ALGEBRAIC METHODS IN THE THEORY OF GENERALIZED HARISH-CHANDRA MODULES

IVAN PENKOV AND GREGG ZUCKERMAN

Abstract. This paper is a review of results on generalized Harish-Chandra modules in the framework of cohomological induction. The main results, obtained during the last 10 years, concern the structure of the fundamental series of $(\mathfrak{g}, \mathfrak{k})-$modules, where $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{k}$ is an arbitrary algebraic reductive in $\mathfrak{g}$ subalgebra. These results lead to a classification of simple $(\mathfrak{g}, \mathfrak{k})-$modules of finite type with generic minimal $\mathfrak{k}-$types, which we state. We establish a new result about the Fernando-Kac subalgebra of a fundamental series module. In addition, we pay special attention to the case when $\mathfrak{k}$ is an eligible $r-$subalgebra (see the definition in section 4) in which we prove stronger versions of our main results. If $\mathfrak{k}$ is eligible, the fundamental series of $(\mathfrak{g}, \mathfrak{k})-$modules yields a natural algebraic generalization of Harish-Chandra's discrete series modules.

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Introduction

Generalized Harish-Chandra modules have now been actively studied for more than 10 years. A *generalized Harish-Chandra module M* over a finite-dimensional reductive Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}-$module M for which there is a reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ such that as a $\mathfrak{k}-$module, M is the direct sum of finite-dimensional generalized $\mathfrak{k}-$isotypic components. If M is irreducible, $\mathfrak{k}$ acts necessarily semisimply on M, and in what follows we restrict ourselves to the study of generalized Harish-Chandra modules on which $\mathfrak{k}$ acts semisimply; see [Z] for an introduction to the topic.

In this paper we present a brief review of results obtained in the past 10 years in the framework of algebraic representation theory, more specifically in the framework of cohomological induction, see [KV] and [Z]. In fact, generalized Harish-Chandra modules have been studied also with geometric methods, see for instance [PSZ] and [PS1], [PS2], [PS3], [Pe], but the geometric point of view remains beyond the scope of the current review. In addition, we restrict ourselves to finite-dimensional Lie algebras $\mathfrak{g}$ and do not review the paper [PZ4], which deals with the case of locally finite Lie algebras. We omit the proofs of most results which have already appeared.

The cornerstone of the algebraic theory of generalized Harish-Chandra modules so far is our work [PZ2]. In this work we define the notion of simple generalized Harish-Chandra modules with generic minimal $\mathfrak{k}-$type and provide a classification of such modules. The result extends in part the Vogan-Zuckerman classification of simple Harish-Chandra modules. It leaves open the questions of existence and classification of simple $(\mathfrak{g}, \mathfrak{k})-$modules of finite type whose minimal $\mathfrak{k}-$types are not generic. While the classification of such modules presents the main open problem in the theory of generalized Harish-Chandra modules, in the note [PZ3] we establish the existence of simple $(\mathfrak{g}, \mathfrak{k})-$modules with arbitrary given minimal $\mathfrak{k}-$type.

In the paper [PZ5] we establish another general result, namely the fact that each module in the fundamental series of generalized Harish-Chandra modules has finite length. We then consider in detail the case when $\mathfrak{k} = \mathfrak{sl}(2)$. In this case the highest weights of $\mathfrak{k}-$types are just non-negative integers μ and the genericity condition is the inequality $\mu \geq \Gamma$, Γ being a bound depending on the pair $(\mathfrak{g}, \mathfrak{k})$. In [PZ5] we improve the bound Γ to an, in general, much lower bound Λ. Moreover, we show that in a number of low dimensional examples the bound Λ is sharp in the sense that the our classification results do not hold for simple $(\mathfrak{g}, \mathfrak{k})-$ modules with minimal $\mathfrak{k}-$type $V(\mu)$ for μ lower than Λ. In [PZ5] we also conjecture that the Zuckerman functor establishes an equivalence of a certain subcategory of the thickening of category O and a subcategory of the category of $(\mathfrak{g}, \mathfrak{k} \simeq \mathfrak{sl}(2))-$modules.

Sections 2 and 3 of the present paper are devoted to a brief review of the above results. We also establish some new results in terms of the algebra $\tilde{\mathfrak{k}} := \mathfrak{k} + C(\mathfrak{k})$ (where $C(\cdot)$ stands for centralizer in $\mathfrak{g}$). A notable such result is Corollary 2.10 which gives a sufficient condition on a simple $(\mathfrak{g}, \mathfrak{k})-$module M for $\tilde{\mathfrak{k}}$ to be a maximal reductive subalgebra of $\mathfrak{g}$ which acts locally finitely on M.

The idea of bringing $\tilde{\mathfrak{k}}$ into the picture leads naturally to considering a preferred class of reductive subalgebras $\mathfrak{k}$ which we call eligible: they satisfy the condition $C(\mathfrak{t}) = \mathfrak{t} + C(\mathfrak{k})$ where $\mathfrak{t}$ is Cartan subalgebra of $\mathfrak{k}$. In section 5 we study a natural generalization of Harish-Chandra's discrete series to the case of an eligible subalgebra $\mathfrak{k}$. A key statement here is that under the assumption of eligibility of $\mathfrak{k}$, the isotypic component of the minimal $\mathfrak{k}-$type of a generalized discrete series module is an irreducible $\tilde{\mathfrak{k}}-$module (Theorem 5.1).

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1. Notation and preliminary results

We start by recalling the setup of [PZ2] and [PZ5].

1.1. **Conventions.** The ground field is $\mathbb{C}$, and if not explicitly stated otherwise, all vector spaces and Lie algebras are defined over $\mathbb{C}$. The sign $\otimes$ denotes tensor product over $\mathbb{C}$. The superscript * indicates dual space. The sign $\in$ stands for semidirect sum of Lie algebras (if $\mathfrak{l} = \mathfrak{l}' \in \mathfrak{l}''$, then $\mathfrak{l}'$ is an ideal in $\mathfrak{l}$ and $\mathfrak{l}'' \cong \mathfrak{l}/\mathfrak{l}'$). $H^{\cdot}(\mathfrak{l}, M)$ stands for the cohomology of a Lie algebra $\mathfrak{l}$ with coefficients in an $\mathfrak{l}$-module M, and $M^{\mathfrak{l}} = H^0(\mathfrak{l}, M)$ stands for space of $\mathfrak{l}$-invariants of M. By $Z(\mathfrak{l})$ we denote the center of $\mathfrak{l}$, and by $\mathfrak{l}_{ss}$ we denote the semisimple part of $\mathfrak{l}$ when $\mathfrak{l}$ is reductive. $\Lambda^{\cdot}(\cdot)$ and $S^{\cdot}(\cdot)$ denote respectively the exterior and symmetric algebra.

If $\mathfrak{l}$ is a Lie algebra, then $U(\mathfrak{l})$ stands for the enveloping algebra of $\mathfrak{l}$ and $Z_{U(\mathfrak{l})}$ denotes the center of $U(\mathfrak{l})$. We identify $\mathfrak{l}$-modules with $U(\mathfrak{l})$-modules. It is well known that if $\mathfrak{l}$ is finite dimensional and M is a simple $\mathfrak{l}$-module (or equivalently a simple $U(\mathfrak{l})$-module), $Z_{U(\mathfrak{l})}$ acts on M via a $Z_{U(\mathfrak{l})}$-*character*, i.e. via an algebra homomorphism $\theta_M : Z_{U(\mathfrak{l})} \to \mathbb{C}$, see Proposition 2.6.8 in [Dix].

We say that an $\mathfrak{l}$-module M is *generated* by a subspace $M' \subseteq M$ if $U(\mathfrak{l}) \cdot M' = M$, and we say that M is *cogenerated* by $M' \subseteq M$, if for any non-zero homomorphism $\psi : M \to \bar{M}$, $M' \cap \ker \psi \neq \{0\}$.

By Soc$M$ we denote the socle (i.e. the unique maximal semisimple submodule) of an $\mathfrak{l}$-module M. If $\omega \in \mathfrak{l}^*$, we put $M^\omega := \{m \in M \mid \mathfrak{l} \cdot m = \omega(\mathfrak{l}) m \ \forall l \in \mathfrak{l}\}$. By $\mathrm{supp}_{\mathfrak{l}} M$ we denote the set $\{\omega \in \mathfrak{l}^* \mid M^\omega \neq 0\}$.

A finite *multiset* is a function f from a finite set D into $\mathbb{N}$. A *submultiset* of f is a multiset f' defined on the same domain D such that $f'(d) \leq f(d)$ for any $d \in D$. For any finite multiset f, defined on a subset D of a vector space, we put $\rho_f := \frac{1}{2} \sum_{d \in D} f(d) d$.

If $\dim M < \infty$ and $M = \bigoplus_{\omega \in \mathfrak{l}^*} M^\omega$, then M determines the finite multiset $\mathrm{ch}_{\mathfrak{l}} M$ which is the function $\omega \mapsto \dim M^\omega$ defined on $\mathrm{supp}_{\mathfrak{l}} M$.

1.2. **Reductive subalgebras, compatible parabolics and generic $\mathfrak{k}$-types.** Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. By $\mathfrak{g}$-mod we denote the category of $\mathfrak{g}$-modules. Let $\mathfrak{k} \subset \mathfrak{g}$ be an algebraic subalgebra which is reductive in $\mathfrak{g}$. We set $\tilde{\mathfrak{k}} = \mathfrak{k} + C(\mathfrak{k})$ and note that $\tilde{\mathfrak{k}} = \mathfrak{k}_{ss} \oplus C(\mathfrak{k})$ where $C(\cdot)$ stands for centralizer in $\mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and let $\mathfrak{h}$ denote an as yet unspecified Cartan subalgebra of $\mathfrak{g}$. Everywhere, but in subsection 1.3 below, we assume that $\mathfrak{t} \subseteq \mathfrak{h}$, and hence that $\mathfrak{h} \subseteq C(\mathfrak{t})$. By Δ we denote the set of $\mathfrak{h}$-roots of $\mathfrak{g}$, i.e. $\Delta = \{\mathrm{supp}_{\mathfrak{h}} \mathfrak{g}\} \setminus \{0\}$. Note that, since $\mathfrak{k}$ is reductive in $\mathfrak{g}$, $\mathfrak{g}$ is a $\mathfrak{t}$-weight module, i.e. $\mathfrak{g} = \bigoplus_{\eta \in \mathfrak{t}^*} \mathfrak{g}^\eta$. We set $\Delta_{\mathfrak{t}} := \{\mathrm{supp}_{\mathfrak{t}} \mathfrak{g}\} \setminus \{0\}$. Note also that the $\mathbb{R}$-span of the roots of $\mathfrak{h}$ in $\mathfrak{g}$ fixes a real structure on $\mathfrak{h}^*$, whose projection onto $\mathfrak{t}^*$ is a well-defined real structure on $\mathfrak{t}^*$. In what follows, we denote by Reη the real part of an element $\eta \in \mathfrak{t}^*$. We fix also a Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subseteq \mathfrak{k}$ with $\mathfrak{b}_{\mathfrak{k}} \supseteq \mathfrak{t}$. Then $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{t} \supset\!\!\!\!+ \mathfrak{n}_{\mathfrak{k}}$, where $\mathfrak{n}_{\mathfrak{k}}$ is the nilradical of $\mathfrak{b}_{\mathfrak{k}}$. We set $\rho := \rho_{\mathrm{ch}_{\mathfrak{t}} \mathfrak{n}_{\mathfrak{k}}}$. The quartet $\mathfrak{g}, \mathfrak{k}, \mathfrak{b}_{\mathfrak{k}}, \mathfrak{t}$ will be fixed throughout the paper. By W we denote the Weyl group of $\mathfrak{g}$.

As usual, we parametrize the characters of $Z_{U(\mathfrak{g})}$ via the Harish-Chandra homomorphism. More precisely, if $\mathfrak{b}$ is a given Borel subalgebra of $\mathfrak{g}$ with $\mathfrak{b} \supset \mathfrak{h}$ ($\mathfrak{b}$ will be specified below), the $Z_{U(\mathfrak{g})}$-character corresponding to $\zeta \in \mathfrak{h}^*$ via the Harish-Chandra homomorphism defined by $\mathfrak{b}$ is denoted by θ_ζ ($\theta_{\rho_{\mathrm{ch}_\mathfrak{h}\mathfrak{b}}}$ is the trivial $Z_{U(\mathfrak{g})}$-character). Sometimes we consider a reductive subalgebra $\mathfrak{l} \subset \mathfrak{g}$ instead of $\mathfrak{g}$ and apply this convention to the characters of $Z_{U(\mathfrak{l})}$. In this case we write $\theta^{\mathfrak{l}}_\zeta$ for $\zeta \in \mathfrak{h}^*_{\mathfrak{l}}$, where $\mathfrak{h}_\mathfrak{l}$ is a Cartan subalgebra of $\mathfrak{l}$.

By $\langle \cdot \cdot \rangle$ we denote the unique $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}^*$ such that $\langle \alpha, \alpha \rangle = 2$ for any long root of a simple component of $\mathfrak{g}$. The form $\langle \cdot , \cdot \rangle$ enables us to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then $\mathfrak{h}$ is identified with $\mathfrak{h}^*$, and $\mathfrak{k}$ is identified with $\mathfrak{k}^*$. We sometimes consider $\langle \cdot , \cdot \rangle$ as a form on $\mathfrak{g}$. The superscript $\perp$ indicates orthogonal space. Note that there is a canonical $\mathfrak{k}$-module decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$ and a canonical decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{t}^\perp$ with $\mathfrak{t}^\perp \subseteq \mathfrak{k}^\perp$. We also set $\| \zeta \|^2 := \langle \zeta, \zeta \rangle$ for any $\zeta \in \mathfrak{h}^*$.

We say that an element $\eta \in \mathfrak{t}^*$ is $(\mathfrak{g}, \mathfrak{k})$-*regular* if $\langle \mathrm{Re}\eta, \sigma \rangle \neq 0$ for all $\sigma \in \Delta_\mathfrak{t}$. To any $\eta \in \mathfrak{t}^*$ we associate the following parabolic subalgebra $\mathfrak{p}_\eta$ of $\mathfrak{g}$:

$$\mathfrak{p}_\eta = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_\eta} \mathfrak{g}^\alpha),$$

where $\Delta_\eta := \{\alpha \in \Delta \mid \langle \mathrm{Re}\eta, \alpha \rangle \geq 0\}$. By $\mathfrak{m}_\eta$ and $\mathfrak{n}_\eta$ we denote respectively the reductive part of $\mathfrak{p}$ (containing $\mathfrak{h}$) and the nilradical of $\mathfrak{p}$. In particular $\mathfrak{p}_\eta = \mathfrak{m}_\eta \supsetplus \mathfrak{n}_\eta$, and if η is $\mathfrak{b}_\mathfrak{t}$-dominant, then $\mathfrak{p}_\eta \cap \mathfrak{k} = \mathfrak{b}_\mathfrak{k}$. We call $\mathfrak{p}_\eta$ a $\mathfrak{t}$-*compatible parabolic subalgebra*. Note that

$$\mathfrak{p}_\eta = C(\mathfrak{t}) \oplus (\bigoplus_{\beta \in \Delta^+_{\mathfrak{t},\eta}} \mathfrak{g}^\beta)$$

where $\Delta^+_{\mathfrak{t},\eta} := \{\beta \in \Delta_\mathfrak{t} \,|\, \langle \mathrm{Re}\eta, \beta \rangle > 0\}$. Hence, $\mathfrak{p}_\eta$ depends upon our choice of $\mathfrak{t}$ and η, but not upon the choice of $\mathfrak{h}$.

A $\mathfrak{t}$-compatible parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$ (i.e. $\mathfrak{p} = \mathfrak{p}_\eta$ for some $\eta \in \mathfrak{t}^*$) is $\mathfrak{t}$-*minimal* (or simply *minimal*) if it does not properly contain another $\mathfrak{t}$-compatible parabolic subalgebra. It is an important observation that if $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$ is minimal, then $\mathfrak{t} \subseteq Z(\mathfrak{m})$. In fact, a $\mathfrak{t}$-compatible parabolic subalgebra $\mathfrak{p}$ is minimal if and only if $\mathfrak{m}$ equals the centralizer $C(\mathfrak{t})$ of $\mathfrak{t}$ in $\mathfrak{g}$, or equivalently if and only if $\mathfrak{p} = \mathfrak{p}_\eta$ for a $(\mathfrak{g}, \mathfrak{k})$-regular $\eta \in \mathfrak{t}^*$. In this case $\mathfrak{n} \cap \mathfrak{k} = \mathfrak{n}_\mathfrak{k}$.

Any $\mathfrak{t}$-compatible parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_\eta$ has a well-defined opposite parabolic subalgebra $\bar{\mathfrak{p}} := \mathfrak{p}_{-\eta}$; clearly $\mathfrak{p}$ is minimal if and only if $\bar{\mathfrak{p}}$ is minimal.

A $\mathfrak{k}$-*type* is by definition a simple finite-dimensional $\mathfrak{k}$-module. By $V(\mu)$ we denote a $\mathfrak{k}$-type with $\mathfrak{b}_\mathfrak{k}$-highest weight μ. The weight μ is then $\mathfrak{k}$-integral (or, equivalently, $\mathfrak{k}_{ss}$−integral) and $\mathfrak{b}_\mathfrak{k}$-dominant.

Let $V(\mu)$ be a $\mathfrak{k}$-type such that $\mu + 2\rho$ is $(\mathfrak{g}, \mathfrak{k})$-regular, and let $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$ be the minimal compatible parabolic subalgebra $\mathfrak{p}_{\mu+2\rho}$. Put $\tilde{\rho}_\mathfrak{n} := \rho_{\mathrm{ch}_\mathfrak{h}\mathfrak{n}}$ and $\rho_\mathfrak{n} := \rho_{\mathrm{ch}_\mathfrak{t}\mathfrak{n}}$. Clearly $\rho_\mathfrak{n} = \tilde{\rho}_\mathfrak{n}|_\mathfrak{t}$. We define $V(\mu)$ to be *generic* if the following two conditions hold:

(1) $\langle \mathrm{Re}\mu + 2\rho - \rho_\mathfrak{n}, \alpha \rangle \geq 0 \ \forall \alpha \in \mathrm{supp}_\mathfrak{t}\mathfrak{n}_\mathfrak{k}$;
(2) $\langle \mathrm{Re}\mu + 2\rho - \rho_S, \rho_S \rangle > 0$ for every submultiset S of $\mathrm{ch}_\mathfrak{t}\mathfrak{n}$.

It is easy to show that there exists a positive constant C depending only on $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{p}$ such that $\langle \mathrm{Re}\mu + 2\rho, \alpha \rangle > C$ for every $\alpha \in \mathrm{supp}_\mathfrak{t}\mathfrak{n}$ implies $\mathfrak{p}_{\mu+2\rho} = \mathfrak{p}$ and that $V(\mu)$ is generic.

1.3. **Generalities on $\mathfrak{g}$−modules.** Suppose M is a $\mathfrak{g}$−module and $\mathfrak{l}$ is a reductive subalgebra of $\mathfrak{g}$. M is *locally finite over* $Z_{U(\mathfrak{l})}$ if every vector in M generates a finite-dimensional $Z_{U(\mathfrak{l})}$−module. Denote by $\mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$ the full subcategory of $\mathfrak{g}$−modules which are locally finite over $Z_{U(\mathfrak{l})}$.

Suppose $M \in \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$ and θ is a $Z_{U(\mathfrak{l})}$−character. Denote by $P(\mathfrak{l}, \theta)(M)$ the generalized θ−eigenspace of the restriction of M to $\mathfrak{l}$. The $Z_{U(\mathfrak{l})}$−*spectrum* of M is the set of characters θ of $Z_{U(\mathfrak{l})}$ such that $P(\mathfrak{l}, \theta)(M) \neq 0$. Denote the $Z_{U(\mathfrak{l})}$ spectrum of M by $\sigma(\mathfrak{l}, M)$. We say that θ is a *central character of* $\mathfrak{l}$ *in* M if $\theta \in \sigma(\mathfrak{l}, M)$. The following is a standard fact.

Lemma 1.1. *If* $M \in \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$, *then*

$$M = \bigoplus_{\theta \in \sigma(\mathfrak{l}, M)} P(\mathfrak{l}, \theta)(M).$$

A $\mathfrak{g}$−module M is *locally Artinian over* $\mathfrak{l}$ if for every vector $v \in M$, $U(\mathfrak{l}) \cdot v$ is an $\mathfrak{l}$−module of finite length.

Lemma 1.2. *If M is locally Artinian over* $\mathfrak{l}$, *then* $M \in \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$.

Proof The statement follows from the fact that $Z_{U(\mathfrak{l})}$ acts via a character on any simple $\mathfrak{l}$−module. □

If $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, by a $(\mathfrak{g}, \mathfrak{p})$−*module* M we mean a $\mathfrak{g}$−module M on which $\mathfrak{p}$ acts locally finitely. By $\mathcal{M}(\mathfrak{g}, \mathfrak{p})$ we denote the full subcategory of $\mathfrak{g}$−modules which are $(\mathfrak{g}, \mathfrak{p})$−modules.

In the remainder of this subsection we assume that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}_{\mathfrak{l}} := \mathfrak{h} \cap \mathfrak{l}$ is a Cartan subalgebra of $\mathfrak{l}$, and that $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{p}$ and $\mathfrak{p} \cap \mathfrak{l}$ is a parabolic subalgebra of $\mathfrak{l}$. By M we denote a $\mathfrak{g}$−module from $\mathcal{M}(\mathfrak{g}, \mathfrak{p})$.

Lemma 1.3. *The set* $\mathrm{supp}_{\mathfrak{h}} M$ *is independent of the choice of* $\mathfrak{h} \subseteq \mathfrak{p}$.

Proof Suppose $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Cartan subalgebras of $\mathfrak{g}$ such that $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{p}$. Let $\mathfrak{m}_j$ be the maximal reductive subalgebra of $\mathfrak{p}$ such that $\mathfrak{h}_j \subseteq \mathfrak{m}_j$, $j = 1, 2$. There exits an inner automorphism $\Psi(\mathfrak{m}_1) = \mathfrak{m}_2$. Then, $\Psi(\mathfrak{h}_1)$ and $\mathfrak{h}_2$ are Cartan subalgebras of $\mathfrak{m}_2$. There exists an inner automorphism Φ of $\mathfrak{m}_2$ such that $\Phi(\Psi(\mathfrak{h}_1)) = \mathfrak{h}_2$. Hence, for any finite dimensional $\mathfrak{p}$−module W, $\mathrm{supp}_{\mathfrak{h}_1} W = \mathrm{supp}_{\mathfrak{h}_2} W$. By assumption M is a union of finite-dimensional $\mathfrak{p}$−modules. □

Proposition 1.4. *M is locally Artinian over $\mathfrak{l}$.*

Proof We apply Proposition 7.6.1 in [Dix] to the pair $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{p})$. In particular, if $v \in M$, then $U(\mathfrak{l}) \cdot v$ has finite length as an $\mathfrak{l}$−module. □

Corollary 1.5. $M \in \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})})$.

Lemma 1.6. $\sigma(\mathfrak{l}, M) \subseteq \{\theta^{\mathfrak{l}}_{(\eta|_{\mathfrak{h}_{\mathfrak{l}}})+\rho_{\mathfrak{l}}} \mid \eta \in \mathrm{supp}_{\mathfrak{h}} M\}$.

Proof The simple $\mathfrak{l}$−subquotients of M are $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{p})$−modules, and our claim follows the well-known relationship between the highest weight of a highest weight module and its central character. □

Let N be a $\mathfrak{g}$−module, and let $\mathfrak{g}[N]$ be the set of elements $x \in \mathfrak{g}$ that act locally finitely in N. Then $\mathfrak{g}[N]$ is a Lie subalgebra of $\mathfrak{g}$, the *Fernando-Kac subalgebra associated to N*. The fact has been proved independently by V. Kac in [K] and by S. Fernando in [F].

Theorem 1.7. *Let M_1 be a non-zero subquotient of M. Assume that $\eta|_{\mathfrak{h}_{\mathfrak{l}}}$ is non-integral relative to $\mathfrak{l}$ for all $\eta \in \mathrm{supp}_{\mathfrak{h}} M$. Then $\mathfrak{l} \not\subseteq \mathfrak{g}[M_1]$.*

Proof By Lemma 1.6, no central character of $\mathfrak{l}$ in M_1 is $\mathfrak{l}$−integral. Therefore, no non-zero $\mathfrak{l}$−submodule of M_1 is finite dimensional. But $M_1 \neq 0$. Hence, $\mathfrak{l} \not\subseteq \mathfrak{g}[M_1]$. □

In agreement with [PZ2], we define a $\mathfrak{g}$-module M to be a $(\mathfrak{g}, \mathfrak{k})$-*module* if M is isomorphic as a $\mathfrak{k}$-module to a direct sum of isotypic components of $\mathfrak{k}$-types. If M is a $(\mathfrak{g}, \mathfrak{k})$-module, we write $M[\mu]$ for the $V(\mu)$-isotypic component of M, and we say that $V(\mu)$ is a $\mathfrak{k}$-*type of M* if $M[\mu] \neq 0$. We say that a $(\mathfrak{g}, \mathfrak{k})$-module M is *of finite type* if $\dim M[\mu] \neq \infty$ for every $\mathfrak{k}$-type $V(\mu)$ of M. Sometimes, we also refer to $(\mathfrak{g}, \mathfrak{k})$-modules of finite type as *generalized Harish-Chandra modules*.

Note that for any $(\mathfrak{g}, \mathfrak{k})$−module of finite type M and any $\mathfrak{k}$−type $V(\sigma)$ of M, the finite-dimensional $\mathfrak{k}$−module $M[\sigma]$ is a $\tilde{\mathfrak{k}}$−module. In particular, M is a $(\mathfrak{g}, \tilde{\mathfrak{k}})$−module of finite type. We will write $M\langle\delta\rangle$ for the $\tilde{\mathfrak{k}}$−isotypic components of M where $\delta \in (\mathfrak{h} \cap \tilde{\mathfrak{k}})^*$.

If M is a module of finite length, a $\mathfrak{k}$-type $V(\mu)$ of M is *minimal* if the function $\mu' \mapsto \| \mathrm{Re}\mu' + 2\rho \|^2$ defined on the set $\{\mu' \in \mathfrak{t}^* \mid M[\mu'] \neq 0\}$ has a minimum at μ. Any non-zero $(\mathfrak{g}, \mathfrak{k})$-module M of finite length has a minimal $\mathfrak{k}$-type.

1.4. **Generalities on the Zuckerman functor.** Recall that the *functor of $\mathfrak{k}$-finite vectors* $\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}}$ is a well-defined left-exact functor on the category of $(\mathfrak{g}, \mathfrak{t})$-modules with values in $(\mathfrak{g}, \mathfrak{k})$-modules,

$$\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}}(M) := \sum_{M' \subset M, \dim M' = 1, \dim U(\mathfrak{k}) \cdot M' < \infty} M'.$$

By $R^{\cdot}\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}} := \bigoplus_{i \geq 0} R^i \Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}}$ we denote as usual the total right derived functor of $\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}}$, see [Z] and the references therein.

Proposition 1.8. *If $\mathfrak{l}$ is any reductive subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$, then there is a natural isomorphism of $\mathfrak{l}$−modules*

$$R^{\cdot}\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}}\,(N) \cong R^{\cdot}\Gamma^{\mathfrak{l},\mathfrak{t}}_{\mathfrak{l},\mathfrak{k}}\,(N). \quad (1)$$

Proof See Proposition 2.5 in [PZ4]. □

Proposition 1.9. *If* $\tilde{N} \in \mathcal{M}(\mathfrak{l}, \mathfrak{t}, Z_{U(\mathfrak{l})}) := \mathcal{M}(\mathfrak{l}, Z_{U(\mathfrak{l})}) \cap \mathcal{M}(\mathfrak{l}, \mathfrak{t})$*, then*

$$R^{\cdot}\Gamma^{\mathfrak{l},\mathfrak{t}}_{\mathfrak{l},\mathfrak{k}}\ (\tilde{N}) \in \mathcal{M}(\mathfrak{l}, \mathfrak{k}, Z_{U(\mathfrak{l})}).$$

Moreover,

$$\sigma(\mathfrak{l}, R^{\cdot}\Gamma^{\mathfrak{l},\mathfrak{t}}_{\mathfrak{l},\mathfrak{k}}\ (\tilde{N})) \subset \sigma(\mathfrak{l}, \tilde{N}).$$

Proof See Proposition 2.12 and Corollary 2.8 in [Z]. □

Corollary 1.10. *If* $N \in \mathcal{M}(\mathfrak{g}, \mathfrak{t}, Z_{U(\mathfrak{l})}) := \mathcal{M}(\mathfrak{g}, Z_{U(\mathfrak{l})}) \cap \mathcal{M}(\mathfrak{g}, \mathfrak{t})$*, then*

$$R^{\cdot}\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}}\ (N) \in \mathcal{M}(\mathfrak{g}, \mathfrak{k}, Z_{U(\mathfrak{l})}).$$

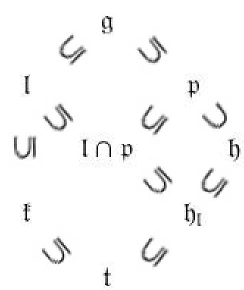
Moreover,

$$\sigma(\mathfrak{l}, R^{\cdot}\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}}\ (N)) \subseteq \sigma(\mathfrak{l}, N).$$

Proof Apply Propositions 1.8 and 1.9. □

Note that the isomorphism (1) enables us to write simply $\Gamma_{\mathfrak{k},\mathfrak{t}}$ instead of $\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}}$.

For $\mathfrak{g} \supseteq \mathfrak{l} \supseteq \mathfrak{k} \supseteq \mathfrak{t}$ as above, let $\mathfrak{p}$ be a $\mathfrak{t}-$compatible parabolic subalgebra of $\mathfrak{g}$. It follows immediately that $\mathfrak{l} \cap \mathfrak{p}$ is a $\mathfrak{t}-$compatible parabolic subalgebra of $\mathfrak{l}$. Let $\mathfrak{h}_{\mathfrak{l}} \subset \mathfrak{l} \cap \mathfrak{p}$ be a Cartan subalgebra of $\mathfrak{l}$ containing $\mathfrak{t}$, and let $\mathfrak{h} \subset \mathfrak{p}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}_{\mathfrak{l}} = \mathfrak{h} \cap \mathfrak{l}$. We have the following diagram of subalgebras:



In this setup we have the following result.

Theorem 1.11. *Suppose* $N \in \mathcal{M}(\mathfrak{g}, \mathfrak{p}) \cap \mathcal{M}(\mathfrak{g}, \mathfrak{t})$*,* M *is a non-zero subquotient of* $R^{\cdot}\Gamma_{\mathfrak{k},\mathfrak{t}}\ (N)$ *and* $\eta|_{\mathfrak{h}_{\mathfrak{l}}}$ *is not* $\mathfrak{l}-$*integral for all* $\eta \in \operatorname{supp}_{\mathfrak{h}} N$*. Then* $\mathfrak{l} \not\subseteq \mathfrak{g}[M]$.

Proof Every central character of $\mathfrak{l}$ in M is a central character of $\mathfrak{l}$ in N. This follows from Corollary 2.8 in [Z]. By our assumptions, no central character of $\mathfrak{l}$ in N is $\mathfrak{l}-$integral. Hence, no $\mathfrak{l}-$submodule of M is finite dimensional, and thus $\mathfrak{l} \not\subseteq \mathfrak{g}[M]$. □

2. The fundamental series: main results

We now introduce one of our main objects of study: the fundamental series of generalized Harish-Chandra modules.

We start by fixing some more notation: if $\mathfrak{q}$ is a subalgebra of $\mathfrak{g}$ and J is a $\mathfrak{q}$-module, we set $\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{q}} J := U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} J$ and $\operatorname{pro}^{\mathfrak{g}}_{\mathfrak{q}} J := \operatorname{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), J)$. For a finite-dimensional $\mathfrak{p}$- or $\bar{\mathfrak{p}}$-module E we set $N_{\mathfrak{p}}(E) := \Gamma_{\mathfrak{t},0}(\operatorname{pro}^{\mathfrak{g}}_{\mathfrak{p}}(E \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n})))$, $N_{\bar{\mathfrak{p}}}(E^*) := \Gamma_{\mathfrak{t},0}(\operatorname{pro}^{\mathfrak{g}}_{\bar{\mathfrak{p}}}(E^* \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}^*)))$. One can show that both $N_{\mathfrak{p}}(E)$ and $N_{\bar{\mathfrak{p}}}(E^*)$ have simple socles as long as E itself is simple.

The *fundamental series* of $(\mathfrak{g}, \mathfrak{k})$-modules of finite type $F^{\cdot}(\mathfrak{k}, \mathfrak{p}, E)$ is defined as follows. Let $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$ be a minimal compatible parabolic subalgebra, E be a simple finite dimensional $\mathfrak{p}$-module on which $\mathfrak{t}$ acts via the weight $\omega \in \mathfrak{t}^*$, and $\mu := \omega + 2\rho^{\perp}_{\mathfrak{n}}$ where $\rho^{\perp}_{\mathfrak{n}} := \rho_{\mathfrak{n}} - \rho$. Set

$$F^{\cdot}(\mathfrak{k}, \mathfrak{p}, E) := R^{\cdot}\Gamma_{\mathfrak{k},\mathfrak{t}}(N_{\mathfrak{p}}(E)).$$

In the rest of the paper we assume that $\mathfrak{h} \cap \tilde{\mathfrak{k}}$ is a Cartan subalgebra of $\tilde{\mathfrak{k}}$.

Theorem 2.1. *a)* $F^{\cdot}(\mathfrak{k}, \mathfrak{p}, E)$ *is a* $(\mathfrak{g}, \mathfrak{k})$*-module of finite type and* $Z_{U(\mathfrak{g})}$ *acts on* $F^{\cdot}(\mathfrak{p}, E)$ *via the* $Z_{U(\mathfrak{g})}$*-character* $\theta_{\nu+\tilde{\rho}}$ *where* $\tilde{\rho} := \rho_{\mathrm{ch}_{\mathfrak{h}}\mathfrak{b}}$ *for some Borel subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}$ *with* $\mathfrak{b} \supset \mathfrak{h}$*,* $\mathfrak{b} \subset \mathfrak{p}$ *and* $\mathfrak{b} \cap \mathfrak{k} = \mathfrak{b}_{\mathfrak{k}}$*, and where* ν *is the* $\mathfrak{b}$*-highest weight of* E *(note that* $\nu|_{\mathfrak{t}} = \omega$*).*

b) $F^{\cdot}(\mathfrak{k}, \mathfrak{p}, E)$ *is a* $(\mathfrak{g}, \mathfrak{k})$*-module of finite length.*
c) There is a canonical isomorphism

$$F^{\cdot}(\mathfrak{k}, \mathfrak{p}, E) \simeq R^{\cdot}\Gamma_{\mathfrak{k}, \tilde{\mathfrak{k}} \cap \mathfrak{m}}(\Gamma_{\tilde{\mathfrak{k}} \cap \mathfrak{m}, 0}(\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{p}}(E \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n})))). \tag{2}$$

Proof Part a) is a recollection of Theorem 2, a) in [PZ2]. Part b) is a recollection of Theorem 2.5 in [PZ5]. Part c) follows from the comparison principle (Proposition 2.6) in [PZ4]. □

Corollary 2.2. $F^{\cdot}(\mathfrak{k}, \mathfrak{p}, E)$ *is a* $(\mathfrak{g}, \tilde{\mathfrak{k}})$*-module of finite type.*

Proof As we observed in subsection 1.3, every $(\mathfrak{g}, \mathfrak{k})-$module of finite type is a $(\mathfrak{g}, \tilde{\mathfrak{k}})-$module of finite type. □

Corollary 2.3. *Let* $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$ *be two algebraic reductive subalgebras such that* $\tilde{\mathfrak{k}_1} = \tilde{\mathfrak{k}_2}$*. Suppose that* $\mathfrak{p}$ *is a parabolic subalgebra which is both* $\mathfrak{t}_1-$ *and* $\mathfrak{t}_2-$*compatible and* $\mathfrak{t}_1-$ *and* $\mathfrak{t}_2-$*minimal for some Cartan subalgebras* $\mathfrak{t}_1$ *of* $\mathfrak{k}_1$ *and* $\mathfrak{t}_2$ *of* $\mathfrak{k}_2$*. Then there exists a canonical isomorphism*

$$F^{\cdot}(\mathfrak{k}_1, \mathfrak{p}, E) \simeq F(\mathfrak{k}_2, \mathfrak{p}, E).$$

Proof Consider the isomorphism (2) for $\mathfrak{k}_1$ and $\mathfrak{k}_2$, and notice that

$$R^{\cdot}\Gamma_{\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}} \cap \mathfrak{m}}(\Gamma_{\tilde{\mathfrak{k}} \cap \mathfrak{m}, 0}(\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{p}}(E \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}))))$$

depends only on $\tilde{\mathfrak{k}}$ and $\mathfrak{p}$, but not on $\mathfrak{k}_1$ and $\mathfrak{k}_2$. □

Corollary 2.4. *Let* M *be any non-zero subquotient of* $F^{\cdot}(\mathfrak{k}, \mathfrak{p}, E)$*. If the* $\mathfrak{b}-$*highest weight* $\nu \in \mathfrak{h}^*$ *of* E *is non-integral after restriction to* $\mathfrak{h} \cap \mathfrak{l}$ *for any reductive subalgebra* $\mathfrak{l}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{l} \supset \tilde{\mathfrak{k}}$*, then* $\tilde{\mathfrak{k}}$ *is a maximal reductive subalgebra of* $\mathfrak{g}[M]$*.*

Proof Corollary 2.2 shows that $\tilde{\mathfrak{k}} \subseteq \mathfrak{g}[M]$. Theorem 1.11 shows that if $\mathfrak{l}$ is a reductive subalgebra of $\mathfrak{g}$ such that $\mathfrak{l}$ is strictly larger than $\tilde{\mathfrak{k}}$, then $\mathfrak{l} \nsubseteq \mathfrak{g}[M]$. The assumption on ν implies that all weights in $\mathrm{supp}_{\mathfrak{h} \cap \mathfrak{l}}(N_{\mathfrak{p}}(E))$ are non-integral with respect to $\mathfrak{l}$. □

Example. Here is an example to Corollary 2.4. Let $\mathfrak{g} = F_4, \mathfrak{k} \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(6)$. Then $\mathfrak{k} = \tilde{\mathfrak{k}}$. By inspection, there is only one proper intermediate subalgebra $\mathfrak{l}$, $\tilde{\mathfrak{k}} \subset \mathfrak{l} \subset \mathfrak{g}$, and $\mathfrak{l}$ is isomorphic to $\mathfrak{so}(9)$. We have $\mathfrak{t} = \mathfrak{h}$, and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ is a standard basis of $\mathfrak{h}^*$, see [Bou]. A weight $\nu = \sum_{i=1}^4 m_i \varepsilon_i$ is $\mathfrak{k}-$integral iff $m_1 \in \mathbb{Z}$ or $m_1 \in \mathbb{Z} + \frac{1}{2}$, and $(m_2, m_3, m_4) \in \mathbb{Z}^3$ or $(m_2, m_3, m_4) \in \mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. On the other hand, ν is $\mathfrak{l}-$integral if $(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4$ or $(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. So if the $\mathfrak{b}-$highest weight ν of E is not $\mathfrak{l}-$integral, Corollary 2.4 implies that $\mathfrak{g}[M] = \tilde{\mathfrak{k}}$ for any simple subquotient M of $F^{\cdot}(\mathfrak{k}, \mathfrak{p}, E)$.

Remark.

a) In [PZ1] another method, based on the notion of a small subalgebra introduced by Willenbring and Zuckerman in [WZ], for computing maximal reductive subalgebras of simple subquotients of $F^{\cdot}(\mathfrak{k}, \mathfrak{p}, E)$ is suggested. Note that the subalgebra $\mathfrak{k} \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(6)$ of F_4 considered in the above example is not small in $\mathfrak{so}(9)$, so the above conclusion that $\mathfrak{g}[M] = \mathfrak{k}$ does not follow from [PZ1]. On the other hand, if one replaces $\mathfrak{k}$ in the example by $\mathfrak{k}' \simeq \mathfrak{so}(5) \oplus \mathfrak{so}(4)$, then a conclusion similar to that of the example can be reached both by the method of [PZ1] and by Corollary 2.4.
b) There are pairs $(\mathfrak{g}, \mathfrak{k})$ to which neither the method of [PZ1] nor Corollary 2.4 apply. Such an example is a pair $(\mathfrak{g} = F_4, \mathfrak{k} \simeq \mathfrak{so}(8))$. The only proper intermediate subalgebra in this case is $\mathfrak{l} \simeq \mathfrak{so}(9)$; however $\mathfrak{so}(8)$ is not small in $\mathfrak{so}(9)$ and any $\mathfrak{k} = \tilde{\mathfrak{k}}-$integrable weight is also $\mathfrak{l}-$integrable.

If M is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type, then $\Gamma_{\mathfrak{k},0}(M^*)$ is a well-defined $(\mathfrak{g}, \mathfrak{k})$-module of finite type and $\Gamma_{\mathfrak{k},0}(\cdot^*)$ is an involution on the category of $(\mathfrak{g}, \mathfrak{k})$-modules of finite type. We put $\Gamma_{\mathfrak{k},0}(M^*) := M^*_{\mathfrak{k}}$. There is an obvious $\mathfrak{g}$-invariant non-degenerate pairing $M \times M^*_{\mathfrak{k}} \to \mathbb{C}$.

The following five statements are recollections of the main results of [PZ2] (Theorem 2 through Corollary 4).

Theorem 2.5. *Assume that* $V(\mu)$ *is a generic* $\mathfrak{k}$*-type and that* $\mathfrak{p} = \mathfrak{p}_{\mu+2\rho}$ *(*μ *is necessarily* $\mathfrak{b}_{\mathfrak{k}}$*-dominant and* $\mathfrak{k}$*-integral).*

a) $F^i(\mathfrak{k}, \mathfrak{p}, E) = 0$ *for* $i \neq s := \dim \mathfrak{n}_{\mathfrak{k}}$ *.*
b) There is a $\mathfrak{k}$*-module isomorphism*

$$F^s(\mathfrak{k}, \mathfrak{p}, E)[\mu] \cong \mathbb{C}^{\dim E} \otimes V(\mu),$$

and $V(\mu)$ *is the unique minimal* $\mathfrak{k}$*-type of* $F^s(\mathfrak{k}, \mathfrak{p}, E)$*.*

c) *Let* $\bar{F}^s(\mathfrak{k},\mathfrak{p},E)$ *be the* $\mathfrak{g}$*-submodule of* $F^s(\mathfrak{k},\mathfrak{p},E)$ *generated by* $F^s(\mathfrak{k},\mathfrak{p},E)[\mu]$*. Then* $\bar{F}^s(\mathfrak{k},\mathfrak{p},E)$ *is simple and* $\bar{F}^s(\mathfrak{k},\mathfrak{p},E) = \mathrm{Soc}F^s(\mathfrak{k},\mathfrak{p},E)$*. Moreover,* $F^s(\mathfrak{k},\mathfrak{p},E)$ *is cogenerated by* $F^s(\mathfrak{k},\mathfrak{p},E)[\mu]$*. This implies that* $F^s(\mathfrak{k},\mathfrak{p},E)^*_{\mathfrak{t}}$ *is generated by* $F^s(\mathfrak{k},\mathfrak{p},E)^*_{\mathfrak{t}}[w_m(-\mu)]$*, where* $w_m \in W_{\mathfrak{k}}$ *is the element of maximal length in the Weyl group* $W_{\mathfrak{k}}$ *of* $\mathfrak{k}$*.*

d) *For any non-zero* $\mathfrak{g}$*-submodule* M *of* $F^s(\mathfrak{k},\mathfrak{p},E)$ *there is an isomorphism of* $\mathfrak{m}$*-modules*

$$H^r(\mathfrak{n},M)^{\omega} \cong E.$$

Theorem 2.6. *Let* M *be a simple* $(\mathfrak{g},\mathfrak{k})-$*module of finite type with minimal* $\mathfrak{k}-$*type* $V(\mu)$ *which is generic . Then* $\mathfrak{p} := \mathfrak{p}_{\mu+2\rho} = \mathfrak{m}\supset\!\!\!\!+\mathfrak{n}$ *is a minimal compatible parabolic subalgebra. Let* $\omega := \mu - 2\rho_n^{\perp}$ *(recall that* $\rho_{\mathfrak{n}}^{\perp} = \rho_{\mathrm{ch}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^{\perp})}$*), and let* E *be the* $\mathfrak{p}-$*module* $H^r(\mathfrak{n},M)^{\omega}$ *with trivial* $\mathfrak{n}-$*action, where* $r = \dim(\mathfrak{n}\cap\mathfrak{k}^{\perp})$*. Then* E *is a simple* $\mathfrak{p}-$*module, the pair* $(\mathfrak{p},E)$ *satisfies the hypotheses of Theorem 2.5, and* M *is canonically isomorphic to* $\bar{F}^s(\mathfrak{p},E)$ *for* $s = \dim(\mathfrak{n}\cap\mathfrak{k})$*.*

Corollary 2.7. *(Generic version of a theorem of Harish-Chandra). There exist at most finitely many simple* $(\mathfrak{g},\mathfrak{k})-$*modules* M *of finite type with a fixed* $Z_{U(\mathfrak{g})}-$*character such that a minimal* $\mathfrak{k}-$*type of* M *is generic. (Moreover, each such* M *has a unique minimal* $\mathfrak{k}-$*type by Theorem 2.5 b).)*

Proof By Theorems 2.1 a) and 2.6, if M is a simple $(\mathfrak{g},\mathfrak{k})-$module of finite type with generic minimal $\mathfrak{k}-$type $V(\mu)$ for some μ, then the $Z_{U(\mathfrak{g})}-$character of M is $\theta_{\nu+\tilde{\rho}}$. There are finitely many Borel subalgebras $\mathfrak{b}$ as in Theorem 2.1 a); thus, if $\theta_{\nu+\tilde{\rho}}$ is fixed, there are finitely many possibilities for the weight ν (as $\theta_{\nu+\tilde{\rho}}$ determines $\nu+\tilde{\rho}$ up to a finite choice). Therefore, up to isomorphism, there are finitely many possibilities for the $\mathfrak{p}-$module E, and hence, up to isomorphism, there are finitely many possibilities for M. □

Theorem 2.8. *Assume that the pair* $(\mathfrak{g},\mathfrak{k})$ *is regular, i.e.* $\mathfrak{t}$ *contains a regular element of* $\mathfrak{g}$*. Let* M *be a simple* $(\mathfrak{g},\mathfrak{k})-$*module (a priori of infinite type) with a minimal* $\mathfrak{k}-$*type* $V(\mu)$ *which is generic. Then* M *has finite type, and hence by Theorem 2.6,* M *is canonically isomorphic to* $\bar{F}^s(\mathfrak{p},E)$ *(where* $\mathfrak{p}, E$ *and* s *are as in Theorem 2.6).*

Corollary 2.9. *Let the pair* $(\mathfrak{g},\mathfrak{k})$ *be regular.*

a) *There exist at most finitely many simple* $(\mathfrak{g},\mathfrak{k})-$*modules* M *with a fixed* $Z_{U(\mathfrak{g})}-$*character, such that a minimal* $\mathfrak{k}-$*type of* M *is generic. All such* M *are of finite type (and have a unique minimal* $\mathfrak{k}-$*type by Theorem 2.5 b)).*

b) *(Generic version of Harish-Chandra's admissibility theorem). Every simple* $(\mathfrak{g},\mathfrak{k})-$*module with a generic minimal* $\mathfrak{k}-$*type has finite type.*

Proof The proof of a) is as the proof of Corollary 2.7 but uses Theorem 2.8 instead of Theorem 2.6, and b) is a direct consequence of Theorem 2.8. □

The following statement follows from Corollary 2.4 and Theorem 2.6.

Corollary 2.10. *Let* M *be as in Theorem 2.6. If the* $\mathfrak{b}-$*highest weight of* E *is not* $\mathfrak{l}-$*integral for any reductive subalgebra* $\mathfrak{l}$ *with* $\tilde{\mathfrak{k}} \subset \mathfrak{l} \subseteq \mathfrak{g}$*, then* $\tilde{\mathfrak{k}}$ *is a maximal reductive subalgebra of* $\mathfrak{g}[M]$*.*

Definition 2.11. *Let* $\mathfrak{p} \supset \mathfrak{b}_{\mathfrak{k}}$ *be a minimal* $\mathfrak{t}-$*compatible parabolic subalgebra and let* E *be a simple finite dimensional* $\mathfrak{p}-$*module on which* $\mathfrak{t}$ *acts by* ω*. We say that the pair* $(\mathfrak{p},E)$ *is allowable if* $\mu = \omega + 2\rho_{\mathfrak{n}}^{\perp}$ *is dominant integral for* $\mathfrak{k}$*,* $\mathfrak{p}_{\mu+2\rho} = \mathfrak{p}$*, and* $V(\mu)$ *is generic.*

Theorem 2.6 provides a classification of simple $(\mathfrak{g},\mathfrak{k})-$modules with generic minimal $\mathfrak{k}-$type in terms of allowable pairs. Note that for any minimal $\mathfrak{t}-$compatible parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}_{\mathfrak{k}}$, there exists a $\mathfrak{p}-$module E such that $(\mathfrak{p},E)$ is allowable.

3. The case $\mathfrak{k} \simeq \mathfrak{sl}(2)$

Let $\mathfrak{k} \simeq \mathfrak{sl}(2)$. In this case there is only one minimal $\mathfrak{t}-$compatible parabolic subalgebra $\mathfrak{p} = \mathfrak{m}\supset\!\!\!\!+\mathfrak{n}$ of $\mathfrak{g}$ which contains $\mathfrak{b}_{\mathfrak{k}}$. Furthermore, we can identify the elements of $\mathfrak{t}^*$ with complex numbers, and the $\mathfrak{b}_{\mathfrak{k}}-$dominant integral weights of $\mathfrak{t}$ in $\mathfrak{n}\cap\mathfrak{k}^{\perp}$ with non-negative integers. It is shown in [PZ2] that in this case the genericity assumption on a $\mathfrak{k}-$type $V(\mu)$, $\mu \geq 0$, amounts to the condition $\mu \geq \Gamma := \tilde{\rho}(h) - 1$ where $h \in \mathfrak{h}$ is the semisimple element in a standard basis e, h, f of $\mathfrak{k} \simeq \mathfrak{sl}(2)$.

In our work [PZ5] we have proved a different sufficient condition for the main results of [PZ2] to hold when $\mathfrak{k} \simeq \mathfrak{sl}(2)$. Let λ_1 and λ_2 be the maximum and submaximum weights of $\mathfrak{t}$ in $\mathfrak{n}\cap\mathfrak{k}^{\perp}$ (if λ_1 has multiplicity at least two in $\mathfrak{n}\cap\mathfrak{k}^{\perp}$, then $\lambda_2 = \lambda_1$; if $\dim\mathfrak{n}\cap\mathfrak{k}^{\perp} = 1$, then $\lambda_2 = 0$). Set $\Lambda := \frac{\lambda_1+\lambda_2}{2}$.

Theorem 3.1. *If $\mathfrak{k} \simeq \mathfrak{sl}(2)$, all statements of section 2 from Theorem 2.5 through Corollary 2.9 hold if we replace the assumption that μ is generic by the assumption $\mu \geq \Lambda$. As a consequence, the isomorphism classes of simple $(\mathfrak{g}, \mathfrak{k})$−modules whose minimal $\mathfrak{k}$−type is $V(\mu)$ with $\mu \geq \Lambda$ are parameterized by the isomorphism classes of simple $\mathfrak{p}$−modules E on which $\mathfrak{t}$ acts via $\mu - 2\rho_{\mathfrak{n}}^{\perp}$.*

The $\mathfrak{sl}(2)$−subalgebras of a simple Lie algebra are classified (up to conjugation) by Dynkin in [D]. We will now illustrate the computation of the bound Λ as well as the genericity condition on μ in examples.

We first consider three types of $\mathfrak{sl}(2)$−subalgebras of a simple Lie algebra: long root−$\mathfrak{sl}(2)$, short root−$\mathfrak{sl}(2)$ and principal $\mathfrak{sl}(2)$ (of course, there are short roots only for the series B, C and for G_2 and F_4). We compare the bounds Λ and Γ in the following table.

	long root	short root	principal
$A_n, n \geq 2$	$\Gamma = n - 1 \geq 1 = \Lambda$	not applicable	$\Gamma = \frac{n(n+1)(n+2)}{6} - 1 \geq 2n - 1 = \Lambda$
$B_n, n \geq 2$	$\Gamma = 2n - 3 \geq 1 = \Lambda$	$\Gamma = 2n - 2 \geq 2 = \Lambda$	$\Gamma = \frac{n(n+1)(4n-1)}{6} - 1 > 4n - 3 = \Lambda$
$C_n, n \geq 3$	$\Gamma = n - 1 > 1 = \Lambda$	$\Gamma = 2n - 2 > 2 = \Lambda$	$\Gamma = \frac{n(n+1)(2n+1)}{3} - 1 > 4n - 3 = \Lambda$
$D_n, n \geq 4$	$\Gamma = 2n - 4 > 1 = \Lambda$	not applicable	$\Gamma = \frac{2(n-1)n(n+1)}{3} - 1 > 4n - 7 = \Lambda$
E_6	$\Gamma = 10 > 1 = \Lambda$	not applicable	$\Gamma = 155 > 21 = \Lambda$
E_7	$\Gamma = 16 > 1 = \Lambda$	not applicable	$\Gamma = 398 > 33 = \Lambda$
E_8	$\Gamma = 28 > 1 = \Lambda$	not applicable	$\Gamma = 1239 > 57 = \Lambda$
F_4	$\Gamma = 7 > 1 = \Lambda$	$\Gamma = 10 > 2 = \Lambda$	$\Gamma = 109 > 21 = \Lambda$
G_2	$\Gamma = 2 > 1 = \Lambda$	$\Gamma = 4 > 3 = \Lambda$	$\Gamma = 15 > 9 = \Lambda$

Table A

Let's discuss the case $\mathfrak{g} = F_4$ in more detail. Recall that the *Dynkin index* of a semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is the quotient of the normalized $\mathfrak{g}$−invariant summetic bilinear form on $\mathfrak{g}$ restricted to $\mathfrak{s}$ and the normalized $\mathfrak{s}$−invariant symmetric bilinear form on $\mathfrak{s}$, where for both $\mathfrak{g}$ and $\mathfrak{s}$ the square length of a long root equals 2. According to Dynkin [D], the conjugacy class of an $\mathfrak{sl}(2)$−subalgebra $\mathfrak{k}$ of F_4 is determined by the Dynkin index of $\mathfrak{k}$ in F_4. Moreover, for $\mathfrak{g} = F_4$ the following integers are Dynkin indices of $\mathfrak{sl}(2)$−subalgebras: 1(long root), 2(short root), $3, 4, 6, 8, 9, 10, 11, 12, 28, 35, 36, 60, 156$. The bounds Λ and Γ are given in the following table.

Dynkin index	1	2	3
	$\Gamma = 7 > 1 = \Lambda$	$\Gamma = 10 > 2 = \Lambda$	$\Gamma = 14 > 3 = \Lambda$
Dynkin index	4	6	8
	$\Gamma = 15 > 3 = \Lambda$	$\Gamma = 16 > 4 = \Lambda$	$\Gamma = 17 > 4 = \Lambda$
Dynkin index	9	10	11
	$\Gamma = 25 > 5 = \Lambda$	$\Gamma = 26 > 5 = \Lambda$	$\Gamma = 28 > 6 = \Lambda$
Dynkin index	12	28	35
	$\Gamma = 29 > 6 = \Lambda$	$\Gamma = 45 > 9 = \Lambda$	$\Gamma = 50 > 10 = \Lambda$
Dynkin index	36	60	156
	$\Gamma = 51 > 10 = \Lambda$	$\Gamma = 67 > 13 = \Lambda$	$\Gamma = 109 > 21 = \Lambda$

Table B

We conclude this section by recalling a conjecture from [PZ5]. Let $C_{\bar{\mathfrak{p}},\mathfrak{t},n}$ denote the full subcategory of $\mathfrak{g}$−mod consisting of finite-length modules with simple subquotients which are $\bar{\mathfrak{p}}$−locally finite $(\mathfrak{g}, \mathfrak{t})$−modules N whose $\mathfrak{t}$−weight spaces N^{β}, $\beta \in \mathbb{Z}$, satisfy $\beta \geq n$. Let $C_{\mathfrak{k},n}$ be the full subcategory of $\mathfrak{g}$−mod consisting of finite length modules whose simple subquotients are $(\mathfrak{g}, \mathfrak{k})$−modules with minimal $\mathfrak{k} \simeq \mathfrak{sl}(2)$−type $V(\mu)$ for $\mu \geq n$. We show in [PZ5] that the functor $R^1\Gamma_{\mathfrak{k},\mathfrak{t}}$ is a well-defined fully faithful functor from $C_{\mathfrak{p},\mathfrak{t},n+2}$ to $C_{\mathfrak{k},n}$ for $n \geq 0$. Moreover, we make the following conjecture.

Conjecture 3.2. *Let $n \geq \Lambda$. Then $R^1\Gamma_{\mathfrak{k},\mathfrak{t}}$ is an equivalence between the categories $\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ and $\mathcal{C}_{\mathfrak{k},n}$.*

We have proof of this conjecture for $\mathfrak{g} \simeq \mathfrak{sl}(2)$ and, jointly with V. Serganova, for $\mathfrak{g} \simeq \mathfrak{sl}(3)$.

4. Eligible subalgebras

In what follows we adopt the following terminology. A *root subalgebra* of $\mathfrak{g}$ is a subalgebra which contains a Cartan subalgebra of $\mathfrak{g}$. An *r-subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{l}$ whose root spaces (with respect to a Cartan subalgebra of $\mathfrak{l}$) are root spaces of $\mathfrak{g}$. The notion of *r*-subalgebra goes back to [D]. A root subalgebra is, of course, an *r*-subalgebra.

We now give the following key definition.

Definition 4.1. *An algebraic reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ is eligible if $C(\mathfrak{t}) = \mathfrak{t} + C(\mathfrak{k})$.*

Note that in the above definition one can replace $\mathfrak{t}$ with any Cartan subalgebra of $\mathfrak{k}$. Furthermore, if $\mathfrak{k}$ is eligible then $\mathfrak{h} \subset C(\mathfrak{t}) = \mathfrak{t} + C(\mathfrak{k}) \subset \tilde{\mathfrak{k}} = \mathfrak{k} + C(\mathfrak{k})$, i.e. $\mathfrak{h}$ is a Cartan subalgebra of both $\tilde{\mathfrak{k}}$ and $\mathfrak{g}$. In particular, $\tilde{\mathfrak{k}}$ is a reductive root subalgebra of $\mathfrak{g}$. As $\mathfrak{k}$ is an ideal in $\tilde{\mathfrak{k}}$, $\mathfrak{k}$ is an *r*-subalgebra of $\mathfrak{g}$.

Proposition 4.2. *Assume $\mathfrak{k}$ is an r−subalgebra of $\mathfrak{g}$. The following three conditions are equivalent:*

(i) $\mathfrak{k}$ *is eligible;*
(ii) $C(\mathfrak{k})_{ss} = C(\mathfrak{t})_{ss}$*;*
(iii) $\dim C(\mathfrak{k})_{ss} = \dim C(\mathfrak{t})_{ss}$.

Proof The implications (i)⇒(ii)⇒(iii) are obvious. To see that (iii) implies (i), observe that if $\mathfrak{k}$ is an r−subalgebra of $\mathfrak{g}$, then $\mathfrak{h} \subseteq \mathfrak{t} + C(\mathfrak{k}) \subseteq C(\mathfrak{t})$. Therefore the inclusion $\mathfrak{t} + C(\mathfrak{k}) \subseteq C(\mathfrak{t})$ is proper if and only if $\mathfrak{g}^{\pm\alpha} \in C(\mathfrak{t})\backslash C(\mathfrak{k})$ for some root $\alpha \in \Delta$, or, equivalently, if the inclusion $C(\mathfrak{k})_{ss} \subseteq C(\mathfrak{t})_{ss}$ is proper. □

An algebraic, reductive in $\mathfrak{g}$, *r*-subalgebra $\mathfrak{k}$ may or may not be eligible. If $\mathfrak{k}$ is a root subalgebra, then $\mathfrak{k}$ is always eligible. If $\mathfrak{g}$ is simple of types A, C, D and $\mathfrak{k}$ is a semisimple *r*-subalgebra, then $\mathfrak{k}$ is necessarily eligible. In general, a semisimple *r*-subalgebra is eligible if and only if the roots of $\mathfrak{g}$ which vanish on $\mathfrak{t}$ are strongly orthogonal to the roots of $\mathfrak{k}$. For example, if $\mathfrak{g}$ is simple of type B and $\mathfrak{k}$ is a simple *r*-subalgebra of type B of rank less or equal than $\mathrm{rk}\mathfrak{g} - 2$, then $C(\mathfrak{k})_{ss}$ is simple of type D whereas $C(\mathfrak{t})_{ss}$ is simple of type B. Hence in this case $\mathfrak{k}$ is not eligible.

Note, however that any semisimple *r*-subalgebra $\mathfrak{k}'$ can be extended to an eligible subalgebra $\mathfrak{k}$ just by setting $\mathfrak{k} := \mathfrak{k}' + \mathfrak{h}_{C(\mathfrak{k}')}$ where $\mathfrak{h}_{C(\mathfrak{k}')}$ is a Cartan subalgebra of $C(\mathfrak{k}')$. Finally, note that if x is any algebraic regular semisimple element of $C(\mathfrak{k}')$, then $\mathfrak{k} := \mathfrak{k}' \oplus Z(C(\mathfrak{k}')) + \mathbb{C}x$ is an eligible subalgebra of $\mathfrak{g}$. Indeed, if $\mathfrak{t}' \subseteq \mathfrak{k}'$ is a Cartan subalgebra of $\mathfrak{k}'$, and $\mathfrak{h}_{\mathfrak{k}} := \mathfrak{t}' \oplus Z(C(\mathfrak{k}')) + \mathbb{C}x$ is the corresponding Cartan subalgebra of $\mathfrak{k}$, then $C(\mathfrak{h}_{\mathfrak{k}})$ is a Cartan subalgebra of $\mathfrak{g}$. Hence,

$$C(\mathfrak{h}_{\mathfrak{k}}) = \mathfrak{h}_{\mathfrak{k}} + C(\mathfrak{k}) \tag{3}$$

as the right-hand side of (3) necessarily contains a Cartan subalgebra of $\mathfrak{g}$.

To any eligible subalgebra $\mathfrak{k}$ we assign a unique weight $\varkappa \in \mathfrak{h}^*$ (the "canonical weight associated with $\mathfrak{k}$"). It is defined by the conditions $\varkappa|_{(\mathfrak{h}\cap\mathfrak{k}_{ss})} = \rho$, $\varkappa|_{(\mathfrak{h}\cap C(\mathfrak{t}))} = 0$.

5. The generalized discrete series

In what follows we assume that $\mathfrak{k}$ is eligible and $\mathfrak{h} \subset \tilde{\mathfrak{k}}$. In this case $\mathfrak{h}$ is a Cartan subalgebra both of $\tilde{\mathfrak{k}}$ and $\mathfrak{g}$. Let $\lambda \in \mathfrak{h}^*$ and set $\gamma := \lambda|_{\mathfrak{t}}$. Assume that $\mathfrak{m} := \mathfrak{m}_\gamma = C(\mathfrak{t})$. Assume furthermore that λ is $\mathfrak{m}$-integral and let E_λ be a simple finite-dimensional $\mathfrak{m}$-module with $\mathfrak{b}$−highest weight λ. Then

$$D(\mathfrak{k}, \lambda) := F^s(\mathfrak{k}, \mathfrak{p}_\gamma, E_\lambda \otimes \Lambda^{\dim \mathfrak{n}_\gamma}(\mathfrak{n}_\gamma^*))$$

is by definition a *generalized discrete series module*.

Note that since $D(\mathfrak{k}, \lambda)$ is a fundamental series module, Theorem 2.1 applies to $D(\mathfrak{k}, \lambda)$. In the case when $\mathfrak{k}$ is a root subalgebra and λ is regular, we have $\lambda = \gamma$ and $\mathfrak{p}_\gamma$ is a Borel subalgebra of $\mathfrak{g}$ which we denote by $\mathfrak{b}_\lambda$. Then $D(\mathfrak{k}, \lambda) = R^s\Gamma_{\mathfrak{k},\mathfrak{h}}(\Gamma_{\mathfrak{h}}(\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{b}_\lambda} E_\lambda))$, i.e. $D(\mathfrak{k}, \lambda)$ is cohomologically co-induced from a 1−dimensional $\mathfrak{b}_\lambda$−module. If in addition, $\mathfrak{k}$ is a symmetric subalgebra, λ is $\mathfrak{k}$-integral, and $\lambda - \tilde{\rho}$ is $\mathfrak{b}_\lambda$−dominant regular, then $D(\mathfrak{k}, \lambda)$ is a $(\mathfrak{g}, \mathfrak{k})$-module in Harish-Chandra's discrete series, see [KV], Ch.XI.

Suppose $\mathfrak{k}$ is eligible but $\mathfrak{k}$ is not a root subalgebra. Suppose further that $\tilde{\mathfrak{k}}$ is symmetric. Any simple subquotient M of $D(\mathfrak{k}, \lambda)$ is a $(\mathfrak{g}, \tilde{\mathfrak{k}})-$module and thus a Harish-Chandra module for $(\mathfrak{g}, \tilde{\mathfrak{k}})$. However, M may or may not be in the discrete series of $(\mathfrak{g}, \tilde{\mathfrak{k}})-$modules. This becomes clear in Theorem 5.6 below.

Our first result is a sharper version of the main result of [PZ3] for an eligible $\mathfrak{k}$.

Theorem 5.1. *Let $\mathfrak{k} \subseteq \mathfrak{g}$ be eligible. Assume that $\lambda - 2\varkappa$ is $\tilde{\mathfrak{k}}$-integral and dominant. Then, $D(\mathfrak{k}, \lambda) \neq 0$. Moreover, if we set $\mu := (\lambda - 2\varkappa)|_{\mathfrak{t}}$, then $V(\mu)$ is the unique minimal $\mathfrak{k}-$type of $D(\mathfrak{k}, \lambda)$. Finally, there are isomorphisms of simple finite-dimensional $\tilde{\mathfrak{k}}-$modules*

$$D(\mathfrak{k}, \lambda)[\mu] \cong D(\mathfrak{k}, \lambda)\langle \lambda - 2\varkappa \rangle \simeq V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa).$$

Proof Note that $\mu = \gamma - 2\rho$. By Lemma 2 in [PZ3]

$$\dim \mathrm{Hom}_{\mathfrak{k}}(V(\mu), D(\mathfrak{k}, \lambda)) = \dim E_\lambda,$$

and hence $D(\mathfrak{k}, \lambda) \neq 0$. In addition, $V(\mu)$ is the unique minimal $\mathfrak{k}-$type of $D(\mathfrak{k}, \lambda)$. By construction, $D(\mathfrak{k}, \lambda)[\mu]$ is a finite-dimensional $\tilde{\mathfrak{k}}-$module. We will use Theorem 2.1 c) to compute $D(\mathfrak{k}, \lambda)[\mu]$ as a $\tilde{\mathfrak{k}}-$module. Since $\mathfrak{k}$ is eligible, we have $\mathfrak{m} = \mathfrak{t} + C(\mathfrak{k})$. As $[\mathfrak{t}, C(\mathfrak{k})] = 0$ and $\mathfrak{t}$ is toral, the restriction of E_λ to $C(\mathfrak{k})$ is simple. We have

$$\tilde{\mathfrak{k}} = \mathfrak{k}_{ss} \oplus C(\mathfrak{k}),$$

and hence there is an isomorphism of $\tilde{\mathfrak{k}}-$modules

$$V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa) \cong (V(\mu)|_{\mathfrak{k}_{ss}}) \boxtimes E_\lambda.$$

Consequently, we have isomorphisms of $C(\mathfrak{k})-$modules

$$\mathrm{Hom}_{\mathfrak{k}}(V(\mu), V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa)) \cong \mathrm{Hom}_{\mathfrak{k}_{ss}}((V(\mu)|_{\mathfrak{k}_{ss}}), V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa)) \cong E_\lambda. \quad (4)$$

Write $\mathfrak{p}_\gamma = \mathfrak{p}$ and note that $\tilde{\mathfrak{k}} \cap \mathfrak{m} = \mathfrak{m}$. By Theorem 2.1 c), we have a canonical isomorphism

$$D(\mathfrak{k}, \lambda) \cong R^s\Gamma_{\tilde{\mathfrak{k}},\mathfrak{m}}(\Gamma_{\mathfrak{m},0}(\mathrm{pro}_{\mathfrak{p}}^{\mathfrak{g}} E_\lambda)).$$

According to the theory of the bottom layer [KV], Ch.V, Sec.6, $D(\mathfrak{k}, \lambda)$ contains the $\tilde{\mathfrak{k}}-$module

$$R^s\Gamma_{\tilde{\mathfrak{k}},\mathfrak{m}}(\Gamma_{\mathfrak{m},0}(\mathrm{pro}_{\tilde{\mathfrak{k}}\cap\mathfrak{p}}^{\tilde{\mathfrak{k}}} E_\lambda))$$

which is in turn isomorphic to $V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa)$.

By the above argument, we have a sequence of injections

$$V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa) \hookrightarrow D(\mathfrak{k}, \lambda)\langle \lambda - 2\varkappa \rangle \hookrightarrow D(\mathfrak{k}, \lambda)[\mu].$$

We conclude from (4) that the above sequence of injections is in fact a sequence of isomorphisms of simple $\tilde{\mathfrak{k}}-$modules. □

Corollary 5.2. *Under the assumptions of Theorem 5.1, there exists a simple $(\mathfrak{g}, \mathfrak{k})-$module M of finite type over $\mathfrak{k}$, such that if $V(\mu_M)$ is a minimal $\mathfrak{k}-$type of M, then $V(\mu_M)$ is the unique minimal $\mathfrak{k}-$type of M and there is an isomorphism of finite-dimensional $\tilde{\mathfrak{k}}-$modules*

$$M[\mu_M] \cong V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa).$$

In particular, $M[\mu_M]$ is a simple $\tilde{\mathfrak{k}}-$submodule of M.

Proof First we construct a module M as required. Let $\bar{D}(\mathfrak{k}, \lambda)$ be the $U(\mathfrak{g})-$submodule of $D(\mathfrak{k}, \lambda)$ generated by the $\tilde{\mathfrak{k}}-$isotypic component $D(\mathfrak{k}, \lambda)\langle \lambda - 2\varkappa \rangle$. Suppose N is a proper $\mathfrak{g}-$submodule of $\bar{D}(\mathfrak{k}, \lambda)$. Since $D(\mathfrak{k}, \lambda)\langle \lambda - 2\varkappa \rangle$ is simple over $\tilde{\mathfrak{k}}$,

$$N \cap (D(\mathfrak{k}, \lambda)\langle \lambda - 2\varkappa \rangle) = 0.$$

Thus, if $N(\mathfrak{k}, \lambda)$ is the maximum proper submodule of $\bar{D}(\mathfrak{k}, \lambda)$, the quotient module

$$M = \bar{D}(\mathfrak{k}, \lambda)/N(\mathfrak{k}, \lambda)$$

is a simple $(\mathfrak{g}, \tilde{\mathfrak{k}})-$module, and M has finite type over $\mathfrak{k}$. Writing $\mu_M = \mu = \gamma - 2\rho$, we see that M has unique minimal $\mathfrak{k}-$type $V(\mu_M)$. Finally, by Theorem 5.1, we have an isomorphism of finite-dimensional $\tilde{\mathfrak{k}}-$modules,

$$M[\mu_M] \cong V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa).$$

□

If $\mathfrak{k}$ is symmetric (and hence $\tilde{\mathfrak{k}}$ is a root subalgebra due to the eligibility of $\mathfrak{k}$), Theorem 5.1 and Corollary 5.2 go back to [V] (where they are proven by a different method).

The following two statements are consequences of the main results of section 2 and Theorem 5.1.

Corollary 5.3. *Let $\mathfrak{k}$ be eligible, $\lambda \in \mathfrak{h}^*$ be such that $\lambda - 2\varkappa$ is $\tilde{\mathfrak{k}}$-integral and $V(\mu)$ is generic for $\mu := \lambda|_{\mathfrak{t}} - 2\rho$.*

a) $\mathrm{Soc}\, D(\mathfrak{k}, \lambda)$ *is a simple $(\mathfrak{g}, \mathfrak{k})$-module with unique minimal $\mathfrak{k}$-type $V(\mu)$.*

b) There is a canonical isomorphism of $C(\mathfrak{k})$-modules

$$\mathrm{Hom}_{\mathfrak{k}}(V(\mu), \mathrm{Soc}\, D(\mathfrak{k}, \lambda)) \simeq E_\lambda.$$

c) There is a canonical isomorphism of $\tilde{\mathfrak{k}}$-modules

$$V(\mu) \otimes \mathrm{Hom}_{\mathfrak{k}}(V(\mu), \mathrm{Soc}\, D(\mathfrak{k}, \lambda)) \simeq V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa),$$

i.e. the $V(\mu)$-isotypic component of $\mathrm{Soc}D(\mathfrak{k}, \lambda)$ *is a simple $\tilde{\mathfrak{k}}$-module isomorphic to $V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa)$.*

d) If $\lambda - 2\varkappa$ is not $\mathfrak{l}-$integral for any reductive subalgebra $\mathfrak{l}$ such that $\tilde{\mathfrak{k}} \subset \mathfrak{l} \subseteq \mathfrak{g}$, then $\tilde{\mathfrak{k}}$ is a maximal reductive subalgebra of $\mathfrak{g}[M]$ for any subquotient M of $D(\mathfrak{k}, \lambda)$, in particular of $\mathrm{Soc}\, D(\mathfrak{k}, \lambda)$.

Proof

a) Observe that $\mathfrak{p}_\gamma = \mathfrak{p}_{\mu+2\rho}$, and $D(\mathfrak{k}, \lambda) = F^s(\mathfrak{k}, \mathfrak{p}_{\mu+2\rho}, E_\lambda \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}^*))$. So, a) follows from Theorem 2.5 c).

b) By Theorem 2.5 c), $\mathrm{Hom}_{\mathfrak{k}}(V(\mu), \mathrm{Soc}\, D(\mathfrak{k}, \lambda)) = \mathrm{Hom}_{\mathfrak{k}}(V(\mu), D(\mathfrak{k}, \lambda))$, which in turn is isomorphic to $\mathrm{Hom}_{\mathfrak{k}}(V(\mu), V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa))$ by Theorem 5.1. The desired isomorphism follows now from (4).

c) This follows from the isomorphism in b) and the isomorphism $V(\mu) \otimes E_\lambda \cong V_{\tilde{\mathfrak{k}}}(\lambda - 2\varkappa)$ of $\tilde{\mathfrak{k}}-$modules.

d) Follows from Corollary 2.4. Note that, since $\mathfrak{k}$ is eligible, $\tilde{\mathfrak{k}}$ is a root subalgebra and the condition that $\lambda - 2\varkappa$ be not $\mathfrak{l}-$integral involves only finitely many subalgebras $\mathfrak{l}$. □

Corollary 5.4. *Let $\mathfrak{k}$ be eligible and let $V(\mu)$ be a generic $\mathfrak{k}-$type.*

a) Let M be a simple $(\mathfrak{g}, \mathfrak{k})-$module of finite type with minimal $\mathfrak{k}-$type $V(\mu)$. Then $M[\mu]$ is a simple finite-dimensional $\tilde{\mathfrak{k}}-$module isomorphic to $V_{\tilde{\mathfrak{k}}}(\lambda)$ for some weight $\lambda \in \mathfrak{h}^*$ such that $\lambda|_{\mathfrak{t}} = \mu + 2\rho$ and $\mu - 2\varkappa$ is $\tilde{\mathfrak{k}}-$integral. Moreover,

$$M \cong \mathrm{Soc}\, D(\mathfrak{k}, \lambda).$$

If in addition λ is not $\mathfrak{l}-$integral for any reductive subalgebra $\mathfrak{l}$ with $\tilde{\mathfrak{k}} \subset \mathfrak{l} \subseteq \mathfrak{g}$, then $\tilde{\mathfrak{k}}$ is a unique maximal reductive subalgebra of $\mathfrak{g}[M]$.

b) If $\mathfrak{k}$ is regular in $\mathfrak{g}$, then a) holds for any simple $(\mathfrak{g}, \mathfrak{k})-$module with generic minimal $\mathfrak{k}-$type $V(\mu)$. In particular M has finite type over $\mathfrak{k}$.

Proof

a) We apply Theorem 2.6. Since $V(\mu)$ is generic, $\mathfrak{p} = \mathfrak{p}_{\mu+2\rho} = \mathfrak{m} \supset\!\!\!\!+ \mathfrak{n}$ is a minimal $\mathfrak{t}-$compatible parabolic subalgebra. Let $\omega := \mu - 2\rho_{\mathfrak{n}}^\perp$ (recall that $\rho_{\mathfrak{n}}^\perp = \rho_{\mathfrak{n}} - \rho$) and let Q be the $\mathfrak{m}-$module $H^r(\mathfrak{n}, M)^\omega$ where $r = \dim(\mathfrak{k}^\perp \cap \mathfrak{n})$.

Observe that Q is a simple $\mathfrak{m}-$module and M is canonically isomorphic to $\bar{F}^s(\mathfrak{p}, Q) = \mathrm{Soc}\, F^s(\mathfrak{p}, Q)$. Let $\lambda \in \mathfrak{h}^*$ be so that $\lambda - 2\tilde{\rho}_{\mathfrak{n}}$ is an extreme weight of $\mathfrak{h}$ in Q. Thus, $F^s(\mathfrak{p}, Q) = F^s(\mathfrak{p}, E_\lambda \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n}^*)) = D(\mathfrak{k}, \lambda)$. Finally, $M \cong \mathrm{Soc}\, D(\mathfrak{k}, \lambda)$, and $\lambda|_{\mathfrak{t}} = \mu + 2\rho$. It follows that $\lambda - 2\varkappa$ is both $\mathfrak{k}-$integral and $C(\mathfrak{k})-$integral. Hence, the weight $\lambda - 2\varkappa$ is $\tilde{\mathfrak{k}}-$integral.

b) We apply Theorem 2.8. □

Corollary 5.5. *If $\mathfrak{k} \simeq \mathfrak{sl}(2)$, the genericity assumption on $V(\mu)$ in Corollaries 5.3 and 5.4 can be replaced by the assumption $\mu \geq \Lambda$.*

Proof The statement follows directly from Theorem 3.1. □

We conclude this paper by discussing in more detail an example of an eligible $\mathfrak{sl}(2)-$subalgebra. Note first that if $\mathfrak{g}$ is any simple Lie algebra and $\mathfrak{k}$ is a long root $\mathfrak{sl}(2)-$subalgebra, then the pair $(\mathfrak{g}, \tilde{\mathfrak{k}})$ is a symmetric pair. This is a well-known fact and it implies in particular that any $(\mathfrak{g}, \mathfrak{k})-$module of finite type and of finite length is a Harish-Chandra module for the pair $(\mathfrak{g}, \tilde{\mathfrak{k}})$. The latter modules are classified under the assumption of simplicity see [KV], Ch.XI; however, in general, it is an open problem to determine which simple $(\mathfrak{g}, \tilde{\mathfrak{k}})-$modules have finite type over $\mathfrak{k}$. Without having been explicitly stated, this problem has been discussed in the literature, see [OW] and the references therein. On the other hand, in this case $\Lambda = 1$, hence Corollaries 5.4 and 5.5 provide a classification of simple $(\mathfrak{g}, \mathfrak{k})-$modules of finite type with minimal $\mathfrak{k}-$types $V(\mu)$ for $\mu \geq 1$. So the above problem reduces to matching the above two classifications in the case $\mu \geq 1$, and finding all simple $(\mathfrak{g}, \mathfrak{k})-$modules of finite type whose minimal $\mathfrak{k}-$type equals $V(0)$ among the simple Harish-Chandra modules for the pair $(\mathfrak{g}, \tilde{\mathfrak{k}})$. We do this here in a special case.

Let $\mathfrak{g} = \mathfrak{sp}(2n+2)$ for $n \geq 2$. By assumption, $\mathfrak{k}$ is a long root $\mathfrak{sl}(2)-$subalgebra, and $\tilde{\mathfrak{k}} \simeq \mathfrak{sp}(2n) \oplus \mathfrak{k}$. Consider simple $(\mathfrak{g}, \tilde{\mathfrak{k}})-$modules with $Z_{U(\mathfrak{g})}-$character equal to the character of a trivial module. According to the Langlards classification, there are precisely $(n+1)^2$ pairwise non-isomorphic such modules, one of which is the trivial module. Following [Co] (see figure 4.5 on page 93) we enumerate them as σ_t for $0 \leq t \leq n$ and σ_{ij} for $0 \leq i \leq n-1, 1 \leq j \leq 2n, i < j, i+j \leq 2n$. The modules σ_t are discrete series modules. The modules σ_{ij} are Langlands quotients of the principal series (all of them are proper quotients in this case).

We announce the following result which we intend to prove elsewhere.

Theorem 5.6. *Let $\mathfrak{g} = \mathfrak{sp}(2n+2)$ for $n \geq 2$ and $\mathfrak{k}$ be a long root $\mathfrak{sl}(2)-$subalgebra.*

a) Any simple $(\mathfrak{g}, \mathfrak{k})-$module of finite type is isomorphic to a subquotient of the generalized discrete series module $D(\mathfrak{k}, \lambda)$ for some $\tilde{\mathfrak{k}} = \mathfrak{sp}(2n) \oplus \mathfrak{k}-$integral weight $\lambda - 2\varkappa$.

b) The modules σ_0, σ_{0i} for $i = 1, \ldots, 2n, \sigma_{12}$ are, up to isomorphism, all of the simple $(\mathfrak{g}, \mathfrak{k})-$modules of finite type whose $Z_{U(\mathfrak{g})}$-character equals that of a trivial $\mathfrak{g}-$module. Moreover, their minimal $\mathfrak{k}-$types are as follows:

module	***minimal $\mathfrak{k}-$type***
σ_0	$V(2n)$
$\sigma_{0j}, n+1 \leq j \leq 2n$	$V(j-1)$
$\sigma_{0j}, 2 \leq j \leq n$	$V(j-2)$
σ_{01} *(trivial representation)*	$V(0)$
σ_{12}	$V(0)$

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Ivan Penkov
School of Engineering and Science
Jacobs University Bremen
Campus Ring 1, Bremen
28759, Bremen, Germany
E-mail address: `i.penkov@jacobs-university.de`

Gregg Zuckerman
Department of Mathematics
Yale University
10 Hillhouse Avenue, P.O. Box 208283
New Haven, CT 06520-8283, USA
E-mail address: `gregg.zuckerman@yale.edu`