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ON THE EXISTENCE OF ABELIAN SURFACES WITH EVERYWHERE GOOD REDUCTION

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Abstract. Let $D \leq 2000$ be a positive discriminant such that $F = \mathbf{Q}(\sqrt{D})$ has narrow class one, and A/F an abelian surface of GL_2-type with everywhere good reduction. Assuming that A is modular, we show that A is either an F-surface or is a base change from $\mathbf{Q}$ of an abelian surface B such that $\mathrm{End}_{\mathbf{Q}}(B) = \mathbf{Z}$, except for $D = 353, 421, 1321, 1597$ and 1997. In the latter case, we show that there are indeed abelian surfaces with everywhere good reduction over F for $D = 353, 421$ and 1597, which are non-isogenous to their Galois conjugates. These are the first known such examples.

1. Introduction

The following is a well-known result due to Faltings [Fal83, Satz 5] (see also [Fal84]):

Theorem 1.1. *Let F be a number field, S a finite set of places of F, and $g \geq 1$ an integer. Then, the set of isomorphism classes of abelian varieties of dimension g defined over F, with good reduction outside S, is finite.*

Theorem 1.1 can be seen as an analogue of the Hermite-Minkowski theorem. The case when $S = \emptyset$ seems of particular interest since it relates to unramified motives. Fontaine [Fon85] showed that there are *no* nonzero abelian varieties over $\mathbf{Q}$ with everywhere good reduction, thus proving Theorem 1.1 for $F = \mathbf{Q}$, $S = \emptyset$ and all $g \geq 1$. Fontaine's result is very striking for two reasons at least. Indeed, not only is this one of the handful cases where one can explicitly determine the set of isomorphism classes of abelian varieties predicted by Theorem 1.1. But also, it shows that, for $F = \mathbf{Q}$ and $S = \emptyset$, this set is empty in every dimension $g \geq 1$. However, this non-existence result seems to be the exception rather than the norm. Indeed, Schoof [Sch03] proved that, for $f \geq 1$ an integer not in $\{1, 3, 4, 5, 7, 8, 9, 11, 12, 15\}$, there exist non-zero abelian varieties with everywhere good reduction over the cyclotomic field $\mathbf{Q}(\zeta_f)$. Similarly, Moret-Bailly [MB01, Corollaire 5.9] asserts that the stack $\mathscr{A}_g$ parametrising all principally polarised abelian schemes of dimension $g \geq 2$ over $\mathbf{Z}$ has a point over $\overline{\mathbf{Z}}$. This means that, for every integer $g \geq 2$, there exist a number field F, and an abelian variety A of dimension g over F with every good reduction. Therefore, in order to elucidate the set of isomorphism classes predicted in Theorem 1.1 for $S = \emptyset$, we might start with the following question:

Question 1.2. Given a number field F and an integer $g \geq 1$, does there exist an abelian variety A of dimension g defined over F, with everywhere good reduction?

It is extremely difficult to give a purely arithmetic-geometric answer to Question 1.2 in general, even for $g = 1$, where there is a great deal of work over quadratic fields (see [Cre92, Elk14, Kag97, Kag01, KK97, Pin82, Set81, Str83] for example). When F is a real quadratic field, work of Freitas-Le Hung-Siksek [FLHS15] shows that all elliptic curves defined over F are modular. So, in this case, the set of isogeny

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classes of elliptic curves with trivial conductor over F, corresponds to a subset of the set of Hilbert newforms of weight 2, level (1) and trivial central character on F, with integer Hecke eigenvalues. Similarly, the Eichler-Shimura conjecture predicts that the latter set injects into the former, meaning that there is in fact a conjectural bijection between the two sets. So, for F real quadratic field and $g = 1$, one can provide an effective answer to Question 1.2 by first determining the set of Hilbert newforms of weight 2, level (1) and trivial central character on F, with integer Hecke eigenvalues.

The Modularity conjecture for GL_2-type abelian varieties and the Eichler-Shimura conjecture make similar predictions in every dimension $g \geq 1$. By making use of this, Kumar and the author found many examples of abelian surfaces with everywhere good reduction over real quadratic fields of narrow class number one and discriminant ≤ 1000 in [DK16], thus providing an answer to Question 1.2 for most of those fields for $g = 2$. In this paper, we extend those results. More specifically, we proved the following result (Theorem 5.1).

Theorem. *Let F be a real quadratic field of narrow class number one and discriminant $D \leq 2000$. Let A be a modular abelian surface of GL_2-type defined over F, with everywhere good reduction. Then, except for $D = 353, 421, 1321, 1597$ or 1997, we have one of the following:*

(i) A is an F-surface, i.e. there is an abelian fourfold B of GL_2-type defined over $\mathbf{Q}$ such that $B \times_{\mathbf{Q}} F$ is isogenous to $A \times {}^{\sigma}A$, where $\mathrm{Gal}(F/\mathbf{Q}) = \langle \sigma \rangle$; or

(ii) There is an abelian surface B defined over $\mathbf{Q}$ such that $\mathrm{End}_{\mathbf{Q}}(B) = \mathbf{Z}$ and $B \times_{\mathbf{Q}} F$ is isogenous to A.

For the exceptional discriminants $D = 353, 421$ and 1597, we showed that there are indeed abelian surfaces defined over F with everywhere good reduction, which are non-isogenous to their Galois conjugates; they are the first known such examples, and are dimension 2 analogue of the elliptic curves of trivial conductor over $\mathbf{Q}(\sqrt{509})$ found by Pinch [Pin82]. In [DK16], the abelian surfaces were obtained by searching for rational points on Hilbert modular surfaces using explicit models in [EK14]. In this paper, we employed a height search in 2-torsion fields, which in turn we used to refine the search methods in [DK16].

Due to the nature of our approach, all our abelian surfaces are of GL_2-type. It would be interesting to find a real quadratic field F, and an abelian surface A defined over F such that A has trivial conductor with $\mathrm{End}_F(A) = \mathbf{Z}$. Such an abelian surface would be conjecturally attached to a Hilbert-Siegel eigenform of genus 2, weight 2 and level (1), with integer Hecke eigenvalues. There are no methods for computing such forms yet, an added difficulty being that the weight 2 is non-cohomological. However, recently, Chenevier [Che18] proved a Hermite-Minkowski type theorem for automorphic forms over GL_n. We believe that one could adapt his approach to GSp_4 (and appropriate quadratic fields) to find the unramified Hilbert Siegel newforms needed to locate those abelian surfaces A of trivial conductor, with $\mathrm{End}_F(A) = \mathbf{Z}$.

The outline of the paper is as follows. In Section 2, we start by revisiting the Doyle-Krumm algorithm for computing algebraic numbers of bounded height; we give a vastly improved version of the algorithm which could be of independent interest on its own. In Section 3, we review some background material on 2-torsion of abelian surfaces, and in Section 4, we recall the Fontaine bounds for the root discriminants for the splitting fields of finite flat p-group schemes. In Section 5, we describe all modular abelian surfaces of GL_2-type over real quadratic fields with discriminant at most 2000 and narrow class number one. Finally, in Sections 6, 7, 8 and 9, we discussed the missing surfaces.

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2. Algebraic numbers of bounded height

In this section, we revisit the algorithm for computing algebraic numbers of bounded height described in the beautiful paper [DK15] of Doyle-Krumm. We propose a refinement which makes the algorithm significantly faster.

Let K be a number field, and $\mathcal{O}_K$ the ring of integers of K. Let $\sigma_1, \dots, \sigma_{r_1}$ be the real embeddings of K, and $\tau_1, \bar{\tau}_1, \dots, \tau_{r_2}, \bar{\tau}_{r_2}$ the complex embeddings, so that $[K : \mathbf{Q}] = r_1 + 2r_2$. For each of these embeddings σ, the absolute $|\ |_\sigma$ is given by $|x|_\sigma = |x|_{\mathbf{C}}$, where $|\ |_{\mathbf{C}}$ is the usual absolute value over $\mathbf{C}$, and that $|\ |_{\tau_i} = |\ |_{\bar{\tau}_i}$, $i = 1, \dots, r_2$. We let M_K^∞ be the set of archimedian absolute values.

For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, let $v_{\mathfrak{p}} : K \to \mathbf{Z} \cup \{\infty\}$ be the discrete valuation at $\mathfrak{p}$. We recall that, for $x \in \mathcal{O}_K$ nonzero, $v_{\mathfrak{p}}(x)$ is the largest integer $n \geq 0$ such that $\mathfrak{p}^n$ divides the ideal (x). The absolute $|\ |_{\mathfrak{p}}$ on K is defined by $|x|_{\mathfrak{p}}^{-v_{\mathfrak{p}}(x)/(e_{\mathfrak{p}} f_{\mathfrak{p}})}$, where $e_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$ are the ramification index and inertia degree respectively; it extends the p-adic absolute value on $\mathbf{Q}$ where p is the unique prime below $\mathfrak{p}$. We let M_K^0 be the set of absolute values $|\ |_{\mathfrak{p}}$, and $M_K = M_K^\infty \cup M_K^0$.

For $v \in M_K$, let K_v be the completion of K at v, and $\mathbf{Q}_v$ the completion of $\mathbf{Q}$ at the restriction of v to $\mathbf{Q}$. We let $n_v := [K_v : \mathbf{Q}_v]$ be the local degree at v. If v is a real place, then $K_v = \mathbf{Q}_v$ hence $n_v = 1$. If v is a complex place, then $K_v = \mathbf{C}$ and $\mathbf{Q}_v = \mathbf{R}$, hence $n_v = 2$. If $v = v_{\mathfrak{p}}$ is a non-archimedean place, then $\mathbf{Q}_v = \mathbf{Q}_p$, where p is the prime below $\mathfrak{p}$. In that case, $n_v = e_{\mathfrak{p}} f_{\mathfrak{p}}$.

We define the *height* function $H_K : K \to \mathbf{R}_{>0}$ by

$$H_K(x) = \prod_{v \in M_K} \max\{|x|_v^{n_v}, 1\}.$$

The function H_K satisfies the following properties:

- For all $a, b \in K$, with $b \neq 0$,
$$H_K(a/b) = \prod_{v \in M_K} \max\{|a|_v^{n_v}, |b|_v^{n_v}\};$$
- For all $a, b \in K$, with $b \neq 0$,
$$H_K(a/b) = \mathrm{N}_{K/\mathbf{Q}}(a,b)^{-1} \prod_{v \in M_K} \max\{|a|_v^{n_v}, |b|_v^{n_v}\},$$
where $\mathrm{N}_{K/\mathbf{Q}}(a,b)$ is the norm of the ideal generated by a and b.
- For all $a, b \in K$,
$$H_K(ab) \leq H_K(a) H_K(b).$$
- For any $x \in K^\times$,
$$H_K(x) = H_K(1/x).$$
- For any $x \in K$, and $\zeta \in \mu_K$ a root of unity,
$$H_K(\zeta x) = H_K(x).$$

For $x \in K$ non-zero, we define the *numerator ideal* and *denominator ideal* of x

$$\mathfrak{a} := \prod_{v_{\mathfrak{p}}(x) > 0} \mathfrak{p}^{v_{\mathfrak{p}}(x)}, \text{ and } \mathfrak{b} := \prod_{v_{\mathfrak{p}}(x) < 0} \mathfrak{p}^{-v_{\mathfrak{p}}(x)},$$

respectively. We note that $\gcd(\mathfrak{a},\mathfrak{b}) = 1$, and that $\mathfrak{a}$ and $\mathfrak{b}$ belong to the same ideal class since $(x) = \mathfrak{a}\mathfrak{b}^{-1}$.

Lemma 2.1. *Let $x \in K^\times$, and let $\mathfrak{a}$ and $\mathfrak{b}$ the numerator and denominator ideals of x respectively. If $H_K(x) \le B$, then we have $\mathrm{N}_{K/\mathbf{Q}}(\mathfrak{a}), \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{b}) \le B$.*

Proof. By definition, we have

$$\begin{aligned} H_K(x) = \prod_v \max\{1, |x|_v^{n_v}\} &= \prod_{v|\infty} \max\{1, |x|_v^{n_v}\} \prod_{v\nmid\infty} \max\{1, |x|_v^{n_v}\} \\ &= \prod_{v|\infty} \max\{1, |x|_v^{n_v}\} \prod_{\substack{\mathfrak{p}\nmid\infty \\ v_\mathfrak{p}(x)<0}} \max\{1, \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{p})^{-v_\mathfrak{p}(x)}\} \\ &\qquad \times \prod_{\substack{\mathfrak{p}\nmid\infty \\ v_\mathfrak{p}(x)>0}} \max\{1, \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{p})^{-v_\mathfrak{p}(x)}\} \\ &= \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{b}) \prod_{v|\infty} \max\{1, |x|_v^{n_v}\}. \end{aligned}$$

Therefore, $H_K(x) \le B$ implies that $\mathrm{N}_{K/\mathbf{Q}}(\mathfrak{b}) \le B$. Since $H_K(x) = H_K(1/x)$, we also get that $H_K(x) \le B$ implies that $\mathrm{N}_{K/\mathbf{Q}}(\mathfrak{a}) \le B$. □

For a given positive real number B, the algorithm below computes all the elements $x \in K$ such that $H_K(x) \le B$.

Algorithm 2.2. Given a number field K, and a positive real number B, this output all elements $x \in K$ such that $H_K(x) \le B$.

(1) Compute the list $\mathscr{L}$ of all integral ideals $\mathfrak{c}$ such that $\mathrm{N}_{K/\mathbf{Q}}(\mathfrak{c}) \le B$;
(2) Initiate the list $\mathscr{G} = \emptyset$. For each $\mathfrak{a}, \mathfrak{b} \in \mathscr{L}$ such that $\mathfrak{a}\mathfrak{b}^{-1}$ is principal and $\gcd(\mathfrak{a},\mathfrak{b}) = 1$, find a generator ξ and append to $\mathscr{G}$;
(3) Let $B_0 := \max\{H_K(\xi) : \xi \in \mathscr{G}\}$, and compute the set $\mathscr{U}$ of units $u \in \mathcal{O}_K^\times$ such that $H_K(u) \le B \cdot B_0$;
(4) Initiate $\mathscr{S} = \{0\}$. For each $\xi \in \mathscr{G}$ such that $H_K(u\xi) \le B$, append $u\xi$ to $\mathscr{S}$.
(5) Return $\mathscr{S}$.

Theorem 2.3. *Given as input a number field K, and a positive integer B, Algorithm 2.2 outputs the set $\mathscr{S}$ of all elements $x \in K$ such that $H_K(x) \le B$.*

Proof. Let $x \in K^\times$ such that $H_K(x) \le B$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be the numerator and denominator ideals of x. Then by Lemma 2.1, we have $\mathrm{N}_{K/\mathbf{Q}}(\mathfrak{a}), \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{b}) \le B$. So $\mathfrak{a}$ and $\mathfrak{b}$ belong to $\mathscr{L}$. Let ξ be the generator of $\mathfrak{a}\mathfrak{b}^{-1}$ contained in $\mathscr{G}$. Then, $(x) = (\xi) = \mathfrak{a}\mathfrak{b}^{-1}$. Therefore, there is a unit $u \in \mathcal{O}_K^\times$ such that $x = u\xi$. It remains to show that $H_K(x) \le B$ implies that $H_K(u) \le B \cdot B_0$. This follows from $u = x\xi^{-1}$, and the third and fourth properties of height functions. □

Algorithm 2.2 is a slight variation on Algorithm 1 and its refinements described in [DK15]. In [DK15], one computes the set of units $\mathscr{U}$ by enumerating rational points in a polytope. However, that process becomes extremely slow as soon as the degree of the field exceeds 4. One can substantially improve on this by using the following lemma. Let $r = r_1 + r_2 - 1$, and $\lambda : \mathcal{O}_K^\times \to \mathbf{R}^{r+1}$ be defined by

$$\lambda(u) := (\log|u|_{\sigma_1}, \ldots, \log|u|_{\sigma_{r_1}}, \log|u|_{\tau_1}^2, \ldots, \log|u|_{\tau_{r_2}}^2).$$

Let $\|\cdot\| : \mathbf{R}^{r+1} \to \mathbf{R}_{\ge 0}$ denote the Euclidean norm.

Lemma 2.4. *Let $x \in \mathcal{O}_K^\times$ be such that $H_K(u) \le B$. Then, we have*

$$\|\lambda(u)\|^2 \le 2(\log B)^2.$$

Proof. We have

$$\log H_K(u) = \sum_{\substack{v|\infty \\ |u|_v>1}} \log |u|_v^{n_v} = - \sum_{\substack{v|\infty \\ |u|_v<1}} \log |u|_v^{n_v}.$$

So, if $H_K(u) \leq B$, then we have that

$$\|\lambda(u)\|^2 = \sum_{v|\infty} (\log |u|_v^{n_v})^2 \leq 2 \left(\sum_{\substack{v|\infty \\ |u|_v>1}} \log |u|_v^{n_v} \right)^2 \leq 2(\log B)^2.$$

□

Remark 2.5. First, let $\mathfrak{c}_i$, $i = 1, \dots, h$, be a complete set of representatives for the classes in $\mathrm{Cl}(K)$. If, $\mathfrak{c}$ is an integral ideal such that $\mathrm{N}_{K/\mathbf{Q}}(\mathfrak{c}) \leq B$, then there exists $\xi_i \in K^\times$, which is $\mathfrak{c}_i$-integral, such that $\mathfrak{c} = \xi_i \mathfrak{c}_i$ and $\mathrm{N}_{K/\mathbf{Q}}(\xi_i) \leq \frac{B}{\mathrm{N}_{K/\mathbf{Q}}(\mathfrak{c}_i)}$. So, in practice, one does a class group precomputation for more efficiency. Then one combines Steps (1) and (2) by listing all the $\xi \in K^\times$ such that $\mathrm{N}_{K/\mathbf{Q}}(\xi) \leq B$, and ξ is $\mathfrak{c}_i$-integral for some $i = 1, \dots, h$.

Second, the output set $\mathscr{S}$ of Algorithm 2.2 tends to be big as the degree of the field K or the height bound B grows. In practice, we found that it was more useful to have a variant of the algorithm which enumerates elements with a fixed denominator ideal $\mathfrak{b}$. One can then vary the denominator if needed.

Finally, by Lemma 2.4, the problem of enumerating the unit set $\mathscr{U}$ in Step (3) becomes one of enumerating lattice points. This can be done very efficiently by using LLL algorithms, leading to substantial improvements in Step (3) of Algorithm 2.2, which make the overall algorithm much faster. There is a great level of care and details in the algorithms described in [DK15] and implemented in `Sage`. The variations we proposed here will add significantly to their efficiency as demonstrated by our own implementation.

3. Galois representations attached to abelian surfaces

In this section, we recall some useful results on the Galois representation on the 2-torsion points of an abelian surface. We start with the following well-known lemma whose proof we couldn't find in the literature.

Lemma 3.1. *Let $\mathbf{F}_2[\varepsilon] = \mathbf{F}_2[x]/(x^2)$. Then, there is a group isomorphism ϕ : $\mathrm{SL}_2(\mathbf{F}_2[\varepsilon]) \simeq \mathbf{Z}/2\mathbf{Z} \times S_4$, which identifies $\ker(\mathrm{SL}_2(\mathbf{F}_2[\varepsilon]) \to \mathrm{SL}_2(\mathbf{F}_2))$ with the unique normal subgroup $N \simeq (\mathbf{Z}/2\mathbf{Z})^3$ of $\mathbf{Z}/2\mathbf{Z} \times S_4$ of order 8.*

Proof. As a subgroup of S_6, $\mathbf{Z}/2\mathbf{Z} \times S_4$ is generated by the two permutations $\sigma := (1, 2, 4, 5)$ and $\tau := (1, 3)(4, 6)$. Similarly, the group $\mathrm{SL}_2(\mathbf{F}_2[\varepsilon])$ is generated by the matrices

$$A := \begin{pmatrix} \varepsilon + 1 & \varepsilon \\ 1 & 1 \end{pmatrix}, \ B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The map $(\sigma \mapsto A, \tau \mapsto B)$ gives the desired isomorphism. □

The following result is often attributed to Mumford, though it was already known in the 19th century.

Theorem 3.2. *Let $C : y^2 = R(x)$ be a curve of genus 2, and $A = Jac(C)$ its Jacobian. Let $\bar{\rho}_{A,2} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{Sp}_4(\mathbf{F}_2)$ be the mod 2 Galois representation attached to A, and L the fixed field of $\ker(\bar{\rho}_{A,2})$. Then, L is the splitting field of the polynomial $R(x)$.*

Theorem 3.3. *Let A be an abelian surface defined over a field k of characteristic zero, which has RM by the maximal order $\mathcal{O}$ of some quadratic field. Also let $\bar{\rho}_{A,2} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(\mathcal{O} \otimes \mathbf{F}_2)$ be the mod 2 Galois representation attached to A, and $G = \mathrm{im}(\bar{\rho}_{A,2})$. Then, we have the following:*

- *(i) If 2 is inert in $\mathcal{O}$ then $G \hookrightarrow A_5$.*
- *(ii) If 2 is split in $\mathcal{O}$ then $G \hookrightarrow S_3 \times S_3$.*
- *(iii) If 2 is ramified in $\mathcal{O}$, then G is a subgroup of $S_4 \times \mathbf{Z}/2\mathbf{Z}$, and there is an exact sequence $0 \to H \to G \to H' \to 0$ where $H \hookrightarrow (\mathbf{Z}/2\mathbf{Z})^3$ and $H' \hookrightarrow S_3$. In fact, we have $G \simeq H \rtimes H'$.*

Proof. The first and second statements follow from Wilson [Wil98, Corollary 4.3.4]. To prove the third, we note that, since 2 is ramified in $\mathcal{O}$, $\mathcal{O} \otimes \mathbf{Z}_2 \simeq \mathbf{F}_2[\varepsilon]$, with $\varepsilon^2 = 0$. Then, we conclude by combining Lemma 3.1 and [Wil98, Corollary 4.3.4]. □

4. Fontaine bound for finite group schemes

The following theorem plays an important rôle in the proof of Theorem 1.1 for $F = \mathbf{Q}$ and $S = \emptyset$ by Fontaine. It will be essential to us through out the paper.

Theorem 4.1 (Fontaine [Fon85])**.** *Let $p \geq 2$ be a prime, F a number field and A an abelian variety over F. Assume that A has everywhere good reduction. Let $K = F(A[p])$ be the field generated by the p-torsion points of A. Then, we have*

$$\delta_K < \delta_F p^{1+\frac{1}{p-1}},$$

where δ_F and δ_K are the root discriminants of F and K.

Remark 4.2. When F is Galois, a much stronger statement than Theorem 4.1 is true. In that case, let L be the normal closure of $K = F(A[p])$. Then, we have

$$\delta_L < \delta_F p^{1+\frac{1}{p-1}}.$$

This is proved in the same way as [Fon85, Lemme 3.4.2].

5. Abelian surfaces with everywhere good reduction

From now on, F is a real quadratic field, with ring of integers $\mathcal{O}_F$. Let $\mathfrak{N}$ be an integral ideal of F, and f be a Hilbert newform of weight 2 and level $\mathfrak{N}$, with Hecke eigenvalue field $K_f = \mathbf{Q}(a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F)$, where $a_{\mathfrak{m}}(f)$ is the Hecke eigenvalue of the Hecke operator at $\mathfrak{m}$. We recall that for every $\tau \in \mathrm{Hom}(K_f, \overline{\mathbf{Q}})$, there is a Hilbert newform f^τ determined by its Hecke eigenvalues by

$$a_{\mathfrak{m}}(f^\tau) := \tau(a_{\mathfrak{m}}(f)).$$

Similarly, there exists a Hilbert newform ${}^\sigma f$ of weight 2 and level $\sigma(\mathfrak{N})$ determined by its Hecke eigenvalues by

$$a_{\mathfrak{m}}({}^\sigma f) := a_{\sigma(\mathfrak{m})}(f).$$

We recall that the L-series of f is given by

$$L(f, s) := \sum_{\mathfrak{m} \subseteq \mathcal{O}_F} \frac{a_{\mathfrak{m}}(f)}{\mathrm{N}(\mathfrak{m})^s}.$$

We recall that an abelian surface A is said to be of GL_2-*type* if there exists a quadratic field K such that $\mathrm{End}_F(A) \otimes \mathbf{Q} \simeq K$. In that case, A is said to be *modular* if there exists a Hilbert newform of weight 2 and level $\mathfrak{N}$ such that

$$L(A, s) = \prod_{\tau : K \hookrightarrow \mathbf{C}} L(f^\tau, s).$$

For more background on Hilbert modular forms, see [DV13, Hid88, Shi78].

In this section, we prove the following theorem:

TABLE 1. The discriminants D for which there are Hilbert newforms of weight $(2,2)$ and level (1) over $\mathbf{Q}(\sqrt{D})$, with quadratic coefficient field $\mathbf{Q}(\sqrt{D'})$.

D	D'	D	D'	D	D'	D	D'
53	8	373	93	677	13, 29, 85	1013	21, 53
61	12	389	8	709	5	1109	5, 5, 53, 77
73	5	397	24	797	8, 29	1277	5, 29
193	17	409	13	809	5	**1321**	$\mathbf{5^{(2)}}$
233	17	**421**	$5, \mathbf{5^{(2)}}$	821	44	1493	5, 65
277	29	433	12	853	21	**1597**	$\mathbf{5^{(2)}}$
349	21	461	29	929	13	**1997**	$\mathbf{8^{(2)}}$
353	$\mathbf{5^{(2)}}$	613	21	997	13		

Theorem 5.1. *Let F be a real quadratic field of narrow class number one and discriminant $D \leq 2000$. Let A be a modular abelian surface of GL_2-type defined over F, with everywhere good reduction. Then, except for $D = 353, 421, 1321, 1597$ or 1997, we have one of the following:*

(i) A is an F-surface, i.e. there is an abelian fourfold B of GL_2-type defined over $\mathbf{Q}$ such that $B \times_{\mathbf{Q}} F$ is isogenous to $A \times {}^{\sigma}A$, where $\mathrm{Gal}(F/\mathbf{Q}) = \langle \sigma \rangle$; or

(ii) There is an abelian surface B defined over $\mathbf{Q}$ such that $\mathrm{End}_{\mathbf{Q}}(B) = \mathbf{Z}$ and $B \times_{\mathbf{Q}} F$ is isogenous to A.

Proof. In Table 1, we have listed all the discriminants $D \leq 2000$ where $F = \mathbf{Q}(\sqrt{D})$ has narrow class number one, and there is a newform f of weight 2 and level (1) over F whose coefficient field is the quadratic field K of discriminant D'. The notation $D'^{(2)}$ means that f and its $\mathrm{Gal}(F/\mathbf{Q})$-conjugate ${}^{\sigma}f$ are not in the same Hecke constituent. (This table was computed using the Hilbert Modular Forms Package in `magma` [BCP97].) By assumption, if A is a surface satisfying the conditions of Theorem 5.1, then A is defined over some $F = \mathbf{Q}(\sqrt{D})$ for some D in Table 1, and has RM by one of the associated D'.

Let D be such a discriminant, and f a newform over $F = \mathbf{Q}(\sqrt{D})$ with coefficients in $K = \mathbf{Q}(\sqrt{D'})$. Except for $D = 353, 421, 1321, 1597$ or 1997, the Hecke constituent of f is unique in its $\mathrm{Gal}(F/\mathbf{Q})$-orbit. So f satisfies one of the following conditions:

(i) f is a base change from $\mathbf{Q}$, in which case

$$a_{\sigma(\mathfrak{p})}(f) = a_{\mathfrak{p}}(f), \text{ for all primes } \mathfrak{p}.$$

(ii) ${}^{\sigma}f = f^{\tau}$, where $\mathrm{Gal}(K/\mathbf{Q}) = \langle \tau \rangle$, in which case

$$a_{\sigma(\mathfrak{p})}(f) = \tau(a_{\mathfrak{p}}(f)), \text{ for all primes } \mathfrak{p}.$$

In Case (i), the form f is a base change of a newform in $g \in S_2(D, (\frac{D}{\cdot}))$, the space of classical forms of weight 2 and level $\Gamma_1(D)$ with character $(\frac{D}{\cdot})$. Let B_g be the fourfold associated to g by the Eichler-Shimura construction [Shi94]. From [Shi94, §§7.7], we have

$$B_g \times_{\mathbf{Q}} F \sim A_f \times {}^{\sigma}A_f.$$

Hence A_f is a base change (in the automorphic sense). In Case (ii), assume that there is an abelian surface A_f attached to f. Then, by [CD17, Theorem 5.4] (see also [DK16]), the isogeny class of A_f descends to $\mathbf{Q}$. So, there exists a surface B defined over $\mathbf{Q}$, with $\mathrm{End}_{\mathbf{Q}}(B) = \mathbf{Z}$, such that $B \times_{\mathbf{Q}} F \sim A_f$. □

Remark 5.2. There are several restrictions in Theorem 5.1 that are non-essential. For example, the assumption that F has narrow class number one can be removed given that the Hilbert Modular Forms Package in `magma` [BCP97] can compute Hilbert modular forms without restriction on the class group. Also, it is possible to go well beyond our bound on the discriminant. However, our goal was to convey the general philosophy of our approach rather than doing extensive computations.

Remark 5.3. In [DK16], the authors give several methods for constructing the surfaces with satisfy the conditions of Theorem 5.1. In particular, they found most of the surfaces for $D \leq 1000$. However, they couldn't find the surfaces for the discriminants $D = 353$ and 421, which are non-base change. The remaining sections are devoted to dealing with the exceptional discriminants listed in Theorem 5.1. We found explicit equations for all of them, except for $D = 1321$ and 1997. (See Remark 8.4 for a discussion on the the missing examples.)

6. The abelian surface for the discriminant $D = 353$

6.1. The field of 2-torsion.

Let f be the newform listed in Table 2. We recall that the $\mathrm{Gal}(F/\mathbf{Q})$-conjugate ${}^{\sigma}f$ is determined by the relation

$$a_{\mathfrak{p}}({}^{\sigma}f) = a_{\sigma(\mathfrak{p})}(f),$$

for $\mathfrak{p} \subset \mathcal{O}_F$ prime. From this and the Hecke eigenvalues in Table 2, it is easy to see that f and ${}^{\sigma}f$ are not in the same Hecke constituent. Assume that there is an abelian surface A_f attached to f; so that the Galois conjugate ${}^{\sigma}A_f$ is attached to ${}^{\sigma}f$. Let $\rho_{f,2} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be the 2-adic Galois representation attached to f, and $\rho_{A_f,2} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_2)$ be the 2-adic representation in the Tate module of A_f. By construction, we have that $\rho_{A_f,2} \simeq \rho_{f,2}$. So reducing modulo 2, we have $\bar{\rho}_{A_f,2} \simeq \bar{\rho}_{f,2}$. Preliminary computations using the Chebotarev density theorem suggest that the image of $\bar{\rho}_{f,2}$ is S_3. But we cannot certify this given that the analogues of the Sturm bound would be impractical in this case. That motivates the following lemma.

Lemma 6.1. *Let $K = F(A_f[2])$ be the field of 2-torsion of the abelian surface A_f, and $L/\mathbf{Q}$ is normal closure. If L is a solvable extension, then it is the splitting field of the polynomial $h = x^6 - 2x^5 + x^4 + 19x^3 - 19x^2 + 2$; and we have*

$$\mathrm{Gal}(L/\mathbf{Q}) = \mathrm{Gal}(K/F)^2 \rtimes \mathbf{Z}/2\mathbf{Z} \simeq S_3^2 \rtimes \mathbf{Z}/2\mathbf{Z}.$$

Proof. We keep the above notations. Form Table 2, we see that the ring of integers of the coefficient $K_f = \mathbf{Q}(\sqrt{5})$ is $\mathcal{O}_f = \mathbf{Z}[e]$, where $e = \frac{1+\sqrt{5}}{2}$. So we have $\rho_{f,2} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(\mathbf{Z}_2[e])$, and its reduction $\bar{\rho}_{f,2} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(\mathbf{F}_4)$. By the modularity assumption, we have $\bar{\rho}_{A_f,2} = \bar{\rho}_{f,2}$.

We now compute a bound on the root discriminant δ_K of K. In Table 2, we have listed the Hecke eigenvalues $a_{\mathfrak{p}}(f) \bmod 2$ for all primes of norm up to 19. For each of those primes, we have computed $o_2(\mathfrak{p})$ the order of the image of $\mathrm{Frob}_{\mathfrak{p}}$ modulo unipotents under the projectivisation $\mathrm{P}\bar{\rho}_{f,2}$ of $\bar{\rho}_{f,2}$.

Let $\mathfrak{p} = (9 + w)$ and $\sigma(\mathfrak{p}) = (10 - w)$ be the two primes above 2. From Table 2, we see that $a_{\mathfrak{p}}(f) = -2e + 1 = 1 \in \mathbf{F}_4$ and $a_{\sigma(\mathfrak{p})}(f) = \alpha \in \mathbf{F}_4$, where $\alpha^2 + \alpha + 1 = 0$. So f is ordinary at $\mathfrak{p}$ and $\sigma(\mathfrak{p})$. Hence, the mod 2 representation restricted to the decomposition group $D_{\mathfrak{p}}$ at $\mathfrak{p}$ is of the form

$$\bar{\rho}_{f,2}|_{D_{\mathfrak{p}}} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 2,$$

TABLE 2. The Frobenius data for the two non Galois conjugate Hilbert newforms of weight 2 and level (1) over $F = \mathbf{Q}(\sqrt{353})$. (Here $e = \frac{1+\sqrt{5}}{2}$ and α is a cyclic generator of $\mathbf{F}_4^\times$.)

$\mathrm{N}\mathfrak{p}$	$\mathfrak{p}$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f) \bmod 2$	$o_2(\mathfrak{p})$	$a_\mathfrak{p}(f) \bmod \sqrt{5}$	$o_{\sqrt{5}}(\mathfrak{p})$
2	$-w-9$	$2e-1$	1	–	0	2
2	$w-10$	$-e+1$	α	–	3	4
9	3	$-2e-2$	0	1	2	3
11	$-10w+99$	$2e+3$	1	3	4	3
11	$10w+89$	$-2e+2$	0	1	1	3
17	$-66w+653$	$-4e+2$	0	1	0	2
17	$-66w-587$	3	1	3	3	4
19	$-28w+277$	2	0	1	2	3
19	$-28w-249$	$2e-3$	1	3	3	3

Similarly, the mod 2 representation restricted to the decomposition group $D_{\sigma(\mathfrak{p})}$ at $\sigma(\mathfrak{p})$ is of the form

$$\bar{\rho}_{f,2}|_{D_{\sigma(\mathfrak{p})}} \simeq \begin{pmatrix} \alpha & * \\ 0 & 1 \end{pmatrix} \mod 2.$$

So, we can use the same argument as in [Dem09] to show that $\delta_K \le 4 \cdot 353^{1/2} = 75.1531....$ (Note that this is the same as the Fontaine bound in Theorem 4.1.)

The Galois extension K/F is unramified outside $\mathfrak{p}$ and $\sigma(\mathfrak{p})$. Assuming that this extension is solvable, the Frobenius data shows that $\mathrm{Gal}(K/F)$ is either C_3, S_3 or A_4. Since the $N = C_2 \times C_2$ is the only non trivial proper normal subgroup of A_4, the latter is only possible if F admits a cyclic cubic extension. However, since F is real quadratic, and has class number one, it cannot have a cyclic cubic extension whose conductor is supported at $\mathfrak{p}$ and $\sigma(\mathfrak{p})$ only. This excludes both C_3 and A_4. So $\mathrm{Gal}(K/F) = S_3$, and it must contain a quadratic extension E'/F ramified at $\mathfrak{p}$ and $\sigma(\mathfrak{p})$ only. The field E' is given by an element in $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 = \{-1, \epsilon\}$, where ϵ is the fundamental unit in $\mathbf{Z}[\frac{1+\sqrt{353}}{2}]$.

The extension $E' = F(\sqrt{-1}) = \mathbf{Q}(\sqrt{353}, \sqrt{-1})$ is totally complex and unramified outside $\mathfrak{p}$ and $\sigma(\mathfrak{p})$, with $\mathrm{Cl}(E') = \mathbf{Z}/8\mathbf{Z}$. There are no cyclic cubic extension of E' unramified outside the primes above $\mathfrak{p}$ and $\sigma(\mathfrak{p})$. So $\mathbf{Q}(\sqrt{353}, \sqrt{-1})$ cannot be the quadratic subfield of K. Therefore, we must have $E' = F(\sqrt{\epsilon})$.

The field $E' = F(\sqrt{\epsilon})$ has 2 real places, and one complex place. It is unramified outside $\mathfrak{p}$ and $\sigma(\mathfrak{p})$; and we have $\mathrm{Cl}(E') = \mathbf{Z}/3\mathbf{Z}$. Letting $H_{E'}$ be the Hilbert class field of E', we see that $H_{E'}$ is the only possible cubic extension of E'. Further, a direct calculation shows that its Frobenius data matches that of the form f listed in Table 2. From this we obtain that $K = H_{E'}$ is the splitting field of the polynomial $g = x^3 - x^2 - w + 10 \in F[x]$. Its normal closure L is given by the polynomial $h = x^6 - 2x^5 + x^4 + 19x^3 - 19x^2 + 2 \in \mathbf{Q}[x]$, with $\mathrm{Gal}(L/\mathbf{Q}) \simeq S_3^2 \ltimes \mathbf{Z}/2\mathbf{Z}$. Since L is solvable, we can compute its root discriminant using local class field theory, which gives that $\delta_L = 2 \cdot 353^{1/2} = 37.5765....$

Alternatively, we can use the Jones-Roberts Tables [JR14] to find the field L. Indeed, assuming that L is solvable, it will be given by a polynomial of degree 6 or 8. In the latter case, we have $\mathrm{Gal}(K/F) = A_4$. The tables are proven to be complete for all solvable polynomials of degree 6 and 8 such that the root discriminant of the normal closure is less than 75.1531.... Only the polynomial h listed in Lemma 6.1 matches the Frobenius data of the form f given in Table 2.

□

6.2. **The search method.** Let us assume that there is an abelian surface $A = A_f$ associated to the Hecke constituent of the form f of level (1) and weight 2 over $F = \mathbf{Q}(\sqrt{353})$ in Table 2. Then, the surface A has RM by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$ where $\frac{1+\sqrt{5}}{2}$ is a unit of norm -1 in $\mathbf{Q}(\sqrt{5})$. Therefore, by [GGR05, Proposition 3.11], A is principally polarisable. Let C be a genus 2 curve defined over F such that $A = \mathrm{Jac}(C)$. Then, there is a curve $C' : y^2 = h'(x)$ where $h'(x) \in F[x]$ has degree 5 or 6, such that $A' = \mathrm{Jac}(C')$ is isomorphic to A over F. By using the Hecke eigenvalues $a_{\mathfrak{p}} = a_{\mathfrak{p}}(f)$ in Table 2, we obtain that

$$\#A(\mathbf{F}_{\mathfrak{p}}) = \mathrm{N}_{\mathbf{Q}(\sqrt{5})/\mathbf{Q}}(\mathrm{N}_{F/\mathbf{Q}}(\mathfrak{p}) + 1 - a_{\mathfrak{p}}) = \mathrm{N}_{\mathbf{Q}(\sqrt{5})/\mathbf{Q}}(2 + 1 - e) = 5,$$

for the prime $\mathfrak{p} = (w - 10)$ above 2. Since $A(F)_{tors}$ injects into $A(\mathbf{F}_{\mathfrak{p}})$, this implies that A does not have a point of order 2 defined over F. By combining this with Lemma 6.1 and Theorem 3.3, we see that the polynomial h' is of the form $h' = h_\alpha h_{\alpha'}$ where $K = F[c]$ is the cubic extension defined by the cubic factor $g := x^3 - x^2 - w + 10$ of h, and $h_\alpha, h_{\alpha'}$ are the minimal polynomials of some elements $\alpha, \alpha' \in K \setminus F$.

By making a search over the integral elements in K using Algorithm 2.2 described in Section 2, we obtain the pair (α, α') with

$$\alpha := \frac{-22c^5 + 35c^4 - 10c^3 - 419c^2 + 235c + 83}{17},$$
$$\alpha' := \frac{-36c^5 + 48c^4 - 4c^3 - 698c^2 + 264c + 74}{17},$$

with $H_K(\alpha) = 64.0000...$ and $H_K(\alpha') = 1856.3958....$ This gives the polynomial

$$h'(x) := x^6 + 8x^5 + (w - 64)x^4 + (20w - 80)x^3 + (-44w + 240)x^2 + (-96w + 1088)x - 64w + 576.$$

From this, wee obtain the global minimal model C displayed in Theorem 6.2.

6.3. **The surfaces.**

Theorem 6.2. *Let* $F = \mathbf{Q}(\sqrt{353})$, *and* $w = \frac{1+\sqrt{353}}{2}$, *and define the curve* $C : y^2 + Q(x)y = P(x)$ *by*

$$P(x) := -(15w + 149)x^6 - (1119w + 9948)x^5 - (36545w + 325090)x^4 - (636332w + 5659370)x^3 - (6227174w + 55387985)x^2 - (32480001w + 288869715)x - 70532813w - 627353458;$$
$$Q(x) := (w + 1)x^3 + x^2 + wx + w + 1.$$

Let ${}^\sigma C$ *denote the Galois conjugate of* C, A *and* ${}^\sigma A$ *the Jacobians of* C *and* ${}^\sigma C$ *respectively. Then, we have the followings:*

(a) The discriminant of the curve C *is* $\mathrm{disc}(C) = -\epsilon^4$, *where* ϵ *is the fundamental unit. So* C, ${}^\sigma C$ *and the surfaces* A, ${}^\sigma A$ *have everywhere good reduction.*

(b) A *and* ${}^\sigma A$ *have real multiplication by* $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$.

(c) A *and* ${}^\sigma A$ *are modular. They correspond to the two Hecke constituents of* $f, {}^\sigma f \in S_2(1)$ *of dimension 2, where* $S_2(1)$ *is the space of Hilbert modular forms of level* (1) *and weight 2 over* F *(see Table 2).*

(d) A *and* ${}^\sigma A$ *are non-isogenous.*

Proof. (a) This is just an easy calculation.

(b) It is enough to prove this for the surface A. Using the equation of the Humbert surface for the discriminant $D = 5$ given in [EK14], we find that A is a twist of the surface corresponding to the point

$$(g, h) = \left(-\frac{12w + 209}{726}, \frac{742w + 6611}{161051}\right).$$

(c) From (b), we know that A and ${}^{\sigma}A$ are of GL_2-type. In Table 2, we have listed the Hecke eigenvalues $a_{\mathfrak{p}}(f) \bmod \sqrt{5}$ for all primes of norm up to 19. For each of those primes, we have computed $o_{\sqrt{5}}(\mathfrak{p})$ the order of the image of $\mathrm{Frob}_{\mathfrak{p}}$ modulo unipotents under the projectivisation $\mathrm{P}\bar{\rho}_{f,\sqrt{5}}$ of $\bar{\rho}_{f,\sqrt{5}}$. From the orders of the elements, it follows that the projective image of $\bar{\rho}_{f,\sqrt{5}}$ is either S_4 or $\mathrm{PGL}(2,\mathbf{F}_5)$. By computing the Euler factors of A then factoring over $\mathbf{Q}(\sqrt{5})$, we check that for each prime $\mathfrak{p}$ in that table, we have

$$\mathrm{Tr}(\rho_{A,\sqrt{5}}(\mathrm{Frob}_{\mathfrak{p}})) = a_{\mathfrak{p}}(f) \text{ or } \tau(a_{\mathfrak{p}}(f)),$$

where $\mathrm{Gal}(\mathbf{Q}(\sqrt{5})/\mathbf{Q}) = \langle\tau\rangle$. Up to Galois conjugation, these agree with those of the form f. So, the projective image of $\bar{\rho}_{A,\sqrt{5}}$ is either S_4 or $\mathrm{PGL}(2,\mathbf{F}_5)$. Hence $\bar{\rho}_{A,\sqrt{5}}$ cannot be dihedral.

By [SBT97, Theorem 1.2], there is an elliptic curve E/F such that $\bar{\rho}_{A,\sqrt{5}} \simeq \bar{\rho}_{E,5}$. The curve E is modular by [FLHS15, Theorem 1]. Thus $\bar{\rho}_{A,\sqrt{5}}$ is modular. Since A is an abelian surface, it is clear that $\rho_{A,\sqrt{5}}$ satisfies the remaining hypotheses of [KT17, Theorem 1.1]. So, we conclude that $\rho_{A,\sqrt{5}}$, and hence A, is modular.

(d) The surfaces A and ${}^{\sigma}A$ correspond to different Hecke constituents. Therefore, they have different isogeny classes by Faltings [Fal83, Korollar 2]. □

Remark 6.3. The field $F = \mathbf{Q}(\sqrt{353})$ appears to be the real quadratic field of narrow class number one, with the smallest discriminant such that there is an abelian surface with RM and everywhere good reduction that is non-isogenous to its Galois conjugate. In that sense, this example would be the analogue in dimension 2 of the elliptic curve of conductor (1) over $\mathbf{Q}(\sqrt{509})$ discovered by Pinch [Pin82]. However, we cannot prove this without assuming modularity.

7. The abelian surface for the discriminant $D = 421$

7.1. **The field of 2-torsion.** In this example, the defining polynomial for the 2-torsion field cannot be obtained via class field theory. Indeed, an inspection of the Frobenius data for the mod 3 representation leads us to the following lemma.

Lemma 7.1. *Assume that there is an abelian surface A_f attached to the form f listed in Table 3. Let $K = F(A_f[2])$ be the field of 2-torsion of A_f, and $L/\mathbf{Q}$ the normal closure of K. Then L is unramified outside 2 and 421, with Galois group*

$$\mathrm{Gal}(L/\mathbf{Q}) \simeq A_5^2 \rtimes \mathbf{Z}/2\mathbf{Z},$$

and we have $\delta_L < 82.0731\ldots$.

Proof. In Table 3, we have listed the Hecke eigenvalues $a_{\mathfrak{p}}(f) \bmod 2$ for all primes of norm up to 11. For each of those primes, we have computed $o_2(\mathfrak{p})$ the order of the image of $\mathrm{Frob}_{\mathfrak{p}}$ modulo unipotents under the projectivisation $\mathrm{P}\bar{\rho}_{f,2}$ of $\bar{\rho}_{f,2}$. The first part of the lemma follows by inspection of that Frobenius data. The second part concerning the root discriminant uses the same argument as in [Dem09]. □

The following polynomial was obtained by an extensive search:

$$\begin{aligned} h = {} & x^{12} - 18x^{10} + 26x^9 + 58x^8 - 212x^7 - 40x^6 + 766x^5 + 268x^4 - 1030x^3 \\ & - 989x^2 - 366x - 99. \end{aligned}$$

It was kindly provided by Eric Driver, John Jones and David P. Roberts as a potential candidate for the defining polynomial for the field L in Lemma 7.1. Theorem 7.2 below confirms that their prediction was accurate, although the search they did was not exhaustive.

TABLE 3. The Frobenius data for the two non Galois conjugate Hilbert newforms of weight 2 and level (1) over $F = \mathbf{Q}(\sqrt{421})$. (Here $e = \frac{1+\sqrt{5}}{2}$ and α is a cyclic generator of $\mathbf{F}_4^\times$.)

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f) \bmod 2$	$o_2(\mathfrak{p})$	$a_\mathfrak{p}(f) \bmod \sqrt{5}$	$o_{\sqrt{5}}(\mathfrak{p})$
3	$4w-43$	$-2e+1$	1	3	0	2
3	$-4w-39$	$2e$	0	1	1	4
4	2	$e-2$	α^2	–	1	1
5	$w-11$	$e-2$	α^2	5	1	–
5	$-w-10$	3	1	3	3	–
7	$-54w-527$	3	1	3	3	4
7	$54w-581$	$e-2$	α^2	5	1	6
11	$25w+244$	0	0	1	0	2
11	$-25w+269$	$-e+5$	α	5	2	1

7.2. **The search method.** Let us assume that there is an abelian surface $A = A_f$ associated to the Hecke constituent of the form f of level 1 and weight 2 over $F = \mathbf{Q}(\sqrt{421})$ in Table 3. Then, the surface A has RM by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$ where $\frac{1+\sqrt{5}}{2}$ is a unit of norm -1 in $\mathbf{Q}(\sqrt{5})$. Therefore, by [GGR05, Proposition 3.11], A is principally polarisable. Let C be a genus 2 curve defined over F such that $A = \mathrm{Jac}(C)$. Then, there is a curve $C' : y^2 = h'(x)$ where $h'(x) \in F[x]$ has degree 5 or 6, such that $A' = \mathrm{Jac}(C')$ is isomorphic to A over F. By using the Hecke eigenvalues in Table 3, we obtain that

$$\#A(\mathbf{F}_\mathfrak{p}) = \mathrm{N}_{\mathbf{Q}(\sqrt{5})/\mathbf{Q}}(\mathrm{N}_{F/\mathbf{Q}}(\mathfrak{p}) + 1 - a_\mathfrak{p}) = \mathrm{N}_{\mathbf{Q}(\sqrt{5})/\mathbf{Q}}(5+1-3) = 9,$$

for the prime $\mathfrak{p} = (-w-10)$ above 5. Hence A does not have a point of order 2 defined over F. By combining this with Lemma 7.1 and Theorem 3.3, we see that the polynomial h' is of the form $h' = h_\alpha$ where $K = F[c]$ is the sextic extension defined by $g := x^6 - 9x^4 + (-2w+14)x^3 + (3w-13)x^2 + (2w+10)x + w + 2$, and h_α the minimal polynomial of some element $\alpha \in K \setminus F$.

By making a search over the integral elements in K using a variant of the Algorithm 2.2 described in Section 2, we obtain an element α with $H_K(\alpha) = 416759.3936...$ which we do not display here. By clearing denominators, we this gives the polynomial with integral coefficients

$$\begin{aligned} h'(x) := x^6 &+ (-824w + 6396)x^5 + (-5950152w + 15262668)x^4 \\ &+ (15357307104w + 189762599664)x^3 \\ &+ (-200691458540784w + 1196268593295456)x^2 \\ &+ (2252755307891174440w - 11533514346058631004)x \\ &+ 88664091615566887507 2w - 101739762213311351986 56. \end{aligned}$$

From this, wee obtain a global minimal for C displayed in Theorem 7.2.

7.3. **The surfaces.**

Theorem 7.2. *Let* $F = \mathbf{Q}(\sqrt{421})$, *and* $w = \frac{1+\sqrt{421}}{2}$, *and define the curve* $C : y^2 + Q(x)y = P(x)$ *by*

$$\begin{aligned} P := (13w+77)x^6 &+ (593w+6772)x^5 + (15049w+131460)x^4 \\ &+ (163829w+1727293)x^3 + (1167345w+10787410)x^2 \\ &+ (3985370w+40412781)x + 6111237w + 58050373; \\ Q := wx^3 &+ x^2 + (w+1)x + w + 1. \end{aligned}$$

Let ${}^{\sigma}C$ denote the Galois conjugate of C, A and ${}^{\sigma}A$ the Jacobians of C and ${}^{\sigma}C$ respectively. Then, we have the following:

(a) The discriminant of the curve C is $\operatorname{disc}(C) = \epsilon^8(25w+244)^{22}$, where ϵ is the fundamental unit and $(25w+244)$ is one of the primes above 11. The surfaces A and ${}^{\sigma}A$ have everywhere good reduction.

(b) A and ${}^{\sigma}A$ have real multiplication by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$.

(c) A and ${}^{\sigma}A$ are modular. They correspond to the two Hecke constituents of $f, {}^{\sigma}f \in S_2(1)$ of dimension 2, where $S_2(1)$ is the space of Hilbert modular forms of level (1) and weight 2 over F (see Table 3).

(d) A and ${}^{\sigma}A$ are non-isogenous.

Proof. Again (a) is an easy calculation; in this case, one shows that the conductor of A at the prime $(25w+244)$ is trivial. To prove (b), we use the equation of the Humbert surface for the discriminant $D = 5$ in [EK14]. We find that A is a twist of the surface corresponding to the point

$$(g,h) = \left(\frac{-16w+147}{54}, \frac{-12229w+129846}{243}\right).$$

In Table 3, we have listed the Hecke eigenvalues $a_{\mathfrak{p}}(f) \bmod \sqrt{5}$ for all primes of norm up to 11. For each of those primes, we have computed $o_{\sqrt{5}}(\mathfrak{p})$ the order of the image of $\mathrm{Frob}_{\mathfrak{p}}$ modulo unipotents under the projectivisation $\mathrm{P}\bar{\rho}_{f,\sqrt{5}}$ of $\bar{\rho}_{f,\sqrt{5}}$. From those orders, we see that the projective image of $\bar{\rho}_{f,\sqrt{5}}$ is $\mathrm{PGL}_2(\mathbf{F}_5)$. A similar argument as in the proof of Theorem 6.2 shows that $\bar{\rho}_{A,\sqrt{5}}$ is modular and surjective. So, we conclude (c) by using [KT17, Theorem 1.1]. Finally, the surfaces A and ${}^{\sigma}A$ are non-isogenous since the forms f and ${}^{\sigma}f$ are not in the same Hecke constituent. This concludes (d). □

Remark 7.3. We note that the surface A in Theorem 7.2 has good reduction at the prime $\mathfrak{p} = (25w+244)$ above 11 even though the curve C has bad reduction at that prime. In this case, the reduction of A is isomorphic to the product of two elliptic curves. This forces the Euler factor of A at $\mathfrak{p}$ to be the square of that of an elliptic curve. For this reason, the corresponding Hecke eigenvalue will be an integer. Here, we have $a_{\mathfrak{p}} = 0$.

From this discussion, we see that the set of primes of bad reduction for the curve, which are primes of good reduction for the Jacobian, is contained the set of primes where the Hecke eigenvalues are integers. Unfortunately, the latter set is infinite, and cannot be used to bound the former. This makes the search for the curve C above much trickier.

8. The abelian surface for the discriminant $D = 1597$

8.1. **The field of 2-torsion.** In this case, the knowledge of the Frobenius data for the form (see Table 4) allows us to prove the following lemma. But we were unable to obtain the field of 2-torsion via a search.

Lemma 8.1. *Assume that there is an abelian surface A_f attached to the form f listed in Table 4. Let $K = F(A_f[2])$ be the field of 2-torsion of A_f, and $L/\mathbf{Q}$ the normal closure of K. Then L is unramified outside 2 and 1597, with Galois group*

$$\mathrm{Gal}(L/\mathbf{Q}) \simeq A_5^2 \rtimes \mathbf{Z}/2\mathbf{Z},$$

and we have $\delta_L < 159.8499\ldots$.

Proof. In Table 4, we have listed the Hecke eigenvalues $a_{\mathfrak{p}}(f) \bmod 2$ for all primes of norm up to 19. For each of those primes, we have computed $o_2(\mathfrak{p})$ the order of the image of $\mathrm{Frob}_{\mathfrak{p}}$ modulo unipotents under the projectivisation $\mathrm{P}\bar{\rho}_{f,2}$ of $\bar{\rho}_{f,2}$. The proof of the lemma follows that of Lemma 7.1. □

Table 4. The Frobenius data for the two non Galois conjugate Hilbert newforms of weight 2 and level (1) over $F = \mathbf{Q}(\sqrt{1597})$. (Here $e = \frac{1+\sqrt{5}}{2}$ and α is a cyclic generator of $\mathbf{F}_4^\times$.)

$\mathbf{N}\mathfrak{p}$	$\mathfrak{p}$	$a_\mathfrak{p}(f)$	$a_\mathfrak{p}(f) \bmod 2$	$o_2(\mathfrak{p})$	$a_\mathfrak{p}(f) \bmod \sqrt{5}$	$o_{\sqrt{5}}(\mathfrak{p})$
3	$-2w-39$	$2e$	0	1	1	4
3	$-2w+41$	$-e+2$	α^2	5	4	4
4	2	$e-2$	α^2	–	1	1
7	$27w-553$	$3e$	α^2	5	4	6
7	$27w+526$	$2e-3$	1	3	3	4
17	$-773w+15832$	6	0	1	1	6
17	$773w+15059$	$5e-1$	α	5	4	6
19	$-w-19$	$-2e+5$	1	3	4	1
19	$-w+20$	$3e-2$	α^2	5	2	3

8.2. **The search method.** Let us assume that there is an abelian surface $A = A_f$ associated to the Hecke constituent of the form f of level (1) and weight 2 over $F = \mathbf{Q}(\sqrt{1597})$ in Table 4. Then, the surface A has RM by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$ where $\frac{1+\sqrt{5}}{2}$ is a unit of norm -1 in $\mathbf{Q}(\sqrt{5})$. Therefore, by [GGR05, Proposition 3.11], A is principally polarisable. Let C be a genus 2 curve defined over F such that $A = \mathrm{Jac}(C)$. Then, there is a curve $C' : y^2 = h'(x)$ where $h'(x) \in F[x]$ has degree 5 or 6, such that $A' = \mathrm{Jac}(C')$ is isomorphic to A over F.

In this case, we use the search method described in [DK16]. Let $Y_-(5)$ be the Hilbert modular surface of discriminant 5 which parametrises all principally polarised abelian surfaces with RM by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$. In [EK14, Theorem 16], the surface $Y_-(5)$ is described as a double cover of the weighted projective space $\mathbf{P}^2_{g,h}$:

$$z = 2(6250h^2 - 4500g^2h - 1350gh - 108h - 972g^5 - 324g^4 - 27g^3).$$

For the abelian surface A attached to the rational point (g, h), the Igusa-Clebsch invariants of A are given by [EK14, Corollary 15]. In this case, we are looking for a surface $A = \mathrm{Jac}(C)$ such that the discriminant of C is a scalar multiple of h^2. However, the fundamental unit $\epsilon = -2518525w - 49063993$ has height 100646511.0000.... So a naive height search as in [DK16] will not work. So, we scale the parameters by setting

$$(g, h) = \left(\frac{\epsilon^m g'}{6u^2}, \frac{\epsilon^n h'}{u^5}\right),$$

where $m, n \in \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$ and g', h', u are integral elements with small height. Letting $m = 1, n = 2$, $g' = 114w - 2335, h' = 1$ and $u = -27w - 526$, we get

$$(g, h) = \left(\frac{16w + 259}{294}, \frac{2913w + 56749}{16807}\right).$$

We get the curve C' using the same approach as in [DK16]. By reduction, this yields the global model for C displayed in Theorem 8.3.

Remark 8.2. The scaling trick introduced in the search method in Subsection 8.2 was fine tuned using the curves we found for the discriminants $D = 353$ and 421. In this case, the trick was successful because the curve C has a unit discriminant. In general, the scaling must take into account the set of primes of bad reduction for C that are primes of good reduction for its Jacobian A. However, as indicated in Remark 7.3, it is not possible to predict those primes even though we know they are contained in the set of primes where the Hilbert newform associated to A has integer

Hecke eigenvalues. Therefore it is very difficult to use this trick to find curves such as the one in Theorem 7.2.

8.3. **The surfaces.**

Theorem 8.3. *Let $F = \mathbf{Q}(\sqrt{1597})$, and $w = \frac{1+\sqrt{1597}}{2}$, and define the curve $C : y^2 + Q(x)y = P(x)$ by*

$$\begin{aligned} P &:= (14154412w + 275745514)x^6 - (489014393w + 9526607332)x^5 \\ &\quad + (7039395048w + 137136152764)x^4 - (54043428224w + 1052833060832)x^3 \\ &\quad + (233382395752w + 4546578743807)x^2 - (537510739916w + 10471376373574)x \\ &\quad + 515810377784w + 10048626384323; \end{aligned}$$

$$Q := x^3 + wx^2 + (w+1)x + w + 1$$

Let ${}^{\sigma}C$ denote the Galois conjugate of C, A and ${}^{\sigma}A$ the Jacobians of C and ${}^{\sigma}C$ respectively. Then, we have the followings:

(a) The discriminant of the curve C is $\mathrm{disc}(C) = \bar{\epsilon}^6$, where ϵ is the fundamental unit. So C, ${}^{\sigma}C$ and the surfaces A, ${}^{\sigma}A$ have everywhere good reduction.

(b) A and ${}^{\sigma}A$ have real multiplication by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$.

(c) A and ${}^{\sigma}A$ are modular. They correspond to the two Hecke constituents of $f, {}^{\sigma}f \in S_2(1)$ of dimension 2, the space of Hilbert modular forms of level (1) and weight 2 over F (see Table 4).

(d) A and ${}^{\sigma}A$ are non-isogenous.

Proof. During the search we described above, we showed that A is a twist of the surface corresponding to the point

$$(g, h) = \left(\frac{16w + 259}{294}, \frac{2913w + 56749}{16807}\right).$$

This shows that A has RM by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$. In Table 4, we have listed the Hecke eigenvalues $a_{\mathfrak{p}}(f) \bmod \sqrt{5}$ for all primes of norm up to 19. For each of those primes, we have computed $o_{\sqrt{5}}(\mathfrak{p})$ the order of the image of $\mathrm{Frob}_{\mathfrak{p}}$ modulo unipotents under the projectivisation $\mathrm{P}\bar{\rho}_{f,\sqrt{5}}$ of $\bar{\rho}_{f,\sqrt{5}}$. From the orders, we see that the residual representation $\bar{\rho}_{A,\sqrt{5}}$ is surjective. So, the rest of the proof of the theorem follows as in Theorem 7.2. □

Remark 8.4. For the discriminants $D = 1321, 1997$, we could not find the corresponding surfaces. In both cases, we strongly believe that the surfaces are Jacobians of curves whose minimal models have primes of bad reduction despite the surfaces themselves having everywhere good reduction. As explained in Remark 8.2, the scaling techniques introduced in Subsection 8.2 cannot be applied in those situations.

9. The abelian surfaces for the discriminant $D = 1997$

9.1. **The field of 2-torsion.**

Lemma 9.1. *Assume that there is an abelian surface A_f attached to f Table 5, and let K be the Galois closure of the field $F(A_f[\sqrt{2}])$. Then K is the splitting field of the polynomial $h' = x^6 - 25x^4 - 50x^3 - 343x^2 - 1372x - 1372$. We have $\mathrm{Gal}(K/\mathbf{Q}) \simeq S_3^2 \ltimes \mathbf{Z}/2\mathbf{Z}$, and $\delta_K = 2^{3/2} \cdot 1997^{1/2} = 126.396\ldots$*

Proof. Let $\rho_{f,\sqrt{2}} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(\mathbf{Z}_2[\sqrt{2}])$ be the $\sqrt{2}$-adic representation attached to f, and $\bar{\rho}_{f,\sqrt{2}} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_2(\mathbf{F}_2)$ its reduction modulo $(\sqrt{2})$. By assumption, we have $\bar{\rho}_{A_f,\sqrt{2}} = \bar{\rho}_{f,\sqrt{2}}$, where $\bar{\rho}_{A_f,\sqrt{2}} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}(A_f[\sqrt{2}])$ the mod $\sqrt{2}$ attached to A_f.

We now compute a bound on the root discriminant δ_K of K. In Table 5, we have listed the Hecke eigenvalues $a_{\mathfrak{p}}(f)$ mod 2 for all primes of norm up to 29. For each of those primes, we have computed $o_2(\mathfrak{p})$ the order of the image of $\mathrm{Frob}_{\mathfrak{p}}$ modulo unipotents under the projectivisation $\mathrm{P}\bar{\rho}_{f,2}$ of $\bar{\rho}_{f,2}$. From that data, we see that $a_{(2)}(f) = -1-\sqrt{2} = 1 \,(\mathrm{mod} \sqrt{2})$. So f is ordinary at (2). Hence, the mod $\sqrt{2}$-representation restricted to the decomposition group at (2) is of the form

$$\bar{\rho}_{f,\sqrt{2}}|_{D_{(2)}} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad \mathrm{mod}\ \sqrt{2}.$$

So, we can use the same argument as in [Dem09] to show that $\delta_K \leq 4 \cdot 1997^{1/2} = 178.751\ldots$. (This is the same as the Fontaine bound in Theorem 4.1.)

The field $F(A_f[\sqrt{2}])$ is a Galois extension of F which is unramified outside the prime (2). The Frobenius data shows that $[F(A_f[\sqrt{2}]) : F]$ is either 3 or 6. Since F has narrow class number one, $F(A_f[\sqrt{2}])$ is unramified outside (2), it cannot be a cyclic cubic extension. So $[F(A_f[\sqrt{2}]) : F] = 6$, and it must contain a quadratic extension E'/F ramified at (2) only. The field E' is given by an element in $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 = \{-1, \epsilon\}$, where ϵ is the fundamental unit in $\mathbf{Z}[\frac{1+\sqrt{1997}}{2}]$.

The extension $E' = F(\sqrt{-1})$ is a totally complex field in which the prime (2) ramifies. It has no cubic extension whose conductor is a power of the prime above 2. However, the class group of $\mathrm{Cl}(F(\sqrt{-1})) = \mathbf{Z}/21\mathbf{Z} = \langle \eta \rangle$. The Frobenius data of the form f does not match that arising from the extension associated to η^7. So, we must have $E' = F(\sqrt{\epsilon})$.

The field $F(\sqrt{\epsilon})$ has 2 real places, and one complex place; and the prime (2) ramifies. Again, there are no cubic extension of $F(\sqrt{\epsilon})$ whose conductor is a power of the prime above 2. This means that $F(A_f[\sqrt{2}])/F(\sqrt{\epsilon})$ must be an unramified cubic extension. (Note that this is consistent with the fact that $[F(A_f[\sqrt{2}]) : F] = 6$.) The class group of $F(\sqrt{\epsilon})$ is cyclic of order 3. So, its Hilbert class field is an unramified cubic extension of $F(\sqrt{\epsilon})$ generated by the polynomial $g' = x^6 + (-w-3)x^4 + (2w+51)x^2 - w + 15$ over F. So, we have that $F(A_f[\sqrt{2}]) = F[x]/(g'(x))$, and its normal closure K is given by the polynomial $g = x^{12} - 7x^{10} - 383x^8 + 1662x^6 - 393x^4 + 3505x^2 - 289$, which is the norm of g'. There are three subfields of degree 6 in the field K, and they are all isomorphic. One of them is given by the polynomial h'. So, by construction, we obtain that K is the splitting field of h'. By explicit calculations, one gets that $\mathrm{Gal}(K/\mathbf{Q}) \simeq S_3^2 \ltimes \mathbf{Z}/2\mathbf{Z}$. Since K is solvable, we can compute its root discriminant using local class field theory, which gives that $\delta_K = 2^{3/2}1997^{1/2} = 126.396\ldots$.

Alternatively, we can also look it up in [JR14] using the Frobenius data in Table 5. There is a unique polynomial h whose Frobenius data matches the one in the table, and such that the root discriminant of the splitting field satisfies the Fontaine bound in Theorem 4.1. The tables are complete in this case. □

Theorem 9.2. *Assume that there is an abelian surface A_f over $F = \mathbf{Q}(\sqrt{1997})$ attached to the form f listed in Table 5. Let L be the normal closure of the field of 2-torsion $F(A_f[2])$. Then there is a polynomial h in Table 6 such that L is the splitting field of h. In that case, we have*

$$\mathrm{Gal}(L/\mathbf{Q}) \simeq S_3^2 \rtimes \mathbf{Z}/2\mathbf{Z}, \text{ or } \mathrm{Gal}(L/\mathbf{Q}) \simeq S_4^2 \rtimes \mathbf{Z}/2\mathbf{Z}.$$

Proof. Let $G = \mathrm{Gal}(F(A_f[2])/F)$. By Lemma 9.1 and Theorem 3.3, there is a subgroup $H \leq (\mathbf{Z}/2\mathbf{Z})^8$, the unique normal subgroup of $\mathrm{SL}_2(\mathbf{F}_2[\varepsilon])$ of order 8, such that $G \simeq H \rtimes S_3$. Since $F(A_f[\sqrt{2}])$ is Galois over F, we see that $F(A_f[2])$ is the compositum of r quadratic extensions $E/F(A_f[\sqrt{2}])$, where r be the rank of H

Table 5. The Frobenius data for the two non Galois conjugate Hilbert newforms of weight 2 and level (1) over $F = \mathbf{Q}(\sqrt{1997})$. (Here $e = \sqrt{2}$).

$\mathrm{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$a_{\mathfrak{p}}(f)$ mod 2	$o_2(\mathfrak{p})$
4	2	$-e-1$	1	–
7	$w+22$	$-e+2$	0	1
7	$-w+23$	-3	1	3
9	3	$-e$	0	1
17	$-6w+137$	$-3e-2$	0	1
17	$-6w-131$	6	0	1
25	5	$2e-5$	1	3
29	$7w+153$	$-3e+2$	0	1
29	$-7w+160$	$-3e+2$	0	1

as an $\mathbf{F}_2$-space. Each quadratic extension $E/F(A_f[\sqrt{2}])$ is unramified outside the primes above 2.

Let us write

$$\mathrm{Gal}(F(A_f[\sqrt{2}])/F) = D_3 = \langle \tau, \sigma | \tau^2 = \sigma^3 = 1, \sigma^2\tau = \tau\sigma \rangle.$$

There are three primes $\mathfrak{P}_1$, $\mathfrak{P}_2, \mathfrak{P}_3$ above 2 in $F(A_f[\sqrt{2}])$, which are permuted transitively by $\mathrm{Gal}(F(A_f[\sqrt{2}])/F)$, with $\mathrm{N}_{F(A_f[\sqrt{2}])/\mathbf{Q}}(\mathfrak{P}_1) = 4$. Up to relabelling those primes, we can write $(2) = (\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^2$, where $\mathfrak{P}_2 = \sigma(\mathfrak{P}_1)$, $\mathfrak{P}_3 = \sigma^2(\mathfrak{P}_1)$, and $\tau(\mathfrak{P}_1) = \mathfrak{P}_1$, $\tau(\mathfrak{P}_2) = \mathfrak{P}_3$.

Let $E/F(A_f[\sqrt{2}])$ be a quadratic extension unramified outside $S := \{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3\}$. Then, the conductor of the compositum of the fields in the $\mathrm{Gal}(F(A_f[\sqrt{2}])/F)$-orbit of E is of the form $(\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^{s_0}$ for some $s_0 \geq 0$. Therefore, the root discriminant of $F(A_f[2])$ is equal to

$$\delta_{F(A_f[2])} = \delta_{F(A_f[\sqrt{2}])}\mathrm{N}_{F(A_f[\sqrt{2}])/\mathbf{Q}}(\mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3)^{\frac{s}{12\cdot 2^r}} = \delta_{F(A_f[\sqrt{2}])}2^{\frac{s}{2^{r+1}}},$$

for some $s \geq 0$. We recall that $\delta_{F(A_f[\sqrt{2}])} = 2 \cdot 1997^{1/2} = 89.3756....$ So, we have $\delta_{F(A_f[2])} < 4 \cdot 1997^{1/2} = 178.7512...$ if and only if one of the following holds:

- $r = 0$, $s \leq 1$;
- $r = 1$, $s \leq 3$;
- $r = 2$, $s \leq 7$;
- $r = 3$, $s \leq 15$.

However, we want $\delta_L < 4 \cdot 1997^{1/2} = 178.7512....$ We will show that this in fact implies that $s \leq 2$ for $r = 0, 1, 2, 3$.

Let $K(S, 2) = K_S^\times/(K_S^\times)^2$ be the 2-Selmer group, where $K_S^\times$ is the subset of $K = F(A_f[\sqrt{2}])$ of that are S-integral. Then $K(S, 2) \simeq (\mathbf{Z}/2\mathbf{Z})^{12}$ is an $\mathbf{F}_2$-module equipped with a $\mathrm{Gal}(F(A_f[\sqrt{2}])/F)$-action. We view $K(S, 2)$ as an $\mathbf{F}_2[D_3]$-module under this action. Every quadratic extension $E/F(A_f[\sqrt{2}])$ unramified outside S corresponds to an element in $K(S, 2)$. It is not hard to see that there is a bijection between the following two sets:

(a) All compositums K' of quadratic extensions $E/F(A_f[\sqrt{2}])$ unramified outside S, which are Galois over F;

(b) $\mathbf{F}_2[D_3]$-submodules V of $K(S, 2)$.

For K' a compositum as in (a), $[K' : F] = 2^r$, where V is the $\mathbf{F}_2[D_3]$-submodule corresponding to K' and $r = \dim_{\mathbf{F}_2}(V)$. The 2-torsion field $F(A_f[2])$ and its normal closure L arise from an $\mathbf{F}_2[D_3]$-submodule with $r = \dim_{\mathbf{F}_2}(V) \leq 3$, and we must have $\delta_L < 178.7512....$

Table 6. The possible polynomials giving the 2-torsion field of the abelian surface A_f defined over $F = \mathbf{Q}(\sqrt{1997})$ with everywhere good reduction attached to the form in Table 5. (Here $G = \mathrm{Gal}(F(A_f[2])/F)$.)

h	G	$\mathrm{Gal}(L/\mathbf{Q})$
$x^6 - 25x^4 - 50x^3 - 343x^2 - 1372x - 1372$	S_3	$S_3^2 \rtimes \mathbf{Z}/2\mathbf{Z}$
$x^{12} + 7x^{10} - 383x^8 - 1662x^6 - 393x^4 - 3505x^2 - 289$	D_6	$S_3^2 \rtimes \mathbf{Z}/2\mathbf{Z}$
$x^{12} - 90x^9 + 1923x^8 - 3932x^7 + 4050x^6 - 664x^5 + 2573x^4 - 3554x^3 + 1922x^2 + 868x + 196$		
$\begin{cases} x^{12} + 17x^{10} - 151x^8 - 1656x^6 + 10988x^4 - 1104x^2 + 16 \\ x^{12} - 17x^{10} - 151x^8 + 1656x^6 + 10988x^4 + 1104x^2 + 16 \end{cases}$	S_4 $S_4 \times \mathbf{Z}/2\mathbf{Z}$	$S_4^2 \rtimes \mathbf{Z}/2\mathbf{Z}$
$\begin{cases} x^{12} - 49x^{10} - 118x^8 + 371x^6 - 442x^4 + 219x^2 + 1 \\ x^{12} + 49x^{10} - 118x^8 - 371x^6 - 442x^4 - 219x^2 + 1 \end{cases}$		
$\begin{cases} x^{12} - 63x^{10} + 597x^8 - 3284x^6 + 2956x^4 - 416x^2 + 16 \\ x^{12} + 63x^{10} + 597x^8 + 3284x^6 + 2956x^4 + 416x^2 + 16 \end{cases}$		
$\begin{cases} x^{12} - 58x^{10} + 1101x^8 - 7548x^6 + 9144x^4 - 1040x^2 + 16 \\ x^{12} + 58x^{10} + 1101x^8 + 7548x^6 + 9144x^4 + 1040x^2 + 16 \end{cases}$		
$\begin{cases} x^{12} - 49x^{10} + 602x^8 - 1293x^6 + 2390x^4 - 501x^2 + 1 \\ x^{12} + 49x^{10} + 602x^8 + 1293x^6 + 2390x^4 + 501x^2 + 1 \end{cases}$		
$\begin{cases} x^{12} - 123x^{10} - 904x^8 - 3079x^6 - 7948x^4 - 7791x^2 - 289 \\ x^{12} + 123x^{10} - 904x^8 + 3079x^6 - 7948x^4 + 7791x^2 - 289 \end{cases}$		
$\begin{cases} x^{12} + 2x^{10} + 130x^8 + 352x^6 - 110x^4 - 594x^2 - 49 \\ x^{12} - 2x^{10} + 130x^8 - 352x^6 - 110x^4 + 594x^2 - 49 \end{cases}$		

For $r = 0$, Lemma 9.1 shows that there is a unique field L. In that case, $F(A_f[2]) = F(A_f[2])$. For $r = 1, 2, 3$, we determine all possible fields L as follows:

(1) For each non-trivial element $v \in K(S, 2)$, we find the corresponding quadratic extension $E/F(A_f[\sqrt{2}])$. Then we compute the $\mathrm{Gal}(F(A_f[\sqrt{2}])/F)$-orbit of E, and the submodule V generated by the $\mathrm{Gal}(F(A_f[\sqrt{2}])/F)$-orbit of v;

(2) If $\dim_{\mathbf{F}_2}(V) = 1, 2, 3$, then we compute the root discriminant δ_E of E. If $\delta_E < 178.7512...$, then we compute the root discriminant of the normal closure L of E. We note that this can be done from the defining polynomial of E as a local computation without having the compute the field L itself, which in this case is extremely big.

(3) If $\delta_L < 178.7512...$, then we compute all degree 12 subfields whose normal closure is L.

In total, we obtain that there are:

(1) 3 fields L with $\mathrm{Gal}(L/\mathbf{Q}) \simeq S_3^2 \rtimes \mathbf{Z}/2\mathbf{Z}$; and

(2) 7 fields L with $\mathrm{Gal}(L/\mathbf{Q}) \simeq S_4^2 \rtimes \mathbf{Z}/2\mathbf{Z}$.

In Table 6, we give some polynomials of degree 12 whose splitting fields are the fields L. We also give the Galois group of the relative extension $F(A_f[2])/F$. Each of the field L such that $\mathrm{Gal}(L/\mathbf{Q}) \simeq S_4^2 \rtimes \mathbf{Z}/2\mathbf{Z}$ is given by a pair of polynomials (h_1, h_2) such that, letting g_1 (resp. g_2) be an irreducible factor of h_1 (resp. h_2) over F, we have $\mathrm{Gal}(K_1/F) \simeq S_4$ and $\mathrm{Gal}(K_2/F) \simeq S_4 \times \mathbf{Z}/2\mathbf{Z}$. So in those cases, there are two possibilities for the extension $F(A_f[2])/F$. □

9.2. **The surfaces.** Let $h(x)$ be one of the polynomial in Table 6, and $g(x)$ an irreducible factor over F. Let $Y_-(8)$ be the Hilbert modular surface of discriminant 8. We recall that $Y_-(8)$ parametrises principally polarised abelian surfaces with RM by

$\mathbf{Z}[\sqrt{2}]$. Let $\bar{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{SL}_2(\mathbf{F}_2[e])$, with $e^2 = 0$, be the mod 2 representation associated to g. We let $Y_-(8)^{\bar{\rho}}$ be the twist of $Y_-(8)$ by $\bar{\rho}$, this parametrises all pairs (A, ϕ) where A is a principally polarised abelian surface defined over F with RM by $\mathbf{Z}[\sqrt{2}]$, and $\phi : A[2] \simeq \bar{\rho}$ an isomorphism.

Theorem 9.3. *Assume that there is an abelian surface A_f over $F = \mathbf{Q}(\sqrt{1997})$ attached to the Hilbert newform f of level (1) and weight 2 listed in Table 5. Then, there exists a polynomial h in Table 6, and an irreducible factor $g \in F[x]$ such that A corresponds to an F-rational on the surface $Y_-(8)^{\bar{\rho}}$, where $\bar{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{SL}_2(\mathbf{F}_2[e])$, with $e^2 = 0$, is the mod 2 representation associated to g.*

Proof. This follows directly from Theorem 9.2. □

Remark 9.4. As we explained earlier in Remark 8.4, we were unable to find the surface A_f attached to the newform f of level (1) over $F = \mathbf{Q}(\sqrt{1997})$ in Table 5, using both the search methods described in Subsections 6.2, 7.2 and 8.2. However, as Theorem 9.3 indicates, A_f must correspond to an F-rational point on a twist of $Y_-(8)^{\bar{\rho}}$, where $\bar{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{SL}_2(\mathbf{F}_2[e])$ is a mod 2 representation arising from one of the polynomials in Table 6. So, it is possible that a search on the twists $Y_-(8)^{\bar{\rho}}$ might be successful. The question of finding explicit equations for twists of Hilbert modular surfaces is one that is interesting in its own right. Indeed although such twists have been used in modularity lifting methods (see [Ell05, SBT97] for example), they have never been approached algorithmically. So, we hope to return to studying them in the future, and use them to find the surface A_f.

Remark 9.5. Assume that there is an abelian surface A_f over $F = \mathbf{Q}(\sqrt{1997})$ attached to the Hilbert newform f of level (1) and weight 2 listed in Table 5. Then, A_f is isomorphic to the Jacobian A' of some Richelot curve $C' : y^2 = \tilde{h}(x)$. By [BD11, Lemma 4.1], there is a cubic extension K'/F and a quadratic polynomial $Q(x) \in K'[x]$ such that C' is of the form

$$C' : y^2 = \tilde{h}(x) = \mathrm{Norm}_{K'[x]/F[x]}(Q(x)).$$

By the uniqueness of the field K in Lemma 9.1, we see K'/F is one the cubic extension of F contained in K. So, up to Galois conjugation, it is defined by a cubic factor of the polynomial $h'(x) = x^6 - 25x^4 - 50x^3 - 343x^2 - 1372x - 1372$. Bending [Ben99, Theorem 4.1] gives a parametrisation of abelian surfaces with RM by $\mathbf{Z}[\sqrt{2}]$ using the fact that they are Jacobians of Richelot curves. However, his family does not seem to be very suitable for height search.

Remark 9.6. There are six *pairwise* non-isogenous elliptic curves over F with trivial conductor. Their $\mathrm{Gal}(F/\mathbf{Q})$-conjugacy classes are represented by the curves:

$$\begin{aligned} E_1 &: y^2 + wxy = x^3 + (w+1)x^2 + (111w + 5401)x + (2406w + 81112); \\ E_2 &: y^2 + (w+1)xy + (w+1)y = x^3 - wx^2 + (19636w + 434383)x \\ &\qquad + (5730650w + 125261893); \\ E_3 &: y^2 + wxy + (w+1)y = x^3 - x^2 + (9370w - 208733)x \\ &\qquad + (2697263w - 61535794). \end{aligned}$$

The fields $F(E_1[2])$ and $F(E_2[2])$ have the same Galois closure, which is the field K in Lemma 9.1. Assuming that the surface A_f in Theorem 9.3 exists, then $A_f[2]$ is an extension of $E_1[2]$ or $E_2[2]$ in the category of finite flat 2-group schemes defined over $\mathcal{O}_F$ up to Galois conjugation. Theorem 9.2 shows that there are least 10 possibilities for the generic fibre of $A_f[2]$. This makes it harder to pin down $A_f[2]$, and hence the 2-adic Tate module of A_f. In part, this explains the extra difficulties we experienced in finding A_f using the same height search as in Subsection 6.2.

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