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2-REPRESENTATIONS OF SMALL QUOTIENTS OF SOERGEL BIMODULES IN INFINITE TYPES

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Abstract. We determine for which Coxeter types the associated small quotient of the 2-category of Soergel bimodules is finitary and, for such a small quotient, classify the simple transitive 2-representations (sometimes under the additional assumption of gradability). We also describe the underlying categories of the simple transitive 2-representations. For the small quotients of general Coxeter types, we give a description for the cell 2-representations.

1. Introduction and description of the results

In this paper we fix the ground field $\mathbb{C}$ of complex numbers. Let (W, S) be a finitely generated Coxeter system and $\mathfrak{h}$ a reflections faithful W-module in the sense of [So2, Definition 1.5]. With this datum one associates a 2-category $\mathscr{S} = \mathscr{S}_{W,S,\mathfrak{h}}$ of Soergel bimodules, see [So2]. By [So1] (for finite Weyl groups) and [EW] (in the general case), the Grothendieck decategorification of $\mathscr{S}$ is isomorphic to the Hecke algebra $\mathbf{H} = \mathbf{H}_{W,S}$ of W.

The paper [KMMZ] studies a certain quotient $\underline{\mathscr{S}}$ of $\mathscr{S}$, called *the small quotient*, in the case when the Coxeter system (W, S) is finite. The main result of [KMMZ] is a classification of all simple transitive 2-representations of $\underline{\mathscr{S}}$ in all finite Coxeter types but $I_2(12)$, $I_2(18)$ and $I_2(30)$. Under the additional assumption of gradability, classification of simple transitive 2-representations in these three exceptional cases was completed in [MT].

The setup of finite Coxeter systems, considered in [KMMZ], is motivated by the fact that, in this setup, the 2-category $\mathscr{S}$ (and hence also the 2-category $\underline{\mathscr{S}}$) can be defined over the coinvariant algebra of W and is *finitary* in the sense of [MM1]. At the same time, for infinite Coxeter types, it might still happen that the 2-category $\underline{\mathscr{S}}$ is finitary despite the fact that the 2-category $\mathscr{S}$ no longer is.

In this note, we determine for which Coxeter systems the corresponding small quotient of the 2-category of Soergel bimodules is finitary, and, in that case, classify all simple transitive 2-representations of this small quotient. Our argument is an application of the main result of [MMMZ] which reduces our classification problem to the classification results of [KMMZ, MT]. Consequently, for some special cases, we need to work under the additional assumption of gradability. We also determine the quiver and relations for the underlying categories of our 2-representations. In the Coxeter type where the small quotient is not finitary, we do not have a classification for the simple transitive 2-representations but still give the quiver description for a distinguished subclass of it, namely the cell 2-representations. It turns out that, in all cases, the underlying category corresponds to the zig-zag algebra (cf. [HK, Du]) of a certain combinatorially described tree.

The paper is organized as follows: Section 2 studies Kazhdan-Lusztig cell combinatorics of the small Kazhdan-Lusztig cell of an arbitrary Coxeter system. Section 3 is devoted to classification of simple transitive 2-representations of finitary small quotients of Soergel bimodules. In Section 4 one finds a description of the quiver and relations for the underlying category of these 2-representations.

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2. Combinatorics of the small Kazhdan-Lusztig cell

2.1. **Kazhdan-Lusztig cells.** We consider the *Hecke algebra* $\mathbf{H} = \mathbf{H}_{W,S}$ of W in the normalization of [So2]. It has the *standard basis* $\{H_w : w \in W\}$ and the *Kazhdan-Lusztig (KL) basis* $\{\underline{H}_w : w \in W\}$, cf [KL].

We define the *left preorder* $\leq_L$ on W as follows: for $x, y \in W$, we have $x \leq_L y$ provided that there is $w \in W$ such that $\underline{H}_y$ appears with a non-zero coefficient when expanding $\underline{H}_w \underline{H}_x$ in the KL basis. Equivalence classes for this preorder are called *left cells*. The fact that $x, y \in W$ belong to the same left cell is written $x \sim_L y$. Similarly one defines the *right preorder* $\leq_R$ and the right cells $\sim_R$ using $\underline{H}_x \underline{H}_w$, and also the *two-sided preorder* $\leq_J$ and the right cells $\sim_J$ using $\underline{H}_w \underline{H}_x \underline{H}_{w'}$.

2.2. **The small cell.** The following statement is fundamental for our results. It can be easily obtained from [Lu1, Proposition 3.8], see also [Do, Lu2, Lu3, Lu4]. A detailed argument from [KMMZ, Lemma 3 and Proposition 4], which is formally written in the setup of finite Coxeter groups, works in the general case.

Proposition 1. *Let (W, S) be a finitely generated indecomposable Coxeter system.*

(*i*) *All simple reflections $s \in S$ belong to the same two-sided cell, called the* small cell *and denoted $\mathcal{J}$.*

(*ii*) *The map $\mathcal{L} \mapsto \mathcal{L} \cap S$ is a bijection between the set of all left cells in $\mathcal{J}$ and S.*

(*iii*) *The map $\mathcal{R} \mapsto \mathcal{R} \cap S$ is a bijection between the set of all right cells in $\mathcal{J}$ and S.*

(*iv*) *An element $e \neq w \in W$ belongs to $\mathcal{J}$ if and only if w has a unique reduced expression.*

(*v*) *If $\mathcal{L}$ is a left cell in $\mathcal{J}$ with $s = \mathcal{L} \cap S$ and $\mathcal{R}$ is a right cell in $\mathcal{J}$ with $t = \mathcal{R} \cap S$, then $\mathcal{L} \cap \mathcal{R}$ consists of all those element $w \in W$ with unique reduced expression for which this unique expression is of the form $t \dots s$.*

For $s \in S$, we denote by $\mathcal{L}_s$ and $\mathcal{R}_s$ the left and the right cells containing s, respectively.

2.3. **Indecomposable Coxeter systems with finite small cells.** Let (W, S) be a finitely generated indecomposable Coxeter system and $\Gamma = (\Gamma_0, \Gamma_1, m)$ is the associated Coxeter-Dynkin diagram. Here $\Gamma_0 = S$ and $m : \Gamma_1 \to \{3, 4, \dots, \infty\}$ is the labeling function of the Coxeter presentation of W, that is, for any $\{s, t\} \in \Gamma_1$, we have $(st)^{m_{s,t}}$ in the Coxeter presentation of W. As usual, for $s, t \in S$, we have $\{s, t\} \notin \Gamma_1$ if and only if $st = ts$. We also omit the label 3 and refer to an edge with label 3 as *unlabeled*.

Indecomposability of (W, S) is equivalent to the condition that Γ is connected. The following statement can be found in [Lu1, Proposition 3.8(h)] without proof, we include a proof for completeness.

Proposition 2. *Let (W, S) be a finitely generated indecomposable Coxeter system. Then the small cell $\mathcal{J}$ is finite if and only if the following conditions are satisfied:*

(*a*) *Γ is a tree.*

(*b*) *Γ has at most one labeled vertex.*

(*c*) *All labels of Γ are finite.*

Proof. If Γ contains a cycle, we can just take consecutive products of generators walking as long as we wish along this cycle and we never hit any of the relations in W. Therefore all elements of W obtained in this way would belong to $\mathcal{J}$ by Proposition 1(iv), making $\mathcal{J}$ infinite. Therefore condition (a) is necessary for finiteness of $\mathcal{J}$.

If Γ contains a subgraph $s \overset{\infty}{\text{———}} t$, then all elements of the form $stst \dots tsts$ contain no relations in W and hence belong to $\mathcal{J}$ by Proposition 1(iv). This again makes $\mathcal{J}$ infinite and means that condition (c) is necessary for finiteness of $\mathcal{J}$.

If Γ contains a connected subgraph of the form

$$s_1 \overset{m}{\text{———}} s_2 \text{———} s_3 \text{———} s_4 \text{———} \dots \text{———} s_{k-1} \overset{n}{\text{———}} s_k$$

(with distinct vertices), where $k \geq 3$ and $m, n > 3$, then all elements of the form

$$(s_1 s_2 s_3 \dots s_{k-1} s_k s_{k-1} \dots s_3 s_2)^l, \quad \text{where} \quad l \in \{1, 2, 3, \dots\},$$

contain no relations in W and hence belong to $\mathcal{J}$ by Proposition 1(iv). This again makes $\mathcal{J}$ infinite and means that condition (b) is necessary for finiteness of $\mathcal{J}$.

Let us now show that conditions (a), (b) and (c) together are sufficient for finiteness of $\mathcal{J}$. Let $w \in \mathcal{J}$ with the unique reduced expression $t_k t_{k-1} \dots t_1$ (here a repetition of simple reflections is allowed). To prove our claim we just need to establish some bound on k. As the reduced expression of w is unique, no pair of consecutive elements in this expression can commute. This means that all pairs $\{t_2, t_1\}$, $\{t_3, t_2\}$, an so on, are edges in Γ and hence we can view w as a walk in Γ starting at t_1 in the obvious way. As Γ contains no cycles by (a), for big k some edge $\{a, b\}$ has to be walked along in the way aba. To keep the reduced expression unique, the edge $\{a, b\}$ then must be labeled.

If $t_k t_{k-1} \dots t_1$ contains a subexpression of the form $abauaba$, where neither a nor b appears in u, then all edges appearing in aua are unlabeled by (b). Again, as Γ contains no cycles by (a), if $u \neq e$, one of the edges in u will have to be walked along in the way $a'b'a'$, contradicting uniqueness of the reduced expression. Hence $u = e$ and this shows that all appearances of a and b in w are next to each other.

The length of a subexpression of w of the form $abab \dots$ is bounded by the finite label of $\{a, b\}$, given by (c). By the argument above, the lengths of the subwords on both sides of this subexpression are bounded by the total number of edges. This implies that k is bounded, and the claim follows. □

2.4. **Combinatorics of finite small cells.** In this subsection we assume that (W, S) is a finitely generated indecomposable Coxeter system such that the small cell $\mathcal{J}$ is finite. Our aim here is to describe the intersections $\mathcal{L}_s \cap \mathcal{R}_t$, for $s, t \in S$.

Corollary 3. *Let (W, S) be a finitely generated indecomposable Coxeter system. If Γ is an unlabeled tree, then $|\mathcal{L}_s \cap \mathcal{R}_t| = 1$, for all $s, t \in S$.*

Proof. We view $w \in \mathcal{L}_s \cap \mathcal{R}_t$ as a walk in Γ similarly to the proof of Proposition 2. As all edges in Γ are unlabeled, the argument in the proof of Proposition 2 implies that no edge can be repeated during this walk. Therefore, by Proposition 1(v), the only possibility for w is to be the unique shortest path from s to t of the form $t \dots s$. The claim follows. □

Let us now assume that Γ is a tree with a unique labeled edge $\{s, t\}$ of finite label $n \in \{4, 5, \dots\}$. Let $\Gamma^{(s)}$ denote the connected component of $\Gamma \setminus \{t\}$ containing s (or, equivalently, the full subgraph of Γ consisting of all vertices r for which there is a walk from r to s which does not pass through t). Let, similarly, $\Gamma^{(t)}$ denote the connected component of $\Gamma \setminus \{s\}$ containing t. Define the function $\pi : S \to \{s, t\}$ by sending vertices in $\Gamma^{(s)}$ to s and vertices in $\Gamma^{(t)}$ to t.

Proposition 4. *Assume that Γ is a tree with a unique labeled edge $\{s, t\}$. Then, for $p, q \in S$, there is a bijection between $\mathcal{L}_{\pi(p)} \cap \mathcal{R}_{\pi(q)}$ and $\mathcal{L}_p \cap \mathcal{R}_q$ given by sending $w \in \mathcal{L}_{\pi(p)} \cap \mathcal{R}_{\pi(q)}$ to $u_1 w u_2$, where u_1 is the unique shortest path from $\pi(q)$ to q and u_2 is the unique shortest path from p to $\pi(p)$.*

Proof. Using Proposition 1(v), it is easy to see that the map from the set $\mathcal{L}_{\pi(p)} \cap \mathcal{R}_{\pi(q)}$ to the set $\mathcal{L}_p \cap \mathcal{R}_q$ described in the formulation is well-defined. It is obviously injective. Furthermore, it is surjective due to the argument in the proof of Proposition 2. The claim follows. □

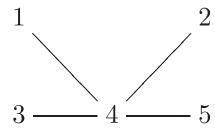
In the setup of Proposition (4), let $(W^{\{s,t\}}, \{s, t\})$ be the parabolic Coxeter subsystem of (W, S) corresponding to $\{s, t\} \subset S$. We will add the superscript $\{s, t\}$ to objects associated with this Coxeter group, for example, $\mathcal{L}_s^{\{s,t\}}$ means the left cell of $W^{\{s,t\}}$ containing s.

Corollary 5. *Assume that Γ is a tree with a unique labeled edge $\{s, t\}$. Then, for $p, q \in \{s, t\}$, we have*

$$\mathcal{L}_p \cap \mathcal{R}_q = \mathcal{L}_p^{\{s,t\}} \cap \mathcal{R}_q^{\{s,t\}}.$$

Proof. That every element in $\mathcal{L}_p^{\{s,t\}} \cap \mathcal{R}_q^{\{s,t\}}$ belongs to $\mathcal{L}_p \cap \mathcal{R}_q$ is clear from the definitions. That every element in $\mathcal{L}_p \cap \mathcal{R}_q$ belongs to $\mathcal{L}_p^{\{s,t\}} \cap \mathcal{R}_q^{\{s,t\}}$ follows from the argument in the proof of Proposition 2. □

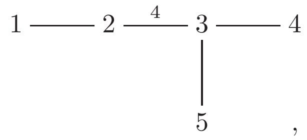
2.5. **Examples.** If Γ is given by



then the cell $\mathcal{J}$ has the following structure (here columns are left cells and rows are right cells indexed by the corresponding simple reflections):

	1	2	3	4	5
1	1	142	143	14	145
2	241	2	243	24	245
3	341	342	3	34	345
4	41	42	43	4	45
5	541	542	543	54	5

If Γ is given by



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then the cell $\mathcal{J}$ has the following structure (here columns are left cells and rows are right cells indexed by the corresponding simple reflections):

	1	2	3	4	5
1	$1, 12321$	$12, 1232$	123	1234	1235
2	$21, 212321$	$2, 232$	23	234	235
3	321	32	$3, 323$	$34, 3234$	$35, 3235$
4	4321	432	$43, 4323$	$4, 43234$	$435, 43235$
5	5321	532	$53, 5323$	$534, 53234$	$5, 53235$

3. Small quotients of Soergel bimodules and their 2-representations

3.1. **Small quotients of Soergel bimodules.** Below, by *graded* we always mean $\mathbb{Z}$-*graded*. We also work over $\mathbb{C}$.

Let $\mathscr{S}^{\mathrm{gr}} := \mathscr{S}^{\mathrm{gr}}(W, S, \mathfrak{h})$ denote the monoidal category of (graded) Soergel bimodules over the polynomial algebra $\mathbb{C}[\mathfrak{h}^*]$ associated to (W, S) and $\mathfrak{h}$, see [So2, EW]. By [So2, EW], the split Grothendieck ring of $\mathscr{S}^{\mathrm{gr}}$ is isomorphic to $\mathbf{H}$ and, for $w \in W$, we denote by B_w the unique (up to isomorphism) indecomposable Soergel bimodule which corresponds to the element $\underline{H}_w$ under this isomorphism. Our normalization of grading shifts is chosen such that the full subcategory $\mathcal{P}$ of $\mathscr{S}^{\mathrm{gr}}$ with objects $\{B_w : w \in W\}$ is positively graded. The graded category $\mathscr{S}^{\mathrm{gr}}$ has finite dimensional graded components, that is, the morphism space between any two indecomposable 1-morphisms is finite dimensional.

Lemma 6. *The monoidal category $\mathscr{S}^{\mathrm{gr}}$ has a unique tensor ideal $\mathscr{I}$ which is maximal in the set of all graded tensor ideals of $\mathscr{S}^{\mathrm{gr}}$ having the property that they*

(1) *do not contain any* id_{B_w}, *where* $w \in \mathcal{J}$.

Proof. Due to positivity of the grading on $\mathcal{P}$, the sum of any family of ideals having property (1) also has property (1). Therefore $\mathscr{I}$ is just the sum of all tensor ideals having property (1). □

The quotient monoidal category $\underline{\mathscr{S}}^{\mathrm{gr}} := \mathscr{S}^{\mathrm{gr}}/\mathscr{I}$ is called the *small quotient* of $\mathscr{S}^{\mathrm{gr}}$, cf. [KMMZ]. We denote by $\mathscr{S}$ and $\underline{\mathscr{S}}$ the ungraded versions of $\mathscr{S}^{\mathrm{gr}}$ and $\underline{\mathscr{S}}^{\mathrm{gr}}$, respectively.

By construction, the images of B_w, where $w \in \mathcal{J} \cup \{e\}$, form a complete and irredundant list of representatives of the isomorphism classes of indecomposable objects in $\underline{\mathscr{S}}$.

3.2. 2-**categories and** 2-**representations.** Although it is natural to define $\mathscr{S}^{\mathrm{gr}}$ as a tensor category, we would like to adapt to the setup of 2-representations of 2-categories considered in [MM1, MM3]. For this we use the coherence theorem for monoidal categories and consider a strictification of $\mathscr{S}^{\mathrm{gr}}$ which we will denote by the same symbol, abusing notation. We also identify a strict monoidal category with the corresponding 2-category with one object.

Recall, from [MM1], that a finitary 2-category (over $\mathbb{C}$) is a 2-category $\mathscr{C}$ such that

- $\mathscr{C}$ has finitely many objects;
- each $\mathscr{C}(\mathtt{i},\mathtt{j})$ is equivalent to the category of projective modules over some finite dimensional $\mathbb{C}$-algebra;
- all compositions are biadditive and $\mathbb{C}$-bilinear whenever this makes sense and all identity 1-morphisms are indecomposable.

The 2-category $\mathscr{S}$ is never finitary as homomorphism spaces between Soergel bimodules are infinite dimensional vector spaces. However, the small quotient $\underline{\mathscr{S}}$ is finitary in some cases, as we will see in Subsection 3.4 below.

A 2-representation of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to an appropriate 2-category, see [MM3]. This can also be viewed as a functorial action of $\mathscr{C}$ on suitable categories. All 2-representation of $\mathscr{C}$ form a 2-category in a natural way, see [MM3] for details. For every object $\mathtt{i}\in\mathscr{C}$, we have the corresponding *principal* 2-representation $\mathscr{C}(\mathtt{i},{}_-)$.

3.3. **A version of graded "abelianization".** There is a natural diagrammatic abelianization 2-functor for finitary 2-categories, see [MM2, Subsection 4.2]. Here we will need a slight modification of this functor due to the fact that $\mathscr{S}^{\mathrm{gr}}$ is not finitary. We only need to abelianize the principal 2-representation of $\mathscr{S}^{\mathrm{gr}}$, so we will try to present the object we need with a minimum amount of technicalities. For $i\in\mathbb{Z}$, we denote by $\langle i\rangle$ the corresponding functor of grading shift (which maps elements of degree n to elements of degree $n-i$, for all $n\in\mathbb{N}$). Here we view $\mathscr{S}^{\mathrm{gr}}$ as a monoidal category rather than a 2-category and define its abelianization as a 1-category.

We denote by $\overline{\mathscr{S}^{\mathrm{gr}}}$ the category whose endomorphisms are diagrams of the form

$$\Big(\prod_{i\leq i_0}\mathrm{F}_i\langle i\rangle\Big)\to\mathrm{G},\ \text{ where } i_0\in\mathbb{Z},$$

in $\mathscr{S}^{\mathrm{gr}}$, and whose morphisms are given by the commutative solid squares in $\mathscr{S}^{\mathrm{gr}}$

$$\begin{array}{ccc}
\Big(\prod\limits_{i\leq i_0}\mathrm{F}_i\langle i\rangle\Big) & \longrightarrow & \mathrm{G}\\
\downarrow & \overset{\beta}{\swarrow} & \downarrow{\scriptstyle\alpha}\\
\Big(\prod\limits_{j\leq i_0'}\mathrm{F}_j'\langle j\rangle\Big) & \longrightarrow & \mathrm{G}'
\end{array}$$

modulo those in which α factors through some β. It is important to note that, due to our positive grading and the bound on the grading shift, for each i, the 1-morphism $\mathrm{F}_i\langle i\rangle$ (as well as the 1-morphism G) has a non-zero homomorphism only to finitely many 1-morphisms $\mathrm{F}_j'\langle j\rangle$.

By construction, the category $\overline{\mathscr{S}^{\mathrm{gr}}}$ is equivalent to the category of finitely generated graded $\mathcal{P}^{\mathrm{op}}$-modules. In particular, $\overline{\mathscr{S}^{\mathrm{gr}}}$ is abelian whenever $\mathcal{P}^{\mathrm{op}}$ is noetherian. We do not know when $\mathcal{P}^{\mathrm{op}}$ is noetherian, but we do not need $\overline{\mathscr{S}^{\mathrm{gr}}}$ to be abelian in what follows.

The left regular action of $\mathscr{S}^{\mathrm{gr}}$ on itself extends, in the obvious way, to an action of $\mathscr{S}^{\mathrm{gr}}$ on $\overline{\mathscr{S}^{\mathrm{gr}}}$.

3.4. **Finitary small quotients of Soergel bimodules.**

Proposition 7. *Assume that (W, S) is indecomposable. Then the 2-category $\underline{\mathscr{S}}$ is finitary if and only if $\mathcal{J}$ is finite.*

Proof. If $\underline{\mathscr{S}}$ is finitary, it must contain only finitely many isomorphism classes of indecomposable objects. By construction, the latter are indexed by $\mathcal{J} \cup \{e\}$. Therefore the condition $|\mathcal{J}| < \infty$ is necessary.

To prove that the condition $|\mathcal{J}| < \infty$ is sufficient, we assume that this condition is satisfied and we need to show that all homomorphism spaces in $\underline{\mathscr{S}}$ are finite dimensional.

Let $\overline{\mathscr{S}^{\mathrm{gr}}}$ be the diagrammatic abelianization of $\mathscr{S}^{\mathrm{gr}}$ as in Subsection 3.3. For $w \in W$, we denote by L_w the simple top of the projective object B_w in $\overline{\mathscr{S}^{\mathrm{gr}}}$. Fix some $s \in S$ and consider the full subcategory $\mathcal{A}$ of $\overline{\mathscr{S}^{\mathrm{gr}}}$ given by the additive closure of the objects of the form $B_w L_s$, where $w \in \mathcal{L}_s$ (up to graded shift). Similarly to [MM1, Lemma 12], one shows that, for $w \in W$, the inequality $B_w L_s \neq 0$ implies $w \in \mathcal{L}_s \cup \{e\}$. Consequently, the action of $\mathscr{S}^{\mathrm{gr}}$ on $\overline{\mathscr{S}^{\mathrm{gr}}}$ restricts to $\mathcal{A}$.

Our next observation is that each $B_x L_s$, where $x \in \mathcal{L}_s$, is finite dimensional in the sense that the direct sum

$$\bigoplus_{y \in W, i \in \mathbb{Z}} \mathrm{Hom}_{\overline{\mathscr{S}^{\mathrm{gr}}}}(B_y\langle i\rangle, B_x L_s)$$

is finite dimensional. Indeed, for $y \in W$ and $i \in \mathbb{Z}$, using adjunction we have

$$\mathrm{Hom}_{\overline{\mathscr{S}^{\mathrm{gr}}}}(B_y\langle i\rangle, B_x L_s) \cong \mathrm{Hom}_{\overline{\mathscr{S}^{\mathrm{gr}}}}(B_{x^{-1}} B_y\langle i\rangle, L_s).$$

if the right hand side is non-zero, then $y \in \mathcal{L}_s \cup \{e\}$ and $|i|$ must be bounded by the length of x. This leaves us with finitely many choices for both x and i. Furthermore, all graded components of homomorphism spaces in $\overline{\mathscr{S}^{\mathrm{gr}}}$ are finite dimensional.

As each $B_x L_s$, where $x \in \mathcal{L}_s$, is finite dimensional, it has finitely many indecomposable summands. Therefore, up to grading shift, $\mathcal{A}$ has finitely many indecomposable objects. In other words, the action of $\mathscr{S}^{\mathrm{gr}}$ on $\mathcal{A}$ is an action on some category which is equivalent to the category of graded projective modules over a finite dimensional graded algebra. Let $\mathscr{J}$ be the kernel (in $\mathscr{S}^{\mathrm{gr}}$) of this action. Note also that this action is given by exact functors as all 1-morphisms in $\mathscr{S}^{\mathrm{gr}}$ have biadjoints. This implies that all morphism spaces between 1-morphisms in the ungraded version of $\mathscr{S}^{\mathrm{gr}}/\mathscr{J}$ are finite dimensional.

Note that the ideal $\mathscr{J}$ is graded by construction and that the identity 2-morphism on B_s does not belong to $\mathscr{J}$ as $B_s(B_s L_s) \neq 0$. Hence $\mathscr{J} \subset \mathscr{I}$ by the maximality of $\mathscr{I}$. Consequently, all morphism spaces between 1-morphisms in $\underline{\mathscr{S}}$ are finite dimensional. This completes the proof. □

3.5. **Simple transitive 2-representations of finitary small quotients of Soergel bimodules.** Following [MM5], we are interested in classification of *simple transitive 2-representations* of $\underline{\mathscr{S}}$ in case the latter 2-category is finitary. Recall, from [MM5], that a simple transitive 2-representation of $\underline{\mathscr{S}}$ is a functorial action of $\underline{\mathscr{S}}$ on a small category $\mathcal{C}$ equivalent to B-proj, for some finite dimensional algebra B, such that $\mathcal{C}$ has no proper $\underline{\mathscr{S}}$-invariant ideals.

Examples of simple transitive 2-representations of $\underline{\mathscr{S}}$ include the so-called *cell 2-representation* $\mathbf{C}_{\mathcal{L}}$ associated to a left cell $\mathcal{L}$, cf. [MM1, MM2].

Proposition 8. *Assume that Γ is an unlabeled tree. Then every simple transitive 2-representations of $\underline{\mathscr{S}}$ is equivalent to a cell 2-representation.*

Proof. Thanks to Corollary 3, [MM6, Proposition 1] and [KM, Corollary 19], in case Γ is an unlabeled tree, the 2-category $\underline{\mathscr{S}}$ satisfies the assumptions of [MM5, Theorem 18] and hence the assertion of the proposition follows from [MM5, Theorem 18]. $\square$

The case when Γ is a tree with one labeled edge requires some preparation. Assume that the labeled edge of gamma is

$$s \overset{n}{\text{——}} t\,, \tag{2}$$

where $3 < n < \infty$. Let $\widetilde{S} = \{s, t\}$ and $\widetilde{W}$ be the parabolic Coxeter subgroup of W generated by $\widetilde{S}$. Let $\widetilde{\mathfrak{h}}$ be the 2-dimensional subspace of $\mathfrak{h}$ generated by the unique (up to scalar) eigenvector of s with eigenvalue -1 and the unique (up to scalar) eigenvector of t with eigenvalue -1. Then we have the corresponding 2-categories $\widetilde{\mathscr{S}}^{\mathrm{gr}}$ and $\underline{\widetilde{\mathscr{S}}}$.

Theorem 9. *Assume that Γ is a tree with one labeled edge of the form* (2)*. Then there is a bijection between the sets of equivalence classes of simple transitive 2-representations of the 2-categories $\underline{\mathscr{S}}$ and $\underline{\widetilde{\mathscr{S}}}$.*

Note that, for $n \neq 12, 18, 30$, simple transitive 2-representations of $\underline{\widetilde{\mathscr{S}}}$ are classified in [KMMZ, Theorem 1]. For $n = 12, 18, 30$, simple transitive 2-representations of $\underline{\widetilde{\mathscr{S}}}$ are classified in [MT, Theorem II] (with some classes of 2-representations constructed in [KMMZ, Theorem 1]).

Proof. Our proof of this theorem is based crucially on an application of [MMMZ, Theorem 15]. Note that $\underline{\mathscr{S}}$ is finitary by Propositions 2 and 7. Define $\mathscr{C}$ as the quotient of the 2-subcategory of $\underline{\mathscr{S}}$ generated by B_w, where $w \in \mathcal{L}_s \cap \mathcal{R}_s$, modulo the unique maximal two-sided 2-ideal which does not contain any non-zero identity 2-morphisms. By [MMMZ, Theorem 15], there is a bijection between the sets of equivalence classes of simple transitive 2-representations of the 2-categories $\underline{\mathscr{S}}$ and $\mathscr{C}$.

Define $\widetilde{\mathscr{C}}$ as the quotient of the 2-subcategory of $\underline{\widetilde{\mathscr{S}}}$ generated by B_w, where $w \in \mathcal{L}_s \cap \mathcal{R}_s$, modulo the unique maximal two-sided 2-ideal which does not contain any non-zero identity 2-morphisms. By [MMMZ, Theorem 15], there is a bijection between the sets of equivalence classes of simple transitive 2-representations of the 2-categories $\underline{\widetilde{\mathscr{S}}}$ and $\widetilde{\mathscr{C}}$.

At the same time, from Corollary 5 and the definition of Soergel bimodules it follows easily that the 2-categories $\mathscr{C}$ and $\widetilde{\mathscr{C}}$ are biequivalent. Therefore there is a bijection between the sets of equivalence classes of simple transitive 2-representations of the 2-categories $\mathscr{C}$ and $\widetilde{\mathscr{C}}$. The claim follows. $\square$

The best way to explicitly define a bijection given by Theorem 9 is to use the approach to simple transitive 2-representations via (co)algebra objects as developed in [MMMT, MMMZ].

4. Quiver and relations for the underlying category

4.1. **Zig-zag categories associated to graphs.** Let Ω be an unoriented graph without loops. Let $\widetilde{\Omega}$ be the oriented graph obtained from Ω by replacing each unoriented edge of Ω by a pair of oppositely oriented edges between the same vertices. Here is an example:

$$\Omega = \; 1 \text{——} 2 \;=\; 3\,, \qquad \widetilde{\Omega} = \; 1 \rightleftarrows 2 \;\overset{\rightrightarrows}{\underset{\leftleftarrows}{}}\; 3\,.$$

We denote by $\mathcal{A}^{\Omega}$ the quotient of the path category of $\widetilde{\Omega}$ by the following relations:

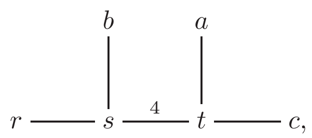
- any path of length three is zero;
- any path of length two between different vertices is zero;
- all paths of length two that start and end at the same vertex are equal.

The category $\mathcal{A}^{\Omega}$ corresponds to the classical *zig-zag* algebra associated to $\widetilde{\Omega}$, cf. [HK, Du, ET]. The category $\mathcal{A}^{\Omega}$ is graded by path lengths. Note that the path algebra of $\mathcal{A}^{\Omega}$ is not unital if Ω has infinitely many vertices.

4.2. **Cell 2-representations.** In this subsection we assume that (W,S) is of any type. For a fixed $s \in S$, consider an unoriented graph $\Lambda^{(s)}$ whose set of vertices is $\Lambda_0^{(s)} := \mathcal{L}_s$ and whose set of unoriented arrows is

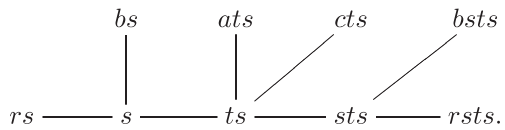
$$\Lambda_1^{(s)} := \{\{u,v\} \in \mathcal{L}_s \times \mathcal{L}_s \,:\, u = tv > v, \text{ for some } t \in S\}.$$

Note that the graph $\Lambda^{(s)}$ is an unlabeled tree which might be infinite. We denote by $\mathcal{A}^{(s)}$ the category $\mathcal{A}^{\Lambda^{(s)}}$. For example, if Γ is the graph



then the associated graph $\Lambda^{(s)}$ is as follows:

(3)



Proposition 10. *The category $\mathcal{A}^{(s)}$ is isomorphic to the underlying category of the cell 2-representation $\mathbf{C}_{\mathcal{L}_s}$ of $\mathscr{S}$ (and hence also of $\underline{\mathscr{S}}$).*

Proof. Let $\mathcal{B}^{(s)}$ denote the underlying category of $\mathbf{C}_{\mathcal{L}_s}$. For $w \in \mathcal{L}_s$, we denote by P_w the indecomposable $\mathcal{B}^{(s)}$-projective corresponding to w. We also denote by L_w the simple top of P_w.

Recall (see e.g. [Ir, Corollary 5.2.4]) that, for $t \in S$ and $w \in W$, we have

$$\underline{H}_t \underline{H}_w = \begin{cases} v\underline{H}_w + v^{-1}\underline{H}_w, & tw < w; \\ \underline{H}_{tw} + \sum\limits_{x<w, tx<x} \mu(x,w)\underline{H}_x, & tw > w; \end{cases}$$

where $\mu(x,w)$ is the *Kazhdan-Lusztig μ-function*, see [KL, Subsection 2.2]. By construction of $\mathbf{C}_{\mathcal{L}_s}$, this implies that, for $t \in S$ and $x, w \in \mathcal{L}_s$, the multiplicity $m_{x,w}^{(t)}$ of P_x as a direct summand of $B_t \cdot P_w$ is given by

$$m_{x,w}^{(t)} = \begin{cases} 2, & x = w \text{ and } tw < w; \\ 1, & x = tw > w; \\ \mu(x,w), & tw > w, x < w, tx < x; \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

In the graded picture, we additionally have that, for $x = w$ and $tw < w$, the two summands P_w appearing in $B_t \cdot P_w$ are, in fact, $P_w\langle -1\rangle$ and $P_w\langle 1\rangle$ and all other appearing summands have no grading shift. By adjunction, see e.g. [AM, Lemma 8], $m_{x,w}^{(t)}$ coincides with the the composition multiplicity of L_w in $B_t \cdot L_x$.

Now, if $x \in \mathcal{L}_s$, then there is a unique $t \in S$ such that $tx < x$. The computation above implies that $B_t \cdot L_x$ has two simple subquotients isomorphic to $L_x\langle -1\rangle$ and $L_x\langle 1\rangle$ while all other summands L_w are in degree zero and are killed by B_t. We have for such w

$$\mathrm{Hom}(B_t \cdot L_x, L_w) \cong \mathrm{Hom}(L_x, B_t \cdot L_w) = 0,$$

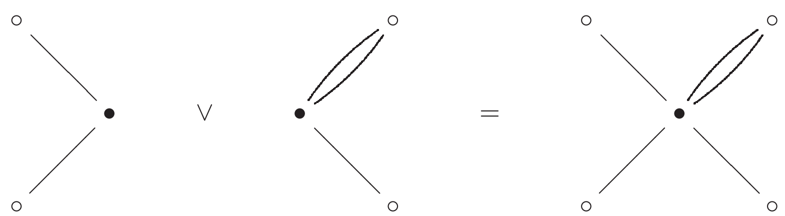
thus $L_x\langle 1\rangle$ is the simple top of $B_t \cdot L_x$. Similarly, $L_x\langle -1\rangle$ is the simple socle of $B_t \cdot L_x$. Note that the module $B_t \cdot L_x$ is both projective and injective by [KMMZ, Theorem 2]. As we have just established that $B_t \cdot L_x$ has simple top, we obtain $B_t \cdot L_x \cong P_x\langle 1\rangle$.

As $\mathcal{P}$ is positively graded, the degree zero part of $B_t \cdot L_x$ is semi-simple. Hence the number of arrows from x to w in $\mathcal{B}^{(s)}$ coincides with the number of arrows from w to x in $\mathcal{B}^{(s)}$ and this also coincides with the multiplicity of L_w in $B_t \cdot L_x$. If $w = rx$, for some $r \in S$, then the multiplicity of L_w in $B_t \cdot L_x$ is equal to 1 by formula (4).

If the multiplicity of L_w in $B_t \cdot L_x$ is non-zero and $w \neq rx$, for any $r \in S$, then $x < w$ by formula (4). In that case, the multiplicity of L_x in $B_{t'} \cdot L_w$, where $t' \in S$ is the unique element such that $B_{t'} \cdot L_w \neq 0$, is also non-zero. Then $w < x$ by formula (4) implying $w = x$, a contradiction.

To sum up, we established that the only L_w which appear in $B_t \cdot L_x$ are those for which $w = rx$, for some $r \in S$, and these simples appear with multiplicity 1. This implies that the Cartan matrices of $\mathcal{B}^{(s)}$ and $\mathcal{A}^{(s)}$ coincide and now it is easy to construct inductively an explicit isomorphism between $\mathcal{B}^{(s)}$ and $\mathcal{A}^{(s)}$ by rescaling, if necessary, all arrows, starting the induction from the initial vertex s. The claim follows. □

4.3. **Rooted graphs and their pointed union.** Here we introduce the notion of the one point union of rooted graphs, which we use in Subsection 4.4. By a *rooted graph*, we mean a pair (Ξ, a) of a graph Ξ and a *root* $a \in \Xi_0$. Let (Ξ, a) and (Ξ', a') be two rooted graphs. The *one point union* $(\Xi, a) \vee (\Xi', a')$ of (Ξ, a) and (Ξ', a') is the graph obtained from the disjoint union $(\Xi, a) \coprod (\Xi', a')$ of (Ξ, a) and (Ξ', a') by identifying the roots a and a'. The graph $(\Xi, a) \vee (\Xi', a')$ is naturally rooted with the root being the identified vertex $a = a'$. Here is an example, where $\bullet$ denotes a root and $\circ$ an ordinary vertex:



4.4. **Simple transitive 2-representations for finitary small quotients.** We assume that (W, S) is indecomposable and that $\underline{\mathscr{S}}$ is finitary. The purpose of the subsection is to describe the underlying category of each gradable simple transitive 2-representation that is not covered in Subsection 4.2. They still corresponds to the zig-zag algebras of certain graphs. We explicitly determine this graph in terms of the Coxeter-Dynkin diagram Γ of (W, S).

If Γ is an unlabeled tree, then any simple transitive 2-representation of $\underline{\mathscr{S}}$ is a cell 2-representation by Proposition 8 and its underlying category is described by Proposition 10. Because of this, in what follows we assume that Γ has a full subgraph of the form (2). Then we have the 2-category $\widetilde{\underline{\mathscr{S}}}$ as in Subsection 3.5.

Let $\mathbf{M}$ be a gradable simple transitive 2-representation of $\underline{\mathscr{S}}$ and $\mathbf{N}$ the corresponding simple transitive 2-representation of $\widetilde{\underline{\mathscr{S}}}$ given by Theorem 9. From the proofs of

Theorem 9 and [MMMZ, Theorem 15] it follows that the underlying category of $\mathbf{N}$ is isomorphic to a full subcategory of the underlying category of $\mathbf{M}$. As $\mathbf{N}$ is gradable, the underlying category of $\mathbf{N}$ is explicitly described in [KMMZ, MT] and is known to be of the form $\mathcal{A}^{\Omega}$, where Ω is a simply laced Dynkin diagram. There is also an additional datum on Ω, given by considering Ω as a bipartite graph $\Omega_0 = \Omega_0^{(s)} \coprod \Omega_0^{(t)}$ where, for $r \in \{s,t\}$, we have $u \in \Omega_0^{(r)}$ if and only if $B_r \cdot L_u \neq 0$.

By deleting the labeled edge $\{s,t\}$, the connected graph Γ splits into a disjoint union of two connected graphs: the graph $\Gamma^{(s)}$ which is the connected component containing s, and the graph $\Gamma^{(t)}$ which is the connected component containing t. For $r \in \{s,t\}$, we denote by $S^{(r)}$ the set of vertices in $\Gamma^{(r)}$, by $W^{(r)}$ the parabolic subgroup of W generated by $S^{(r)}$, and by $\underline{\mathscr{S}^{(r)}}$ the small quotient 2-category of Soergel bimodules associated with $(W^{(r)}, S^{(r)})$. Let $\mathcal{L}_r^{(r)}$ be the left cell in $W^{(r)}$ containing r.

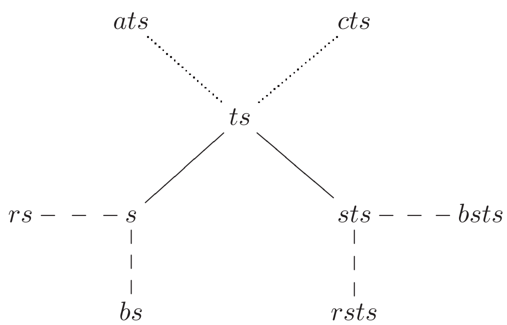
Note that $\Gamma^{(r)}$ is an unlabeled tree and hence any simple transitive 2-representation of $\underline{\mathscr{S}^{(r)}}$ is a cell 2-representation by Proposition 8 and its underlying category is described by Proposition 10. The cell-2 representation for the left cell $\mathcal{L}_r^{(r)}$ is of the form $\mathcal{A}^{\Lambda^{(r)}}$, where $\Lambda^{(r)}$ is as in Subsection 4.2 with (W,S) replaced by $(W^{(r)}, S^{(r)})$ (which is in this case isomorphic to $\Gamma^{(r)}$).

We now construct a graph Θ via a sequence of pointed unions (see Subsection 4.3). Consider $\Lambda^{(r)}$ as a rooted graph with root r. Letting $\Omega_0^{(s)} = \{u_1, u_2, \ldots, u_p\}$ and $\Omega_0^{(t)} = \{w_1, w_2, \ldots, w_q\}$,

- set $\Theta(0) := \Omega$ and consider it as a rooted graph with root u_1;
- for i from 1 to p, define $\Theta(i)$ as the one point union of $\Lambda^{(s)}$ and $\Theta(i-1)$ and then change the root of $\Theta(i)$ to u_{i+1}, if $i < p$, or to w_1, if $i = p$;
- for j from 1 to q, define $\Theta(p+j)$ as the one point union of $\Lambda^{(t)}$ and $\Theta(p+j-1)$ and then change the root of $\Theta(p+j)$ to w_{j+1}, if $j < q$, or forget the root if $j = q$;
- define the graph Θ as the unrooted graph $\Theta(p+q)$.

Informally, the graph Θ is obtained from Ω by attaching at each vertex in $\Omega_0^{(s)}$ a copy of $\Lambda^{(s)}$ at $s \in \Lambda^{(s)}$ and attaching at each vertex in $\Omega_0^{(t)}$ a copy of $\Lambda^{(t)}$ at t.

As an example, below we rearrange the graph (3) such that the solid part depicts Ω, the dashed parts show the attached copies of $\Lambda^{(s)}$ and the dotted part shows the attached copy of $\Lambda^{(t)}$.



Theorem 11. *The category $\mathcal{A}^{\Theta}$ is isomorphic to the underlying category of the 2-representation $\mathbf{M}$ of $\underline{\mathscr{S}}$.*

Proof. Let $\mathcal{B}$ be the the underlying category of the 2-representation $\mathbf{M}$ of $\underline{\mathscr{S}}$. Since $\mathcal{B}$ is graded, the same argument as the one used in the proof of Proposition 10 implies that $\mathcal{B}$ is isomorphic to $\mathcal{A}^{\Theta'}$, for some graph Θ'. So, we just need to check that this Θ' is isomorphic to Θ constructed above.

The subgraph of Θ' which is isomorphic to Ω is uniquely determined by the fact that $\mathbf{N}$ is isomorphic to a full subcategory of the underlying category of $\mathbf{M}$. See above.

Take now some vertex $u \in \Omega_0^{(s)}$, viewed as a vertex of Θ' and consider the additive closure $\mathcal{C}$ of all $B_w \cdot L_u$, where $w \in \mathcal{L}_s^{(s)}$. The action of $\underline{\mathscr{S}^{(s)}}$ preserves $\mathcal{C}$ by construction and it is easy to see that the corresponding 2-representation of $\underline{\mathscr{S}^{(s)}}$ is the cell 2-representation corresponding to $\mathcal{L}_s^{(s)}$. Note that the underlying category of this 2-representation is isomorphic to $\mathcal{A}^{\Lambda^{(s)}}$. This argument works for any $u \in \Omega_0^{(s)}$ and a similar argument (with $\mathcal{A}^{\Lambda^{(s)}}$ replaced by $\mathcal{A}^{\Lambda^{(t)}}$) works for any $u \in \Omega_0^{(t)}$. Consequently, there is a natural embedding of Θ into Θ', and we are left to show that Θ' has no extra edges.

Let $u_1, u_2 \in \Omega_0^{(s)}$ be two different vertices and $\mathcal{C}_1$ and $\mathcal{C}_2$ the corresponding 2-representations of $\underline{\mathscr{S}^{(s)}}$ constructed in the previous paragraph. Existence of an edge in Θ' connecting a vertex in $\mathcal{C}_1$ and a vertex in $\mathcal{C}_2$ implies existence of a non-trivial discrete self-extension for the cell 2-representation of $\underline{\mathscr{S}^{(s)}}$ in the sense of [CM, Subsection 5.2]. This is, however, prohibited by [CM, Theorem 25].

Let $u \in \Omega_0^{(s)}$ and $v \in \Omega_0^{(t)}$, and let p be a vertex in the copy of $\Gamma^{(s)}$ in Θ attached to u and q be a vertex in the copy of $\Gamma^{(t)}$ attached to v. Suppose there is an edge in Θ' between p and q. Letting p' be the element in $\mathcal{L}_s$ corresponding to p and q' be the elements in $\mathcal{L}_t$ corresponding to q, we have

$$\mathrm{Hom}_{\mathcal{B}}(B_{(q')^{-1}} B_{p'} L_u, L_v) \cong \mathrm{Hom}_{\mathcal{B}}(B_{p'} L_u, B_{q'} L_v) \supseteq \mathrm{Hom}_{\mathcal{B}}(P_p, P_q) \neq 0,$$

since $B_{p'} L_u$ contains P_p, the projective at p in $\mathcal{B}$ and $B_{q'} L_v$ contains P_q. In particular, $B_{(q')^{-1}} B_{p'} L_u \neq 0$ and $(q')^{-1} p' \in \mathcal{J}$. Writing $(q')^{-1} p' = txys$, where x is in the parabolic subgroup generated by $S^{(t)} \setminus t$ and y is in the parabolic subgroup generated by $S^{(s)} \setminus s$, we see that $(q')^{-1} p' \in \mathcal{J}$ implies $x = y = e$. Therefore, the edge between p and q comes from an edge in Ω. This completes the proof. □

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