d) \to X_{d'}$ in $\mathcal{C}(d')$ for each morphism $f\colon d \to d'$ in D, which are compatible with composition.

We make a particular case of interest explicit.

Example 2.6. Let $T \xrightarrow{\theta} S \xleftarrow{\varphi} R$ be a diagram of monoids in a monoidal model category. If we assume that S is cofibrant as an R-module, then using the projective model structure in all three cases, Lemma 2.3 ensures that

$$\mathrm{Mod}_T \xrightarrow{\theta_*} \mathrm{Mod}_S \xrightarrow{\varphi^*} \mathrm{Mod}_R$$

is a diagram of model categories. The category of sections consists of the following data:

- a T-module L;
- an S-module M;
- an R-module N;
- an S-module map $\theta_* L \to M$;
- an R-module map $\varphi^* M \to N$.

The category of sections of $\mathcal{C}$ supports a 'diagram injective' model structure under some combinatorial conditions on the indexing category. Recall that D is called an *inverse category* if, informally, the morphisms in D^{op} only 'go in one direction'. We refer the reader to [32, 5.1.1] for the definition, and here instead give two main examples:

$$\bullet \to \bullet \leftarrow \bullet \quad \text{and} \quad \bullet \to \bullet \to \bullet.$$

We have the following result.

Theorem 2.7 ([26, 3.1])**.** *Let $\mathcal{C}$ be a diagram of model categories of shape D, where D is an inverse category. There is a diagram injective model structure on the category of sections of $\mathcal{C}$ with weak equivalences and cofibrations determined objectwise. Moreover if each $\mathcal{C}(d)$ is cellular and proper then so is the diagram injective model structure on $\mathcal{C}$.*

Notation 2.8. We will often consider diagrams of module categories as in Example 2.6. The category of sections admits a `dimp` (diagram injective, module projective) model structure and a `dimi` (diagram injective, module injective) model structure.

3. Tensor-triangulated categories and their enhancements

3.1. **Balmer spectrum.** We recall the necessary background from [5].

Definition 3.1. A *tensor triangulated category* consists of a triangulated category T and a symmetric monoidal tensor product $\otimes \colon \mathsf{T} \times \mathsf{T} \to \mathsf{T}$ which is exact in each variable.

A full subcategory of T is *thick* if it is closed under completing triangles and taking retracts. It is *localizing* if it is thick and closed under arbitrary sums. Finally, it is is an *ideal* if it contains zero and it is closed under tensoring with arbitrary objects of T. For a set of objects $\mathcal{K}$ in T, we write $\mathrm{Thick}(\mathcal{K})$ for the smallest thick subcategory of T containing $\mathcal{K}$, and $\mathrm{Loc}(\mathcal{K})$ for the smallest localizing subcategory of T containing $\mathcal{K}$. We write $\mathrm{Thick}_\otimes(\mathcal{K})$ for the smallest thick subcategory of T containing $\mathcal{K}$ and closed under tensoring with compact objects. We write $\mathrm{Loc}_\otimes(\mathcal{K})$ for the smallest localizing subcategory of T containing $\mathcal{K}$ and closed under tensoring with arbitrary objects. If $\mathbb{1}$ is a compact generator for T then $\mathrm{Thick}(\mathcal{K}) = \mathrm{Thick}_\otimes(\mathcal{K})$ and $\mathrm{Loc}(\mathcal{K}) = \mathrm{Loc}_\otimes(\mathcal{K})$.

To avoid set theoretical issues we will restrict to essentially small tensor triangulated categories. Concretely, this means that given a tensor triangulated category T, we will restrict to the full subcategory T^ω of compact objects; that is, those objects X of T for which the functor $[X, -]$ preserves arbitrary sums. Recall that an object X of T is said to be dualizable if the natural map $\underline{\mathrm{Hom}}(X, \mathbb{1}) \otimes Y \to \underline{\mathrm{Hom}}(X, Y)$ is an isomorphism for all $Y \in \mathsf{T}$. In general T^ω will not always be essentially small; however, it is when T is *rigidly-compactly generated*. This means that T has a set of compact and dualizable objects $\mathcal{G}$ that generates the whole category in the sense that $\mathrm{Loc}(\mathcal{G}) = \mathsf{T}$, and the monoidal unit $\mathbb{1}$ is compact. It follows that the compact objects and the dualizable objects in T coincide, see [33, 2.1.3(d)].

We list a few key examples of rigidly-compactly generated tensor triangulated categories.

Examples 3.2.

(a) The derived category $\mathsf{D}(R)$ of a commutative ring is a tensor triangulated category with respect to the (derived) tensor product of R-modules. It is compactly generated by R, and the full subcategory of compact objects coincides with that of perfect complexes, i.e., those complexes that are quasi-isomorphic to a bounded complex of finitely generated projectives.

(b) The stable homotopy category $h\mathrm{Sp}$ is a tensor triangulated category with respect to the (derived) smash product. It is compactly generated by the sphere spectrum S^0, and the full subcategory of compact objects coincides with that of finite spectra, that is, spectra of the form $\Sigma^n A$ where A is a homotopy retract of a finite based CW-complex and n is an integer.

(c) For G a compact Lie group, the equivariant stable homotopy category $h\mathrm{Sp}_G$ is a tensor triangulated category with respect to the (derived) smash product. It is compactly generated by the set of cosets G/H_+ as H ranges over the closed subgroups of G. The compact objects are those of the form $S^{-V} \wedge \Sigma^\infty A$ for V a G-representation and A a homotopy retract of a based finite G-CW complex [34, I.4.7]. One category of particular interest in this paper comes from considering the rational version of this category, that is, the category obtained by localizing with respect to the rational equivalences.

(d) Let G be a finite group and k be a field with characteristic dividing the order of G. The stable module category $\mathrm{StMod}(kG)$ is a tensor triangulated category with respect to the tensor product $- \otimes_k -$ with diagonal G-action. It is compactly generated by the simple modules, and the full subcategory of compact objects coincides with the class of finitely generated kG-modules.

Definition 3.3. Let T be a tensor triangulated category.

- A *prime ideal* is a proper thick ideal $\mathfrak{p}$ with the property that
$$x \otimes y \in \mathfrak{p} \Rightarrow x \in \mathfrak{p} \text{ or } y \in \mathfrak{p}.$$

- The *Balmer spectrum* $\mathrm{Spc}^\omega(\mathsf{T})$ is the set of all prime ideals of T^ω.

- We endow the Balmer spectrum with a *Zariski topology* generated by the closed sets
$$\mathrm{supp}(X) = \{\mathfrak{p} \in \mathrm{Spc}^\omega(\mathsf{T}) \mid X \notin \mathfrak{p}\}$$
where X ranges over the compact objects of T. In particular, the *specialization closure* of a prime $\mathfrak{p}$ is given by $\overline{\{\mathfrak{p}\}} = \{\mathfrak{q} \mid \mathfrak{q} \subseteq \mathfrak{p}\}$.

We recall the following important result that we will use repetitively throughout.

Proposition 3.4 ([5, 2.6])**.** *The support of compact objects has the following properties:*

(a) $\mathrm{supp}(0) = \emptyset$ *and* $\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}^\omega(\mathsf{T})$*;*
(b) $\mathrm{supp}(X \oplus Y) = \mathrm{supp}(X) \cup \mathrm{supp}(Y)$*;*
(c) $\mathrm{supp}(\Sigma X) = \mathrm{supp}(X)$*;*
(d) $\mathrm{supp}(Y) \subseteq \mathrm{supp}(X) \cup \mathrm{supp}(Z)$ *for any distinguished triangle* $X \to Y \to Z \to \Sigma X$*;*
(e) $\mathrm{supp}(X \otimes Y) = \mathrm{supp}(X) \cap \mathrm{supp}(Y)$*.*

Moreover, two compact objects have equal support if and only if they generate the same thick ideal. In particular, if an object has empty support it is equivalent to the zero object.

We will only consider Balmer spectra which are *Noetherian* spaces, that is those which satisfy the ascending chain condition for open subsets. By [7, 7.14], the Balmer spectrum is Noetherian if and only if each prime $\mathfrak{p}$ is *visible*, i.e., there exists a compact object $K_\mathfrak{p}$ in T for which $\mathrm{supp}(K_\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$. We will call $K_\mathfrak{p}$ a *Koszul object for* $\mathfrak{p}$.

The *dimension* of a prime is the length of a maximal length chain of primes down to a closed point (= minimal prime). The dimension of the Balmer spectrum is the supremum of the dimensions of all primes. We note that such a chain always exists as the space is spectral [15].

3.2. **Sufficiently nice model categories.** Our methods require that our tensor-triangulated category is the homotopy category of a suitable stable monoidal model category. Many of the finer details will be omitted since they are not relevant for the remainder of the paper, but we recall essentials from [3, §4].

Let $\mathcal{C}$ be a model category. The first requirement is that the homotopy category $h\mathcal{C}$ is tensor-triangulated. If $\mathcal{C}$ is a stable, symmetric monoidal model category this is true [32, 4.3.2, 6.6.4, §7.1]. We will moreover assume that the model structure on $\mathcal{C}$ satisfies the *monoid axiom* which ensures that categories of modules over monoids in $\mathcal{C}$ support a projective model structure, see [35]. Next, we need to be able to perform left Bousfield localizations and cellularizations (i.e., right Bousfield localizations). In order to do this, we assume that $\mathcal{C}$ is proper and cellular. It then follows by [30, 4.1.1, 5.1.1] that left Bousfield localizations at sets of maps, and cellularizations at sets of objects exist.

In order to ensure that the homotopy category $h\mathcal{C}$ is rigidly-compactly generated, we assume that $\mathcal{C}$ is compactly generated in the sense of [39, 1.2.3.4], has unit $\mathbb{1}$ cofibrant and cell-compact, and has generating cofibrations of the form $S^n \otimes \mathcal{G} \to \Delta^{n+1} \otimes \mathcal{G}$ where $\mathcal{G}$ is a suitable set of cell-compact cofibrant objects which will become the compact generators of $h\mathcal{C}$. The fact that these assumptions guarantee that $h\mathcal{C}$ is rigidly-compactly generated follows from [39, 1.2.3.7, 1.2.3.8]. These conditions also ensure that homological localizations exist [3, §6.A].

Definition 3.5. We say that a stable symmetric monoidal model category $\mathcal{C}$ is *rigidly-compactly generated* if it satisfies the above conditions.

In summary we have the following.

Theorem 3.6. *Let $\mathcal{C}$ be a rigidly-compactly generated model category, then its homotopy category $h\mathcal{C}$ is a rigidly-compactly generated tensor-triangulated category.*

As already alluded to in the previous section, we do not wish to consider all tensor-triangulated categories, but only those whose Balmer spectrum has a particularly tame form. This leads us to the final definition of this section.

Definition 3.7. A rigidly-compactly generated model category $\mathcal{C}$ is said to be a *finite-dimensional Noetherian* model category if $\mathrm{Spc}^\omega(h\mathcal{C})$ is a finite-dimensional Noetherian space.

We refer to Section 5 for a discussion of some examples of finite-dimensional Noetherian model categories.

4. Localization, completion and support

4.1. **Localization, completion and support at a single prime.** First we define localization, torsion and completion functors at each prime of the Balmer spectrum, and use it to define support. Additional details can be found in [3].

Let $\mathcal{C}$ be a finite-dimensional Noetherian model category, and write $\mathcal{G}$ for a set of compact and dualizable generators. By the Noetherian condition, for each prime $\mathfrak{p}$ we may choose a Koszul object $K_{\mathfrak{p}}$ (i.e., compact and with support $\overline{\{\mathfrak{p}\}}$). These are not canonical, but any two generate the same thick tensor ideal, so that constructions below will not depend on this choice. The theory of supports shows

$$\mathfrak{p} = \mathrm{Thick}_\otimes(K_{\mathfrak{q}} \mid \mathfrak{p} \not\subseteq \mathfrak{q}).$$

For each prime $\mathfrak{p}$ there are associated torsion, completion and localization functors, independent of the choice of the Koszul object $K_{\mathfrak{p}}$.

Definition 4.1.

- The *torsion functor* $\Gamma_{\mathfrak{p}}$ is the cellularization at $K_{\mathfrak{p}} \otimes \mathcal{G} = \{K_{\mathfrak{p}} \otimes G \mid G \in \mathcal{G}\}$.
- The *localization functor* $L_{\mathfrak{p}}$ is the nullification of $\mathfrak{p}$.
- The *completion functor* $\Lambda_{\mathfrak{p}}$ is the homological localization at $K_{\mathfrak{p}}$.

Remark 4.2.

(a) The apparent asysmmetry between $\Gamma_{\mathfrak{p}}$ and $\Lambda_{\mathfrak{p}}$ is due to the fact that Bousfield localization is defined in terms of an enrichment in spaces (i.e., $\mathrm{map}(K_{\mathfrak{p}}, X)$ is a space rather than an object of $\mathcal{C}$), whereas that of a homological localization is defined in terms of the tensor product, and $K_{\mathfrak{p}} \otimes X$ is an object of $\mathcal{C}$.

(b) We remark that [3, 6.7] is therefore incorrect unless the category is compactly generated by its tensor unit, and should instead refer to cellularization at $K_{\mathfrak{p}} \otimes \mathcal{G}$.

(c) Since the homotopy category of a cellularization at a set of compact objects $\mathcal{K}$ is the localizing subcategory generated by $\mathcal{K}$ (see [24, 2.5, 2.6]), one sees that the homotopy category of $\mathfrak{p}$-torsion objects is the localizing tensor ideal generated by $K_{\mathfrak{p}}$. This puts us in the context of [19].

Using these functors we can define a notion of support for non-compact objects following [11].

Definition 4.3. We define the *support* of on object $M \in \mathcal{C}$ by

$$\mathrm{supp}(M) = \{\mathfrak{p} \in \mathrm{Spc}^{\omega}(h\mathcal{C}) \mid \Gamma_{\mathfrak{p}} L_{\mathfrak{p}} M \not\simeq 0\}.$$

Recall the notion of support for compact objects from Definition 3.3. We record the following key facts.

Proposition 4.4 ([7, 7.17])**.** *Let $\mathcal{C}$ be a finite-dimensional Noetherian model category. The support defined above satisfies the following properties:*

(a) *The two notions of support coincide for any compact objects of $\mathcal{C}$.*

(b) *The support defined above satisfies the following properties:*

  (1) *For any set of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$ we have $\mathrm{supp}(\oplus_{i \in I} X_i) = \cup_{i \in I}\mathrm{supp}(X_i)$.*

  (2) *For all $X \in \mathcal{C}$ we have $\mathrm{supp}(\Sigma X) = \mathrm{supp}(X)$.*

  (3) *For any $X, Y \in \mathcal{C}$ there is an inclusion $\mathrm{supp}(X \otimes Y) \subseteq \mathrm{supp}(X) \cap \mathrm{supp}(Y)$. This is an equality if X and Y are compact objects.*

  (4) *Given a distinguished triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{C}$ then there is an inclusion $\mathrm{supp}(Y) \subseteq \mathrm{supp}(X) \cup \mathrm{supp}(Z)$.*

(c) *An object has empty support if and only if it is zero.*

4.2. **Counterparts for open and closed sets.** Given a collection of prime ideals $\mathcal{S}$ we let $\mathcal{K}(\mathcal{S})$ denote a set of Kozsul objects for $\mathcal{S}$, i.e., $\mathcal{K}(\mathcal{S}) = \{K_{\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{S}\}$. A collection of prime ideals $\mathcal{S}$ is said to be *a family* if $\mathfrak{p} \in \mathcal{S}$ and $\mathfrak{q} \subseteq \mathfrak{p}$ implies that $\mathfrak{q} \in \mathcal{S}$, and *cofamily* if $\mathfrak{p} \in \mathcal{S}$ and $\mathfrak{p} \subseteq \mathfrak{q}$ implies that $\mathfrak{q} \in \mathcal{S}$.

Remark 4.5. We have adopted the terminology from equivariant topology for brevity. The standard geometric terminology is that a family is 'a specialization closed set' and that a cofamily is 'a generalization closed set'. It is helpful to observe that a Zariski closed set is a family, and a Zariski open set is a cofamily.

In particular, we will consider the cones below and above a fixed prime $\mathfrak{p}$:

$$\wedge(\mathfrak{p}) = \{\mathfrak{q} \mid \mathfrak{q} \subseteq \mathfrak{p}\} \quad \text{and} \quad \vee(\mathfrak{p}) = \{\mathfrak{q} \mid \mathfrak{p} \subseteq \mathfrak{q}\}.$$

By definition $\wedge(\mathfrak{p})$ is a family and $\vee(\mathfrak{p})$ is a cofamily.

Definition 4.6. Let V be a family with complement V^c the associated cofamily.

- The *torsion functor* Γ_V is the cellularization at $\mathcal{K}(V) \otimes \mathcal{G}$.
- The *localization functor* L_{V^c} is the nullification of $\mathcal{K}(V)$.
- The *completion functor* Λ_V is the homological localization at $\bigoplus_{\mathfrak{p} \in V} K_{\mathfrak{p}}$.

Remark 4.7. Note that the versions for a single prime and for (co)families are related by $\Gamma_{\mathfrak{p}} = \Gamma_{\wedge(\mathfrak{p})}$, $L_{\mathfrak{p}} = L_{\vee(\mathfrak{p})}$ and $\Lambda_{\mathfrak{p}} = \Lambda_{\wedge(\mathfrak{p})}$. It is important to note that $\vee(\mathfrak{p})$ is *not* the complement of $\wedge(\mathfrak{p})$.

It follows from the definition of the torsion and localization functors, that for any object X of $\mathcal{C}$, there is a natural cofibre sequence

$$\Gamma_V X \to X \to L_{V^c} X$$

with

$$\operatorname{supp}(\Gamma_V X) = \operatorname{supp}(X) \cap V \quad \text{and} \quad \operatorname{supp}(L_{V^c} X) = \operatorname{supp}(X) \setminus V.$$

Example 4.8. Let V consist of primes of dimension less than or equal to n. In this case we write $\Gamma_V = \Gamma_{\leq n}$, $L_{V^c} = L_{\geq n+1}$ and $\Lambda_V = \Lambda_{\leq n}$. The notation is chosen as such for the effect on the supports; for example $\operatorname{supp}(\Gamma_{\leq n} \mathbb{1})$ is the collection of primes of dimension less than or equal to n and $\operatorname{supp}(L_{\geq n+1} \mathbb{1})$ is the collection of primes of dimension greater than or equal to $n+1$.

4.3. **Properties of the functors.** The following proposition records key facts and properties which we will use throughout this paper.

Proposition 4.9. *Let V be a family.*

(a) *The functors L_{V^c} and Λ_V preserve monoid objects as endofunctors on $\mathcal{C}$.*
(b) *The functors Γ_V and L_{V^c} are smashing (i.e., $\Gamma_V X \simeq \Gamma_V \mathbb{1} \otimes X$ and $L_{V^c} X \simeq L_{V^c} \mathbb{1} \otimes X$), but Λ_V is usually not smashing.*
(c) *If R denotes fibrant replacement in $\mathcal{C}$, then $\operatorname{Hom}(\Gamma_V \mathbb{1}, RX) \simeq \Lambda_V X$.*
(d) *The natural maps $\Gamma_V X \to \Gamma_V \Lambda_V X$ and $\Lambda_V \Gamma_V X \to \Lambda_V X$ are equivalences for all X. We call this the* MGM *equivalence.*

Proof. For (a), by [41, 3.2] it is sufficient to prove that $L_{V^c}\mathcal{C}$ and $\Lambda_V \mathcal{C}$ satisfy the monoid axiom, since both L_{V^c} and Λ_V are monoidal Bousfield localizations [3, 5.6, 6.3]. Recall that $\mathcal{C}$ satisfies the monoid axiom by our standing assumptions. Therefore, $\Lambda_V \mathcal{C}$ also satisfies the monoid axiom since it is a homological localization, see for instance [8, 3.8]. To show that $L_{V^c}\mathcal{C}$ satisfies the monoid axiom, we note that we can view it as a homological localization with regard to the object $L_{V^c}\mathbb{1}$, and so part (a) follows. Part (b) is well known; see for instance [33, 3.3.3, 3.3.5] and [3, 5.5]. Part (c) follows from the fact that every $\mathcal{K}(V)$-equivalence is a $\Gamma_V \mathbb{1}$-equivalence. The first equivalence of part (d) is [19, 2.3], and the other follows similarly. $\square$

Certain composites are immediate from the geometric descriptions.

Proposition 4.10. *Suppose that $V' \subseteq V$ is an inclusion of families and $U' \supseteq U$ an inclusion of cofamilies. For all objects X of $\mathcal{C}$ we have natural equivalences*

(a) $L_U L_{U'} X \xleftarrow{\sim} L_U X \xrightarrow{\sim} L_{U'} L_U X$
(b) $\Gamma_V \Gamma_{V'} X \xrightarrow{\sim} \Gamma_{V'} X \xleftarrow{\sim} \Gamma_{V'} \Gamma_V X$
(c) $\Lambda_{V'} \Lambda_V X \xleftarrow{\sim} \Lambda_{V'} X \xrightarrow{\sim} \Lambda_V \Lambda_{V'} X.$

Extreme cases also simplify.

Lemma 4.11. *Let $\mathcal{C}$ be a finite-dimensional Noetherian model category.*

(a) *If the Balmer spectrum of $h\mathcal{C}$ is irreducible (i.e., there is a single generic point $\mathfrak{g}$) then the completion $\Lambda_{\mathfrak{g}}$ is isomorphic to the identity.*

(b) *If $\mathfrak{m}$ is a closed point of the Balmer spectrum, then there is a weak equivalence $\Lambda_{\mathfrak{m}}\mathbb{1} \xrightarrow{\sim} L_{\mathfrak{m}}\Lambda_{\mathfrak{m}}\mathbb{1}$.*

4.4. **Splittings.** Recall from Example 4.8 that we write $\Gamma_{\leq 0}$ and $\Lambda_{\leq 0}$ for the torsion and completion functor associated to the family of primes of dimension zero. We use subscripts to indicate the dimension of primes. For example, $\mathfrak{p}_0$ indicates a prime of dimension zero, and $\bigoplus_{\mathfrak{p}_0}$ is shorthand for the sum over all primes of dimension zero.

Proposition 4.12. *For all $X \in \mathcal{C}$, there are equivalences*

(a) $\Gamma_{\leq 0}X \xleftarrow{\sim} \bigoplus_{\mathfrak{p}_0} \Gamma_{\mathfrak{p}_0}X,$

(b) $\Lambda_{\leq 0}X \xrightarrow{\sim} \prod_{\mathfrak{p}_0} \Lambda_{\mathfrak{p}_0}X.$

Proof. We prove part (a) and note that part (b) follows from applying Proposition 4.9. Write V_0 for the family of zero dimensional primes. We must show that $\bigoplus_{\mathfrak{p}_0} \Gamma_{\mathfrak{p}_0}X$ has the universal property of the $(\mathcal{K}(V_0) \otimes \mathcal{G})$-cellularization. In other words, we must show that $\bigoplus_{\mathfrak{p}_0} \Gamma_{\mathfrak{p}_0}X$ is in $\mathrm{Loc}_\otimes(K_{\mathfrak{q}_0} \mid \dim\mathfrak{q}_0 = 0)$ and that the natural map $\bigoplus_{\mathfrak{p}_0} \Gamma_{\mathfrak{p}_0}X \to X$ is a $(K_{\mathfrak{q}_0} \otimes \mathcal{G})$-cellular equivalence for each $\mathfrak{q}_0$. The first condition is clear from the fact that $\Gamma_{\mathfrak{p}_0}X \in \mathrm{Loc}_\otimes(K_{\mathfrak{p}_0})$ by definition. By compactness of $K_{\mathfrak{q}_0} \otimes G$, the natural map $\bigoplus_{\mathfrak{p}_0} \Gamma_{\mathfrak{p}_0}X \to X$ is a $K_{\mathfrak{q}_0} \otimes \mathcal{G}$-cellular equivalence if and only if

$$\bigoplus_{\mathfrak{p}_0}[K_{\mathfrak{q}_0} \otimes G, \Gamma_{\mathfrak{p}_0}X] \to [K_{\mathfrak{q}_0} \otimes G, X]$$

is an equivalence for all zero-dimensional primes $\mathfrak{q}_0$ and $G \in \mathcal{G}$.

If $\mathfrak{q}_0 \neq \mathfrak{p}_0$ we claim that $[K_{\mathfrak{q}_0} \otimes G, \Gamma_{\mathfrak{p}_0}X] = 0$. Since $\Gamma_{\mathfrak{q}_0}K_{\mathfrak{q}_0} \to K_{\mathfrak{q}_0}$ is an equivalence, $[K_{\mathfrak{q}_0} \otimes G, \Gamma_{\mathfrak{p}_0}X] = [K_{\mathfrak{q}_0} \otimes G, \Lambda_{\mathfrak{q}_0}\Gamma_{\mathfrak{p}_0}X]$. The map $\Gamma_{\mathfrak{p}_0}X \to 0$ is a $K_{\mathfrak{q}_0}$-equivalence since the support of $\Gamma_{\mathfrak{p}_0}K_{\mathfrak{q}_0}$ is empty, and hence $\Lambda_{\mathfrak{q}_0}\Gamma_{\mathfrak{p}_0}X \simeq 0$. Therefore, $\bigoplus_{\mathfrak{p}_0}[K_{\mathfrak{q}_0} \otimes G, \Gamma_{\mathfrak{p}_0}X] \to [K_{\mathfrak{q}_0} \otimes G, X]$ is an equivalence if and only if $[K_{\mathfrak{q}_0} \otimes G, \Gamma_{\mathfrak{q}_0}X] \to [K_{\mathfrak{q}_0} \otimes G, X]$ is an equivalence, which is true by the definition of $\Gamma_{\mathfrak{q}_0}$. $\square$

It is worth making explicit some consequences for the torsion part.

Corollary 4.13.

(a) *If the support of X consists of primes of dimension i then*

$$X \simeq \bigoplus_{\mathfrak{p}_i}\Gamma_{\mathfrak{p}_i}X \simeq \bigoplus_{\mathfrak{p}_i}L_{\mathfrak{p}_i}X.$$

(b) *If the support of X consists of primes of dimension $\geq i$ then there is an equivalence*

$$\Gamma_{\leq i}X \xleftarrow{\sim} \bigoplus_{\mathfrak{p}_i}\Gamma_{\mathfrak{p}_i}X.$$

(c) *If the support of X consists of primes of dimension $\leq i$ then there is an equivalence*

$$L_{\geq i}X \xleftarrow{\sim} \bigoplus_{\mathfrak{p}_i}L_{\mathfrak{p}_i}X.$$

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5. Examples

We now illustrate the theory described in the previous sections with several examples. We will suppress notation for underlying (co)fibrant replacements in the construction of the functors Γ, L and Λ, in order to not unnecessarily overburden the notation. For instance, we will often write $\Lambda X = F(\Gamma \mathbb{1}, X)$ where the target X should be implicitly interpreted as fibrantly replaced as in Proposition 4.9.

Notation 5.1. We emphasize that we label primes by their *dimension* and write this as a subscript, see for instance Example 4.8. Therefore $\mathfrak{p}_i$ denotes a Balmer prime of dimension i. We also draw Balmer spectra with dimension increasing up the page, so that closed points are at the bottom. Arrows indicate inclusion.

5.1. **Derived commutative algebra.** Let R be a Noetherian commutative ring. We write Mod_R for the category of unbounded chain complexes of R-modules. There is a an order reversing homeomorphism which is given by

$$\mathrm{Spec}(R) \to \mathrm{Spc}^\omega(\mathsf{D}(R)) \qquad \wp \mapsto \{M \in \mathsf{D}(R)^\omega \mid M_\wp \simeq 0\}.$$

If R is a finite-dimensional commutative Noetherian ring, then Mod_R is a finite-dimensional Noetherian model category when equipped with the projective model structure.

Writing $\wp$ in place of the corresponding Balmer prime, the functors $\Gamma_\wp$, $\Lambda_\wp$ and $L_\wp$ correspond to the usual functors of derived torsion, derived completion and localization in algebra. There are explicit models for these functors as follows. Suppose that $\wp = (x_1, \ldots, x_n)$. We define the stable Koszul complex by

$$K_\infty(\wp) = (R \to R[1/x_1]) \otimes_R \cdots \otimes_R (R \to R[1/x_n])$$

and the Čech complex $\check{C}_\wp R$ is the suspension of the kernel of the natural map $K_\infty(\wp) \to R$. Up to equivalence this does not depend on the chosen generators of $\wp$. Then we have the following identifications for an R-module X:

$$\boxed{\Gamma_\wp X \simeq K_\infty(\wp) \otimes_R X \qquad \Lambda_\wp X \simeq \mathrm{Hom}_R(K_\infty(\wp), X) \qquad L_\wp X \simeq X_\wp}$$

As usual $X_\wp$ is the localization *at* the ideal $\wp$ so it inverts the multiplicative set $R \setminus \wp$. On the other hand, $L_{(\wedge\wp)^c}$ is the localization *away* from $\wp$ so can be calculated as $\check{C}_\wp R \otimes_R X$. For more details see for instance [17, 22].

For example, take $R = \mathbb{Z}$, $\wp = (p)$: we may take $K_{(p)} = \mathbb{Z}/p$. Thus $\Gamma_p \mathbb{Z} \simeq \Sigma^{-1} \mathbb{Z}/p^\infty$, $L_p \mathbb{Z} \simeq \mathbb{Z}_{(p)}$ and $\Lambda_p \mathbb{Z} \simeq \mathbb{Z}_p$, the p-adic integers.

5.2. **Chromatic homotopy theory.** Let $\mathrm{Sp}_{(p)}$ denote the category of p-local spectra equipped with the stable model structure. Let

$$P(n) = \{X \in h\mathrm{Sp}_{(p)}^\omega \mid K(n)_*(X) = 0\}$$

where $K(0)$ is rational homology, $K(\infty)$ is mod p homology and for finite $n \geq 1$, $K(n)$ is the nth mod p Morava K-theory. By the Künneth theorem the $P(n)$ are prime, and Hopkins–Smith [31] showed that the Balmer spectrum is linear:

$$P(0) \supset P(1) \supset P(2) \supset \cdots \supset P(n) \supset \cdots \supset P(\infty).$$

Note that this Balmer spectrum is not finite-dimensional (and worse still, non-Noetherian). As such, we use the smashing localization $L_n = L_{K(0) \vee \cdots \vee K(n)}$ to obtain a finite-dimensional Noetherian model category with Balmer spectrum

$$P(0) \supset P(1) \supset P(2) \supset \cdots \supset P(n).$$

so that the prime $P(i)$ is of dimension $n - i$ in the L_n-local category.

5.3. **Rational equivariant spectra.** Let G be a compact Lie group. We write Sp_G for the category of rational G-equivariant spectra. The second author [20] showed that the Balmer spectrum of rational G-spectra is in bijection with the set of conjugacy classes of closed subgroups of G, with inclusion corresponding to cotoral inclusion; recall that $K \hookrightarrow H$ is *cotoral* if K is normal in H and the quotient H/K is a torus. The bijection is given by sending a conjugacy class (H) to

$$\mathfrak{p}_H = \{X \mid \Phi^H X \simeq_1 *\}.$$

In general the Balmer spectrum of rational G-spectra is not Noetherian. However, when G is an r-torus the Balmer spectrum is r-dimensional and Noetherian, so the topology is determined by the poset structure.

For example, if $\mathbb{T}$ is the circle group, then the Balmer spectrum is given by
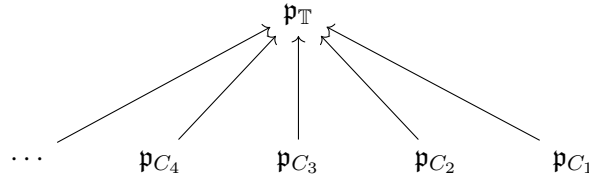
where the arrows indicate cotoral inclusions.

Note that the support of a rational G-spectrum X coincides with its geometric isotropy $\mathcal{I}_g(X) = \{H \mid \Phi^H X \not\simeq 0\}$. Any family $\mathcal{F}$ of subgroups of G defines a family of primes $\{\wp_H \mid H \in \mathcal{F}\}$, that by abuse of notation we still denote by $\mathcal{F}$. We write $E\mathcal{F}$ for the universal G-space characterized by its fixed points

$$(E\mathcal{F})^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ \emptyset & \text{if } H \notin \mathcal{F} \end{cases}$$

and $\widetilde{E}\mathcal{F}$ for the cofibre of the nonzero map $E\mathcal{F}_+ \to S^0$. Then we have the following identifications

$$\boxed{\Gamma_{\mathcal{F}} X \simeq E\mathcal{F}_+ \wedge X \qquad \Lambda_{\mathcal{F}} X \simeq F(E\mathcal{F}_+, X) \qquad L_{\mathcal{F}^c} M \simeq \widetilde{E}\mathcal{F} \wedge X.}$$

6. The standard model

Throughout we fix a finite-dimensional Noetherian model category $\mathcal{C}$ and a family of primes V in $\mathrm{Spc}^{\omega}(h\mathcal{C})$. This gives a set of Koszul objects $\mathcal{K}(V)$ and associated functors $\Gamma_V, L_{V^c}, \Lambda_V$ as defined in Section 4. When no confusion is likely to arise we will drop the reference to the family V.

Remark 6.1. The machinery developed in this paper applies to any rigidly-compactly generated model category $\mathcal{C}$ equipped with functors Γ, L and Λ satisfying the properties listed in Section 4. For example, these functors can be defined whenever $\mathcal{C}$ is part of a local duality context [10] or a Tate context [19]. We have decided to present this material from the perspective of the Balmer spectrum, as this approach better suits our long-term goal of constructing a torsion model for a tensor-triangulated category via the dimension filtration [4].

Definition 6.2. We write $T^{\lrcorner}$ for the diagram

$$\begin{array}{ccc} & & L\mathbb{1} \\ & & \downarrow{\scriptstyle\theta} \\ \Lambda\mathbb{1} & \xrightarrow{\ \varphi\ } & L\Lambda\mathbb{1} \end{array}$$

which we shall refer to as the *Tate cospan*.

Lemma 6.3 ([19, 2.3])**.** *The homotopy pullback of $T^{\lrcorner}$ is weakly equivalent to $\mathbb{1}$. Accordingly, for any X, the square*

$$
\begin{array}{ccc}
X & \longrightarrow & L\mathbb{1} \otimes X \\
\downarrow & & \downarrow \\
\Lambda\mathbb{1} \otimes X & \longrightarrow & L\Lambda\mathbb{1} \otimes X
\end{array}
$$

is a homotopy pullback.

Proof. The fibres of $\mathbb{1} \to L\mathbb{1}$ and $\Lambda\mathbb{1} \to L\Lambda\mathbb{1}$ are both equivalent to $\Gamma\mathbb{1}$ and hence $\mathbb{1}$ is the homotopy pullback. $\qquad\square$

By Proposition 4.9 we know that the objects appearing in the Tate cospan are monoid objects in $\mathcal{C}$. As such we can construct module categories over them.

Definition 6.4. We define the *pre-standard* category $\mathcal{C}_{\text{pre-s}}$ to be the category of sections of the diagram category

$$
\begin{array}{ccc}
& & \text{Mod}_{L\mathbb{1}} \\
& & \downarrow{\scriptstyle \theta_*} \\
\text{Mod}_{\Lambda\mathbb{1}} & \xrightarrow[\varphi_*]{} & \text{Mod}_{L\Lambda\mathbb{1}}
\end{array}
$$

where the arrows are the extensions of scalars coming from the ring maps $\theta\colon L\mathbb{1} \to L\Lambda\mathbb{1}$ and $\varphi\colon \Lambda\mathbb{1} \to L\Lambda\mathbb{1}$. We equip this category with the `dimp` model structure where the weak equivalences and cofibrations are determined objectwise in the projective model structure on modules, see Notation 2.8.

Recall from Definition 2.5 that the category $\mathcal{C}_{\text{pre-s}}$ has objects which are quintuples (V, N, W, f, g) such that:

- $V \in \text{Mod}_{L\mathbb{1}}$;
- $N \in \text{Mod}_{\Lambda\mathbb{1}}$;
- $W \in \text{Mod}_{L\Lambda\mathbb{1}}$;
- $f\colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to W$ in $\text{Mod}_{L\Lambda\mathbb{1}}$;
- $g\colon L\Lambda\mathbb{1} \otimes_{\Lambda\mathbb{1}} N \to W$ in $\text{Mod}_{L\Lambda\mathbb{1}}$.

We will usually draw such an object as a diagram

$$
\begin{array}{ccc}
& & V \\
& & \vdots{\scriptstyle f} \\
& & \downarrow \\
N & \dashrightarrow[g] & W
\end{array}
\tag{1}
$$

where the dashed arrow indicates the existence of a map after extension of scalars.

We define a functor $(-)_{\text{stan}}\colon \mathcal{C} \to \mathcal{C}_{\text{pre-s}}$ by tensoring with the Tate cospan:

$$
X_{\text{stan}} = \left(
\begin{array}{ccc}
& & L\mathbb{1} \otimes X \\
& & \vdots \\
& & \downarrow \\
\Lambda\mathbb{1} \otimes X & \dashrightarrow & L\Lambda\mathbb{1} \otimes X
\end{array}
\right).
$$

This functor admits a right adjoint which is given by taking the pullback. More precisely, consider an object in the pre-standard category as in Equation (1). By adjunction, the data of $L\Lambda\mathbb{1}$-module maps $f\colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to W$ and $g\colon L\Lambda\mathbb{1} \otimes_{\Lambda\mathbb{1}} N \to W$ is equivalent to the data of maps

$f^\#\colon V \to \theta^*W$ and $g^\#\colon N \to \varphi^*W$ of $L\mathbb{1}$-modules and $\Lambda\mathbb{1}$-modules respectively. Therefore, by applying the forgetful functors, we can view the cospan

$$N \xrightarrow{g^\#} W \xleftarrow{f^\#} V$$

as a diagram in $\mathcal{C}$. Taking the pullback of this diagram gives a functor $\lim\colon \mathcal{C}_{\text{pre-s}} \to \mathcal{C}$.

Theorem 6.5. *The adjunction*

$$\mathcal{C} \xrightleftharpoons[\lim]{(-)_{\text{stan}}} \mathcal{G}_{\text{stan}}\text{-cell-}\mathcal{C}_{\text{pre-s}}$$

is a Quillen equivalence, where $\mathcal{G}$ is a set of compact generators for $\mathcal{C}$.

Proof. The compact generators $\mathcal{G}$ of $\mathcal{C}$ get sent to compact objects of $\mathcal{C}_{\text{pre-s}}$ as in [26, 4.1]. Since the derived unit is a weak equivalence by Lemma 6.3, the Cellularization Principle [24, 2.7] applies to give the result. $\square$

The following theorem is the one-step case of the result of [3, 9.5].

Theorem 6.6. *A bifibrant object $(V, N, W, f, g) \in \mathcal{C}_{\text{pre-s}}$ is bifibrant in $\mathcal{G}_{\text{stan}}\text{-cell-}\mathcal{C}_{\text{pre-s}}$ if and only if $f\colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to W$ and $g\colon L\Lambda\mathbb{1} \otimes_{\Lambda\mathbb{1}} N \to W$ are weak equivalences. As such, $\mathcal{G}_{\text{stan}}\text{-cell-}\mathcal{C}_{\text{pre-s}}$ is the strict homotopy limit of the diagram category in the sense of [12].*

As a warm up for our approach to the torsion model in Section 8, we observe that one can improve upon the above theorem as we now describe.

Definition 6.7. The category $\mathcal{C}^e_{\text{pre-s}}$ of *extended pre-standard* objects is the full subcategory of $\mathcal{C}_{\text{pre-s}}$ consisting of objects (V, N, W, f, g) for which the map $f\colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to W$ is an isomorphism. Diagrammatically, the objects are those of the form

$$
\begin{array}{c}
V \\
\vdots \\
\bullet \\
\downarrow \\
N \dashrightarrow[g] W
\end{array}
$$

where the dot on the arrow indicates that the map is an isomorphism after extension of scalars. By adjunction an object of $\mathcal{C}^e_{\text{pre-s}}$ is equivalently a $\Lambda\mathbb{1}$-module map $\beta\colon N \to L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V$.

There is an evident inclusion $\mathcal{C}^e_{\text{pre-s}} \hookrightarrow \mathcal{C}_{\text{pre-s}}$. This admits a right adjoint $\Gamma_e\colon \mathcal{C}_{\text{pre-s}} \to \mathcal{C}^e_{\text{pre-s}}$ which is defined by

$$
\Gamma_e\left(
\begin{array}{c}
V \\
\vdots\, f \\
\downarrow \\
N \dashrightarrow[g] W
\end{array}
\right)
=
\begin{array}{c}
V \\
\vdots \\
\bullet \\
\downarrow \\
N' \longrightarrow L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V
\end{array}
$$

where N' is the pullback

$$
\begin{array}{ccc}
N' & \longrightarrow & L\Lambda\mathbb{1} \otimes_{\Lambda\mathbb{1}} V \\
\downarrow & & \downarrow f \\
N & \xrightarrow{g^\#} & W
\end{array}
$$

in the category of $\Lambda\mathbb{1}$-modules.

There exists a `dimp` model structure on $\mathcal{C}^e_{\text{pre-s}}$, and a simple application of the Cellularization Principle [24] shows that $\mathcal{G}_{\text{stan}}\text{-cell-}\mathcal{C}_{\text{pre-s}} \simeq_Q \mathcal{G}_{\text{stan}}\text{-cell-}\mathcal{C}^e_{\text{pre-s}}$.

Definition 6.8. The *standard model* for $\mathcal{C}$ is the model category $\mathcal{C}_s = \mathcal{G}_{\text{stan}}\text{-cell-}\mathcal{C}^e_{\text{pre-s}}$.

Combining this with the previous results of this section we obtain the following.

Theorem 6.9. *There is a zig-zag of Quillen equivalences*

$$\mathcal{C} \simeq_Q \mathcal{C}_s.$$

Moreover an object in the homotopy category of $\mathcal{C}_s$ is determined by the data of a $\Lambda\mathbb{1}$-module map $\beta \colon N \to L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V$ such that $L\Lambda\mathbb{1} \otimes_{\Lambda\mathbb{1}} \beta$ is an equivalence, where N is a $\Lambda\mathbb{1}$-module and V is an $L\mathbb{1}$-module.

7. The pre-torsion model

We now move towards the torsion model. We note that the setup remains the same as in Section 6. Implicitly we have fixed a family of Balmer primes V and a set of Kozsul objects $\mathcal{K} = \mathcal{K}(V)$.

7.1. Enriching the splicing maps.
Torsion modules over a complete ring are equivalent to torsion modules in the underlying category. This is the crucial difference between the adelic and Zariski models. The following lemma gives a version for our context.

Lemma 7.1. *There is a Quillen equivalence $(\mathcal{K} \otimes \mathcal{G})\text{-cell-}\mathcal{C} \simeq_Q (\Lambda\mathbb{1} \otimes \mathcal{K} \otimes \mathcal{G})\text{-cell-}\mathrm{Mod}_{\Lambda\mathbb{1}}(\mathcal{C})$.*

Proof. There is a ring map $\theta \colon \mathbb{1} \to \Lambda\mathbb{1}$ which induces a Quillen adjunction

$$\Lambda\mathbb{1} \otimes - \colon \mathcal{C} \rightleftarrows \mathrm{Mod}_{\Lambda\mathbb{1}} : \theta^*.$$

We apply the Cellularization Principle [24, 2.7] at the set of compact objects $\mathcal{K} \otimes \mathcal{G}$ in $\mathcal{C}$. The objects $\Lambda\mathbb{1} \otimes K \otimes G$ are compact as $\Lambda\mathbb{1}$-modules for all $K \in \mathcal{K}$ and $G \in \mathcal{G}$ since the restriction of scalars functor preserves sums. The derived unit $K \otimes G \to \Lambda\mathbb{1} \otimes K \otimes G$ is an equivalence for all $K \in \mathcal{K}$ and $G \in \mathcal{G}$ by the MGM equivalence, since each K is torsion and Γ is smashing. Therefore the stated Quillen equivalence holds. $\qquad\square$

In light of the previous lemma, when considering torsion modules we can work over $\Lambda\mathbb{1}$.

Definition 7.2. We define the *pre-torsion* category $\mathcal{C}_{\text{pre-t}}$ to be the category of sections of the diagram category

$$
\begin{array}{c}
\mathrm{Mod}_{L\mathbb{1}} \\
\theta_* \downarrow \\
\mathrm{Mod}_{L\Lambda\mathbb{1}} \xrightarrow{\ \varphi^*\ } \mathrm{Mod}_{\Lambda\mathbb{1}}.
\end{array}
$$

The vertical map θ_* is the extension of scalars $L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} -$ along the ring map $\theta \colon L\mathbb{1} \to L\Lambda\mathbb{1}$, while the horizontal map is restriction of scalars along the ring map $\varphi \colon \Lambda\mathbb{1} \to L\Lambda\mathbb{1}$.

Remark 7.3. The categories $\mathcal{C}_{\text{pre-s}}$ and $\mathcal{C}_{\text{pre-t}}$ are defined as the categories of sections over two very similar diagrams. The crucial difference lies in the map $\varphi \colon \Lambda\mathbb{1} \to L\Lambda\mathbb{1}$: in the first case we extend scalars along φ whereas in the second we restrict scalars along φ.

By Definition 2.5 an object in $\mathcal{C}_{\text{pre-t}}$ is a quintuple (V, W, T, f, g) where:

- $V \in \mathrm{Mod}_{L\mathbb{1}}$;
- $W \in \mathrm{Mod}_{L\Lambda\mathbb{1}}$;
- $T \in \mathrm{Mod}_{\Lambda\mathbb{1}}$;
- $f \colon \theta_* V \to W$ in $\mathrm{Mod}_{L\Lambda\mathbb{1}}$;
- $g \colon \varphi^* W \to T$ in $\mathrm{Mod}_{\Lambda\mathbb{1}}$.

The morphisms in $\mathcal{C}_{\text{pre-t}}$ are triples $(\alpha\colon V \to V', \beta\colon W \to W', \gamma\colon T \to T')$ such that the evident squares commute.

As in the pre-standard model, we will often denote this object by

$$
\begin{array}{ccc}
V & & \\
f\big\downarrow & & \\
W & \xrightarrow{\ g\ } & T
\end{array}
$$

where we have not made the horizontal morphism dashed as we will rarely make any reference to the restriction of scalars.

In order to equip $\mathcal{C}_{\text{pre-t}}$ with a model structure, we require both θ_* and φ^* to both be Quillen of the same handedness. In general, θ_* will not be a right adjoint, and so we view both the functors as left adjoints.

Recall from Lemma 2.3 that in the projective model structure on modules, θ_* is always left Quillen, whereas φ^* is left Quillen if and only if $L\Lambda\mathbb{1}$ is cofibrant as an $\Lambda\mathbb{1}$-module. We now describe how to pick models for $L\mathbb{1}$, $\Lambda\mathbb{1}$ and $L\Lambda\mathbb{1}$ with this property. This will allow us to endow $\mathcal{C}_{\text{pre-t}}$ with the `dimp` model structure from Notation 2.8. There are two approaches depending on whether the commutativity of the rings is important.

7.2. **Non-commutative case.** We may choose a model for $L\mathbb{1}$ which is cofibrant as a ring, and therefore as an object of $\mathcal{C}$ by [35, 4.1(3)]. Tensoring the map $\mathbb{1} \to L\mathbb{1}$ with $\Lambda\mathbb{1}$ gives a ring map $\Lambda\mathbb{1} \to L\mathbb{1} \otimes \Lambda\mathbb{1}$ which is a cofibration of $\Lambda\mathbb{1}$-modules. As $\Lambda\mathbb{1}$ is a cofibrant $\Lambda\mathbb{1}$-module, this shows that $L\mathbb{1} \otimes \Lambda\mathbb{1}$ is a cofibrant $\Lambda\mathbb{1}$-module. Since L is smashing, $L\mathbb{1} \otimes \Lambda\mathbb{1}$ is a model for $L\Lambda\mathbb{1}$, and since it is a cofibrant $\Lambda\mathbb{1}$-module, the restriction of scalars functor $\varphi^*\colon \mathrm{Mod}_{L\Lambda\mathbb{1}} \to \mathrm{Mod}_{\Lambda\mathbb{1}}$ is left Quillen as desired.

We emphasise that since we took cofibrant replacements as rings/algebras, the resulting objects need not admit a commutative multiplication. However, we give an alternative approach in the next subsection.

7.3. **Commutative case.** In many cases of interest, the rings at play in the torsion model are actually *commutative*. For example, if $\mathbb{T}$ is the circle group, and $\mathcal{F}$ denotes the family of finite subgroups of $\mathbb{T}$, then $\widetilde{E}\mathcal{F}$ is a (genuinely) commutative ring $\mathbb{T}$-spectrum [21]. If $L\mathbb{1}$ and $\Lambda\mathbb{1}$ are commutative, and the category supports a *convenient* model structure, then we can preserve commutativity in the torsion model.

Definition 7.4. Let $\mathcal{C}$ be a monoidal model category and suppose that there exists another model structure $\widetilde{\mathcal{C}}$ on the same underlying category as $\mathcal{C}$, which has the same weak equivalences and for which the identity functor $\widetilde{\mathcal{C}} \to \mathcal{C}$ is left Quillen. We say that $(\mathcal{C}, \widetilde{\mathcal{C}})$ is *convenient* if the forgetful functor $\mathrm{CAlg}_S(\widetilde{\mathcal{C}}) \to \mathrm{Mod}_S(\mathcal{C})$ preserves cofibrant objects, for all commutative monoids $S \in \mathcal{C}$.

Examples 7.5. The following pairs of model structures are all convenient:

(1) (projective, projective) on chain complexes over a field of characteristic zero, see for instance [40, §5.1].
(2) (stable, positive stable) model structure on spectra is *not* convenient. However, Shipley [37, 4.1] constructs a flat and positive flat model structure on spectra so that (flat, positive flat) is convenient. We note that there is a left Quillen equivalence from the stable model structure to the flat model structure.
(3) The previous example also works for equivariant spectra. More precisely, the pair of model structures (flat, positive flat) on equivariant spectra constructed in [14, 4.2.15] is convenient.

In order to ensure that the restriction of scalars functor φ^* is left Quillen, we first take a model of $L\mathbb{1}$ which is cofibrant as a commutative ring in the model structure right lifted from $\widetilde{\mathcal{C}}$. If $(\mathcal{C}, \widetilde{\mathcal{C}})$ is convenient then this model of $L\mathbb{1}$ is also cofibrant as an object of $\mathcal{C}$. Therefore the ring map $\Lambda\mathbb{1} \to L\mathbb{1} \otimes \Lambda\mathbb{1}$ is a cofibration of $\Lambda\mathbb{1}$-modules, and as such the restriction of scalars along this map is left Quillen.

7.4. **Constructing the pre-torsion model.** We now define an adjunction between $\mathcal{C}$ and $\mathcal{C}_{\text{pre-t}}$ and show that after a suitable cellularization this gives a Quillen equivalence. This is the first step towards the torsion model constructed in the next section.

We define a functor $(-)_{\text{tors}} \colon \mathcal{C} \to \mathcal{C}_{\text{pre-t}}$ by

$$
X_{\text{tors}} = \left(
\begin{array}{c}
L\mathbb{1} \otimes X \\
\vdots \\
\downarrow \\
L\Lambda\mathbb{1} \otimes X \longrightarrow \Sigma\Gamma\mathbb{1} \otimes \Lambda\mathbb{1} \otimes X
\end{array}
\right).
$$

Remark 7.6. In light of Lemma 7.1 and the MGM equivalence $\Gamma \simeq \Gamma\Lambda$, any torsion module is equivalent to a $\Lambda\mathbb{1}$-module. As such, the object $\Sigma\Gamma\mathbb{1} \otimes \Lambda\mathbb{1} \otimes X$ is equivalent to $\Sigma\Gamma X$.

The functor $(-)_{\text{tors}} \colon \mathcal{C} \to \mathcal{C}_{\text{pre-t}}$ has a right adjoint $\mathsf{R} \colon \mathcal{C}_{\text{pre-t}} \to \mathcal{C}$ defined by sending the object

$$
\begin{array}{c}
V \\
f \Big\downarrow \\
W \xrightarrow{\ g\ } T
\end{array}
$$

to the pullback of the cospan

$$
\begin{array}{c}
V \\
\Big\downarrow{\scriptstyle f^\#} \\
\mathrm{fib}(g \colon W \to T) \longrightarrow W.
\end{array}
$$

Note that $f \colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to W$ is adjunct of a map $f^\# \colon V \to W$ of $L\mathbb{1}$-modules. Therefore, applying the forgetful functors we view the cospan as a diagram in $\mathcal{C}$ and take the pullback in $\mathcal{C}$.

Lemma 7.7. *The objects in $\mathcal{G}_{\text{tors}}$ are compact in the homotopy category of $\mathcal{C}_{\text{pre-t}}$.*

Proof. The evaluation functors $\mathcal{C}_{\text{pre-t}} \to \mathrm{Mod}_{L\mathbb{1}}$ and $\mathcal{C}_{\text{pre-t}} \to \mathrm{Mod}_{\Lambda\mathbb{1}}$ have left adjoints defined by

$$
a(V) = \left(
\begin{array}{c}
V \\
\vdots \\
\downarrow \\
L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \longrightarrow L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V
\end{array}
\right) \quad \text{and} \quad b(T) = \left(
\begin{array}{c}
0 \\
\vdots \\
\downarrow \\
0 \longrightarrow T
\end{array}
\right)
$$

respectively. It is clear from the definitions of a and b that they preserve cofibrations and weak equivalences since these are defined objectwise. Therefore a and b are left Quillen. Note that the derived functors of a and b preserve compact objects since the derived evaluation preserves sums. For all $G \in \mathcal{G}$, the cofibre sequence

$$
b(\Lambda\mathbb{1} \otimes G) \to a(L\mathbb{1} \otimes G) \to G_{\text{tors}}
$$

shows that G_{tors} is compact in the homotopy category of $\mathcal{C}_{\text{pre-t}}$. $\square$

We can now prove the main theorem of this section.

Theorem 7.8. *The adjunction $(-)_{\text{tors}} \colon \mathcal{C} \rightleftarrows \mathcal{G}_{\text{tors}}\text{-cell-}\mathcal{C}_{\text{pre-t}} \colon \mathsf{R}$ is a Quillen equivalence.*

Proof. We first verify that this is a Quillen adjunction and then apply the Cellularization Principle [24] at the set $\mathcal{G}$ of compact generators of $\mathcal{C}$. Recall that $\mathcal{C}_{\text{pre-t}}$ is equipped with the `dimp` model structure, so we must check that $(-)_{\text{tors}}$ sends cofibrations (resp., acyclic cofibrations) in $\mathcal{C}$ to objectwise cofibrations (resp., acyclic cofibrations). It is clear for the $L\mathbb{1}$ and $L\Lambda\mathbb{1}$ coordinates since extension of scalars is left Quillen. In the final coordinate $(-)_{\text{tors}}$ is given by tensoring with $\Sigma\Gamma\mathbb{1} \otimes \Lambda\mathbb{1} \otimes -$. Therefore we must show that $\Sigma\Gamma\mathbb{1} \otimes \Lambda\mathbb{1} \otimes -$ sends cofibrations (resp., acyclic cofibrations) in $\mathcal{C}$ to cofibrations (resp., acyclic cofibrations) of $\Lambda\mathbb{1}$-modules. As extension of scalars along $\mathbb{1} \to \Lambda\mathbb{1}$ is left Quillen, it suffices to show that $\Gamma\mathbb{1} \otimes -$ is left Quillen as an endofunctor of $\mathcal{C}$. This is true since $\Gamma\mathbb{1}$ is cofibrant in $\mathcal{C}$ by definition of the functor Γ. Therefore, $(-)_{\text{tors}} : \mathcal{C} \rightleftarrows \mathcal{C}_{\text{pre-t}} : \mathsf{R}$ is a Quillen adjunction.

By Lemma 7.7, $\mathcal{G}_{\text{tors}}$ is a set of compact objects of the homotopy category of $\mathcal{C}_{\text{pre-t}}$. The derived unit of the Quillen adjunction

$$(-)_{\text{tors}} : \mathcal{C} \rightleftarrows \mathcal{C}_{\text{pre-t}} : \mathsf{R}$$

is easily seen to be a weak equivalence since the Tate square is a homotopy pullback, see Lemma 6.3. Therefore the Cellularization Principle shows that the stated Quillen equivalence holds. $\qquad\square$

7.5. **Relation to the pre-standard model.** The adjunction $(-)_{\text{tors}} : \mathcal{C} \rightleftarrows \mathcal{C}_{\text{pre-t}} : \mathsf{R}$ factors through the pre-standard category $\mathcal{C}_{\text{pre-s}}$ as follows. We define a functor $\text{hcof} : \mathcal{C}_{\text{pre-s}} \to \mathcal{C}_{\text{pre-t}}$ by

$$\text{hcof}\begin{pmatrix} & V \\ & \vdots \\ & \downarrow \\ N \dashrightarrow W \end{pmatrix} = \begin{matrix} V \\ \vdots \\ \downarrow \\ W \longrightarrow \text{cof}(N \to W) \end{matrix}$$

by taking the horizontal cofibre in the category of $\Lambda\mathbb{1}$-modules. Note that by adjunction, the map $L\Lambda\mathbb{1} \otimes_{\Lambda\mathbb{1}} N \to W$ defines a map of $\Lambda\mathbb{1}$-modules $N \to W$, and it is this map whose horizontal cofibre we take. Similarly, one defines a functor $\text{hfib} : \mathcal{C}_{\text{pre-t}} \to \mathcal{C}_{\text{pre-s}}$ by taking the horizontal fibre as $\Lambda\mathbb{1}$-modules. One notes that the left and right adjoint of the composite adjunction

$$\mathcal{C} \xrightarrow[\underset{\lim}{\longleftarrow}]{(-)_{\text{stan}}} \mathcal{C}_{\text{pre-s}} \xrightarrow[\underset{\text{hfib}}{\longleftarrow}]{\text{hcof}} \mathcal{C}_{\text{pre-t}}.$$

are $(-)_{\text{tors}} = (\text{hcof}) \circ (-)_{\text{stan}}$ and $\mathsf{R} = (\lim) \circ (\text{hfib})$.

It is also easy to see that these functors give a Quillen equivalence $\text{hcof} : \mathcal{C}_{\text{pre-s}} \rightleftarrows \mathcal{C}_{\text{pre-t}} : \text{hfib}$ and hence a Quillen equivalence

$$\text{hcof} : \mathcal{G}_{\text{stan}}\text{-cell-}\mathcal{C}_{\text{pre-s}} \rightleftarrows \mathcal{G}_{\text{tors}}\text{-cell-}\mathcal{C}_{\text{pre-t}} : \text{hfib}$$

relating the models for $\mathcal{C}$ described in Theorems 6.5 and 7.8.

8. The torsion model

In this section we provide a Quillen equivalence between the model category $\mathcal{G}_{\text{tors}}\text{-cell-}\mathcal{C}_{\text{pre-t}}$ considered in the previous section and a cellularization of a certain subcategory $\mathcal{C}^e_{\text{pre-t}}$. The cellularization of $\mathcal{C}^e_{\text{pre-t}}$ is the *torsion model* $\mathcal{C}_t$. This description will allow us to understand the homotopy category of the torsion model.

Definition 8.1. The category $\mathcal{C}^e_{\text{pre-t}}$ of *extended pre-torsion* objects is the full subcategory of $\mathcal{C}_{\text{pre-t}}$ consisting of objects (V, W, T, f, g) for which the map $f : L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to W$ is an isomorphism.

Diagramatically, the objects are those of the form

$$
\begin{array}{ccc}
V & & \\
\vdots & & \\
\bullet & & \\
\downarrow & & \\
W & \xrightarrow{\ g\ } & T
\end{array}
$$

where the dot on the arrow indicates that the map is an isomorphism after extension of scalars. For brevity, and in line with [18], we will often say that an object of the extended pre-torsion model is a $\Lambda\mathbb{1}$-module map $\beta\colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to T$. In this notation a map is a commutative square

$$
\begin{array}{ccc}
L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V & \xrightarrow{\ \beta\ } & T \\
{\scriptstyle L\Lambda\mathbb{1}\otimes_{L\mathbb{1}}\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\
L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} W & \xrightarrow{\ \gamma\ } & U.
\end{array}
$$

There is an adjunction

$$
\mathcal{C}_{\text{pre-t}} \underset{\Gamma_e}{\overset{i}{\xleftrightarrow{\hspace{2em}}}} \mathcal{C}^e_{\text{pre-t}}
$$

where i is the evident inclusion and Γ_e is defined by

$$
\Gamma_e\left(
\begin{array}{ccc}
V & & \\
{\scriptstyle f}\vdots & & \\
\downarrow & & \\
W & \xrightarrow{\ g\ } & T
\end{array}
\right)
=
\begin{array}{ccc}
V & & \\
\vdots & & \\
\bullet & & \\
\downarrow & & \\
L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V & \xrightarrow{\ gf\ } & T.
\end{array}
$$

We note that $\mathcal{C}^e_{\text{pre-t}}$ has all limits and colimits. Colimits are calculated in $\mathcal{C}_{\text{pre-t}}$ whereas limits are obtained by applying the functor Γ_e to the limits in $\mathcal{C}_{\text{pre-t}}$. By [28, 2.1], we may endow the category $\mathcal{C}^e_{\text{pre-t}}$ with the `dimp` model structure, see Notation 2.8.

Lemma 8.2. *There is a Quillen equivalence*

$$
\mathcal{G}_{\text{tors}}\text{-cell-}\mathcal{C}_{\text{pre-t}} \underset{\Gamma_e}{\overset{i}{\xleftrightarrow{\hspace{2em}}}} \mathcal{G}_{\text{tors}}\text{-cell-}\mathcal{C}^e_{\text{pre-t}}.
$$

Proof. Since the weak equivalences and cofibrations are determined objectwise in the diagram injective model structure, it is clear that the inclusion i preserves them. Therefore, the adjunction (without the cellularizations) is a Quillen adjunction. Since i sends $\mathcal{G}_{\text{tors}}$ to $\mathcal{G}_{\text{tors}}$, the Quillen adjunction passes to the cellularizations. By Lemma 7.7, $\mathcal{G}_{\text{tors}}$ is a set of compact objects in $\mathcal{C}_{\text{pre-t}}$, and it follows similarly that the objects of $\mathcal{G}_{\text{tors}}$ are also compact in $\mathcal{C}^e_{\text{pre-t}}$. Therefore by the Cellularization Principle [24] it is enough to show that the derived unit is a weak equivalence on $\mathcal{G}_{\text{tors}}$ which is clear. $\square$

Definition 8.3. The *torsion model* of $\mathcal{C}$ is the model category

$$
\mathcal{C}_t = \mathcal{G}_{\text{tors}}\text{-cell-}\mathcal{C}^e_{\text{pre-t}}.
$$

Combining the results of Theorem 7.8 and Lemma 8.2 gives the following statement.

Theorem 8.4. *There is a zig-zag of Quillen equivalences $\mathcal{C} \simeq_Q \mathcal{C}_t$.*

We now turn our attention to identifying the homotopy category of the torsion model $\mathcal{C}_t$. Throughout we will use the following observation.

Remark 8.5. If $\mathcal{L}$ is a set of compact objects, then the homotopy category of $\mathcal{L}$-cell-$\mathcal{C}$ can be identified with the localizing subcategory of $h\mathcal{C}$ generated by $\mathcal{L}$, see [24, 2.5, 2.6].

Lemma 8.6. *An object $X \in \mathcal{C}$ is $(\Gamma\mathbb{1} \otimes \mathcal{G})$-cellularly acyclic if and only if it is $(\mathcal{K} \otimes \mathcal{G})$-cellularly acyclic. Therefore, the cellularizations of $\mathcal{C}$ at the sets $(\Gamma\mathbb{1} \otimes \mathcal{G})$ and $(\mathcal{K} \otimes \mathcal{G})$ are equal as model categories.*

Proof. Since $\Gamma\mathbb{1}$ is torsion the reverse implication is clear. Conversely suppose that $[\Gamma\mathbb{1} \otimes G, X] \simeq 0$ for all $G \in \mathcal{G}$. This means that $[G, \mathrm{Hom}(\Gamma\mathbb{1}, X)] = 0$ for all $G \in \mathcal{G}$ and so $\mathrm{Hom}(\Gamma\mathbb{1}, X) \simeq 0$ in $\mathcal{C}$. Then $[K \otimes G, \mathrm{Hom}(\Gamma\mathbb{1}, X)] \simeq 0$ for each $K \in \mathcal{K}$, and since $\Gamma K \simeq K$ we conclude that $[K \otimes G, X] = 0$ as required. The second claim then follows since the cellularly acyclic objects determine the weak equivalences in the cellularization. $\square$

Theorem 8.7. *A bifibrant object $(\beta\colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to T)$ of $\mathcal{C}^e_{\mathrm{pre\text{-}t}}$ is bifibrant in $\mathcal{C}_t$ if and only if the natural map $\Gamma T \to T$ is a weak equivalence in $\mathcal{C}$.*

Proof. Let $\mathcal{L} = \mathcal{L}(\mathcal{K}, \mathcal{G})$ be the set of objects of $\mathcal{C}^e_{\mathrm{pre\text{-}t}}$ of the form

$$(0 \to \Lambda\mathbb{1} \otimes K \otimes G) \quad \text{and} \quad (L\Lambda\mathbb{1} \otimes G \to 0)$$

where K ranges over $\mathcal{K}$ and G ranges over the generators $\mathcal{G}$. Firstly, we show that $\mathcal{C}_t = \mathcal{L}$-cell-$\mathcal{C}^e_{\mathrm{pre\text{-}t}}$ as model categories. Note that there is a triangle

$$(0 \to \Sigma\Gamma\mathbb{1} \otimes \Lambda\mathbb{1} \otimes G) \to (L\Lambda\mathbb{1} \otimes G \to \Sigma\Gamma\mathbb{1} \otimes \Lambda\mathbb{1} \otimes G) \to (L\Lambda\mathbb{1} \otimes G \to 0)$$

and combining this with Lemma 8.6 shows that an object of $\mathcal{C}^e_{\mathrm{pre\text{-}t}}$ is $\mathcal{G}_{\mathrm{tors}}$-cellularly acyclic if and only if it is $\mathcal{L}$-cellularly acyclic. Therefore the $\mathcal{G}_{\mathrm{tors}}$-cellularization and the $\mathcal{L}$-cellularization give the same model category.

We now claim that the objects in $\mathcal{L}$ are compact. Since $\Lambda\mathbb{1} \otimes K \otimes G$ is compact as a $\Lambda\mathbb{1}$-module and sums in $\mathcal{C}^e_{\mathrm{pre\text{-}t}}$ are computed levelwise, we immediately see that $(0 \to \Lambda\mathbb{1} \otimes K \otimes G) \in \mathcal{L}$ is compact. To see that $(L\Lambda\mathbb{1} \otimes G \to 0)$ is compact, we show that any map in $\mathcal{C}^e_{\mathrm{pre\text{-}t}}$

$$\begin{array}{ccc}
L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} (L\mathbb{1} \otimes G) & \longrightarrow & 0 \\
{\scriptstyle L\Lambda\mathbb{1}\otimes\varphi}\downarrow & & \downarrow \\
L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} (\bigoplus_i V_i) & \longrightarrow & \bigoplus_i T_i
\end{array}$$

factors through a finite stage. This is clear since the image of φ is nonzero only for finitely many indices i as $L\mathbb{1} \otimes G$ is a compact $L\mathbb{1}$-module.

Since the objects in $\mathcal{L}$ are compact, the bifibrant objects in $\mathcal{L}$-cell-$\mathcal{C}^e_{\mathrm{pre\text{-}t}}$ are exactly the objects of the localizing subcategory $\mathrm{Loc}(\mathcal{L})$. From this it is clear that $(\beta\colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to T) \in \mathcal{C}^e_{\mathrm{pre\text{-}t}}$ is bifibrant in the cellularization if and only if $\Gamma T \to T$ is an equivalence. $\square$

Remark 8.8. By Theorem 8.7, an object in the homotopy category of the torsion model $\mathcal{C}_t$ is represented by the following data:

    (i) a $L\mathbb{1}$-module V;
    (ii) a $\Lambda\mathbb{1}$-module T which is torsion in the sense that the natural map $\Gamma T \to T$ is a weak equivalence;
    (iii) a $\Lambda\mathbb{1}$-module map $\beta\colon L\Lambda\mathbb{1} \otimes_{L\mathbb{1}} V \to T$.

Since extension of scalars along $\mathbb{1} \to \Lambda\mathbb{1}$ does not affect cellularization by Lemma 7.1, we may assume T is a $\Lambda\mathbb{1}$-module and can replace (ii) and (iii) with

    (ii') an object $T \in \mathcal{C}$ such that $\Gamma T \to T$ is a weak equivalence;

(iii') a map $\beta' : V \to T$ in $\mathcal{C}$.

We call (i), (ii), (iii) *adelic data* and (i), (ii)', (iii)' *Zariski data*. The message of this paper is that one should use the adelic data because they are closer to algebra than the Zariski data, and we think of the adelic data as an enrichment of the Zariski data. For example with rational $SO(2)$-spectra, in the Zariski formulation we are working with modules over the $SO(2)$-equivariant rational sphere which has a very complicated algebraic structure. However, in the adelic formulation we work over $DE\mathcal{F}_+$, which is formal and $\pi_*^{\mathbb{T}}(DE\mathcal{F}_+) = \mathcal{O}_{\mathcal{F}}$ is a product of polynomial rings. See Section 10 for more details.

Example 8.9. Let us consider the torsion model for $\mathrm{Mod}_{\mathbb{Z}}$ with the family of closed points, i.e., the family of primes corresponding to the algebraic ideals (p) for $p > 0$ prime. Recall that a maximal prime in $\mathrm{Spec}(\mathbb{Z})$ becomes minimal in the Balmer spectrum. The set $\{\mathbb{Z}/p \mid p \text{ prime}\}$ is a set of Koszul objects for this family of primes. Using Proposition 4.12 and Section 5.1 we calculate:

$$L\mathbb{1} = \mathbb{Q} \qquad \Gamma\mathbb{1} = \bigoplus_{p \text{ prime}} \Sigma^{-1}\mathbb{Z}/p^\infty \qquad \Lambda\mathbb{1} = \prod_{p \text{ prime}} \mathbb{Z}_p.$$

Furthermore by Proposition 4.12, a complex $T \in \mathrm{Mod}_{\mathbb{Z}}$ satisfies $\Gamma T \simeq T$ if and only if it decomposes as $T \simeq \bigoplus_p T_p$ where T_p is derived p-torsion. Therefore the data in the Zariski torsion model for $\mathrm{Mod}_{\mathbb{Z}}$ consists of:

- a rational chain complex $V \in \mathrm{Mod}_{\mathbb{Q}}$;
- a derived p-torsion complex T_p for each prime number p;
- a chain map $\beta : V \to \bigoplus_p T_p$.

The adelic torsion model instead consists of:

- a rational chain complex $V \in \mathrm{Mod}_{\mathbb{Q}}$;
- a derived p-torsion complex T_p for each prime number p but now seen as a module over $\mathbb{Z}_p$;
- a map $(\prod_p \mathbb{Z}_p) \otimes V \to \bigoplus_p T_p$ of $(\prod_p \mathbb{Z}_p)$-modules.

Remark 8.10. Finally let us record the precise relationship between the standard model and the torsion model. Combining the results of Section 7.5 with Lemma 8.2, one sees that there is a zig-zag of Quillen equivalences

$$\mathcal{C}_s \underset{\Gamma_e}{\overset{}{\rightleftarrows}} \mathcal{G}_{\text{stan}}\text{-cell-}\mathcal{C}_{\text{pre-s}} \underset{\text{hfib}}{\overset{\text{hcof}}{\rightleftarrows}} \mathcal{G}_{\text{tors}}\text{-cell-}\mathcal{C}_{\text{pre-t}} \underset{\Gamma_e}{\overset{}{\rightleftarrows}} \mathcal{C}_t$$

between the standard model and torsion model for $\mathcal{C}$.

9. Cellular skeleton theorems

The cellularization of the model structure present in the torsion model forces one of the objects to be derived torsion, see Theorem 8.7. If one is in a suitably algebraic situation, one can internalize this cellularization and prove a Quillen equivalence to a category of differential objects in an abelian category where this object is now torsion at the abelian level. We call such a Quillen equivalence a *cellular skeleton theorem*, and in this section we prove such a result. We then apply this to simplify the torsion model for the derived category of a regular local ring of Krull dimension 1. In Section 10 the cellular skeleton theorem will provide the final step in proving the algebraic torsion model for rational $\mathbb{T}$-spectra.

9.1. **Torsion functors.** There are two types of relevant torsion functor: with respect to a multiplicative set and with respect to an ideal.

Definition 9.1. Let R be a commutative ring and consider an ideal I and a multiplicative closed set $\mathcal{S}$ of R.

- The I-power torsion functor $\Gamma_I \colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ is defined by

$$\Gamma_I N = \{n \in N \mid I^a n = 0 \text{ for some } a > 0\}.$$

  If the natural map $\Gamma_I N \to N$ is an isomorphism, we say that N is I-*power torsion.*
- The $\mathcal{S}$-torsion functor $\Gamma_{\mathcal{S}} \colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ is defined by sending an R-module N to the kernel of the map $N \to \mathcal{S}^{-1}N$. If the natural map $\Gamma_{\mathcal{S}} N \to N$ is an isomorphism, we say that N is $\mathcal{S}$-*torsion.*

Remark 9.2. The $\mathcal{S}$-torsion functor is more elementary and easily expressed in terms of the torsion functors for principal ideals

$$\Gamma_{\mathcal{S}} N = \mathrm{colim}_{a \in \mathcal{S}} \Gamma_a N.$$

Notation 9.3. By a *torsion functor* for a commutative ring R we mean Γ_I for some ideal I of R, or $\Gamma_{\mathcal{S}}$ for some multiplicative set $\mathcal{S}$ in R. We write $\mathbb{\Gamma}$ to denote a torsion functor and $\mathrm{Mod}_R^{\mathbb{\Gamma}\text{-tors}}$ for the full subcategory of Mod_R consisting of the $\mathbb{\Gamma}$-torsion modules.

We will be working under the following additional assumption.

Hypothesis 9.4. The torsion functor $\mathbb{\Gamma} \colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ preserves injective modules.

If R is Noetherian then this hypothesis holds as a consequence of the Baer criterion, see [13, 2.1.4]. One immediately sees that $\mathrm{Mod}_R^{\mathbb{\Gamma}\text{-tors}}$ is an abelian category with enough injectives. This hypothesis ensures that the injective dimension of a torsion module is the same whether we think of it in the category of $\mathbb{\Gamma}$-torsion modules or in the category of R-modules.

9.2. **The cellular skeleton theorem.** Suppose we have a diagram of commutative rings $S \to \mathcal{E}^{-1}R \leftarrow R$ where $\mathcal{E}$ is a multiplicative set in R. Furthermore, we assume that R is equipped with a torsion functor $\mathbb{\Gamma}$ in the sense of the previous section.

Consider the sections of the diagram category

$$\begin{array}{c} \mathrm{Mod}_S \\ \vert \\ \bullet \\ \downarrow \\ \mathrm{Mod}_{\mathcal{E}^{-1}R} \longrightarrow \mathrm{Mod}_R \end{array}$$

where the vertical is extension of scalars along the ring map $S \to \mathcal{E}^{-1}R$ and the horizontal is restriction of scalars along the ring map $R \to \mathcal{E}^{-1}R$. We denote the sections of this category by $\widehat{\mathcal{A}}_t$. An object of $\widehat{\mathcal{A}}_t$ is given by the data of an R-module map $\mathcal{E}^{-1}R \otimes_S V \to N$ where V is an S-module and N is an R-module.

We are interested in situations when $\widehat{\mathcal{A}}_t$ is cellularized in such a way that the map $\mathbb{\Gamma}N \to N$ is forced to be a quasi-isomorphism, so that in particular, the homology of N is $\mathbb{\Gamma}$-torsion. The goal of this section is to prove that one can strictify this at the abelian level by replacing N with a $\mathbb{\Gamma}$-torsion module.

Definition 9.5. We define the category $\mathcal{A}_t$ to have objects R-module maps $\mathcal{E}^{-1}R \otimes_S V \to T$ where V is an S-module and T is a Γ-torsion R-module. Morphisms in $\mathcal{A}_t$ are commutative squares

$$
\begin{array}{ccc}
\mathcal{E}^{-1}R \otimes_S V & \longrightarrow & T \\
{\scriptstyle 1 \otimes \varphi} \downarrow & & \downarrow {\scriptstyle \psi} \\
\mathcal{E}^{-1}R \otimes_S W & \longrightarrow & U.
\end{array}
$$

The category $\mathcal{A}_t$ is abelian with kernels and cokernels defined levelwise.

Lemma 9.6.

    (a) *The abelian category $\mathcal{A}_t$ has enough injectives.*
    (b) *The category $d\mathcal{A}_t$ of differential objects in $\mathcal{A}_t$ supports a* `dimi` *model structure in which the weak equivalences are the objectwise quasi-isomorphisms and the cofibrations are the objectwise monomorphisms.*
    (c) *Suppose that S is a field. The injective dimension of $\mathcal{A}_t$ is equal to $\mathrm{injdim}(\mathrm{Mod}_R) + 1$.*

Proof. For an S-module V and an Γ-torsion R-module T, we put

$$
e^t(V) = (\mathcal{E}^{-1}R \otimes_S V \to 0) \quad \text{and} \quad f^t(T) = (\mathcal{E}^{-1}R \otimes_S \mathrm{Hom}_R(\mathcal{E}^{-1}R, T) \to T)
$$

where the second map is the evaluation. A straightforward calculation shows that e^t and f^t are right adjoint to the respective forgetful functors. The forgetful functors are exact as (co)kernels in $\mathcal{A}_t$ are defined levelwise, and therefore e^t and f^t preserve injectives. Consider an object $X = (\mathcal{E}^{-1}R \otimes_S V \to T) \in \mathcal{A}_t$ and choose a monomorphism $T \to I$ where I is an injective Γ-torsion module. Then there is a monomorphism $X \to f^t(I) \oplus e^t(V)$ and so part (a) follows. Part (b) then follows from [18, Appendix B].

For part (c), we write r for the injective dimension of the category of R-modules. Consider an object $X = (\mathcal{E}^{-1}R \otimes_S V \to T)$ of $\mathcal{A}_t$ and choose an injective resolution

$$
0 \to T \xrightarrow{\alpha_0} I_0 \xrightarrow{\alpha_1} I_1 \to \ldots \to I_{r-1} \xrightarrow{\alpha_r} I_r \to 0
$$

of T. We put $V_i = \mathrm{Hom}_R(\mathcal{E}^{-1}R, I_i)$ for all $i \geq 0$. Then one checks that that there is an exact sequence

$$
0 \to X \to e^t(V) \oplus f^t(I_0) \to e^t(V_0) \oplus f^t(I_1) \to \ldots \to e^t(V_{r-1}) \oplus f^t(I_r) \to e^t(V_r) \to 0
$$

which provides an injective resolution of X in $\mathcal{A}_t$. This shows that the injective dimension is less than or equal to $r + 1$. Finally we note that the bound can be achieved. For example, using the notation above

$$
\mathrm{Ext}_{\mathcal{A}_t}^{r+1}(e^t(S), (0 \to T)) = \mathrm{Hom}_{\mathcal{A}_t}(e^t(S), e^t(V_r)) = V_r = \mathrm{Ext}_R^r(\mathcal{E}^{-1}R, T)
$$

which need not be zero. $\qquad\square$

Remark 9.7. The category $\widehat{\mathcal{A}}_t$ supports both a `dimi` and `dimp` model structure, but the category $d\mathcal{A}_t$ supports only a `dimi` model structure since there are enough torsion injectives, but in general no torsion projectives.

We now construct an adjunction (i, Γ_t) between the categories $\widehat{\mathcal{A}}_t$ and $d\mathcal{A}_t$ described above. There is an evident inclusion $i \colon d\mathcal{A}_t \to \widehat{\mathcal{A}}_t$. We now claim that this has a right adjoint Γ_t defined as

follows. Given $(\alpha \colon \mathcal{E}^{-1} R \otimes_S V \to N) \in \widehat{\mathcal{A}}_t$ we construct the pullback

$$
\begin{array}{ccc}
W' & \xrightarrow{\ \gamma\ } & \mathbb{\Gamma} N \\
\downarrow & \lrcorner & \uparrow \\
\mathcal{E}^{-1} R \otimes_S V & \xrightarrow{\ \alpha\ } & N
\end{array}
$$

in the category of R-modules. By applying the $\mathcal{E}$-torsion functor $\Gamma_{\mathcal{E}}$ to the pullback above, we see that $\Gamma_{\mathcal{E}} W' \hookrightarrow \Gamma_{\mathcal{E}}(\mathcal{E}^{-1} R \otimes_S V) = 0$ since $\Gamma_{\mathcal{E}}$ preserves monomorphisms. Thus $\mathcal{E}$ acts by isomorphisms on W' and so W' can be promoted to a $\mathcal{E}^{-1} R$-module. We now construct a second pullback in the category of S-modules

$$
\begin{array}{ccccc}
V' & \xrightarrow{\ \delta^{\sharp}\ } & W' & \xrightarrow{\ \gamma\ } & \mathbb{\Gamma} N \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \uparrow \\
V & \xrightarrow{\ \eta \otimes V\ } & \mathcal{E}^{-1} R \otimes_S V & \xrightarrow{\ \alpha\ } & N.
\end{array}
$$

By adjunction, the map $\delta^{\sharp}$ corresponds to a $\mathcal{E}^{-1} R$-map $\delta \colon \mathcal{E}^{-1} R \otimes_S V' \to W'$. We finally define

$$
\Gamma_t(\mathcal{E}^{-1} R \otimes_S V \xrightarrow{\alpha} N) = (\mathcal{E}^{-1} R \otimes_S V' \xrightarrow{\gamma \delta} \mathbb{\Gamma} N).
$$

One can then check that (i, Γ_t) forms an adjoint pair.

Lemma 9.8. *Let $\mathcal{L} = \{(\mathcal{E}^{-1} R \to 0), (0 \to K)_{K \in \mathcal{K}}\}$ where $\mathcal{K}$ is a set of compact generators for the derived category of $\mathbb{\Gamma}$-torsion R-modules. A bifibrant object $(\alpha \colon \mathcal{E}^{-1} R \otimes_S V \to N) \in \widehat{\mathcal{A}}_t$ is bifibrant in the $\mathcal{L}$-cellularization if and only if the canonical R-module map $\mathbb{\Gamma} N \to N$ is a quasi-isomorphism.*

Proof. We argue that the cells $(\mathcal{E}^{-1} R \to 0)$ and $(0 \to K)$ are compact in $\widehat{\mathcal{A}}_t$ so we can identify the homotopy category of the cellularization with the localizing subcategory generated by the cells. The desired result then follows by a standard localizing subcategory argument.

It is clear by assumption that the cell $(0 \to K)$ is compact in $\widehat{\mathcal{A}}_t$, so it remains to show that $(\mathcal{E}^{-1} R \to 0)$ is compact. Consider a map from $(\mathcal{E}^{-1} R \to 0)$ into a direct sum $\bigoplus_i (\mathcal{E}^{-1} R \otimes_S V_i \xrightarrow{\alpha_i} N_i)$. Note that any such map is uniquely determined by a map of S-modules $S \to \bigoplus_i \ker(\alpha)$ which factors through a finite stage since S is a compact S-module. It follows that the cell $(\mathcal{E}^{-1} R \to 0)$ is compact and this concludes the proof. $\qquad \square$

We now state and prove the cellular skeleton theorem.

Theorem 9.9. *Let $\mathcal{L} = \{(\mathcal{E}^{-1} R \to 0), (0 \to K)_{K \in \mathcal{K}}\}$ where $\mathcal{K}$ is a set of compact generators for the derived category of $\mathbb{\Gamma}$-torsion R-modules. The adjunctions*

$$
\mathcal{L}\text{-cell-}\widehat{\mathcal{A}}_t^{\mathtt{dimp}} \underset{1}{\overset{1}{\rightleftarrows}} \mathcal{L}\text{-cell-}\widehat{\mathcal{A}}_t^{\mathtt{dimi}} \underset{\Gamma_t}{\overset{i}{\leftrightarrows}} d\mathcal{A}_t^{\mathtt{dimi}}
$$

are both Quillen equivalences.

Proof. For the first adjunction, without the cellularization the result is clear since the identity is a left Quillen equivalence from the projective to the injective model structure on modules. The cellularization preserves the Quillen equivalence.

For the second adjunction, we firstly we verify that it is a Quillen adjunction. Note that

$$
\Gamma_t : \widehat{\mathcal{A}}_t \leftrightarrows d\mathcal{A}_t : i
$$

is a Quillen adjunction when both sides are equipped with the `dimi` model structure. Therefore, by the dual result to Dugger [16, A.2], to check that

$$\Gamma_t : \mathcal{L}\text{-cell-}\widehat{\mathcal{A}}_t^{\mathtt{dimi}} \leftrightarrows d\mathcal{A}_t^{\mathtt{dimi}} : i$$

is a Quillen adjunction it is sufficient to check that i sends objects of $d\mathcal{A}_t$ to cellular objects. This is clear by Lemma 9.8. To see that this Quillen adjunction is moreover a Quillen equivalence, firstly note that the unit is always an equivalence. The counit is an equivalence on objects of the form identified in Lemma 9.8 so the result follows. $\qquad\square$

9.3. **Application to regular local rings of dimension 1.** We now apply the cellular skeleton theorem to simplify the torsion model for the derived category of a regular local ring of Krull dimension 1. Suppose that R is a regular local ring which has Krull dimension 1 (i.e., R is a discrete valuation ring), and write $\mathfrak{m}$ for its maximal ideal. Take the family of prime ideals which consists of the maximal algebraic prime $\mathfrak{m}$ (i.e., the minimal Balmer prime). Since R is regular and has Krull dimension 1, the maximal ideal $\mathfrak{m}$ is generated by a single element, say x. We can now identify the functors at play in the torsion model as $L\mathbb{1} = R[x^{-1}]$, $\Gamma\mathbb{1} = \Sigma^{-1}R/x^\infty$ and $\Lambda\mathbb{1} = R_\mathfrak{m}^\wedge$, see Section 5.1 and Remark 9.10 for more details. Therefore the torsion model (Theorem 8.4) gives a Quillen equivalence

$$\mathrm{Mod}_R \simeq_Q \text{cell-}\left(\begin{array}{c} \mathrm{Mod}_{R[x^{-1}]} \\ {\scriptstyle|} \\ {\bullet} \\ {\downarrow} \\ \mathrm{Mod}_{R_\mathfrak{m}^\wedge[x^{-1}]} \longrightarrow \mathrm{Mod}_{R_\mathfrak{m}^\wedge} \end{array} \right)$$

where the cellularization can be taken at the set of cells $\{(0 \to R/x), (R_\mathfrak{m}^\wedge[x^{-1}] \to 0)\}$ by Theorem 8.7.

Define $\mathcal{A}_t(R)$ to be the category whose objects are $R_\mathfrak{m}^\wedge$-module maps $R_\mathfrak{m}^\wedge[x^{-1}] \otimes_{R[x^{-1}]} V \to T$ where V is a $R[x^{-1}]$-module and T is an $\mathfrak{m}$-power torsion module. By Lemma 9.6 this is an abelian category with enough injectives. The object R/x is a compact generator for the derived category of $\mathfrak{m}$-power torsion modules, and therefore applying the cellular skeleton theorem (Theorem 9.9) shows that we have a zig-zag of Quillen equivalences

$$\mathrm{Mod}_R \simeq_Q d\mathcal{A}_t(R).$$

Remark 9.10. We restrict to the case of regular local rings of dimension 1 so that $L\mathbb{1}$ is a ring, rather than a DGA. In general, for a local ring $(R, \mathfrak{m})$ the localization $L\mathbb{1} = \check{C}_\mathfrak{m}(R)$ can be additively obtained from $K_\infty(\mathfrak{m})$ by deleting R from degree 0 and regrading. Therefore, if $\mathfrak{m} = (x)$ is principal then $K_\infty(x) = (R \to R[x^{-1}])$ and $\check{C}_\mathfrak{m}(R) = R[x^{-1}]$.

10. The torsion model for rational $\mathbb{T}$-spectra

We now apply the theory developed in the previous sections to establish an algebraic torsion model for the category of rational $\mathbb{T}$-spectra: Theorem 10.8 promotes the derived equivalence of [18] to a Quillen equivalence.

In this part of the paper we will be working rationally, i.e., all spectra are rationalized without further comment, and all homology and cohomology theories will be unreduced with rational coefficients.

10.1. **Preliminaries.** Unless stated otherwise, we write $\mathbb{T}$ for the circle group and $\mathrm{Sp}_{\mathbb{T}}$ for the category of orthogonal $\mathbb{T}$-spectra with the rational $\mathbb{T}$-stable model structure. The Balmer spectrum was discussed in Section 5.3.

We let z denote the natural representation of $\mathbb{T}$, so that z^n is a one dimensional representation with kernel F cyclic of order n. The cohomology ring $H^*(B\mathbb{T}/F) = \mathbb{Q}[c_F]$ is polynomial on the Euler class c_F of z^n.

Definition 10.1.

- The family of finite subgroups of $\mathbb{T}$ will be denoted $\mathcal{F}$.
- We denote by $\mathcal{O}_{\mathcal{F}}$ the coefficient ring
$$\mathcal{O}_{\mathcal{F}} = \prod_{F \in \mathcal{F}} H^*(B\mathbb{T}/F).$$
- For a $\mathbb{T}$-representation V, the Euler class $e(V) \in \mathcal{O}_{\mathcal{F}}$ is the element with F-component $e(V)_F = e(V^F) \in H^{|V^F|}(B\mathbb{T}/F)$, and we write $\mathcal{E}_{\mathbb{T}} = \{e(V) \mid V^{\mathbb{T}} = 0\}$.
- We define the $\mathcal{F}$-Tate ring by $t_{\mathcal{F}} = \mathcal{E}_{\mathbb{T}}^{-1}\mathcal{O}_{\mathcal{F}}$.

Recall from Section 5.3 that we have functors $L_{\mathcal{F}^c}, \Gamma_{\mathcal{F}}$ and $\Lambda_{\mathcal{F}}$ associated to the family $\mathcal{F}$. Unravelling the definitions we see that the diagram $T_{\mathcal{F}}^{\lrcorner}$ from Definition 6.2 corresponds to the Tate diagram

$$
\begin{array}{c}
\widetilde{E}\mathcal{F} \\
\downarrow \\
DE\mathcal{F}_+ \longrightarrow DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}.
\end{array}
$$

Proposition 10.2. *By applying the $\mathbb{T}$-homotopy groups functor to the diagram $T_{\mathcal{F}}^{\lrcorner}$ above we obtain the following diagram of commutative rings*

$$
T_{\mathcal{F}}^a = \quad
\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
\mathcal{O}_{\mathcal{F}} \longrightarrow t_{\mathcal{F}}
\end{array}
$$

with the obvious maps. Furthermore, the diagram $T_{\mathcal{F}}^a$ is intrinsically formal in the sense that any cospan of commutative DGAs $A \to B \leftarrow C$ with homology $T_{\mathcal{F}}^a$ is quasi-isomorphic to $T_{\mathcal{F}}^a$.

Proof. The $\mathbb{T}$-homotopy groups of $DE\mathcal{F}_+$ can be calculated using [18, 2.2.3, 2.4.1]. It follows from [18, §5.2] that $\pi_*^{\mathbb{T}}(DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}) = t_{\mathcal{F}}$. Finally formality follows from [27, 9.4, §9.D]. $\qquad\square$

10.2. **Euler torsion modules.** We would like to apply the theory of Section 9 to the diagram of rings $\mathcal{O}_{\mathcal{F}} \to t_{\mathcal{F}} \leftarrow \mathbb{Q}$ equipped with the torsion functor associated to the set $\mathcal{E}_{\mathbb{T}}$ of Euler classes. The Euler torsion functor is the composite of two functors which both preserve injective objects by [18, 17.3.3, 17.3.5]. Therefore by the results in Section 9 there exists an abelian category $\mathcal{A}_t(\mathbb{T})$ of injective dimension 2, whose objects have the form $(t_{\mathcal{F}} \otimes V \to T)$ where T is an Euler torsion $\mathcal{O}_{\mathcal{F}}$-module and V is a $\mathbb{Q}$-module. In particular, the category $d\mathcal{A}_t(\mathbb{T})$ of differential objects in $\mathcal{A}_t(\mathbb{T})$ supports a `dimi` model structure.

In order to apply the cellular skeleton theorem we need to understand the compact generators in the derived category of Euler torsion modules. Let $e_F \in \mathcal{O}_{\mathcal{F}}$ be the idempotent given by projection onto the F-factor. Then we have the following useful criterion.

Lemma 10.3 ([18, 4.6.6]). *The following conditions on a $\mathcal{O}_{\mathcal{F}}$-module M are equivalent:*

(a) M is Euler torsion;

(b) $M = \bigoplus_{F \in \mathcal{F}} e_F M$ and for each $m \in M$ there exist $N > 0$ such that $c^N m = 0$, where c is the total Chern class.

Lemma 10.4. *The derived category of x-power torsion $\mathbb{Q}[x]$-modules is generated by any non-zero module. In particular, it is compactly generated by $\mathbb{Q}$.*

Corollary 10.5. *The derived category of Euler torsion $\mathcal{O}_{\mathcal{F}}$-modules is compactly generated by the modules $H^*(B\mathbb{T}/F)/(c_F) = \mathbb{Q}_F$ for finite subgroups F.*

Proof. Let $\alpha_F \colon \mathbb{T} \to \mathbb{T}$ be a representation with kernel exactly F, and consider $e(\alpha_F) \in H^2(B\mathbb{T}/F)$. Lemma 10.3 tells us that an $\mathcal{O}_{\mathcal{F}}$-module M is Euler torsion if and only if it decomposes as $M = \bigoplus_F e_F M$ with $e_F M$ an $e(\alpha_F)$-torsion $H^*(B\mathbb{T}/F)$-module. It follows that the functor $M \mapsto (e_F M)_F$ defines an equivalence between the category of Euler torsion $\mathcal{O}_{\mathcal{F}}$-modules and the category

$$\coprod_{F \in \mathcal{F}} e(\alpha_F)\text{-tors-}H^*(B\mathbb{T}/F)\text{-mod.}$$

Therefore the claim follows from Lemma 10.4 by recalling that there is a canonical isomorphism of rings $\mathbb{Q}[c_F] \to H^*(B\mathbb{T}/F)$ sending c_F to $e(\alpha_F)$. □

10.3. **Fixed points and commutativity.** In this section we describe how to ensure a good interplay between commutativity and cofibrancy when taking categorical fixed points. This is harder than it first appears, due to the fact that categorical fixed points is not a Quillen functor between the flat model structures.

Suppose that we have a map $S \to R$ of commutative ring G-spectra. In addition, suppose that R is a cofibrant S-module. Note that this can be forced if we work in the flat model structure, see Section 7.3 for more details. We wish to choose a suitable cofibrant replacement for R^G, denoted by $Q_f(R^G)$, with the following properties:

(a) there is a map of commutative ring spectra $\varphi \colon S^G \to Q_f(R^G)$;

(b) φ exhibits $Q_f(R^G)$ as a cofibrant S^G-module.

Condition (b) ensures that the restriction of scalars functor φ^* is left Quillen by Lemma 2.3. This in turn guarantees that the sections of the required diagram category has a `dimp` model structure. Condition (a) is enforced to ensure that after applying Shipley's theorem [38, 1.2] we obtain commutative DGAs, and as such, may apply formality arguments.

We now show that a cofibrant replacement satisfying conditions (a) and (b) does exist. In addition, the replacement we construct will have other convenient properties which we state as Lemmas 10.6 and 10.7.

Taking categorical G-fixed points, we obtain a map of commutative ring spectra $S^G \to R^G$ since $(-)^G$ is lax symmetric monoidal. We now replace R^G as a commutative S^G-algebra in the flat model structure, which gives a commutative diagram

$$
\begin{array}{ccc}
S^G & \longrightarrow & Q_f(R^G) \\
 & \searrow & \downarrow{\scriptstyle \sim} \\
 & & R^G
\end{array}
$$

of commutative ring spectra. Then condition (a) is satisfied and (b) follows from [37, 4.1].

We note that one could also cofibrantly replace R^G as an S^G-algebra in the stable model structure to give $\beta \colon Q(R^G) \to R^G$, but that it may not be possible to satisfy both (a) and (b) with such a replacement. Nonetheless, this alternative cofibrant replacement acts as a useful stepping stone

between G-spectra and non-equivariant spectra. This will play an important role in the next section, via the following two lemmas.

Lemma 10.6. *There is a composite Quillen adjunction*

$$\mathrm{Mod}_R^{\mathrm{flat}}(\mathrm{Sp}_G) \xrightarrow[\;\;1\;\;]{\;\;1\;\;} \mathrm{Mod}_R^{\mathrm{stable}}(\mathrm{Sp}_G) \xleftarrow[\;(-)^G\;]{\;R\otimes_{\inf R^G}\inf(-)\;} \mathrm{Mod}_{R^G}^{\mathrm{stable}}(\mathrm{Sp}) \xleftarrow[\;\beta^*\;]{\;\beta_*\;} \mathrm{Mod}_{Q(R^G)}^{\mathrm{stable}}(\mathrm{Sp}).$$

Moreover, this composite Quillen adjunction is a Quillen equivalence if and only if the middle adjunction is a Quillen equivalence.

Proof. The middle adjunction arises from the adjoint lifting theorem since $(-)^G$ is lax monoidal; see [36, §3.3] or [25] for more details. The left hand adjunction is a Quillen equivalence since the weak equivalences in the stable and flat model structure are the same, and the right hand adjunction is a Quillen equivalence since β is a weak equivalence. $\square$

Lemma 10.7. *There is a map of ring spectra $\alpha\colon Q(R^G) \to Q_f(R^G)$ which is a weak equivalence. Therefore there is a Quillen equivalence*

$$\mathrm{Mod}_{Q(R^G)}^{\mathrm{stable}} \xrightarrow[\;\alpha^*\;]{\;\alpha_*\;} \mathrm{Mod}_{Q_f(R^G)}^{\mathrm{flat}}$$

given by extension and restriction of scalars.

Proof. There is a commutative square

$$
\begin{array}{ccc}
S^G & \longrightarrow & Q_f(R^G) \\
{\scriptstyle\mathrm{stable}}\big\uparrow & & \sim\big\downarrow{\scriptstyle\mathrm{flat}} \\
Q(R^G) & \xrightarrow[\;\mathrm{stable}\;]{\;\sim\;} & R^G
\end{array}
$$

of ring spectra. Since the right hand vertical is an acyclic flat fibration and hence an acyclic stable fibration, there is a lift $Q(R^G) \to Q_f(R^G)$ in the square as a map of rings. Moreover, this is a weak equivalence by the 2-out-of-3 property. The claimed Quillen equivalence follows since the identity functor is a right Quillen equivalence from the flat model structure to the stable model structure. $\square$

10.4. **The Quillen equivalences.** The goal of this section is to prove the following result.

Theorem 10.8. *There is a Quillen equivalence*

$$\mathrm{Sp}_{\mathbb{T}} \simeq_Q d\mathcal{A}_t(\mathbb{T}).$$

We divide the proof into several steps. Since the formality argument is a vital step and it relies upon *commutativity*; we must use a convenient model structure, see the discussion in Section 7.3. Therefore, we will work with the flat model structure on equivariant spectra as in [14]. This introduces some complications since the categorical fixed points functor is not Quillen as a functor $\mathrm{Sp}_G^{\mathrm{flat}} \to \mathrm{Sp}^{\mathrm{flat}}$. We shall use the results of Section 10.3 to overcome these complications.

Step 1. Consider the category $\mathrm{Sp}_{\mathbb{T}}$ with $L = L_{\mathcal{F}^c}$ the $\mathcal{F}$-nullification functor given by $LX = \widetilde{E}\mathcal{F} \wedge X$. We apply Theorem 8.4 to deduce a Quillen equivalence

$$\mathrm{Sp}_{\mathbb{T}} \simeq_Q \text{cell-}\begin{pmatrix} \mathrm{Mod}_{\widetilde{E}\mathcal{F}} \\ \vert \\ \bullet \\ \downarrow \\ \mathrm{Mod}_{DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}} \longrightarrow \mathrm{Mod}_{DE\mathcal{F}_+} \end{pmatrix}$$

where the right hand side uses the flat model structure.

Step 2. The next step is to take categorical fixed points to remove equivariance. Since the categorical $\mathbb{T}$-fixed points functor $\mathrm{Sp}_{\mathbb{T}} \to \mathrm{Sp}$ is *not* right Quillen between the flat model structures we must employ the strategy described in Section 10.3. Loosely speaking we first neglect the commutativity to ensure that the relevant functors are Quillen. Once we have then passed to the non-equivariant world, one can reclaim commutativity of the rings.

Firstly we recall that the following result.

Lemma 10.9. *There are Quillen equivalences*

- [25, 4.3, 9.1]
$$\mathrm{Mod}_{DE\mathcal{F}_+} \simeq_Q \mathrm{Mod}_{(DE\mathcal{F}_+)^{\mathbb{T}}}$$

- [25, 7.1]
$$\mathrm{Mod}_{\widetilde{E}\mathcal{F}} \simeq_Q \mathrm{Mod}_{S^0}$$

- [25, 7.1]
$$\mathrm{Mod}_{DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}} \simeq_Q \mathrm{Mod}_{(DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F})^{\mathbb{T}}}$$

We now use the notation of Section 10.3 for the necessary cofibrant replacements. There is a Quillen equivalence

$$\begin{pmatrix} \mathrm{Mod}_{\widetilde{E}\mathcal{F}} \\ \vert \\ \bullet \\ \downarrow \\ \mathrm{Mod}_{DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}} \longrightarrow \mathrm{Mod}_{DE\mathcal{F}_+} \end{pmatrix} \leftrightarrows \begin{pmatrix} \mathrm{Mod}_{S^0} \\ \vert \\ \bullet \\ \downarrow \\ \mathrm{Mod}_{(DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F})^{\mathbb{T}}} \longrightarrow \mathrm{Mod}_{Q(DE\mathcal{F}_+^{\mathbb{T}})} \end{pmatrix}$$

where the left hand side is equipped with the flat model structure, and the right hand side is equipped with the stable model structure. This Quillen equivalence follows from Lemmas 10.6 and 10.9.

Next, we claim that there is a Quillen equivalence

$$\begin{pmatrix} \mathrm{Mod}_{S^0} \\ \vert \\ \bullet \\ \downarrow \\ \mathrm{Mod}_{(DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F})^{\mathbb{T}}} \longrightarrow \mathrm{Mod}_{Q(DE\mathcal{F}_+^{\mathbb{T}})} \end{pmatrix} \rightleftarrows \begin{pmatrix} \mathrm{Mod}_{S^0} \\ \vert \\ \bullet \\ \downarrow \\ \mathrm{Mod}_{(DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F})^{\mathbb{T}}} \longrightarrow \mathrm{Mod}_{Q_f(DE\mathcal{F}_+^{\mathbb{T}})} \end{pmatrix}$$

where the left hand side has the stable model structure, but the right hand side has the flat model structure. The functors are the identity on the leftmost vertices and as such are Quillen equivalences since the identity functor from the stable to the flat model structure is a left Quillen equivalence. In the bottom right vertex, applying Lemma 10.7 shows that we have a Quillen equivalence. We emphasize that each of the rings appearing on the right hand side is commutative; this is vital for the next step.

Step 3. We now apply Shipley's theorem [38, 1.2] which says that given a commutative $H\mathbb{Q}$-algebra A, there is a commutative DGA ΘA and a Quillen equivalence $\mathrm{Mod}_A \simeq_Q \mathrm{Mod}_{\Theta A}$, and that moreover $H_*(\Theta A) = \pi_* A$. Therefore we have a Quillen equivalence

$$
\left(
\begin{array}{c}
\mathrm{Mod}_{S^0} \\
\vdots \\
\downarrow \\
\mathrm{Mod}_{(DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F})^{\mathbb{T}}} \longrightarrow \mathrm{Mod}_{Q_f(DE\mathcal{F}_+^{\mathbb{T}})}
\end{array}
\right)
\simeq_Q
\left(
\begin{array}{c}
\mathrm{Mod}_{\Theta(S^0)} \\
\vdots \\
\downarrow \\
\mathrm{Mod}_{\Theta((DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F})^{\mathbb{T}})} \longrightarrow \mathrm{Mod}_{\Theta(Q_f(DE\mathcal{F}_+^{\mathbb{T}}))}
\end{array}
\right)
$$

where each of the DGAs appearing is commutative. By Proposition 10.2, we know that the cospan $\mathcal{O}_\mathcal{F} \to t_\mathcal{F} \leftarrow \mathbb{Q}$ is intrinsically formal as a cospan of commutative DGAs. This gives a Quillen equivalence

$$
\left(
\begin{array}{c}
\mathrm{Mod}_{\Theta(S^0)} \\
\vdots \\
\downarrow \\
\mathrm{Mod}_{\Theta((DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F})^{\mathbb{T}})} \longrightarrow \mathrm{Mod}_{\Theta(Q_f(DE\mathcal{F}_+^{\mathbb{T}}))}
\end{array}
\right)
\simeq_Q
\left(
\begin{array}{c}
\mathrm{Mod}_{\mathbb{Q}} \\
\vdots \\
\downarrow \\
\mathrm{Mod}_{t_\mathcal{F}} \longrightarrow \mathrm{Mod}_{\mathcal{O}_\mathcal{F}}
\end{array}
\right).
$$

We note that Shipley's algebraicization theorem still holds in the flat model structure by [43]. To simplify notation we call the (underlying) category on the right $\widehat{\mathcal{A}}_t(\mathbb{T})$.

Step 4. Combining the results of the previous steps, we see that we have a zig-zag of Quillen equivalences
$$
\mathrm{Sp}_\mathbb{T} \simeq_Q \text{cell-}\widehat{\mathcal{A}}_t(\mathbb{T}).
$$
We now describe the effect of the cellularization on $\widehat{\mathcal{A}}_t(\mathbb{T})$ and use the cellular skeleton theorem to internalize the cellularization. Recall that an object in $\widehat{\mathcal{A}}_t(\mathbb{T})$ consists of a $\mathbb{Q}$-module V, a $t_\mathcal{F}$-module W and an $\mathcal{O}_\mathcal{F}$-module N, together with module maps $\alpha \colon t_\mathcal{F} \otimes V \to W$ and $\beta \colon W \to N$. We also required the $t_\mathcal{F}$-module map α to be an isomorphism. As this data is equivalent to an $\mathcal{O}_\mathcal{F}$-module map $t_\mathcal{F} \otimes V \to N$, we will use this shorthand notation to denote an object in the category $\widehat{\mathcal{A}}_t(\mathbb{T})$.

Lemma 10.10. *The cellularization of $\widehat{\mathcal{A}}_t(\mathbb{T})$ can be taken at the set of cells*
$$
\{(t_\mathcal{F} \to 0), (0 \to \mathbb{Q}_F)_{F \in \mathcal{F}}\}.
$$

Proof. We first determine the homotopy of the cells. For $H \in \mathcal{F}$, we put $\mathbb{I}(H) = \Sigma^{-2}\mathbb{Q}[c_H, c_H^{-1}]/\mathbb{Q}[c_H]$ which is a torsion injective $H^*(B\mathbb{T}/H)$-module, and write $A(H)$ for the H-Burnside ring which is an Euler torsion $\mathcal{O}_\mathcal{F}$-module. We use [18, 2.4.3] to see that $\pi_*^\mathbb{T}(\Sigma E\mathcal{F}_+) = \bigoplus_{F \in \mathcal{F}} \Sigma^2 \mathbb{I}(F)$. The isotropy separation sequence and the tom Dieck splitting show that $\pi_*^\mathbb{T}(\Sigma E\mathcal{F}_+ \wedge \mathbb{T}/H_+) = \pi_*^\mathbb{T}(\Sigma \mathbb{T}/H_+) = \Sigma^2 A(H)$. It follows that the cells have homotopy

$$
\pi_*^\mathbb{T}((\mathbb{T}/\mathbb{T}_+)_{\mathrm{tors}}) = (t_\mathcal{F} \to \bigoplus_{F \in \mathcal{F}} \Sigma^2 \mathbb{I}(F)) \quad \text{and} \quad \pi_*^\mathbb{T}((\mathbb{T}/H_+)_{\mathrm{tors}}) = (0 \to \Sigma^2 A(H))
$$

for $H \in \mathcal{F}$. Using Corollary 10.5, one can see that the localizing subcategory generated by the cells $\{\pi_*^\mathbb{T}((\mathbb{T}/H_+)_{\mathrm{tors}}) \mid H \leq \mathbb{T}\}$ is the same as that generated by $(t_\mathcal{F} \to 0)$ and $(0 \to \mathbb{Q}_F)_{F \in \mathcal{F}}$ so they have the same cellularization. $\square$

Since $\{(0 \to \mathbb{Q}_F)\}_{F \in \mathcal{F}}$ is a set of compact generators for Euler torsion modules by Corollary 10.5, applying the cellular skeleton theorem (Theorem 9.9) then gives a Quillen equivalence
$$
\text{cell-}\widehat{\mathcal{A}}_t(\mathbb{T}) \simeq_Q d\mathcal{A}_t(\mathbb{T})
$$
which completes the proof of Theorem 10.8.

References

[1] M. Abbasirad. Homotopy theory of differential graded modules and adjoints of restriction of scalars. University of Sheffield PhD thesis, 2014.

[2] S. Balchin, T. Barthel, and J. P. C. Greenlees. Prismatic decompositions and rational G-spectra. In preparation.

[3] S. Balchin and J. P. C. Greenlees. Adelic models of tensor-triangulated categories. *Advances in Mathematics*, 375:107339, 2020.

[4] S. Balchin, J. P. C. Greenlees, L. Pol, and J. Williamson. Torsion models for tensor-triangulated categories. In preparation.

[5] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[6] P. Balmer. Supports and filtrations in algebraic geometry and modular representation theory. *Amer. J. Math.*, 129(5):1227–1250, 2007.

[7] P. Balmer and G. Favi. Generalized tensor idempotents and the telescope conjecture. *Proc. Lond. Math. Soc. (3)*, 102(6):1161–1185, 2011.

[8] D. Barnes. Splitting monoidal stable model categories. *J. Pure Appl. Algebra*, 213(5):846–856, 2009.

[9] D. Barnes, J. P. C. Greenlees, M. Kędziorek, and B. Shipley. Rational $SO(2)$-equivariant spectra. *Algebr. Geom. Topol.*, 17(2):983–1020, 2017.

[10] T. Barthel, D. Heard, and G. Valenzuela. Local duality in algebra and topology. *Adv. Math.*, 335:563–663, 2018.

[11] D. Benson, S. B. Iyengar, and H. Krause. Local cohomology and support for triangulated categories. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(4):573–619, 2008.

[12] J. E. Bergner. Homotopy limits of model categories and more general homotopy theories. *Bull. Lond. Math. Soc.*, 44(2):311–322, 2012.

[13] M. P. Brodmann and R. Y. Sharp. *Local cohomology. An algebraic introduction with geometric applications*, volume 136 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2013.

[14] M. Brun, B. I. Dundas, and M. Stolz. Equivariant Structure on Smash Powers. arXiv:1604.05939, 2016.

[15] A. B. Buan, H. Krause, and Ø. Solberg. Support varieties: an ideal approach. *Homology Homotopy Appl.*, 9(1):45–74, 2007.

[16] D. Dugger. Replacing model categories with simplicial ones. *Trans. Amer. Math. Soc.*, 353(12):5003–5027, 2001.

[17] W. G. Dwyer and J. P. C. Greenlees. Complete modules and torsion modules. *Amer. J. Math.*, 124(1):199–220, 2002.

[18] J. P. C. Greenlees. Rational S^1-equivariant stable homotopy theory. *Mem. Amer. Math. Soc.*, 138(661):xii+289, 1999.

[19] J. P. C. Greenlees. Tate cohomology in axiomatic stable homotopy theory. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 149–176. Birkhäuser, Basel, 2001.

[20] J. P. C. Greenlees. The Balmer spectrum of rational equivariant cohomology theories. *J. Pure Appl. Algebra*, 223(7):2845–2871, 2019.

[21] J. P. C. Greenlees. Couniversal spaces which are equivariantly commutative ring spectra. *Homology Homotopy Appl.*, 22(1):69–75, 2020.

[22] J. P. C. Greenlees and J. P. May. Completions in algebra and topology. In *Handbook of algebraic topology*, pages 255–276. North-Holland, Amsterdam, 1995.

[23] J. P. C. Greenlees and B. Shipley. An algebraic model for free rational G-spectra for connected compact Lie groups G. *Math. Z.*, 269(1-2):373–400, 2011.

[24] J. P. C. Greenlees and B. Shipley. The cellularization principle for Quillen adjunctions. *Homology Homotopy Appl.*, 15(2):173–184, 2013.

[25] J. P. C. Greenlees and B. Shipley. Fixed point adjunctions for equivariant module spectra. *Algebr. Geom. Topol.*, 14(3):1779–1799, 2014.

[26] J. P. C. Greenlees and B. Shipley. Homotopy theory of modules over diagrams of rings. *Proc. Amer. Math. Soc. Ser. B*, 1:89–104, 2014.

[27] J. P. C. Greenlees and B. Shipley. An algebraic model for rational torus-equivariant spectra. *J. Topol.*, 11(3):666–719, 2018.

[28] T. Haraguchi. On model structure for coreflective subcategories of a model category. *Math. J. Okayama Univ.*, 57:79–84, 2015.

[29] K. Hess, M. Kędziorek, E. Riehl, and B. Shipley. A necessary and sufficient condition for induced model structures. *J. Topol.*, 10(2):324–369, 2017.

[30] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[31] M. J. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.

[32] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[33] M. Hovey, J. H. Palmieri, and N. P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.

[34] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[35] S. Schwede and B. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

[36] S. Schwede and B. Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol.*, 3:287–334, 2003.

[37] B. Shipley. A convenient model category for commutative ring spectra. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 473–483. Amer. Math. Soc., Providence, RI, 2004.

[38] B. Shipley. $H\mathbb{Z}$-algebra spectra are differential graded algebras. *Amer. J. Math.*, 129(2):351–379, 2007.

[39] B. Toën and G. Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.

[40] D. White. Model structures on commutative monoids in general model categories. *J. Pure Appl. Algebra*, 221(12):3124–3168, 2017.

[41] D. White. Monoidal Bousfield localizations and algebras over operads. arXiv:1404.5197, 2018.

[42] J. Williamson. Algebraic models of change of groups functors in (co)free rational equivariant spectra. arXiv:2003.12412, 2020.

[43] J. Williamson. Flatness and Shipley's algebraicization theorem. *Homology Homotopy Appl.*, 23(1):191–218, 2021.

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