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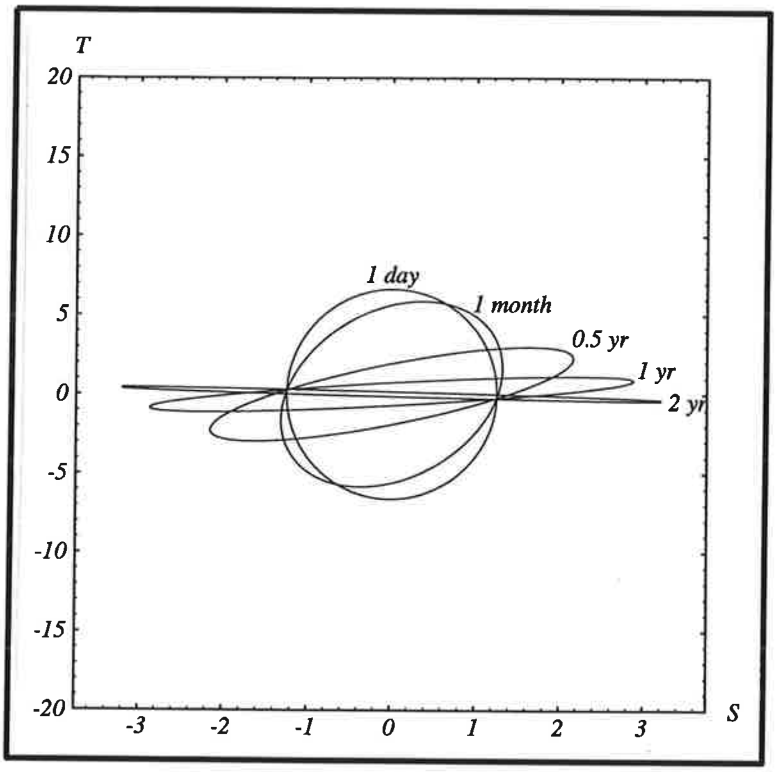

PART I: WHY ARE SALINITY ANOMALIES OF GREAT IMPORTANCE FOR LONG-TIME CLIMATE VARIABILITY? ANALYSIS OF SIMPLE CLIMATE MODELS

PART II: EXACT SOLUTION OF JACOBI TYPE EVOLUTION EQUATIONS

by

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Exact Solution of Jacobi Type Evolution Equations

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Abstract

We present a class of exactly solvable nonlinear evolution equations that arise in the context of the stability of the ocean's thermohaline circulation. Using Lyapunov techniques we obtain the solution of this type of equations by isolating their invariant subsets in phase space. It is shown that some solutions have finite escape time. In extension, the method is applicable to the analysis of partial differential equations of similar structure.

Key words: Climate models, nonlinear ordinary differential equations, Lyapunov techniques
AMS subject classifications: 34C20, 58F07, 58F10, 58F40, 47A99

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1 Introduction

The representation of elementary aspects of the oceanic large-scale circulation in terms of box models has attracted some studies of climate sensitivity (among others: Stommel, 1961; Marotzke, 1994; Ruddick and Zhang, 1996). Following Stommel (1961), these studies consider primarily the possibilities of multiple equilibria of the large-scale thermohaline circulation. This topic was boosted by Bryan's (1986) study, finding that complex numerical ocean circulation models exhibit multiple equilibria under the same atmospheric forcing conditions. To understand the interactions of the thermohaline circulation with other climate components, energy balance models proposed by Budyko (1969) and Sellers (1969) have been rediscovered. These models parameterize the meridional heat transport in terms of diffusion (North, 1975 a b), more recent energy balance models include furthermore a hydrological cycle (Jentsch, 1991; Chen et al., 1995; Chu and Ledley, 1995).

As the simplest possible conceptual model for the thermohaline circulation, moist atmospheric energy balance models have been coupled to Stommel's (1961) ocean model (Nakamura et al., 1994; Marotzke and Stone, 1995; Lohmann et al., 1996). Here, we will present the analytical solution of this class of simplified models which are based on Stommel's (1961) box model. In the following, the derivation of the analytical solution for this class of simplified models will be discussed. An extensive analysis of the physical implications of this solution for climate relevant aspects of the oceanic thermohaline circulation is given elsewhere (Lohmann and Schneider, 1997).

The ordinary differential equation is transformed into a linear differential equation which can be solved analytically. For this transformation, we analyze the invariant subsets of the evolution equation. The method of obtaining invariant subsets is quite a general one: A polynomial ansatz for a Lyapunov function is used which is similar to the well known method of Zubov (1964).

2 Solving evolution equations of Jacobi-type

Let U, V be linear spaces, H a Hilbert space and X and Y some sets. We use the symbols:

$L(\mathrm{U},\mathrm{V})$	The set of linear continuous mappings from U to V.
$\langle\cdot,\cdot\rangle$	The scalar product of the Hilbert space H.
$(\mathrm{X} \longrightarrow \mathrm{Y})$	The set af all mappings from X to Y.

Let $n \in \mathsf{N}$, $x \in \left(\mathsf{R}_0^+ \longrightarrow \mathsf{R}^n\right)$, $a \in \mathsf{R}^n$ and $A \in L(\mathsf{R}^n, \mathsf{R}^n)$. The climate models under consideration (among others: Stommel, 1961; Marotzke and Stone, 1995; Lohmann et al., 1996; Ruddick and Zhang, 1996; Lohmann and Schneider, 1997) are of the following structure:

$$\frac{d}{dt}x = Ax + \langle a, x\rangle\, x, \qquad x(0) = x_0 \in \mathsf{R}^n \tag{1}$$

where $\langle\cdot,\cdot\rangle : \mathsf{R}^n \times \mathsf{R}^n \longrightarrow \mathsf{R}$ is the canonical scalar product in R^n. Elimination of the time t in this equation yields a *Jacobian* differential equation as discussed by Stephanow (1982, Chapter I, § 5).

In Prop. 2.1, we obtain invariant sets in the phase space of ordinary differential equations using Lyapunov techniques:

Proposition 2.1 *Suppose that for every solution $t \longmapsto y(t)$ of a differential equation in R^n*

$$\frac{d}{dt} y = f(t, y), \quad y(t_0) = y_0, \qquad t, t_0 \in \mathsf{R}, \; y(t) \in \mathsf{R}^n \tag{2}$$

and some Lyapunov function $F \in (\mathsf{R}^n \longrightarrow \mathsf{R})$, one has

$$\frac{d}{dt} F(y(t)) = F(y(t)) \cdot \psi(y(t)), \qquad t > 0, \tag{3}$$

where $t \longmapsto F(y(t))$ is absolutely continuous and $t \longmapsto \psi(y(t))$ is locally integrable.
Then the sets

$$\begin{aligned} \mathcal{F}_0 &:= \{x \in \mathsf{R}^n \mid F(x) = 0\} \\ \mathcal{F}_> &:= \{x \in \mathsf{R}^n \mid F(x) > 0\} \\ \mathcal{F}_< &:= \{x \in \mathsf{R}^n \mid F(x) < 0\} \end{aligned}$$

are invariant sets of the differential equation (2).

Proof:

For $t, t_0 \in \mathsf{R}$ (3) yields

$$\frac{d}{dt}\left[F(y(t)) \cdot \exp\left(-\int_{t_0}^{t} \psi(y(\tau))\, d\tau\right)\right] = 0 \qquad ,$$

thus

$$F(y(t)) = F(y(t_0)) \cdot \exp\left(\int_{t_0}^{t} \psi(y(\tau))\, d\tau\right)$$

and the operator $F \in (\mathsf{R}^n \longrightarrow \mathsf{R})$ conserves the sign of $y(t) \in \mathsf{R}^n$.

$\square$

For our differential equation (1), we try the simplest possible form for F , a function that is the sum of a nonzero constant and a linear part. Because of notational reasons, a proper choice for $F \in (\mathsf{R}^n \longrightarrow \mathsf{R})$ is:

$$F(x) = \langle \omega, x \rangle - 1 \quad , \qquad \text{with } \omega \in \mathsf{R}^n . \tag{4}$$

It follows that

$$\begin{aligned} \frac{d}{dt} F(x) &= \langle \omega, Ax \rangle + \langle \omega, x \rangle \langle a, x \rangle \\ &= \langle A^* \omega, x \rangle + \langle a, x \rangle + F(x) \langle a, x \rangle \\ &= \langle A^* \omega + a, x \rangle + F(x) \langle a, x \rangle \qquad , \end{aligned}$$

where A^* is the (Hilbert space) adjoint of A. Therefore, we obtain using Prop. 2.1:

Proposition 2.2 *Let $\omega \in \mathsf{R}^n$ be defined as some solution of the equation $A^* \omega + a \stackrel{!}{=} 0$. Using the function F of (4) the sets (see Prop. 2.1) $\mathcal{F}_0$, $\mathcal{F}_>$ and $\mathcal{F}_<$ are invariant subsets of the differential equation (1). It is*

$$\frac{d}{dt} F(x) = \langle a, x \rangle F(x) \qquad .$$

This means especially, that for $x_0 \in \mathcal{F}_>$ or $x_0 \in \mathcal{F}_<$ one may apply the transformation

$$p := \frac{x}{F(x)} = \frac{x}{\langle \omega, x \rangle - 1} \tag{5}$$

to the differential equation (1), which maps the invariant subspace $\mathcal{F}_0$ to infinity. Inversion of that transformation yields

$$x = \frac{p}{F(p)} = \frac{p}{\langle \omega, p \rangle - 1} \qquad . \tag{6}$$

Using (1) and Prop. 2.2, we get

$$\begin{aligned}
\dot{p} &= \frac{\dot{x}}{F(x)} - \frac{x}{F(x)}\frac{\dot{F}(x)}{F(x)} \\
&\overset{\text{(Prop. 2.2)}}{=} \frac{\dot{x}}{F(x)} - \frac{x}{F(x)}\langle a, x\rangle \\
&\overset{(1)}{=} A\frac{x}{F(x)} = Ap \qquad .
\end{aligned}$$

Thus, p satisfies the linear differential equation

$$\frac{d}{dt}p = Ap, \qquad p(0) = p_0 \overset{(5)}{=} \frac{x_0}{\langle \omega, x_0\rangle - 1} \tag{7}$$

with solution

$$p(t) = \exp(At)\, p_0\,, \qquad t \in \mathsf{R}\,. \tag{8}$$

Equations (6) and (7) yield therefore the solution of our original differential equation (1):

$$x(t) = \frac{\exp(At)\, x_0}{\langle \omega, \exp(At)\, x_0\rangle - \langle \omega, x_0\rangle + 1}, \qquad A^{\dagger}\omega + a \overset{!}{=} 0. \tag{9}$$

The considerations leading to (9) show that this solution is valid only for $x_0 \in \mathcal{F}_< \cup \mathcal{F}_>$. However, (9) is also defined for $x_0 \in \mathcal{F}_0$. Substitution of $x(\cdot)$ from (9) into (1) and using $A^{*}\omega + a = 0$ shows, that this function solves the differential equation also in the remaining case $x \in \mathcal{F}_0$.

Some solutions of the differential equation (1) exist according to (9) only for a finite time, because these solutions escape to infinity in a finite amount of time. We denote the denominator in (9) as

$$\gamma(t; x_0) := \langle \omega, \exp(At)\, x_0\rangle - \langle \omega, x_0\rangle + 1 \qquad . \tag{10}$$

It is $\gamma(0) = 1$ initially. For the initial states $x_0 \in \mathsf{R}^n$, we define the boundary times

$$
\begin{aligned}
t_-(x_0) &:= \inf\{\tau \in (-\infty, 0] \mid \gamma(t; x_0) > 0 \quad \text{for all } t \in (\tau, 0]\} \\
t_+(x_0) &:= \sup\{\tau \in [0, +\infty) \mid \gamma(t; x_0) > 0 \quad \text{for all } t \in [0, \tau)\} \ ,
\end{aligned}
\tag{11}
$$

such that (1) exists in the time interval $(t_-(x_0), t_+(x_0))$. Summing up, we obtain the following result:

Theorem 2.1 *Let there exist a solution* ω *to the equation* $A^*\omega + a = 0$. *Then the solution of the differential equation* (1) *is*

$$
x(t) = \frac{\exp(At)\, x_0}{\langle \omega, \exp(At)\, x_0 \rangle - \langle \omega, x_0 \rangle + 1} \ ,
$$

$$
x_0 \in \mathsf{R}^n, \quad t \in (t_-(x_0), t_+(x_0)) \ .
$$

From (10), (11) and Th. 2.1 we obtain the following stability condition:

Proposition 2.3 *The solution* $\mathsf{R}_0^+ \ni t \longmapsto x(t) \in \mathsf{R}^n$ *starting at* $x_0 \in \mathsf{R}^n$ *does not escape to infinity in finite time* $\Longleftrightarrow$

$$
\gamma(t; x_0) > 0 \qquad \text{for all } t \in \mathsf{R}_0^+ \ .
$$

Bounded solutions of this type are of interest for the application of (1) to climate sensitivity studies. Problems of wave-breaking or the onset of turbulence, on the other hand, are particularly concerned with the escape-time determined by solutions of (1) with $\gamma(t; x_0) \leq 0$.

Naturally, there arises the question weather we have simply solved a particular type of equation or we have found a method of solving more general classes of equations. The next theorem gives us an answer:

Theorem 2.2 *Consider the differential equation in* R^n

$$\frac{d}{dt}x = Ax + f(x), \qquad x(0) = x_0 \in \mathsf{R}^n,$$
$$f(x) = o(\|x\|), \qquad x \longrightarrow 0 \quad ,$$

where $\|.\|$ *denotes a norm in* R^n. *There exists a transformation* $p = \sigma(x)\,x$ *with* $\sigma \in (\mathsf{R}^n \longrightarrow \mathsf{R})$ *transforming the nonlinear differential equation into the linearized equation* $\dot{p} = Ap \iff$

$$f \text{ is of the form} \quad f(x) = \rho(x)\,x \quad \text{with } \rho \in (\mathsf{R}^n \longrightarrow \mathsf{R}) \quad .$$

Proof:

Let $p := \sigma(x)\,x$ such that $\dot{p} = Ap$. Using the differential equation for x, one obtains with the abbreviation $q(x) := \ln(\sigma(x))$ the relation

$$f(x) + \langle \nabla q(x), Ax + f(x) \rangle\, x = 0 \quad ,$$

and thus $f(x) = \rho(x)\,x$ with $\rho(x) = \langle \nabla q(x), Ax + f(x) \rangle$.

On the other hand, let $f(x) = \rho(x)\,x$ and consider the linear partial differential equation for $\mathsf{R}^n \ni x \longmapsto q(x) \in \mathsf{R}$

$$\rho(x) + \langle \nabla q(x), (A + \rho(x))\,x \rangle = 0.$$

It possesses a solution q with characteristics leading to the original differential equation from which we obtain the transformation $p = e^{q(x)}x$. Inserting gives

$$\dot{p} = e^q \dot{x} + \dot{q}p = e^q \dot{x} - \rho p = Ap \quad .$$

$\square$

Therefore, our transformation may be applicable for the solution of a specific class of equations where the nonlinearity is directed towards the variable $x \in \mathsf{R}^n$. However, the first step of our approach, the determination of invariant subsets using the Lyapunov-Zubov method, is valid in a more general context. Without solving the equations, the method yields important information about the phase space structure to find appropriate coordinates to be used as a starting point for perturbation methods and numerical work.

Furthermore, the above algorithm is readily generalized to infinite dimensional systems. Then, we define our differential equation as an evolution equation in a convenient functional analytic setting (Showalter, 1979; Temam, 1988). As an example consider the integro differential equation

$$\frac{\partial \psi}{\partial t} = a\,\Delta\psi + \psi \cdot \int_{\mathsf{R}} \alpha(x)\,\psi(x)\; d\tau(x) \quad , \quad \psi \in \mathrm{H} \, , \quad a \in \mathsf{C},$$

where Δ denotes the Laplacian. The integral with kernel $\alpha(x)$ and measure $\tau(x)$ defines a linear operator on the Hilbert space H. Using a Hilbert space formulation and the above described method, this equation can be transformed into a solvable equation. It is conceivable that such integro differential equations can be used for nonlinear phenomena with a self-interaction or for modeling processes having a finite time interval. Such types of evolution equations provide a good starting point for further investigations.

3 Concluding Remarks

We were able to solve a class of nonlinear evolution equations in R^n by reducing it to a linear differential equation. The key step was to find a corresponding transformation. This was accomplished by using Lyapunov techniques, determing purely algorithmically the invariant subsets in phase space. Application of this procedure to a simple climate model provides a conceptual framework for the study of the thermohaline circulation (Lohmann and Schneider, 1997). Moreover, our method presented here is valuable for a large class of nonlinear evolution equations.

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