NEW MODULI COMPONENTS OF RANK 2 BUNDLES ON PROJECTIVE SPACE

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ABSTRACT. We present a new family of monads whose cohomology is a stable rank two vector bundle on $\mathbb{P}^3$. We also study the irreducibility and smoothness together with a geometrical description of some of these families. These facts are used to construct a new infinite series of rational moduli components of stable rank two vector bundles with trivial determinant and growing second Chern class. We also prove that the moduli space of stable rank two vector bundles with trivial determinant and second Chern class equal to 5 has exactly three irreducible rational components.
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1. INTRODUCTION

In [33] Maruyama proved that the rank r stable vector bundles on a projective scheme X with fixed Chern classes $c_1, ..., c_r$ can be parametrized by an algebraic quasi-projective variety, denoted by $\mathcal{B}_X(r, c_1, ..., c_r)$. Although this result has been known for almost 40 years, there are just a few concrete examples and established facts about such varieties, even for cases like $X = \mathbb{P}^3$ and $r = 2$. For instance, $\mathcal{B}_{\mathbb{P}^3}(2, 0, 1)$ was studied by Barth in [3], $\mathcal{B}_{\mathbb{P}^3}(2, 0, 2)$ was described by Harthorne in [19], $\mathcal{B}_{\mathbb{P}^3}(2, -1, 2)$ studied by Harthorne and Sols in [22] and by Manolache in [32], while $\mathcal{B}_{\mathbb{P}^3}(2, -1, 4)$ was described by Banica and Manolache in [1]. This probably happened due to the fact that the questions of irreducibility (solved in [36] and [37]), and smoothness (answered in [28]) of the so-called *instanton component* of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, 0, c_2)$ for all $c_2 \in \mathbb{Z}_+$ remained open until 2014.

In this paper, we continue the study of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, 0, n)$, which we will simply denote by $\mathcal{B}(n)$ from now on, with the goal of providing new examples of families of vector bundles, and understanding their geometry. It is more or less clear from the table in [21, Section 5.3] that $\mathcal{B}(1)$ and $\mathcal{B}(2)$ should be irreducible, while $\mathcal{B}(3)$ and $\mathcal{B}(4)$ should have exactly two irreducible components; see [16] and [11], respectively, for the proof of the statements about $\mathcal{B}(3)$ and $\mathcal{B}(4)$. As for $\mathcal{B}(5)$, a description of all its irreducible components had been a challenge since 1980ies. In the paper, we give a complete answer to this problem - see Main Theorem 2.

For $n \geq 5$, two families of irreducible components have been studied, namely the *instanton components*, whose generic point corresponds to an instanton bundle, and the *Ein components*, whose generic point corresponds to a bundle given as cohomology of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0$$

where $b \geq a \geq 0$ and $c > a + b$. All of the components of $\mathcal{B}(n)$ for $n \leq 4$ are of either of these types; here we focus on a new family of bundles that appear as soon as $n \geq 5$.

More precisely, we study the set of vector bundles in $\mathcal{B}(a^2 + k)$ for each $a \geq 2$ and $k \geq 1$ which arise as cohomologies of monads of the form:

$$(1) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \to V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

which will be denoted by $\mathcal{G}(a, k)$. We provide a bijection between such monads and monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

where $\tilde{E}$ is a symplectic rank 4 instanton bundle of charge k. When $k = 1$, these facts are used to prove our first main result. (See Theorem 20 below.)

Main Theorem 1. *For each $a \geq 2$ not equal to 3, $\mathcal{G}(a, 1)$ is a nonsingular dense subset of a rational irreducible component of $\mathcal{B}(a^2 + 1)$ of dimension*

$$4 \cdot \binom{a+3}{3} - a - 1.$$

Our second main result provides a complete description of all the irreducible components of $\mathcal{B}(5)$.

Main Theorem 2. *The moduli space $\mathcal{B}(5)$ has exactly 3 rational irreducible components, namely: the* instanton component, *of dimension 37, which is nonsingular and consists of those bundles given as cohomology of monads of the form*

$$(2) \qquad 0 \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{12} \otimes \mathcal{O}_{\mathbb{P}^3} \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \text{ and}$$

$$(3) \qquad 0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \to V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(2) \to 0;$$

the Ein component, *nonsingular of dimension 40, which consists of those bundles given as cohomology of monads of the form*

$$(4) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0;$$

the closure of the set $\mathcal{G}(2,1)$, of dimension 37, which consists of those bundles given as cohomology of monads of the form

$$(5) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0$$

and

$$(6) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0.$$

Hartshorne and Rao proved in [21] that every stable rank 2 bundle $\mathcal{E}$ on $\mathbb{P}^3$ with Chern classes $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 5$ is the cohomology of one of the monads listed above. Rao showed in [35] that bundles given as cohomology of monads of the form (3) lie in the closure of the family of instanton bundles of charge 5, which was shown to be irreducible firstly by Coanda, Tikhomirov and Trautmann in [13]; see also [36]. The irreducibility of the family of bundles which arise as cohomology of monads of the form (4) was established by Ein in [15].

The fact that the closure of $\mathcal{G}(2,1)$ is an irreducible rational component of $\mathcal{B}(5)$ is the particular case $a = 2$ of Main Theorem 1. Finally, we show that the set of bundles given by the monads of the form (6) lies in the closure of $\mathcal{G}(2,1)$.

We now give a breaf sketch of the contents of the paper. In Section 2 we recall some general properties of monads and of symplectic instanton bundles on $\mathbb{P}^3$. We especially treat the rank 4 symplectic instantons of charge 1. Any such bundle E is described as a middle term of an exact triple with a rank 2 trivial bundle at the left hand and a null correlation rank 2 sheaf at the right hand. In Section 3 we study the set $\mathcal{G}(a,k)$ of (the isomorphism classes of) the so-called modified instanton bundles which are rank 2 bundles that arise as cohomology bundles of monads of the form (1) with $a \geq 2$ and $k \geq 1$. We show that each modified instanton appears as cohomology bundle of a monad of the form

$$(7) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \to E \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

where E is a rank 4 symplectic instanton of charge k. In the case $k = 1$, this relation will be essential for the further constructions.

In Section 4 we study the set $\mathcal{G}(a,1)$. We construct three families of symplectic monads of the form (7). The first one is the universal family, with the base scheme S, of monads with E splitting as $E = \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus N$ where N is a null correlation bundle. The second is a family, with the base scheme $\widetilde{S}$ containing S as a dense open subset, of monads E a general symplectic rank 4 instanton of charge 1. The third is a family of monads with E splitting as in the first one, but with a new base Y. All the three families inherit universal cohomology sheaves, and it is shown that the images of their corresponding modular morphisms to $\mathcal{B}(a^2 + 1)$ have the same closure $\overline{\mathcal{G}(a,1)}$ - see Propositions 14 and 15. In Section 5 the three families mentioned above are used to prove the Main Theorem 1 - see Theorem 20.

Sections 6 and 7 are devoted to the study of the monads of the form (6). In Section 6 we show that the cohomology sheaves $\mathcal{E}$ of those among such monads that are not reduced to the monads of the form (5) are closely related (by two subsequent elementary transformations - see Proposition 25) to rank 2 reflexive sheaves with Chern classes $(0, 2, 2k)$, $0 \leq k \leq 3$. A complete classification of the moduli components of these reflexive sheaves performed in Section 7 - see Propositions 26 and 27 - leads to the dimension estimate, given in Theorem 21, for the subset of the bundles $\mathcal{E}$ specified above. It follows that this subset is not a component of $\mathcal{B}(5)$, and we use this in Section 8 to prove the Main Theorem 2.

Notation and Conventions.

- In this work, $\mathbf{k}$ is an algebraically closed field of characteristic zero.
- V_n, respectively, U_n denotes a $\mathbf{k}$-vector space of dimension n.
- $\langle v \rangle$ the 1-dimensional subspace of V_n spanned by a nonzero vector $v \in V_n$.
- $\mathbf{P}(F) := \mathrm{Proj}(\mathrm{Sym}^{\bullet}_{\mathcal{O}_X} F)$ the projective spectrum of F, for a coherent $\mathcal{O}_X$-sheaf F on a given scheme X.
- $\mathcal{O}_{\mathbf{P}(F)}(1)$ the Grothendieck sheaf on $\mathbf{P}(F)$.
- $\mathbf{V}(F) := \mathrm{Spec}(\mathrm{Sym}^{\bullet}_{\mathcal{O}_X} F)$, for X and F as above.
- $\mathbb{P}^3 := P(U_4)$ the projective 3-space.
- $\mathbf{Isom}(V_n \otimes \mathcal{O}_X, F) \to X$ the principal $GL(n, \mathbf{k})$-bundle of frames of a rank n locally free $\mathcal{O}_X$-sheaf F.
- $\mathbf{X} := \mathbb{P}^3 \times X$, for a given scheme X.
- $p_X : \mathbf{X} \to X$ the projection onto the second factor, for $\mathbf{X}$ and X as above.
- $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ the morphism induced by the morphism of schemes $f : X \to Y$.
- $F_X := f^* F$, $\varphi_X := f^* \varphi : F_X \to G_X$, $\mathbf{E}_{\mathbf{X}} := \mathbf{f}^* \mathbf{E}$, for a given $\mathcal{O}_Y$-sheaf F, a given morphism $\varphi : F \to G$ of $\mathcal{O}_Y$-sheaves, a given $\mathcal{O}_{\mathbf{Y}}$-sheaf (or, a complex of sheaves) $\mathbf{E}$, and $f : X \to Y$ and $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ as above.
- $\mathbf{E}(a, 0) := \mathbf{E} \otimes \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes \mathcal{O}_X$, for X and $\mathbf{E}$ as above, and $a \in \mathbb{Z}$.
- $X \xleftarrow{g_X} X \times_Z Y \xrightarrow{f_Y} Y$ the projections of the fibre product $X \times_Z Y$ induced by the morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$.
- $H^i(F)$ the i-th cohomology group of the sheaf F on $\mathbb{P}^3$.
- $Gr(n, V_k)$ the grassmannian variety of n-dimensional subspaces of V_k.
- Variety means an integral (i. e., reduced and irreducible) scheme.
- Since we are working with rank 2 vector bundles on $\mathbb{P}^3$, and Gieseker stability is equivalent to μ-stability, we will not make any distinction between these two concepts.
- We will not make any distinction between vector bundles and locally free sheaves.
- $[E]$ the isomorphism class of a given sheaf on $\mathbb{P}^3$; in case $\mathcal{E}$ is a rank 2 stable sheaf on $\mathbb{P}^3$, $[\mathcal{E}]$ is also considered as a point in the moduli space M of stable rank 2 sheaves on $\mathbb{P}^3$.
- $\Phi_X : X \to M$, $x \mapsto [\mathbf{E}|_{\mathbb{P}^3 \times \{x\}}]$ the morphism defined by the $\mathcal{O}_{\mathbf{X}}$-sheaf $\mathbf{E}$ which is a family of stable rank 2 vector bundles on $\mathbb{P}^3$ with base X, for M as above. We call Φ_X the *modular morphism defined by the family* $\mathbf{E}$.
- $\mathcal{R}(e, n, m)$ the set of isomorphism classes of rank 2 reflexive sheaves on $\mathbb{P}^3$ with Chern classes $(c_1, c_2, c_3) = (e, n, m)$.
- $\ell(Y) := h^0(\mathcal{O}_Y)$ the length of a 0-dimensional scheme Y.
- $\mathrm{H}^1_*(E) = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^1(E(i))$ the graded cohomology module over the graded ring $\Gamma_*(\mathcal{O}_{\mathbb{P}^3}) := \bigoplus_{j \geq 0} \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(j))$.
- $(s)_0 := \{x \in X | s(x) = 0\}$ the scheme of zeroes of a section s of a given vector bundle on a scheme X.
- $\mathrm{Sp}(\mathcal{E})$ the spectrum of a vector bundle $[\mathcal{E}] \in \mathcal{B}(5)$, i. e., the nondecreasing sequence of integers $(a_1, a_2, a_3, a_4, a_5)$ uniquely defined by $\mathcal{E}$ - see [5], [20, Section 7].
- All the commutative diagrams of sheaves below which do not contain monads are assumed to have exact rows and columns. In these diagrams, the arrows $F \rightarrowtail G$, resp., $F \twoheadrightarrow G$ are shortenings for $0 \to F \to G$, resp., $F \to G \to 0$.

2. Monads and symplectic instanton bundles

Recall that a monad is a complex of vector bundles of the form:

$$(8) \qquad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

such that α is injective, and β is surjective. We call the sheaf $E := \ker \beta / \operatorname{im} \alpha$ the cohomology of the monad (8). When α is locally left invertible (i. e., it is a subbundle morphism), then E is a vector bundle.

The notion of monad is important in the study of vector bundles on $\mathbb{P}^3$ because Horrocks proved in [23] that every vector bundle on $\mathbb{P}^3$ is cohomology of a monad of the form (8) with A, B and C being sums of line bundles.

For completeness, we include in this section some useful results about monads that will be required in this work. The following lemma gives a relation between isomorphism classes of monads and its cohomology vector bundles; a proof can be found in [34, Lemma 4.1.3].

Lemma 1. *Let E and E' be, respectively, cohomology of the following monads:*

$$(9) \qquad M: \quad 0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0 \;,$$

$$(10) \qquad\qquad M' : \quad 0 \longrightarrow A' \xrightarrow{\ a'\ } B' \xrightarrow{\ b'\ } C' \longrightarrow 0 \ .$$

If one has that $\mathrm{Hom}(B, A') = \mathrm{Hom}(C, B') = \mathrm{Ext}^1(C, A') = \mathrm{Ext}^1(B, A') = \mathrm{Ext}^1(C, B') = \mathrm{Ext}^2(C, A') = 0$, *then there exists a bijection between the set of all morphisms from* E *to* E' *and the set of all morphisms of monads from (9) to (10).*

The following important corollary will be used several times in what follows, and a proof can also be found in [34, Lemma 4.1.3, Corollary 2].

Corollary 2. *Consider the monad* M *and its dual monad* $M^\vee$, *where:*

$$M : \quad 0 \longrightarrow A \xrightarrow{\ a\ } B \xrightarrow{\ b\ } C \longrightarrow 0 \ , \qquad\qquad M^\vee : \quad 0 \longrightarrow C^\vee \xrightarrow{\ b^\vee\ } B^\vee \xrightarrow{\ a^\vee\ } A^\vee \longrightarrow 0.$$

If these monads satisfy the hypothesis of Lemma 1, and there exists an isomorphism $f : E \to E^\vee$ *between its cohomology bundles such that* $f^\vee = -f$, *then there are isomorphisms* $h : C \to A^\vee$, *and* $q : B \to B^\vee$, *such that* $q^\vee = -q$, *and* $h \circ b = a^\vee \circ q$.

Recall that every locally free sheaf E on $\mathbb{P}^3$ is the cohomology of a monad of the form [23]:

$$(11) \qquad\qquad 0 \to \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(a_i) \to \oplus_{j=1}^{s} \mathcal{O}_{\mathbb{P}^3}(b_j) \to \oplus_{k=1}^{t} \mathcal{O}_{\mathbb{P}^3}(c_k) \to 0.$$

In this work we will be interested in rank 2 locally free sheaves with vanishing first Chern class. Under these conditions, we have $E^\vee \simeq E$, and this implies that $t = r$, $s = 2r + 2$, and $\{a_i\} = \{-c_k\}$. In addition, the middle entry of the monad is also self dual, so that (11) reduces to

$$0 \to \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(a_i) \to \oplus_{j=1}^{r+1} (\mathcal{O}_{\mathbb{P}^3}(b_j) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_j)) \to \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(-a_i) \to 0.$$

Finally, recall also that r coincides with the number of generators of $\mathrm{H}_*^1(E) = \bigoplus_{p \in \mathbb{Z}} \mathrm{H}^1(E(p))$ as a graded module over the ring of homogeneous polynomials in four variables, while a_i are the degrees of these generators, cf. [27, Thm. 2.3].

Instanton bundles are a particularly important class of stable rank 2 vector bundles due to their many remarkable properties and applications in mathematical physics. Besides this, instanton bundles form the only known irreducible component of the moduli space $\mathcal{B}(c)$ for every $c \in \mathbb{N}$.

In the remaining part of this section we will present the main results concerning instanton sheaves that will be used below. We start by recalling the definition of instanton sheaves on $\mathbb{P}^3$; see [25, Introduction] for further information on these objects.

Definition 3. *An* instanton sheaf *on* $\mathbb{P}^3$ *is a torsion free coherent sheaf* E *with* $c_1(E) = 0$ *satisfying the following cohomological conditions:*

$$(12) \qquad\qquad h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

The integer $n := c_2(E)$ *is called the* charge *of* E. *When* E *is locally free, we say that* E *is an* instanton bundle.

We remark that instanton bundles of rank $r > 2$ and non locally free instanton sheaves of rank $r \geq 2$ on $\mathbb{P}^3$ are not μ-semistable in general, and also the vanishing of $h^1(E(-2))$ does not imply the vanishing of $h^2(E(-2))$. The definition above is the right generalization of the usual definition of an instanton vector bundle in the sense that, applying the Beilinson spectral sequence [34, Ch. II, Thm. 3.1.4]

$$(13) \qquad\qquad \mathrm{E}_1^{pq} = H^q(E(-p-1) \otimes \Omega_{\mathbb{P}^3}^{-p}) \otimes \mathcal{O}_{\mathbb{P}^3}(p+1) \Rightarrow \mathrm{E}_\infty^{p+q} = \left\{ \begin{array}{ll} E, & p+q = 0, \\ 0, & p+q \neq 0, \end{array} \right.$$

to an arbitrary rank r instanton sheaf E of charge k, the vanishing (12) yields that E is the cohomology of a monad of the form

$$(14) \qquad\qquad 0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{r+2k} \otimes \mathcal{O}_{\mathbb{P}^3} \to V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

Note that, conversely, the cohomology of a monad as above is an instanton sheaf as defined in Definition 3, see [25, Thm. 3].

The cokernel N of any monomorphism of sheaves $\mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega_{\mathbb{P}^3}^1(1)$ is called a *null correlation sheaf*:

$$(15) \qquad\qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\ s\ } \Omega_{\mathbb{P}^3}^1(1) \to N \to 0.$$

Such sheaves are precisely the rank 2 instanton sheaves of charge 1, and are parametrized by the projective space $\mathbb{P}H^0(\Omega_{\mathbb{P}^3}^1(2)) \simeq \mathbb{P}^5$. If N is locally free, we say that N is a *null correlation bundle*. The set of non locally free null

correlation sheaves are parametrized by the Grassmanian of lines in $\mathbb{P}^3$: given a line $l \subset \mathbb{P}^3$ the corresponding null correlation sheaf N_l is defined up to an isomorphism by the exact sequence

$$(16) \qquad 0 \to N_l \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\varepsilon} \mathcal{O}_l(1) \to 0.$$

For the purposes of this paper, it is important to study rank 4 instanton bundles of charge 1. Some of the following facts might be well known, but for lack of a reference we include proofs here.

Lemma 4. *Every rank 4 instanton bundle E of charge 1 over $\mathbb{P}^3$ fits into an exact sequence:*

$$(17) \qquad 0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \to E \to N \to 0,$$

where N is a null correlation sheaf. If N is a null correlation bundle, then sequence (17) *splits. In addition,*

$$(18) \qquad h^0(E) = 2, \qquad h^i(E) = 0, \quad i \geq 1.$$

Proof. As observed in the paragraph right below Definition 3, E can be obtained as cohomology of a monad (14) for $r = 4$ and $k = 1$:

$$(19) \qquad M_E: \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0.$$

Without loss of generality, we can choose homogeneous coordinates $[x : y : z : w]$ in $\mathbb{P}^3$ and a basis in V_6, such that the map β can be written as

$$(20) \qquad \beta := \begin{pmatrix} x & y & z & w & 0 & 0 \end{pmatrix}.$$

Hence using the display of the above monad, we have that E fits into the following short exact sequence

$$(21) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \Omega(1) \to E \to 0.$$

From the above short exact sequence we can build up the following commutative diagramm

The rightmost column is the desired sequence.

If N is locally free, then $\mathrm{Ext}^1(N, \mathcal{O}_{\mathbb{P}^3}) \simeq H^1(N) = 0$, so the sequence in display (17) splits. The equality (18) follows from (17). $\qquad \square$

Remark. Assume that a bundle E is the cohomology bundle of the monad (19). Then an easy cohomological computation shows that E is a rank 4 instanton bundle of charge 1.

Note that, substituting N instead of E into the Beilinson spectral sequence (13) yields the monad for N:

$$(22) \qquad M_N: \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\overline{\alpha}} V_4 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\overline{\beta}} \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \qquad N = \ker \overline{\beta}/\mathrm{im}\, \overline{\alpha},$$

fitting together with the monad (19) in the commutative diagram

$$(23)$$

In this diagram the exact middle column is obtained from the exact triple $0 \to V_2 \to V_6 \to V_4 \to 0$ arising as the cohomology sequence of the exact triple $0 \to V_2 \otimes \Omega_{\mathbb{P}^3} \to E \otimes \Omega_{\mathbb{P}^3} \to N \otimes \Omega_{\mathbb{P}^3} \to 0$ induced by the triple (17). In addition, from (23) and (20) we obtain

$$(24) \qquad \overline{\beta} = \begin{pmatrix} x & y & z & w \end{pmatrix}.$$

Proposition 5. *Let E be a rank 4 instanton bundle E of charge 1 over $\mathbb{P}^3$, then $h^0(S^2E) = 3$, $h^1(S^2E) = 5$, $h^2(S^2E) = 0$.*

Proof. Taking the symmetric power of the sequence in display (21), we obtain that S^2E fits into the following short exact sequence:

$$0 \longrightarrow V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \Omega \longrightarrow (S^2V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) \oplus (V_2 \otimes \Omega(1)) \oplus S^2\Omega(2) \longrightarrow S^2E \longrightarrow 0.$$

From the long exact sequence of cohomology we have

$$0 \to S^2V_2 \to H^0(S^2E) \to \mathbf{k} \to \Lambda^2W^\vee \to H^1(S^2E) \to 0,$$

where W is the 4-dimensional $\mathbf{k}$−vector space such that $\mathbb{P}^3 = \mathbb{P}(W)$, and

$$0 \to H^2(S^2E) \to 0.$$

From which we conclude that $H^2(S^2E) = 0$. The map $\mathbf{k} \to \Lambda^2W^\vee$ is given by the skew-form corresponding to the morphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega(1)$, in the definition of E, and in particular is non-zero, which implies that $\mathbf{k} \to \Lambda^2W^\vee$ is injective, and therefore

$$H^0(S^2E) \simeq S^2V_2 \text{ and } H^1(S^2E) \simeq \Lambda^2W^\vee/\mathbf{k}$$

from which our result follows. $\qquad\square$

In the remaining part of this section we will discuss the existence of a symplectic structure on an arbitrary rank 4 instanton bundle of charge 1. Recall that a locally free sheaf E is said to be *symplectic* if it admits a symplectic structure, that is, there exists an isomorphism $\varphi : E \to E^\vee$, such that $\varphi^\vee = -\varphi$. A *symplectic instanton bundle* is a pair (E, φ) consisting of an instanton bundle E together with a symplectic structure φ on it; two symplectic instanton bundles (E, φ) and (E', φ') are isomorphic if there exists a bundle isomorphism $g : E \xrightarrow{\sim} E'$ such that $\varphi = g^\vee \circ \varphi' \circ g$.

Proposition 6. *Any rank 4 instanton bundle E of charge 1 admits a symplectic structure. In particular, if E splits as $E = V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N$ where N is a null correlation bundle, then any symplectic structure φ on E splits as $\varphi = \varphi_1 \oplus \varphi_2$ where φ_1 and φ_2 are symplectic structures on $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}$ and N, respectively.*

Proof. Let E be an instanton rank 4 bundle. If E splits as $E = V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N$, where N is a null correlation bundle, then $\det(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) = \det N = \mathcal{O}_{\mathbb{P}^3}$, hence both rank 2 bundles $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}$ and N admit symplectic structures, say,

$$(25) \qquad \varphi_1 : V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\simeq} V_2^\vee \otimes \mathcal{O}_{\mathbb{P}^3}, \qquad \varphi_2 : N \xrightarrow{\simeq} N^\vee.$$

Then

$$(26) \qquad \varphi = \varphi_1 \oplus \varphi_2 : E \xrightarrow{\simeq} E^\vee$$

is a symplectic structure on E. Since

$$(27) \qquad \operatorname{Hom}(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}, N) = \operatorname{Hom}(N, V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) = 0,$$

it follows immediately that any symplectic stucture on E splits as in (26).

Note also that, in view of (15)

$$(28) \qquad \operatorname{Ext}^i(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}, N) = \operatorname{Ext}^i(N, V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) = 0, \quad i \geq 1.$$

Now let E be a non-splitting instanton, i. e. $E/V_2 \otimes \mathcal{O}_{\mathbb{P}^3}$ is a null correlation sheaf N_l which is not locally free at the points of the line l given by the equations, say, $\{x = y = 0\}$. This means that the morphism $\overline{\alpha}$ in the monad (22) for $N = N_l$ is vanishes at l, so that

$$(29) \qquad \overline{\alpha} = A\begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = (\alpha_{ij}), \quad 1 \leq i \leq 4, \ 1 \leq j \leq 2,$$

where A is a (4×2)-matrix of rank 2. The condition that $\overline{\beta} \circ \overline{\alpha}$ in (22) is the zero morphism together with (29) and (24) implies that all the coefficients α_{ij} of the matrix A, except α_{12} and α_{21}, vanish and $\alpha_{12} + \alpha_{21} = 0$. Thus, taking without loss of generality $\alpha_{12} = 1$, we obtain

$$(30) \qquad \overline{\alpha} = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix}.$$

Since the cohomology sheaf of the middle monad in (23) is locally free, the morphism α in that diagram is a subbundle morphism. This together with (30) implies, again without loss of generality, that there exists a (2×2)-matrix $C = (c_{ij})$ such that

$$
(31) \qquad \alpha = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ c_{11}x + c_{12}y + z \\ c_{21}x + c_{22}y + w \end{pmatrix}.
$$

It now follows from (31) and (20) that the skew-symmetric (6×6)-matrix J of the following (2×2)-block form

$$
J = \begin{pmatrix} Q & \mathbb{O} & -C^t \\ \mathbb{O} & \mathbb{O} & -\mathbb{1} \\ C & \mathbb{1} & \mathbb{O} \end{pmatrix}, \qquad \text{where} \qquad Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

satisfies the condition $\alpha = J\beta^t$. This means that, taking $-J$ for the matrix of the symplectic form $q : V_6 \to V_6^\vee$ with respect to the above choice of the basis in V_6, we obtain that α and β as morphisms satisfy the condition $\beta = \alpha^\vee \circ q$. In other words, the monad (19) is symplectic. Then by Corollary 2 its cohomology bundle E also admits a symplectic structure. $\qquad \square$

3. MODIFIED INSTANTON MONADS

We will now study monads of the form (1), with $a \geq 2$ and $k \geq 1$:

$$
(32) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,
$$

which we call *modified instanton monads*. The set of isomorphism classes of bundles arising as cohomology of such monads will be denoted by $\mathcal{G}(a, k)$. Note that, so far, $\mathcal{G}(a, k)$ could possibly be empty.

Proposition 7. *For each $a \geq 2$ and $k \geq 1$, the family $\mathcal{G}(a, k)$ is non-empty and contains stable bundles, while every $[\mathcal{E}] \in \mathcal{G}(a, k)$ is μ-semistable. In addition, every $[\mathcal{E}] \in \mathcal{G}(a, 1)$ is stable.*

Proof. Let F be an rank 2 instanton bundle of charge k. Let $a \geq 2$ and take $\sigma \in \mathrm{H}^0(F(2a))$ such that its zero locus $X := (\sigma)_0$ is a curve; such σ always exists if F is a 't Hooft instanton bundle, for instance. Let Y be a complete intersection curve given by the intersection of two surfaces of degree a such that $X \cap Y = \emptyset$. According to [21, Lemma 4.8], there exists a bundle E and a section $\tau \in \mathrm{H}^0(E(a))$ such that $(\tau)_0 = Y \cup X$ which is given as cohomology of a monad of the form (32). In addition, since F is stable, X is not contained in any surface of degree a, hence neither is $Y \cup X$, and $\mathcal{E}$ is also stable.

It is straightforward to check that every $[\mathcal{E}] \in \mathcal{G}(a, k)$ satisfies $h^0(\mathcal{E}(-1)) = 0$, thus $\mathcal{E}$ is μ-semistable.

Now fix $k = 1$, and assume that there is $[\mathcal{E}] \in \mathcal{G}(a, 1)$ satisfying $h^0(\mathcal{E}) \neq 0$. Setting $K := \ker \beta$, it follows that $h^0(K) \neq 0$, hence the quotient $K' := K/\mathcal{O}_{\mathbb{P}^3}$ fits into the following exact sequence

$$
0 \to K' \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta'} \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0.
$$

By [8, Thm. 2.7] K' is μ-stable. However, the monomorphism $\alpha : \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to K$ induces a monomorphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to K'$; by the μ-stability of K', we should have

$$
-1 < \mu(K') = -\frac{a+1}{3} \implies a < 2,
$$

providing the desired contradiction. $\qquad \square$

Remark. Note that the space X of monads (32) is a locally closed subscheme of the affine space $A = \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1), V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3}) \times \mathrm{Hom}(V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(a) \oplus V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1))$ defined as $X = \{(\alpha, \beta) \in A \mid \alpha$ is a subbundle morphism, β is an epimorphism and $\beta \circ \alpha = 0\}$, and there is the universal cohomology bundle $\mathcal{E}$ on $\mathbf{X}$. In case $k = 1$, it follows from Proposition 7 that $\mathcal{G}(a, 1)$ is the image of X under the modular morphism $\Phi_X : X \to \mathcal{B}(a^2 + 1)$, $x \mapsto [\mathbf{E}|_{\mathbb{P}^3 \times \{x\}}]$. Thus, $\mathcal{G}(a, 1)$ is a constructible set, i. e., a disjoint union of locally closed subsets of $\mathcal{B}(a^2 + 1)$.

Next, we provide a cohomological characterization for modified instanton bundles.

Proposition 8. *A vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ is the cohomology of a monad of the form (32) if and only if $\mathrm{H}_*^1(\mathcal{E})$ has one generator in degree $-a$ and k generators in degree -1, and its Chern classes are $c_1(\mathcal{E}) = 0$, and $c_2(\mathcal{E}) = a^2 + k$.*

Proof. The "only if" part is straightforward. If $\mathcal{E}$ is a self dual vector bundle on $\mathbb{P}^3$ with one generator in degree $-a$ and k generators in degree -1, then by [27, Thm. 2.3], $\mathcal{E}$ is cohomology of a monad of the type:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \oplus_{i=1}^{2k+4} \mathcal{O}_{\mathbb{P}^3}(k_i) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

Computing the Chern class give us $c_2(\mathcal{E}) = a^2 + k - \sum_{i=1}^6 k_i^2$, since $c_2(\mathcal{E}) = a^2 + k$, we have $k_i = 0$ for all i.   □

The modified instanton bundles are also related to usual instanton bundles of higher rank in a very important way. The precise relationship is outlined in the next couple of lemmas, and then summarized in Proposition 12 below.

Lemma 9. *(i) Given a vector bundle $[\mathcal{E}] \in \mathcal{G}(a, k)$, there exists a rank 4 instanton bundle E of charge k, and sections $\sigma \in H^0(E(a))$, $\tau \in H^0(E^\vee(a))$ such that the complex:*

$$(33) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

is a monad whose cohomology coincides with $\mathcal{E}$.
(ii) The construction of the monad (33) is functorial in the sense that, if $\mathcal{E} \xrightarrow{\sim} \mathcal{E}'$, then the induced isomorphism $E \xrightarrow{\sim} E'$ extends to an isomorphism of monads

$$(34) \qquad \begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\ \sigma\ } & E & \xrightarrow{\ \tau\ } & \mathcal{O}_{\mathbb{P}^3}(a) \\ {\scriptstyle \simeq}\downarrow{\scriptstyle f} & & {\scriptstyle \simeq}\downarrow{\scriptstyle g} & & {\scriptstyle \simeq}\downarrow{\scriptstyle h} \\ \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\ \sigma'\ } & E' & \xrightarrow{\ \tau'\ } & \mathcal{O}_{\mathbb{P}^3}(a). \end{array}$$

Proof. (i) Since $a \geq 2$, there is the canonical subbundle morphism $i : V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ which, together with the morphisms α and β from the monad (32), yields a subbundle morphism $\alpha_1 := \alpha \circ i : V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3}$ and an epimorphism $\beta_1 := i^\vee \circ \beta : V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \to V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1)$. We thus obtain a new monad of type (14) with $r = 4$:

$$(35) \qquad 0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_1} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta_1} V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

the cohomology bundle

$$(36) \qquad E = \frac{\ker(\beta_1)}{\operatorname{im}(\alpha_1)}$$

of which is a rank-4 instanton, according to a remark after (14). The monads (32) and (35) fit in a commutative diagram with exact columns

$$(37)$$

Now a standard diagram chasing with diagram (37) using (36) and the relation $\mathcal{E} = \frac{\ker(\beta)}{\operatorname{im}(\alpha)}$ yields a subbundle morphism $\mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} E$ and an epimorphism $E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a)$ fitting in the monad (33) with the cohomology bundle $\mathcal{E}$.

(ii) Again, since $a \geq 2$, it follows immediately from (35) and (36) that $\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}(a), E') = \operatorname{Hom}(E, \mathcal{O}_{\mathbb{P}^3}(-a)) = \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(a), \mathcal{O}_{\mathbb{P}^3}(-a)) = \operatorname{Ext}^1(E, \mathcal{O}_{\mathbb{P}^3}(-a)) = \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(a), E') = \operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3}(a), \mathcal{O}_{\mathbb{P}^3}(-a)) = 0$ for the rank-4 instanton bundles E and E' of charge k. The statement (ii) now follows from [34, Lemma 4.1.3].   □

Lemma 10. *Given a monad (33) with E being a rank 4 instanton bundle of charge k, there is a monad of the form (32) whose cohomology coincides with the cohomology of the above monad.*

Proof. This is a diagram chasing. Namely, by (14), E is the cohomology of a monad of the form

$$(38) \qquad 0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_1} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta_1} V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

This monad can be splitted to the exact triples of bundles

$$(39) \qquad 0 \to E \to \operatorname{coker}(\alpha_1) \to V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

(40) $$0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_1} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\varepsilon} \operatorname{coker}(\alpha_1) \to 0.$$

Respectively, the monad (33) splits into the exact triples

(41) $$0 \to \ker(\tau) \to E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0, \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \to \ker(\tau) \xrightarrow{\delta} \mathcal{E} \to 0,$$

where $\mathcal{E}$ is the cohomology bundle of the monad (33). The triple (39) and the first triple (41), together with the vanishing of $\operatorname{Ext}^1(V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(a))$, yields by push-out the exact triple $0 \to \ker(\tau) \to \operatorname{coker}(\alpha_1) \xrightarrow{\gamma} V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0$ which, together with (40), yields a commutative diagram in which we set $K := \ker(\gamma \circ \varepsilon)$:

$$
\begin{array}{ccccc}
V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrowtail & K & \twoheadrightarrow & \ker(\tau) \\
\| & & \downarrow & & \downarrow \\
V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrowtail & V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\ \varepsilon\ } \twoheadrightarrow & \operatorname{coker}(\alpha_1) \\
& & \downarrow{\scriptstyle \gamma \circ \varepsilon} & & \downarrow{\scriptstyle \gamma} \\
& & V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) & = & V_k' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a).
\end{array}
$$

Similarly, the upper horizontal triple of this diagram, together with the second triple (41), yield the exact triple $0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \to K \to \mathcal{E} \to 0$ which, being combined with the middle vertical triple in this diagram, yields the monad (32) with the cohomology bundle $\mathcal{E}$. $\qquad \square$

Next, we argue that the instanton bundle E obtained in Lemma 9 comes with a natural symplectic structure.

Lemma 11. *If E is a rank 4 instanton bundle of charge k that fits in a monad of the form (33), such that its cohomology sheaf $\mathcal{E}$ is a vector bundle, then E admits a symplectic structure, and τ is determined by σ.*

Proof. Since $\mathcal{E}$ is a rank 2 vector bundle with $c_1(\mathcal{E}) = 0$, there is a (unique up to scaling) symplectic isomorphism $\varphi : \mathcal{E} \xrightarrow{\simeq} \mathcal{E}^\vee$. Now, repeating the proof of Lemma 9(ii) for $\mathcal{E}' = \mathcal{E}^\vee$, we obtain an isomorphism of monads:

$$
\begin{array}{ccccc}
\mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\ \sigma\ } \rightarrowtail & E & \xrightarrow{\ \tau\ } \twoheadrightarrow & \mathcal{O}_{\mathbb{P}^3}(a) \\
{\scriptstyle \simeq} \downarrow {\scriptstyle g} & & {\scriptstyle \simeq} \downarrow {\scriptstyle \varphi} & & {\scriptstyle \simeq} \downarrow {\scriptstyle h} \\
\mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\ \tau^\vee\ } \rightarrowtail & E^\vee & \xrightarrow{\ \sigma^\vee\ } \twoheadrightarrow & \mathcal{O}_{\mathbb{P}^3}(a)
\end{array}
$$

such that $\varphi^\vee = -\varphi$, so (E, φ) is a symplectic instanton bundle, and $\tau = \sigma^\vee \circ \varphi$. $\qquad \square$

Putting Lemmas 9, 10 and 11 together, we obtain the following statement.

Proposition 12. *A rank 2 bundle $\mathcal{E}$ belongs to $\mathcal{G}(a,k)$, i. e., $\mathcal{E}$ is the cohomology of a monad of the form (32) if and only if it is also the cohomology $\mathcal{E} = \mathcal{H}^0(A_{E,\varphi,\sigma})$ of a monad of the form:*

(42) $$A_{E,\varphi,\sigma} : \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} E \xrightarrow{\sigma^\vee \circ \varphi} \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

where (E, φ) is a rank 4 symplectic instanton bundle of charge k.

4. Set $\mathcal{G}(a,1)$ and related families of sheaves

We introduce a piece of notation which we will use below. Denote by $\mathcal{I}(k)$ the set of isomorphism classes of symplectic rank 4 instanton bundles with $c_2 = k$. As before, let V_k and V_{2k+4} be the fixed vector spaces of dimensions k and $2k+4$, respectively, and let $(\wedge^2 V_{2k+4}^\vee)^0$ be an open subset of the vector space $\wedge^2 V_{2k+4}^\vee$ consisting of nondegenerate symplectic forms on V_{2k+4}. Next, for a given morphism $\tilde{\alpha} : V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3}$ we denote by a the homomorphism $V_k \otimes U_4 \to V_{2k+4}$ corresponding to the morphism $\tilde{\alpha}$ under the isomorphism $\operatorname{Hom}(V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1), V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3}) \cong W := \operatorname{Hom}(V_k \otimes U_4, V_{2k+4})$, where $U_4 := H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee$. We will call $\tilde{\alpha}$ the *morphism associated to $a \in W$.*

Recall the description of symplectic rank 4 instantons (E, φ) in terms of symplectic monads (43) below. Namely, for a given point

$$m = (a, q) \in W \times (\wedge^2 V_{2k+4}^\vee)^0$$

consider the monad (38) in which $\tilde{\alpha}$ the morphism associated to the homomorphism a, and the morphism $\tilde{\beta}$ is such that $\tilde{\beta} = \tilde{\alpha}^t(q)$, where $\tilde{\alpha}^t(q)$ is the composition $V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{q \otimes \mathrm{id}_{\mathcal{O}_{\mathbb{P}^3}}} V_{2k+4}^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\alpha}^\vee} V_k^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$:

(43) $$A_m : \quad 0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\alpha}^t(q)} V_k^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

We call A_m a *symplectic monad*. We also will denote by $\mathcal{H}^0(A_m)$ the cohomology bundle of the monad A_m.

Consider the set $\mathcal{M}(k)$ of symplectic monads (43):

$$(44) \qquad \mathcal{M}(k) = \{(a,q) \in W \times (\wedge^2 V_{2k+4}^\vee)^0 \mid (a,q) \text{ satisfies the conditions (i)-(ii)}\}$$

where:

(i) the morphism $\tilde{\alpha}$ associated to a is a subbundle morphism,

(ii) the composition $\tilde{\alpha}^t(q) \circ \tilde{\alpha}$ is the zero morphism.

Since W is a vector space, and the condition (i), resp., (ii) is an open, resp., closed condition on the point $a \in W$, it follows that $\mathcal{M}(k)$ has a natural structure of a locally closed subscheme of the affine space $W \times \wedge^2 V_{2k+4}^\vee$.

From now on we will restrict to the case $k = 1$. Set $\widetilde{M} := \mathcal{M}(1)$. Note that the condition (i) of the definition of $\mathcal{M}(k)$ is empty in the case $k = 1$, since in this case the the vanishig of $\wedge^2(V_1^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1))$ clearly implies $\alpha^t(q) \circ \alpha = 0$. Hence, $\widetilde{M}$ is a nonempty open (hence dense) subset of the affine space $W \times \wedge^2 V_6^\vee$, where $W = \mathrm{Hom}(V_1 \otimes U_4, V_6) \simeq \mathbf{k}^{24}$. In particular, $\widetilde{M}$ is irreducible and

$$(45) \qquad \dim \widetilde{M} = \dim W + \dim \wedge^2 V_6^\vee = 45.$$

Proposition 13. *Any rank 4 instanton of charge 1 appears as a cohomology bundle of a symplectic monad*

$$(46) \qquad A_m: \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\alpha}^t(q)} \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

for some $m \in \widetilde{M}$.

Proof. Let E be a rank 4 instanton of charge 1. According to Proposition (6), E admits a symplectic structure $\varphi: E \xrightarrow{\sim} E^\vee$. It then known from [10, Section 3] that, under the condition $h^0(E) = h^1(-2) = 0$ on a symplectic bundle E, this bundle is a cohomology of a symplectic monad from $\widetilde{M}$. However, the proof given therein, works without changes under the slightly weaker conditions (12) used in the Definition 3. $\qquad\square$

On $\widetilde{\mathbf{M}} = \mathbb{P}^3 \times \widetilde{M}$ there is the universal symplectic monad $\mathcal{A}_{\widetilde{\mathbf{M}}}: \; 0 \to \mathcal{O}_{\widetilde{\mathbf{M}}}(-1,0) \xrightarrow{\boldsymbol{\alpha}} V_6 \otimes \mathcal{O}_{\widetilde{\mathbf{M}}} \xrightarrow{\boldsymbol{\alpha}^t} \mathcal{O}_{\widetilde{\mathbf{M}}}(1,0) \to 0$ with the cohomology sheaf $\widetilde{\mathbf{E}} = \ker \boldsymbol{\alpha}^t / \mathrm{im} \boldsymbol{\alpha}$. Here $\boldsymbol{\alpha}^t = \boldsymbol{\alpha}^\vee \circ \mathbf{q}_{\widetilde{\mathbf{M}}}$ and $\mathbf{q}_{\widetilde{\mathbf{M}}}: V_6 \otimes \mathcal{O}_{\widetilde{\mathbf{M}}} \xrightarrow{\sim} V_6^\vee \otimes \mathcal{O}_{\widetilde{\mathbf{M}}}$ is the tautological symplectic structure on $V_6 \otimes \mathcal{O}_{\widetilde{\mathbf{M}}}$. From now on we fix an isomorphism of the monad $\mathcal{A}_{\widetilde{\mathbf{M}}}$ with its dual monad $\mathcal{A}_{\widetilde{\mathbf{M}}}^\vee$ by the following diagram:

$$
\begin{array}{ccccc}
\mathcal{A}_{\widetilde{\mathbf{M}}}: & \mathcal{O}_{\widetilde{\mathbf{M}}}(-1,0) & \xrightarrow{\boldsymbol{\alpha}} V_6 \otimes \mathcal{O}_{\widetilde{\mathbf{M}}} & \xrightarrow{\boldsymbol{\alpha}^t} \mathcal{O}_{\widetilde{\mathbf{M}}}(1,0) \\
& {\scriptstyle -\mathrm{id}} \downarrow {\scriptstyle \simeq} & {\scriptstyle \mathbf{q}_{\widetilde{\mathbf{M}}}} \downarrow {\scriptstyle \simeq} & {\scriptstyle \mathrm{id}} \downarrow {\scriptstyle \simeq} \\
\mathcal{A}_{\widetilde{\mathbf{M}}}^\vee: & \mathcal{O}_{\widetilde{\mathbf{M}}}(-1,0) & \xrightarrow{(\boldsymbol{\alpha}^t)^\vee} V_6^\vee \otimes \mathcal{O}_{\widetilde{\mathbf{M}}} & \xrightarrow{\boldsymbol{\alpha}^\vee} \mathcal{O}_{\widetilde{\mathbf{M}}}(1,0).
\end{array}
$$

This isomorphism induces the symplectic structure

$$(47) \qquad \boldsymbol{\varphi}_{\widetilde{\mathbf{M}}}: \widetilde{\mathbf{E}} \xrightarrow{\sim} \widetilde{\mathbf{E}}^\vee, \quad \text{and} \quad E_m = \widetilde{\mathbf{E}}|_{\mathbb{P}^3 \times \{m\}}, \quad \varphi_m = \boldsymbol{\varphi}_{\widetilde{\mathbf{M}}}|_{\mathbb{P}^3 \times \{m\}}: E_m \xrightarrow{\sim} E_m^\vee, \quad m \in \widetilde{M},$$

i. e. (E_m, φ_m) is a symplectic rank 4 instanton of charge 1. Note that, by the universality of the space $\widetilde{M}$, for any symplectic rank 4 instanton (E, φ), there exists a unique point $m \in \widetilde{M}$ such that $(E, \varphi) = (E_m, \varphi_m)$, where E_m and φ_m are given by (47). It follows from (18) and the Base Change that the $\mathcal{O}_{\widetilde{M}}$-sheaf $\widetilde{\mathbf{U}} := p_{\widetilde{M}*} \widetilde{\mathbf{E}}$ is a rank 2 locally free sheaf and there is an exact triple on $\widetilde{\mathbf{M}}$, where $\mathbf{ev}$ is the canonical morphism:

$$(48) \qquad 0 \to \widetilde{\mathbf{U}}_{\widetilde{\mathbf{M}}} \xrightarrow{\mathbf{ev}} \widetilde{\mathbf{E}} \to \widetilde{\mathbf{N}} \to 0, \qquad \widetilde{\mathbf{N}} := \mathrm{coker}(\mathbf{ev}),$$

and, for any $m \in \widetilde{M}$, the restriction of this triple onto $\mathbb{P}^3 \times \{m\}$ coincides with the triple (17) for $E = E_m$. We thus have a map $\Psi: \widetilde{M} \to \mathbb{P}^5 = P(\wedge^2 V_4^\vee)$, $m \mapsto [\widetilde{\mathbf{N}}|_{\mathbb{P}^3 \times \{m\}}]$. The map Ψ has the following explicit description. Given a point $m = (a,q) \in \widetilde{M}$, consider a homomorphism $f(a,q): V_4 \xrightarrow{a} V_6 \xrightarrow{q} V_6^\vee \xrightarrow{a^\vee} V_4^\vee$. It is clearly skew-symmetric: $f(a,q) \in \wedge^2 V_4^\vee$. An easy diagram chasing with the display of the monad $\mathcal{A}_{\widetilde{\mathbf{M}}}|_{\mathbb{P}^3 \times \{m\}}$ (i. e., equivalently, of the monad (46)) using (48) shows that

$$(49) \qquad \Psi(m) = \langle f(a,q) \rangle \in P(\wedge^2 V_4^\vee),$$

so that Ψ is a well-defined morphism. By the universality of the monad $\mathcal{A}_{\widetilde{\mathbf{M}}}$, Ψ is surjective.

We next consider the set

$$M := \{m \in \widetilde{M} \mid \widetilde{\mathbf{N}}|_{\mathbb{P}^3 \times \{m\}} \text{ is locally free}\}.$$

From the definition of M it follows that it is a nonempty open subset of $\widetilde{M}$, hence it is irreducible since $\widetilde{M}$ is irreducible. Denote

$$(50) \qquad \mathbf{E} := \widetilde{\mathbf{E}}_{\mathbf{M}}, \qquad \boldsymbol{\varphi}_{\mathbf{M}} := (\varphi_{\widetilde{\mathbf{M}}})_{\mathbf{M}} : \mathbf{E} \xrightarrow{\simeq} \mathbf{E}^{\vee}, \qquad \mathbf{U} := \widetilde{\mathbf{U}}_M, \qquad \mathbf{N} := \widetilde{\mathbf{N}}_{\mathbf{M}},$$

where $\boldsymbol{\varphi}_{\widetilde{\mathbf{M}}}$ is the symplectic structure (47). Note that, by Lemma 4, for any $m \in M$ the triple (48) restricted onto $\mathbb{P}^3 \times \{m\}$ splits:

$$(51) \qquad E_m \simeq \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus N_m, \qquad m \in M,$$

where N_m is a null correlation bundle. We now show that these splittings globalize to the splitting of the triple $0 \to \mathbf{U} \to \mathbf{E} \to \mathbf{N} \to 0$ obtained from (48) by restriction onto $\mathbf{M}$:

$$(52) \qquad \mathbf{E} = \mathbf{U} \oplus \mathbf{N}.$$

Indeed, the last triple considered as an extension is given by the element in $\mathrm{Ext}^1(\mathbf{N}, \mathbf{U})$. By (27), (28) and the Base Change [31, Thm. 1.4], the sheaves $\mathcal{E}xt_{p_M}^i(\mathbf{N}, \mathbf{U})$, $i = 0, 1$, vanish, and the exact sequence relating global and relative Ext [31, (1)] yields $\mathrm{Ext}^1(\mathbf{N}, \mathbf{U}) = 0$.

Now, for $a \geq 2$ and any $m \in M$, the triple (17) twisted by $\mathcal{O}_{\mathbb{P}^3}(a)$, in which we set $E = E_m$, yields:

$$(53) \qquad h^0(E_m(a)) = 4\binom{a+3}{3} - a - 2, \qquad h^i(E_m(a)) = 0, \quad i > 0.$$

Formulas (47), (53) and the Base Change show that the sheaf

$$(54) \qquad F = p_{M*}(\mathbf{E}(a, 0))$$

is a locally free $\mathcal{O}_M$-sheaf of rank $r = h^0(E_m(a))$. Consider the scheme $T = \mathbf{P}(F^{\vee})$. By the above, T is set-theoretically described as

$$(55) \qquad T = \{(m, \langle \sigma \rangle) \mid m \in M, \ 0 \neq \sigma \in H^0(E_m(a))\},$$

and the natural projection $\rho : T \to M$, $(m, \langle \sigma \rangle) \mapsto m$ is a locally trivial $\mathbb{P}^{r-1}$-bundle. Note that, since M is an open subset of the affine space W, it follows that T is a variety, and from (45) and (53) we have

$$(56) \qquad \dim T = h^0(E_m(a)) - 1 + \dim M = 4\binom{a+3}{3} - a + 42.$$

On T and $\mathbf{M}$ we have canonical morphisms $F_T^{\vee} \xrightarrow{\mathrm{ev}} L$ and $F_{\mathbf{M}} \xrightarrow{\mathrm{can}} \mathbf{E}(a, 0)$, respectively, where $L = \mathcal{O}_{\mathbf{P}(F^{\vee})}(1)$ is the Grothendieck sheaf. Consider the composition of morphisms

$$(57) \qquad \boldsymbol{\sigma} : \mathcal{O}_{\mathbb{P}^3} \boxtimes L^{\vee} \xrightarrow{\mathrm{ev}_{\mathbf{T}}^{\vee}} F_{\mathbf{T}} \xrightarrow{\mathrm{can}_{\mathbf{T}}} \mathbf{E}_{\mathbf{T}}(a, 0).$$

By definition, for any point $(m, \mathbf{k}\sigma) \in T$ the restriction $\boldsymbol{\sigma}|_{\mathbb{P}^3 \times \{(m, \mathbf{k}\sigma)\}}$ coincides, up to a twist by $\mathcal{O}_{\mathbb{P}^3}(-a)$, with the morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m$. In view of (51) we may represent σ as $\sigma = (\sigma_1, \sigma_2)$, $\sigma_1 \in H^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(a))$, $\sigma_2 \in H^0(N_m(a))$. For the pair $\sigma = (\sigma_1, \sigma_2) \neq (0, 0)$ we will adopt in the sequel, together with the notation $\langle \sigma \rangle$, the following equivalent notation:

$$(58) \qquad [\sigma_1 : \sigma_2] := \langle \sigma \rangle = \{(\lambda \sigma_1, \lambda \sigma_2) | \lambda \in \mathbf{k}^{\times}\},$$

and also understand $[\sigma_1 : \sigma_2]$ as a point of the projective space $P(H^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(a)) \oplus H^0(N_m(a)))$. Under this notation, define an open subset S of T as

$$(59) \qquad \begin{aligned} S := \{(m, [\sigma_1 : \sigma_2]) \in T \mid (i) \ \sigma = (\sigma_1, \sigma_2) : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m \simeq \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus N_m \\ \text{is a subbundle morphism and } (ii) \ \sigma_1, \sigma_2 \neq 0\}. \end{aligned}$$

The subset S is clearly open in T. Moreover, it is nonempty. Indeed, for any point $m \in M$, E_m decomposes as in (51). Take any $a \geq 2$. Since the direct summand N_m is a null correlation bundle, it follows quickly from the triple (15) for $N = N_m$, twisted by $\mathcal{O}_{\mathbb{P}^3}(a)$, that $N_m(a)$ is generated by global sections. From this it follows easily (cf. [19, Proof of Prop. 1.4]) that a general section $\sigma_2 \in H^0(N_m(a))$ has 1-dimensional zero-locus $(\sigma_2)_0$. Next, since a general section $\sigma_1 \in H^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(a))$ has for its zero locus a complete intersection curve $(\sigma_1)_0 = D_1 \cap D_2$ for two surfaces D_1, D_2 of degree a, it follows that for general D_1 and D_2 we have $(\sigma_1)_0 \cap (\sigma_2)_0 = \emptyset$. Hence, the section $\sigma = (\sigma_1, \sigma_2) \in H^0(E_m(a))$ has no zeroes and therefore defines a subbundle morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m$.

It follows that S is irreducible and dense in T since T is irreducible. The morphism $\boldsymbol{\sigma}_{\mathbf{S}}$ is included in the monad $\boldsymbol{\mathcal{A}} := (\mathcal{A}_{\widetilde{\mathbf{M}}})_{\mathbf{S}}$ on $\mathbf{S}$:

$$(60) \qquad \boldsymbol{\mathcal{A}} : \ 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L^{\vee} \xrightarrow{\boldsymbol{\sigma}_{\mathbf{S}}} \mathbf{E}_{\mathbf{S}} \xrightarrow{\boldsymbol{\sigma}_{\mathbf{S}}^t} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L \to 0$$

where $\boldsymbol{\sigma}_{\mathbf{S}}^{t}$ is the composition $\mathbf{E_S} \xrightarrow{\boldsymbol{\varphi}_{\mathbf{S}}} \mathbf{E_S^\vee} \xrightarrow{\boldsymbol{\sigma}_{\mathbf{S}}^\vee} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L$. By construction, for any point $(m, \langle \sigma \rangle) \in S$, the restriction of the monad $\boldsymbol{\mathcal{A}}$ onto $\mathbb{P}^3 \times \{(m, \langle \sigma \rangle)\}$ is isomorphic to the monad $A_{E_m, \varphi_m, \sigma}$ in (42). Hence,

(61) $$\mathcal{H}^0(\boldsymbol{\mathcal{A}})|_{\mathbb{P}^3 \times \{(m, \langle \sigma \rangle)\}} = \mathcal{H}^0(A_{E_m, \varphi_m, \sigma}), \qquad (m, \langle \sigma \rangle) \in S.$$

In (63)-(65) below we will extend the constructions (54)-(55), (59)-(61) of the data F, T, S, $\boldsymbol{\mathcal{A}}$ and $\mathcal{H}^0(\boldsymbol{\mathcal{A}})$ over M to the constructions of the corresponding data $\widetilde{F}$, $\widetilde{T}$, $\widetilde{S}$, $\widetilde{\boldsymbol{\mathcal{A}}}$, $\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})$ over $\widetilde{M}$. As a consequence, it will follow:

(62) $$F = \widetilde{F}_{\mathbf{M}}, \qquad T = M \times_{\widetilde{M}} \widetilde{T}, \qquad S \xhookrightarrow{\text{open dense}} \widetilde{S}, \qquad \boldsymbol{\mathcal{A}} = (\widetilde{\boldsymbol{\mathcal{A}}})_{\mathbf{S}}, \qquad \mathcal{H}^0(\boldsymbol{\mathcal{A}}) = (\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}}))_{\mathbf{S}}.$$

For this, we first set

(63) $$\widetilde{F} := p_{\widetilde{M}*}(\widetilde{\mathbf{E}}(a, 0)), \qquad \widetilde{T} := \mathbf{P}(\widetilde{F}^\vee),$$

and remark that formulas (53) are still true for any $m \in \widetilde{M}$, so that the sheaf $\widetilde{F}$ is a locally free $\mathcal{O}_{\widetilde{M}}$-sheaf of rank $r = h^0(E_m(a))$ given by (53), and the scheme $\widetilde{T} := \mathbf{P}(\widetilde{F}^\vee)$ is set-theoretically described as $\widetilde{T} = \{(m, \langle \sigma \rangle) \mid m \in \widetilde{\mathcal{M}}, \; 0 \neq \sigma \in H^0(E_m(a))\}$. The natural projection $\widetilde{\rho} : \widetilde{T} \to \widetilde{M}$, $(m, \langle \sigma \rangle) \mapsto m$ is a locally trivial $\mathbb{P}^{r-1}$-bundle, so that, since $\widetilde{M}$ is an open subset of the affine space W, it follows that $\widetilde{T}$ is an irreducible variety of dimension

(64) $$\dim \widetilde{T} = h^0(E_m(a)) - 1 + \dim \widetilde{M} = 4\binom{a+3}{3} - a + 42.$$

Here, in accordance with (56), $\widetilde{T}$ and T have the same dimension. Next, we have an open subset $\widetilde{S}$ of $\widetilde{T}$ defined as $\widetilde{S} := \{(m, \langle \sigma \rangle) \in \widetilde{T} \mid \sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m$ is a subbundle morphism.$\}$ Since the condition (ii) in (59) is open, comparing the definition of $\widetilde{S}$ with (59) we obtain that S is an open subset of $T \cap \widetilde{S}$, where the intersection is taken in $\widetilde{T}$. Since S is nonempty and $\widetilde{T}$ is irreducible, the inclusion $S \xhookrightarrow{\text{open dense}} \widetilde{S}$ in (62) follows and, moreover, $\widetilde{\rho}_S : S \to M$ coincides with the projection ρ.

Next, we have the extension of the universal monad (60) from $\mathbf{S}$ to $\widetilde{\mathbf{S}}$: $\widetilde{\boldsymbol{\mathcal{A}}} : \; 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L^\vee \xrightarrow{\boldsymbol{\sigma}} \widetilde{\mathbf{E}}_{\widetilde{\mathbf{S}}} \xrightarrow{\boldsymbol{\sigma}^t} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L \to 0$, satisfying the relation similar to (61):

(65) $$\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})|_{\mathbb{P}^3 \times \{(m, \langle \sigma \rangle)\}} = \mathcal{H}^0(A_{E_m, \varphi_m, \sigma}), \qquad (m, \langle \sigma \rangle) \in \widetilde{S}.$$

Whence, the relations (62) follow from (50), (63) and the Base Change.

Consider the modular morphisms

(66) $$\Phi_S : S \to \mathcal{B}(a^2 + 1), \qquad \Phi_{\widetilde{S}} : \widetilde{S} \to \mathcal{B}(a^2 + 1),$$

defined by the families of sheaves $\mathcal{H}^0(\boldsymbol{\mathcal{A}})$ and $\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})$, respectively. The relations (62), (65), and Proposition 12 together with the irreducibility of $\widetilde{S}$ yield

Proposition 14. *(i) For $a \geq 2$, the set $\mathcal{G}(a, 1)$ of isomorphism classes of cohomology sheaves of monads (32) for $k = 1$ is the image of the modular morphism*

$$\Phi_{\widetilde{S}} : \; \widetilde{S} \to \mathcal{B}(a^2 + 1), \qquad (m, \langle \sigma \rangle) \mapsto [\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})|_{\mathbb{P}^3 \times \{(m, \langle \sigma \rangle)\}}],$$

defined by the family $\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})$ of sheaves over $\widetilde{S}$. Its closure $\overline{\mathcal{G}(a, 1)}$ in $\mathcal{B}(a^2 + 1)$ is an irreducible scheme.
(ii) The set $\mathcal{G}(a, 1)_0 := \Phi_S(S)$ is dense in $\overline{\mathcal{G}(a, 1)}$.

In the remaining part of this section we will construct a new family of monads $\mathbf{A_Y}$ on $\mathbb{P}^3$, with base Y and cohomology sheaves belonging to $\mathcal{G}(a, 1)$, for which the related modular morphism

$$\Phi_Y : \; Y \to \mathcal{B}(a^2 + 1), \qquad y \mapsto [\mathcal{H}^0(\mathbf{A_Y})|_{\mathbb{P}^3 \times \{y\}}]$$

has $\mathcal{G}(a, 1)_0$ as its image (see Proposition 15 below). This family will be used in the next Section to prove one of the main results of the paper - the rationality of $\overline{\mathcal{G}(a, 1)}$.

To construct the variety Y, consider the moduli space of $B := \mathcal{B}(1)$ of locally free null correlation bundles on $\mathbb{P}^3$. This is well known to be isomorphic to $\mathbb{P}^5 \smallsetminus G(2, 4)$, where $G(2, 4)$ is the Plücker hyperquadric (see, e.g., [34, Thm. 4.3.4]). Moreover, on $\mathbf{B} = \mathbb{P}^3 \times B$ there is the universal family $\boldsymbol{\mathcal{N}}$ of null correlation bundles. Consider the vector bundle $\boldsymbol{\mathcal{E}} = V_2 \otimes \mathcal{O}_{\mathbf{B}} \oplus \boldsymbol{\mathcal{N}}$ and denote $E_b = \boldsymbol{\mathcal{E}}|_{\mathbb{P}^3 \times \{b\}}$, $N_b = \boldsymbol{\mathcal{N}}|_{\mathbb{P}^3 \times \{b\}}$, $b \in B$, so that

(67) $$E_b = V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N_b, \qquad b \in B.$$

By linear algebra, there are canonical isomorphisms $\boldsymbol{\varphi}_{(1)} : V_2 \otimes \mathcal{O}_{\mathbf{B}} \xrightarrow{\simeq} V_2^\vee \otimes \wedge^2 V_2 \otimes \mathcal{O}_{\mathbf{B}}$ and $\boldsymbol{\varphi}_{(2)} : \boldsymbol{\mathcal{N}} \xrightarrow{\simeq} \boldsymbol{\mathcal{N}}^\vee \otimes \wedge^2 \boldsymbol{\mathcal{N}}$. The sheaf $\boldsymbol{\mathcal{N}}$ fits in the exact triple $0 \to \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_B(-1) \to \Omega^1_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_B \to \boldsymbol{\mathcal{N}} \to 0$ globalizing (15), so that $\wedge^2 \boldsymbol{\mathcal{N}} \simeq \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_B(1)$. (Here we set $\mathcal{O}_B(\pm 1) := \mathcal{O}_{G(2,4)}(\pm 1)|_B$.) Consider the varieties $B_1 := \mathbf{V}(\wedge^2 V_2^\vee \otimes \mathcal{O}_B) \smallsetminus$

$\{0 - \text{section}\} \xrightarrow{\pi_1} B$ and $B_2 := \mathbf{V}(\mathcal{O}_B(-1)) \smallsetminus \{0 - \text{section}\} \xrightarrow{\pi_2} B$. Note that the pullback of a line bundle onto its total space with the 0-section removed trivializes this bundle, we obtain $\pi_2^* \mathcal{O}_B(1) \simeq \mathcal{O}_{B_2}$, hence $(\wedge^2 \mathcal{N})_{B_2} \simeq \mathcal{O}_{B_2}$. Similarly, $(\wedge^2 V_2 \otimes \mathcal{O}_B)_{B_1} \simeq \mathcal{O}_{B_1}$. Thus, we obtain the symplectic structures

$$\boldsymbol{\varphi}_{B_1} := (\boldsymbol{\varphi}_{(1)})_{B_1} : V_2 \otimes \mathcal{O}_{B_1} \xrightarrow{\simeq} V_2^\vee \otimes \mathcal{O}_{B_1}, \qquad \boldsymbol{\varphi}_{B_2} := (\boldsymbol{\varphi}_{(2)})_{B_2} : \mathcal{N}_{B_2} \xrightarrow{\simeq} \mathcal{N}_{B_2}^\vee.$$

Consider the variety $\widetilde{B} := B_1 \times_B B_2$. On $\widetilde{\mathbf{B}}$ we obtain from $\mathcal{E}$ a vector bundle $\mathcal{E}_{\widetilde{\mathbf{B}}}$ with the symplectic structure $\boldsymbol{\varphi}_{\widetilde{\mathbf{B}}}$, where

$$\tag{68} \mathcal{E}_{\widetilde{\mathbf{B}}} = V_2 \otimes \mathcal{O}_{\widetilde{\mathbf{B}}} \oplus \mathcal{N}_{\widetilde{\mathbf{B}}}, \qquad \boldsymbol{\varphi}_{\widetilde{\mathbf{B}}} = \boldsymbol{\varphi}_1 \oplus \boldsymbol{\varphi}_2 : \mathcal{E}_{\widetilde{\mathbf{B}}} \to \mathcal{E}_{\widetilde{\mathbf{B}}}^\vee,$$

and $\boldsymbol{\varphi}_1 := (\boldsymbol{\varphi}_{B_1})_{\widetilde{\mathbf{B}}} : V_2 \otimes \mathcal{O}_{\widetilde{\mathbf{B}}} \xrightarrow{\simeq} V_2^\vee \otimes \mathcal{O}_{\widetilde{\mathbf{B}}}$, $\boldsymbol{\varphi}_2 := (\boldsymbol{\varphi}_{B_2})_{\widetilde{\mathbf{B}}} : \mathcal{N}_{\widetilde{\mathbf{B}}} \xrightarrow{\simeq} \mathcal{N}_{\widetilde{\mathbf{B}}}^\vee$. By the above, we have the following description of the varieties B_1, B_2 and $\widetilde{B}$:

$$\tag{69} \begin{aligned} B_1 &= \{(b, \varphi_1) \mid b \in B, \ \varphi_1 : V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\simeq} V_2^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \text{ is a symplectic structure}\}, \\ B_2 &= \{(b, \varphi_2) \mid b \in B, \ \varphi_2 : N_b \xrightarrow{\simeq} N_b^\vee \text{ is a symplectic structure}\}, \\ \widetilde{B} &= \{(b, \varphi_1, \varphi_2) \mid (b, \varphi_i) \in B_i, \ i = 1, 2\}. \end{aligned}$$

The following constructions (see (71)-(76)) are parallel to the constructions (59)-(61). Twisting the equality (67) by $\mathcal{O}_{\mathbb{P}^3}(a)$, we obtain as in (53): $h^0(E_b(a)) = 4\binom{a+3}{3} - a - 2$, $h^i(E_b(a)) = 0$, $i > 0$. Thus, as in (54), the sheaf $F_B = p_{B*}(\mathcal{E}(a, 0))$ is a locally free $\mathcal{O}_B$-sheaf of rank $r = h^0(E_b(a))$. Consider the variety $\mathcal{T} := \mathbf{P}(F_B^\vee)$. Similarly to (55) we have

$$\tag{70} \mathcal{T} = \{(b, \langle \sigma \rangle) \mid b \in B, \ 0 \neq \sigma \in H^0(E_b(a))\}.$$

For any point $(b, \langle \sigma \rangle) \in \mathcal{T}$ in view of (67) we may represent σ as a pair $\sigma = (\sigma_1, \sigma_2)$, $\sigma_1 \in H^0(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(a))$, $\sigma_2 \in H^0(N_b(a))$. Thus, using the notation (58) we can rewrite (70) as $\mathcal{T} = \{(b, [\sigma_1 : \sigma_2]) \mid b \in B, \ [\sigma_1 : \sigma_2] \in P(H^0(E_b(a)))\}$. On the other hand, representing σ as a morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_b$, we see that, when $(b, \langle \sigma \rangle)$ runs through $\mathcal{T}$, the morphisms σ, as in (57), globalize to a morphism $\boldsymbol{\sigma}_{\mathcal{T}} : \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L_{\mathcal{T}}^\vee \to \mathcal{E}_{\mathcal{T}}$ on $\mathcal{T}$, where $L_{\mathcal{T}}$ is the Grothendieck sheaf $\mathcal{O}_{\mathcal{T}/B}(1)$. Next, similar to (59), we define an open subset $\mathcal{S}$ of $\mathcal{T}$ as

$$\tag{71} \mathcal{S} := \{(b, [\sigma_1 : \sigma_2]) \in \mathcal{T} \mid (i) \ (\sigma_1, \sigma_2) : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m \text{ is a subbundle morphism and } (ii) \ \sigma_1, \sigma_2 \neq 0\}.$$

Note that $\mathcal{S}$ is a nonempty set. (The proof mimics that of nonemptiness of the subset M of T given in paragraph after (59).) By the Base Change, the sheaf $F_{\widetilde{B}} = p_{\widetilde{B}*}(\mathcal{E}_{\widetilde{\mathbf{B}}}(a, 0))$ is isomorphic to the sheaf $(F_B)_{\widetilde{B}}$. Therefore, from the definition of $\mathcal{T}$ it follows that the variety $\widetilde{Y} := \mathbf{P}(F_{\widetilde{B}}^\vee)$ is isomorphic to $\widetilde{B} \times_B \mathcal{T}$:

$$\tag{72} \widetilde{Y} \simeq \widetilde{B} \times_B \mathcal{T}.$$

Thus by (69) and (70) we have $\widetilde{Y} = \{(b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \mid (b, \varphi_1, \varphi_2) \in \widetilde{B}, \ [\sigma_1 : \sigma_2] \in P(H^0(E_b(a)))\}$, and the natural projection $\widetilde{Y} \to \widetilde{B}$, $(\beta, \langle \sigma \rangle) \mapsto \beta$ is a locally trivial $\mathbb{P}^{r-1}$-bundle. We now use (72) and the open subset $\mathcal{S}$ of $\mathcal{T}$ to define an open subset Y of $\widetilde{Y}$ as

$$\tag{73} Y := \widetilde{B} \times_B \mathcal{S}.$$

Here, Y is a nonempty open in $\widetilde{Y}$ since $\mathcal{S}$ is nonempty. It follows that Y is irreducible and dense in $\widetilde{Y}$ since $\widetilde{Y}$ is irreducible. In addition, using (71) and the above description of $\widetilde{Y}$ we obtain:

(74) $Y = \{(b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \in \widetilde{Y} \mid (i) \ (\sigma_1, \sigma_2) : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m \text{ is a subbundle morphism and } (ii) \ \sigma_1, \sigma_2 \neq 0\}.$

The morphism $\boldsymbol{\sigma}_{\mathbf{Y}} := (\boldsymbol{\sigma}_{\mathcal{T}})_{\mathbf{Y}}$ is included in the universal monad on $\mathbf{Y}$:

$$\tag{75} \mathbf{A}_{\mathbf{Y}} : \ 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L_{\mathbf{Y}}^\vee \xrightarrow{\boldsymbol{\sigma}_{\mathbf{Y}}} \mathcal{E}_{\mathbf{Y}} \xrightarrow{\boldsymbol{\sigma}_{\mathbf{Y}}^t} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L_{\mathbf{Y}} \to 0,$$

where $L_Y = (L_{\mathcal{T}})_Y$ and $\boldsymbol{\sigma}_{\mathbf{Y}}^t$ is the composition $\mathcal{E}_{\mathbf{Y}} \xrightarrow{\boldsymbol{\varphi}_{\mathbf{Y}}} \mathcal{E}_{\mathbf{Y}}^\vee \xrightarrow{\boldsymbol{\sigma}_{\mathbf{Y}}^\vee} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L_Y$. By construction, for any point $(\beta, \langle \sigma \rangle) \in Y$, $\beta = (b, \varphi_1, \varphi_2)$, the restriction of the monad $\mathbf{A}_{\mathbf{Y}}$ onto $\mathbb{P}^3 \times \{(\beta, \langle \sigma \rangle)\}$ is isomorphic to the monad $A_{E_b, \varphi_1 \oplus \varphi_2, \sigma}$ in (42). Hence,

$$\tag{76} \mathcal{H}^0(\mathbf{A}_{\mathbf{Y}})|_{\mathbb{P}^3 \times \{(\beta, \langle \sigma \rangle)\}} = \mathcal{H}^0(A_{E_b, \varphi_1 \oplus \varphi_2, \sigma}), \qquad (\beta, \langle \sigma \rangle) \in Y, \qquad \beta = (b, \varphi_1, \varphi_2).$$

Now consider the rank 2 the vector bundle $\mathbf{U}$ on M defined in (50) and its associated principal frame bundle

$$I := \mathbf{Isom}(V_2 \otimes \mathcal{O}_M, \mathbf{U}) \xrightarrow{\xi} M$$

together with the tautological isomorphism $V_2 \otimes \mathcal{O}_I \xrightarrow{\sim} \mathbf{U}_I$. Using this isomorphism and applying to (52) the functor $\boldsymbol{\xi}^*$ we obtain an isomorphism

$$(77) \qquad \mathbf{E_I} \cong V_2 \otimes \mathcal{O_I} \oplus \mathbf{N_I}.$$

Besides, by (50), we have a symplectic structure $\boldsymbol{\varphi_I} := (\boldsymbol{\varphi_M})_I : \mathbf{E_I} \xrightarrow{\cong} \mathbf{E_I^\vee}$ on $\mathbf{E_I}$. This symplectic structure in view of (77) splits into a direct sum of two symplectic structures

$$(78) \qquad \boldsymbol{\varphi_I} = \boldsymbol{\varphi_{I,1}} \oplus \boldsymbol{\varphi_{I,2}}, \qquad \boldsymbol{\varphi_{I,1}} : V_2 \otimes \mathcal{O_I} \xrightarrow{\cong} V_2^\vee \otimes \mathcal{O_I}, \qquad \boldsymbol{\varphi_{I,2}} : \mathbf{N_I} \xrightarrow{\cong} \mathbf{N_I^\vee}.$$

Remark that, by the defscription of the morphism Ψ given in (49), we have $\Psi(M) = B$. Now, comparing (68)-(69) with (77)-(78), we obtain a morphism

$$(79) \qquad \Gamma : I \to \widetilde{B}, \qquad x \mapsto (b, \varphi_1, \varphi_2), \qquad b = \Psi(\xi(x)), \qquad \varphi_i = \boldsymbol{\varphi_{I,i}}|_{\mathbb{P}^3 \times \{x\}}, \ i = 1, 2,$$

such that

$$(80) \qquad \mathbf{E_I} \cong (\boldsymbol{\mathcal{E}_{\widetilde{B}}})_{\mathbf{I}}, \qquad \boldsymbol{\varphi_I} \cong (\boldsymbol{\varphi_{\widetilde{B}}})_{\mathbf{I}},$$

and these isomorphisms are compatible with the direct sum decompositions (77), (78) and (68). From (79) and the surjectivity of Ψ it follows that Γ is also surjective. Set

$$(81) \qquad X := I \times_M S, \qquad Y \xleftarrow{\Gamma_Y} X \xrightarrow{\xi_S} S, \qquad F_I := p_{I*}(\mathbf{E_I}(a, 0)).$$

From (54), (80), the isomorphism $F_{\widetilde{B}} \simeq (F_B)_{\widetilde{B}}$ and the Base Change we obtain $F_I \simeq (F_{\widetilde{B}})_I$, so that, in view of (72) and the equality $T = \mathbf{P}(F^\vee)$, the variety $\widetilde{X} := \mathbf{P}(F_X^\vee)$ satisfies the isomorphisms

$$(82) \qquad I \times_M T \simeq \widetilde{X} \simeq I \times_{\widetilde{B}} \widetilde{Y}.$$

The definition of X (see (81)) and the left isomorphism (82) imply that there exists an open embedding $X \hookrightarrow \widetilde{X}$ such that $X = \widetilde{X} \times_T S$. Therefore, comparing the descriptions (74) and (59) of Y and S and using the right isomorphism (82), we obtain:

$$(83) \qquad X \simeq I \times_{\widetilde{B}} Y.$$

This together with (80) implies that $\mathbf{E_X} \cong (\boldsymbol{\mathcal{E}_Y})_{\mathbf{X}}$. Moreover, since $X = I \times_M S$, we have

$$(84) \qquad \boldsymbol{\mathcal{A}_X} \cong (\mathbf{A_Y})_{\mathbf{X}},$$

where the monads $\boldsymbol{\mathcal{A}}$ and $\mathbf{A_Y}$ were defined in (60) and (75), respectively. Consider the modular morphisms

$$(85) \qquad \Phi_X : X \to \mathcal{B}(a^2 + 1), \qquad \Phi_Y : Y \to \mathcal{B}(a^2 + 1),$$

defined by the (families of) sheaves $\mathcal{H}^0(\boldsymbol{\mathcal{A}_X})$, $\mathcal{H}^0(\mathbf{A_Y})$, respectively. From (84), (83) and (81) it follows that Φ_X factors through Γ_Y and through ξ_S as: $\Phi_X = \Phi_Y \circ \Gamma_Y = \Phi_S \circ \xi_S$. Here, $\Phi_S : S \to \mathcal{B}(a^2 + 1)$ is the modular morphism (66), ξ_S in (81) is surjective by the surjectivity of ξ, and Γ_Y is surjective as Γ is surjective. Hence,

$$(86) \qquad \mathcal{G}(a, 1)_0 = \Phi_S(S) = \Phi_Y(Y).$$

On the other hand, by Proposition 14, $\mathcal{G}(a, 1)_0$ is dense in $\overline{\mathcal{G}(a, 1)}$. We thus obtain

Proposition 15. *Let $\Phi_Y : Y \to \mathcal{B}(a^2 + 1)$ be the modular morphism defined by the family of sheaves $\mathcal{H}^0(\mathbf{A_Y})$, where $\mathbf{A_Y}$ is the monad (75). Then $\Phi_Y(Y)$ is dense in $\overline{\mathcal{G}(a, 1)}$.*

5. SERIES OF RATIONAL IRREDUCIBLE COMPONENTS OF THE MODULI SPACES $\mathcal{B}(a^2 + 1)$

Consider the variety Y defined in (73). We first will relate to Y a new variety $\mathcal{P}_a$, together with a natural projection $\pi : Y \to \mathcal{P}_a$. In this section we will relate the morphism π to the modular morphism $\Phi_Y : Y \to \mathcal{B}(a^2+1)$ (for the precise formulation see Theorem 18). For this, take any point $y \in Y$. By (74), y is a collection of data

$$y = (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]),$$

where (i) $b \in B$, (ii) $\varphi_1 : V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\cong} V_2^\vee \otimes \mathcal{O}_{\mathbb{P}^3}$ and $\varphi_2 : N_b \xrightarrow{\cong} N_b^\vee$ are symplectic isomorphisms:

$$(87) \quad \varphi_1 \in H^0(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbb{P}^3})^\vee) \smallsetminus \{0\} = \wedge^2 V_2^\vee \smallsetminus \{0\} \cong \mathbf{k}^\times, \qquad \varphi_2 \in H^0(\wedge^2 N_b^\vee) \smallsetminus \{0\} = H^0(\mathcal{O}_{\mathbb{P}^3}) \smallsetminus \{0\} \cong \mathbf{k}^\times,$$

(iii) σ_1 and σ_2 are:

$$(88) \qquad 0 \neq \sigma_1 \in H^0(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(a)) = \mathrm{Hom}(V_2^\vee, W_a), \qquad W_a := H^0(\mathcal{O}_{\mathbb{P}^3}(a)), \qquad 0 \neq \sigma_2 \in H^0(N_b(a)),$$

(iv) $\sigma = (\sigma_1, \sigma_2) : \mathcal{O}_{\mathbb{P}^3}(-a) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N_b$ is a subbundle morphism. In $\mathrm{Hom}(V_2^\vee, W_a)$ consider an open subset $\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) := \{\sigma_1 \in \mathrm{Hom}(V_2^\vee, W_a) \mid \sigma_1 : V_2^\vee \to W_a \text{ is a monomorphism}\}$. One can easily see (use the argument in paragraph after (59)) that

$$\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) = \{\sigma_1 \in \mathrm{Hom}(V_2^\vee, W_a) \mid \dim(\sigma_1)_0 = 1\}.$$

Besides, note that the group $GL(V_2)$ naturally acts on $\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a)$ via its action on $V_2^\vee$, and we have an isomorphism

(89)
$$\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a)/GL(V_2) \xrightarrow{\simeq} Gr(2, W_a)$$

and the factorization morphism

(90)
$$\tau_1 : \mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) \to Gr(2, W_a), \quad \sigma_1 \mapsto \mathrm{im}(\sigma_1 : V_2^\vee \hookrightarrow W_a).$$

Next, as it was mentioned in Section 4 (see paragraph after (59)), the set $H^0(N_b(a))^* := \{\sigma_2 \in H^0(N_b(a)) \mid \dim(\sigma_2)_0 = 1\}$ is open dense in $H^0(N_b(a))$. Besides, it is clearly invariant under the action of the group $\mathrm{Aut}(N_b(a)) = \mathbf{k}^\times$. (Recall that the null correlation bundle N_b is stable and therefore simple, i. e., $\mathrm{End}(N_b(a)) = \mathbf{k} \cdot \mathrm{id}$.) Hence,

(91)
$$P(H^0(N_b(a)))^* = H^0(N_b(a))^*/\mathrm{Aut}(N_b(a)) \overset{\mathrm{open}}{\hookrightarrow} P(H^0(N_b(a))) \simeq \mathbb{P}^r,$$

where $r = 2\binom{a+3}{3} - a - 3$, and we have the factorization morphism

(92)
$$\tau_2 : H^0(N_b(a))^* \to P(H^0(N_b(a)))^*, \qquad \sigma_2 \mapsto \langle \sigma_2 \rangle.$$

Now the above condition (iv) imposed on (σ_1, σ_2) can be rewritten in the form:

(93)
$$(\sigma_1, \sigma_2) \in H_{b,a} := \{(\sigma_1, \sigma_2) \in \mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) \times H^0(N_b(a))^* \mid (\sigma_1)_0 \cap (\sigma_2)_0 = \emptyset\}.$$

Clearly, $H_{b,a}$ is a dense open subset of $\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) \times H^0(N_b(a))^*$. This subset is invariant under the action of the group $\mathbf{k}^\times$ by homotheties. Therefore, denoting $P(H_{b,a}) := H_{b,a}/\mathbf{k}^\times$ and using (90) and (92), we obtain the factorization morphism

(94)
$$\tau : \ P(H_{b,a}) \to Gr(2, W_a) \times \mathbb{P}(H^0(N_b(a)))^*, \qquad [\sigma_1 : \sigma_2] \mapsto (\tau_1(\sigma_1), \tau_2(\sigma_2)).$$

To globalize the above pointwise (w.r.t. $b \in B$) constructions over B, set $\mathcal{K} := p_{B*}(\mathcal{N}(a, 0))$. The variety $\mathbf{P}(\mathcal{K}^\vee)$ has the description $\mathbf{P}(\mathcal{K}^\vee) = \{(b, \langle \sigma_2 \rangle) \mid b \in B, \ \langle \sigma_2 \rangle \in P(H^0(N_b(a)))\}$. Consider its dense open subset

$$\Pi_a := \{(b, \langle \sigma_2 \rangle) \in \mathbf{P}(\mathcal{K}^\vee) \mid \langle \sigma_2 \rangle \in \mathbb{P}(H^0(N_b(a)))^*\}$$

and set

(95)
$$\mathcal{G}_a := Gr(2, W_a) \times \Pi_a, \qquad \mathcal{G}_a = \{(b, V, \langle \sigma_2 \rangle) \mid V \in Gr(2, W_a), (b, \langle \sigma_2 \rangle) \in \Pi_a\}.$$

By construction, $\mathcal{G}_a$ is a rational variety. Next, remark that, comparing the definitions (71) and (93) of $\mathcal{S}$ and $H_{b,a}$, we obtain

$$\mathcal{S} = \{(b, [\sigma_1 : \sigma_2]) \mid b \in B, \ [\sigma_1 : \sigma_2] \in \mathbb{P}(H_{b,a})\}.$$

Thus, by (94), we have a well-defined morphism

(96)
$$\tau : \ \mathcal{S} \to \mathcal{G}_a, \ (b, [\sigma_1 : \sigma_2]) \mapsto (b, \tau_1(\sigma_1), \tau_2(\sigma_2)).$$

Consider the group $\tilde{G} = GL(V_2) \times \mathbf{k}^\times$, its normal subgroup $G' = \{(\rho \cdot \mathrm{id}_{V_2}, \rho) \mid \rho \in \mathbf{k}^\times\}$, and let

(97)
$$G = \tilde{G}/G'$$

be the factor group. We will use the following notation for elements of G: $[g_1 : \lambda] := (g_1, \lambda)G' = \{(\rho g_1, \rho\lambda) \mid \rho \in \mathbf{k}^\times\}$, $(g_1, \lambda) \in \tilde{G}$. The group G naturally acts on $\mathcal{S}$ as:

(98)
$$a_\mathcal{S} : \ \mathcal{S} \times G \to \mathcal{S}, \quad ((b, [\sigma_1 : \sigma_2]), [g_1 : \lambda]) \mapsto (b, [g_1 \circ \sigma_1 : \lambda\sigma_2]),$$

and formulas (89)-(96) show that $\mathcal{G}_a = \mathcal{S}/G$ and the morphism $\tau : \ \mathcal{S} \to \mathcal{G}_a$ in (96) is the quotient morphism for this action and it is a principal G-bundle. Therefore in view of (53) we have:

(99)
$$\dim \mathcal{G}_a = \dim P(H_{b,a}) + \dim B - \dim G = 4\binom{a+3}{3} - a - 2.$$

The principal G-bundle $\mathcal{S} \xrightarrow{\tau} \mathcal{G}_a$ by construction is locally trivial, hence there exists an open dense subset U of $\mathcal{G}_a$ and a section $U \xhookrightarrow{s} \mathcal{S}$ of the projection $\tau|_{\tau^{-1}(U)} : \tau^{-1}(U) \to U$:

(100)

$$
\begin{array}{ccc}
 & & \mathcal{S} \\
 & \nearrow^{s} & \downarrow{\tau} \\
U & \xhookrightarrow{\text{open}} & \mathcal{G}_a.
\end{array}
$$

Here U is rational since $\mathcal{G}_a$ is rational as it was mentioned above.

Now consider the variety $\mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbf{B}}) \oplus \wedge^2 \mathcal{N})$ together with the embeddings

$$\mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbf{B}})) \hookrightarrow \mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbf{B}}) \oplus \wedge^2 \mathcal{N}) \hookleftarrow \mathbf{P}(\wedge^2 \mathcal{N})$$

and denote $P\widetilde{B} := \mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbf{B}}) \oplus \wedge^2 \mathcal{N}) \setminus \{\mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbf{B}})) \sqcup \mathbf{P}(\wedge^2 \mathcal{N})\}$. By construction, the natural projection $P\widetilde{B} \to B$ is a locally trivial fibration with fiber

(101)
$$\mathbf{F} \simeq \mathbb{P}^1 \setminus \{2 \text{ points}\}.$$

Using the description (69) of the varieties B_1, B_2 and the notation (58) in which we put φ_1, φ_2 in place of σ_1, σ_2, we obtain $\mathbb{P}\widetilde{B} = \{(b, [\varphi_1 : \varphi_2]) \mid (b, \varphi_i) \in B_i, \; i = 1, 2\}$. Remark that the group $\mathbf{k}^\times$ naturally acts on $\widetilde{B}$ as

(102)
$$\widetilde{B} \times \mathbf{k}^\times \to \widetilde{B}, \qquad ((b, \varphi_1, \varphi_2), \lambda) \mapsto (b, \lambda\varphi_1, \lambda\varphi_2),$$

(here we use the description (69) of $\widetilde{B}$), so that

(103)
$$P\widetilde{B} = \widetilde{B}/\mathbf{k}^\times,$$

and we have the factorization morphism

(104)
$$\pi_{\widetilde{B}} : \; \widetilde{B} \to P\widetilde{B}, \qquad (b, \varphi_1, \varphi_2) \mapsto (b, [\varphi_1 : \varphi_2]).$$

Consider the varieties $PY := P\widetilde{B} \times_B \mathcal{S} = \{(b, [\varphi_1 : \varphi_2], [\sigma_1 : \sigma_2]) \mid (b, [\varphi_1 : \varphi_2]) \in P\widetilde{B}, \; (b, [\sigma_1 : \sigma_2]) \in \mathcal{S}\}$ and $\mathcal{P}_a := P\widetilde{B} \times_B \mathcal{G}_a = \{(b, [\varphi_1 : \varphi_2], V, \langle\sigma_2\rangle) \mid (b, [\varphi_1 : \varphi_2]) \in P\widetilde{B}, \; (b, V, \langle\sigma_2\rangle) \in \mathcal{G}_a\}$, where $\mathcal{G}_a$ was defined in (95). From (99) and (101) we have

(105)
$$\dim \mathcal{P}_a = \dim \mathcal{G}_a + \dim \mathbf{F} = 4\binom{a+3}{3} - a - 1.$$

Note that the local triviality of the fibration $P\mathcal{B} \to B$ implies that the natural projection

(106)
$$pr_{\mathcal{Y}} : \; PY \to \mathcal{S}$$

is a locally trivial fibration with fiber $\mathbf{F}$ given in (101).

The morphism $\pi_{\widetilde{B}}$ in (104) induces the morphism

(107)
$$\pi_Y : \; Y \to PY, \; (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \mapsto (b, [\varphi_1 : \varphi_2], [\sigma_1 : \sigma_2]),$$

and from (102)-(104) it follows that π_Y is a factorization morphism of the following $\mathbf{k}^\times$-action on Y:

(108)
$$a_Y : \; Y \times \mathbf{k}^\times \to Y, \qquad ((b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]), \lambda) \mapsto (b, \lambda\varphi_1, \lambda\varphi_2, [\sigma_1 : \sigma_2]).$$

Respectively, the morphism $\tau : Y_a \to \mathcal{G}_a$ defined in (96) induces a morphism

(109)
$$\tau_Y : \; PY \to \mathcal{P}_a, \; (b, [\varphi_1 : \varphi_2], [\sigma_1 : \sigma_2]) \mapsto (b, [\varphi_1 : \varphi_2], \tau_1(\sigma_1), \tau_2(\sigma_2)).$$

Define the morphism $\pi : Y \to \mathcal{P}_a$ as the composition

(110)
$$\pi = \tau_Y \circ \pi_Y : \; Y \to \mathcal{P}_a, \qquad (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \mapsto (b, [\varphi_1 : \varphi_2], \tau_1(\sigma_1), \tau_2(\sigma_2)).$$

We will now proceed to the study of the fibers of the morphism π.

Definition 16. *Introduce on Y the following equivalence relation:*

(111)
$$y = (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \; \sim \; (\tilde{b}, \tilde{\varphi}_1, \tilde{\varphi}_2, [\tilde{\sigma}_1 : \tilde{\sigma}_2]) = \tilde{y}$$

if there exists an isomorphism of symplectic monads A_y and $A_{\tilde{y}}$, i. e., a commutative diagram with rows A_y and $A_{\tilde{y}}$:

(112)
$$
\begin{array}{ccccc}
A_y : & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{(\sigma_1, \sigma_2)} & V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N_b & \xrightarrow{(\sigma_1^\vee \circ \varphi_1, \sigma_2^\vee \circ \varphi_2)} & \mathcal{O}_{\mathbb{P}^3}(a) \\
 & h_- \downarrow \simeq & & (g_1, g_2) \downarrow \simeq & & h_+ \downarrow \simeq \\
A_{\tilde{y}} : & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{(\tilde{\sigma}_1, \tilde{\sigma}_2)} & V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N_{\tilde{b}} & \xrightarrow{(\tilde{\sigma}_1^\vee \circ \tilde{\varphi}_1, \tilde{\sigma}_2^\vee \circ \tilde{\varphi}_2)} & \mathcal{O}_{\mathbb{P}^3}(a).
\end{array}
$$

We denote by $[y] = [b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]]$ *the equivalence class of a point* $y = (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \in Y$ *under this equivalence relation.*

Note that, in diagram (112), one has

(113) $$g_1 \in \mathrm{Isom}(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}, V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) \cong GL(V_2);$$

and $g_2 \in \mathrm{Isom}(N_b, N_{\tilde{b}})$ which in view of the stability of N_b implies that

(114) $$b = \tilde{b}, \qquad g_2 = \lambda \cdot \mathrm{id}_{N_b}, \qquad \lambda \in \mathbf{k}^\times;$$

besides, the isomorphisms h_-, h_+ are multiplications by some constants μ, $\nu \in \mathbf{k}^\times$, respectively:

(115) $$h_- = \mu \cdot \mathrm{id}_{\mathcal{O}_{\mathbb{P}^3}(-a)}, \qquad h_+ = \nu \cdot \mathrm{id}_{\mathcal{O}_{\mathbb{P}^3}(a)}.$$

Furthermore, in view of (87), (113), (114) and the symplecticity of φ_1, φ_2, we have in (112)

(116) $$\tilde{\varphi}_1 = \lambda_1 \varphi_1, \quad \tilde{\varphi}_2 = \lambda_2 \varphi_2, \quad \lambda_1, \lambda_2 \in \mathbf{k}^\times, \quad g_1^\vee \circ \varphi_1 \circ g_1 = \det(g_1)\varphi_1, \quad g_2^\vee \circ \varphi_2 \circ g_2 = \lambda^2 \varphi_2.$$

The leftmost square of diagram (112) together with (115) impliies:

(117) $$\tilde{\sigma}_1 = \frac{1}{\mu} g_1 \circ \sigma_1, \qquad \tilde{\sigma}_2 = \frac{\lambda}{\mu} \sigma_2.$$

Respectively, the rightmost square of diagram (112) yields $\nu \sigma_1^\vee \circ \varphi_1 = \tilde{\sigma}_1^\vee \circ \tilde{\varphi}_1 \circ g_1$, $\nu \sigma_2^\vee \circ \varphi_2 = \lambda \tilde{\sigma}_2^\vee \circ \tilde{\varphi}_2$. Substituting (115)-(117) into the last equalities we obtain the relations $\nu = \frac{\lambda_1 \det(g_1)}{\mu}$ and $\nu = \frac{\lambda_2 \lambda^2}{\mu}$. Whence $\lambda_1 \det(g_1) = \lambda_2 \lambda^2$. This relation shows that the G-action (98) on $\mathcal{S}$ lifts to the following G-action on PY:

(118) $$a_{PY}: \; PY \times G \to PY, \quad ((b, [\varphi_1 : \varphi_2], [\sigma_1 : \sigma_2]), [g_1 : \lambda]) \mapsto (b, [\frac{\varphi_1}{\det(g_1)} : \frac{\varphi_2}{\lambda^2}], [g_1 \circ \sigma_1 : \lambda \sigma_2]).$$

Thus, $\mathcal{P}_a = PY/G$ and the morphism

(119) $$\tau_Y: \; PY \to \mathcal{P}_a$$

in (109) is the quotient morphism for this action and it is a locally trivial principal G-bundle. We therefore have a commutative diagram

$$
\begin{array}{ccc}
PY & \xrightarrow{\;\tau_Y\;} & \mathcal{P}_a \\
\downarrow{\scriptstyle pr_\mathcal{Y}} & & \downarrow{\scriptstyle pr_\mathcal{G}} \\
\mathcal{S} & \xrightarrow{\;\tau\;} & \mathcal{G}_a,
\end{array}
$$

where $pr_\mathcal{G}$ is a natural projection. Since by (106) the morphism $pr_\mathcal{Y}: \; PY \to \mathcal{S}$ is a locally trivial fibration with fibre $\mathbf{F}$, the open subset U of $\mathcal{G}_a$ and the section $U \xhookrightarrow{s} \mathcal{S}$ in the diagram (100), after possible shrinking U, can be lifted to an open section $\mathbf{F} \times U \xhookrightarrow{\tilde{s}} PY$ of the projection $\tau_Y : PY \to \mathcal{P}_a$:

$$
\begin{array}{ccc}
 & & PY \\
 & \nearrow{\scriptstyle \tilde{s}} & \downarrow{\scriptstyle \tau_Y} \\
\mathbf{F} \times U & \xhookrightarrow{\text{open}} & \mathcal{P}_a.
\end{array}
$$

Since $\mathbf{F}$ is rational by (101) and U is rational, it follows that

(120) $$\mathcal{P}_a \quad \text{is rational.}$$

Next, from (107)-(108), (118) and (119) it follows that the morphism $\pi : Y \to \mathcal{P}_a$ in (110) is the quotient morphism of the following action of the group $\overline{G} := G \times \mathbf{k}^\times$ on Y, where $G = \widetilde{G}/G'$ was defined in (97):

(121) $$a_Y: \; Y \times \overline{G} \to Y, \quad ((b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]), ([g_1 : \lambda], \mu)) \mapsto (b, \frac{\mu \varphi_1}{\det(g_1)}, \frac{\mu \varphi_2}{\lambda^2}, [g_1 \circ \sigma_1 : \lambda \sigma_2]), \qquad \overline{G} = G \times \mathbf{k}^\times.$$

Moreover,

(122) $$\pi : Y \to \mathcal{P}_a = Y/\overline{G} \quad \text{is a principal } \overline{G}\text{-bundle,}$$

and computations (113)-(118) show that the equivalence class $[y]$ of any point $y \in Y$ is the $\overline{G}$-orbit of y:

(123) $$[y] = a_Y(\{y\} \times \overline{G}) = \pi^{-1}(\pi(y)), \qquad y \in Y.$$

In other words, $\mathcal{P}_a$ is the set of equivalence classes of points of Y:

(124) $$\mathcal{P}_a = \{[y] \mid y \in Y\}.$$

Remark that, by Corollary 2, the equality $[y] = [\tilde{y}]$, i. e. the isomorphism of symplectic monads A_y and $A_{\tilde{y}}$ in (112) is equivalent to the isomorphism of their cohomology rank 2 bundles as symplectic bundles $(\mathcal{H}^0(A_y), \psi_y)$ and $(\mathcal{H}^0(A_{\tilde{y}}), \psi_{\tilde{y}})$, i. e., to the commutativity of the diagram

(125)
$$
\begin{array}{ccc}
\mathcal{H}^0(A_y) & \xrightarrow[\simeq]{\psi_y} & \mathcal{H}^0(A_y)^\vee \\
{\scriptstyle f}\downarrow{\scriptstyle\simeq} & & {\scriptstyle f^\vee}\uparrow{\scriptstyle\simeq} \\
\mathcal{H}^0(A_{\tilde{y}}) & \xrightarrow[\simeq]{\psi_{\tilde{y}}} & \mathcal{H}^0(A_{\tilde{y}})^\vee.
\end{array}
$$

Here ψ_y, respectively, $\psi_{\tilde{y}}$, is a symplectic isomorphism induced by the symplectic isomorphism of the monad A_y with its dual $A_y^\vee$, respectively, of $A_{\tilde{y}}$ with $A_{\tilde{y}}^\vee$. Thus, denoting by $[\mathcal{H}^0(A_y), \psi_y]$ the isomorphism class of the pair $(\mathcal{H}^0(A_y), \psi_y)$, we have:

(126)
$$[y] = [\mathcal{H}^0(A_y), \psi_y] = [\mathcal{H}^0(A_y)].$$

This together with (122)-(124) shows that the modular morphism

$$\Phi_Y : \ Y \to \mathcal{B}(a^2 + 1), \quad y \mapsto [\mathcal{H}^0(A_y)]$$

factors through an injective map $\Theta : \mathcal{P}_a \to \mathcal{B}(a^2 + 1)$, i. e.

(127)
$$\Phi_Y = \Theta \circ \pi.$$

Since Y is clearly smooth, the map Θ is actually a morphism. This outcomes from the following well known general result. (For the convenience of the reader we give its proof here.)

Lemma 17. *Let X, Y, Z be quasiprojective varieties with Y smooth, and let $a : X \to Y$ and $b : X \to Z$ be morphisms such that a is surjective and b is constant on the fibers of a. Then there exists a morphism $f : Y \to Z$ such that $b = f \circ a$.*

Proof. Consider the morphism $g : X \to Y \times Z$, $x \mapsto (a(x), b(x))$, and let $Y \xleftarrow{a'} Y \times Z \xrightarrow{b'} Z$ be the projections onto factors so that $a = a' \circ g$ and $b = b' \circ g$. Since b is constant on the fibers of p, it follows that $\tilde{a} := a'|_{g(X)} : g(X) \to Y$ is a bijection. Therefore, as Y is smooth, $\tilde{a}$ is an isomorphism (see, e.g., [S, Ch.2, Section 4.4, Thm. 2.16]). The desired morphism f is now the composition $f = b' \circ \tilde{a}^{-1}$. $\square$

Now Proposition 15 together with (105), (120), (122) and (127) yields

Theorem 18. *There exists an injective morphism $\Theta : \mathcal{P}_a \hookrightarrow \mathcal{B}(a^2 + 1)$ such that the modular morphism $\Phi_Y : Y \to \mathcal{B}(a^2 + 1)$ factorizes as*

(128)
$$\Phi_Y : \ Y \xrightarrow{\pi} \mathcal{P}_a \xrightarrow{\Theta} \mathcal{B}(a^2 + 1),$$

where $\pi : Y \to \mathcal{P}_a$ is a principal $\overline{G}$-bundle with the group $\overline{G}$ defined in (121). The variety $\overline{\mathcal{G}(a,1)}$ containing the rational variety $\mathcal{G}(a,1)_0 = \Theta(\mathcal{P}_a)$ as a dense subset is rational of dimension $4\binom{a+3}{3} - a - 1$.

We next obtain the following important formula.

Lemma 19. *For every $[\mathcal{E}] \in \mathcal{G}(a,1)_0$ with $a \geq 2$, it holds*

$$h^1(\mathcal{E}nd(\mathcal{E})) = 4 \cdot \binom{a+3}{3} - a - 1 + \varepsilon(a),$$

where $\varepsilon(a) = 1$ when $a = 3$, and $\varepsilon(a) = 0$ when $a \neq 3$.

Proof. Since $\mathcal{E}$ is a self dual rank 2 bundle, we have $\mathcal{E}nd(\mathcal{E}) \simeq S^2\mathcal{E} \oplus \Lambda^2\mathcal{E} = S^2\mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^3}$, thus $h^1(\mathcal{E}nd(\mathcal{E})) = h^1(S^2\mathcal{E})$. We will compute the latter.

By the definition of $\mathcal{G}(a,1)_0$ (see Proposition 14.(ii), (59) and (61)), $\mathcal{E}$ is the cohomology of a complex $M^\bullet$ with terms $M^{-1} = \mathcal{O}_{\mathbb{P}^3}(-a)$, $M^0 = E \simeq \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus N$, $M^1 = \mathcal{O}_{\mathbb{P}^3}(a)$. Proceed to the double complex $M^\bullet \otimes M^\bullet$ and to its total complex $T^\bullet$. The symmetric part of $T^\bullet$ is the monad $0 \to E(-a) \to S^2E \oplus \mathcal{O}_{\mathbb{P}^3} \to E(a) \to 0$, whose cohomology sheaf is isomorphic to $S^2\mathcal{E}$. Therefore this monad can be broken into two short exact sequences

$$0 \to K \to S^2E \oplus \mathcal{O}_{\mathbb{P}^3} \to E(a) \to 0 \quad \text{and} \quad 0 \to E(-a) \to K \to S^2\mathcal{E} \to 0.$$

Since $h^0(E(-a)) = h^0(S^2\mathcal{E}) = 0$, it follows that $h^0(K) = 0$; in addition, $h^1(E(a)) = h^2(S^2E \oplus \mathcal{O}_{\mathbb{P}^3}) = 0$ (use Proposition 5) implies that $h^2(K) = 0$. It then follows in view of the splitting $E \simeq \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus N$ that

(129)
$$h^1(S^2\mathcal{E}) = h^1(K) + h^2(E(-a)) = h^1(K) + \varepsilon(a), \qquad \varepsilon(a) := h^2(N(-a)),$$

since $h^1(E(-a)) = 0$ for $a \geq 2$.

To complete our calculation, consider the exact sequence

$$0 \to H^0(S^2 E \oplus \mathcal{O}_{\mathbb{P}^3}) \to H^0(E(a)) \to H^1(K) \to H^1(S^2 E \oplus \mathcal{O}_{\mathbb{P}^3}) \to 0.$$

Since $h^0(S^2 E \oplus \mathcal{O}_{\mathbb{P}^3}) = 4$ and $h^1(S^2 E \oplus \mathcal{O}_{\mathbb{P}^3}) = 5$ by Proposition 5, we conclude that

$$h^1(K) = h^0(E(a)) + 1 = h^0(N(a)) + V_2 \otimes h^0(\mathcal{O}_{\mathbb{P}^3}(a)) + 1,$$

which, together with the equality in equation (129), yields the desired formula. □

It is interesting to observe that the right hand side of the formula in Lemma 19 yields the value of $h^1(\mathcal{E}nd(\mathcal{E}))$ expected by the deformation theory when $a = 2$ and $a = 3$, respectively 37 and 77; when $a \geq 4$, one can check that $4 \cdot \binom{a+3}{3} - a - 1 > 8(a^2 + 1) - 3$.

Noting that, in view of Theorem 18, the dimension of $\overline{\mathcal{G}(a,1)}$ equals $h^1(\mathcal{E}nd(E))$ for $a = 2$ and $a \geq 4$, as calculated in Lemma 19, and using Proposition 14, we have therefore completed the proof of the first main result of this paper.

Theorem 20. *For $a = 2$ and $a \geq 4$, the rank 2 bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

fill out a dense subset of a rational irreducible component of $\mathcal{B}(a^2 + 1)$ of dimension

$$4 \cdot \binom{a+3}{3} - a - 1.$$

In particular, for the case $a = 2$, we conclude that rank 2 bundles given as cohomology of monads of the form (5) yield a dense subset of an irreducible component of $\mathcal{B}(5)$ with expected dimension 37.

6. COHOMOLOGY BUNDLE $\mathcal{E}$ OF THE MONAD OF TYPE (6) AND THE RELATED REFLEXIVE SHEAF $\mathcal{F}$

Consider the set

$$\mathcal{H} = \{[\mathcal{E}] \in \mathcal{B}(5) \mid \mathcal{E} \text{ is cohomology of a monad of type (6)}\}.$$

It is known that $\mathcal{H} \neq \emptyset$ - see [21, Table 5.3, $c_2 = 5$, Case (2).ii]. Note that the set $\mathcal{H}$ is a constructible subset of $\mathcal{B}(5)$, as well as $\mathcal{G}(2,1)$ (see Remark after Proposition 7). The aim of this and the subsequent sections is to prove

Theorem 21. *The set $\mathcal{H}$ satisfies the condition $\dim(\mathcal{H} \smallsetminus (\mathcal{G}(2,1) \cap \mathcal{H})) \leq 36$. Its closure in $\mathcal{B}(5)$ does not constitute a component of $\mathcal{B}(5)$.*

In this section we will relate the vector bundle $[\mathcal{E}] \in \mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H})$ to a rank 2 reflexive sheaf $\mathcal{F}$ with Chern classes $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = 2$ and $c_3(\mathcal{F}) = 2k$, $0 \leq k \leq 6$, which appears as a middle cohomology of a left-exact complex $K^\bullet$ (see (154)) induced by the monad of type (6) defining $\mathcal{E}$. This relation will be established in Proposition 25. We will then use it in Section 7 to prove Theorem 21.

Let $[\mathcal{E}] \in \mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H})$ be the cohomology bundle of the monad of the form (6):

$$M^\bullet: \quad 0 \to M^{-1} \xrightarrow{\alpha} M^0 \xrightarrow{\beta} M^1 \to 0,$$

(130) $\quad M^{-1} = \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \quad M^0 = \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1),$

$\quad M^1 = V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2).$

Since the bundle $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ is a uniquely defined subbundle of the bundle M^{-1} (respectively, $V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ is a uniquely defined quotient bundle of M^1), we obtain a commutative diagram in which α_0 and β_0 are the induced morphisms:

(131)

Here the induced monad

$$(132) \qquad 0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_0} M^0 \xrightarrow{\beta_0} V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

has the rank 4 cohomology bundle

$$(133) \qquad E = \frac{\ker \beta_0}{\operatorname{im} \alpha_0}.$$

Mimicking now the argument with diagram (37), we obtain that there exist a subbunle morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-2) \to E$ and an epimorphism $\tau : E \to \mathcal{O}_{\mathbb{P}^3}(2)$ which yield the monad the the cohomology bundle $\mathcal{E}$:

$$(134) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\sigma} E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(2) \to 0, \qquad \mathcal{E} = \ker \tau / \operatorname{im} \sigma.$$

Since there is a uniquely defined (up to a scalar multiple) quotient morphism $M^0 \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)$, we have a well-defined morphism

$$(135) \qquad \tilde{\alpha} : \ V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_0} M^0 \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)$$

and, dually, a well-defined morphism

$$(136) \qquad \tilde{\beta} : \ \mathcal{O}_{\mathbb{P}^3}(1) \hookrightarrow M^0 \xrightarrow{\beta_0} V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1).$$

Assume that both $\tilde{\alpha}$ and $\tilde{\beta}$ are nonzero morphisms. Then a standard diagram chasing shows that, in the monad (132), one can split out a direct summand $\mathcal{O}_{\mathbb{P}^3}(-1)$ from $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ and M^0, respectively, split out a direct summand $\mathcal{O}_{\mathbb{P}^3}(1)$ from M^0 and $V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1)$, without changing its cohomology bundle E. Thus, the monad (132) reduces to a monad

$$(137) \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha'} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta'} \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \qquad E = \frac{\ker \beta'}{\operatorname{im} \alpha'}.$$

Now by the remark after Lemma 4, E is a rank 4 instanton bundle, so that, by (134) and Lemma 10, $\mathcal{E}$ is the cohomology bundle of the monad (32) for $a = 2$ and $k = 1$. This means that $\mathcal{E} \in \mathcal{G}(2,1) \cap \mathcal{H}$, contrary to the assumption on $\mathcal{E}$.

We thus may assume that either (a) $\tilde{\alpha} = 0$, $\tilde{\beta} \neq 0$, or (b) $\tilde{\alpha} = \tilde{\beta} = 0$. (We omit the case $\tilde{\alpha} \neq 0$, $\tilde{\beta} = 0$, since it is completely similar to the case (a).)

(a) Case $\tilde{\alpha} = 0$, $\tilde{\beta} \neq 0$. We are going to show that this case is impossible.

First, note that, since $\tilde{\beta} \neq 0$, we may as above split out a direct summand $\mathcal{O}_{\mathbb{P}^3}(1)$ from the middle term and the righthand term of the monad (132), without changing its cohomology bundle E. Thus, this monad reduces to a monad

$$(138) \qquad 0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha'} \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta'} \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \qquad E = \frac{\ker \beta'}{\operatorname{im} \alpha'}.$$

Next, the condition $\tilde{\alpha} = 0$ means that the subbundle morphism α' in (132) factors through a subbundle morphism α'' in the commutative diagram

$$(139)$$

where $F_4 := \operatorname{coker} \alpha''$ and $F_5 := \operatorname{coker} \alpha'$ are vector bundles of rank 4 and 5, respectively. From this diagram it follows immediately that $\mathcal{O}_{\mathbb{P}^3}(-1)$ splits out as a direct summand of F_5:

$$(140) \qquad F_5 \cong \mathcal{O}_{\mathbb{P}^3}(-1) \oplus F_4.$$

The monad (138) and the diagram (139) yield a commutative diagram

(141)

$$
\begin{array}{ccccc}
F_3 & \rightarrowtail & F_4 & \xrightarrow{\eta \circ \lambda} & A \\
\downarrow & & \downarrow{\scriptstyle \lambda} & & \downarrow \\
E & \xrightarrow{\nu} & F_5 & \xrightarrow{\eta} & \mathcal{O}_{\mathbb{P}^3}(1) \\
\downarrow{\scriptstyle \mu \circ \nu} & & \downarrow{\scriptstyle \mu} & & \downarrow \\
B & \rightarrowtail & \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\bar{\eta}} & C,
\end{array}
$$

where $F_3 := \ker(\eta \circ \lambda)$, $A := F_4/F_3$, $B := E/F_3$, $C := \mathcal{O}_{\mathbb{P}^3}(1)/A$. Here $A \neq 0$, since otherwise $C \simeq \mathcal{O}_{\mathbb{P}^3}(1)$, and then $\bar{\eta}$ is not surjective, contrary to (141). Hence, C is a torsion sheaf, and A, B and F_3 are torsion free sheaves of rank 1, 1 and 4, respectively. Therefore, the diagram (141) implies that $c_1(F_4) - c_1(E) = 2c_1(\mathcal{O}_{\mathbb{P}^3}(1))$. On the other hand, in view of (140) we have a well-defined injective morphism $\rho : E \xrightarrow{\nu} F_5 \xrightarrow{pr_2} F_4$ such that, by the Snake Lemma, $Q := \mathrm{coker}\rho \cong A/B$ is a torsion sheaf. In addition, by the above equality, $c_1(Q) = 2c_1(\mathcal{O}_{\mathbb{P}^3}(1)) \neq 0$, i. e., $Q \neq 0$. However, (141) and the Snake Lemma yield a commutative diagram

(142)

$$
\begin{array}{ccccc}
E & \xrightarrow{\rho} & F_4 & \twoheadrightarrow & Q \\
\| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \bar{i}} \\
E & \xrightarrow{\nu} & \mathcal{O}_{\mathbb{P}^3}(-1) \oplus F_4 & \xrightarrow{\eta} & \mathcal{O}_{\mathbb{P}^3}(1) \\
& & \downarrow & & \downarrow \\
& & \mathcal{O}_{\mathbb{P}^3}(-1) & = & \mathcal{O}_{\mathbb{P}^3}(1)/\bar{i}(Q),
\end{array}
$$

where i is the inclusion of the direct summand and $\bar{i}$ is the induced morphism. But the torsion sheaf Q is not a subsheaf of $\mathcal{O}_{\mathbb{P}^3}(1)$, and we obtain a contradiction, as claimed.

Summarizing the above arguments, we see that the bundle $[\mathcal{E}] \in \mathcal{H}$ is the cohomology $\mathcal{H}^0(M^\bullet)$ of a monad $M^\bullet$ of the form (6) satisfying the condition (a): $(\tilde{\alpha}, \tilde{\beta}) \neq (0,0)$, then $M^\bullet$ is reducible to a monad of the form (5), i. e. $[\mathcal{E}] \in \mathcal{H} \cap \mathcal{G}(2,1)$. Thus, denoting

(143) $\qquad \mathcal{H}_0 := \{ [\mathcal{E}] \in \mathcal{H} \mid \mathcal{E} = \mathcal{H}^0(M^\bullet), \text{where } M^\bullet \text{ satisfies the condition (b): } \tilde{\alpha} = \tilde{\beta} = 0 \},$

we obtain

(144) $$\mathcal{H} \smallsetminus (\mathcal{H} \cap \mathcal{G}(2,1)) \subset \mathcal{H}_0.$$

We thus proceed to the study of the case $\tilde{\alpha} = \tilde{\beta} = 0$.

(b) Case $\tilde{\alpha} = \tilde{\beta} = 0$.

First, consider the commutative diagram

(145)

$$
\begin{array}{ccccc}
\mathcal{O}_{\mathbb{P}^3}(1) & = & \mathcal{O}_{\mathbb{P}^3}(1) & & \\
\downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} & & \\
V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) & \xrightarrow{i_0} & M^0 & \xrightarrow{g_0} & \mathcal{O}_{\mathbb{P}^3}(1) \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & & \| \\
V_6 \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{i} & V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{g} & \mathcal{O}_{\mathbb{P}^3}(1)
\end{array}
$$

and the exact triples following from (132) and (133)

(146) $$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_0} M^0 \xrightarrow{c_0} C_0 \to 0,$$

(147) $$0 \to E \xrightarrow{d_0} C_0 \xrightarrow{e_0} V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \qquad C_0 := \mathrm{coker}\, \alpha_0, \qquad \beta_0 = e_0 \circ c_0.$$

The condition $\tilde{\alpha} = 0$ implies that there exists a subbundle morphism $0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_1} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ such that

(148) $$\alpha_0 = i_0 \circ \alpha_1.$$

Setting $C := \operatorname{coker}(h_0 \circ \alpha_0)$, $C_1 := \operatorname{coker} \alpha_1$, $\alpha_2 := h_1 \circ \alpha_1$, $C_2 := \operatorname{coker} \alpha_2$, we obtain from (145)-(146) and (148) an induced commutative diagram

(149)

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(1) & === & \mathcal{O}_{\mathbb{P}^3}(1) \\
\downarrow{\scriptstyle \bar{j}_1} & & \downarrow{\scriptstyle \bar{j}_0} \\
C_1 \xrightarrow{\ \bar{i}_0\ } & C_0 \xrightarrow{\ \bar{g}_0\ } & \mathcal{O}_{\mathbb{P}^3}(-1) \\
\downarrow{\scriptstyle \bar{h}_1} & \downarrow{\scriptstyle \bar{h}_0} & \| \\
C_2 \xrightarrow{\ \bar{i}\ } & C \xrightarrow{\ \bar{g}\ } & \mathcal{O}_{\mathbb{P}^3}(-1)
\end{array}
$$

and an exact triple

(150)
$$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_2} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{c_2} C_2 \to 0.$$

From the condition $\tilde{\beta} = 0$ and diagram chasing it follows that there exists an injective morphism $j: \mathcal{O}_{\mathbb{P}^3}(1) \to E$ such that $\bar{j}_0 = d_0 \circ j$. From this relation and (147), (149) and (150) by diagram chasing we obtain the folowing data:

1) an exact triple

(151)
$$0 \to \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{j} E \xrightarrow{h} E_3 \to 0, \qquad E_3 := \operatorname{coker} j,$$

2) a commutative diagram

(152)

$$
\begin{array}{ccc}
\mathcal{F} \longrightarrow & E_3 \xrightarrow{\ \varepsilon\ } & \mathcal{O}_{\mathbb{P}^3}(-1) \\
\downarrow & \downarrow{\scriptstyle d} & \| \\
C_2 \xrightarrow{\ \bar{i}\ } & C \xrightarrow{\ \bar{g}\ } & \mathcal{O}_{\mathbb{P}^3}(-1) \\
\downarrow{\scriptstyle \bar{e}} & \downarrow{\scriptstyle e} & \\
V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) & === & V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1),
\end{array}
$$

where d and e are the induced morphisms, $\varepsilon := d \circ \bar{g}$, $\bar{e} := e \circ \bar{i}$,

3) a sheaf

(153)
$$\mathcal{F} := \ker \varepsilon,$$

and a left-exact complex

(154)
$$K^\bullet: \ 0 \to K^{-1} \xrightarrow{\alpha_2} K^0 \xrightarrow{\beta_2} K^1 \to 0, \quad \beta_2 := \bar{e} \circ c_2,$$
$$K^{-1} = V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \quad K^0 = V_6 \otimes \mathcal{O}_{\mathbb{P}^3}, \quad K^1 = V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1),$$

such that

(155)
$$\mathcal{H}^0(K^\bullet) = \mathcal{F}, \qquad \mathcal{H}^1(K^\bullet) = \operatorname{coker} \varepsilon.$$

From (134), (154) and the vanishing of $\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(-2))$ follows the commutative diagram

(156)

$$
\begin{array}{ccc}
& \mathcal{O}_{\mathbb{P}^3}(1) === \mathcal{O}_{\mathbb{P}^3}(1) & \\
& \downarrow{\scriptstyle j} \qquad\qquad \downarrow{\scriptstyle j'} & \\
\mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\ \sigma\ } & E \longrightarrow & \operatorname{coker} \sigma \\
\| & \downarrow{\scriptstyle h} & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\ h \circ \sigma\ } & E_3 \longrightarrow & \mathcal{L}
\end{array}
$$

where $\mathcal{L} := \operatorname{coker}(h \circ \sigma)$ and j' is an induced morphism which is nonzero, hence injective, since coker σ is locally free by the exact triple $0 \to \mathcal{E} \to \operatorname{coker} \sigma \to \mathcal{O}_{\mathbb{P}^3}(2) \to 0$ following from (134). Since $\mathcal{E}$ is stable by assumption, so

that $h^0(\mathcal{E}(-1)) = 0$ (see [34]), the last triple and (156) yield a commutative diagram

(157)

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(1) & =\!\!=\!\!= & \mathcal{O}_{\mathbb{P}^3}(1) \\
\downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle \bar{\beta}} \\
\mathcal{E} \rightarrowtail \text{coker } \sigma & \longrightarrow\!\!\!\!\rightarrow & \mathcal{O}_{\mathbb{P}^3}(2) \\
\| \quad\quad \downarrow & & \downarrow \\
\mathcal{E} \rightarrowtail \mathcal{L} & \longrightarrow\!\!\!\!\rightarrow & \mathcal{O}_{\mathbb{P}^2}(2),
\end{array}
$$

where $\mathbb{P}^2 = \text{Supp}(\text{coker } \bar{\beta})$ is a projective plane in $\mathbb{P}^3$. Note that, in this diagram, $\bar{\beta}$ is the composition $\bar{\beta}$: $\mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\text{can}} M^0 \xrightarrow{\beta} M^1$, and im $\bar{\beta} \hookrightarrow \mathcal{O}_{\mathbb{P}^3}(2)$ since $\tilde{\beta} = 0$. Thus, $\mathbb{P}^2$ is uniquely defined by the morphism β in the monad $M^\bullet$. In a similar way, since $\tilde{\alpha} = 0$, the morphism α in $M^\bullet$ uniquely defines a morphism $\bar{\alpha} : \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3}(-1)$, hence a projective plane $\mathbb{P}_0^2 = \text{Supp}(\text{coker } \bar{\alpha})$. For these two planes we will use the notation

(158) $$\mathbb{P}^2 = \mathbb{P}^2(M^\bullet, \beta), \ \mathbb{P}_0^2 = \mathbb{P}^2(M^\bullet, \alpha).$$

Consider the lower horizontal triple in (157):

(159) $$0 \to \mathcal{E} \xrightarrow{\theta} \mathcal{L} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^2}(2) \to 0, \qquad \mathbb{P}^2 = \mathbb{P}^2(M^\bullet, \beta).$$

Lemma 22. *The sheaf $\mathcal{L}$ in (159) is a stable reflexive rank 2 sheaf on $\mathbb{P}^3$, $[\mathcal{L}] \in \mathcal{R}(1, 4, 6)$.*

Proof. First, show that the triple (159) doesn't split. Indeed, otherwise, the lower horizontal triple in (156) extends to a commutative push-out diagram

(160)

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(-2) \rightarrowtail & E_3 & \longrightarrow\!\!\!\!\rightarrow \mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^2}(2) \\
\| & \uparrow & \uparrow \\
\mathcal{O}_{\mathbb{P}^3}(-2) \rightarrowtail \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) & \longrightarrow\!\!\!\!\rightarrow & \mathcal{O}_{\mathbb{P}^2}(2),
\end{array}
$$

where the lower triple splits since $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^3}(-2)) = 0$. This yields a nonzero morphism $\delta : \mathcal{O}_{\mathbb{P}^2}(2) \to E_3$ which being composed with the morphism ε in (153) is the zero morphism $\mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{O}_{\mathbb{P}^3}(-1)$. Hence, in (153), δ factors through a nonzero morphism

(161) $$\delta' : \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{F}.$$

On the other hand, (154) and (155) yield an exact triple $0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \text{ker } \beta_2 \to \mathcal{F} \to 0$ which, together with (161), extends to a push-out diagram similar to (160):

(162)

$$
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^3}(-2) \rightarrowtail & \text{ker } \beta_2 & \longrightarrow\!\!\!\!\rightarrow \mathcal{F} \\
\| & \uparrow{\scriptstyle \delta''} & \uparrow{\scriptstyle \delta'} \\
\mathcal{O}_{\mathbb{P}^3}(-2) \rightarrowtail \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(2),
\end{array}
$$

where $\delta''|_{\mathcal{O}_{\mathbb{P}^2}(2)}$ is nonzero. However, this is impossible since ker β_2 by definition is torsion free as a subsheaf of a locally free sheaf $V_6 \otimes \mathcal{O}_{\mathbb{P}^3}$.

Next, since $\mathcal{E} \cong \mathcal{E}^\vee$ is locally free and $\mathcal{E}xt^1(\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_{\mathbb{P}^2}(-1)$, then, applying the functor $\mathcal{E}xt^\bullet(-, \mathcal{O}_{\mathbb{P}^3})$ to the triple (159) we have an exact sequence

(163) $$0 \to \mathcal{L}^\vee \xrightarrow{\theta^\vee} \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\varphi} {}^1\mathcal{L} \to 0, \qquad {}^1\mathcal{L} := \mathcal{E}xt^1(\mathcal{L}, \mathcal{O}_{\mathbb{P}^3}).$$

Let $d = \dim({}^1\mathcal{L})$. Consider the three possible cases: (a) $d = 2$, (b) $d = 1$, and (c) $d = 0$. We will show that the cases (a) and (b) lead to a contradiction.

(a) $d = 2$. In this case $\dim \text{Sing } \mathcal{L} = 2$, i. e. the torsion subsheaf $\mathcal{T}ors(\mathcal{L})$ of $\mathcal{L}$ has dimension 2. This necessarily implies that the composition $\mathcal{T}ors(\mathcal{L}) \hookrightarrow \mathcal{L} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^2}(2)$ is an isomorphism giving the splitting of the triple (159), contrary to the above.

(b) $d = 1$. In this case ${}^1\mathcal{L} = \mathcal{O}_Z(-1)$ for Z a subscheme of $\mathbb{P}^2$, of dimension $\dim Z = 1$. Hence ker $\varphi \hookrightarrow \mathcal{O}_{\mathbb{P}^2}(-1-k)$ for some $k \geq 2$. By (163) the sheaf ker φ is the quotient of $\mathcal{E}$, it follows that $h^0(\mathcal{E}_{\mathbb{P}^2}(-1-k)) \neq 0$, $k \geq 1$, and so

$h^0(\mathcal{E}_{\mathbb{P}^2}(-2)) \neq 0$. On the other hand, since $\mathcal{E}$ is the cohomology of (130), by [21, Table 5.3, case 5(2.ii)] it has the spectrum $\mathrm{Sp}(\mathcal{E}) = (-1, 0, 0, 0, 1)$, and then it follows that

$$(164) \qquad\qquad h^1(\mathcal{E}(-3)) = 0, \quad h^1(\mathcal{E}(-2)) = 1.$$

Thus, by the first equality in (164), the inequality $h^0(\mathcal{E}_{\mathbb{P}^2}(-2)) \neq 0$ contradicts to the cohomology sequence of the exact triple

$$(165) \qquad\qquad 0 \to \mathcal{E}(-3) \to \mathcal{E}(-2) \to \mathcal{E}_{\mathbb{P}^2}(-2) \to 0$$

as $h^0(\mathcal{E}(-2)) = 0$ by the stability of E. Note also that we have proved here the equality

$$(166) \qquad\qquad H^0(\mathcal{E}_{\mathbb{P}^2}(-2)) = 0, \qquad \forall\, \mathbb{P}^2 \subset \mathbb{P}^3.$$

(c) $d = 1$. In this case ${}^1\mathcal{L} = \mathcal{O}_Z(-1)$ for Z a subscheme of $\mathbb{P}^2$ of dimension $\dim Z = 0$, and the sequence $0 \to \mathcal{L}^\vee \xrightarrow{\theta^\vee} \mathcal{E} \to \mathcal{I}_{Z,\mathbb{P}^2}(-1) \to 0$, and $Z = (s)_0$, $0 \neq s \in H^0(\mathcal{E}(-1)|_{\mathbb{P}^2})$. Since $\dim Z = 0$, it follows that $\mathcal{E}xt^1(\mathcal{I}_{Z,\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^3}) = \mathcal{E}xt^1(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_{\mathbb{P}^2}(2)$. Thus, applying the functor $\mathcal{E}xt^\bullet(-, \mathcal{O}_{\mathbb{P}^3})$ to the last triple, since $\mathcal{E} \cong \mathcal{E}^\vee$ we obtain an exact triple $0 \to \mathcal{E} \xrightarrow{\theta^{\vee\vee}} \mathcal{L}^{\vee\vee} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^2}(2) \to 0$. Comparing this triple with (159) and taking into account that, by construction, the composition $\mathcal{E} \xrightarrow{\theta} \mathcal{L} \xrightarrow{\mathrm{can}} \mathcal{L}^{\vee\vee}$ coincides with $\theta^{\vee\vee}$ we obtain that $\mathcal{L} \xrightarrow{\mathrm{can}} \mathcal{L}^{\vee\vee}$ is an isomorphism, i. e. $\mathcal{L}$ is reflexive.

Next, as $c_t(\mathcal{E}) = 1 + 5t^2$, formulas for Chern classes of $\mathcal{L}$ follow from (159). In particular, $\mathcal{L}^\vee \cong \mathcal{L}(-1)$ has $c_1(\mathcal{L}(-1)) = -1$, and since $h^0(\mathcal{E}) = 0$, it follows that

$$(167) \qquad\qquad h^0(\mathcal{L}(-1)) = 0.$$

Thus, $\mathcal{L}$ is stable by [20, Lemma 3.1]. Lemma is proved. $\qquad\qquad\square$

We now proceed to the more close study of the sheaf $\mathcal{F}$. Consider the upper horizontal triple of the diagram (152) which extends to an exact sequence:

$$(168) \qquad\qquad 0 \to \mathcal{F} \to E_3 \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\bar{Y}}(-1) \to 0, \qquad \bar{Y} \subset \mathbb{P}^3.$$

Lemma 23. *The sheaf $\mathcal{F}$ defined in* (153) *is a reflexive rank 2 sheaf on $\mathbb{P}^3$ fitting in an exact triple*

$$(169) \qquad\qquad 0 \to \mathcal{F} \xrightarrow{\zeta} \mathcal{L} \to \mathcal{I}_{\bar{Y}, \mathbb{P}^2_0}(-1) \to 0,$$

and in its dual

$$(170) \qquad\qquad 0 \to \mathcal{L}(-1) \to \mathcal{F} \xrightarrow{\rho} \mathcal{I}_{\bar{Z}, \mathbb{P}^2_0}(2) \to 0,$$

where $\mathbb{P}^2_0 = \mathbb{P}^2(M^\bullet, \alpha)$, $\bar{Y}$, $\bar{Z} \subset \mathbb{P}^2_0$, $\dim \bar{Y} \leq 0$, $\dim \bar{Z} \leq 0$, and

$$(171) \qquad\qquad \ell(\bar{Y}) + \ell(\bar{Z}) = 6.$$

Chern classes of $\mathcal{F}$ are $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = 2$, $0 \leq c_3(\mathcal{F}) = 2\ell(\bar{Y}) \leq 12$.

Proof. We first show that $\mathrm{rk}\,\mathcal{F} = 2$. Indeed, if ε in (168) is the zero morphism, then the diagram (152) and the Snake Lemma yield an epimorphism $V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)$ which is impossible. Hence, $\varepsilon \neq 0$ and (153) implies that $\mathrm{rk}\,\mathcal{F} = 2$ and, moreover, that $\bar{Y} \subsetneq \mathbb{P}^3$, i. e., $\bar{Y}$ is a proper subscheme of $\mathbb{P}^3$. Note also that, by (132) and (133), $c_1(E) = 0$, hence $c_1(E_3) = -1$ in view of (151). Thus, (168) implies that $c_1(\mathcal{F}) = c_1(\mathcal{O}_{\bar{Y}}(-1)) \geq 0$.

Next, consider the lower exact triple in (156):

$$(172) \qquad\qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{h \circ \sigma} E_3 \to \mathcal{L} \to 0.$$

If the composition $f := \varepsilon \circ h \circ \sigma$ is zero, then (168) and (172) imply that there exist injective morphisms $\mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{f_1} \mathcal{F}$ and $\mathrm{coker}(f_1) \xrightarrow{f_2} \mathcal{L}$. Since $\mathrm{rk}\,\mathcal{F} = 2$, $c_1(\mathcal{F}) \geq 0$ and $\mathcal{L}$ is reflexive by Lemma 22, it follows that $\mathrm{coker}(f_1)$ is a rank 1 torsion free sheaf with $c_1(\mathrm{coker}(f_1)) \geq 2$. Thus, the injectivity of f_2 shows that $h^0(\mathcal{L}(-2)) \neq 0$, contrary to the stability of $\mathcal{L}$ (see Lemma 22). It follows that $f \neq 0$, so that (168) and (172) extend to a comutative diagram

$$(173)$$

where $\mathbb{P}_0^2$ is some projective plane in $\mathbb{P}^3$. If $\bar{\delta}$ is an isomorphism, then $\operatorname{coker}(\varepsilon) \simeq \mathcal{O}_{\mathbb{P}_0^2}(-1)$, so that the diagram (152) and the Snake Lemma yield an epimorphism $V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \twoheadrightarrow \mathcal{O}_{\mathbb{P}_0^2}(-1)$ which is impossible. Hence, $\bar{Y} \subsetneqq \mathbb{P}_0^2$, i. e., $\bar{Y}$ is a proper subscheme of $\mathbb{P}_0^2$, $\dim \bar{Y} \leq 1$, and (173) yields an exact triple (169).

Show that the case $\dim \bar{Y} = 1$ is impossible. Indeed, in this case $\bar{Y}$ contains a divisor $D \subset \mathbb{P}_0^2$ of degree $k \geq 1$ as a subscheme, and this yields an epimorphim $\mathcal{O}_{\bar{Y}}(-1) \overset{b}{\to} \mathcal{O}_D(-1)$. On the other hand, the middle horizontal exact sequence in (173), together with diagram (152) and the Snake Lemma, yield an epimorphism $V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \twoheadrightarrow \mathcal{O}_{\bar{Y}}(-1)$. This epimorphism composed with the above epimorphism b gives an epimorphism $V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \twoheadrightarrow \mathcal{O}_D(-1)$ which is impossible, since $h^0(\mathcal{O}_D(-2)) = 0$, as follows from the cohomology of the exact triple $0 \to \mathcal{O}_{\mathbb{P}_0^2}(-2-k) \to \mathcal{O}_{\mathbb{P}_0^2}(-k) \to \mathcal{O}_D(-2) \to 0$.

Hence, $\dim \bar{Y} \leq 0$ and therefore, denoting ${}^i\mathcal{I} := \mathcal{E}xt^i(\mathcal{I}_{\bar{Y},\mathbb{P}_0^2}(-1), \mathcal{O}_{\mathbb{P}^3})$, $i \geq 1$, we obtain ${}^1\mathcal{I} = \mathcal{O}_{\mathbb{P}^2}(2)$, $\dim {}^2\mathcal{I} \leq 0$, ${}^3\mathcal{I} = 0$. Besides, set ${}^i\mathcal{F} := \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_{\mathbb{P}^3})$, ${}^i\mathcal{L} := \mathcal{E}xt^i(\mathcal{L}, \mathcal{O}_{\mathbb{P}^3})$, $i \geq 1$, and remark that, for the reflexive sheaf $\mathcal{L}$, $\dim {}^1\mathcal{L} = 0$, ${}^i\mathcal{L} = 0$, $i = 2, 3$ (see [20, proof of Thm. 2.5]). Now, applying to (169) the functor $\mathcal{E}xt^\bullet(-, \mathcal{O}_{\mathbb{P}^3})$ and using the above relations we obtain the equalities ${}^i\mathcal{F} = 0$, $i = 2, 3$, and an exact sequence $0 \to \mathcal{L}^\vee \overset{\zeta^\vee}{\longrightarrow} \mathcal{F}^\vee \to \mathcal{O}_{\mathbb{P}^2}(2) \overset{\mu}{\to} {}^1\mathcal{L} \to {}^1\mathcal{F} \to {}^2\mathcal{I} \to 0$, wherefrom $\dim {}^1\mathcal{F} \leq 0$ and $\ker \mu \simeq \mathcal{I}_{Z,\mathbb{P}_0^2}(2)$ for some subscheme Z of $\mathbb{P}_0^2$, of dimension $\dim Z \leq 0$. We thus obtain an exact triple $0 \to \mathcal{L}^\vee \overset{\zeta^\vee}{\longrightarrow} \mathcal{F}^\vee \to \mathcal{I}_{Z,\mathbb{P}_0^2}(2) \to 0$ and the relation $\mathcal{E}xt^1(\mathcal{I}_{Y,\mathbb{P}_0^2}(2), \mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_{\mathbb{P}^2}(-1)$. Next, applying to the last triple the functor $\mathcal{E}xt^\bullet(-, \mathcal{O}_{\mathbb{P}^3})$ yields an exact sequence $0 \to \mathcal{F}^{\vee\vee} \overset{\zeta^{\vee\vee}}{\longrightarrow} \mathcal{L}^{\vee\vee} \to \mathcal{O}_{\mathbb{P}^2}(-1) \overset{\nu}{\to} \mathcal{E}xt^1(\mathcal{F}^\vee, \mathcal{O}_{\mathbb{P}^3})$. By [20, Cor. 1.2] $\mathcal{F}^\vee$ is a reflexive rank 2 sheaf, hence $\dim \mathcal{E}xt^1(\mathcal{F}^\vee, \mathcal{O}_{\mathbb{P}^3}) \leq 0$ by [20, Rem. 2.7.1], and therefore $\ker \nu \simeq \mathcal{I}_{W,\mathbb{P}_0^2}(-1)$ for some subscheme W of $\mathbb{P}_0^2$, of dimension $\dim W \leq 0$. Thus the last sequence leads to an exact triple $0 \to \mathcal{F}^{\vee\vee} \overset{\zeta^{\vee\vee}}{\longrightarrow} \mathcal{L}^{\vee\vee} \to \mathcal{I}_{W,\mathbb{P}_0^2}(-1) \to 0$ which together with (169) fits in a commutative diagram

Besides, the above stated relations ${}^i\mathcal{F} = 0$, $i = 2, 3$, $\dim {}^1\mathcal{F} \leq 0$ show that the sheaf $\mathcal{F}$ is locally free outside the set of dimension ≤ 0, and this shows that the sheaf $\kappa = \operatorname{coker}(\mathcal{F} \overset{\mathrm{can}}{\longrightarrow} \mathcal{F}^{\vee\vee})$ has dimension ≤ 0 and by the Snake Lemma κ is a subsheaf of $\ker c$. However, the sheaf has no subsheaves of dimension 0. Hence, $\kappa = 0$ and $\mathcal{F} \overset{\mathrm{can}}{\longrightarrow} \mathcal{F}^{\vee\vee}$ is an isomorphism, i. e. $\mathcal{F}$ is reflexive. A standard computation with the triple (169) yields the values of Chern classes of $\mathcal{F}$, The triple (170) and the equality (171) are obtained by applying to (169) the functor $\mathcal{E}xt^\bullet(-, \mathcal{O}_{\mathbb{P}^3})$ and using formulas for Chern classes of $\mathcal{F}$ and $\mathcal{L}$. The inequality $0 \leq c_3(\mathcal{F}) \leq 12$ follows from (171). $\qquad\square$

Lemma 24. *The projective planes $\mathbb{P}^2$ and $\mathbb{P}_0^2$ defined in (158) coincide.*

Proof. The middle horizontal triple $0 \to \mathcal{E} \to \operatorname{coker} \sigma \to \mathcal{O}_{\mathbb{P}^3}(2) \to 0$ in (157) as an extension is defined by a nonzero element in $\operatorname{Ext}^1(\mathcal{E}, \mathcal{O}_{\mathbb{P}^2}(2)) \simeq H^1(\mathcal{E}(-2))$. Since $h^1(\mathcal{E}(-2)) = 1$ by (164), it follows that the sheaf $\operatorname{coker} \sigma$ is defined by $\mathcal{E}$ uniquely up to an isomorphism. Since $h^0(\mathcal{L}(-1)) = 0$ as $\mathcal{L}$ is stable by Lemma 22, the twisted by $\mathcal{O}_{\mathbb{P}^3}(-1)$ middle vertical triple $0 \to \mathcal{O}_{\mathbb{P}^3} \to \operatorname{coker}\sigma(-1) \to \mathcal{L}(-1) \to 0$ in (157) shows that $h^0(\operatorname{coker}\sigma(-1)) = 1$. Hence, $\mathcal{L} = \mathcal{L}(M^\bullet)$ is uniquely up to an isomorphism defined by $\operatorname{coker}\sigma$ (and therefore by $\mathcal{E}$) as

$$(174) \qquad \mathcal{L}(M^\bullet) = (\operatorname{coker}\sigma(-1)/\mathcal{O}_{\mathbb{P}^3})(1).$$

Then the lower horizontal triple in (157) shows that the plane $\mathbb{P}^2 = \mathbb{P}^2(M^\bullet, \beta)$ is determined uniquely by $\mathcal{E}$ as

$$(175) \qquad \mathbb{P}^2(M^\bullet, \beta) = \operatorname{Supp}(\mathcal{L}(M^\bullet)/\mathcal{E}).$$

Next, it follows from (158) that

$$(176) \qquad \mathbb{P}^2(M^\bullet, \alpha) = \mathbb{P}^2(M^{\bullet\vee}, \beta^\vee), \quad \text{resp.,} \quad \mathbb{P}^2(M^\bullet, \beta) = \mathbb{P}^2(M^{\bullet\vee}, \alpha^\vee),$$

where $M^{\bullet\vee} : \ 0 \to (M^1)^\vee \overset{\beta^\vee}{\longrightarrow} (M^0)^\vee \overset{\alpha^\vee}{\longrightarrow} (M^{-1})^\vee \to 0$ is the monad dual to $M^\bullet$. The monad $M^{\bullet\vee}$ defines the monad dual to (134): $0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \overset{\tau^\vee}{\to} E \overset{\sigma^\vee}{\to} \mathcal{O}_{\mathbb{P}^3}(2) \to 0$ with $\mathcal{E}^\vee = \ker(\sigma^\vee)/\operatorname{im}(\tau^\vee)$, and the argument dual to the above yields the formulas dual to (174) and (175): $\mathbb{P}^2(M^{\bullet\vee}, \beta^\vee) = \operatorname{Supp}(\mathcal{L}(M^{\bullet\vee})/\mathcal{E}^\vee)$, $\mathcal{L}(M^{\bullet\vee}) = (\operatorname{coker}(\tau^\vee)(-1)/\mathcal{O}_{\mathbb{P}^3})(1)$. Since $\mathcal{E}^\vee \simeq \mathcal{E}$ these formulas mean in view of (176) that the plane $\mathbb{P}_0^2 = \mathbb{P}^2(M^\bullet, \alpha)$ is uniquely defined by $\mathcal{E}$ via the same construction as above, hence it coincides with $\mathbb{P}^2 = \mathbb{P}^2(M^\bullet, \beta)$. $\qquad\square$

Let $\mathcal{F} \in \mathcal{R}(0, 2, 2k)$ be the reflexive sheaf defined in (153), where $0 \leq k \leq 6$ by Lemma 23, i. e.,

(177) $$[\mathcal{F}] \in \bigsqcup_{0 \leq k \leq 6} \mathcal{R}_k, \qquad \mathcal{R}_k := \mathcal{R}(0, 2, 2k).$$

Formulas (143), (144) and Lemmas 22, 23 and 24 yield

Proposition 25. *There is an inclusion*

(178) $$\mathcal{H} \smallsetminus (\mathcal{H} \cap \mathcal{G}(2, 1)) \subset \bigsqcup_{0 \leq k \leq 6} \mathcal{H}_k, \qquad where$$

(179)
$$\mathcal{H}_k = \{[\mathcal{E}] \in \mathcal{B}(5) \mid \mathcal{E} \text{ is obtained from } \mathcal{F}, \text{where } [\mathcal{F}] \in \mathcal{R}_k,$$
$$\text{by the two subsequent elementary transformations (180) below}\},$$

(180)
$$0 \to \mathcal{L}(-1) \to \mathcal{F} \xrightarrow{\rho} \mathcal{I}_{\bar{Z}, \mathbb{P}^2}(2) \to 0, \qquad (step\ 1)$$
$$0 \to \mathcal{E} \to \mathcal{L} \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^2}(2) \to 0, \qquad (step\ 2)$$

where $\mathbb{P}^2$ is some plane in $\mathbb{P}^3$, $\bar{Z} \subset \mathbb{P}^2$, $\dim \bar{Z} \leq 0$, $\ell(\bar{Z}) = 6 - k$, and $\mathcal{L}$ is a stable reflexive sheaf from $\mathcal{R}(1, 4, 6)$.

7. GEOMETRIC PROPERTIES OF SHEAVES $\mathcal{F}$ AND MODULI OF COHOMOLOGY BUNDLES $\mathcal{E}$ OF MONADS (6)

In this section we explore in detail the geometry of the reflexive sheaves $\mathcal{F}$ described in Lemma 23. The main result of this study will be the upper estimates for the dimensions of the moduli space of sheaves $\mathcal{F}$ and sheaves $\mathcal{L}$ obtained from $\mathcal{F}$ by the elementary transformation (170). These estimates are obtained in Propositions 26 and 27 below. This will eventually lead to the proof of Theorem 21.

Denote
$$\mathcal{R}_k^u := \{[\mathcal{F}] \in \mathcal{R}_k \mid \mathcal{F} \text{ is unstable}\}, \qquad \mathcal{R}_k^s := \{[\mathcal{F}] \in \mathcal{R}_k \mid \mathcal{F} \text{ is stable}\},$$
$$\mathcal{H}_k^u := \{[\mathcal{E}] \in \mathcal{H}_k \mid \mathcal{E} \text{ is obtained from } \mathcal{F} \text{ in (179)}, \text{where } [\mathcal{F}] \in \mathcal{R}_k^u\},$$
$$\mathcal{H}_k^s := \{[\mathcal{E}] \in \mathcal{H}_k \mid \mathcal{E} \text{ is obtained from } \mathcal{F} \text{ in (179)}, \text{where } [\mathcal{F}] \in \mathcal{R}_k^s\},$$

where $0 \leq k \leq 6$. Thus, $\mathcal{R}_k = \mathcal{R}_k^u \sqcup \mathcal{R}_k^s$ and (144) and (178) yield:

(181) $$\mathcal{H} \smallsetminus (\mathcal{H} \cap \mathcal{G}(2, 1)) \subset \bigsqcup_{0 \leq k \leq 6} (\mathcal{H}_k^u \sqcup \mathcal{H}_k^s).$$

The estimate for the dimension of $\mathcal{H} \smallsetminus (\mathcal{H} \cap \mathcal{G}(2, 1))$ will eventually follow from the computations of dimensions of $\mathcal{H}_k^u$ and $\mathcal{H}_k^s$ which we will give below. For this, we start with an explicit description of the spaces $\mathcal{R}_k^u$ and $\mathcal{R}_k^s$.

Proposition 26. *(i) $\mathcal{R}_k^u \neq \emptyset$ only for $0 \leq k \leq 3$, and any sheaf $\mathcal{F}$ from $\mathcal{R}_k^u$ fits in an exact triple*

(182) $$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} \mathcal{F} \xrightarrow{u} \mathcal{I}_{C, \mathbb{P}^3} \to 0,$$

where $C = \mathrm{Sing}(\mathcal{F}/\mathcal{O}_{\mathbb{P}^3})$ is a l.c.i. curve of degree 2 in $\mathbb{P}^3$, $\chi(\mathcal{O}_C) = 4 - \frac{1}{2} c_3(\mathcal{F}) = 4 - k$.
(ii) If C is reduced, then either $c_3(\mathcal{F}) = 4$ and C is a disjoint union $l_1 \sqcup l_2$ of two projective lines in $\mathbb{P}^3$, or $c_3(\mathcal{F}) = 6$, then C is a plane conic in $\mathbb{P}^3$.
(iii) If C is nonreduced then C is the scheme structure of multiplicity two on a projective line l in $\mathbb{P}^3$ defined by an exact sequence

(183) $$0 \to \mathcal{I}_{C, \mathbb{P}^3} \to \mathcal{I}_{l, \mathbb{P}^3} \to \mathcal{O}_l(m) \to 0, \qquad -1 \leq m = 2 - k \leq 2.$$

(iv) The moduli spaces $\mathcal{R}_k^u$ are varieties of dimensions

(184) $$\dim \mathcal{R}_0^u = \dim \mathcal{R}_3^u = 14, \qquad \dim \mathcal{R}_1^u = \dim \mathcal{R}_2^u = 13,$$

and they are fine.

Proof. (i)-(iii). By Lemma 23, we have $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = 2$. Since $\mathcal{F}$ is unstable, it follows from [20, Lemma 3.1] that $H^0(\mathcal{F}) \neq 0$. Besides, from (167) and the triple (169) twisted by $\mathcal{O}_{\mathbb{P}^3}(-1)$ we obtain $H^0(\mathcal{F}(-1)) = 0$. Take a section $0 \neq s \in H^0(\mathcal{F})$ and define a subscheme C in $\mathbb{P}^3$ by the ideal sheaf $\mathcal{I}_{C, \mathbb{P}^3} = \mathrm{im}(u : \mathcal{F} \xrightarrow[\simeq]{\mathrm{can}} \mathcal{F}^\vee \xrightarrow{s^\vee} \mathcal{O}_{\mathbb{P}^3})$.
(The canonical isomorphism can : $\mathcal{F} \xrightarrow{\simeq} \mathcal{F}^\vee$ follows since $c_1(\mathcal{F}) = 0$.) From the equality $H^0(\mathcal{F}(-1)) = 0$, by [20, Thm 4.1] we obtain that:
(a) C is a Cohen-Macaulay curve in $\mathbb{P}^3$ satisfying the triple (182), so that $\deg C = c_2(\mathcal{F}) = 2$, and

(b) the triple (182) is exact, and the equality $\chi(\mathcal{O}_C) = 4-k$ follows from this triple and [20, Thm. 2.3]; moreover, (182) defines an extension

(185) $$\xi \in \mathrm{Ext}^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) \simeq H^0(\mathcal{E}xt^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3})) \simeq H^0(\mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3})).$$

(Here we use standard isomorphisms relatig global Ext-groups and $\mathcal{E}xt$-sheaves - see [20, Sec. 4].) If C is a reduced curve then, since $\deg C = 2$, C is either a disjoint union $l_1 \sqcup l_2$ of lines, or a conic. If C is nonreduced, then C is the scheme structure of multiplicity two on a projective line l (in the sense of [12, Definition on p. 58]). Moreover, since C is Cohen-Macaulay, the sheaf $\mathcal{I}_{l,\mathbb{P}^3}/\mathcal{I}_{C,\mathbb{P}^3}$ has no 0-dimensional torsion. Hence, by [12, Claim on p. 59], the exact triple (183) follows and, moreover, C is a locally complete intersection. The triples (183) and (182) yield the equality $m = 2 - \frac{1}{2}c_3(\mathcal{F}) = 2\text{-k}$. Furthermore, (183) and the isomorphism

(186) $$\mathcal{I}_{l,\mathbb{P}^3}|_l \simeq N_{l/\mathbb{P}^3}^\vee \simeq \mathcal{O}_l(-1)^{\oplus 2}$$

imply $m \geq -1$. Besides, $2 - m = k = \frac{1}{2}c_3(\mathcal{F}) \geq 0$, as $\mathcal{F}$ is reflexive. Thus, $-1 \leq m \leq 2$ and therefore $0 \leq k \leq 3$.
(iv) Consider the varieties $\mathcal{C}_k = \{C \mid C \text{ is a l.c.i. curve of degree 2 in } \mathbb{P}^3, \chi(\mathcal{O}_C) = 4 - k\}$, $0 \leq k \leq 3$. From (i)-(iii) and [12, Remark 1.3] it follows that $\mathcal{C}_k$ are rational varieties of dimensions

(187) $$\dim \mathcal{C}_0 = 11, \quad \dim \mathcal{C}_1 = 9, \quad \dim \mathcal{C}_2 = \dim \mathcal{C}_3 = 8.$$

Note that (183) yields an exact triple $0 \to \mathcal{O}_l(2-k) \to \mathcal{O}_C \to \mathcal{O}_l \to 0$, $k = \frac{1}{2}c_3(\mathcal{F})$. Applying to it the functor $\mathcal{E}xt^2(\mathcal{O}_l, \mathcal{O}_{\mathbb{P}^3})$ and using the relations $\mathcal{E}xt^2(\mathcal{O}_l, \mathcal{O}_{\mathbb{P}^3}) \simeq \det(N_{l/\mathbb{P}^3}) \simeq \mathcal{O}_l(2)$, $\mathcal{E}xt^i(\mathcal{O}_l, \mathcal{O}_{\mathbb{P}^3}) = 0$, $i = 1, 3$, (see [34, pp. 49-50]) we obtain an exact triple $0 \to \mathcal{O}_l(2) \to \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3}) \xrightarrow{\epsilon} \mathcal{O}_l(k) \to 0$ which, together with (185), yields

(188) $$\dim \mathrm{Hom}(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) = 1, \ \mathrm{Ext}^{\geq 2}(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) = 0,$$
$$\dim \mathrm{Ext}^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) = h^0(\mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^3})) = k + 4,$$

Now, by (i)-(iii), for $0 \leq k \leq 3$, the spaces $\mathcal{R}_k^u$ are described as: $\mathcal{R}_k^u = \{([\mathcal{F}], \langle\xi\rangle) \mid [\mathcal{F}] \in \mathcal{R}_k, \ \mathcal{F} \text{ fits in (182)}, \ \langle\xi\rangle \in \mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}))\} = \{(C, \langle\xi\rangle) \mid C \in \mathcal{C}_k, \ \langle\xi\rangle \in \mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_{C,\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}))\}$. This, together with (188), shows that $\mathcal{R}_k$ is a projective fibration with fibre $\mathbb{P}^{k+3}$ over $\mathcal{C}_k$, and (187) yields (184). Note that there exist universal flat families $\Gamma \subset \boldsymbol{\mathcal{C}}_k$ of curves C, and in view of (188) and [31, Thm. 1.4] the sheaves $\mathcal{E}xt^i_{p_{\mathcal{C}_k}}(\mathcal{I}_{\Gamma, \boldsymbol{\mathcal{C}}_k}, \mathcal{O}_{\boldsymbol{\mathcal{C}}_k})$ commute with the base change. Hence, by [31, Prop. 4.2] there exist universal sheaves $\boldsymbol{\mathcal{F}}$ on $\boldsymbol{\mathcal{R}}_k^u$, i. e., $\mathcal{R}_k^u$ are fine moduli spaces. $\qquad\square$

Proposition 27. *Suppose that $[\mathcal{F}] \in \mathcal{R}_k^s$. Then the following statements hold.*
(i) $\mathcal{R}_k^s \neq \emptyset$ only for $0 \leq k \leq 2$.
(ii) $\dim \mathcal{R}_k^s = 13$, $k = 0, 1, 2$.
(iii) For $0 \leq k \leq 2$ and any $[\mathcal{F}] \in \mathcal{R}_k^s$, $\dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) = 13$, $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$.
(iv) For any $\mathbb{P}^2 \subset \mathbb{P}^3$, $h^0(\mathcal{F}_{\mathbb{P}^2}(2)) = 10$, $h^1(\mathcal{F}_{\mathbb{P}^2}(2)) = 0$.

Proof. Statements (i)-(iii) are proved in [12, Sec. 2]. The equalities in statement (iv) follow from the exact triple $0 \to \mathcal{F}(1) \to \mathcal{F}(2) \to \mathcal{F}_{\mathbb{P}^2}(2) \to 0$ and [12, Tables 2.8.1 and 2.12.2] for $k = 2, 4$ and, respectively, from [19, §9] for $k = 0$. $\qquad\square$

We next proceed to a detailed description of the relation between the spaces $\mathcal{H}_k^u$ and $\mathcal{R}_k^u$ for $0 \leq k \leq 3$ and the spaces $\mathcal{H}_k^s$ and $\mathcal{R}_k^s$ for $0 \leq k \leq 2$ given by steps 1 and 2 of formula (180). Denote

$$^2S_{\mathcal{F}}^u := \{\mathbb{P}^2 \in \check{\mathbb{P}}^3 \mid \dim(C \cap \mathbb{P}^2) = 1, \ C \subset \mathbb{P}^2, \ \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^3}/(\mathcal{F}/\mathcal{O}_{\mathbb{P}^3})\}, \ [\mathcal{F}] \in \mathcal{R}_3^u,$$
$$^2S_{\mathcal{F}}^u := \emptyset, \ [\mathcal{F}] \in \mathcal{R}_k^u, \quad k \leq 2,$$
$$^1S_{\mathcal{F}}^u := \{\mathbb{P}^2 \in \check{\mathbb{P}}^3 \mid \dim(C \cap \mathbb{P}^2) = 1, \ \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^3}/(\mathcal{F}/\mathcal{O}_{\mathbb{P}^3}), \ C \not\subset \mathbb{P}^2\}, \ [\mathcal{F}] \in \mathcal{R}_k^u,$$
$$^2S_{\mathcal{F}}^s = {}^1S_{\mathcal{F}}^s := \emptyset, \quad \text{if } [\mathcal{F}] \in \mathcal{R}_k^s,$$
$$^0S_{\mathcal{F}}^u := \{\mathbb{P}^2 \in \check{\mathbb{P}}^3 \smallsetminus ({}^1S_{\mathcal{F}}^u \cup {}^2S_{\mathcal{F}}^u) \mid \mathrm{Sing}\mathcal{F} \cap \mathbb{P}^2 \neq \emptyset\}, \ [\mathcal{F}] \in \mathcal{R}_k^u,$$
$$^0S_{\mathcal{F}}^s := \{\mathbb{P}^2 \in \check{\mathbb{P}}^3 \smallsetminus ({}^1S_{\mathcal{F}}^s \cup {}^2S_{\mathcal{F}}^s) \mid \mathrm{Sing}\mathcal{F} \cap \mathbb{P}^2 \neq \emptyset\}, \ [\mathcal{F}] \in \mathcal{R}_k^s,$$
$$^{-1}S_{\mathcal{F}}^u := \check{\mathbb{P}}^3 \smallsetminus ({}^0S_{\mathcal{F}}^u \cup {}^1S_{\mathcal{F}}^u \cup {}^2S_{\mathcal{F}}^u),$$
$$^{-1}S_{\mathcal{F}}^s := \check{\mathbb{P}}^3 \smallsetminus ({}^0S_{\mathcal{F}}^s \cup {}^1S_{\mathcal{F}}^s \cup {}^2S_{\mathcal{F}}^s),$$
$$\mathcal{D}_k^u := \mathcal{R}_k^u \times \check{\mathbb{P}}^3, \quad {}^i\mathcal{D}_k^u := \{([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k^u \mid \mathbb{P}^2 \in {}^iS_{\mathcal{F}}^u\}, \quad -1 \leq i \leq 2,$$
$$\mathcal{D}_k^s := \mathcal{R}_k^s \times \check{\mathbb{P}}^3, \quad {}^i\mathcal{D}_k^s := \{([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k^s \mid \mathbb{P}^2 \in {}^iS_{\mathcal{F}}^s\}, \quad -1 \leq i \leq 2,$$
$$\mathcal{D}_k := \mathcal{D}_k^u \sqcup \mathcal{D}_k^s = \bigsqcup_{-1 \leq i \leq 2} {}^i\mathcal{D}_k, \quad \text{where } {}^i\mathcal{D}_k := {}^i\mathcal{D}_k^u \sqcup {}^i\mathcal{D}_k^s, \quad -1 \leq i \leq 2.$$

Clearly, ${}^{i}\mathcal{D}_k^u$ (respectively, ${}^{i}\mathcal{D}_k^s$) are locally closed in $\mathcal{D}_k^u$ (respectively, $\mathcal{D}_k^s$) and

$$\text{(189)} \qquad \mathcal{D}_k^u = \bigsqcup_{-1 \leq i \leq 2} {}^{i}\mathcal{D}_k^u, \qquad \mathcal{D}_k^s = \bigsqcup_{-1 \leq i \leq 2} {}^{i}\mathcal{D}_k^s,$$

$$\text{(190)} \qquad \dim {}^{i}\mathcal{D}_k^u \leq \dim \mathcal{R}_k^u + 2 - i, \quad \dim {}^{i}\mathcal{D}_k^s \leq \dim \mathcal{R}_k^s + 2 - i, \quad -1 \leq i \leq 2.$$

Next, denote

$$\text{(191)} \qquad \begin{aligned} &\Pi(\mathcal{F}, \mathbb{P}^2) := \{\langle \rho \rangle \in \mathbb{P}(\mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}(2))))|\mathrm{im}(\rho : \mathcal{F} \to \mathcal{O}_{\mathbb{P}^2}(2)) = \mathcal{I}_{Z,\mathbb{P}^2}(2) \\ &\text{for a subscheme } Z \subset \mathbb{P}^2, \ \dim Z \leq 0, \ \ell(Z) = 6 - k\}, \quad ([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k, \end{aligned}$$

$$\text{(192)} \qquad \begin{aligned} &K_{([\mathcal{F}],\mathbb{P}^2)} := \{\langle \rho \rangle \in \Pi(\mathcal{F}, \mathbb{P}^2) \mid [\mathcal{L}_\rho := (\ker \rho)(1)] \in \mathcal{R}(1, 4, 6) \text{ is stable}\}, \\ &([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k, \mathcal{Q}_k := \{([\mathcal{F}], \mathbb{P}^2, \langle \rho \rangle) \mid ([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k, \ \langle \rho \rangle \in K_{([\mathcal{F}],\mathbb{P}^2)}\}, \\ &\mathcal{Q}_k \xrightarrow{p_{1k}} \mathcal{D}_k \text{ is the forgetful map,} \ \ p_{1k}^{-1}([\mathcal{F}], \mathbb{P}^2) = K_{([\mathcal{F}],\mathbb{P}^2)}, \end{aligned}$$

$$\text{(193)} \qquad \begin{aligned} &\Sigma_{([\mathcal{F}],\mathbb{P}^2,\langle \rho \rangle)} := \{\langle \gamma \rangle \in \mathbb{P}(\mathrm{Hom}(\mathcal{L}_\rho, \mathcal{O}_{\mathbb{P}^2}(2))) \mid \gamma : \ \mathcal{L}_\rho \to \mathcal{O}_{\mathbb{P}^2}(2) \text{ is an} \\ &\text{epimorphism and } \ker \gamma \text{ is locally free}\}, \qquad ([\mathcal{F}], \mathbb{P}^2, \langle \rho \rangle) \in \mathcal{Q}_k, \\ &T_k := \{([\mathcal{F}], \mathbb{P}^2, \langle \rho \rangle, \langle \gamma \rangle) \mid ([\mathcal{F}], \mathbb{P}^2, \langle \rho \rangle) \in \mathcal{Q}_k, \ \langle \gamma \rangle \in \Sigma_{([\mathcal{F}],\mathbb{P}^2,\langle \rho \rangle)}\}, \\ &T_k \xrightarrow{p_{2k}} \mathcal{Q}_k \text{ is the forgetful map,} \ \ p_{2k}^{-1}([\mathcal{F}], \mathbb{P}^2, \langle \rho \rangle) = \Sigma_{([\mathcal{F}],\mathbb{P}^2,\langle \rho \rangle)}, \end{aligned}$$

$$\text{(194)} \qquad \begin{aligned} &\mathcal{Q}_k^u := p_{1k}^{-1}(\mathcal{D}_k^u \cap p_{1k}(\mathcal{Q}_k)), \quad T_k^u := p_{2k}^{-1}(\mathcal{Q}_k^u \cap p_{2k}(T_k)), \\ &\mathcal{Q}_k^s := p_{1k}^{-1}(\mathcal{D}_k^s \cap p_{1k}(\mathcal{Q}_k)), \quad T_k^s := p_{2k}^{-1}(\mathcal{Q}_k^s \cap p_{2k}(T_k)). \end{aligned}$$

(Here $0 \leq k \leq 3$ and $0 \leq k \leq 2$ in unstable case and stable case, respectively.) Since $\mathcal{D}_k = \mathcal{D}_k^u \sqcup \mathcal{D}_k^s$, it follows that

$$\text{(195)} \qquad \mathcal{Q}_k = \mathcal{Q}_k^u \sqcup \mathcal{Q}_k^s, \qquad T_k = T_k^u \sqcup T_k^s, \qquad 0 \leq k \leq 3.$$

(For consistency, in (195) and below we set $\mathcal{Q}_3^s = T_3^s = \emptyset$.) Since the stability of the sheaf $\mathcal{L}_\rho$ is an open property in flat families [24, Prop. 2.3.1] it follows that

$$\text{(196)} \qquad p_{1k}^{-1}([\mathcal{F}], \mathbb{P}^2) = K_{([\mathcal{F}],\mathbb{P}^2)} \xhookrightarrow{\text{open}} \Pi(\mathcal{F}, \mathbb{P}^2), \qquad ([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k.$$

Take any point $([\mathcal{F}], \mathbb{P}^2, \langle \rho \rangle, \langle \gamma \rangle)$ Since by definition $[\mathcal{L}_\rho] \in \mathcal{R}(1, 4, 6)$ is stable and $\mathcal{E} = \ker \gamma$ is a vector bundle, we obtain from the second triple (180) that $[\mathcal{E}] \in \mathcal{R}(0, 5, 0)$ is also stable, i. e. $[\mathcal{E}] \in \mathcal{B}(5)$. Thus, we obtain a natural map

$$\text{(197)} \qquad f_k : \ T_k \to \mathcal{B}(5), \quad ([\mathcal{F}], \mathbb{P}^2, \langle \rho \rangle, \langle \gamma \rangle) \mapsto [\ker \gamma]$$

and by Proposition 25, $\mathcal{H}_k^u \subset f_k(T_k^u)$, $\mathcal{H}_k^s \subset f_k(T_k^s)$. This, together with (181) and and the second formula (195) yields

$$\text{(198)} \qquad \mathcal{H} \smallsetminus (\mathcal{H} \cap \mathcal{G}(2,1)) \subset \bigsqcup_{0 \leq k \leq 3} f_k(T_k).$$

It will follow from computations below that T_k are disjoint unions of schemes and f_k are morphisms for each of these schemes and all admissible values of k.

Lemma 28. *Let* $([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k$ *and* $\Pi(\mathcal{F}, \mathbb{P}^2) \neq \emptyset$. *Then:*

(i) there is an open embedding $j : \ \Pi(\mathcal{F}, \mathbb{P}^2) \xhookrightarrow{\text{open}} \mathbb{P}(H^0((\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2)))$ *and for any* $\langle \rho \rangle \in \Pi(\mathcal{F}, \mathbb{P}^2)$ *there exists a subscheme* $W(\rho)$ *of* $\mathbb{P}^2$, $\dim W(\rho) \leq 0$, *and an exact triple*

$$\text{(199)} \qquad 0 \to \mathcal{F}_{\mathbb{P}^2}(2) \xrightarrow{\text{can}} (\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2) \to \mathcal{O}_{W(\rho)} \to 0, \qquad \ell_{W(\rho)} = k;$$

(ii) if $\Sigma_{([\mathcal{F}],\mathbb{P}^2,\langle \rho \rangle)} \neq \emptyset$ *for* $([\mathcal{F}], \mathbb{P}^2, \langle \rho \rangle) \in \mathcal{Q}_k$, *then there is an open embedding* $\Sigma_{([\mathcal{F}],\mathbb{P}^2,\langle \rho \rangle)} \xhookrightarrow{\text{open}} \mathbb{P}(\mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10}$;

(iii) if $([\mathcal{F}], \mathbb{P}^2) \in {}^{-1}\mathcal{D}_k$, *then* $k = 0$, $h^0(\mathcal{F}_{\mathbb{P}^2}^{\vee\vee}(2)) = h^0(\mathcal{F}_{\mathbb{P}^2}(2)) = 10$;

(iv) if ${}^{0}\mathcal{D}_k \neq \emptyset$ *and* $([\mathcal{F}], \mathbb{P}^2) \in {}^{0}\mathcal{D}_k$, *then* $1 \leq k \leq 2$, $h^0(\mathcal{F}_{\mathbb{P}^2}(2)) = 10$ *and*

$$\text{(200)} \qquad h^0((\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2)) = 10 + k;$$

(v) if $^1\mathcal{D}_k \cup {}^2\mathcal{D}_k \neq \emptyset$ and $([\mathcal{F}], \mathbb{P}^2) \in {}^1\mathcal{D}_k \cup {}^2\mathcal{D}_k$, then the equalities (200) *hold for $k = 1, 2, 3$ and*

$$(201) \qquad h^0((\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2)) = h^0((\mathcal{F}_{\mathbb{P}^2})(2)) = 11, \quad if \quad k = 0.$$

Proof. (i) Take any $\langle \rho \rangle \in \Pi(\mathcal{F}, \mathbb{P}^2)$. By the definition of $\Pi(\mathcal{F}, \mathbb{P}^2)$ we may consider ρ as a composition $\rho : \mathcal{F} \xrightarrow{\otimes \mathcal{O}_{\mathbb{P}^2}} \mathcal{F}_{\mathbb{P}^2} \xrightarrow{\bar{\rho}} \mathcal{I}_{Z,\mathbb{P}^2}(2)$ with $\dim Z \leq 0$, $\ell_Z = 6 - k$. As $\mathcal{F}$ is reflexive, $\mathcal{F}_{\mathbb{P}^2}$ has no torsion as a $\mathcal{O}_{\mathbb{P}^2}$-sheaf [20, §1]. Hence, $\ker \bar{\rho}$ is a rank 1 torsion free $\mathcal{O}_{\mathbb{P}^2}$-sheaf. Since $c_1(\mathcal{F}_{\mathbb{P}^2}) = 0$, $c_2(\mathcal{F}_{\mathbb{P}^2}) = 2$, $\ker \bar{\rho} \simeq \mathcal{I}_{W,\mathbb{P}^2}(-2)$, where $\dim W \leq 0$, and there is an exact triple

$$(202) \qquad 0 \to \mathcal{I}_{W,\mathbb{P}^2} \xrightarrow{\theta_\rho} \mathcal{F}_{\mathbb{P}^2}(2) \xrightarrow{\bar{\rho}} \mathcal{I}_{Z,\mathbb{P}^2}(4) \to 0.$$

The monomorphism $\theta = \theta_\rho$ in this triple extends to a commutative square

$$(203) \qquad \begin{array}{ccc} \mathcal{I}_{W,\mathbb{P}^2} & \xrightarrow{\theta_\rho} & \mathcal{F}_{\mathbb{P}^2}(2) \\ {\scriptstyle \mathrm{can}} \downarrow & & \downarrow {\scriptstyle \mathrm{can}} \\ \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\theta_\rho^{\vee\vee}} & (\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2), \end{array}$$

and we obtain a morphism $j : \Pi(\mathcal{F}, \mathbb{P}^2) \to \mathbb{P}(H^0((\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2)))$, $\langle \rho \rangle \mapsto \theta_\rho^{\vee\vee}$. To construct the inverse to j morphism $\psi : (\mathrm{im} j) \to \Pi(\mathcal{F}, \mathbb{P}^2)$, take any $(\tilde{\theta} : \mathcal{O}_{\mathbb{P}^2} \to (\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2)) \in \mathrm{im} j$. The morphism $\theta : \mathcal{I}_{W,\mathbb{P}^2} \to \mathcal{F}_{\mathbb{P}^2}(2)$ such that $\tilde{\theta} = \theta^{\vee\vee}$ is recovered from $\tilde{\theta}$ as $\tilde{\theta}|_{\mathcal{I}_{W,\mathbb{P}^2}}$, where $\mathcal{I}_{W,\mathbb{P}^2} = \mathrm{can}^{-1}(\tilde{\theta}(\mathcal{O}_{\mathbb{P}^2}) \cap \mathrm{can}(\mathcal{F}_{\mathbb{P}^2}(2)))$. Then $\bar{\theta}$ defines via θ a morphism $\bar{\rho}$ as the quotient morphism $\mathcal{F}_{\mathbb{P}^2}(2) \to \mathrm{coker}\theta \simeq \mathcal{I}_{Z,\mathbb{P}^2}(4)$, and we set $\psi(\langle \bar{\theta} \rangle) := \langle \bar{\rho} \circ (- \otimes \mathcal{O}_{\mathbb{P}^2}) \rangle$. The openness of j follows from the openness of the condition $\rho : \mathcal{F} \to \mathcal{O}_{\mathbb{P}^2}(2)$ to be surjective.

Next, remark that, in (203), the $\mathcal{O}_{\mathbb{P}^2}$-sheaf $\mathrm{coker}\theta \simeq \mathcal{I}_{Z,\mathbb{P}^2}(4)$ has no torsion, hence there is no nonzero morphism $\mathcal{O}_W = \mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_{W,\mathbb{P}^2} \to \mathrm{coker}\theta$, since $\dim W \leq 0$. Thus, (203) and the Snake Lemma yield an exact triple (199) with $W(\rho) = W$.

(ii) The injection $\Sigma_{([\mathcal{F}],\mathbb{P}^2,\langle \rho \rangle)} \hookrightarrow \mathbb{P}(\mathrm{Hom}(\mathcal{L}_\rho, \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10}$ is an open embedding since the condition that $\gamma : \mathcal{L} := \mathcal{L}_\rho \to \mathcal{O}_{\mathbb{P}^2}(2)$ is an epimorphism and $\ker \gamma$ is locally free is open on $\langle \gamma \rangle \in \mathbb{P}(\mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^2}(2)))$. We thus have to show that $\dim \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^2}(2)) = 11$. Consider the epimorphism $\bar{\gamma} = \gamma|_{\mathbb{P}^2} : \mathcal{L}_{\mathbb{P}^2} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)$. Since by definition $[\mathcal{L}] \in \mathcal{R}(1, 4, 6)$, it follows that $\ker \bar{\gamma} \simeq \mathcal{I}_{Y,\mathbb{P}^2}(-1)$ for some subscheme Y of $\mathbb{P}^2$, $\dim Y = 0$, $\ell_Y = 6$. This yields an exact triple $0 \to \mathcal{I}_{Y,\mathbb{P}^2}(-1) \to \mathcal{L}_{\mathbb{P}^2} \xrightarrow{\bar{\gamma}} \mathcal{O}_{\mathbb{P}^2}(2) \to 0$. Applying to it the functor $\mathcal{E}xt^\bullet_{\mathcal{O}_{\mathbb{P}^2}}(-, \mathcal{O}_{\mathbb{P}^2}(2))$ we obtain an exact triple $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{H}om(\mathcal{L}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2)) \to \mathcal{O}_{\mathbb{P}^2}(3) \to 0$ which implies $\dim \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathbb{P}^2}(2)) = \dim \mathrm{Hom}(\mathcal{L}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2)) = h^0(\mathcal{H}om(\mathcal{L}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2))) = 11$.

(iii) Since $([\mathcal{F}], \mathbb{P}^2) \in {}^{-1}\mathcal{D}_k$, $(\mathcal{F}_{\mathbb{P}^2})^{\vee\vee} \simeq \mathcal{F}_{\mathbb{P}^2}$ is locally free $\mathcal{O}_{\mathbb{P}^2}$-sheaf, and (199) implies $k = 0$. Now, if $\mathcal{F}$ is unstable, then applying to (182) the functor $- \otimes \mathcal{O}_{\mathbb{P}^2}(2)$ we have an exact triple

$$(204) \qquad 0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{F}_{\mathbb{P}^2} \to \mathcal{I}_{Y,\mathbb{P}^2}(2) \to 0, \qquad \dim Y = 0, \qquad \ell_Y = \deg C = 2,$$

and this triple yields the desired values of $h^i((\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2)) = h^i(\mathcal{F}_{\mathbb{P}^2}(2))$. If $\mathcal{F}$ is stable, then these values are given by Proposition 27.(iv).

(iv) Since $([\mathcal{F}], \mathbb{P}^2) \in {}^0\mathcal{D}_k \cup {}^2\mathcal{D}_k \neq \emptyset$, the morphism can in (199) is not an isomorphism, hence $k = \ell_{W(\rho)} \geq 1$. On the other hand, $k \leq 3$ by Propositions 27(i) and 26(i). As above, if $\mathcal{F}$ is unstable, the triple (204) is true, which yields the equalities $h^0(\mathcal{F}_{\mathbb{P}^2}(2)) = 10$, $h^1(\mathcal{F}_{\mathbb{P}^2}(2)) = 0$. Respectively, if $\mathcal{F}$ is stable, these equalities follow from Proposition 27.(iv). Whence, by (199), we have (200).

We only have to show that, in case $\mathcal{F}$ is unstable, $k \leq 2$. By the definition of the sets $^i\mathcal{D}_k^u$, $i = 0, 1$, the condition $([\mathcal{F}], \mathbb{P}^2) \in {}^0\mathcal{D}_k^u$ implies that $\mathbb{P}^2 \notin {}^1S_{\mathcal{F}}^u$. This means that the exact triple (204) is true, with $\dim Y = 0$, $\ell_Y = \deg C = 2$. Dualizing this $\mathcal{O}_{\mathbb{P}^2}$-triple we easily obtain an inequality $h^0(\mathcal{E}xt^1(\mathcal{F}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2})) \leq h^0(\mathcal{E}xt^2(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^2})) = \ell_Y = 2$ and an exact tripe $0 \to \mathcal{O}_{\mathbb{P}^2} \to (\mathcal{F}_{\mathbb{P}^2})^\vee \to \mathcal{I}_{Z,\mathbb{P}^2}(2) \to 0$ for some scheme $Z \subset \mathbb{P}^2$ with $\dim Z \leq 0$, $\ell_Z = 2 - h^0(\mathcal{E}xt^1(\mathcal{F}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}))$. This triple, together with the triple (204) and the isomorphism $(\mathcal{F}_{\mathbb{P}^2})^\vee \simeq (\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}$, yields an exact triple $0 \to \mathcal{F}_{\mathbb{P}^2}(2) \xrightarrow{\mathrm{can}} (\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2) \to K \to 0$, where K is an artinian sheaf of length $h^0(K) = h^0(\mathcal{E}xt^1(\mathcal{F}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2})) \leq 2$. Comparing this triple with (199) we obtain $K \simeq \mathcal{O}_{W(\rho)}$ and $k = \ell_{W(\rho)} \leq 2$.

(v) From the condition $([\mathcal{F}], \mathbb{P}^2) \in {}^1\mathcal{D}_k \neq \emptyset$ and Proposition 26 it follows that $\mathbb{P}^2 \cap \mathrm{Sing}\mathcal{F} = l$ is a line, if $k = 0, 1, 2$; respectively, $\mathbb{P}^2 \cap \mathrm{Sing}\mathcal{F} = C$ is a conic, if $k = 3$. Thus, applying to the triples (182) and (183) the functor $- \otimes \mathcal{O}_{\mathbb{P}^2}(2)$ and using the resolution $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^2} \to 0$, we obtain the following exact

triples, where $\dim W = 0$ and if $k = 0$ or 1, then $W \subset l$:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{F}_{\mathbb{P}^2}(2) \to \mathcal{I}_{C,\mathbb{P}^3}(2)|_{\mathbb{P}^2} \to 0,$$

$$0 \to \mathcal{O}_l(3-k) \to \mathcal{I}_{C,\mathbb{P}^3}(2)|_{\mathbb{P}^2} \to \mathcal{I}_{W,\mathbb{P}^2}(1) \to 0, \ \ell_W = 3 - k, \ k \leq 2,$$

(205)

$$0 \to \mathcal{O}_l \to \mathcal{I}_{C,\mathbb{P}^3}(2)|_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0, \ k = 3, \ \ C \not\subset \mathbb{P}^2, \ \ \mathbb{P}^2 \cap C = l,$$

$$0 \to \mathcal{O}_C(1) \to \mathcal{I}_{C,\mathbb{P}^3}(2)|_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2} \to 0, \ k = 3, \ \ C \subset \mathbb{P}^2.$$

Since $\mathcal{F}$ is locally free for $k = 0$, $h^i((\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2)) = h^i(\mathcal{F}_{\mathbb{P}^2}(2))$, from (205) we obtain (201). Respectively, for $k = 1, 2, 3$, (205) and (199) imply (200). □

For $0 \leq k \leq 3$, let $B \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$ be the graph of incidence, $\mathcal{O}_B(2) = \mathcal{O}_{\mathbb{P}^3}(2) \boxtimes \mathcal{O}_{\check{\mathbb{P}}^3}|_B$, and let $pr_0 : \mathcal{D}_k^u \to \mathcal{D}_k^u$, $pr_1 : \mathcal{D}_k^u \to \mathcal{R}_k^u$, $\mathcal{D}_k^u \to \mathbb{P}^3 \times \check{\mathbb{P}}^3$ be the projections. For each $m \geq 0$ consider the set

(206) $$Y = Y_{k,m}^u := \{([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k^u \mid \dim \mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}(2)) = m\}, \quad m \geq 0,$$

and set $\mathbf{Y} = pr_0^{-1}(Y)$, $q_i = pr_i|_{\mathbf{Y}}$, $i = 0, 1, 2$, $L = \mathcal{E}xt_{q_0}(q_1^*\mathcal{F}, q_2^*\mathcal{O}_B(2))$, where $\mathcal{F}$ is the universal sheaf on $\mathcal{R}_k^u$ which exists by Proposition 26.(iv), $\mathcal{Y} = \mathbf{Y} \times_Y \mathbf{P}(L^\vee)$, and let $\mathbf{P}(L^\vee) \overset{\lambda}{\leftarrow} \mathcal{Y} \overset{\pi}{\to} \mathbf{Y}$ and $\mathcal{Y} \overset{\mu}{\to} \mathbf{P}(L^\vee) \overset{\nu}{\to} Y$ be the projections. By [2, Satz 3], $Y = Y_{k,m}^u$ is locally closed in $\mathcal{D}_k^u$ and the sheaf L is a rank m locally free sheaf on Y which commutes with the base change, i. e., for $y = ([\mathcal{F}], \mathbb{P}^2) \in Y$, one has $L|_y = \mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}(2))$. On $\mathcal{Y}$ there is a universal morphism $\boldsymbol{\rho} : (q_1 \circ \pi)^*\mathcal{F} \to (q_2 \circ \pi)^*\mathcal{O}_B(2) \otimes \mu^*\mathcal{O}_{\mathbf{P}(L^\vee)}(1)$. Consider the set

(207) $$X = X_{k,m}^u := \{([\mathcal{F}], \mathbb{P}^2) \in Y_{k,m}^u \mid K_{([\mathcal{F}],\mathbb{P}^2)} \neq \emptyset\}.$$

From this definition it follows that the sheaf $\mathrm{im}\boldsymbol{\rho}$ is flat over $\mathbf{P}(L^\vee)$ at any point $x \in \nu^{-1}(X)$. This implies that X is an open (possibly, empty) subset of Y, hence it is locally closed in $\mathcal{D}_k^u$. Therefore, since in view of Proposition 26.(iv) $\mathcal{D}_k^u$ are varieties, the set $\Phi_k^u = \{m \in \mathbb{Z}_{\geq 0} \mid X_{k,m}^u \neq \emptyset\}$ is finite. By the definitions (191), (192), (194) and (207) we have

(208) $$X_k^u := \bigsqcup_{m \in \Phi_k^u} X_{k,m}^u = p_{1k}(\mathcal{Q}_k^u), \qquad \mathcal{Q}_k^u = p_{1k}^{-1}(X_k^u).$$

Denoting ${}^iX_k^u = {}^i\mathcal{D}_k^u \cap X_k^u$, ${}^i\mathcal{Q}_k^u = p_{1k}^{-1}({}^iX_k^u)$, $-1 \leq i \leq 2$, we find from the first equality (189) that

(209) $$\mathcal{Q}_k^u = \bigsqcup_{-1 \leq i \leq 2} {}^i\mathcal{Q}_k^u.$$

The inclusion (196) and Lemma 28.(i) yield that the projection $p_{1k} : {}^i\mathcal{Q}_k^u \to {}^iX_k^u$ decomposes as

(210) $$p_{1k} : {}^i\mathcal{Q}_k^u \overset{\mathrm{open}}{\hookrightarrow} {}^i\tilde{\mathcal{Q}}_k^u \overset{\tilde{p}_{1k}}{\longrightarrow} {}^iX_k^u, \qquad -1 \leq i \leq 2, \quad 0 \leq k \leq 3,$$

where ${}^i\tilde{\mathcal{Q}}_k^u \overset{\tilde{p}_{1k}}{\longrightarrow} {}^iX_k^u$ is the projective fibration with fibre $\mathbb{P}(H^0((\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2)))$ over an arbitrary point $([\mathcal{F}], \mathbb{P}^2) \in {}^iX_k^u$. Here by (208) each ${}^iX_k^u$ is a disjoint union of schemes, This shows that each ${}^i\mathcal{Q}_k^u$ is a disjoint union of schemes. Since ${}^iX_k^u \subset \mathcal{D}_k^u$, it follows from (190) and Lemma 28.(iii)-(v) that

(211) $$\dim {}^i\mathcal{Q}_k^u \leq \dim {}^i\mathcal{R}_k^u + 11 - i + k, \qquad -1 \leq i \leq 2, \quad 0 \leq k \leq 3.$$

Thus, in view of (184), we obtain $\dim {}^i\mathcal{Q}_k^u \leq 26$ for all possible i, k, hence (209) yields

(212) $$\dim \mathcal{Q}_k^u \leq 26, \qquad 0 \leq k \leq 3.$$

To obtain a similar estimate for dimensions of $\mathcal{Q}_k^s$, we define similarly to (206) the locally closed subsets $Y_{k,m}^s := \{([\mathcal{F}], \mathbb{P}^2) \in \mathcal{D}_k^s \mid \dim \mathrm{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}(2)) = m\}$, $m \geq 0$, of $\mathcal{D}_k^s$. Next, note that apriori there is no universal sheaf $\mathcal{F}$ on $\mathcal{R}_k^s$. However, by Proposition 27, $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$ for any $[\mathcal{F}] \in \mathcal{R}_k^s$, $0 \leq k \leq 2$. This means that the deformation theory for $\mathcal{R}_k^s$ is unobstructed, so there exists an open cover $\mathcal{R}_k^s = \bigcup_{j \in J} \mathcal{U}_j$ and universal sheaves $\mathcal{F}_j$ on $\mathcal{U}_j$ (see, e. g., [9, Appendix A1-A2], [17, Ch. 6]). The existence of these local universal sheaves is enough to show that the sets $X_{k,m}^s$ defined similarly to (207) as $X_{k,m}^s := \{([\mathcal{F}], \mathbb{P}^2) \in Y_{k,m}^s \mid K_{([\mathcal{F}],\mathbb{P}^2)} \neq \emptyset\}$, are locally closed subsets of $\mathcal{D}_k^s$. We then have, similarly to (208)-(209), a finite dijoint unions of schemes $X_k^s := \bigsqcup_{m \in \Phi_k^s} X_{k,m}^s = p_{1k}(\mathcal{Q}_k^s)$ and relations $\mathcal{Q}_k^s = p_{1k}^{-1}(X_k^s)$. Denoting ${}^iX_k^s = {}^i\mathcal{D}_k^s \cap X_k^s$, ${}^i\mathcal{Q}_k^s = p_{1k}^{-1}({}^iX_k^u)$, $-1 \leq i \leq 2$, and mimicking the argument in (209)-(211) with u substituted by s, we obtain that ${}^i\mathcal{Q}_k^s$, respectively, $\mathcal{Q}_k^s$ are disjoint unions of schemes satisfying the inequalities $\dim {}^i\mathcal{Q}_k^s \leq \dim {}^i\mathcal{R}_k^s + 11 - i + k$, $-1 \leq i \leq 2$, $0 \leq k \leq 2$. These formulas and Proposition 27.(ii) imply the inequalities $\dim \mathcal{Q}_k^s \leq 26$, $0 \leq k \leq 2$, which, together with (212) and (195) yield

(213) $$\dim \mathcal{Q}_k \leq 26, \qquad 0 \leq k \leq 3.$$

(Remind that, as in (195), we set $\mathcal{Q}_3^s = T_3^s = \emptyset$.) Now from Lemma 28.(ii) and the last formula in the display (193), similarly to (210), we obtain that T_k are disjoint unions of schemes and projections $p_{2k} : T_k \to \mathcal{Q}_k$ are morphisms which decompose as $p_{2k} : T_k \xhookrightarrow{\text{open}} \tilde{T}_k \xrightarrow{\tilde{p}_{2k}} \mathcal{Q}_k$, where $\tilde{T}_k \xrightarrow{\tilde{p}_{2k}} {}^i\mathcal{Q}_k$ is the projective fibration with fibre $\mathbb{P}(H^0((\mathcal{F}_{\mathbb{P}^2})^{\vee\vee}(2))) \simeq \mathbb{P}^{10}$ over an arbitrary point $([\mathcal{F}], \mathbb{P}^2, \langle\rho\rangle) \in \mathcal{Q}_k$, $0 \leq k \leq 3$. This together with (213) yields

$$(214) \qquad \dim T_k \leq 36, \qquad 0 \leq k \leq 3.$$

Proof of Theorem 21. It is clear from the above that the maps $f_k : T_k \to \mathcal{B}(5)$ defined in (197) are morphisms. The inequality $\dim(\mathcal{H} \smallsetminus (\mathcal{G}(2,1) \cap \mathcal{H})) \leq 36$ now follows from (198) and (214). However, by [20, Remark 3.4.1], any irreducible component of $\mathcal{B}(5)$ has dimension at least 37. Hence, Theorem 21 follows. $\qquad\square$

8. Components of $\mathcal{B}(5)$

We finally have at hand all the results needed to complete the proof of our second main result, namely, the characterization of the irreducible components of $\mathcal{B}(5)$ given by Main Theorem 2. This entire section will be devoted to this goal.

Proof of Main Theorem 2. The first ingredient of the proof is the fact, proved by Hartshorne and Rao, that every bundle in $\mathcal{B}(5)$ is cohomology of one of the monads (2)-(6), cf. [21, Table 5.3, case 5.(1)-(4)].

Recall that for each stable rank 2 bundle E on $\mathbb{P}^3$ with vanishing first Chern class, the number $\alpha(E) := h^1(E(-2))$ mod 2 is called the Atiyah–Rees α-invariant of E, see [19, Definition on p. 237]. Hartshorne showed [19, Corollary 2.4] that this number is invariant on the connected components of the moduli space of stable vector bundles on $\mathbb{P}^3$. One can easily check that the cohomologies of monads of the form (2) and (3) have α-invariant equal to 0, while the cohomologies of the other three types of monads have α-invariant equal to 1.

Rao showed in [35] that the family of bundles obtained as cohomology of monads of the form (3) is irreducible, of dimension 36, and it lies in a unique component of $\mathcal{B}(5)$. Since instanton bundles of charge 5, i. e. the cohomologies of monads of the form (2), yield an irreducible family of dimension 37, it follows that the set

$$(215) \qquad \mathcal{I} := \{[E] \in \mathcal{B}(5) \mid \alpha(E) = 0\}$$

forms a single irreducible component of $\mathcal{B}(5)$, of dimension 37, whose generic point corresponds to an instanton bundle. In addition, every $[E] \in \mathcal{I}$ satisfies $\mathrm{H}^1(\mathcal{E}nd(E)) = 37$; this was originaly proved by Katsylo and Ottaviani for instanton bundles [30], and by Rao for the cohomologies of monads of the form (3) [35, Section 3]. Therefore, we also conclude that $\mathcal{I}$ is nonsingular. This completes the proof of the first part of the Main Theorem.

Our next step is to analyse those bundles with Atiyah–Rees invariant equal to 1.

Hartshorne proved in [20, Theorem 9.9] that the family $\mathcal{K}$ of stable rank 2 bundles E with $c_1(E) = 0$ and $c_2(E) = 5$ whose spectrum is $(-2, -1, 0, 1, 2)$ is an irreducible, nonsigular family of dimension 40, and from the definition of spectrum one has

$$(216) \qquad h^1(\mathcal{E}(-2)) = 3, \qquad [\mathcal{E}] \in \mathcal{K}.$$

The bundles from $\mathcal{K}$ are precisely those given as cohomologies of monads of the form (4), cf. [21, Table 5.3, case 5.(4)], which is a particular case of a class of monads studied by Ein in [15]. It is shown in [15] that the closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ in $\mathcal{B}(5)$ is an irreducible component of $\mathcal{B}(5)$ of dimension 40.

We proved in Main Theorem 1, case $a = 2$, that the bundles arising as cohomology of monads of the form (5) form a dense subset $\mathcal{G}(2,1)$ of a rational irreducible component of dimension 37. Consider the set $\mathcal{H}$ of bundles arising as cohomology of monads of the form (6). Since the bundles from $\mathcal{G}(2,1) \cup \mathcal{H}$ have the spectrum $(-1, 0, 0, 0, 1)$ by [21, Table 5.3, case 5.(2)], by we have (cf. (164))

$$(217) \qquad h^1(\mathcal{E}(-2)) = 1, \qquad [\mathcal{E}] \in \mathcal{G}(2,1) \cup \mathcal{H},$$

so that $\alpha(\mathcal{E}) = 1$, and therefore, in view of (215), $\mathcal{H} \cap \mathcal{I} = \emptyset$. Since, by Theorem 21, $\mathcal{H}$ does not constitute a component in $\mathcal{B}(5)$, it then follows from the above that $\mathcal{H} \subset \overline{\mathcal{G}(2,1)} \cup \overline{\mathcal{K}}$.

Proposition 29. $\mathcal{H} \subset \overline{\mathcal{G}(2,1)}$ *and* $\overline{\mathcal{K}} = \mathcal{K}$.

Proof. We only have to show that $(\mathcal{G}(2,1) \cup \mathcal{H}) \cap \overline{\mathcal{K}} = \emptyset$. Suppose by contradiction that there exists a vector bundle $[\mathcal{E}] \in (\mathcal{G}(2,1) \cup \mathcal{H}) \cap \overline{\mathcal{K}}$. By (216) and the inferior semi-continuity of the dimension of the cohomology groups of coherent sheaves, one has that $h^1(\mathcal{E}(-2)) \geq 3$, contrary to (217). $\qquad\square$

This last proposition finally concludes the proof of Main Theorem 2. $\qquad\square$

We summarize all the information in the Main Theorem 2, and the discrete invariants of stable rank 2 bundles with $c_1 = 0$ and $c_2 = 5$ in the following table.

TABLE 1. Irreducible components of $\mathcal{B}(5)$

Component	Dimension	Monads	Spectra	α-invariant
Instanton	37	(2)	(0,0,0,0,0)	0
		(3)	(-1,-1,0,1,1)	
Ein	40	(4)	(-2,-1,0,1,2)	1
Modified Instanton	37	(5)	(-1,0,0,0,1)	1
		(6)		

Remark. Inspired by the techniques introduced in the present paper, the authors of [38] construct another infinite series of irreducible components of $\mathcal{B}(0, n)$ whose special point corresponds to a bundle obtained as the cohomology of a monad similar to the one in display (24), just substituting a direct sum of two rank 2 instantons bundles for the rank 4 instanton bundle of charge 1 in middle term.

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