SURJECTIVE SEPARATING MAPS ON NONCOMMUTATIVE L^p-SPACES

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Abstract. Let $1 \leq p < \infty$ and let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded map between noncommutative L^p-spaces. If T is bijective and separating (i.e., for any $x, y \in L^p(\mathcal{M})$ such that $x^*y = xy^* = 0$, we have $T(x)^*T(y) = T(x)T(y)^* = 0$), we prove the existence of decompositions $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$, $\mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and maps $T_1\colon L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1)$, $T_2\colon L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2)$, such that $T = T_1 + T_2$, T_1 has a direct Yeadon type factorisation and T_2 has an anti-direct Yeadon type factorisation. We further show that T^{-1} is separating in this case. Next we prove that for any $1 \leq p < \infty$ (resp. any $1 \leq p \neq 2 < \infty$), a surjective separating map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded (resp. completely bounded) if and only if there exists a decomposition $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ such that $T|_{L^p(\mathcal{M}_1)}$ has a direct Yeadon type factorisation and $\mathcal{M}_2$ is subhomogeneous.

1. Introduction

This paper deals with separating maps between noncommutative L^p-spaces, $1 \leq p < \infty$. These operators were investigated recently in [1, 4, 5], to which we refer for background, motivation and historical facts. Recall that a bounded map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ between two noncommutative L^p-spaces is called separating if for any $x, y \in L^p(\mathcal{M})$, the condition $x^*y = xy^* = 0$ implies that $T(x)^*T(y) = T(x)T(y)^* = 0$. It was shown in [4, Proposition 3.11] and [1, Theorem 3.3 & Remark 3.4] that $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is separating if and only if there exists a w^*-continuous Jordan homomorphism $J\colon \mathcal{M} \to \mathcal{N}$, a positive operator B affiliated with $\mathcal{N}$ and commuting with the range of J, as well as a partial isometry $w \in \mathcal{N}$ such that $w^*w = s(B) = J(1)$ and

$$T(x) = wBJ(x), \qquad (x \in \mathcal{M} \cap L^p(\mathcal{M})).$$

Such a factorization (which is necessarily unique) is called a Yeadon type factorization in [4, 5]. We further say that T admits a direct Yeadon type factorization if the Jordan homomorphism J in this factorization is a $*$-homomorphism. It is proved in [5, Proposition 4.4] and [1, Theorem 3.6] that any separating map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ with a direct Yeadon type factorization is necessarily completely bounded. It is also proved in [5, Proposition 4.5] that any such map is S^1-bounded (see Section 2 below for the definition). The main purpose of the present paper is to establish a form of converse of these results for surjective maps. More precisely, we prove the following characterizations.

Theorem. Let $1 \leq p < \infty$, let $\mathcal{M}, \mathcal{N}$ be semifinite von Neumann algebras and let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a surjective separating map. The following are equivalent :

(i) T is S^1-bounded;

Key words and phrases. von Neumann algebras, noncommutative L^p-spaces, separating maps, operator spaces.

(ii) There exists a direct sum decomposition $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ such that the restriction of T to $L^p(\mathcal{M}_1)$ has a direct Yeadon type factorization and $\mathcal{M}_2$ is subhomogeneous.

Moreover if $p \neq 2$, then (ii) is also equivalent to :

(iii) T is completely bounded.

These results will be proved in Section 4. We also provide an example showing that the surjectivity assumption cannot be dropped. In section 3, we establish a general decomposition result for bijective separating maps which plays a key role in the above characterization results. We prove in passing that the inverse of any bijective separating map is separating as well. Section 2 is preparatory.

2. Background

In this section we recall some necessary background on semifinite noncommutative L^p-spaces and subhomogeneous von Neumann algebras.

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal semifinite faithful trace $\tau_{\mathcal{M}}$. Assume that $\mathcal{M} \subset B(\mathcal{H})$ acts on some Hilbert space $\mathcal{H}$. Let $L^0(\mathcal{M})$ denote the $*$-algebra of all closed densely defined (possibly unbounded) operators on $\mathcal{H}$, which are $\tau_{\mathcal{M}}$-measurable. Then for any $1 \leq p < \infty$, the noncommutative L^p-space associated with $\mathcal{M}$ can be defined as

$$L^p(\mathcal{M}) := \big\{x \in L^0(\mathcal{M}) \,:\, \tau_{\mathcal{M}}(|x|^p) < \infty\big\}.$$

We set $\|x\|_p := \tau_{\mathcal{M}}(|x|^p)^{\frac{1}{p}}$ for any $x \in L^p(\mathcal{M})$. Then $L^p(\mathcal{M})$ equipped with $\|\cdot\|_p$ is a Banach space. The reader may consult [3,8,12] and the references therein for details and further properties.

We let S^p, $1 \leq p < \infty$, denote the noncommutative L^p-space built upon $B(\ell^2)$ with its usual trace; this is in fact the Schatten p-class of operators on ℓ^2. For any $m \geq 1$, we let S^p_m denote the Schatten p-class of $m \times m$ matrices. Whenever E is an operator space, we let $S^p_m[E]$ denote the E-valued Schatten space introduced in [6, Chapter1].

Recall that we may identify $L^p(\mathcal{M} \otimes M_m)$ with $L^p(\mathcal{M}) \otimes S^p_m$ in a natural way. Let $\mathcal{N}$ be, possibly, another semifinite von Neumann algebra. We say that an operator $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is completely bounded if there exists a constant $K \geq 0$ such that

$$\|T \otimes I_{S^p_m} : L^p(\mathcal{M} \otimes M_m) \to L^p(\mathcal{N} \otimes M_m)\| \leq K,$$

for any $m \geq 1$. In this case, the completely bounded norm of T is the smallest such uniform bound and is denoted by $\|T\|_{cb}$. We further say that T is a complete isometry if $T \otimes I_{S^p_m}$ is an isometry for any $m \geq 1$.

In [5, Section 3], we introduced S^1-valued noncommutative L^p-spaces, which naturally extend previous constructions from [2,6]. We recall this definition here.

For $1 \leq p < \infty$, the S^1-valued noncommutative L^p-space, $L^p(\mathcal{M}; S^1)$, is the space of all infinite matrices $[x_{ij}]_{i,j \geq 1}$ in $L^p(\mathcal{M})$ for which there exist families $(a_{ik})_{i,k \geq 1}$ and $(b_{kj})_{k,j \geq 1}$ in $L^{2p}(\mathcal{M})$ such that $\sum_{i,k} a_{ik} a_{ik}^*$ and $\sum_{k,j} b_{kj}^* b_{kj}$ converge in $L^p(\mathcal{M})$ and for all $i, j \geq 1$,

$$x_{ij} = \sum_{k=1}^{\infty} a_{ik} b_{kj}.$$

We equip $L^p(\mathcal{M};S^1)$ with the following norm

$$\|[x_{ij}]\|_{L^p(\mathcal{M};S^1)} = \inf\left\{\Big\|\sum_{i,k=1}^{\infty} a_{ik}a_{ik}^*\Big\|_p^{\frac{1}{2}}\Big\|\sum_{k,j=1}^{\infty} b_{kj}^*b_{kj}\Big\|_p^{\frac{1}{2}}\right\}, \tag{1}$$

where the infimum is taken over all families $(a_{ik})_{i,k\geq 1}$ and $(b_{kj})_{k,j\geq 1}$ as above. The space $L^p(\mathcal{M};S^1)$ endowed with this norm is a Banach space.

For any integer $m \geq 1$, we let $L^p(\mathcal{M};S^1_m)$ be the subspace of $L^p(\mathcal{M};S^1)$ of matrices $[x_{ij}]_{i,j\geq 1}$ with support in $\{1,\ldots,m\}^2$.

Following [5, Definition 3.8], we say that a bounded operator $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded if there exists a constant $K \geq 0$ such that

$$\|T \otimes I_{S^1_m} : L^p(\mathcal{M};S^1_m) \longrightarrow L^p(\mathcal{N};S^1_m)\| \leq K,$$

for any $m \geq 1$. In this case, the S^1-bounded norm of T is the smallest such uniform bounded and is denoted by $\|T\|_{S^1}$. We further say that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is an S^1-isometry if for each $m \geq 1$, $T \otimes I_{S^1_m}$ is an isometry.

We proved in [5] that for any $n \geq 1$, $L^p(M_n;S^1_m) = S^p_n[S^1_m]$ isometrically. Further, if $\mathcal{M},\mathcal{N}$ are hyperfinite, then $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded if and only if it is regular in the sense of [7].

We note that any direct sum $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ induces isometric identifications $L^p(\mathcal{M}) = L^p(\mathcal{M}_1) \overset{p}{\oplus} L^p(\mathcal{M}_2)$ and $L^p(\mathcal{M};S^1) = L^p(\mathcal{M}_1;S^1) \overset{p}{\oplus} L^p(\mathcal{M}_2;S^1)$ (see [5, Lemma 5.2] for the last identification).

Recall that a C^*-algebra $\mathcal{A}$ is called subhomogeneous of degree $\leq N$ if all irreducible representations of $\mathcal{A}$ are of maximum dimension N. If $\mathcal{A}$ is subhomogeneous of degree $\leq N$, for some N, we simply say that $\mathcal{A}$ is subhomogeneous. It is well-known (see for example [9, Theorem 7.1.1]) that $\mathcal{M}$ is a subhomogeneous von Neumann algebra of degree $\leq N$ if and only if there exist $r \geq 1$, integers $1 \leq n_1 \leq n_2 \leq \ldots \leq n_r \leq N$ and abelian von Neumann algebras $L^\infty(\Omega_1),\ldots,L^\infty(\Omega_r)$ such that

$$\mathcal{M} \simeq \overset{\infty}{\underset{1\leq j\leq r}{\oplus}} L^\infty(\Omega_j;M_{n_j}). \tag{2}$$

If a von Neumann algebra $\mathcal{M}$ is not subhomogeneous of degree $\leq N$, it is well-known that there is a non zero $*$-homomorphism $\gamma : M_{N+1} \to \mathcal{M}$. Lemma 2.1 below makes this more explicit in the semifinite case.

Lemma 2.1. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $N \geq 1$. If $\mathcal{M}$ is not subhomogeneous of degree $\leq N$, then there is a complete isometry from S^p_{N+1} into $L^p(\mathcal{M})$ that is also an S^1-isometry.*

Proof. Let $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ be the direct sum decomposition of $\mathcal{M}$ into a type I summand $\mathcal{M}_1$ and a type II summand $\mathcal{M}_2$ (see e.g. [11, Section 5]).

Assume that $\mathcal{M}_2 \neq \{0\}$. Following the same lines as in [5, Lemma 2.3], there is a projection e in $\mathcal{M}_2$, a trace preserving von Neumann algebra identification

$$\mathcal{M}_2 \simeq M_{N+1}\overline{\otimes}(e\mathcal{M}_2e) \tag{3}$$

and a finite trace projection ε in $e\mathcal{M}_2e$ such that the mapping

$$\gamma : M_{N+1} \to \mathcal{M}_2 \subset \mathcal{M}; \qquad \gamma(a) = a \otimes \varepsilon$$

is a non zero $*$-homomorphism taking values in $L^1(\mathcal{M})$, and therefore $L^p(\mathcal{M})$.

For every $[a_{ij}]_{1\leq i,j\leq m}$ in $S^p_{N+1} \otimes S^p_m$ we have that

$$\|[a_{ij} \otimes \varepsilon]\|_{L^p(\mathcal{M}_2 \otimes M_m)} = \|\varepsilon\|_p \|[a_{ij}]\|_{L^p(M_{N+1}\otimes M_m)},$$

and therefore $\|\varepsilon\|_p^{-1}\gamma$ is a complete isometry from S^p_{N+1} into $L^p(\mathcal{M})$. By [5, Lemma 5.1],

$$\|[a_{ij} \otimes \varepsilon]\|_{L^p(\mathcal{M}_2;\, S^1_m)} = \|\varepsilon\|_p \|[a_{ij}]\|_{S^p_{N+1}[S^1_m]},$$

and therefore $\|\varepsilon\|_p^{-1}\gamma$ is also an S^1-isometry from S^p_{N+1} into $L^p(\mathcal{M})$.

If $\mathcal{M}_2 = \{0\}$, then $\mathcal{M}$ is of type I. Since $\mathcal{M}$ is not subhomogeneous of degree $\leq N$, it follows from [11, Theorem V.1.27] that there exist a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq N+1$ and an abelian von Neumann algebra W such that $\mathcal{M}$ contains $B(\mathcal{H})\overline{\otimes}W$ as a summand. Using this summand instead of (3) and arguing as above we obtain the result in this case as well. $\square$

3. Bijective separating maps and their inverses

The goal of this section is to provide a decomposition for bijective separating maps that facilitates their study. We apply this decomposition to show that the inverse of a bijective separating map is separating as well.

First we recall some terminologies and results that we will use. A Jordan homomorphism between von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ is a linear map $J : \mathcal{M} \to \mathcal{N}$ such that

$$J(x^*) = J(x)^* \quad \text{and} \quad J(xy + yx) = J(x)J(y) + J(y)J(x)$$

for all x and y in $\mathcal{M}$. It is plain that $*$-homomorphisms and anti-$*$-homomorphisms are Jordan homomorphisms. In fact, every Jordan homomorphism is a sum of a $*$-homomorphism and an anti-$*$-homomorphism, as we recall here.

Let $J : \mathcal{M} \to \mathcal{N}$ be a Jordan homomorphism and let $\mathcal{D} \subset \mathcal{N}$ be the w^*-closed C^*-algebra generated by $J(\mathcal{M})$. Then $J(1)$ is the unit of $\mathcal{D}$. By e.g. [10, Theorem 3.3], there exist projections e and f in the center of $\mathcal{D}$ such that $e + f = J(1)$, $x \mapsto J(x)e$ is a $*$-homomorphism, and $x \mapsto J(x)f$ is an anti-$*$-homomorphism. Let $\mathcal{N}_1 = e\mathcal{N}e$ and $\mathcal{N}_2 = f\mathcal{N}f$. Define $\pi : \mathcal{M} \to \mathcal{N}_1$ and $\sigma : \mathcal{M} \to \mathcal{N}_2$ by $\pi(x) = J(x)e$ and $\sigma(x) = J(x)f$, for all $x \in \mathcal{M}$. Then J is valued in $\mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and $J(x) = \pi(x) + \sigma(x)$, for all $x \in \mathcal{M}$.

Assume that $\mathcal{M}$ and $\mathcal{N}$ are semifinite von Neumann algebras and let $1 \leq p < \infty$. In [4], inspired by Yeadon's fundamental description of isometries between noncommutative L^p-spaces, we say that a bounded operator $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ has a Yeadon type factorization if there exist a w^*-continuous Jordan homomorphism $J\colon \mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a positive operator B affiliated with $\mathcal{N}$, which satisfy the following conditions:

(a) $w^*w = J(1) = s(B)$, the support projection of B;
(b) every spectral projection of B commutes with $J(x)$, for all $x \in \mathcal{M}$;
(c) $T(x) = wBJ(x)$ for all $x \in \mathcal{M} \cap L^p(\mathcal{M})$.

We call (w, B, J) the Yeadon triple associated with T. This triple is unique. Following [5], if J is a $*$-homomorphism (respectively, anti-$*$-homomorphism), we say that T has a direct (respectively, anti-direct) Yeadon type factorization.

Following [4], we say that a bounded operator $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is separating if for every $x, y \in L^p(\mathcal{M})$ such that $x^*y = xy^* = 0$, we have that $T(x)^*T(y) = T(x)T(y)^* = 0$. The following characterization has a fundamental role in the study of separating maps.

Theorem 3.1. *([1, Theorem 3.3], [4, Theorem 3.5]) A bounded operator $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ admits a Yeadon type factorization if and only if it is separating.*

It is easy to see that for a separating map $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ with Yeadon triple (w, B, J), we have that

$$T(z^*) = wT(z)^*w \qquad (z \in L^p(\mathcal{M})). \tag{4}$$

Also, if T has a direct (respectively, anti-direct) Yeadon type factorization, we get that

$$T(zm) = T(z)J(m) \ \text{ (respectively, } T(mz) = T(z)J(m)\text{)}\,, \tag{5}$$

for every $z \in L^p(\mathcal{M})$ and $m \in \mathcal{M}$.

Remark 3.2. Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a separating map with Yeadon triple (w, B, J). We observe that if T is surjective, then w is a unitary. Indeed on the one hand, we see that T is valued in $wL^p(\mathcal{N})$. Since $ww^*w = w$, this implies that T is valued in $ww^*L^p(\mathcal{N})$. Hence, if T is surjective, we have $ww^*L^p(\mathcal{N}) = L^p(\mathcal{N})$, which implies that $ww^* = 1$. On the other hand, $T(x) = T(x)J(1)$, for any $x \in L^p(\mathcal{M})$. Hence, T is valued in $L^p(\mathcal{N})J(1)$. Hence, if T is surjective, we have $L^p(\mathcal{N})J(1) = L^p(\mathcal{N})$, which implies $w^*w = J(1) = 1$.

Proposition 3.3. *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a separating map that is bijective. Then there exist direct sum decompositions*

$$\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2, \qquad \textit{and} \qquad \mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2,$$

and bounded bijective separating maps $T_1 : L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1)$ with a direct Yeadon type factorization and $T_2 : L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2)$ with an anti-direct Yeadon type factorization such that $T = T_1 + T_2$.

Proof. Assume that $w = 1$. Consider a decomposition for J, induced by central projections e and f, as recalled above. As detailed in [5, Remark 4.3], this induces a decomposition $\mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and separating maps

$$T_1\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{N}_1), \qquad T_1(x) = T(x)e,$$

with Yeadon triple (e, Be, π), and hence a direct Yeadon type factorization, and

$$T_2\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{N}_2), \qquad T_2(x) = T(x)f,$$

with Yeadon triple (f, Bf, σ), and hence an anti-direct Yeadon type factorization, such that $T = T_1 + T_2$.

Let $\mathcal{M}_1 := \ker(\sigma)$ and $\mathcal{M}_2 := \ker(\pi)$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are w^*-closed ideals of $\mathcal{M}$, there exist central projections $\alpha, \beta \in \mathcal{M}$ such that $\mathcal{M}_1 = \alpha\mathcal{M}$, and $\mathcal{M}_2 = \beta\mathcal{M}$. Set $\mathcal{M}_3 := (1-\alpha)(1-\beta)\mathcal{M}$. Note that $\alpha\beta \in \ker(\sigma) \cap \ker(\pi)$, and therefore $J(\alpha\beta) = 0$. Since T is one-to-one, by [4, Remark 3.14(a)], J is one-to-one and therefore we must have that $\alpha\beta = 0$. Hence,

$$1 = \alpha + \beta + (1-\alpha)(1-\beta).$$

Consequently, $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2 \overset{\infty}{\oplus} \mathcal{M}_3$, and so we have the decomposition

$$L^p(\mathcal{M}) = L^p(\mathcal{M}_1) \overset{p}{\oplus} L^p(\mathcal{M}_2) \overset{p}{\oplus} L^p(\mathcal{M}_3).$$

The result will follow if we can show that

$$L^p(\mathcal{M}_1) = \ker(T_2), \quad L^p(\mathcal{M}_2) = \ker(T_1) \quad \text{and} \quad \mathcal{M}_3 = \{0\}.$$

To see that $L^p(\mathcal{M}_1) \subseteq \ker(T_2)$, let $x \in \mathcal{M}_1 \cap L^p(\mathcal{M}_1)$, then

$$T_2(x) = B\sigma(x) = 0.$$

Hence, $\mathcal{M}_1 \cap L^p(\mathcal{M}_1) \subset \ker(T_2)$ and therefore $L^p(\mathcal{M}_1) \subset \ker(T_2)$. Now suppose that x belongs to $\ker(T_2)$. For any $n \geq 1$, let $p_n = \chi_{[-n,n]}(|x^*|)$, the projection associated with the indicator function of $[-n, n]$ in the Borel functional calculus of $|x^*|$, and $x_n := p_n x$. Then, using (5), we have that

$$T_2(x_n) = T_2(x)\sigma(p_n) = 0.$$

Hence, $B\sigma(x_n) = 0$. Since $s(B) = 1$, this implies that $\sigma(x_n) = 0$, that is x_n is in $\mathcal{M}_1$. Now because $x_n \to x$ in $L^p(\mathcal{M})$, we obtain that x belongs to $L^p(\mathcal{M}_1)$. Hence,

$$L^p(\mathcal{M}_1) = \ker(T_2).$$

Similarly, we can show that $L^p(\mathcal{M}_2) = \ker(T_2)$.

Finally, we show that $\mathcal{M}_3 = \{0\}$. Let $x \in L^p(\mathcal{M})$. By surjectivity of T, there is y in $L^p(\mathcal{M})$ such that $T(y) = T_1(x)$. Writing $T(y) = T_1(y) + T_2(y)$, we obtain that $T_1(x - y) = 0$ and $T_2(y) = 0$, that is $x - y$ belongs to $\ker(T_1) = L^p(\mathcal{M}_2)$ and y belongs to $\ker(T_2) = L^p(\mathcal{M}_1)$, thus x belongs to $L^p(\mathcal{M}_1) \overset{p}{\oplus} L^p(\mathcal{M}_2)$. Hence, $\mathcal{M}_3 = \{0\}$. This completes the proof in the case $w = 1$.

In the general case, consider the map $\widetilde{T} := w^*T(\cdot)$, which takes any $x \in \mathcal{M} \cap L^p(\mathcal{M})$ to $BJ(x)$. By Remark 3.2, $\widetilde{T}$ is also a bijective separating map. Its Yeadon triple is $(1, B, J)$. We may apply the above decomposition to the map $\widetilde{T}$ to obtain decompositions $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$, $\mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and bounded bijective separating maps $\widetilde{T_1} : L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1)$ with a direct Yeadon type factorization and $\widetilde{T_2} : L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2)$ with an anti-direct Yeadon type factorization such that $\widetilde{T} = \widetilde{T_1} + \widetilde{T_2}$. Since $w\widetilde{T} = T$, we obtain the result. □

Proposition 3.4. *Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is a bijective separating map, then*

(i) *$T^{-1} : L^p(\mathcal{N}) \to L^p(\mathcal{M})$ is separating.*
(ii) *If $J : \mathcal{M} \to \mathcal{N}$ is the Jordan homomorphism associated with T, then J is invertible and $J^{-1} : \mathcal{N} \to \mathcal{M}$ is the Jordan homomorphism associated with T^{-1}.*

Proof. Using the decomposition given in Proposition 3.3, it is enough to show parts (i) and (ii) for a bijective separating map with a direct Yeadon type factorization. So, throughout the proof we assume that this is the case. Note that by Remark 3.2, $J(1) = 1$.

(i) Suppose that $a, b \in L^p(\mathcal{N})$ such that $a^*b = ab^* = 0$. We show that $T^{-1}(a)^*T^{-1}(b) = T^{-1}(a)T^{-1}(b)^* = 0$. Let $x = T^{-1}(a)$ and $y = T^{-1}(b)$. Set $p_n := \chi_{[-n,n]}(|y|)$, for any

$n \geq 1$. We have that

$$\begin{aligned} T(x^*yp_n)B &= T(x^*)J(yp_n)B && \text{by (5)} \\ &= T(x^*)w^*T(yp_n) \\ &= wT(x)^*T(yp_n) && \text{by (4)} \\ &= wT(x)^*T(y)J(p_n) && \text{by (5)} \\ &= wa^*bJ(p_n) = 0. \end{aligned}$$

Since $s(B) = J(1) = 1$, we obtain $T(x^*yp_n) = 0$. Because T is one-to-one, we have that $x^*yp_n = 0$. Now, since $yp_n \to y$, we get that $x^*y = 0$. A similar argument using $ab^* = 0$ implies that $xy^* = 0$. Hence T^{-1} must be separating.

(*ii*) By part (*i*), T^{-1} is separating. We let J' denote the Jordan homomorphism of its Yeadon triple. Let $e \in \mathcal{N}$ be a projection with finite trace. For any $y \in e\mathcal{N}e$, we have that $T^{-1}(y) = T^{-1}(e)J'(y)$. Applying (5), we deduce that

$$y = TT^{-1}(y) = T\left(T^{-1}(e)J'(y)\right) = TT^{-1}(e)\ JJ'(y) = e\ JJ'(y).$$

Using the w^*-continuity of J and J', and the w^*-density of the union of the $e\mathcal{N}e$, for $\tau_{\mathcal{N}}(e) < \infty$, we deduce that $y = JJ'(y)$ for any $y \in \mathcal{N}$. By [4, Remark 3.14(a)], since T is one-to-one, J must be one-to-one. Hence, J is invertible with $J^{-1} = J'$. □

Remark 3.5. Part (*ii*) of Proposition 3.4 shows that a separating invertible map $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ admits a direct Yeadon type factorization if and only if T^{-1} does.

4. A CHARACTERIZATION OF COMPLETELY/S^1-BOUNDED SURJECTIVE SEPARATING MAPS

In this section we show that a separating map can always be reduced to a one-to-one separating map and therefore we may confine ourself to the study of separating maps that are surjective rather than bijective. The goal of the section is to provide a characterization for surjective separating maps that are completely bounded (when $p \neq 2$) or S^1-bounded. We show that the surjectivity assumption is essential.

We require [5, Propositions 4.4 & 4.5] later on in our arguments in this section. We recall the statements for convenience.

Proposition 4.1. *Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded operator with a direct Yeadon type factorization. Then T is completely bounded and $\|T\|_{cb} = \|T\|$.*

Proposition 4.2. *Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded operator with a direct Yeadon type factorization. Then T is S^1-bounded and $\|T\|_{S^1} = \|T\|$.*

Lemma 4.3. *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a separating map. Then there exists a direct sum decomposition $\mathcal{M} = \mathcal{M}_0 \overset{\infty}{\oplus} \widetilde{\mathcal{M}}$ such that* $\ker(T) = L^p(\mathcal{M}_0)$.

Proof. Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a separating map and $J : \mathcal{M} \to \mathcal{N}$ be the Jordan homomorphism associated with T via its Yeadon type factorization. Let $\mathcal{M}_0 := \ker(J)$. Then $\mathcal{M}_0$ is an ideal. Since J is w^*-continuous, $\mathcal{M}_0$ is w^*-closed. Hence we have a direct sum decomposition

$$\mathcal{M} = \mathcal{M}_0 \overset{\infty}{\oplus} \widetilde{\mathcal{M}}.$$

It is clear that $L^p(\mathcal{M}_0) \subset \ker T$. Further $J|_{\widetilde{\mathcal{M}}}$ is one-to-one. By [4, Remark 3.14(a)] this implies that $T|_{L^p(\widetilde{\mathcal{M}})}$ is one-to-one. This yields the result. $\square$

For any von Neumann algebra $\mathcal{M}$, we let $\mathcal{M}^{op}$ denote its opposite von Neumann algebra. Recall that the underlying dual Banach space structure and involution on $\mathcal{M}^{op}$ are the same as on $\mathcal{M}$ but the product of x and y is defined by yx rather than xy. Note that the Banach spaces $L^p(\mathcal{M})$ and $L^p(\mathcal{M}^{op})$ are the same. It is evident that, for von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, $J : \mathcal{M} \to \mathcal{N}$ is a $*$-homomorphism if and only if

$$J^{op} : \mathcal{M}^{op} \to \mathcal{N}; \quad x \mapsto J(x),$$

is an anti-$*$-homomorphism. Hence, a separating map $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ has a direct Yeadon type factorization if and only if

$$T^{op} : L^p(\mathcal{M}^{op}) \to L^p(\mathcal{N}); \quad x \mapsto T(x),$$

has an anti-direct Yeadon type factorization.

Lemma 4.4 below is the principal ingredient of our characterization theorems. Its proof relies on the relation between the completely bounded norm or S^1-norm of the identity map

$$I^{op} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})$$

and the norms of the transformations

$$[x_{ij}]_{1\leq i,j\leq m} \mapsto [x_{ji}]_{1\leq i,j\leq m}$$

either on $L^p(\mathcal{M} \otimes M_m)$ or on $L^p(\mathcal{M}; S^1_m)$, in particular in the specific case when $\mathcal{M} = M_n$. We will use the fact that for any $n \geq 1$, we have $L^p(M_n \otimes M_m) \simeq S^p_m[S^p_n]$, isometrically, provided that S^p_n is equipped with the operator space structure given in [6].

Let t_m denote the transposition map on scalar $m \times m$ matrices. Assume that $\mathcal{M}$ is semifinite. The map

$$I_{\mathcal{M}^{op}} \otimes t_m : \mathcal{M}^{op} \otimes M_m \to \mathcal{M}^{op} \otimes M_m^{op}$$

is a trace preserving $*$-homomorphism, and so

$$I_{L^p(\mathcal{M}^{op})} \otimes t_m \colon L^p(\mathcal{M}^{op} \otimes M_m) \longrightarrow L^p(\mathcal{M}^{op} \otimes M_m^{op})$$

is an isometry. Moreover $\mathcal{M}^{op} \otimes M_m^{op} = (\mathcal{M} \otimes M_m)^{op}$, hence $L^p(\mathcal{M}^{op} \otimes M_m^{op}) = L^p(\mathcal{M} \otimes M_m)$ isometrically. For any $[x_{ij}]_{1\leq i,j\leq m}$ in $L^p(\mathcal{M}) \otimes S^p_m$, since $I_{L^p(\mathcal{M}^{op})} \otimes t_m$ maps $[x_{ij}]$ to $[x_{ji}]$, we get that

$$\big\|[x_{ij}]\big\|_{L^p(\mathcal{M}^{op} \otimes M_m)} = \big\|[x_{ji}]\big\|_{L^p(\mathcal{M} \otimes M_m)}. \tag{6}$$

We now show that similarly, for any $[x_{ij}]_{1\leq i,j\leq m}$ in $L^p(\mathcal{M}) \otimes S^1_m$,

$$\big\|[x_{ij}]\big\|_{L^p(\mathcal{M}^{op};\, S^1_m)} = \big\|[x_{ji}]\big\|_{L^p(\mathcal{M};\, S^1_m)}. \tag{7}$$

To verify the identity (7), assume that $\|[x_{ij}]\|_{L^p(\mathcal{M}^{op};\, S^1_m)} < 1$. Taking into account the opposite product and (1), we can write

$$x_{ij} = \sum_k b_{kj} a_{ik}$$

for some a_{ik}, b_{kj} in $L^{2p}(\mathcal{M})$ such that $\sum_{i,k} a_{ik}^* a_{ik}$ and $\sum_{k,j} b_{kj} b_{kj}^*$ have norm < 1 in $L^p(\mathcal{M})$. This exacly means that $\|[x_{ji}]\|_{L^p(\mathcal{M};\, S^1_m)} < 1$. This shows the inequality $\geq$ in (7). Reversing the argument we find the other inequality.

Identities (6) and (7), respectively, imply

$$\big\|I^{op}\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb} = \sup_{m\geq 1}\big\|I_{L^p(\mathcal{M})}\otimes t_m\colon L^p(\mathcal{M}\otimes M_m) \longrightarrow L^p(\mathcal{M}\otimes M_m)\big\|, \tag{8}$$

and

$$\big\|I^{op}\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{S^1} = \sup_{m\geq 1}\big\|I_{L^p(\mathcal{M})}\otimes t_m\colon L^p(\mathcal{M};\ S^1_m) \longrightarrow L^p(\mathcal{M}; S^1_m)\big\|. \tag{9}$$

When $\mathcal{M} = M_n$, the above identities can be more specific. In fact, as we show below, we have that for any $n \geq 1$,

$$\big\|I^{op}\colon S^p_n \longrightarrow \{S^p_n\}^{op}\big\|_{cb} = \big\|t_n : S^p_n \to S^p_n\big\|_{cb} \tag{10}$$

and

$$\big\|I^{op}\colon S^p_n \longrightarrow \{S^p_n\}^{op}\big\|_{S^1} = \big\|t_n : S^p_n \to S^p_n\big\|_{S^1}. \tag{11}$$

Using (6) applied to $\mathcal{M} = M_n$, to prove (10), it is enough to show that for any $[x_{ij}]_{1\leq i,j\leq m}$ in $S^p_n \otimes S^p_m$,

$$\big\|[t_n(x_{ij})]\big\|_{S^p_m[S^p_n]} = \big\|[x_{ji}]\big\|_{S^p_m[S^p_n]}. \tag{12}$$

This follows from the fact that $t_m \otimes t_n = t_{nm}$ is an isometry on $S^p_m[S^p_n] \simeq S^p_{nm}$, and hence

$$\big\|(t_m \otimes t_n)[t_n(x_{ij})]\big\|_{S^p_m[S^p_n]} = \big\|[t_n(x_{ij})]\big\|_{S^p_m[S^p_n]}.$$

Since $(t_m \otimes t_n)[t_n(x_{ij})] = [x_{ji}]$, this yields (12).

Likewise, using (7) applied to $\mathcal{M} = M_n$, to prove (11), it is enough to show that for any $[x_{ij}]_{1\leq i,j\leq m}$ in $S^p_n \otimes S^1_m$,

$$\big\|[t_n(x_{ij})]\big\|_{S^p_n[S^1_m]} = \big\|[x_{ji}]\big\|_{S^p_n[S^1_m]}. \tag{13}$$

Assume that $\|[t_n(x_{ij})]\|_{S^p_n[S^1_m]} < 1$. According to (1), we can write

$$t_n(x_{ij}) = \sum_k a_{ik}b_{kj}$$

for some a_{ik}, b_{kj} in S^{2p}_n such that $\sum_{i,k} a_{ik}a^*_{ik}$ and $\sum_{k,j} b^*_{kj}b_{kj}$ have norm < 1 in S^p_n. Then we have

$$x_{ij} = \sum_k t_n(a_{ik}b_{kj}) = \sum_k t_n(b_{kj})t_n(a_{ik}),$$

hence

$$x_{ji} = \sum_k t_n(b_{ki})t_n(a_{jk}).$$

Further

$$\sum_{k,j} t_n(a_{jk})^* t_n(a_{jk}) = t_n\Big(\sum_{j,k} a_{jk}a^*_{jk}\Big),$$

and t_n is an isomerty on S^p_n. Consequently, $\sum_{k,j} t_n(a_{kj})^* t_n(a_{jk})$ has norm < 1 in S^p_n. Similarly, $\sum_{i,k} t_n(b_{ki})t_n(b_{ki})^*$ has norm < 1 in S^p_n. This shows that $\|[x_{ji}]\|_{S^p_n[S^1_m]} < 1$. We have thus proved the inequality $\geq$ in (13). Reversing the argument we find the other inequality.

In the sequel, $E(x)$ denotes the integer part of x.

Lemma 4.4. *Suppose that $\mathcal{M}$ is a semifinite von Neumann algebra.*

(i) If $\mathcal{M}$ is subhomogeneous of degree $\leq N$ for some $N \geq 1$, then for all $[x_{ij}] \in M_m \otimes L^p(\mathcal{M})$, $m \geq 1$, we have that

$$\|[x_{ji}]\|_{L^p(\mathcal{M}\otimes M_m)} \leq N^{2|1/2-1/p|}\|[x_{ij}]\|_{L^p(\mathcal{M}\otimes M_m)},$$

and

$$\|[x_{ji}]\|_{L^p(\mathcal{M};S^1_m)} \leq N\|[x_{ij}]\|_{L^p(\mathcal{M};\, S^1_m)}.$$

(ii) Suppose that there exists $K \geq 1$ such that for all $[x_{ij}] \in L^p(\mathcal{M}) \otimes S^p_m$, $m \geq 1$,

$$\|[x_{ji}]\|_{L^p(\mathcal{M}\otimes M_m)} \leq K\|[x_{ij}]\|_{L^p(\mathcal{M}\otimes M_m)}. \tag{14}$$

Then if $p \neq 2$, $\mathcal{M}$ is subhomogeneous of degree $\leq N$ with $N = E\left(K^{\frac{1}{2|1/2-1/p|}}\right)$.

(iii) Suppose that there exists $K \geq 1$ such that for all $[x_{ij}] \in L^p(\mathcal{M}) \otimes S^p_m$, $m \geq 1$,

$$\|[x_{ji}]\|_{L^p(\mathcal{M};S^1_m)} \leq K\|[x_{ij}]\|_{L^p(\mathcal{M};\, S^1_m)}. \tag{15}$$

Then $\mathcal{M}$ is subhomogeneous of degree $\leq N$ with $N = E(K)$.

Proof. (*i*) Assume that $\mathcal{M} = L^\infty(\Omega; M_n)$. Let $m \geq 1$ be given. We have that

$$L^p(\mathcal{M} \otimes M_m) \simeq L^p(\Omega; S^p_m[S^p_n]).$$

By Pisier-Fubini Theorem [6, (3.6)],

$$L^p(\mathcal{M}; S^1_m) \simeq L^p(\Omega; S^p_n[S^1_m]).$$

Consequently,

$$\big\|I_{L^p(\mathcal{M})} \otimes t_m \colon L^p(\mathcal{M} \otimes M_m) \longrightarrow L^p(\mathcal{M} \otimes M_m)\big\| = \big\|t_m \otimes I_{S^p_n} \colon S^p_m[S^p_n] \longrightarrow S^p_m[S^p_n]\big\|. \tag{16}$$

and

$$\big\|I_{L^p(\mathcal{M})} \otimes t_m \colon L^p(\mathcal{M}; S^1_m) \longrightarrow L^p(\mathcal{M}; S^1_m)\big\| = \big\|I_{S^p_n} \otimes t_m \colon S^p_n[S^1_m] \longrightarrow S^p_n[S^1_m]\big\|. \tag{17}$$

Applying (8) to both sides of (16), we deduce

$$\big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb} = \big\|I^{op} \colon S^p_n \longrightarrow \{S^p_n\}^{op}\big\|_{cb},$$

and applying (9) to both sides of (17), we deduce that

$$\big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{S^1} = \big\|I^{op} \colon S^p_n \longrightarrow \{S^p_n\}^{op}\big\|_{S^1}.$$

By [5, Lemma 5.3],

$$\big\|t_n : S^p_n \to S^p_n\big\|_{cb} = n^{2|1/p-1/2|} \quad \text{and} \quad \big\|t_n : S^p_n \to S^p_n\big\|_{S^1} = n,$$

hence we obtain by (10) and (11) that

$$\big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb} = n^{2|1/p-1/2|} \quad \text{and} \quad \big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{S^1} = n.$$

When $\mathcal{M}$ is subhomogeneous of degree $\leq N$, there exist $r \geq 1$, integers $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r \leq N$ and abelian von Neumann algebras $L^\infty(\Omega_1), \ldots, L^\infty(\Omega_r)$ such that (2) holds. Then for any $m \geq 1$, we have that

$$L^p(\mathcal{M} \otimes M_m) \simeq \mathop{\overset{p}{\oplus}}_{1\leq j\leq r} L^p(\Omega_j; S^p_m[S^p_{n_j}]) \quad \text{and} \quad L^p(\mathcal{M}; S^1_m) \simeq \mathop{\overset{p}{\oplus}}_{1\leq j\leq r} L^p(\Omega_j; S^p_{n_j}[S^1_m]).$$

Using our previous argument and direct sums we deduce that

$$\|I^{op} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})\|_{cb} \leq N^{2|\frac{1}{p}-\frac{1}{2}|} \quad \text{and} \quad \|I^{op} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})\|_{S^1} \leq N.$$

The result follows from (6) and (7).

(ii) Suppose that $\mathcal{M}$ is not subhomogeneous of degree $\leq N = E(K^{\frac{1}{2|1/2-1/p|}})$. By Lemma 2.1, there exists a complete isometry

$$S^p_{N+1} \hookrightarrow \mathcal{M}.$$

This embedding implies that for any $m \geq 1$,

$$\big\|t_m \otimes I_{S^p_{N+1}} \colon S^p_m[S^p_{N+1}] \longrightarrow S^p_m[S^p_{N+1}]\big\| \leq \big\|I_{L^p(\mathcal{M})} \otimes t_m \colon L^p(\mathcal{M} \otimes M_m) \longrightarrow L^p(\mathcal{M} \otimes M_m)\big\|.$$

According to (8) and (10), this implies that

$$\big\|t_{N+1} \colon S^p_{N+1} \longrightarrow S^p_{N+1}\big\|_{cb} \leq \big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb}.$$

Hence

$$\big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb} \geq (N+1)^{2|\frac{1}{p}-\frac{1}{2}|}.$$

Comparing this with inequality (14) above and applying (6), we get a contradiction.

(iii) The proof is similar to the proof of part (ii). □

Proposition 4.5. *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be separating. If $\mathcal{M}$ is subhomogeneous then T is completely bounded and S^1-bounded.*

Proof. Changing T to w^*T, we can assume that $w = J(1)$. By [5, Remark 4.3], we can write T as a sum $T = T_1 + T_2$ such that T_1 has a direct Yeadon type factorization and T_2 has an anti-direct Yeadon type factorization. By Propositions 4.1 and 4.2, T_1 is completely bounded and S^1-bounded. Hence it suffices to show that T_2 is completely bounded and S^1-bounded. Let $I^{op} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})$ be the identity map and set $T_2^{op} =: T_2 \circ I^{op-1}$. Since T_2 has an anti-direct Yeadon type factorization, T_2^{op} has a direct Yeadon type factorization. So, by Propositions 4.1 and 4.2, T_2^{op} is completely bounded and S^1-bounded. Since $\mathcal{M}$ is subhomogeneous, part (i) of Lemma 4.4 and its proof show that I^{op} is completely bounded and S^1-bounded. By composition, we obtain that $T_2 = T_2^{op} \circ I^{op}$ is completely bounded and S^1-bounded. □

Proposition 4.6. *Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is a bijective separating map with an anti-direct Yeadon type factorization.*

(i) *If $p \neq 2$ and T is completely bounded then $\mathcal{M}$ is subhomogeneous.*
(ii) *If T is S^1-bounded then $\mathcal{M}$ is subhomogeneous.*

Proof. (i) Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p \neq 2 < \infty$, is a bijective separating map with an anti-direct Yeadon type factorization. Assume that T is completely bounded. Let $I^{op} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})$ be the identity map and set $T^{op} := T \circ I^{op-1}$. Since T is bijective with an anti-direct Yeadon type factorization, T^{op} is bijective with a direct Yeadon type factorization. By part (i) of Proposition 3.4 and Remark 3.5, T^{op-1} is also separating with a direct Yeadon type factorization. Therefore, by Proposition 4.1, T^{op-1} is completely bounded. Hence, $I^{op} := T^{op-1} \circ T$ is completely bounded. It now follows from part (ii) of Lemma 4.4 and (6) that $\mathcal{M}$ is subhomogeneous.

(ii) The same argument as in part (i) with S^1-bounded (norm) replacing completely bounded (norm), Proposition 4.2 replacing Proposition 4.1, part (iii) of Lemma 4.4 replacing its part (ii) and (7) replacing (6) yields the result. □

Remark 4.7. Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p < \infty$, is a surjective separating isometry with an anti-direct Yeadon type factorization. The proof of Proposition 4.6 shows that when T is completely bounded and $p \neq 2$, $\mathcal{M}$ is subhomogeneous of degree $\leq E(\|T\|_{cb}^{\frac{1}{2|1/2-1/p|}})$. When T is S^1-bounded, $\mathcal{M}$ is subhomogeneous of degree $\leq E(\|T\|_{S^1})$.

Theorem 4.8. *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p \neq 2 < \infty$, be a bounded separating map that is surjective. Then the following are equivalent.*

(i) *T is completely bounded.*

(ii) *There exists a decomposition $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ such that $T|_{L^p(\mathcal{M}_1)}$ has a direct Yeadon type factorization and $\mathcal{M}_2$ is subhomogeneous.*

Proof. $(i) \implies (ii)$ Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p \neq 2 < \infty$, is a surjective completely bounded separating map. In view of Lemma 4.3, we may assume T is bijective.

By Proposition 3.3, there exist decompositions $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ and $\mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and surjective separating maps $T_1 : L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1)$ and $T_2 : L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2)$ such that T_1 has a direct Yeadon type factorization, T_2 has an anti-direct Yeadon type factorization and $T = T_1 + T_2$. Since T is completely bounded, T_2 is also completely bounded. By part (i) of Proposition 4.6, $\mathcal{M}_2$ must be subhomogeneous.

$(ii) \implies (i)$ This is a consequence of Propositions 4.1 and 4.5. □

Theorem 4.9. *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p < \infty$, be a separating map that is surjective. Then the following are equivalent.*

(i) *T is S^1-bounded.*

(ii) *There exists a decomposition $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ such that $T|_{L^p(\mathcal{M}_1)}$ has a direct Yeadon type factorization and $\mathcal{M}_2$ is subhomogeneous.*

Proof. The proof is similar to Theorem 4.8, replacing completely bounded with S^1-bounded, part (i) of Proposition 4.6 by its part (ii) and Proposition 4.1 by Proposition 4.2. □

The following example shows the surjectivity assumption in Theorems 4.8 and 4.9 is essential. In fact in this example, on a non-subhomogeneous semifinite von Neumann algebra $\mathcal{M}$ and for a given $\varepsilon > 0$, we construct a separating isometry $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ such that T is not surjective, $\|T\|_{cb} \leq 1 + \varepsilon$, $\|T\|_{S^1} \leq 1 + \varepsilon$ and part (ii) of Theorems 4.8 and 4.9 is not satisfied.

The isometry T in our example is set up between hyperfinite von Neumann algebras and so $\|T\|_{cb} \leq \|T\|_{S^1}$ (see [7, Proposition 2.2] and [5, Proposition 3.11]). Therefore, we only need to verify that for such T we have that $\|T\|_{S^1} \leq 1 + \varepsilon$.

Example 4.10. Let $1 < p < \infty$. Consider the von Neumman algebra

$$\mathcal{M} = \ell^\infty\{M_n\} = \big\{(x_n)_{n\geq 1} \, : \, \forall\, n \geq 1,\ x_n \in M_n \ \text{and} \ \sup_{n\geq 1} \|x_n\|_\infty < \infty\big\},$$

the infinite direct sum of all M_n, $n \geq 1$. Let $\mathcal{N} := \mathcal{M} \overset{\infty}{\oplus} \mathcal{M}$, the direct sum of two copies of $\mathcal{M}$. The noncommutative L^p-space associated with $\mathcal{M}$ is

$$\ell^p\{S_n^p\} = \big\{(x_n)_{n\geq 1} \, : \, \forall\, n \geq 1,\ x_n \in S_n^p \ \text{and} \ \sum_{n\geq 1} \|x_n\|_p^p < \infty\big\},$$

equipped with the norm

$$\big\|(x_n)_{n\geq 1}\big\|_p = \Big(\sum_{n=1}^{\infty} \|x_n\|_p^p\Big)^{\frac{1}{p}},$$

and so the noncommutative L^p-space associated with $\mathcal{N}$ is $\ell^p\{S_n^p\} \overset{p}{\oplus} \ell^p\{S_n^p\}$. Let $(\beta_n)_{n\geq 1}$ be a sequence in the interval $(0,1)$. We may define two operators

$$T_1\colon \ell^p\{S_n^p\} \to \ell^p\{S_n^p\} \qquad \text{and} \qquad T_2\colon \ell^p\{S_n^p\} \to \ell^p\{S_n^p\}$$

by setting

$$T_1\big((x_n)_{n\geq 1}\big) \;=\; \big((1-\beta_n)^{\frac{1}{p}} x_n\big)_{n\geq 1} \qquad \text{and} \qquad T_2\big((x_n)_{n\geq 1}\big) \;=\; \big(\beta_n^{\frac{1}{p}}\, t_n(x_n)\big)_{n\geq 1}$$

for any $x = (x_n)_{n\geq 1} \in \ell^p\{S_n^p\}$. Consider

$$T\colon \ell^p\{S_n^p\} \to \ell^p\{S_n^p\} \overset{p}{\oplus} \ell^p\{S_n^p\}, \qquad T(x) = (T_1(x), T_2(x)).$$

It is plain that T is an isometry. Indeed for any $x = (x_n)_{n\geq 1} \in \ell^p\{S_n^p\}$, we have

$$\begin{aligned}\|T(x)\|_p^p &= \|T_1(x)\|_p^p + \|T_2(x)\|_p^p\\ &= \sum_{n=1}^{\infty}(1-\beta_n)\|x_n\|_p^p + \sum_{n=1}^{\infty}\beta_n\|{}^t x_n\|_p^p = \sum_{n=1}^{\infty}\|x_n\|_p^p = \|x\|_p^p.\end{aligned}$$

Given $\varepsilon > 0$, consider the above construction with

$$\beta_n = \frac{(1+\varepsilon)^p - 1}{n^p - 1}.$$

We show that T is S^1-bounded with $\|T\|_{S^1} \leq 1+\varepsilon$. Indeed consider an integer $m \geq 1$. We have

$$\ell^p\{S_n^p\}\big[S_m^1\big] \;=\; \ell^p\{S_n^p[S_m^1]\},$$

and therefore, we also have that

$$\Big(\ell^p\{S_n^p\} \overset{p}{\oplus} \ell^p\{S_n^p\}\Big)\,\big[S_m^1\big] \;=\; \ell^p\{S_n^p[S_m^1]\} \overset{p}{\oplus} \ell^p\{S_n^p[S_m^1]\}.$$

Now let $x = (x_n)_{n\geq 1} \in \ell^p\{S_n^p[S_m^1]\}$ (here each x_n is an element of $S_n^p[S_m^1]$). Then

$$(I_{S_m^1} \otimes T)(x) = \Big(\big((1-\beta_n)^{\frac{1}{p}} x_n\big)_{n\geq 1}, \big(\beta_n^{\frac{1}{p}}\,(t_n \otimes I_{S_m^1})(x_n)\big)_{n\geq 1}\Big).$$

Consequently,

$$\begin{aligned}\big\|(I_{S_m^1} \otimes T)(x)\big\|_p^p &= \sum_{n=1}^{\infty}(1-\beta_n)\|x_n\|_{S_n^p[S_m^1]}^p + \sum_{n=1}^{\infty}\beta_n\|(t_n \otimes I_{S_m^1})(x_n)\|_{S_n^p[S_m^1]}^p\\ &\leq \sum_{n=1}^{\infty}(1-\beta_n)\|x_n\|_{S_n^p[S_m^1]}^p + n^p\beta_n\|x_n\|_{S_n^p[S_m^1]}^p \qquad \text{by [5, Lemma 5.3 (ii)]}\\ &\leq (1+\varepsilon)^p \sum_{n=1}^{\infty}\|x_n\|_{S_n^p[S_m^1]}^p = (1+\varepsilon)^p\|x\|_p^p.\end{aligned}$$

It is clear that T is separating and that the Jordan homomorphism $J\colon \mathcal{M} \to \mathcal{N}$ in its Yeadon triple is given by

$$J\big((x_n)_{n\geq 1}\big) = \big((x_n)_{n\geq 1}, (t_n(x_n))_{n\geq 1}\big).$$

It follows that whenever $\mathcal{M}_1$ is a non zero summand of $\mathcal{M}$, the Yeadon factorization of the restriction of T to $L^p(\mathcal{M}_1)$ is neither direct nor indirect. A fortiori, T does not satisfy the assertion *(ii)* of Theorem 4.9.

ACKNOWLEDGMENT

The second author would like to gratefully thank "Laboratoire de Mathématiques de Besançon" for hospitality and excellent working condition she received during her visit.

References

[1] G. Hong, S. K. Ray and S. Wang, *Maximal ergodic inequalities for positive operators on noncommutative L^p-spaces*, Preprint 2020, arXiv:1907.12967v5.

[2] M. Junge, *Doob's inequality for non-commutative martingales*, J. Reine Angew. Math. 549 (2002) 149-190.

[3] M. Junge and Q. Xu, *Noncommutative maximal ergodic theorems*, J. Amer. Math. Soc. 20 (2007), 385-439.

[4] C. Le Merdy and S. Zadeh, *ℓ^1-contractive maps on noncommutative L^p-spaces*, to appear in Journal of Operator Theory, arXiv:1907.03995.

[5] C. Le Merdy and S. Zadeh, *On factorization of separating maps on noncommutative L^p-spaces*, preprint 2020, arXiv:2007.04577.

[6] G. Pisier, *Non-commutative vector valued L^p-spaces and completely p-summing maps*, Astérisque, Vol. 247, 1998.

[7] G. Pisier, *Regular operators between non-commutative L_p-spaces*, Bull. Sci. Math. 119 (1995), 95-118.

[8] G. Pisier and Q. Xu, *Non-commutative L^p-spaces*, Handbook of the geometry of Banach spaces, Vol. 2, pp. 1459-1517, North-Holland, Amsterdam, 2003.

[9] V. Runde, *Amenable Banach algebras*, Springer-Verlag, New York, 2020.

[10] E. Størmer, *On the Jordan structure of C^*-algebras*, Trans. Amer. Math. Soc. 120 (1965), 438-447.

[11] M. Takesaki, *Theory of operator algebras. I*, Encyclopaedia of Mathematical Sciences, Vol. 124, Springer-Verlag, Berlin, 2002.

[12] M. Terp, *L_p-spaces associated with von Neumann algebras*, Notes, Math. Institute, Copenhagen Univ., 1981.

[13] F. Y. Yeadon, *Isometries of noncommutative L^p-spaces*, Math. Proc. Cambridge Philos. Soc. 90 (1981), 41-50.

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